Fatigue initiation in heterogeneous brittle materials

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Introduction

The fatigue process can be schematically divided into an initiation stage and the propagation period. In this paper, we consider brittle materials where only the initiation phase needs to be studied, since the propagation is always unstable. We suppose that the initiation is due to some initial randomly distributed defects that grow according to a generalized Paris' law and become critical for a given number of cycles and for a given load level (characterized by a maximum equivalent stress σ over a cycle). The purpose of this paper is to present a statistical approach of the initiation stage in fatigue derived from a *unified framework* of the failure probability in the case of monotonic and cyclic loadings. We will show in particular that the iso-probability curves in a Wöhler diagram, have a shape which is *independent of the initial statistical distribution of defects*. Applications of this study are to be found in the case of engineering ceramics.
The first section of this paper is devoted to the introduction of our model for the evolution of an initial defect subjected to a cyclic loading with a uniform stress. In the second section, this mesoscopic model is applied to the analysis of a structure Ω of volume V subjected to any stress field and divided into a large number of representative volume element V_0.

Mesoscopic model: Defect Distribution and its Evolution

Initial heterogeneities in the material are modelled by randomly distributed defects only characterized by their size, a. No restriction is assumed on the form of the initial distribution of defects, $f_0(a)$. In the case of cyclic loadings, the defects grow. We assume that this evolution is given by a generalized form of Paris' law :

$$\frac{da}{dN} = 0 \qquad \text{if } a < a_{th}$$
$$\frac{da}{dN} = C\,\sigma^n\,(\sqrt{a} - \sqrt{a_{th}})^m \qquad \text{if } a > a_{th} \tag{1}$$

where σ is a maximum equivalent stress over a cycle (e.g. the maximum value over a cycle of the principal tensile stress for ceramics), a is the size of defect (for instance the length of a crack in a two dimensional problem or the radius of a penny-shaped crack in a three dimensional problem (see for instance Ref. [1]), a_{th} the threshold defect size under which no evolution is possible and C, m, n are material parameters (note that m equals n in a classical Paris' law). Let us note here that we implicitly assume in our description that interactions between defects can be neglected, since the evolution law, as well as the criteria for growth and failure, only depend on the defect size and not on its environment. We also discard the possibility of nucleating new cracks during the fatigue process.

Two defect sizes are particularly important in this description: for a given stress the threshold size, a_{th}, below which the defect cannot grow, and the maximum size, a_{max}, the defect can assume before leading to an overall failure of the material. Both sizes can be connected to material intrinsic parameters using a generalized stress intensity factor,

$$K = Y\sigma\sqrt{a} \tag{2}$$

where Y depends on the shape of the modelled defect. Failure of the material is given by K_{max}, whereas no propagation occurs for K less than K_{th}. Both quantities K_{max} and K_{th} are considered as material parameters, whereas a_{max} and a_{th} depend on the imposed stress through Eq.(2).

From these considerations, we can readily write down the expression of two stresses which will bound the evolution of the material under fatigue. Let us call a_M the largest defect size present in a given sample, or volume element, subjected to a uniform stress field. Under monotonic loading, the failure will occur for a value of the stress $\sigma=\sigma_{max}$, such that

$$\sigma_{max} = \frac{K_{max}}{Y\sqrt{a_M}}. \tag{3}$$

Whereas, for cyclic loading, if the maximum stress is smaller than a threshold σ_{th},

$$\sigma_{th} = \frac{K_{th}}{Y\sqrt{a_M}} \tag{4}$$

then no defect will grow __ and thus the material will not fail __ whatever the number of cycles. In the interval $\sigma_{th} < \sigma < \sigma_u$, failure will occur after a finite number of cycles.

In the case of fatigue, the influence of the mean stress on the threshold stress can also be studied. We proposed in the evolution law to take into account the maximum stress over a cycle, i.e. the sum of the mean stress $\bar{\sigma}$ and the half amplitude stress $\Delta\sigma/2$. Thus in a Haig's diagram, the equation of the Haig's curve is given by : $\bar{\sigma} + \Delta\sigma/2 = B$, where B is a constant determined by the half amplitude stress σ_{th} obtained with a mean stress equal to zero. This assumption is supported by experimental results [2].

The defect size distribution is supposed to be characterized by a probability density function f. We do not consider other parameters such as the orientation, or the shape of the defects [3]. These features can be thought of being already integrated through the definition of an *effective* mean defect size. In the following, for the sake of simplicity, we will consider only the case of constant cyclic loadings. Generalization to more complex loadings can be done easily by integration of Eq.(1).

For cyclic loadings, function f evolves with the number of cycles and with the load level σ as parameters : f is denoted by $f_N(a,\sigma)$. Using evolution law (1), it is easy to compute $f_N(a,\sigma)$ and to relate it to the initial distribution. Integration of (1) gives the defect size, $a(N)$, after N cycles, knowing the initial size $a(0)$.

$$a(N)=a(0) \qquad \text{if } a(0)<a_{th} \qquad (5)$$

$$\varphi(\sqrt{a(N)}) - \varphi(\sqrt{a(0)}) = C\sigma^n N \qquad \text{if } a(0)>a_{th} \qquad (5')$$

where the function φ can be written

$$\varphi(x) = \frac{(x-\sqrt{a_{th}})^{1-m}((1-m)x+\sqrt{a_{th}})}{(2-m)(1-m)} \qquad (5'')$$

when $m\neq1$ and $m\neq2$. (Those two special cases have a different expression which can be written without difficulty.) It is useful to introduce a function ψ such that $a(0)= \psi(a(N), \sigma, N)$. ψ can be computed from Eqs.(5 and 5'), although a closed expression cannot be written for a general value of parameter m. Since the evolution equation (1) is deterministic, the probability to find a defect of size a after N cycles is equal to the probability to find initially a defect of size $\psi(a, \sigma, N)$. Since it is assumed that no new crack nucleates, f_N can be related to f_0 by

$$f_N(a, \sigma) = f_0(\psi(a, \sigma, N)) \frac{\partial\psi}{\partial a} \qquad (6)$$

where the coefficient $(\partial\psi/\partial a)$ comes from the change of the measure (from da to $d\psi(a)$).

