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SHARP NON-ASYMPTOTIC CONCENTRATION INEQUALITIES FOR THE APPROXIMATION OF THE INVARIANT MEASURE OF A DIFFUSION

I. HONORÉ

Abstract. For an ergodic Brownian diffusion with invariant measure ν, we consider a sequence of empirical distributions $(\nu_n)_{n\geq 1}$ associated with an approximation scheme with decreasing time step $(\gamma_n)_{n\geq 1}$ along an adapted regular enough class of test functions f such that $f-\nu(f)$ is a coboundary of the infinitesimal generator $\mathcal{A}$. Denote by σ the diffusion coefficient and φ the solution of the Poisson equation $\mathcal{A}\varphi = f-\nu(f)$. When the square norm of $|\sigma^*\nabla\varphi|^2$ lies in the same coboundary class as f, we establish sharp non-asymptotic concentration bounds for suitable normalizations of $\nu_n(f)-\nu(f)$. Our bounds are optimal in the sense that they match the asymptotic limit obtained by Lamberton and Pagès in [LP02], for a certain large deviation regime. In particular, this allows us to derive sharp non-asymptotic confidence intervals. We provide as well a Slutsky like Theorem, for practical applications, where the deviation bounds are also asymptotically independent of the corresponding Poisson problem. Eventually, we are able to handle, up to an additional constraint on the time steps, Lipschitz sources f in an appropriate non-degenerate setting.

1. Introduction

1.1. **Statement of the problem.** Consider the stochastic differential equation

$$dY_t = b(Y_t)dt + \sigma(Y_t)dW_t, \tag{1.1}$$

where $(W_t)_{t\geq 0}$ stands for a Wiener process of dimension $r\in\mathbb{N}$ on a given filtered probability space $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, \mathbb{P})$, $b:\mathbb{R}^d\to\mathbb{R}^d$, and $\sigma:\mathbb{R}^d\to\mathbb{R}^d\otimes\mathbb{R}^r$ are Lipschitz continuous functions and satisfy a Lyapunov condition (see further Assumption $\mathcal{L}_{\mathbf{V}}$) which provides the existence of an invariant measure ν. Throughout the article, uniqueness of the invariant measure ν is assumed. The purpose of this work is to estimate the invariant measure of the diffusion equation (1.1).

In order to make a clear parallel with the objects we will introduce for the approximation of ν, let us first recall some basic facts on $(Y_t)_{t\geq 0}$ and ν.

Introduce for a bounded continuous function f and $t\in\mathbb{R}_+$ the average occupation measure:

$$\nu_t(f) := \frac{1}{t}\int_0^t f(Y_s)ds. \tag{1.2}$$

Foremost, bear in mind the *usual* ergodic theorem which holds under appropriate Lyapunov conditions (see e.g. [KM11]):

$$\nu_t(f) \underset{t\to+\infty}{\overset{a.s.}{\longrightarrow}} \nu(f) := \int f d\nu. \tag{1.3}$$

Under suitable stability and regularity conditions, Bhattacharya [Bha82] then established a corresponding Central Limit Theorem (CLT). Namely, for all smooth enough function f,

$$\sqrt{t}\big(\nu_t(f)-\nu(f)\big) \underset{t\to+\infty}{\overset{\mathcal{L}}{\longrightarrow}} \mathcal{N}\left(0, \int_{\mathbb{R}^d} |\sigma^*\nabla\varphi(x)|^2\nu(dx)\right), \tag{1.4}$$

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where φ is the solution of the Poisson equation $\mathcal{A}\varphi = f - \nu(f)$ and $\mathcal{A}$ stands for the infinitesimal operator of the diffusion (1.1) (see (2.3) below for more details). In the following, we say that f is **coboundary** when there is a smooth solution φ to the Poisson equation $\mathcal{A}\varphi = f - \nu(f)$.

Identity (1.4) is a Central Limit Theorem (CLT) whose asymptotic variance is the integral of the well known **carré du champ** (called energy but we will say, from now on, by abuse of terminology, *carré du champ*), for more precision, see [BGL14] and [Led99]. The *carré du champ* is actually a bilinear operator defined for any smooth functions φ, ψ by $\mathbf{\Gamma}(\varphi, \psi) := \mathcal{A}(\varphi \cdot \psi) - \mathcal{A}\varphi \cdot \psi - \varphi \cdot \mathcal{A}\psi$, and so:

$$\nu\left(\mathbf{\Gamma}\left(\varphi, \varphi\right)\right) = \int_{\mathbb{R}^d} \mathbf{\Gamma}(\varphi, \varphi)\nu(dx) = -2\int_{\mathbb{R}^d} \mathcal{A}\varphi \cdot \varphi \; \nu(dx) = \int_{\mathbb{R}^d} |\sigma^* \nabla \varphi|^2 \nu(dx).$$

Indeed, observe that $\nu\left(\mathcal{A}\left(\varphi \cdot \psi\right)\right) = 0$. This is a consequence of the fact that ν solves in the distributional sense the Fokker-Planck equation $\mathcal{A}^*\nu = 0$. Also, this observation yields that, in order to bypass solving a Poisson equation, a common trick consists in dealing with smooth functions of the form $\mathcal{A}\varphi$.

From a practical point of view, several questions appear: how to approach the process $(Y_t)_{t\geq 0}$, the integral ν_t, and the deviation from the asymptotic measure appearing in (1.4)? Here, the first question is addressed by considering a suitable discretization scheme with decreasing time steps, $(\gamma_k)_{k\geq 1}$. The integral ν_t can then be approximated by the associated empirical measure, whose deviations will be controlled in our main results. In particular, we take advantage of the discrete analogue to (1.4) established by Lamberton and Pagès in [LP02] for the current approximation scheme, to derive sharp non-asymptotic bounds for the empirical measure.

We propose an approximation algorithm based on an Euler like discretization with decreasing time step, first introduced by Lamberton and Pagès in [LP02] who derived related asymptotic limit theorem in the spirit of (1.4), and exploited as well in [HMP17] where some corresponding non-asymptotic bounds are obtained.

For the decreasing step sequence $(\gamma_k)_{k\geq 1}$ and $n \geq 0$, the scheme deriving from (1.1) is defined by:

$$\text{(S)} \qquad \begin{cases} X_{n+1} = X_n + \gamma_{n+1} b(X_n) + \sqrt{\gamma_{n+1}}\sigma(X_n)U_{n+1}, \\ X_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}), \end{cases}$$

where $(U_n)_{n\geq 1}$ is an i.i.d. sequence of random variables on $\mathbb{R}^r$, independent of X_0, and whose moments match with the Gaussian ones up to order 3. In particular, more general innovations than the Brownian increments can be used.

Intuitively, the decreasing steps in (S) allow to be more and more precise when time grows.

The empirical (random) occupation measure of the scheme is defined for all $A \in \mathcal{B}(\mathbb{R}^d)$ (where $\mathcal{B}(\mathbb{R}^d)$ denotes the Borel σ-field on $\mathbb{R}^d$) by:

$$\nu_n(A) := \nu_n(\omega, A) := \frac{\sum_{k=1}^n \gamma_k \delta_{X_{k-1}(\omega)}(A)}{\sum_{k=1}^n \gamma_k}. \tag{1.5}$$

We are interested in the long time approximation, so we need to consider steps $(\gamma_k)_{k\geq 1}$ such that $\Gamma_n := \sum_{k=1}^n \gamma_k \underset{n}{\to} +\infty$.

Under suitable Lyapunov like assumptions, Lamberton and Pagès in [LP02] first proved the following ergodic result: for any $\nu - a.s.$ continuous function f with polynomial growth, $\nu_n(f) \xrightarrow[n]{a.s.} \nu(f) = \int_{\mathbb{R}^d} f(x)\nu(dx)$, which is the discrete analogue of (1.3).

The main benefit of decreasing steps instead of constant ones is thus that the empirical measure directly converges towards the invariant one. Otherwise, taking $\gamma_k = h > 0$ in (S), the previous ergodic theorem must be changed into: $\nu_n(f) \xrightarrow[n]{a.s.} \nu^h(f) = \int_{\mathbb{R}^d} f(x)\nu^h(dx)$, where ν^h is the invariant measure of the scheme. So, an extra study must be carried out, namely the difference $\nu - \nu^h$ should be estimated. For more details about this approach we refer to [TT90], [Tal02] and the

work of Malrieu and Talay [MT06]. This work first addressed the issue of deriving non-asymptotic controls for the deviations of empirical measure of type (1.5) when $\gamma_k = \gamma > 0$ (constant step). The backbone of their approach consisted in establishing a Log Sobolev inequality, which implies Gaussian concentration, for the Euler scheme. In whole generality, functional inequalities (such as the Log Sobolev one) are a powerful tools to get simple controls on the invariant distribution associated with the diffusion process (1.1), see e.g. Ledoux [Led99] or Bakry *et al.* [BGL14]. Withal Log Sobolev, and Poincaré inequalities turn out to be quite *rigid* in the framework of discretization schemes like (S) with or without decreasing steps.

For the CLT associated with stationary Markov chains, we refer to Gordin's Theorem (see [GL78]). Note as well that the variance of the limit Gaussian law is also the *carré du champ* for discrete Poisson equation associated with the generator of the chain.

Let us mention as well some related works. In [BB06], Blower and Bolley establish Gaussian concentration properties for deviations of functional of the path in the case of metric space valued homogeneous Markov chains. Non-asymptotic deviation bounds for the Wasserstein distance between the marginal distributions and the stationary law, in the homogeneous case can be found in [Boi11] (see also Boissard and Le Gouic in [BLG14] for controls on the expectations of this Wasserstein distance). The key point of these works is to demonstrate contraction properties of the transition kernel of the homogeneous Markov chain for a Wasserstein metric, which requires some continuity in this metric for the transition law involved, see e.g. [BB06].

In the current work, we aim to establish an optimal non-asymptotic concentration inequality for $\nu_n(f) - \nu(f)$. When $|\sigma^*\nabla\varphi|^2 - \nu(|\sigma^*\nabla\varphi|^2)$ is a coboundary, we manage to improve the estimates in [HMP17]. Insofar, we better the variance in the upper-bound obtained in Theorem 3 therein: when the time step is such that $\gamma_n \asymp n^{-\theta}$, for $\theta > \frac{1}{3}$, for all $n \in \mathbb{N}$, for a smooth enough function φ s.t. $\mathcal{A}\varphi = f - \nu(f)$, under suitable assumptions (further called Assumptions **(A)**), and if $\|\sigma\|^2$ is coboundary then there exist explicit non-negative sequences $(\widetilde{c}_n)_{n\geq 1}$ and $(\widetilde{C}_n)_{n\geq 1}$, respectively increasing and decreasing for n large enough, with $\lim_n \widetilde{C}_n = \lim_n \widetilde{c}_n = 1$ s.t. for all $n \geq 1$ and $0 < a = o(\sqrt{\Gamma_n})$:

$$\mathbb{P}\big[\sqrt{\Gamma_n}|\nu_n(f) - \nu(f)| \geq a\big] = \mathbb{P}\big[|\sqrt{\Gamma_n}\nu_n(\mathcal{A}\varphi)| \geq a\big] \leq 2\widetilde{C}_n \exp\big(-\frac{\widetilde{c}_n a^2}{2\nu(\|\sigma\|^2)\|\nabla\varphi\|_\infty^2}\big).$$

In fact, we get below the optimal variance bound, namely the *carré du champ* $\nu(|\sigma^*\nabla\varphi|^2)$, instead of the expression $\nu(\|\sigma\|^2)\|\nabla\varphi\|_\infty^2$ as in the previous inequality. Up to the same previously indicated deviation threshold, $a = o(\sqrt{\Gamma_n})$, we derive the optimal Gaussian concentration. Consequently, we are able to derive directly some sharp non-asymptotic confidence intervals.

To establish our non-asymptotic results, we use martingale increment techniques which turn out to be very robust in a rather large range of application fields. Let us for instance mention the work of Frikha and Menozzi [FM12] which establishes non-asymptotic bounds for the regular Monte Carlo error associated with the Euler discretization of a diffusion until a finite time interval $[0, T]$ and for a class of stochastic algorithms of Robbins-Monro type. Still with martingale approach, Dedecker and Gouëzel [DG15] have obtained non-asymptotic deviation bounds for separately bounded functionals of geometrically ergodic Markov chains on a general state space. Eventually, we can refer again to the work [HMP17] in the current setting.

The paper is organized as follows. In Section 2, we state our notations and assumptions as well as some known and useful results related to our approximation scheme. Section 3 is devoted to our main concentration results (for a certain deviation regime that we will call *Gaussian deviations*), we also state therein several technical lemmas whose proofs are postponed to Section 4. Importantly, we also provide a user's guide to the proof which emphasizes the key steps in our approach. Section 4 is the technical core of the paper. In Section 5, in view of practical applications, we provide a Slutsky like result with deviation bounds independent of the invariant measure. We then discuss in Section 6 some regularity issues for the considered test functions. Namely, we recall some

assumptions introduced in [HMP17] which yield appropriate regularity concerning the solution of Poisson equation. We also extend there our main results to test functions f that are Lipschitz continuous, up to some constraints on the step sequence.

We proceed in Section 7 to the explicit optimization of the constants appearing in the concentration bound deriving from our approach. Intrinsically, this procedure conducts to two deviation regimes, the *Gaussian* one up to $a = o(\sqrt{\Gamma_n})$ and the *super Gaussian* one for $a \gg \sqrt{\Gamma_n}$ which deteriorates the concentration rate. Even though awkward at first sight (see e.g. Remark 4 below), this refinement turns out to be useful for some numerical purposes, as it emphasized in Section 8. We conclude there with some numerical results associated with a degenerate diffusion and our Slutsky like result. Some additional technical details needed in Section 7 are gathered in Appendix A.

2. Assumptions and Existing Results

2.1. **General notations.** For all step sequence $(\gamma_n)_{n\geq 1}$, we denote:

$$\forall \ell \in \mathbb{R},\ \Gamma_n^{(\ell)} := \sum_{k=1}^{n} \gamma_k^{\ell},\ \Gamma_n := \sum_{k=1}^{n} \gamma_k = \Gamma_n^{(1)}.$$

Practically, the time step sequence is assumed to have the form: $\gamma_n \asymp \frac{1}{n^\theta}$ with $\theta \in (0,1]$, where for two sequences $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$ the notation $u_n \asymp v_n$ means that $\exists n_0 \in \mathbb{N}$, $\exists C \geq 1$ s.t. $\forall n \geq n_0,\ C^{-1} v_n \leq u_n \leq C v_n$.

We will denote by C a non negative constant, and by $(e_n)_{n\geq 1}, (\mathscr{R}_n)_{n\geq 1}$ deterministic generic sequences s.t. $e_n \underset{n}{\to} 0$ and $\mathscr{R}_n \underset{n}{\to} 1$, that may change from line to line. The constant C depends, uniformly in time, as well as the sequences $(e_n)_{n\geq 1}, (\mathscr{R}_n)_{n\geq 1}$, on known parameters appearing in the assumptions introduced in Section 2.2 (called **(A)** throughout the document). Other possible dependencies will be explicitly specified.

In the following, for any smooth enough function f, for $k \in \mathbb{N}$ we will denote $D^k f$ the tensor of the kth derivatives of f. Namely $D^k f = (\partial_{i_1} \dots \partial_{i_k} f)_{1\leq i_1,\dots,i_k \leq d}$. However, for a multi-index $\alpha \in \mathbb{N}_0^d := (\mathbb{N} \cup \{0\})^d$, we set $D^\alpha f = \partial_{x_1}^{\alpha_1} \dots \partial_{x_d}^{\alpha_d} f : \mathbb{R}^d \to \mathbb{R}$.

For a β-Hölder continuous function $f : \mathbb{R}^d \to \mathbb{R}$, we introduce the notation

$$[f]_\beta := \sup_{x\neq x'} \frac{|f(x) - f(x')|}{|x - x'|^\beta} < +\infty,$$

for its Hölder modulus of continuity. Here, $|x - x'|$ stands for the Euclidean norm of $x - x' \in \mathbb{R}^d$.

We denote, for $(p, m) \in \mathbb{N}^2$, by $\mathcal{C}^p(\mathbb{R}^d, \mathbb{R}^m)$ the space of p-times continuously differentiable functions from $\mathbb{R}^d$ to $\mathbb{R}^m$. Besides, for $f \in \mathcal{C}^p(\mathbb{R}^d, \mathbb{R}^m)$, $p \in \mathbb{N}$, we define for $\beta \in (0,1]$ the Hölder modulus:

$$[f^{(p)}]_\beta := \sup_{x\neq x', |\alpha|=p} \frac{|D^\alpha f(x) - D^\alpha f(x')|}{|x - x'|^\beta} \leq +\infty,$$

where α (viewed as an element of $\mathbb{N}^d$) is a multi-index of length p, i.e. $|\alpha| := \sum_{i=1}^d \alpha_i = p$. Hence, in the above definition, the $|\cdot|$ in the numerator is the usual absolute value. We will as well use the notation $[\![n, p]\!]$, $(n, p) \in (\mathbb{N}_0)^2, n \leq p$, for the set of integers being between n and p.

From now on, we introduce for $k \in \mathbb{N}_0, \beta \in (0,1]$ and $m \in \{1, d, d \times r\}$ the Hölder spaces

$$\mathcal{C}^{k,\beta}(\mathbb{R}^d, \mathbb{R}^m) := \{f \in \mathcal{C}^k(\mathbb{R}^d, \mathbb{R}^m) : \forall \alpha \in \mathbb{N}^d, |\alpha| \in [\![1, k]\!], \sup_{x \in \mathbb{R}^d} |D^\alpha f(x)| < +\infty, [f^{(k)}]_\beta < +\infty\},$$

$$\mathcal{C}_b^{k,\beta}(\mathbb{R}^d, \mathbb{R}^m) := \{f \in \mathcal{C}^{k,\beta}(\mathbb{R}^d, \mathbb{R}^m) : \|f\|_\infty < +\infty\}. \tag{2.1}$$

In the above definition, for a bounded mapping $\zeta : \mathbb{R}^d \to \mathbb{R}^m$, $m \in \{1, d, d \times r\}$, we write $\|\zeta\|_\infty := \sup_{x \in \mathbb{R}^d} \|\zeta\zeta^*(x)\|$ with $\|\zeta(x)\| = \mathrm{Tr}\,(\zeta\zeta^*(x))^{1/2}$, where for $M \in \mathbb{R}^m \otimes \mathbb{R}^m$, $\mathrm{Tr}(M)$ stands for the trace of M. Hence $\|\cdot\|$ is the Fröbenius norm [1].

With these notations, $\mathcal{C}^{k,\beta}(\mathbb{R}^d, \mathbb{R}^m)$ stands for the subset of $\mathcal{C}^k(\mathbb{R}^d, \mathbb{R}^m)$ whose elements have bounded derivatives up to order k and β-Hölder continuous k^{th} derivatives. For instance, the space of Lipschitz continuous functions from $\mathbb{R}^d$ to $\mathbb{R}^m$ is denoted by $\mathcal{C}^{0,1}(\mathbb{R}^d, \mathbb{R}^m)$.

Eventually, for a given Borel function $f : \mathbb{R}^d \to E$, where E can be $\mathbb{R}$, $\mathbb{R}^d$, $\mathbb{R}^d \otimes \mathbb{R}^r, \mathbb{R}^d \otimes \mathbb{R}^d$, we set for $k \in \mathbb{N}_0$:

$$f_k := f(X_k).$$

For $k \in \mathbb{N}_0$, we denote by $\mathcal{F}_k := \boldsymbol{\sigma}\big((X_j)_{j \in [\![0,k]\!]}\big)$ the $\boldsymbol{\sigma}$-algebra generated by the $(X_j)_{j \in [\![0,k]\!]}$.

2.2. **Hypotheses.**

(C1) The first term of the random sequence X_0 is supposed to be sub-Gaussian, i.e. there is a threshold $\lambda_0 > 0$ such that:

$$\forall \lambda < \lambda_0, \quad \mathbb{E}[\exp(\lambda |X_0|^2)] < +\infty.$$

(GC) The innovations $(U_n)_{n \geq 1}$ form an i.i.d. sequence with law μ, we also assume that $\mathbb{E}[U_1] = 0$ and for all $(i, j, k) \in \{1, \cdots, r\}^3$, $\mathbb{E}[U_1^i U_1^j] = \delta_{ij}$, $\mathbb{E}[U_1^i U_1^j U_1^k] = 0$. Moreover, $(U_n)_{n \geq 1}$ and X_0 are independent. Eventually, U_1 satisfies the following standard Gaussian concentration property, i.e. for every $1-$Lipschitz continuous function $g : \mathbb{R}^r \to \mathbb{R}$ and every $\lambda > 0$:

$$\mathbb{E}\big[\exp(\lambda g(U_1))\big] \leq \exp\big(\lambda \mathbb{E}[g(U_1)] + \frac{\lambda^2}{2}\big).$$

In particular, Gaussian and symmetrized Bernoulli random variables (in short r.v.) satisfy this inequality.

Pay attention that a wider class of sub-Gaussian distributions could be considered. Namely, random variables for which there exists $\varpi > 0$ s.t. for all $\lambda > 0$:

$$\mathbb{E}\big[\exp(\lambda g(U_1))\big] \leq \exp\big(\lambda \mathbb{E}[g(U_1)] + \frac{\varpi \lambda^2}{4}\big). \tag{2.2}$$

It is well know that this assumption yields that for all $r \geq 0$, $\mathbb{P}[|U_1| \geq r] \leq 2\exp(-\frac{r^2}{\varpi})$.

(C2) There is a positive constant κ s.t., defining for all $x \in \mathbb{R}^d$, $\Sigma(x) := \sigma\sigma^*(x)$:

$$\sup_{x \in \mathbb{R}^d} \mathrm{Tr}(\Sigma(x)) = \sup_{x \in \mathbb{R}^d} \|\sigma(x)\|^2 \leq \kappa.$$

($\mathcal{L}_\mathbf{V}$) We consider the following Lyapunov like stability condition:

There exists $V : \mathbb{R}^d \longrightarrow [v^*, +\infty[$ with $v^* > 0$ s.t.

i) $V \in \mathcal{C}^2(\mathbb{R}^d, \mathbb{R})$, $\|D^2 V\|_\infty < \infty$, and $\lim_{|x| \to \infty} V(x) = +\infty$.

1. This notation allows to define similarly vector and matrix norms. In fact, $\mathbb{R}^d$ vectors can be regarded as line vectors. Then we define similarly for both cases the uniform norm $\|\cdot\|_\infty$

ii) There exists $C_V \in (0, +\infty)$ s.t. for all $x \in \mathbb{R}^d$:

$$|\nabla V(x)|^2 + |b(x)|^2 \leq C_V V(x).$$

iii) Let $\mathcal{A}$ be the infinitesimal generator associated with the diffusion equation (1.1), defined for all $\varphi \in \mathcal{C}_0^2(\mathbb{R}^d, \mathbb{R})$ and for all $x \in \mathbb{R}^d$ by:

$$\mathcal{A}\varphi(x) = b(x) \cdot \nabla\varphi(x) + \frac{1}{2}\mathrm{Tr}\big(\Sigma(x)D^2\varphi(x)\big), \tag{2.3}$$

where, for two vectors $v_1, v_2 \in \mathbb{R}^d$, the symbol $v_1 \cdot v_2$ stands for the canonical inner product of v_1 and v_2. There exist $\alpha_V > 0$, $\beta_V \in \mathbb{R}^+$ s.t. for all $x \in \mathbb{R}^d$,

$$\mathcal{A}V(x) \leq -\alpha_V V(x) + \beta_V.$$

(U) There is a unique invariant measure ν to equation (1.1).

For $\beta \in (0, 1]$, we introduce:

($\mathbf{T}_\beta$) We choose a test function φ for which

i) φ smooth enough, i.e. $\varphi \in \mathcal{C}^{3,\beta}(\mathbb{R}^d, \mathbb{R})$,

We further assume that there exist $C_\varphi, C_{V,\varphi} > 0$ s.t. for all $x \in \mathbb{R}^d$:

ii) $\|D^2\varphi(x)\| \leq C_\varphi/(1 + |x|)$,

iii) $|\varphi(x)| \leq C_{V,\varphi}(1 + \sqrt{V(x)})$.

(S) We assume that the sequence $(\gamma_k)_{k\geq 1}$ is *small enough*, Namely, we suppose that for all $k \geq 1$:

$$\gamma_k \leq \min\Big(\frac{1}{2\sqrt{C_V\bar{c}}}, \frac{\alpha_V}{2C_V\|D^2V\|_\infty}\Big).$$

The constraint in **(S)** means that the time steps have to be sufficiently *small* w.r.t. the diffusion coefficients and the Lyapunov function.

Remark 1. *The above condition ($\mathcal{L}_V$) actually implies that the drift coefficient b lies, out of a compact set, between two hyperplanes separated from 0. Also, the Lyapunov function is lower than the square norm. In other words, there exist constants $K, \bar{c} > 0$ such that for all $|x| \geq K$,*

$$|V(x)| \leq \bar{c}|x|^2, \ |b(x)| \leq \sqrt{C_V\bar{c}}|x|. \tag{2.4}$$

Observe that we have supposed **(U)** *without imposing any non-degeneracy conditions. Existence of invariant measure follows from* **($\mathcal{L}_\mathbf{V}$)** *(see* [EK86]*). For uniqueness, additional conditions need to be considered ((hypo)ellipticity* [KM11], [PV01], [Vil09] *or confluence* [PP12]*).*

Remark 2. *In* **($\mathbf{T}_\beta$)**, *conditions ii) and iii) are technical assumptions which greatly simplify the analysis (see Appendix in* [Hon18] *for a complete discussion). Hypothesis ii) is natural when there is a function $f \in \mathcal{C}^{1,\beta}(\mathbb{R}^d, \mathbb{R})$ with $\nu(f) = 0$ s.t.*

$$\mathcal{A}\varphi = f. \tag{2.5}$$

If $\sigma \in \mathcal{C}_b^{1,\beta}(\mathbb{R}^d, \mathbb{R}^{d\times r}), b \in \mathcal{C}^{1,\beta}(\mathbb{R}^d, \mathbb{R}^d)$ and $\varphi \in \mathcal{C}^{3,\beta}(\mathbb{R}^d, \mathbb{R})$, differentiating the previous equality (2.5) *yields:*

$$|D^2\varphi b| = \Big|\nabla f - D^1 b^* \nabla\varphi - \mathrm{Tr}\Big(\frac{1}{2}\big(D^1(\Sigma)D^2\varphi + \Sigma D^3\varphi\big)\Big)\Big| \leq \bar{K},$$

for some $\bar{K} > 0$ [2]*. Indeed, each term ($D^1 b, \Sigma, D^1(\Sigma), \nabla\varphi, D^2\varphi, D^3\varphi$) is assumed to be bounded (c.f. i) in* **($\mathbf{T}_\beta$)***). With conditions ($\mathcal{L}_\mathbf{V}$), we see that there is $C > 0$ s.t. for all $x \in \mathbb{R}^d$, $|b(x)| \geq C(1+|x|)$,*

2. For any tensor M and N, we denote by MN the contraction between M and N. In particular, $D^1 b\nabla\varphi$ is a vector with entries $\big(D^1 b^*\nabla\varphi\big)_i = \sum_{j=1}^d \partial_j b_i \nabla\varphi_j$, $i \in [\![1, d]\!]^2$. Also $D^1(\Sigma)D^2\varphi$, and $\Sigma D^3\varphi$ are $\mathbb{R}^d \otimes \mathbb{R}^d$ matrices.

so

$$\|D^2\varphi(x)\| \leq \frac{K}{C(1+|x|)} = \frac{C_\varphi}{1+|x|}.$$

The above bound is consistent with Remark 2.7 in [KP10] *for the parabolic equation associated with* (2.5).

Note that, a priori, a similar control holds almost everywhere under the weaker assumption that $b, \Sigma, D^2\varphi$ *are Lipschitz continuous functions, thanks to the Rademacher theorem, see e.g.* [Ste70].

We say that assumption **(A)** holds whenever **(C1)**, **(GC)**, **(C2)**, $(\mathcal{L}_{\mathbf{V}})$, **(U)**, $(\mathbf{T}_\beta)$ for some $\beta \in (0,1]$ and **(S)** are fulfilled. Except when explicitly indicated, we assume throughout the paper that assumption **(A)** is in force.

Assume the step sequence $(\gamma_k)_{k\geq 1}$ is chosen s.t. $\gamma_k \asymp k^{-\theta}$, $\theta \in (0,1]$. In particular, this implies that, for any $\varepsilon \geq 0$, $\Gamma_n^{(\varepsilon)} \asymp n^{1-\varepsilon\theta}$ if $\varepsilon\theta < 1$, $\Gamma_n^{(\varepsilon)} \asymp \ln(n)$ if $\varepsilon\theta = 1$ and $\Gamma_n^{(\varepsilon)} \asymp 1$ if $\varepsilon\theta > 1$.

2.3. **On some Related Existing Result.** In [LP02], Lamberton and Pagès, proved an asymptotic result with the decreasing step scheme (S). Precisely, they obtain the discrete counterpart of (1.4) established in [Bha82], emphasizing as well some discretization effects, leading to a bias in the limit law, when the time step becomes too coarse. This last case is however the one leading to the highest convergence rates in the CLT. We recall here their main results, Theorem 10 of the above reference, for the sake of completeness.

Theorem 1. *[Asymptotic Limit Results in* [LP02]*] Assume* **(C2)**, $(\mathcal{L}_{\mathbf{V}})$, **(U)** *hold. If* $\mathbb{E}[U_1] = 0$, $\mathbb{E}[U_1^{\otimes 3}] = 0$, *we get the following limit results where* ν_n *stands for the empirical measure defined in* (1.5).

(a) *Fast decreasing step. If* $\theta \in (\frac{1}{3}, 1]$ *and* $\mathbb{E}[|U_1|^6] < +\infty$, *then, for all function* $\varphi \in \mathcal{C}^{2,1}(\mathbb{R}^d, \mathbb{R}) \cap \mathcal{C}^3(\mathbb{R}^d, \mathbb{R})$, *one has:*

$$\sqrt{\Gamma_n}\nu_n(\mathcal{A}\varphi) \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\big(0, \int_{\mathbb{R}^d} |\sigma^*\nabla\varphi|^2 d\nu\big).$$

(b) *Critical decreasing step. If* $\theta = \frac{1}{3}$ *and if* $\mathbb{E}[|U_1|^8] < +\infty$, *then for all function* $\varphi \in \mathcal{C}^{3,1}(\mathbb{R}^d, \mathbb{R}) \cap \mathcal{C}^4(\mathbb{R}^d, \mathbb{R})$, *one gets:*

$$\sqrt{\Gamma_n}\nu_n(\mathcal{A}\varphi) \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\big(\widetilde{\gamma}m, \int_{\mathbb{R}^d} |\sigma^*\nabla\varphi|^2 d\nu\big),$$

where

$$\begin{aligned} \widetilde{\gamma} &:= \lim_{n\to+\infty} \frac{\Gamma_n^{(2)}}{\sqrt{\Gamma_n}}, \; m := -\int_{\mathbb{R}^d} \Big(\mathrm{Tr}\big(\frac{1}{2}D^2\varphi(x)b(x)^{\otimes 2}\big) + \Phi_4(x)\Big)\nu(dx), \\ \Phi_4(x) &:= \int_{\mathbb{R}^r} \mathrm{Tr}\Big(\frac{1}{2}D^3\varphi(x)b(x)(\sigma(x)u)^{\otimes 2} + \frac{1}{24}D^4\varphi(x)(\sigma(x)u)^{\otimes 4}\Big)\mu(du), \end{aligned}$$

recalling that μ *denotes the law of the i.i.d. innovations* $(U_k)_{k\geq 1}$ [3].

(c) *Slowly decreasing step. If* $\theta \in (0, \frac{1}{3})$ *and if* $\mathbb{E}[|U_1|^8] < +\infty$, *then for all globally Lipschitz function* $\varphi \in \mathcal{C}^{3,1}(\mathbb{R}^d, \mathbb{R}) \cap \mathcal{C}^4(\mathbb{R}^d, \mathbb{R})$, *one gets:*

$$\frac{\Gamma_n}{\Gamma_n^{(2)}}\nu_n(\mathcal{A}\varphi) \xrightarrow{\mathbb{P}} m.$$

3. With our tensor notations, $D^2\varphi(x)b(x)^{\otimes 2} \in (\mathbb{R}^d)^{\otimes 2}$, $D^3\varphi(x)b(x)(\sigma(x)u)^{\otimes 2} \in (\mathbb{R}^d)^{\otimes 3}$, and $D^4\varphi(x)(\sigma(x)u)^{\otimes 4} \in (\mathbb{R}^d)^{\otimes 4}$.

It is possible to relax the boundedness condition on σ in **(C2)**, considering $\lim_{|x|\to+\infty}\frac{|\sigma^*\nabla\varphi(x)|^2}{V(x)}=0$ (strictly sublinear diffusion) in case *(a)* and $\sup_{x\in\mathbb{R}^d}\frac{|\sigma^*\nabla\varphi(x)|^2}{V(x)}<+\infty$ (sublinear diffusion) in case *(b)*. We refer to Theorems 9 and 10 in [LP02] for additional details.

Remark 3. *First of all, observe that the normalization is the same as for* (1.4)*. It is the square root of the considered running time, namely* t *for the diffusion and* Γ_n *for the scheme. In other words, a CLT is still available for the discretization procedure. However, by choosing a critical time step, i.e. for the fast convergence* $\theta=\frac{1}{3}$*, a bias is begot. It can be regarded as a discretization effect. Note that, for all* $\theta\geq\frac{1}{3}$*, any step leads to the same asymptotic variance, namely the carré du champ,* $\int_{\mathbb{R}^d}|\sigma^*\nabla\varphi|^2d\nu$*, like in* [Bha82]*. However, for the slow decreasing step,* $\theta<\frac{1}{3}$*, the discretization effect is prominent and "hides" the CLT.*

Let us also mention the work of Panloup [Pan08b], where under similar assumptions for stochastic equation driven by a Lévy process, the convergence of the decreasing time step algorithm towards the invariant measure of the stochastic process is established (see also [Pan08a] for the CLT associated with square integrable Lévy innovations).

In the current diffusive context, i.e. under **(A)**, some non-asymptotic results were successfully established in [HMP17]. It was as well observed there that if we slacken the regularity of the test function φ, a new bias looms. For $\varphi\in\mathcal{C}^{3,\beta}(\mathbb{R}^d,\mathbb{R})$ with $\beta\in(0,1)$, if $\theta=\frac{1}{2+\beta}$ then $\sqrt{\Gamma_n}\nu_n(\mathcal{A}\varphi)$ exhibits deviations similar to the ones of a biased normal law with a different bias than in Theorem 1, c.f. Theorems 2, 3, 4 and 5 in [HMP17]. When $\beta=1$, the two biases correspond. We willingly shirk any discussion about bias appearance, which is discussed in the formerly mentioned article. Our target is to refine Theorem 4 in [HMP17] that we recall:

Theorem 2. *[Non-asymptotic concentration inequalities in* [HMP17]*] Assume* **(A)** *holds, if there is* $\vartheta\in\mathcal{C}^{3,\beta}(\mathbb{R}^d,\mathbb{R})$ *satisfying* $(\mathbf{T}_\beta)$ *and s.t.*

$$\mathcal{A}\vartheta=\|\sigma\|^2-\nu(\|\sigma\|^2).$$

For $\beta\in(0,1]$ *and* $\theta\in(\frac{1}{2+\beta},1]$*, there exist two explicit monotonic sequences* $\tilde{c}_n\leq 1\leq\tilde{C}_n,\ n\geq 1$*, with* $\lim_n\tilde{C}_n=\lim_n\tilde{c}_n=1$ *such that for all* $n\geq 1$ *and* $a>0$*:*

$$\mathbb{P}\big[|\sqrt{\Gamma_n}\nu_n(\mathcal{A}\varphi)|\geq a\big]\leq 2\,\widetilde{C}_n\exp\big(-\frac{\widetilde{c}_n}{2\nu(\|\sigma\|^2)\|\nabla\varphi\|_\infty^2}\,\Phi_n(a)\big),$$

$$\Phi_n(a):=\Big[\Big(a^2\big(1-\frac{2}{1+\sqrt{1+4\,\bar{c}_n^3\,\frac{\Gamma_n}{a^2}}}\big)\Big)\vee\Big(a^{\frac{4}{3}}\Gamma_n^{\frac{1}{3}}\bar{c}_n\big(1-\frac{2}{3}\bar{c}_n\big(\frac{\Gamma_n}{a^2}\big)^{\frac{1}{3}}\big)_+\Big)\Big],\tag{2.6}$$

where $x_+=\max(x,0)$ *and* $\bar{c}_n:=\Big(\frac{[\varphi]_1}{[\vartheta]_1}\Big)^{2/3}\nu(\|\sigma\|^2)\|\sigma\|_\infty^{-2/3}\check{c}_n$ *with* $\check{c}_n$ *being an explicit nonnegative sequence s.t.* $\check{c}_n\downarrow_n 1$*.*

Remark 4. *Observe that two regimes compete in the above bound. From now on, we refer to* ***Gaussian deviations*** *when* $\frac{a}{\sqrt{\Gamma_n}}\underset{n}{\to}0$*. In this case, asymptotically the right hand side of the inequality* (2.6) *is* $2\exp\big(-\frac{a^2}{2\nu(\|\sigma\|^2)\|\nabla\varphi\|_\infty^2}\big)$*. In other words, the empirical measure is sub-Gaussian with asymptotic variance equals to* $\nu(\|\sigma\|^2)\|\nabla\varphi\|_\infty^2$ *which is an upper-bound of the carré du champ,* $\nu(|\sigma^*\nabla\varphi|^2)$ *(asymptotic variance in the limit theorem). Thus, this is not fully satisfactory. Throughout the article, we refer to* ***super Gaussian deviations*** *when* $\frac{a}{\sqrt{\Gamma_n}}\underset{n}{\to}+\infty$*. In this case, a subtle phenomenon appears: the right hand side gives a super Gaussian regime. In particular, the term in the exponential of the r.h.s. of* (2.6) *is bounded from above and below by* $a^{4/3}\Gamma_n^{1/3}$*. We anyhow*

emphasize that Theorem 2 in [HMP17] *provides a non-asymptotic Gaussian concentration for all deviation regimes:*

$$\mathbb{P}\big[|\sqrt{\Gamma_n}\nu_n(\mathcal{A}\varphi)| \geq a\big] \leq 2\,\widetilde{C}_n \exp\big(-\frac{\widetilde{c}_n a^2}{2\|\sigma\|_\infty^2\|\nabla\varphi\|_\infty^2}\big). \tag{2.7}$$

In particular, for super Gaussian deviations, the deviation (2.7) *is asymptotically better. However, it had already been observed in* [HMP17] *that the bound* (2.6) *turned out to be useful for numerical purposes as it led to bounds closer to the empirical realizations. We will derive in Theorem 7 of Section 7 a deviation bound similar to* (2.6) *with an improved variance bound. Namely, we succeed to replace* $\nu(\|\sigma\|^2)\|\nabla\varphi\|_\infty^2$ *by the carré du champ. We then observe in the numerical results of Section 8 that the associated deviation bounds match rather precisely those of the empirical realizations.*

We will also employ the terminology of ***intermediate Gaussian deviations*** *when* $\frac{a}{\sqrt{\Gamma_n}} \underset{n}{\to} C > 0$. *For this regime, we keep a Gaussian regime with deteriorated constants. Again, we first deal with Gaussian deviations, and we postpone the study of super Gaussian deviations to Section 7.*

Remark 5. *Actually, in the proof of Theorem 2, we can only use a map* ϑ *satisfying assumption* $(\mathbf{T}_\beta)$ *s.t.* $\mathcal{A}\vartheta \geq |\sigma^*\nabla\varphi|^2 - \nu(|\sigma^*\nabla\varphi|^2)$. *However, this inequality is equivalent to the coboundary condition* $\nu - a.s.$. *In fact, we set the function* $f := \mathcal{A}\vartheta - |\sigma^*\nabla\varphi|^2 + \nu(|\sigma^*\nabla\varphi|^2) \geq 0$, *and*

$$\nu(f) = \nu(\mathcal{A}\vartheta) = 0. \tag{2.8}$$

As f *is continuous and non negative,* $f = 0$ $\nu - a.s.$ *Hence* $\mathcal{A}\vartheta = |\sigma^*\nabla\varphi|^2 - \nu(|\sigma^*\nabla\varphi|^2)$ $\nu - a.s.$

3. Main results

Our main contribution consists in establishing a concentration inequality whose variance matches asymptotically the *carré du champ*, see (1.4) and Theorem 1 in what we called the regime of *Gaussian deviations*. In the numerical part of [HMP17], we see that changing the bound $\nu(\|\sigma\|^2)\|\nabla\varphi\|_\infty^2$ by the *carré du champ*, leads to bounds much closer to the realizations. Here, we state a simple and "sharp" inequality.