Structural analysis: Expression of the Cumulative Initiation Probability

We consider a structure Ω of volume V, subjected to any stress field, which can be divided into a large number of elements of volume V_0. The latter being a representative volume element subjected to uniform stress fields. The cumulative initiation probability P_{I0} is the probability to find a defect larger than the critical defect size a_{max} . Thus for monotonic loadings,.

$$P_{I0}=\int_{a_{max}}^{\infty} f_0(x)\, dx \ . \tag{7}$$

In the case of cyclic loading we can apply the same argument, using f_N instead of f_0. Using relation (6), P_{I0} can be rewritten as :

$$P_{I0}=\int_{\psi(a_{max},\sigma, N)}^{\infty} f_0(x)\, dx \tag{8}$$

where $\psi(a_{max},\sigma, N)$ represents the initial defect size that, after N cycles and under a stress σ, reaches the critical defect size a_{max}. Hence, relation (8) has a similar form as expression (7) and constitutes a generalization to cyclic loadings [3].

If the propagation phase is unstable (e.g. for brittle materials such as engineering ceramics [4]), the cumulative initiation probability also corresponds to the cumulative failure probability : relation (8) represents whence the correlation between the defect distribution and the cumulative failure probability.

Let us also note that for structures made of brittle material and submitted to cyclic loadings, relation (7) does *not* give a decoupling of the cumulative failure probability P_{I0} in a product of a function that depends only upon the load level by a function that depends only upon the number of cycles N, as suggested in Ref. [5].

On the structural scale, the initiation occurs if one defect within the volume becomes critical. That means that if this defect reaches the critical size, a_{max}, the initiation occurs. Determining the initiation at the structural scale is equivalent to finding the "weakest" link in the structure, which is expected to be a good approximation for brittle materials with a dilute density of defects. Let us also recall that we already used the assumption that the interactions between defects could be neglected during initiation, and the hypothesis of the independence of events ([6], [7]). The cumulative initiation probability P_I of a structure Ω can be related to P_{I0} by :

$$\ln(1-P_I) = \frac{1}{V_0}\int_{\Omega} \ln(1-P_{I0})\ dV \qquad (9)$$

By means of expressions (8) and (9), a general relationship between the defect distribution and the initiation (or failure) probability of a structure Ω can be derived :

$$P_I=1-\exp\left[\frac{1}{V_0}\int_{\Omega} \ln\left\{1-\int_{\psi(a_{max},\sigma,N)}^{\infty} f_0(x)\,dx\right\} dV\right] \qquad \text{for cyclic loadings} \qquad (10)$$

For monotonic loading, the same equation holds with the substitution of $\psi(a_{max}, \sigma, N)$ into a_{max}. It should be emphasized that expression (10) also represents the cumulative failure probability if the propagation process is unstable as for engineering ceramics such as silicon nitride [4].

Iso–Cumulative Initiation Probabilities

In the case of a given fatigue loading, the iso–cumulative initiation probabilities are given, on a structure scale, by $P_I = constant$, thus $F(\psi(a_{max},\sigma,N))=constant$ where F is the cumulative frequency associated to function f_0, since the loading pattern is constant. This condition can simply be rewritten as :

$$\psi(K^2_{max}/Y^2\sigma^2,\ \sigma,\ N)= A \qquad (11)$$

where A is a constant. Let us stress this important result:
The iso-probability curves in the plane (σ, N), *(Wöhler diagram) are independent of the defect size distribution* f_0. *The latter allows to quantify the value of the probability on a given curve, but does not affect its shape.*
The iso-probability curves are solely determined by the evolution equation, Paris' law (1).

The number of cycles N corresponding to a given probability can be derived from equations (5') and (11) :

$$N = \frac{1}{C\sigma^n}\left\{\varphi(K_{max}/Y\sigma)- B\right\} \qquad (12)$$

where B is a constant equal to $B=\varphi(\sqrt{A})$. Expression (12) shows that, whatever the value of m, $dN/d\sigma < 0$, (an increase in the number of cycles, for a given probability, implies a decrease of the load). Let us note that the result (12) does not depend upon the defect distribution.

To compare two iso-probabilities (P_{I1} and P_{I2}) in a Wöhler diagram, one may study the quantity $\log(N_2/N_1)$ at a given load level. This ratio is given by

$$\frac{N_2}{N_1} = \frac{\varphi(K_{max}/Y\sigma) - B_2}{\varphi(K_{max}/Y\sigma) - B_1} \qquad (13)$$

where B_1 and B_2 are two constants ($B_2 < B_1$). Since $d(N_2/N_1)/d\sigma > 0$ whatever the value of m, the iso-probabilities diverge with the applied stress. This result shows that the higher the load level, the larger the scatter of the time to initiation (to failure). It can also be noticed that this result is independent of the defect distribution, and of the power of n in Paris' law (1).

Conclusions

In the study of initiation processes, we have presented a unified framework for brittle failure under monotonic and cyclic loading taking into account the statistical distribution of defect within the material. In the case of materials for which the propagation process is unstable, expression (10) gives a relationship between the defect distribution and the cumulative failure probability. In fatigue, we have obtained the strikingly simple result that the iso-probability curves are independent of the initial statistical defect distribution of the material.

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