Theorem 3 (Sharp non-asymptotic deviation results)**.** *Assume* **(A)** *is in force. Suppose that there exists* $\vartheta \in \mathcal{C}^{3,\beta}(\mathbb{R}^d, \mathbb{R})$ *satisfying* $(\mathbf{T}_\beta)$ *for some* $\beta \in (0,1]$ *s.t.*

$$\mathcal{A}\vartheta = |\sigma^*\nabla\varphi|^2 - \nu(|\sigma^*\nabla\varphi|^2). \tag{3.1}$$

Then, for $\theta \in (\frac{1}{2+\beta}, 1]$, *there exist explicit non-negative sequences* $(c_n)_{n\geq 1}$ *and* $(C_n)_{n\geq 1}$, *respectively increasing and decreasing for* n *large enough, with* $\lim_n C_n = \lim_n c_n = 1$ *s.t. for all* $n \geq 1$, $a > 0$ *satisfying* $\frac{a}{\sqrt{\Gamma_n}} \to 0$ *(Gaussian deviations), the following bound holds:*

$$\mathbb{P}\big[|\sqrt{\Gamma_n}\nu_n(\mathcal{A}\varphi)| \geq a\big] \leq 2\,C_n \exp\big(-c_n\frac{a^2}{2\nu(|\sigma^*\nabla\varphi|^2)}\big).$$

Remark 6. *We obtain the optimal Gaussian bound with the carré du champ as variance which corresponds to CLT. This is asymptotically the sharpest result that we can expect. This inequality is very important for confidence intervals, as in this context* a *is supposed to be "small", i.e. bounded. Under suitable regularity assumptions on* f, *which guarantee that the function* φ *solving* $\mathcal{A}\varphi = f - \nu(f)$ *satisfies* $(\mathbf{T}_\beta)$ *for some* $\beta \in (0,1]$, *it readily follows from Theorem 3 that:*

$$\mathbb{P}\Big[\nu(f) \in \big[\nu_n(f) - \frac{a}{\sqrt{\Gamma_n}}, \nu_n(f) + \frac{a}{\sqrt{\Gamma_n}}\big]\Big] \geq 1 - 2C_n \exp\big(-c_n\frac{a^2}{2\nu(|\sigma^*\nabla\varphi|^2)}\big).$$

The conditions on f *that lead to the required smoothness on* φ *and* ϑ *are discussed in Section 6 (see in particular Theorem 5 and Corollary 1). Briefly, it suffices to consider that, additionally to* **(C2)** *and* $(\mathcal{L}_\mathbf{V})$, Σ *is also uniformly elliptic,* $b \in \mathcal{C}^{1,\beta}(\mathbb{R}^d, \mathbb{R}^d), \sigma \in \mathcal{C}^{1,\beta}(\mathbb{R}^d, \mathbb{R}^d \otimes \mathbb{R}^d)$ *and that the source*

$f \in \mathcal{C}^{1,\beta}(\mathbb{R}^d, \mathbb{R}^d)$. This last assumption on f can be weakened to Lipschitz continuous (see Theorem 6) with some restriction on the steps.

On the other hand, in view of practical implementations of confidence intervals, i.e. to obtain bounds with explicit concentration constants that do not depend on $\nu(|\sigma^*\nabla\varphi|^2)$, we establish in Theorem 4 of Section 5 a Slutsky like result.

3.1. **User's guide to the proof.** Recall that, for a fixed given $n \in \mathbb{N}$ and $\varphi \in \mathcal{C}^{3,\beta}(\mathbb{R}^d, \mathbb{R})$, we want to estimate the quantity

$$\mathbb{P}[\sqrt{\Gamma_n}|\nu_n(\mathcal{A}\varphi)| \geq a], \ \forall a > 0,$$

where $\nu_n(\mathcal{A}\varphi) = \frac{1}{\Gamma_n}\sum_{k=1}^n \gamma_k \mathcal{A}\varphi(X_{k-1})$. We focus below on the term $\mathbb{P}[\sqrt{\Gamma_n}\nu_n(\mathcal{A}\varphi) \geq a]$. Indeed, the contribution $\mathbb{P}[\sqrt{\Gamma_n}\nu_n(\mathcal{A}\varphi) \leq -a]$ can be handled by symmetry.

The first step of the proof consists in writing $\big(\mathcal{A}\varphi(X_{k-1})\big)_{k\in[\![1,n]\!]}$ with a splitting method to isolate the terms depending on the current innovation U_k for $\mathcal{A}\varphi(X_{k-1})$. This is done in Lemma 1 below. Precisely, for all $k \in [\![1,n]\!]$ and $\varphi \in \mathcal{C}^{3,\beta}(\mathbb{R}^d, \mathbb{R})$ we prove that:

$$\begin{aligned}\varphi(X_k) - \varphi(X_{k-1}) &= \gamma_k \mathcal{A}\varphi(X_{k-1}) + \gamma_k^2 \int_0^1 (1-t)\mathrm{Tr}\Big(D^2\varphi(X_{k-1} + t\gamma_k b_{k-1}) b_{k-1} \otimes b_{k-1}\Big)dt \\ &\quad + \frac{1}{2}\gamma_k \,\mathrm{Tr}\Big(\big(D^2\varphi(X_{k-1} + \gamma_k b_{k-1}) - D^2\varphi(X_{k-1})\big)\Sigma_{k-1}\Big) + \psi_k(X_{k-1}, U_k),\end{aligned} \tag{3.2}$$

where

$$\begin{aligned}\psi_k(X_{k-1}, U_k) &= \sqrt{\gamma_k}\sigma_{k-1}U_k \cdot \nabla\varphi(X_{k-1} + \gamma_k b_{k-1}) \\ &\quad + \gamma_k \int_0^1 (1-t)\mathrm{Tr}\Big(D^2\varphi(X_{k-1} + \gamma_k b_{k-1} + t\sqrt{\gamma_k}\sigma_{k-1}U_k)\sigma_{k-1}U_k \otimes U_k\sigma_{k-1}^* \\ &\quad - D^2\varphi(X_{k-1} + \gamma_k b_{k-1})\Sigma_{k-1}\Big)dt.\end{aligned} \tag{3.3}$$

Observe that the term $\psi_k(X_{k-1}, U_k)$ in the r.h.s. of (3.2) is the only term containing the current innovation U_k. Thus, the mapping $u \mapsto \psi_k(X_{k-1}, u)$ is Lipschitz continuous, because φ is.

This property is crucial to proceed with a martingale increment technique. Indeed, introducing the compensated increment $\Delta_k(X_{k-1}, U_k) := \psi_k(X_{k-1}, U_k) - \mathbb{E}[\psi_k(X_{k-1}, U_k)|\mathcal{F}_{k-1}]$, assumption **(GC)** allows to derive:

$$\forall \lambda > 0, \ \mathbb{E}[\exp(-\lambda\Delta_k(X_{k-1}, U_k))|\mathcal{F}_{k-1}] \leq \exp\big(\frac{\lambda^2[\psi(X_{k-1}, \cdot)]_1^2}{2}\big). \tag{3.4}$$

The corner stone of the proof is then to apply recursively this control to the martingale $M_m := \sum_{k=1}^m \Delta_k(X_{k-1}, U_k)$, $m \in [\![1,n]\!]$.

To control the deviation, the first step is an exponential inequality which combined to (3.2) yields:

$$\begin{aligned}\mathbb{P}[\sqrt{\Gamma_n}\nu_n(\mathcal{A}\varphi) \geq a] &\leq \exp\big(-\frac{a\lambda}{\sqrt{\Gamma_n}}\big)\mathbb{E}\Big[\exp(\lambda\nu_n(\mathcal{A}\varphi))\Big] \\ &\leq \exp\big(-\frac{a\lambda}{\sqrt{\Gamma_n}}\big)\mathbb{E}\Big[\exp\big(-\frac{\lambda q M_n}{\Gamma_n}\big)\Big]^{1/q}\mathcal{R}_n,\end{aligned} \tag{3.5}$$

for $\lambda > 0$, $q > 1$ and $\mathcal{R}_n$ is a remainder (whose behaviour is investigated in Lemma 6). The main contribution in the above equation is the one involving M_n which can be analyzed thanks to (3.4). Namely:

$$\begin{aligned}\mathbb{E}\Big[\exp\big(-\frac{q\lambda}{\Gamma_n}M_n\big)\Big] &= \mathbb{E}\Big[\exp\big(-\frac{q\lambda}{\Gamma_n}M_{n-1}\big)\mathbb{E}\Big[\exp\big(-\frac{q\lambda}{\Gamma_n}\Delta_n(X_{n-1}, U_n)\big)\Big|\mathcal{F}_{n-1}\Big]\Big] \\ &\leq \mathbb{E}\Big[\exp\big(-\frac{q\lambda}{\Gamma_n}M_{n-1}\big)\exp\big(\frac{q^2\lambda^2[\psi(X_{n-1}, \cdot)]_1^2}{2\Gamma_n^2}\big)\Big].\end{aligned} \tag{3.6}$$

A first approach in [HMP17], in order to iterate the estimates involving the conditional expectations, consisted in bounding uniformly $[\psi(X_{n-1},\cdot)]_1 \leq \sqrt{\gamma_n}\|\sigma\|_\infty\|\nabla\varphi\|_\infty$ (which is easily deduced from (3.2)). Iterating the procedure led to the estimate

$$\mathbb{P}[\sqrt{\Gamma_n}\nu_n(\mathcal{A}\varphi) \geq a] \leq \exp\big(-\frac{a\lambda}{\sqrt{\Gamma}_n}\big)\exp\big(\frac{q\lambda^2}{2\Gamma_n}\|\sigma\|_\infty^2\|\nabla\varphi\|_\infty^2\big)\mathcal{R}_n. \tag{3.7}$$

Optimizing over λ, letting as well $q \downarrow_n 1$ in a suitable way, gives the deviation upper-bound $C_n\exp(-c_n\frac{a^2}{2\|\sigma\|_\infty^2\|\nabla\varphi\|_\infty^2})$, with $C_n, c_n > 0$ respectively increasing and decreasing to 1 with n (see Theorem 2 of [HMP17] for details).

To obtain the expected variance corresponding to the *carré du champ* $\nu(|\sigma^*\nabla\varphi|^2)$, the key point is to control finely the Lipschitz modulus of $\psi_k(X_{k-1},\cdot)$ in (3.4), (6.4).

From (3.2), we get the following simple expression of the derivative $\nabla_u\psi_k(X_{k-1},u)|_{u=U_k} = \sqrt{\gamma_k}\sigma_{k-1}^*\nabla\varphi(X_k)$. Hence, there is a remainder term $\mathcal{R}(\gamma_k, X_{k-1}, U_k)$ s.t.

$$\nabla_u\psi_k(X_{k-1},u)|_{u=U_k} = \sqrt{\gamma_k}\sigma_{k-1}^*\nabla\varphi_{k-1} + \mathcal{R}(\gamma_k, X_{k-1}, U_k). \tag{3.8}$$

In order to exhibit for each evaluation of the conditional expectations in (3.4), the contribution $\nu(|\sigma^*\nabla\varphi|^2)$, we use the auxiliary Poisson problem:

$$\mathcal{A}\vartheta = |\sigma^*\nabla\varphi|^2 - \nu(|\sigma^*\nabla\varphi|^2). \tag{3.9}$$

We then write for the main term to control in (3.5),

$$\mathbb{E}[\exp(-\frac{\lambda q M_n}{\Gamma_n})] \leq \mathscr{T}_1^{\frac{1}{\rho}}\mathscr{T}_2^{\frac{\rho-1}{\rho}}, \tag{3.10}$$

for $\rho > 1$ where:

$$\begin{aligned}\mathscr{T}_1 &:= \mathbb{E}\exp\big(-\rho\frac{q\lambda}{\Gamma_n}M_n - \frac{\rho^2q^2\lambda^2}{2\Gamma_n^2}\sum_{k=1}^n\gamma_k\mathcal{A}\vartheta(X_{k-1})\big),\\ \mathscr{T}_2 &:= \mathbb{E}\exp\big(\frac{\lambda^2q^2\rho^2}{2(\rho-1)\Gamma_n^2}\sum_{k=1}^n\gamma_k\mathcal{A}\vartheta(X_{k-1})\big).\end{aligned} \tag{3.11}$$

Exploiting (3.9), we can now rewrite

$$\begin{aligned}\mathscr{T}_1 &= \mathbb{E}\exp\big(-\rho\frac{q\lambda}{\Gamma_n}M_n - \frac{\rho^2q^2\lambda^2}{2\Gamma_n^2}\sum_{k=1}^n\gamma_k\big[|\sigma^*\nabla\varphi(X_{k-1})|^2 - \nu(|\sigma^*\nabla\varphi|^2)\big]\big)\\ &= \exp\big(\frac{\rho^2q^2\lambda^2}{2\Gamma_n}\nu(|\sigma^*\nabla\varphi|^2)\big)\mathbb{E}\exp\big(-\rho\frac{q\lambda}{\Gamma_n}M_n - \frac{\rho^2q^2\lambda^2}{2\Gamma_n^2}\sum_{k=1}^n\gamma_k|\sigma^*\nabla\varphi(X_{k-1})|^2\big).\end{aligned} \tag{3.12}$$

The first term in the above r.h.s. yields the expected variance when we optimize over λ for q and ρ going to 1, which is the case in the regime of so called *Gaussian deviations* in Theorem 3. It improves the previous bound (3.7). Introduce now for $m \in [\![1,n]\!]$,

$$S_m := \exp\big(-\frac{\rho q\lambda}{\Gamma_n}M_m - \frac{\rho^2q^2\lambda^2}{2\Gamma_n^2}\sum_{k=1}^m\gamma_k|\sigma^*\nabla\varphi(X_{k-1})|^2\big). \tag{3.13}$$

Bringing to mind that $M_m = \sum_{k=1}^m\Delta_k(X_{k-1},U_k)$, where $\mathbb{E}[\Delta_k(X_{k-1},U_k)|\mathcal{F}_{k-1}] = 0$ and $[\Delta_k(X_{k-1},\cdot)]_1 = [\psi_k(X_{k-1},\cdot)]_1$, we get from (3.8), that, up to the remainder term $(\mathcal{R}(\gamma_k, X_{k-1}, U_k))_{k\in[\![1,n]\!]}$, S_m can be viewed as a super martingale (see Lemma 5 for details). We actually rigorously show that, in the Gaussian regime (i.e. for $\frac{a}{\sqrt{\Gamma_n}} \to 0$), for $\theta \in (1/3, 1)$

$$\mathbb{E}[S_n]^{\frac{1}{\rho q}} \leq \mathscr{R}_n \underset{n\to+\infty}{\longrightarrow} 1.$$

For $\theta = 1$, or for *super Gaussian deviations* (i.e. for $\frac{a}{\sqrt{\Gamma_n}} \to +\infty$, see Section 7) with $\theta \in (1/3, 1)$ we get:

$$\mathbb{E}[S_n]^{\frac{1}{\rho q}} \leq \mathscr{R}_n \exp\Big(\big(\frac{\rho q \lambda^2}{\Gamma_n} + \frac{\rho^3 q^3 \lambda^4}{(\rho-1)\Gamma_n^3}\big) e_n \Big),$$

where $e_n > 0$ decreases to 0 with n and $\mathscr{R}_n > 0$ is still going to 1 with n. The difficulty in the above control is that the optimized λ also depends on n and ρ (see (3.30) below).

The second term $\mathscr{T}_2$ is estimated directly repeating the arguments of the proof of Theorem 2 in [HMP17] which are recalled above (see equations (3.5) to (3.7)). We apply the previous martingale increment technique that previously led to (3.7). Denoting by M_n^ϑ the martingale associated with the $\big(\psi_k^\vartheta(X_{k-1}, U_k)\big)_{k \in [\![1,n]\!]}$ deriving from the expansion of $\mathcal{A}\vartheta$ similarly to (3.2), we obtain:

$$\mathscr{T}_2 \leq \mathbb{E}\Big[\exp\big(- \frac{\lambda^2 q^2 \rho^2 \bar{q} M_n^\vartheta}{2(\rho-1)\Gamma_n^2} \big) \Big]^{1/\bar{q}} \mathcal{R}_n^\vartheta \leq \exp\Big(\frac{\lambda^4 q^4 \rho^4 \bar{q}}{8(\rho-1)^2 \Gamma_n^3} \|\sigma\|_\infty^2 \|\nabla\vartheta\|_\infty^2 \Big) \mathcal{R}_n^\vartheta, \tag{3.14}$$

for $\bar{q} > 1$ and where the superscript ϑ means that we only need to replace φ by ϑ in the previous definitions. Like in (3.5), $\mathcal{R}_n^\vartheta$ is here a remainder.

Eventually, by (3.5), (3.12) and (3.14) with the different controls of $\mathbb{E}[S_n]$ (see Lemma 5):

$$\mathbb{P}[\sqrt{\Gamma_n}\nu_n(\mathcal{A}\varphi) \geq a] \leq \exp\Big(- \frac{a\lambda}{\sqrt{\Gamma_n}} + \frac{\lambda^2}{\Gamma_n} A_n(\rho) + \frac{\lambda^4}{\Gamma_n^3} B_n(\rho) \Big) \mathscr{R}_n, \tag{3.15}$$

with $\mathscr{R}_n \to 1$, $A_n(\rho) := \rho(\frac{q\nu(|\sigma^*\nabla\vartheta|^2)}{2} + e_n)$, $B_n := \frac{\rho^3}{\rho-1}\frac{q^3}{4}(\frac{\bar{q}\|\sigma\|_\infty^2\|\nabla\vartheta\|_\infty^2}{2} + e_n)$ for $e_n > 0$ decreasing to 0 with n.

We perform an optimization over λ with the Cardan method. However, the optimal choice of λ depends on ρ. So an optimization can be done for ρ too. In Lemma 4 below, we choose ρ for the regime of *Gaussian deviations* (i.e. $\frac{a}{\sqrt{\Gamma_n}} \underset{n}{\to} 0$) which yields:

$$\mathbb{P}\big[|\sqrt{\Gamma_n}\nu_n(\mathcal{A}\varphi)| \geq a\big] \leq 2\, C_n \exp\Big(- c_n \frac{a^2}{2\nu(|\sigma^*\nabla\varphi|^2)} \Big),$$

for $c_n, C_n > 0$ respectively decreasing and increasing (for n big enough) to 1 with n.

The optimal choices of λ and ρ for the regime of *super Gaussian deviations* (i.e. $\frac{a}{\sqrt{\Gamma_n}} \underset{n}{\to} +\infty$) is eventually discussed in Section 7. This leads to

$$\mathbb{P}\big[|\sqrt{\Gamma_n}\nu_n(\mathcal{A}\varphi)| \geq a\big] \leq 2\, C_n \exp\Big(- c_n \frac{a^{4/3}\Gamma_n^{1/3}}{2\|\sigma\|_\infty^{2/3}\|\nabla\vartheta\|_\infty^{2/3}} \Big).$$

3.2. **Technical lemmas and Proof of the Main Results.** We first give a decomposition lemma of $\nu_n(\mathcal{A}\varphi)$ which is the starting point of our analysis. Its proof can be found in [HMP17] (see Lemma 1 therein).

Lemma 1 (Decomposition of the empirical measure)**.** *For all $n \geq 1$, $k \in [\![1,n]\!]$ and $\varphi \in \mathcal{C}^2(\mathbb{R}^d, \mathbb{R})$, the identity* (3.2) *holds and we have:*

$$\begin{aligned} \Gamma_n \nu_n(\mathcal{A}\varphi) &= \varphi(X_n) - \varphi(X_0) - \Big[\sum_{k=1}^n \gamma_k^2 \int_0^1 (1-t) \mathrm{Tr}\Big(D^2\varphi(X_{k-1} + t\gamma_k b_{k-1}) b_{k-1} \otimes b_{k-1} \Big) dt \\ &\quad + \frac{1}{2} \sum_{k=1}^n \gamma_k \,\mathrm{Tr}\Big(\big(D^2\varphi(X_{k-1} + \gamma_k b_{k-1}) - D^2\varphi(X_{k-1}) \big) \Sigma_{k-1}^2 \Big) + \sum_{k=1}^n \psi_k(X_{k-1}, U_k) \Big], \end{aligned} \tag{3.16}$$

where $\psi_k(X_{k-1}, U_k)$ is defined in (3.3)*.*

Remark 7. *In spite of the square terms in U_k appearing in the r.h.s. of* (3.3)*, we have that, conditionally to $\mathcal{F}_{k-1}$, $u \mapsto \psi_k(X_{k-1}, u)$ is Lipschitz continuous. Indeed, on the r.h.s. U_k only appears in ψ_k and on the l.h.s. we know that φ is Lipschitz. Hence, for all $(u, u') \in (\mathbb{R}^d)^2$:*

$$|\psi_k(X_{k-1}, u) - \psi_k(X_{k-1}, u')| \leq \sqrt{\gamma_k}\|\sigma_{k-1}\|\|\nabla\varphi\|_\infty |u - u'|. \tag{3.17}$$

Our strategy consists in controlling how far the Lipschitz modulus in (3.17) *is from $|\sigma_{k-1}^* \nabla\varphi_{k-1}|$. The first step is to obtain an explicit derivative of $\psi_k(X_{k-1}, \cdot)$, see* (3.21) *below.*

For notational convenience we introduce, for a given $n \in \mathbb{N}^*$ the following quantities:

$$\begin{aligned} R_n &:= \varphi(X_n) - \varphi(X_0) - \sum_{k=1}^n \gamma_k^2 \int_0^1 (1-t)\,\mathrm{Tr}\Big(D^2\varphi(X_{k-1} + t\gamma_k b_{k-1}) b_{k-1} \otimes b_{k-1}\Big) dt \\ &- \frac{1}{2}\sum_{k=1}^n \gamma_k \mathrm{Tr}\Big(\big(D^2\varphi(X_{k-1} + \gamma_k b_{k-1}) - D^2\varphi(X_{k-1})\big)\Sigma_{k-1}^2\Big), \\ M_n &:= \sum_{k=1}^n \Delta_k(X_{k-1}, U_k),\ \widetilde{R}_n := R_n - \sum_{k=1}^n \mathbb{E}\big[\psi_k(X_{k-1}, U_k) |\, \mathcal{F}_{k-1}\big], \end{aligned} \tag{3.18}$$

where for all $k \in [\![1, n]\!]$:

$$\Delta_k(X_{k-1}, U_k) := \psi_k(X_{k-1}, U_j) - \mathbb{E}\big[\psi_k(X_{k-1}, U_k) |\, \mathcal{F}_{k-1}\big]. \tag{3.19}$$

From these definitions, Lemma 1 can be rewritten:

$$\nu_n(\mathcal{A}\varphi) = \frac{1}{\Gamma_n}(\widetilde{R}_n - M_n), \tag{3.20}$$

where M_n is a martingale. The key idea of the proof is to control more precisely the Lipschitz modulus of $\psi_n(X_{n-1}, \cdot)$ than it was done in [HMP17]. From the definition in (3.2), let us write for all $k \in [\![1, n]\!]$:

$$\psi_k(X_{k-1}, U_k) = \varphi_k - \varphi_{k-1} + R_{k-1,k},$$

where

$$\begin{aligned} R_{k-1,k} &:= -\gamma_k \mathcal{A}\varphi(X_{k-1}) - \gamma_k^2 \int_0^1 (1-t)\,\mathrm{Tr}\Big(D^2\varphi(X_{k-1} + t\gamma_k b_{k-1}) b_{k-1} \otimes b_{k-1}\Big) dt \\ &- \frac{1}{2}\gamma_k \mathrm{Tr}\Big(\big(D^2\varphi(X_{k-1} + \gamma_k b_{k-1}) - D^2\varphi(X_{k-1})\big)\Sigma_{k-1}^2\Big). \end{aligned}$$

Hence, by derivation

$$\nabla_u \psi_k(X_{k-1}, u)|_{u=U_k} = \sqrt{\gamma_k}\sigma_{k-1}^* \nabla\varphi(X_k). \tag{3.21}$$

We will establish that the value of $\nabla_u \psi_k(X_{k-1}, u)|_{u=U_k}$ is not "too far" from $\sqrt{\gamma_k}\sigma_{k-1}^* \nabla\varphi(X_{k-1})$.

3.2.1. *Proof of Theorem 3 for bounded innovations.* We first give the complete proof in this particular case. We will specify the additional required controls for possibly unbounded innovations in the next subsection.

A key tool in the derivation of our main results is the following lemma whose proof is postponed to Section 4 for the sake of clarity.

Lemma 2. *[Remainders from Taylor decomposition] Under* **(A)***, for all $q \geq 1$ and $\lambda > 0$, we have:*

$$\mathbb{P}\big[\sqrt{\Gamma_n}\nu_n(\mathcal{A}\varphi) \geq a\big] \leq \exp\big(-\frac{a\lambda}{\sqrt{\Gamma_n}}\big)\Big(\mathbb{E}\exp\big(-\frac{q\lambda}{\Gamma_n}M_n\big)\Big)^{\frac{1}{q}} \exp(\frac{\lambda^2}{\Gamma_n}e_n)\mathscr{R}_n. \tag{3.22}$$

We will now sharply control the Lipschitz constant of $\psi_k(X_{k-1},\cdot)$, or equivalently $\Delta_k(X_{k-1},\cdot)$, which appears iteratively to handle the martingale term in (3.22).

In case of bounded innovations, we see by assumption $(\mathbf{T}_\beta)$ i) (smoothness of φ) that:

$$\begin{aligned}
&|\nabla_u \Delta_k(X_{k-1},u)|_{u=U_k}| = |\sqrt{\gamma_k}\sigma_{k-1}^*\nabla\varphi(X_k)| \\
&\leq |\sqrt{\gamma_k}\sigma_{k-1}^*\nabla\varphi(X_{k-1})| + |\sqrt{\gamma_k}\sigma_{k-1}^*\left[\nabla\varphi(X_k)-\nabla\varphi(X_{k-1}+\gamma_k b_{k-1})\right]| \\
&+|\sqrt{\gamma_k}\sigma_{k-1}^*\left[\nabla\varphi(X_{k-1}+\gamma_k b_{k-1})-\nabla\varphi(X_{k-1})\right]| \\
&\leq \sqrt{\gamma_k}|\sigma_{k-1}^*\nabla\varphi(X_{k-1})| + \gamma_k\|\sigma_{k-1}\|^2\|D^2\varphi\|_\infty\|U_k\|_\infty \\
&+\gamma_k^{3/2}\|\sigma_{k-1}\|\Big|b_{k-1}\int_0^1 D^2\varphi(X_{k-1}+t\gamma_k b_{k-1})dt\Big|. \qquad (3.23)
\end{aligned}$$

Let us deal with the last term $\gamma_k^{3/2}\|\sigma_{k-1}\|\left|b_{k-1}\int_0^1 D^2\varphi(X_{k-1}+t\gamma_k b_{k-1})dt\right|$. Because we have assumed $(\mathbf{T}_\beta)$ ii), $\|D^2\varphi(x)\|\leq\frac{C}{1+|x|}$. Recall as well from $(\mathcal{L}_\mathbf{V})$ ii) that $|b|\leq C_V^{1/2}V^{1/2}$. We introduce now the partition $\{|X_{k-1}|\leq K\}\cup\{|X_{k-1}|>K\}$ for a given constant $K>0$. The Lyapunov function V_{k-1} is then bounded on the random set $\{|X_{k-1}|\leq K\}$, and for $\{|X_{k-1}|>K\}$ we write:

$$|D^2\varphi(X_{k-1}+t\gamma_k b_{k-1})|\leq\frac{C}{1+|X_{k-1}+t\gamma_k b_{k-1}|}.$$

This control allows to balance the growth of V_{k-1}. Indeed from $(\mathcal{L}_\mathbf{V})$ for a given threshold $K>0$ there are constants $C_V,\bar{c}:=\bar{c}(V,K)>0$ s.t. $\forall|x|\geq K, |V(x)|\leq\bar{c}|x|^2,\ |b(x)|\leq\sqrt{\bar{c}\,C_V}|x|$ (see also Remark 1). Hence, since $t\in[0,1]$,

$$|X_{k-1}+t\gamma_k b_{k-1}|\geq(|X_{k-1}|-\gamma_k|b_{k-1}|)_+\geq|X_{k-1}|(1-\gamma_k\sqrt{\bar{c}\,C_V})_+.$$

From assumption $(\mathbf{S})$, we have that for $k\geq1$, $(1-\gamma_k\sqrt{\bar{c}\,C_V})\geq1/2$ (equilibrium of the steps and the constants). Thus:

$$|D^2\varphi(X_{k-1}+t\gamma_k b_{k-1})|\leq\|D^2\varphi\|_\infty\mathbb{1}_{\{|X_{k-1}|\leq K\}}+\frac{2C}{|X_{k-1}|}\mathbb{1}_{\{|X_{k-1}|>K\}}. \qquad (3.24)$$

Hence,

$$\begin{aligned}
&\gamma_k^{3/2}\|\sigma_k\|\Big|b_{k-1}\int_0^1 D^2\varphi(X_{k-1}+t\gamma_k b_{k-1})dt\Big| \\
&\leq\gamma_k^{3/2}\|\sigma_{k-1}\|\sqrt{C_V}\int_0^1\Big[V^{1/2}(X_{k-1})\|D^2\varphi\|_\infty\mathbb{1}_{\{|X_{k-1}|\leq K\}}+V^{1/2}(X_{k-1})\frac{2C}{|X_{k-1}|}\mathbb{1}_{\{|X_{k-1}|>K\}}\Big]dt \\
&\leq\gamma_k^{3/2}\|\sigma_{k-1}\|\sqrt{C_V}\big(\sup_{|x|\leq K}\big(V^{1/2}(x)\big)\|D^2\varphi\|_\infty+\sqrt{\bar{c}}2C\big)\leq C\gamma_k^{3/2}.
\end{aligned}$$

(3.25)

Plugging in (3.23) the control (3.25), we then derive, recalling that we consider first $\|U_k\|_\infty\leq C$:

$$|\nabla_u\Delta_k(X_{k-1},u)|_{u=U_k}|\leq|\sqrt{\gamma_k}\sigma_{k-1}^*\nabla\varphi(X_{k-1})|+C\gamma_k,$$

then

$$[\Delta_k(X_{k-1},\cdot)]_1^2\leq\gamma_k|\sigma^*\nabla\varphi|^2(X_{k-1})+C\gamma_k^{3/2}. \qquad (3.26)$$

Let us introduce for all $(m,n)\in\mathbb{N}_0^2$, $m\leq n$ and $(\rho,q)\in(1,+\infty)^2$:

$$T_m:=\exp\big(-\frac{\rho q\lambda}{\Gamma_n}\Delta_m(X_{m-1},U_m)-\frac{\rho^2(q\lambda)^2}{2\Gamma_n^2}\gamma_m|\sigma^*\nabla\varphi(X_{m-1})|^2\big). \qquad (3.27)$$

From the definition of S_m in (3.13), we write $S_m=\prod_{k=1}^m T_k$. The coefficients $(T_m)_{m\geq1}$ can be viewed as multiplicative increments of $(S_m)_{m\geq0}$. Inequality (3.26) precisely allows to quantify the

martingality default for $(S_m)_{m\geq 0}$. These factors appear when we exploit the auxiliary Poisson problem (3.9) in the definition of $\mathscr{T}_1$ in (3.11).

$$\mathscr{T}_1 = \exp\big(\frac{\rho^2 q^2\lambda^2}{2\Gamma_n}\nu(|\sigma^*\nabla\varphi|^2)]\big)\mathbb{E}\big[S_{n-1}\mathbb{E}[T_n|\mathcal{F}_{n-1}]\big].$$

Thereby, from the upper-bound (3.26) of the Lipschitz modulus, we directly obtain from (3.27) and **(GC)**

$$\begin{aligned}\mathbb{E}[T_n|\mathcal{F}_{n-1}] &= \exp\big(-\frac{\rho^2(q\lambda)^2}{2\Gamma_n^2}\gamma_n|\sigma^*\nabla\varphi(X_{n-1})|^2\big)\mathbb{E}\Big[\exp\big(\frac{\rho q\lambda}{\Gamma_n}\Delta_n(X_{n-1},U_n)\big)\Big|F_{n-1}\Big]\\ &\leq \exp\big(\frac{\rho^2(q\lambda)^2}{2\Gamma_n^2}\gamma_n^{3/2}C\big).\end{aligned}$$

Hence, iterating:

$$\begin{aligned}\mathscr{T}_1 &\leq \exp\big(\frac{\rho^2(q\lambda)^2}{2\Gamma_n}\nu(|\sigma^*\nabla\varphi|^2)\big)\mathbb{E}[S_{n-1}]\exp\big(\frac{\rho^2(q\lambda)^2}{2\Gamma_n^2}\gamma_n^{3/2}C\big)\\ &\leq \exp\big(\frac{\rho^2(q\lambda)^2}{2\Gamma_n}\nu(|\sigma^*\nabla\varphi|^2)\big)\exp\big(\frac{\rho^2(q\lambda)^2}{2\Gamma_n}\underbrace{\frac{\Gamma_n^{(3/2)}}{\Gamma_n}}_{=e_n}C\big) = \exp\big(\frac{\rho^2(q\lambda)^2}{2\Gamma_n}(\nu(|\sigma^*\nabla\varphi|^2)+e_n)\big),\end{aligned}$$

where $e_n\to 0$. The controls for $\mathscr{T}_2$ are deduced from (3.14). We now gather the previous estimates into (3.10) (we recall $\mathbb{E}[\exp(-\frac{\lambda q M_n}{\Gamma_n})]\leq \mathscr{T}_1^{\frac{1}{\rho}}\mathscr{T}_2^{\frac{\rho-1}{\rho}}$) in the following lemma. Note also that the term $\mathcal{R}_n^{\vartheta}$ appearing in (3.14) is controlled similarly to remainders in Lemma 2.

Lemma 3 (Gaussian concentration term)**.** *With notations of* (3.18)*, under* **(A)***, for a bounded* $\rho>1$*, we have:*

$$\mathbb{E}\exp\big(-\frac{\lambda q}{\Gamma_n}M_n\big)^{\frac{1}{q}}\leq \exp\big(\frac{\lambda^2}{\Gamma_n}\tilde{A}_n+\frac{\lambda^4}{\Gamma_n^3}\tilde{B}_n\big)\mathscr{R}_n,$$

where

$$\tilde{A}_n := \rho\big(\frac{q\nu(|\sigma^*\nabla\varphi|^2)}{2}+e_n\big)\quad \textit{and}\quad \tilde{B}_n := \frac{\rho^3}{\rho-1}\frac{q^3}{4}\big(\frac{\bar{q}\|\sigma\|_\infty^2[\vartheta]_1^2}{2}+e_n\big), \tag{3.28}$$

for some $1<\bar{q}:=\bar{q}(n)\underset{n}{\to}1$*, and with:* $e_n\underset{n\to+\infty}{\longrightarrow}0$*,* $\mathscr{R}_n\underset{n\to+\infty}{\longrightarrow}1$ *uniformly in* λ*.*

As a consequence of the previous Lemmas 2 and 3, we obtain (3.15), namely:

$$\mathbb{P}\big(\sqrt{\Gamma_n}\nu_n(\mathcal{A}\varphi)\geq a\big)\leq C_n\exp\big(P(\lambda)\big), \tag{3.29}$$

with $P(\lambda):=-\frac{a\lambda}{\sqrt{\Gamma_n}}+\frac{\lambda^2}{\Gamma_n}A_n+\frac{\lambda^4}{\Gamma_n^3}B_n$, where $A_n=A_n(\rho)=\rho\widetilde{A}_n$ and $B_n=B_n(\rho):=\frac{\rho^3}{\rho-1}\widetilde{B}_n$ with

$$\widetilde{A}_n=\frac{q\nu(|\sigma^*\nabla\varphi|^2)}{2}+e_n\quad\text{and}\quad \widetilde{B}_n=\frac{q^3}{4}\big(\frac{\bar{q}\|\sigma\|_\infty^2\|\nabla\vartheta\|_\infty^2}{2}+e_n\big).$$

Next, like enunciated at the end of the *User's guide to the proof*, we optimize a fourth order polynomial by the Cardan method, see (3.30) below (and Section 4 in [HMP17]). If $\lambda_n=\arg\min_\lambda P(\lambda)$, then

$$P'(\lambda_n)=-\frac{a}{\sqrt{\Gamma_n}}+\frac{2\lambda_n}{\Gamma_n}A_n+\frac{4\lambda_n^3}{\Gamma_n^3}B_n=0.$$

The Cardan-Tartaglia formula yields only one positive real root. Namely, setting

$$\Phi_n(a,\rho)=\Big(\frac{a}{\sqrt{\Gamma_n}\tilde{B}_n}+\big(\frac{a^2}{\tilde{B}_n^2\Gamma_n}+(\rho-1)\big(\frac{2\tilde{A}_n}{3\tilde{B}_n}\big)^3\big)^{\frac{1}{2}}\Big)^{\frac{1}{3}}+\Big(\frac{a}{\sqrt{\Gamma_n}\tilde{B}_n}-\big(\frac{a^2}{\tilde{B}_n^2\Gamma_n}+(\rho-1)\big(\frac{2\tilde{A}_n}{3\tilde{B}_n}\big)^3\big)^{\frac{1}{2}}\Big)^{\frac{1}{3}},$$

this conducts to take $\lambda = \lambda_n$ with:

$$\lambda_n := \frac{\Gamma_n}{2}\frac{(\rho-1)^{\frac{1}{3}}}{\rho}\Phi_n(a,\rho). \tag{3.30}$$

Moreover, remark from the binomial Newton expansion that:

$$\Phi_n(a,\rho)^3 = \frac{2a}{\sqrt{\Gamma_n}\tilde{B}_n} - \frac{2(\rho-1)^{1/3}\tilde{A}_n}{\tilde{B}_n}\Phi_n(a,\rho).$$

From (3.30) and the above expression, $P(\lambda_n) = P_{\min}(a,\Gamma_n,\rho) = \lambda_n\big(-\frac{a}{\sqrt{\Gamma_n}} + \frac{\lambda_n}{\Gamma_n}A_n + \frac{\lambda_n^3}{\Gamma_n^3}B_n\big)$, we thus obtain

$$P_{\min}(a,\Gamma_n,\rho) := -\frac{\sqrt{\Gamma_n}(\rho-1)^{1/3}\Phi_n(a,\rho)}{2^3\rho}\big(3a - \sqrt{\Gamma_n}(\rho-1)^{1/3}\tilde{A}_n\Phi_n(a,\rho)\big). \tag{3.31}$$

Then, from (3.29):

$$\mathbb{P}\big(\sqrt{\Gamma_n}\nu_n(\mathcal{A}\varphi) \geq a\big) \leq C_n \exp\big(P_{\min}(a,\Gamma_n,\rho)\big), \tag{3.32}$$

which is exactly the same bound appearing in Remark 11 in [HMP17], up to a modification of $\tilde{A}_n$, containing here the expected *carré du champ*.

The optimization over λ leads to study how ρ should asymptotically behave. The following lemma indicates that, when $a = o(\sqrt{\Gamma_n})$, taking $\rho - 1 \asymp \frac{a}{\sqrt{\Gamma_n}}$ yields a Gaussian concentration inequality in (3.29) with the optimal constant.

Lemma 4 (Choice of ρ for the Gaussian concentration regime)**.** *For $P_{\min}(a,\Gamma_n,\rho)$ as in* (3.31)*, there is $\rho := \rho(n,a) > 1$ s.t. If $\frac{a}{\sqrt{\Gamma_n}} \underset{n}{\to} 0$, taking $\rho - 1 \asymp \frac{a}{\sqrt{\Gamma_n}}$*

$$P_{\min}(a,\Gamma_n,\rho) \underset{\frac{a}{\sqrt{\Gamma_n}}\underset{n}{\to}0}{=} -\frac{a^2}{2\nu(|\sigma^*\nabla\varphi|^2)}(1+o(1)).$$

For the sake of clarity, the proof of Lemma 4 is postponed to Section 4.2. From (3.32) and Lemma 4, we conclude the proof of Theorems 3 and 8 for bounded innovations.

3.2.2. *Proof of Theorems 3 for unbounded innovations.* Switching to unbounded innovations requires additional technicalities. Our strategy consists in considering a truncation argument writing $\Delta_k(X_{k-1},U_k) = \Delta_k(X_{k-1},U_k)[\mathbb{1}_{|U_k|\leq\frac{r_{k,n}}{2}} + \mathbb{1}_{|U_k|>\frac{r_{k,n}}{2}}]$, to control the Lipschitz modulus of ψ_{k-1} where $(r_{k,n})_{n\geq1,k\leq n}$ is a suitable sequence specified in (3.37) below. In particular, $r_{k,n} := r_{k,n}((\mathbf{A}),\lambda,\rho)$ where $\lambda > 0, \rho > 1$ are as in the *User's Guide to the Proof*.

For our choice below, we will have that, for all $k \in [\![1,n]\!]$, $r_{k,n} \uparrow_n +\infty$. That choice for $r_{k,n}$ also yields that when $|U_k| \leq r_{k,n}$, our controls behave like for the bounded case. But when $|U_k| > r_{k,n}$, we will handle this large deviation regime by assumption **(GC)**. We indeed know that for all $K > 0$:

$$\mu(\{|x| > K\}) \leq 2\exp(-\frac{K^2}{2}). \tag{3.33}$$

Let us recall from the definition of $\Delta_k(X_{k-1},U_k)$ in (3.3) and (3.18) that:

$$\Delta_k(X_{k-1},U_k) = \sqrt{\gamma_k}\sigma_{k-1}U_k\cdot\nabla\varphi(X_{k-1}+\gamma_k b_{k-1}) + \Xi_k(X_{k-1},U_k), \tag{3.34}$$

where for all $(k,u) \in [\![1,n]\!] \times \mathbb{R}^r$:

$$\begin{aligned}\Xi_k(X_{k-1},u) := \quad & \gamma_k\int_0^1(1-t)\mathrm{Tr}\Big(D^2\varphi(X_{k-1}+\gamma_k b_{k-1}+t\sqrt{\gamma_k}\sigma_{k-1}u)\sigma_{k-1}u\otimes u\sigma_{k-1}^* \\ & -\mathbb{E}\big[D^2\varphi(X_{k-1}+\gamma_k b_{k-1}+t\sqrt{\gamma_k}\sigma_{k-1}U_k)\sigma_{k-1}U_k\otimes U_k\sigma_{k-1}^*|\mathcal{F}_{k-1}\big]\Big)dt.\end{aligned} \tag{3.35}$$

For the terms $(T_k)_{k\leq n}$ defined in (3.27), the lemma below controls the "*super martingality*" default of $S_n = \prod_{k=1}^n T_k$.

Lemma 5. *For all $k \in [\![1,n]\!]$*

$$\mathbb{E}[T_k|\mathcal{F}_{k-1}] \leq \aleph_{k,n}(\lambda,\gamma_k,r_{k,n}),$$

with

$$\aleph_{k,n}(\lambda,\gamma_k,r_{k,n}) := \big(1+2\exp(-Cr_{k,n}^2)\big)\exp\big(\frac{\rho q\lambda}{\Gamma_n}C\gamma_k^{1/2}\exp(-\frac{r_{k,n}^2}{4})\big)\exp\big(\frac{\rho^2q^2\lambda^2}{\Gamma_n^2}C\gamma_k^{3/2}r_{k,n}^2\big), \tag{3.36}$$

and

$$r_{k,n} = r_{k,n}((\mathbf{A}),\lambda,\rho) := \begin{cases} r_n = C(1+\frac{\rho q\lambda}{\Gamma_n})\big(\frac{\Gamma_n}{\Gamma_n^{(3/2)}}\big)^{1/4}, \ \textit{for } \theta\in(\frac{1}{3},1), \\ C(1+\frac{\rho q\lambda}{\Gamma_n})\ln(n+1)^{1/4}\ln(k+1)^{1/2}, \ \textit{for } \theta=1, \end{cases} \tag{3.37}$$

for $C>0$ s.t. $r_{1,1} > \bar{c}\sqrt{\gamma_1}\|\sigma\|_\infty\|\nabla\varphi\|_\infty$ for $\bar{c}$ large enough (every $\bar{c}>8$ works, see the proof of Lemma 5, and equation (4.22)). This choice is briefly explained in Remark 8 below.

Remark 8. *The specific form of the truncation and of the time steps chosen yields*

$$r_{k,n} = \begin{cases} O\big((1+\frac{\rho q\lambda}{\Gamma_n})n^{\frac{1-\theta}{4}}\big), \text{ if } \theta\in(2/3,1), \\ O\big((1+\frac{\rho q\lambda}{\Gamma_n})\ln(n)^{-1/4}n^{\frac{1}{12}}\big), \text{ if } \theta=2/3, \\ O\big((1+\frac{\rho q\lambda}{\Gamma_n})n^{\frac{\theta}{8}}\big), \text{ if } \theta\in(1/3,2/3). \end{cases}$$

Anyhow, we always have for each $k\leq n$, $r_{k,n} \underset{n}{\longrightarrow} +\infty$. Identity (3.38) in the following lemma can give an intuition of our choice in (3.37). This result ensures that the terms $\aleph_{k,n}(\lambda,\gamma_k,r_{k,n})$ can be viewed as remainders (observe indeed that some $e_n \underset{n}{\to} 0$ appear in the exponential (3.38) below). From the optimization in λ performed in the proof Lemma 4 (see equation (4.11)), the contribution $\frac{\rho\lambda}{\Gamma_n}$ appearing in (3.36) will be large in the regime of super Gaussian deviations (see as well Remark 16). The above choice of $r_{k,n}$ actually permits to control the remainders in all the considered regimes.

For $\theta\in(\frac{1}{3},1)$, the choice in (3.37) can seem natural in order to absorb the term $\prod_{k=1}^n\exp\big(\frac{\rho q\lambda}{\Gamma_n}C\gamma_k^{1/2}\exp(-\frac{r_{k,n}^2}{16})\big) = \exp\big(\frac{\rho q\lambda}{\Gamma_n}C\Gamma_n^{(1/2)}\exp(-\frac{r_n^2}{16})\big)$ coming from the iteration of (3.36). For $\theta=1$, the choice is a bit different due to the associated logarithmic explosion rates (i.e. $\Gamma_n \asymp \ln(n)$). Actually, for the Gaussian deviations ($\frac{a}{\sqrt{\Gamma_n}}\to 0$), we have $\frac{\rho q\lambda}{\Gamma_n}\to_n 0$, see again (4.11) and Remark 16 below, and the term $\frac{\rho q\lambda}{\Gamma_n}$ could be removed in (3.37). On the other hand, the contribution $\exp\big(\frac{\rho^2q^2\lambda^2}{\Gamma_n^2}C\gamma_k^{3/2}r_{k,n}^2\big)$ will eventually yield a negligible contribution in the polynomial appearing in Lemma 3. We refer to the proof of Lemma 5 for details.

Lemma 6 (Control of "super martingality default" of S_n)**.** *There exist non negative sequences $(\mathscr{R}_n)_{n\geq1}$, $(e_n)_{n\geq1}$ s.t. $\mathscr{R}_n \underset{n}{\longrightarrow} 1$, $e_n \underset{n}{\longrightarrow} 0$, and for all $n\geq1$:*

$$\mathbb{E}[S_n] \leq \prod_{k=1}^n \aleph_{k,n}(\lambda,\gamma_k,r_{k,n}) = \mathscr{R}_n\exp\Big(\big(\frac{\rho^2q^2\lambda^2}{\Gamma_n}+\frac{\rho^4q^4\lambda^4}{\Gamma_n^3}\big)e_n\Big). \tag{3.38}$$

Observe that a term in λ^4 appears here for the control of $\mathbb{E}[S_n]$. This is specifically due to the unbounded contributions. Namely, the exponential term in (3.38) comes from $\exp(\frac{\rho^2q^2\lambda^2}{\Gamma_n^2}C\gamma_k^{3/2}r_{k,n}^2)$ in (3.36), the definition of $r_{k,n}$ in (3.37), and using as well that $e_n \asymp \frac{\Gamma_n^{(3/2)}}{\Gamma_n} \underset{n\to\infty}{\to} 0$. The other terms in (3.36), corresponding to sub-Gaussian tails, give the remainder $\mathscr{R}_n$.

Note now carefully that, reproducing the arguments of the bounded case and using as well Lemma 6 to control $\mathbb{E}[S_n]$ yields that Lemma 3 remains valid, up to a modification of the remainders e_n. The proof then follows similarly to the bounded case. To sum up, the specificity of the unbounded innovations was to precisely control the Lipschitz constants, considering a suitable truncation, as well as the "martingality default" of S_n appearing in $\mathscr{T}_1$.

4. Proofs of technical lemmas

4.1. Remainders from the Taylor decomposition.

Proof of Lemma 2. For the control of remainders from Taylor expansion in (3.2), and in Lemma 1, we recall a useful result from [HMP17] (see Proposition 1 therein). Under **(A)**, there is a constant $c_V := c_V((\mathbf{A})) > 0$ such that for all $\lambda \in [0, c_V]$, $\xi \in [0,1]$:

$$I_V^{\xi} := \sup_{n\geq 0} \mathbb{E}[\exp(\lambda V_n^{\xi})] < +\infty. \tag{4.1}$$

We also refer to Lemaire (see [Lem05]) for additional integrability results of the Lyapunov functions in a more general framework.

From the notations in (3.18), we recall (3.20): $\nu_n(\mathcal{A}\varphi) = \frac{1}{\Gamma_n}(\widetilde{R}_n - M_n)$. The idea is now to write for $a, \lambda > 0$:

$$\begin{aligned}\mathbb{P}\big[\sqrt{\Gamma_n}\nu_n(\mathcal{A}\varphi) \geq a\big] &\leq \exp\big(-\frac{a\lambda}{\sqrt{\Gamma_n}}\big)\mathbb{E}\Big[\exp\big(\frac{\lambda}{\Gamma_n}(\widetilde{R}_n - M_n)\big)\Big] \\ &\leq \exp\big(-\frac{a\lambda}{\sqrt{\Gamma_n}}\big)\mathbb{E}\Big[\exp\big(-\frac{q\lambda}{\Gamma_n}M_n\big)\Big]^{1/q}\mathbb{E}\Big[\exp\big(\frac{p\lambda}{\Gamma_n}|\widetilde{R}_n|\big)\Big]^{1/p},\end{aligned} \tag{4.2}$$

for $\frac{1}{p} + \frac{1}{q} = 1,\ p, q > 1$. We rewrite the Taylor expansion with the same notations as in [HMP17]: $\widetilde{R}_n = L_n - (D_{2,b,n} + D_{2,\Sigma,n} + \bar{G}_n)$ where:

$$\begin{aligned} D_{2,b,n} &:= \sum_{k=1}^{n}\gamma_k^2\int_0^1(1-t)\,\mathrm{Tr}\Big(D^2\varphi(X_{k-1}+t\gamma_k b_{k-1})b_{k-1}\otimes b_{k-1}\Big)dt, \\ D_{2,\Sigma,n} &:= \frac{1}{2}\sum_{k=1}^{n}\gamma_k\mathrm{Tr}\Big(\big(D^2\varphi(X_{k-1}+\gamma_k b_{k-1}) - D^2\varphi(X_{k-1})\big)\Sigma_{k-1}^2\Big), \\ \bar{G}_n &:= \sum_{k=1}^{n}\mathbb{E}\left[\psi_k(X_{k-1},U_k)|\mathcal{F}_{k-1}\right], \\ L_n &:= \varphi(X_n) - \varphi(X_0). \end{aligned} \tag{4.3}$$

From (4.3), (4.2) and the Cauchy-Schwarz inequality, we get:

$$\begin{aligned}\mathbb{P}\big[\sqrt{\Gamma_n}\nu_n(\mathcal{A}\varphi) \geq a\big] &\leq \exp\big(-\frac{a\lambda}{\sqrt{\Gamma_n}}\big)\Big(\mathbb{E}\exp\big(-\frac{q\lambda}{\Gamma_n}M_n\big)\Big)^{\frac{1}{q}} \times \Big(\mathbb{E}\exp\big(\frac{2p\lambda}{\Gamma_n}|L_n|\big)\Big)^{\frac{1}{2p}} \\ &\times\Big(\mathbb{E}\exp\big(\frac{4p\lambda}{\Gamma_n}|\bar{G}_n|\big)\Big)^{\frac{1}{4p}}\Big(\mathbb{E}\exp\big(\frac{8p\lambda}{\Gamma_n}|D_{2,\Sigma,n}|\big)\Big)^{\frac{1}{8p}}\Big(\mathbb{E}\exp\big(\frac{8p\lambda}{\Gamma_n}|D_{2,b,n}|\big)\Big)^{\frac{1}{8p}}.\end{aligned} \tag{4.4}$$

The term L_n in (4.4) is controlled in Lemma 4 in [HMP17] (for $j = 2$ therein):

$$\Big(\mathbb{E}\exp\big(4p\lambda\frac{|L_n|}{\Gamma_n}\big)\Big)^{\frac{1}{4p}} \leq (I_V^1)^{\frac{1}{4p}}\exp\big(\frac{3pC_{V,\varphi}^2\lambda^2}{c_V\Gamma_n^2} + \frac{c_V}{p}\big) = \mathscr{R}_n\exp\big(\frac{\lambda_n^2}{\Gamma_n}e_n\big), \tag{4.5}$$

for $p = p_n \to_n +\infty$ s.t. $\frac{p}{\Gamma_n} \to_n 0$.

Thanks to Lemma 3 in [HMP17], we obtain:

$$\frac{|\bar{G}_n|}{\sqrt{\Gamma_n}} \leq a_n := \frac{[\varphi^{(3)}]_\beta\big\|\sigma\big\|_\infty^{3+\beta}\mathbb{E}\big[|U_1|^{3+\beta}\big]}{(1+\beta)(2+\beta)(3+\beta)}\frac{\Gamma_n^{(\frac{3+\beta}{2})}}{\sqrt{\Gamma_n}},\ a.s.\ .$$

Moreover, $a_n \underset{n}{\to} a_\infty = 0$ for $\theta \in (\frac{1}{2+\beta}, 1]$. Hence, for all $p > 1$:

$$\Big(\mathbb{E}\exp\big(\frac{4p\lambda}{\Gamma_n}|\bar{G}_n|\big)\Big)^{\frac{1}{4p}} \leq \exp\big(\frac{\lambda}{\sqrt{\Gamma_n}}a_n\big) \leq \exp\big(\frac{\lambda^2}{2\Gamma_n p} + \frac{a_n^2 p}{2}\big) = \mathscr{R}_n\exp\big(\frac{\lambda^2}{\Gamma_n}e_n\big). \tag{4.6}$$

for $p = p_n \to_n +\infty$ s.t. $a_n^2 p \to_n 0$.

We handle the term $D_{2,\Sigma,n}$ in (4.4) by Lemma 5 in [HMP17]: there exists $C_1 := C_1((\mathbf{A}), \varphi) > 0$ such that

$$\Big(\mathbb{E}\exp\big(\frac{4p\lambda_n}{\Gamma_n}\big|D_{2,\Sigma,n}\big|\big)\Big)^{\frac{1}{4p}} \leq \exp\big(C_1\frac{p\lambda_n^2(\Gamma_n^{(2)})^2}{\Gamma_n^2}\big)(I_V^1)^{\frac{1}{4p}} = \mathscr{R}_n \exp\big(\frac{\lambda_n^2}{\Gamma_n}e_n\big), \tag{4.7}$$

for $p = p_n \to_n +\infty$ s.t. $p\frac{(\Gamma_n^{(2)})^2}{\Gamma_n} \to_n 0$ (we recall that for all $\theta \in (\frac{1}{3}, 1]$, $\frac{\Gamma_n^{(2)}}{\sqrt{\Gamma_n}} \underset{n}{\to} 0$).

We deal with the term $D_{2,b,n}$ from (4.4). Because for all $x \in \mathbb{R}^d$, $\big|D^2\varphi(x)\big| \leq \frac{C_\varphi}{1+|x|}$ for some constant $C_\varphi \geq 0$ (the first case considered therein), thanks to Lemma 5 in [HMP17], we know that there exists $C_2 := C_2((\mathbf{A}), \varphi) > 0$ such that:

$$\Big(\mathbb{E}\exp\big(\frac{4p\lambda_n}{\Gamma_n}\big|D_{2,b,n}\big|\big)\Big)^{\frac{1}{4p}} \leq \exp\big(C_2(\frac{3p\lambda_n^2(\Gamma_n^{(2)})^2}{2\Gamma_n^2} + \frac{1}{2p})\big)(I_V^1)^{\frac{1}{4p}} = \mathscr{R}_n \exp\big(\frac{\lambda_n^2}{\Gamma_n}e_n\big), \tag{4.8}$$

also for $p = p_n \to_n +\infty$ s.t. $p\frac{(\Gamma_n^{(2)})^2}{\Gamma_n} \to_n 0$.

We gather (4.5), (4.6), (4.7) and (4.8) into (4.4), which allows us to control the remainder involving $\widetilde{R}_n$ previously decomposed in (4.3). There are non-negative sequences $(\mathscr{R}_n)_{n\geq 1}$, $(e_n)_{n\geq 1}$ s.t. $\lim\limits_{n\to+\infty} \mathscr{R}_n = 1$, $\lim\limits_{n\to+\infty} e_n = 0$ and:

$$\mathbb{P}\big[\sqrt{\Gamma_n}\nu_n(\mathcal{A}\varphi) \geq a\big] \leq \exp\Big(-\frac{a\lambda}{\sqrt{\Gamma_n}}\Big)\Big(\mathbb{E}\exp\Big(-\frac{q\lambda}{\Gamma_n}M_n\Big)\Big)^{\frac{1}{q}}\exp\Big(\frac{\lambda_n^2}{\Gamma_n}e_n\Big)\mathscr{R}_n. \tag{4.9}$$

□

4.2. **Asymptotics in the parameter ρ.** We first begin with the proof of Lemma 4 which is purely analytical and rather independent of our probabilistic setting. We recall that we use it for both bounded and unbounded innovations.

Proof of Lemma 4 . From the expression of $\Phi_n(a,\rho)$ in Theorem 8, remember that

$$\Phi_n(a,\rho) = \Big(\frac{a}{\sqrt{\Gamma_n}\tilde{B}_n} + \big(\frac{a^2}{\tilde{B}_n^2\Gamma_n} + (\rho-1)\big(\frac{2\tilde{A}_n}{3\tilde{B}_n}\big)^3\big)^{\frac{1}{2}}\Big)^{\frac{1}{3}} + \Big(\frac{a}{\sqrt{\Gamma_n}\tilde{B}_n} - \big(\frac{a^2}{\tilde{B}_n^2\Gamma_n} + (\rho-1)\big(\frac{2\tilde{A}_n}{3\tilde{B}_n}\big)^3\big)^{\frac{1}{2}}\Big)^{\frac{1}{3}}.$$

Here, $\rho > 1$ is a *free* parameter. Let us set:

$$\rho - 1 := \xi\frac{27}{8}\frac{\tilde{B}_n a^2}{\tilde{A}_n^3\Gamma_n}, \tag{4.10}$$

for a parameter $\xi := \xi(a,n) > 0$ to optimize. This choice yields

$$\Phi_n(a,\rho) = \frac{a^{1/3}}{\tilde{B}_n^{1/3}\Gamma_n^{1/6}}\big((1+\sqrt{1+\xi})^{1/3} + (1-\sqrt{1+\xi})^{1/3}\big).$$

Hence, from the definition of λ_n in (3.30):

$$\frac{\rho\lambda_n}{\Gamma_n} = \frac{1}{2}(\rho-1)^{1/3}\Phi_n(a,\rho) = \frac{3}{2^2}\frac{a}{\tilde{A}_n\sqrt{\Gamma_n}}\xi^{1/3}\big((1+\sqrt{1+\xi})^{1/3} + (1-\sqrt{1+\xi})^{1/3}\big). \tag{4.11}$$

We point out that $\xi \longmapsto \xi^{1/3}\big((1+\sqrt{1+\xi})^{1/3} + (1-\sqrt{1+\xi})^{1/3}\big)$ is a bounded function from $[0,+\infty)$ to $[0,\frac{2}{3})$. In fact,

$$\begin{aligned} &\xi^{1/3}\big((1+\sqrt{1+\xi})^{1/3} + (1-\sqrt{1+\xi})^{1/3}\big) \\ \underset{\xi\to+\infty}{=}\ &\xi^{1/3}(1+\xi)^{1/6}\big(1 + \frac{1}{3\sqrt{1+\xi}} - (1 - \frac{1}{3\sqrt{1+\xi}}) + o(\frac{1}{\sqrt{1+\xi}})\big) \underset{\xi\to+\infty}{\sim} \frac{2}{3}. \end{aligned} \tag{4.12}$$

From the definition of $P_{\min}(a,\rho,\Gamma_n) = P(\lambda_n) = \min_{x>0} P(x)$ in (3.31) and from the definition of $P(\lambda)$ in (3.29),

$$\begin{aligned}
P(\lambda_n) &= -\frac{(\rho-1)^{1/3}}{\rho}\frac{\sqrt{\Gamma_n}\Phi_n(a,\rho)}{8}(3a-\sqrt{\Gamma_n}(\rho-1)^{1/3}\tilde{A}_n\Phi_n(a,\rho)) \\
&= -\frac{(\rho-1)^{1/3}\Phi_n(a,\rho)}{2}\times\frac{\sqrt{\Gamma_n}}{4\rho}\times(3a-\tilde{A}_n\sqrt{\Gamma_n}(\rho-1)^{1/3}\Phi_n(a,\rho)) \\
&\underset{(4.10),(4.11)}{=} -\frac{3^2}{2^4}\frac{a^2}{\tilde{A}_n(\xi\frac{27\tilde{B}_n a^2}{8\tilde{A}_n^3\Gamma_n}+1)}\xi^{1/3}\big((1+\sqrt{1+\xi})^{1/3}+(1-\sqrt{1+\xi})^{1/3}\big) \\
&\quad\times\Big(1-\frac{\xi^{1/3}}{2}\big((1+\sqrt{1+\xi})^{1/3}+(1-\sqrt{1+\xi})^{1/3}\big)\Big) \\
&=: -\frac{3^2}{2^4\tilde{A}_n}a^2 f_\Psi(\xi), \qquad (4.13)
\end{aligned}$$

for

$$f_\Psi : \xi\in\mathbb{R}_+ \longmapsto \frac{g(\xi)}{\Psi\xi+1}, \qquad (4.14)$$

where

$$g:\xi\longmapsto \xi^{1/3}\big((1+\sqrt{1+\xi})^{1/3}+(1-\sqrt{1+\xi})^{1/3}\big)\Big(1-\frac{\xi^{1/3}}{2}\big((1+\sqrt{1+\xi})^{1/3}+(1-\sqrt{1+\xi})^{1/3}\big)\Big),$$

and

$$\Psi := \frac{27}{8}\frac{\tilde{B}_n a^2}{\tilde{A}_n^3\Gamma_n}\Big(\overset{(4.10)}{=}\frac{\rho-1}{\xi}\Big). \qquad (4.15)$$

We bring to mind that we consider "Gaussian deviations", namely $\frac{a}{\sqrt{\Gamma_n}}\underset{n}{\to}0$.

From (7.11) and the asymptotic of $\tilde{A}_n$ defined in (7.24), i.e. $4\tilde{A}_n\to_n 2\nu(|\sigma^*\nabla\varphi|^2)$, we want to choose $\xi:=\xi(a,n)$ s.t. $\Lambda(\xi):=\frac{3^2}{2^4}f_\Psi(\xi)\underset{\frac{a}{\sqrt{\Gamma_n}}\underset{n}{\to}0}{\to}\frac{1}{4}$. This would indeed yield $P(\lambda_n)\underset{\frac{a}{\sqrt{\Gamma_n}}\underset{n}{\to}0}{\sim}-\frac{a^2}{4\tilde{A}_n}$. From the definition of Ψ in (7.13), we have $\Psi=\frac{27\tilde{B}_n a^2}{8\tilde{A}_n\Gamma_n}\underset{\frac{a}{\sqrt{\Gamma_n}}\to 0}{\longrightarrow}0$. Observe then, that, taking ξ going to infinity such that

$$\xi\Psi\underset{\frac{a}{\sqrt{\Gamma_n}}\to 0}{\to}0 \qquad (4.16)$$

yields $\Lambda(\xi)=\frac{3^2}{2^4}\frac{g(\xi)}{\Psi\xi+1}\underset{\frac{a}{\sqrt{\Gamma_n}}\to 0}{\to}\frac{1}{4}$, noting from (4.12) and the above definition of g that $g(\xi)\underset{\xi\to+\infty}{\to}\big(\frac{2}{3}\big)^2$. □

Remark 9 (Controls of the optimized parameters λ and ρ for *Gaussian deviations*)**.** *We give here some useful estimates to control the remainder terms in the truncation procedure associated with unbounded innovations, see proof of Lemma 6. They specify the behaviour of the quantity* $\frac{\rho\lambda_n}{\Gamma_n}$.

In the regime of Gaussian deviations, we get from (4.11) *and* (4.12)*:*

$$\frac{\rho\lambda_n}{\Gamma_n}\underset{\frac{a}{\sqrt{\Gamma_n}}\to 0}{\longrightarrow}0,\ \lambda_n\underset{\frac{a}{\sqrt{\Gamma_n}}\to 0}{\asymp}a\sqrt{\Gamma_n}. \qquad (4.17)$$

4.3. **Technical Lemmas for Unbounded Innovations.** We proceed with the proof of Lemmas 5 and 6 which are specifically needed for unbounded innovations.

Proof of Lemma 5. We use a partition at the threshold $r_{k,n}$ on the variable U_k, to control finely the term $\Delta_k(X_{k-1},U_k)$:

$$\mathbb{E}\Big[\exp\big(-\frac{\rho q\lambda}{\Gamma_n}\Delta_k(X_{k-1},U_k)\big)|\mathcal{F}_{k-1}\Big]=\mathbb{E}\Big[\exp\big(-\frac{\rho q\lambda}{\Gamma_n}\Delta_n(X_{k-1},U_k)\big)\mathbb{1}_{|U_k|\leq r_{k,n}}|\mathcal{F}_{k-1}\Big]$$

$$+\mathbb{E}\Big[\exp\big(-\frac{\rho q\lambda}{\Gamma_n}\Delta_k(X_{k-1},U_k)\big)\mathbb{1}_{|U_k|>r_{k,n}}|\mathcal{F}_{k-1}\Big]=:T^M_{k,s}+T^M_{k,l}, \tag{4.18}$$

where $T^M_{k,s}$ and $T^M_{k,l}$ stand for the contributions in $\mathbb{E}[T_k|\mathcal{F}_{k-1}]$ associated with the martingale increment $\Delta(X_{k-1},U_k)$ for which the innovation is respectively *small* and *large*.

Let us first write:

$$T^M_{k,s}=\mathbb{E}\Big[\exp\big(-\frac{\rho q\lambda}{\Gamma_n}\Delta_n(X_{k-1},U_k)\big)\mathbb{1}_{|U_k|\leq r_{k,n}}|\mathcal{F}_{k-1}\Big]=\int_{|u|\leq r_{k,n}}\exp\big(-\frac{\rho q\lambda}{\Gamma_n}\Delta_n(X_{k-1},U_k)\big)\mu(du).$$

Observe now that, similarly to the computations of Section 3.2.1 for bounded innovations, see equation (3.26), if $|u|\leq r_{k,n}$, $u\in\mathbb{R}^r\mapsto\Delta_n(X_{k-1},u)$ is s.t.

$$[\Delta_n(X_{k-1},\cdot)]^2_{1,B(0,r_{k,n})}\leq\gamma_k|\sigma^*_{k-1}\nabla\varphi(X_{k-1})|^2+C\gamma_k^{3/2}r^2_{k,n}, \tag{4.19}$$

where $[\Delta_n(X_{k-1},\cdot)]^2_{1,B(0,r_{k,n})}$ denotes the Lipschitz modulus of $\Delta_n(X_{k-1},\cdot)$ restricted on the ball $B(0,r_{k,n})$ of $\mathbb{R}^r$ with radius $r_{k,n}$.

Let us now extend $u\in B(0,r_{k,n})\mapsto\Delta_n(X_{k-1},u)$ into a Lipschitz function on the whole set $\mathbb{R}^r$ which globally verifies the bound of equation (4.19). The easiest way to do so is to consider:

$$u\in\mathbb{R}^r\mapsto\bar{\Delta}(X_{k-1},u)=\Delta\big(X_{k-1},\Pi_{\bar{B}(0,r_{k,n})}(u)\big),$$

where $\Pi_{\bar{B}(0,r_{k,n})}(\cdot)$ denotes the projection on $\bar{B}(0,r_{k,n})$, namely for $u\in\bar{B}(0,r_{k,n})$, $\Pi_{\bar{B}(0,r_{k,n})}(u)=u$, for $u\notin B(0,r_{k,n})$, $\Pi_{\bar{B}(0,r_{k,n})}(u)=\frac{u}{|u|}r_{k,n}$. It is readily seen that:

$$[\bar{\Delta}(X_{k-1},\cdot)]^2_1\leq\gamma_k|\sigma^*_{k-1}\nabla\varphi(X_{k-1})|^2+C\gamma_k^{3/2}r^2_{k,n}.$$

Hence:

$$\begin{aligned}T^M_{k,s}&=\int_{|u|\leq r_{k,n}}\exp\big(-\frac{\rho q\lambda}{\Gamma_n}\bar{\Delta}_n(X_{k-1},U_k)\big)\mu(du)\leq\mathbb{E}\Big[\exp\big(-\frac{\rho q\lambda}{\Gamma_n}\bar{\Delta}_n(X_{k-1},U_k)\big)|\mathcal{F}_{k-1}\Big]\\&\underset{(\mathbf{GC})}{\leq}\exp\big(\frac{\rho^2q^2\lambda^2}{2\Gamma_n^2}[\bar{\Delta}(X_{k-1},\cdot)]^2_1-\frac{\rho q\lambda}{\Gamma_n}\mathbb{E}[\bar{\Delta}(X_{k-1},U_k)]\big)\\&\leq\exp\big(\frac{\rho^2q^2\lambda^2}{2\Gamma_n^2}(\gamma_k|\sigma^*_{k-1}\nabla\varphi(X_{k-1})|^2+C\gamma_k^{3/2}r^2_{k,n})-\frac{\rho q\lambda}{\Gamma_n}\mathbb{E}[\bar{\Delta}(X_{k-1},U_k)]\big).\end{aligned} \tag{4.20}$$

Bearing in mind that $\mathbb{E}[\Delta(X_{k-1},U_k)]=0$, and observe now:

$$\begin{aligned}\mathbb{E}[\bar{\Delta}(X_{k-1},U_k)]&=\mathbb{E}[\Delta(X_{k-1},U_k)\mathbb{1}_{|U_k|<r_{k,n}}]+\mathbb{E}[\bar{\Delta}(X_{k-1},U_k)\mathbb{1}_{|U_k|\geq r_{k,n}}]\\&=-\mathbb{E}[\Delta(X_{k-1},U_k)\mathbb{1}_{|U_k|\geq r_{k,n}}]+\mathbb{E}[\Delta\big(X_{k-1},\Pi_{B(0,r_{k,n})}(U_k)\big)\mathbb{1}_{|U_k|\geq r_{k,n}}]\\&\leq\Big(\mathbb{E}[|\Delta(X_{k-1},U_k)|^2]^{1/2}+\mathbb{E}[|\Delta\big(X_{k-1},\Pi_{B(0,r_{k,n})}(U_k)\big)|^2]^{1/2}\Big)\mathbb{P}[|U_k|\geq r_{k,n}]^{1/2}\\&\leq C\gamma_k^{1/2}\exp(-\frac{r^2_{k,n}}{4}),\end{aligned}$$

exploiting (3.33), (3.34) and (3.35) for the last inequality. Plugging this bound into (4.20) yields:

$$T^M_{k,s}\leq\exp\big(\frac{\rho^2q^2\lambda^2}{2\Gamma_n^2}(\gamma_k|\sigma^*_{k-1}\nabla\varphi(X_{k-1})|^2+C\gamma_k^{3/2}r^2_{k,n})+C\gamma_k^{1/2}\frac{\rho q\lambda}{\Gamma_n}\exp(-\frac{r^2_{k,n}}{4})\big). \tag{4.21}$$

The remaining term in (4.18), involving also the large deviations of the innovation, can be controlled as follows:

$$\begin{aligned} T_{k,l}^M &= \mathbb{E}\Big[\exp\big(-\frac{\rho q\lambda}{\Gamma_n}\Delta_k(X_{k-1},U_k)\big)\mathbb{1}_{|U_k|>r_{k,n}}|\mathcal{F}_{k-1}\Big] \\ &\leq \mathbb{E}\Big[\exp\big(-\frac{2\rho q\lambda}{\Gamma_n}\Delta_k(X_{k-1},U_k)\big)|\mathcal{F}_{k-1}\Big]^{1/2}\mathbb{E}\Big[\mathbb{1}_{|U_k|>r_{k,n}}\Big]^{1/2} \\ &\leq 2\mathbb{E}\Big[\exp\big(-\frac{2\rho q\lambda}{\Gamma_n}\Delta_k(X_{k-1},U_k)\big)|\mathcal{F}_{k-1}\Big]^{1/2}\exp(-\frac{r_{k,n}^2}{4}) \\ &\overset{(\mathbf{GC})}{\leq} 2\exp\big(\frac{\rho^2q^2\lambda^2}{\Gamma_n^2}\gamma_k\|\sigma\|_\infty^2\|\nabla\varphi\|_\infty^2-\frac{r_{k,n}^2}{4}\big). \end{aligned}$$

Let us proceed from our definition of $r_{k,n}=r_{k,n}((\mathbf{A}),\lambda,\rho)$ in (3.37) and write $r_{k,n}=(1+\frac{\rho q\lambda}{\Gamma_n})u_{k,n}$. One has that for all $k\leq n$, $u_{k,n}\underset{n}{\to}+\infty$. In other words, with the previous inequality, recalling $r_{k,n}^2\geq u_{k,n}^2(1+\frac{\rho^2q^2\lambda^2}{\Gamma_n^2})$:

$$\begin{aligned} T_{k,l}^M &\leq 2\exp\big(\frac{q^2\rho^2\lambda^2}{\Gamma_n^2}(\gamma_k\|\sigma\|_\infty^2\|\nabla\varphi\|_\infty^2-\frac{u_{k,n}^2}{4})-\frac{u_{k,n}^2}{4}\big) \\ &\leq 2\exp\big(-C\frac{q^2\rho^2\lambda^2}{\Gamma_n^2}\frac{u_{k,n}^2}{8}-\frac{u_{k,n}^2}{4}\big)\leq 2\exp\big(-Cr_{k,n}^2\big), \end{aligned} \tag{4.22}$$

since $u_{k,n}^2>16\gamma_1\|\sigma\|_\infty^2\|\nabla\varphi\|_\infty^2$ (which explains our choice for the constant $r_{k,n}$ in (3.37)). Plugging (4.21) and (4.22) into (4.18) yields:

$$\begin{aligned} &\mathbb{E}\Big[\exp\big(-\frac{\rho q\lambda}{\Gamma_n}\Delta_k(X_{k-1},U_k)\big)|\mathcal{F}_{k-1}\Big] \\ &\leq \exp\Big(\frac{\rho^2q^2\lambda^2}{2\Gamma_n^2}\big(\gamma_k|\sigma_{k-1}^*\nabla\varphi_{k-1}|^2+C\gamma_k^{3/2}r_{k,n}^2\big)+\frac{\rho q\lambda}{\Gamma_n}C\gamma_k^{1/2}\exp(-\frac{r_{k,n}^2}{4})\Big)+2\exp\big(-Cr_{k,n}^2\big) \\ &\leq \exp\big(\frac{\rho^2q^2\lambda^2}{2\Gamma_n^2}\big(\gamma_k|\sigma_{k-1}^*\nabla\varphi_{k-1}|^2+C\gamma_k^{3/2}r_{k,n}^2\big)+\frac{\rho q\lambda}{\Gamma_n}C\gamma_k^{1/2}\exp(-\frac{r_{k,n}^2}{4})\big)\big(1+2\exp(-Cr_{k,n}^2)\big) \\ &=: \exp\big(\frac{\rho^2q^2\lambda^2}{2\Gamma_n^2}\gamma_k|\sigma_{k-1}^*\nabla\varphi_{k-1}|^2\big)\times\aleph_{k,n}(\lambda,\gamma_k,r_{k,n}), \end{aligned}$$

where

$$\aleph_{k,n}(\lambda,\gamma_k,r_{k,n}):=\Big(1+2\exp\big(-Cr_{k,n}^2\big)\Big)\exp\big(\frac{\rho q\lambda}{\Gamma_n}C\gamma_k^{1/2}\exp(-\frac{r_{k,n}^2}{4})\big)\exp\big(\frac{\rho^2q^2\lambda^2}{\Gamma_n^2}C\gamma_k^{3/2}r_{k,n}^2\big).$$

We have thus isolated the "significant" term $\exp\big(\frac{\rho^2q^2\lambda^2}{2\Gamma_n^2}\gamma_k|\sigma_{k-1}^*\nabla\varphi_{k-1}|^2\big)$. The result follows from the definition of T_k in (3.27). □

Proof of Lemma 6. We recall for convenience the definition of $r_{k,n}$ in (3.37):

$$r_{k,n}=\begin{cases} r_n=C(1+\frac{\rho q\lambda}{\Gamma_n})(\frac{\Gamma_n}{\Gamma_n^{(3/2)}})^{1/4},\ for\ \theta\in(\frac{1}{3},1), \\ C((1+\frac{\rho q\lambda}{\Gamma_n})\ln(n+1)^{1/4}\ln(k+1)^{1/2},\ for\ \theta=1. \end{cases}$$

From the above definition of $\aleph_{k,n}(\lambda,\gamma_k,r_{k,n})$, we introduce two remainders:

$$\begin{aligned}\mathcal{R}_n^1 &:= \prod_{k=1}^n\Big(1+2\exp\big(-Cr_{k,n}^2\big)\Big)\times\prod_{k=1}^n\exp\big(\frac{\rho q\lambda}{\Gamma_n}C\gamma_k^{1/2}\exp(-\frac{r_{k,n}^2}{4})\big)\\ &=: \mathcal{R}_n^{11}\times\mathcal{R}_n^{12},\\ \mathcal{R}_n^2 &:= \exp\big(\frac{\rho^2q^2\lambda^2}{\Gamma_n^2}C\sum_{k=1}^n\gamma_k^{3/2}r_{k,n}^2\big),\end{aligned}\tag{4.23}$$

that naturally appear when we iterate Lemma 5. Precisely:

$$\mathbb{E}[S_n]\le\mathbb{E}[S_{n-1}]\aleph_{n,n}(\lambda,\gamma_n,r_{n,n})\le\prod_{k=1}^n\aleph_{k,n}(\lambda,\gamma_k,r_{k,n})=\mathcal{R}_n^1\times\mathcal{R}_n^2.\tag{4.24}$$

- For $\theta\in(1/3,1)$:

We have chosen $r_{k,n}=r_n$ in (3.37), so, $\mathcal{R}_n^1=\Big(1+2\exp\big(-Cr_n^2\big)\Big)^n\exp\big(\frac{\rho q\lambda}{\Gamma_n}C\Gamma_n^{(1/2)}\exp(-\frac{r_n^2}{4})\big)=\mathcal{R}_n^{11}\mathcal{R}_n^{12}$, and:

$$\begin{aligned}\mathcal{R}_n^{11} &= \Big(1+2\exp\big(-Cr_n^2\big)\Big)^n\le\exp\Big(n\ln\big(1+2\exp(-C(\frac{\Gamma_n}{\Gamma_n^{(3/2)}})^{1/2})\big)\Big)\\ &\le\exp\big(2n\exp(-C(\frac{\Gamma_n}{\Gamma_n^{(3/2)}})^{1/2})\big)\underset{n\to+\infty}{\longrightarrow}1,\end{aligned}$$

as, for $\theta\in(1/3,1)$, $\Big(\frac{\Gamma_n}{\Gamma_n^{(3/2)}}\Big)^{1/2}\ge\eta(n)$ where

$$\eta(n):=C\big(n^{(1-\theta)/2}\mathbb{1}_{\theta\in(2/3,1)}+(n^{1/3}\ln(n)^{-1})^{\frac12}\mathbb{1}_{\theta=2/3}+n^{\theta/4}\mathbb{1}_{\theta\in(1/3,2/3)}\big),\tag{4.25}$$

see also Remark 8. For the remaining of the proof, we will thoroughly exploit this kind of arguments. Precisely, we recall that:

$$\forall\zeta\in\mathbb{R}_+,\ \exists C_\zeta\ge1,\ s.t.\ \forall0\le\beta\le\zeta,\forall x\in\mathbb{R}_+,\ x^\beta\exp(-x^2)\le C_\zeta\exp(-C_\zeta^{-1}x^2).\tag{4.26}$$

Thus, for the remaining term $\mathcal{R}_n^{12}$ in $\mathcal{R}_n^1$, exploiting (4.26), up to a modification of the constant $C>0$ from line to line:

$$\begin{aligned}\mathcal{R}_n^{12} &\le \exp\big(\frac{\rho q\lambda}{\Gamma_n}C\Gamma_n^{(1/2)}\exp(-\frac{r_n^2}{4})\big)\\ &\le \exp\Big(\frac{\rho q\lambda}{\Gamma_n}C\Gamma_n^{(1/2)}\exp\big(-\frac{\rho^2q^2\lambda^2}{\Gamma_n^2}\frac{C}{4}(\frac{\Gamma_n}{\Gamma_n^{(3/2)}})^{1/2}\big)\exp\big(-\frac{C}{4}(\frac{\Gamma_n}{\Gamma_n^{(3/2)}})^{1/2}\big)\Big)\\ &\le \exp\Big(C\Gamma_n^{(1/2)}(\frac{\Gamma_n^{(3/2)}}{\Gamma_n})^{1/4}\exp\big(-\frac{\rho^2q^2\lambda^2}{\Gamma_n^2}\frac{C}{8}(\frac{\Gamma_n}{\Gamma_n^{(3/2)}})^{1/2}\big)\exp(-\frac{C}{4}(\frac{\Gamma_n}{\Gamma_n^{(3/2)}})^{1/2})\Big)\\ &\le \exp\big(C\Gamma_n^{(1/2)}\exp(-\frac{C}{8}(\frac{\Gamma_n}{\Gamma_n^{(3/2)}})^{1/2})\big)\le\exp\big(Cn^{1-\theta/2}e^{-\eta(n)}\big)\overset{(4.25),(4.26)}{\underset{n\to+\infty}{\longrightarrow}}1.\end{aligned}$$

Hence $\mathcal{R}_n^1=\mathcal{R}_n^{11}\mathcal{R}_n^{12}\underset{n}{\to}1$.

Introducing

$$e_n^{\theta<1}:=C\sqrt{\frac{\Gamma_n^{(3/2)}}{\Gamma_n}}\underset{n\to+\infty}{\longrightarrow}0,$$

we get from the definition of $r_{k,n}$ in (3.37) and (4.23)

$$\mathcal{R}_n^2\le\exp\Big(\big(\frac{\rho^2q^2\lambda^2}{\Gamma_n}+\frac{\rho^4q^4\lambda^4}{\Gamma_n^3}\big)e_n^{\theta<1}\Big).$$

Recalling, from (4.24), that $\mathbb{E}[S_n] \leq \mathcal{R}_n^1 \mathcal{R}_n^2$, we thus get from the above computations:

$$\mathbb{E}[S_n] \leq \mathscr{R}_n \exp\big((\frac{\rho^2 q^2 \lambda^2}{\Gamma_n} + \frac{\rho^4 q^4 \lambda^4}{\Gamma_n^3}) e_n\big),$$

which gives the result for $\theta \in (\frac{1}{3}, 1)$.

• For $\theta = 1$, using the previous notations:

$$\begin{aligned}
0 &\leq \ln(\mathcal{R}_n^{11}) \leq \ln\Big(\prod_{k=1}^n \big(1 + 2\exp(-Cr_{k,n}^2)\big)\Big) = \sum_{k=1}^n \ln\Big(1 + 2\exp\big(-Cr_{k,n}^2\big)\Big) \\
&\leq C\sum_{k=1}^n \exp\big(-Cr_{k,n}^2\big) \leq C\sum_{k=1}^n \exp\big(-C\ln(n+1)^{1/2}\ln(k+1)\big) \\
&= C\sum_{k=1}^n (k+1)^{-C\ln(n+1)^{1/2}} \leq C2^{-C\ln(n+1)^{1/2}} + C\int_2^n x^{-C\ln(n+1)^{1/2}} dx \\
&= C2^{-C\ln(n+1)^{1/2}} + C\frac{2^{1-C\ln(n+1)^{1/2}} - n^{1-C\ln(n+1)^{1/2}}}{C\ln(n+1)^{1/2} - 1} \underset{n\to+\infty}{\longrightarrow} 0.
\end{aligned}$$

We deduce that:

$$\mathcal{R}_n^{11} = \prod_{k=1}^n \Big(1 + 2\exp\big(-Cr_{k,n}^2\big)\Big) \underset{n\to+\infty}{\longrightarrow} 1.$$

Like for the case $\theta \in (1/3, 1)$, we get the control:

$$\begin{aligned}
\mathcal{R}_n^{12} &\leq \prod_{k=1}^n \exp\big(\frac{\rho q\lambda}{\Gamma_n} C\gamma_k^{1/2}\exp(-\frac{r_{k,n}^2}{4})\big) \\
&\leq \prod_{k=1}^n \exp\Big(\frac{\rho q\lambda}{\Gamma_n} C\gamma_k^{1/2}\exp(-\frac{C}{4}[1 + (\frac{\rho q\lambda}{\Gamma_n})^2]\ln(n+1)^{1/2}\ln(k+1))\Big) \\
&\underset{(4.26)}{\leq} \prod_{k=1}^n \exp\Big(C\gamma_k^{1/2}\exp\big(-\frac{C}{4}[1 + C(\frac{\rho q\lambda}{\Gamma_n})^2]\ln(n+1)^{1/2}\ln(k+1)\big)\Big) \\
&\leq \prod_{k=1}^n \exp\big(C\frac{1}{k^{1/2}}(k+1)^{-C\ln(n+1)^{1/2}}\big) \underset{n\to+\infty}{\longrightarrow} 1,
\end{aligned}$$

in fact, we have already established for the control of $\mathcal{R}_n^{11}$ that $\sum_{k=1}^n (k+1)^{-C\ln(n+1)^{1/2}} \underset{n}{\longrightarrow} 0$. Thus, we proved that $\mathcal{R}_n^1 = \mathcal{R}_n^{11}\mathcal{R}_n^{12} \underset{n}{\to} 1$.

Let us now turn to the other contribution in (4.23):

$$\begin{aligned}
\mathcal{R}_n^2 &= \exp\big(\frac{\rho^2 q^2 \lambda^2}{\Gamma_n^2} C\sum_{k=1}^n \gamma_k^{3/2} r_{k,n}^2\big) \\
&\leq \exp\Big(2C(\frac{\rho^2 q^2 \lambda^2}{\Gamma_n^2} + \frac{\rho^4 q^4 \lambda^4}{\Gamma_n^4})\ln(n+1)^{1/2}\sum_{k=1}^n \gamma_k^{3/2}\ln(k+1)\Big) \\
&\leq \exp\big(2C(\frac{\rho^2 q^2 \lambda^2}{\Gamma_n} + \frac{\rho^4 q^4 \lambda^4}{\Gamma_n^3})\frac{\sum_{k=1}^n \gamma_k^{3/2}\ln(k+1)}{\sqrt{\Gamma_n}}\big) \\
&= \exp\big((\frac{\rho^2 q^2 \lambda^2}{\Gamma_n} + \frac{\rho^4 q^2 \lambda^4}{\Gamma_n^3}) e_n^{\theta=1}\big),
\end{aligned}$$

with $e_{1,n}^{\theta=1} := 2C\frac{1}{\sqrt{\Gamma_n}}\sum_{k=1}^{n}\frac{\ln(k+1)}{k^{3/2}} \underset{n\to+\infty}{\longrightarrow} 0$, and recalling that $\sqrt{\Gamma_n} \asymp \ln(n+1)^{1/2}$ for the last inequality.

To sum up, for all $\theta \in (\frac{1}{3}, 1]$, thanks to inequality (4.24), we know that there exist non negative sequences $(\mathscr{R}_n)_{n\geq 1}$, $(e_n)_{n\geq 1}$, s.t. $\mathscr{R}_n \underset{n}{\longrightarrow} 1$, $e_n \underset{n}{\longrightarrow} 0$, and

$$\mathbb{E}[S_n] \leq \mathscr{R}_n \exp\Big(\big(\frac{\rho^2 q^2 \lambda^2}{\Gamma_n} + \frac{\rho^4 q^4 \lambda_n^4}{\Gamma_n^3}\big)e_n\Big).$$

□

5. Slutsky type concentration results

When $f^2 - \nu(f^2)$ is coboundary and for f satisfying suitable growth assumptions, we state a Slutsky like Theorem. The following result is a by-product of Theorem 3. The idea is to divide the normalized deviations by the term $\sqrt{\nu_n(f^2) - \nu(f)^2}$ in order to get rid of the contribution $\nu(|\sigma^*\nabla\varphi|^2)$ in the upper-bound. In fact, the purpose of our work is to estimate the invariant measure ν. Upper-bounding the deviation by this very measure is not natural from the practical viewpoint.

Theorem 4 (Slutsky type concentration result)**.** *We suppose that* **(A)** *holds, for $f \in \mathcal{C}^{1,\beta}(\mathbb{R}^d, \mathbb{R})$, $f^2 \in \mathcal{C}^{1,\beta}(\mathbb{R}^d, \mathbb{R})$, $\beta \in (0, 1]$, and $\theta \in (\frac{1}{2+\beta}, 1]$. We assume as well that there are solutions $\varphi, \tilde{\varphi}, \vartheta \in \mathcal{C}^{3,\beta}(\mathbb{R}^d, \mathbb{R})$ to $\mathcal{A}\varphi = f - \nu(f)$, $\mathcal{A}\tilde{\varphi} = f^2 - \nu(f^2)$ and $\mathcal{A}\vartheta = |\sigma^*\nabla\varphi|^2 - \nu(|\sigma^*\nabla\varphi|^2)$ respectively, satisfying* $(\mathbf{T}_\beta)$*. If there are constants $C_{V,f} > 0$, $\eta \in [\frac{1}{4}, \frac{1}{2}]$, s.t. for all $x \in \mathbb{R}^d$:*

$$|f(x)| \leq C_{V,f}(1 + V^{\frac{1}{2}-\eta}(x)), \tag{5.1}$$

and $a = o(\Gamma_n^{\frac{\eta}{1+2\eta}})$, then there exist two explicit monotonic sequences $c_n \leq 1 \leq C_n$, $n \geq 1$, with $\lim_n C_n = \lim_n c_n = 1$ s.t.

$$\begin{aligned}
&\mathbb{P}\big[|\sqrt{\Gamma_n}\frac{\nu_n(f) - \nu(f)}{\sqrt{\nu_n(f^2) - \nu_n(f)^2}}| \geq a\big] \leq 2\, C_n \exp\big(-c_n\frac{a^2}{2}\big),\\
&\mathbb{P}\Big[\nu(f) \in \big[\nu_n(f) - \frac{a\sqrt{\nu_n(f^2) - \nu_n(f)^2}}{\sqrt{\Gamma_n}}, \nu_n(f) + \frac{a\sqrt{\nu_n(f^2) - \nu_n(f)^2}}{\sqrt{\Gamma_n}}\big]\Big] \geq 1 - 2C_n \exp\big(-c_n\frac{a^2}{2}\big).
\end{aligned} \tag{5.2}$$

Remark 10. *We obtain here, Gaussian deviations, up to the threshold $a = o(\Gamma_n^{\frac{\eta}{1+2\eta}})$. This condition is more stringent than in Theorem 3 and Theorem 5 in* [HMP17]*. However this is still perfectly suited for confidence intervals (in fact, a is allowed to go to $+\infty$ as far as $\frac{a}{\Gamma_n^{\frac{\eta}{1+2\eta}}} \to_n 0$).*

The smaller deviation range is here the price to pay to get deviation bounds which are asymptotically independent of the gradient $\nabla\varphi$ of the associated Poisson problem. This improves, up to the previous indicated restriction on a, the result of Theorem 5 in [HMP17] *where an upper-bound of $|\nabla\varphi|$, depending on $[f]_1$ and the Lyapunov conditions, appears. Here, the normalization is explicitly computable and only the remainder terms do depend on the auxiliary Poisson problems. Namely, the C_n, c_n in* (5.2) *depend on $\varphi, \tilde{\varphi}, \vartheta$.*

However, to obtain this Slutsky like theorem, we have to consider a test function f, s.t. f^2 is Lipschitz continuous which is natural in order to have the required smoothness on the corresponding Poisson problem solved by $\tilde{\varphi}$. This imposes the restriction for the growth condition in (5.1)*. Again, to have $\nabla\tilde{\varphi}$ bounded, where $\mathcal{A}\tilde{\varphi} = f^2 - \nu(f^2)$, it is natural to impose that f^2 has linear growth. This still comes from the fact that $\nabla\tilde{\varphi}$ will be controlled by $[f^2]_1$ and the Lyapunov conditions, see Section 6 below (especially Theorem 5 and Corollaries 1 and 2) and Lemma 6 of* [HMP17]*.*

Proof of Theorem 4. Observe, that for all $f \in \mathcal{C}^{1,\beta}(\mathbb{R}^d, \mathbb{R})$, we have:

$$Var_\nu(f) = \nu(f^2) - \nu(f)^2 = \nu(|f - \nu(f)|^2) = \nu(|\sigma^* \nabla \varphi|^2),$$

see for instance [BGL14].

Furthermore, we get that:

$$\begin{aligned} &\nu_n(f^2) - \nu_n(f)^2 - [\nu(f^2) - \nu(f)^2] \\ &= \nu_n(f^2 - \nu(f^2)) - (\nu_n(f) - \nu(f))^2 - 2\nu_n(f)\nu(f) + 2\nu(f)^2 \\ &= \nu_n(\mathcal{A}\widetilde{\varphi}) - \nu_n(\mathcal{A}\varphi)^2 - 2\nu(f)\nu_n(\mathcal{A}\varphi). \end{aligned} \tag{5.3}$$

Hence, like for the proof of Theorem 3 in [HMP17], we proceed foremost with an exponential Bienaymé-Tchebychev inequality. For all $n > 0$, and $\lambda_n > 0$ introduced in (3.30), we have:

$$\begin{aligned} &\mathbb{P}\big[\sqrt{\Gamma_n}\frac{\nu_n(f) - \nu(f)}{\sqrt{\nu_n(f^2) - \nu_n(f)^2}} \geq a\big] = \mathbb{P}\big[\nu_n(f - \nu(f)) \geq \frac{a}{\sqrt{\Gamma_n}}\sqrt{\nu_n(f^2) - \nu(f)^2}\big] \\ &\leq \mathbb{E}\Big[\exp\big(-\frac{a\lambda_n}{\sqrt{\Gamma_n}}\sqrt{\nu_n(f^2) - \nu_n(f)^2}\big)\exp\big(\lambda_n \nu_n(\mathcal{A}\varphi)\big)\Big] = \exp\big(-\frac{a\lambda_n}{\sqrt{\Gamma_n}}\sqrt{\nu(f^2) - \nu(f)^2}\big) \\ &\times \mathbb{E}\Big[\exp\big(-\frac{a\lambda_n}{\sqrt{\Gamma_n}}\big[\sqrt{\nu_n(f^2) - \nu_n(f)^2} - \sqrt{\nu(f^2) - \nu(f)^2}\,\big]\big)\exp\big(\lambda_n \nu_n(\mathcal{A}\varphi)\big)\Big] \\ &= \exp\big(\frac{-a\lambda_n}{\sqrt{\Gamma_n}}\sqrt{\nu(f^2) - \nu(f)^2}\big) \\ &\times \mathbb{E}\Big[\exp\big(\frac{-a\lambda_n}{\sqrt{\Gamma_n}}\frac{\nu_n(f^2) - \nu_n(f)^2 - \nu(f^2) + \nu(f)^2}{\sqrt{\nu_n(f^2) - \nu_n(f)^2} + \sqrt{\nu(f^2) - \nu(f)^2}}\big)\exp\big(\lambda_n \nu_n(\mathcal{A}\varphi)\big)\Big] \\ &\overset{(5.3)}{=} \exp\big(\frac{-a\lambda_n}{\sqrt{\Gamma_n}}\sqrt{\nu(|\sigma^*\nabla\varphi|^2)}\big) \\ &\times \mathbb{E}\Big[\exp\big(\frac{-a\lambda_n}{\sqrt{\Gamma_n}}\frac{\nu_n(\mathcal{A}\widetilde{\varphi}) - \nu_n(\mathcal{A}\varphi)^2 - 2\nu(f)\nu_n(\mathcal{A}\varphi)}{\sqrt{\nu_n(f^2) - \nu_n(f)^2} + \sqrt{\nu(f^2) - \nu(f)^2}}\big)\exp\big(\lambda_n \nu_n(\mathcal{A}\varphi)\big)\Big]. \end{aligned} \tag{5.4}$$

By the Hölder and Cauchy-Schwarz inequalities, we obtain from (5.4) and for $\tilde{p}, \tilde{q} > 1$, s.t. $\frac{1}{\tilde{p}} + \frac{1}{\tilde{q}} = 1$:

$$\begin{aligned} &\mathbb{P}\big[\sqrt{\Gamma_n}\frac{\nu_n(f) - \nu(f)}{\sqrt{\nu_n(f^2) - \nu_n(f)^2}} \geq a\big] \leq \exp\big(-\frac{a\lambda_n}{\sqrt{\Gamma_n}}\sqrt{\nu(|\sigma^*\nabla\varphi|^2)}\big)\mathbb{E}\Big[\exp\big(\lambda_n \tilde{q}\nu_n(\mathcal{A}\varphi)\big)\Big]^{1/\tilde{q}} \\ &\times \mathbb{E}\Big[\exp\big(-\frac{a\tilde{p}\lambda_n}{\sqrt{\Gamma_n}}\frac{2\nu_n(\mathcal{A}\widetilde{\varphi})}{\sqrt{\nu_n(f^2) - \nu_n(f)^2} + \sqrt{\nu(f^2) - \nu(f)^2}}\big)\Big]^{1/(2\tilde{p})} \\ &\times \mathbb{E}\Big[\exp\big(-\frac{a\tilde{p}\lambda_n}{\sqrt{\Gamma_n}}\frac{-4\nu_n(\mathcal{A}\varphi)^2}{\sqrt{\nu_n(f^2) - \nu_n(f)^2} + \sqrt{\nu(f^2) - \nu(f)^2}}\big)\Big]^{1/(4\tilde{p})} \\ &\times \mathbb{E}\Big[\exp\big(-\frac{a\tilde{p}\lambda_n}{\sqrt{\Gamma_n}}\frac{-8\nu(f)\nu_n(\mathcal{A}\varphi)}{\sqrt{\nu_n(f^2) - \nu_n(f)^2} + \sqrt{\nu(f^2) - \nu(f)^2}}\big)\Big]^{1/(4\tilde{p})} \\ &=: \exp\big(-\frac{a\lambda_n}{\sqrt{\Gamma_n}}\sqrt{\nu(|\sigma^*\nabla\varphi|^2)}\big)\mathbb{E}\Big[\exp\big(\lambda_n \tilde{q}\nu_n(\mathcal{A}\varphi)\big)\Big]^{1/\tilde{q}}\Upsilon_1\Upsilon_2\Upsilon_3. \end{aligned} \tag{5.5}$$

The first idea of the following is to get rid of the $\mathcal{F}_n$-dependency of the denominator appearing in the exponential terms $\Upsilon_1, \Upsilon_2, \Upsilon_3$ (namely $\nu_n(f^2) - \nu_n(f)^2$).

Let us deal with the term Υ_1:

$$\begin{aligned}
\Upsilon_1 &= \mathbb{E}\Big[\exp\big(-\frac{a\tilde{p}\lambda_n}{\sqrt{\Gamma_n}}\frac{2\nu_n(\mathcal{A}\widetilde{\varphi})}{\sqrt{\nu_n(f^2)-\nu_n(f)^2}+\sqrt{\nu(f^2)-\nu(f)^2}}\big)\Big]^{1/(2\tilde{p})}\\
&\leq \Big(\mathbb{E}\Big[\exp\big(\frac{a\tilde{p}\lambda_n}{\sqrt{\Gamma_n}}\frac{2\nu_n(\mathcal{A}\widetilde{\varphi})}{\sqrt{\nu(f^2)-\nu(f)^2}}\big)\mathbb{1}_{\nu_n(\mathcal{A}\widetilde{\varphi})\geq 0}\Big]\\
&+\mathbb{E}\Big[\exp\big(-\frac{a\tilde{p}\lambda_n}{\sqrt{\Gamma_n}}\frac{2\nu_n(\mathcal{A}\widetilde{\varphi})}{\sqrt{\nu(f^2)-\nu(f)^2}}\big)\mathbb{1}_{\nu_n(\mathcal{A}\widetilde{\varphi})<0}\Big]\Big)^{1/(2\tilde{p})}\\
&\leq \Big(\mathbb{E}\Big[\exp\big(\frac{a\tilde{p}\lambda_n}{\sqrt{\Gamma_n}}\frac{4\nu_n(\mathcal{A}\widetilde{\varphi})}{\sqrt{\nu(f^2)-\nu(f)^2}}\big)\Big]^{1/2}\mathbb{P}[\nu_n(\mathcal{A}\widetilde{\varphi})\geq 0]^{1/2}\\
&+\mathbb{E}\Big[\exp\big(-\frac{a\tilde{p}\lambda_n}{\sqrt{\Gamma_n}}\frac{4\nu_n(\mathcal{A}\widetilde{\varphi})}{\sqrt{\nu(f^2)-\nu(f)^2}}\big)\Big]^{1/2}\mathbb{P}[\nu_n(\mathcal{A}\widetilde{\varphi})<0]^{1/2}\Big)^{1/(2\tilde{p})}\\
&\leq \mathscr{R}_n\exp\Big(\frac{\tilde{p}4a^2}{\Gamma_n}\frac{\lambda_n^2}{\Gamma_n}(\|\sigma\|_\infty^2\|\nabla\widetilde{\varphi}\|_\infty^2+e_n)\Big)\big(\mathbb{P}[\nu_n(\mathcal{A}\widetilde{\varphi})\geq 0]^{1/2}+\mathbb{P}[\nu_n(\mathcal{A}\widetilde{\varphi})<0]^{1/2}\big)^{1/(2\tilde{p})}.
\end{aligned}$$

The last inequality is a consequence of inequalities (3.5)-(3.7) (see also the proof of Theorem 2 in [HMP17]). We choose $\tilde{p}=\tilde{p}(n)\to+\infty$, s.t. $\frac{\tilde{p}a^2}{2\Gamma_n}\to_n 0$ (we recall that for *Gaussian deviations*, $\frac{a}{\sqrt{\Gamma_n}}\to_n 0$). Consequently, we obtain

$$\big(\mathbb{P}[\nu_n(\mathcal{A}\widetilde{\varphi})\geq 0]^{1/2}+\mathbb{P}[\nu_n(\mathcal{A}\widetilde{\varphi})<0]^{1/2}\big)^{\frac{1}{2\tilde{p}}}\leq 2^{\frac{1}{2\tilde{p}}}\underset{n}{\longrightarrow}1.$$

Then:

$$\Upsilon_1\leq\mathscr{R}_n\exp\Big(\frac{\lambda_n^2}{\Gamma_n}e_n\Big). \tag{5.6}$$

We proceed with the same method for the term Υ_3, to obtain also:

$$\Upsilon_3\leq\mathscr{R}_n\exp\Big(\frac{\lambda_n^2}{\Gamma_n}e_n\Big). \tag{5.7}$$

Let us now deal with Υ_2. Observe that this term leads to new difficulties. Namely we have to handle the exponential of the square of the empirical measure. The control of this term constrains us to assume that f is controlled by the Lyapunov function V. This last choice is made in view of exploiting the exponential integrability of V (see (4.1) and Property 1 in [HMP17]).

• For the sake of clarity, we will first deal with a bounded function f corresponding to the case $\eta=\frac{1}{2}$ in (5.1). This analysis will be a starting point to handle Υ_2 in whole generality.

Let us first try to control this term directly by the boundedness assumption of f.

$$\begin{aligned}
\Upsilon_2 &= \mathbb{E}\Big[\exp\big(\frac{a\tilde{p}\lambda_n}{\sqrt{\Gamma_n}}\frac{4\nu_n(f-\nu(f))^2}{\sqrt{\nu_n(f^2)-\nu_n(f)^2}+\sqrt{\nu(f^2)-\nu(f)^2}}\big)\Big]^{1/(4\tilde{p})}\\
&\leq\exp\big(\frac{a\tilde{p}\lambda_n}{\sqrt{\Gamma_n}}\frac{16\|f\|_\infty^2}{\sqrt{\nu(f^2)-\nu(f)^2}}\big)^{1/(4\tilde{p})}=\exp\big(\frac{a\lambda_n}{\sqrt{\Gamma_n}}\frac{4\|f\|_\infty^2}{\sqrt{\nu(f^2)-\nu(f)^2}}\big),
\end{aligned}$$

observing that $|\nu_n(f)-\nu(f)|\leq 2\|f\|_\infty$. The problem is that there is no reason why $\frac{a\lambda_n}{\sqrt{\Gamma_n}}$ should tend to 0. Actually, for *Gaussian deviations* ($\frac{a}{\sqrt{\Gamma_n}}\underset{n}{\to}0$), we have:

$$\lambda_n\underset{\frac{a}{\sqrt{\Gamma_n}}\to 0}{\asymp}a\sqrt{\Gamma_n},$$

see Lemma 4 and (4.17) in Remark 9. Hence $\frac{a\lambda_n}{\sqrt{\Gamma_n}}\underset{\frac{a}{\sqrt{\Gamma_n}}\to 0}{\asymp}a^2\underset{n}{\not\to}0$.

To get through this problem, we introduce a threshold $\tilde{a} = \tilde{a}(n) > 0$ associated with the empirical measure $\nu_n(\mathcal{A}\varphi)$. The idea is then to dominate, with previously established Gaussian concentration results, the contributions associated with $|\nu_n(\mathcal{A}\tilde{\varphi})| \geq \tilde{a}$. Namely,

$$\begin{aligned}
\Upsilon_2 &\leq \Big(\mathbb{E}\Big[\exp\big(\frac{a\tilde{p}\lambda_n}{\sqrt{\Gamma_n}}\frac{4\nu_n(\mathcal{A}\varphi)^2}{\sqrt{\nu(f^2)-\nu(f)^2}}\big)\mathbb{1}_{|\nu_n(\mathcal{A}\varphi)|\geq\tilde{a}}\Big] \\
&+\mathbb{E}\Big[\exp\big(\frac{a\tilde{p}\lambda_n}{\sqrt{\Gamma_n}}\frac{4\nu_n(\mathcal{A}\varphi)^2}{\sqrt{\nu(f^2)-\nu(f)^2}}\big)\mathbb{1}_{|\nu_n(\mathcal{A}\varphi)|<\tilde{a}}\Big]\Big)^{1/(4\tilde{p})} \\
&\leq \Big(\mathbb{E}\Big[\exp\big(\frac{a\tilde{p}\lambda_n}{\sqrt{\Gamma_n}}\frac{8\nu_n(\mathcal{A}\varphi)^2}{\sqrt{\nu(f^2)-\nu(f)^2}}\big)\Big]^{1/2}\mathbb{P}[|\nu_n(\mathcal{A}\varphi)|\geq\tilde{a}]^{1/2} \\
&+\mathbb{E}\Big[\exp\big(\frac{a\tilde{p}\lambda_n}{\sqrt{\Gamma_n}}\frac{4\tilde{a}^2}{\sqrt{\nu(f^2)-\nu(f)^2}}\big)\Big]\Big)^{1/(4\tilde{p})} \\
&\overset{(3.7)}{\leq} \Big(\exp\big(\frac{a\tilde{p}\lambda_n}{\sqrt{\Gamma_n}}\frac{16\|f\|_\infty^2}{\sqrt{\nu(f^2)-\nu(f)^2}}\big)2\widetilde{C}_n\exp\big(\frac{-\tilde{c}_n\tilde{a}^2\Gamma_n}{4\|\sigma\|_\infty^2\|\nabla\varphi\|_\infty^2}\big) \\
&+\exp\big(\frac{a\lambda_n}{\sqrt{\Gamma_n}}\frac{4\tilde{p}\tilde{a}^2}{\sqrt{\nu(f^2)-\nu(f)^2}}\big)\Big)^{1/(4\tilde{p})},
\end{aligned} \tag{5.8}$$

for explicit non-negative sequences $(\widetilde{c}_n)_{n\geq1}$ and $(\widetilde{C}_n)_{n\geq1}$, respectively increasing and decreasing for n large enough, with $\lim_n \widetilde{C}_n = \lim_n \widetilde{c}_n = 1$, where the last inequality is a consequence of Theorem 2 in [HMP17] (see also (3.7) in the *User's guide*).

Next, we tune the threshold $\tilde{a}$. The first term will be controlled by the Gaussian tail $\exp\big(\frac{-\tilde{c}_n\tilde{a}^2\Gamma_n}{4\|\sigma\|_\infty^2\|\nabla\varphi\|_\infty^2}\big)$ for $\tilde{a}^2\Gamma_n$ big enough. On the other hand, we will show that the explosive contribution for the second term coming from $\tilde{p}$ can be absorbed letting $\frac{a\lambda_n\tilde{a}^2}{\sqrt{\Gamma_n}} \to_n 0$ which is possible for $\tilde{a}$ small enough. In other words, we choose $\tilde{a}$ such that:

$$\frac{a\tilde{p}\lambda_n}{\sqrt{\Gamma_n}}\frac{16\|f\|_\infty^2}{\sqrt{\nu(f^2)-\nu(f)^2}} < \frac{\tilde{c}_n\tilde{a}^2\Gamma_n}{8\|\sigma\|_\infty^2\|\nabla\varphi\|_\infty^2},\ \frac{a\lambda_n}{\sqrt{\Gamma_n}}\tilde{p}\tilde{a}^2 \leq C.$$

By definition of λ_n in (3.30) and Remark 9 we know that for *Gaussian deviations*, $\lambda_n \underset{\frac{a}{\sqrt{\Gamma_n}}\to0}{\asymp} a\sqrt{\Gamma_n}$. So, there is a constant $C_G > 0$ s.t. $\lambda_n \leq C_G a\sqrt{\Gamma_n}$. Then our pick of $\tilde{a}$ is, for a suitable constant C, equivalent to:

$$Ca^2\tilde{p} < \tilde{a}^2\Gamma_n,\ a^2\tilde{p}\tilde{a}^2 \leq C \iff \frac{Ca^2\tilde{p}}{\Gamma_n} < \tilde{a}^2 \leq \frac{C}{a^2\tilde{p}}. \tag{5.9}$$

A choice for $\tilde{a}$ is thus possible as soon as

$$\frac{a^2\tilde{p}}{\Gamma_n} < \frac{C}{a^2\tilde{p}} \iff \frac{a^4\tilde{p}^2}{\Gamma_n} < C.$$

We recall that $\tilde{p} = \tilde{p}(n) \underset{n}{\longrightarrow} +\infty$ (possibly arbitrarily slowly). Hence, our conditions on $\tilde{a}$ can be fulfilled for $\frac{a}{\Gamma_n^{1/4}} \to 0$.

Plugging this choice into (5.8) yields:

$$\begin{aligned}\Upsilon_2 &= \mathbb{E}\Big[\exp\big(\frac{a\tilde{p}\lambda_n}{\sqrt{\Gamma_n}}\frac{4\nu_n(\mathcal{A}\varphi)^2}{\sqrt{\nu(f^2)-\nu(f)^2}}\big)\Big]^{1/(4\tilde{p})}\\ &\le \Big(2\tilde{C}_n\underbrace{\exp\big(\frac{-\tilde{c}_n\tilde{a}^2\Gamma_n}{8\|\sigma\|_\infty^2\|\nabla\varphi\|_\infty^2}\big)}_{\to_n 0,\ \le 1}+\underbrace{\exp\big(\frac{a\lambda_n}{\sqrt{\Gamma_n}}\frac{4\tilde{p}\tilde{a}^2}{\sqrt{\nu(f^2)-\nu(f)^2}}\big)}_{\le C}\Big)^{1/(4\tilde{p})}\\ &\le (2\tilde{C}_n+C)^{1/(4\tilde{p})}=\mathscr{R}_n\underset{n}{\to}1. \end{aligned}\tag{5.10}$$

• We assume now that f satisfies (5.1) for $\eta\in[\frac14,\frac12)$ (i.e. there are a constant $C_{V,f}>0$, s.t. for all $x\in\mathbb{R}^d$: $|f(x)|\le C_{V,f}(1+V^{\frac12-\eta}(x))$). In particular, the function f is not necessarily bounded anymore and η is lower bounded by $\frac14$ which gives that f^2 has linear growth.

To begin with, we proceed like for $\eta=\frac12$ (f bounded). Namely, we use the same threshold $\tilde{a}$ satisfying $\frac{Ca^2\tilde{p}}{\Gamma_n}<\tilde{a}^2\le\frac{C}{a^2\tilde{p}}$. With this hypothesis, we change the last inequality of (5.8) into

$$\begin{aligned}\Upsilon_2 &\le \Big(\mathbb{E}\Big[\exp\big(\frac{a\tilde{p}\lambda_n}{\sqrt{\Gamma_n}}\frac{8\nu_n\big(2+V^{\frac12-\eta}(\cdot)+\nu(V^{\frac12-\eta})\big)^2}{\sqrt{\nu(f^2)-\nu(f)^2}}\big)\Big]^{1/2}2\widetilde{C}_n\exp\big(\frac{-\tilde{c}_n\tilde{a}^2\Gamma_n}{4\|\sigma\|_\infty^2\|\nabla\varphi\|_\infty^2}\big)\\ &+\exp\big(\frac{a\lambda_n}{\sqrt{\Gamma_n}}\frac{4\tilde{p}\tilde{a}^2}{\sqrt{\nu(f^2)-\nu(f)^2}}\big)\Big)^{1/(4\tilde{p})}\\ &\le \Big(\mathbb{E}\Big[\exp\big(\frac{a\tilde{p}\lambda_n}{\sqrt{\Gamma_n}}\frac{C\nu_n(V^{1-2\eta})}{\sqrt{\nu(f^2)-\nu(f)^2}}\big)\Big]\exp\big(\frac{a\tilde{p}\lambda_n}{\sqrt{\Gamma_n}}\frac{C}{\sqrt{\nu(f^2)-\nu(f)^2}}\big)2\widetilde{C}_n\exp\big(\frac{-\tilde{c}_n\tilde{a}^2\Gamma_n}{4\|\sigma\|_\infty^2\|\nabla\varphi\|_\infty^2}\big)\\ &+\exp\big(\frac{a\lambda_n}{\sqrt{\Gamma_n}}\frac{4\tilde{p}\tilde{a}^2}{\sqrt{\nu(f^2)-\nu(f)^2}}\big)\Big)^{1/(4\tilde{p})}\\ &\le \Big(\mathbb{E}\Big[\exp\big(\frac{a\tilde{p}\lambda_n}{\sqrt{\Gamma_n}}\frac{C\nu_n(V^{1-2\eta})}{\sqrt{\nu(f^2)-\nu(f)^2}}\big)\Big]2\widetilde{C}_n\exp\big(\frac{-\tilde{c}_n\tilde{a}^2\Gamma_n}{8\|\sigma\|_\infty^2\|\nabla\varphi\|_\infty^2}\big)+C\Big)^{1/(4\tilde{p})}. \end{aligned}\tag{5.11}$$

The penultimate inequality comes from the Cauchy Schwarz inequality. The previous choice of $\tilde{a}$ yields the last inequality. The difficulty, here, is to establish the exponential integrability of $\exp\big(\frac{a\tilde{p}\lambda_n}{\sqrt{\Gamma_n}}\frac{C\nu_n(V^{1-2\eta})}{\sqrt{\nu(f^2)-\nu(f)^2}}\big)$.

From (4.1) and Property 1 in [HMP17], we still have some freedom for the tuning of the parameters to establish the exponential integrability of $\nu_n(V^{1-2\eta})$. Namely, by the Young inequality, we get for all $\varepsilon>0$:

$$V^{1-2\eta}\le(1-2\eta)\varepsilon^{\frac{1}{1-2\eta}}V+2\eta\varepsilon^{-\frac{1}{2\eta}}.$$

Plugging this control into (5.11) yields:

$$\Upsilon_2\le\Big(\mathbb{E}\Big[\exp\big(\frac{a\tilde{p}\lambda_n}{\sqrt{\Gamma_n}}\frac{C[(1-2\eta)\varepsilon^{\frac{1}{1-2\eta}}\nu_n(V)+2\eta\varepsilon^{-\frac{1}{2\eta}}]}{\sqrt{\nu(f^2)-\nu(f)^2}}\big)\Big]2\widetilde{C}_n\exp\big(\frac{-\tilde{c}_n\tilde{a}^2\Gamma_n}{8\|\sigma\|_\infty^2\|\nabla\varphi\|_\infty^2}\big)+C\Big)^{1/(4\tilde{p})}.\tag{5.12}$$

In particular, by the Jensen inequality:

$$\exp\big(\frac{a\tilde{p}\lambda_n}{\sqrt{\Gamma_n}}\frac{C(1-2\eta)\varepsilon^{\frac{1}{1-2\eta}}\nu_n(V))}{\sqrt{\nu(f^2)-\nu(f)^2}}\big)\le\frac{1}{\Gamma_n}\sum_{k=1}^n\gamma_k\exp\big(\frac{a\tilde{p}\lambda_n}{\sqrt{\Gamma_n}}\frac{C(1-2\eta)\varepsilon^{\frac{1}{1-2\eta}}V(X_{k-1})}{\sqrt{\nu(f^2)-\nu(f)^2}}\big).\tag{5.13}$$

We now aim to pick ε s.t. $\exp\big(\frac{a\tilde{p}\lambda_n}{\sqrt{\Gamma_n}}\frac{C(1-2\eta)\varepsilon^{\frac{1}{1-2\eta}}V(X_{k-1})}{\sqrt{\nu(f^2)-\nu(f)^2}}\big)$ is integrable. Recall now from (4.1) (see Property 1 in [HMP17]), that there is $c_V>0$ s.t. for all $\lambda<c_V$, $I_V^1:=\sup_{n\ge0}\mathbb{E}[\exp(\lambda V_n^1)]<+\infty$.

So, we need to have:

$$\frac{a\tilde{p}\lambda_n}{\sqrt{\Gamma_n}}\frac{C(1-2\eta)\varepsilon^{\frac{1}{1-2\eta}}}{\sqrt{\nu(f^2)-\nu(f)^2}} \leq a^2\tilde{p}C_G\frac{C(1-2\eta)\varepsilon^{\frac{1}{1-2\eta}}}{\sqrt{\nu(f^2)-\nu(f)^2}} \leq c_V,$$

where the constant $C_G > 0$ is s.t. $\lambda_n \leq C_G a\sqrt{\Gamma_n}$. Hence, we choose:

$$\varepsilon = \big(\frac{c_V\sqrt{\nu(f^2)-\nu(f)^2}}{Ca^2C_G(1-2\eta)\tilde{p}}\big)^{1-2\eta}. \tag{5.14}$$

This pick yields thanks to (5.13):

$$\mathbb{E}\Big[\exp\big(\frac{a\lambda_n}{\sqrt{\Gamma_n}}\frac{C\nu_n(V^{1-2\eta})\tilde{p}}{\sqrt{\nu(f^2)-\nu(f)^2}}\big)\Big] \leq \frac{1}{\Gamma_n}\sum_{k=1}^{n}\gamma_k\mathbb{E}\Big[\exp\big(c_V V(X_{k-1})\big)\Big] \leq I_V^1. \tag{5.15}$$

Moreover, we write for the contribution in $\varepsilon^{-\frac{1}{2\eta}}$ in (5.12):

$$\begin{aligned}\exp\big(\frac{a\tilde{p}\lambda_n}{\sqrt{\Gamma_n}}\frac{C2\eta\varepsilon^{-\frac{1}{2\eta}}}{\sqrt{\nu(f^2)-\nu(f)^2}}\big) &\leq \exp\big(C_G a^2\tilde{p}\frac{C2\eta\varepsilon^{-\frac{1}{2\eta}}}{\sqrt{\nu(f^2)-\nu(f)^2}}\big)\\ &\leq \exp\big(Ca^{2+\frac{1-2\eta}{\eta}}\tilde{p}^{1+\frac{1-2\eta}{2\eta}}\big) = \exp\big(Ca^{\frac{1}{\eta}}\tilde{p}^{\frac{1}{2\eta}}\big).\end{aligned} \tag{5.16}$$

Plugging (5.15), (5.16) into (5.12) yields:

$$\Upsilon_2 \leq \Big(I_V^1\exp\big(Ca^{\frac{1}{\eta}}\tilde{p}^{\frac{1}{2\eta}}\big)2\widetilde{C}_n\exp\big(\frac{-\tilde{c}_n\tilde{a}^2\Gamma_n}{8\|\sigma\|_\infty^2\|\nabla\varphi\|_\infty^2}\big)+C\Big)^{1/(4\tilde{p})}. \tag{5.17}$$

Hence, to get the upper-bound converging to 0 (thanks to $\tilde{p}\to_n +\infty$), we need to consider:

$$\frac{Ca^{\frac{1}{\eta}}\tilde{p}^{\frac{1}{2\eta}}}{\Gamma_n} < \tilde{a}^2.$$

But we have already considered in (5.9), $\tilde{a}^2 \leq \frac{C}{a^2\tilde{p}}$ which implies that:

$$\frac{a^{\frac{1}{\eta}}\tilde{p}^{\frac{1}{2\eta}}}{\Gamma_n} < \frac{C}{a^2\tilde{p}} \Longleftrightarrow \frac{a^{\frac{1+2\eta}{\eta}}\tilde{p}^{\frac{1+2\eta}{2\eta}}}{\Gamma_n} < C.$$

In other words, we need to have $a = o(\Gamma_n^{\frac{\eta}{1+2\eta}})$ in order to choose $\tilde{p}$ s.t. $\frac{a^{\frac{1+2\eta}{\eta}}\tilde{p}^{\frac{1+2\eta}{2\eta}}}{\Gamma_n} \to_n 0$.

We obtain directly by this constraint that:

$$\Upsilon_2 \leq \big(2I_V^1C_n\exp\big(-C\tilde{a}^2\Gamma_n\big)+C\big)^{1/(4\tilde{p})} = \mathscr{R}_n \to_n 1. \tag{5.18}$$

Gathering (5.6), (5.7) and (5.10) for $\eta = \frac{1}{2}$ or (5.18) for $\eta \in [\frac{1}{4}, \frac{1}{2})$ into (5.5):

$$\begin{aligned}&\mathbb{P}\big[\sqrt{\Gamma_n}\frac{\nu_n(f)-\nu(f)}{\sqrt{\nu_n(f^2)-\nu_n(f)^2}} \geq a\big]\\ \leq\ & \mathscr{R}_n\exp\big(\frac{\lambda_n^2}{\Gamma_n}e_n\big)\exp\big(-\frac{a\lambda_n}{\sqrt{\Gamma_n}}\sqrt{\nu(|\sigma^*\nabla\varphi|^2)}\big)\times\mathbb{E}\Big[\exp\big(\lambda_n\tilde{q}\nu_n(\mathcal{A}\varphi)\big)\Big]^{1/\tilde{q}}\\ \leq\ & \mathscr{R}_n\exp\big(\frac{\lambda_n^2}{\Gamma_n}\bar{A}_n+\frac{\lambda^4}{\Gamma_n^3}\bar{B}_n\big)\exp\big(-\frac{a\lambda_n}{\sqrt{\Gamma_n}}\sqrt{\nu(|\sigma^*\nabla\varphi|^2)}\big) \underset{n}{=} \mathscr{R}_n\exp\big(-\frac{a^2}{2}(1+o(1))\big).\end{aligned}$$

The last inequality comes from the proof of Theorem 3, where $\bar{A}_n = \tilde{q}A_n$, and $\bar{B}_n = \tilde{q}^3B_n$ (see as well the *User's guide to the proof* where the A_n and B_n are defined). The last equality from the definition of λ_n (see Lemma 4). The non-asymptotic confidence interval result is a direct consequence of the previous result. □

6. Regularity Results and Consequences

This section is devoted to some regularity results for the Poisson problem

$$\mathcal{A}\varphi = f - \nu(f). \tag{6.1}$$

In particular, we state below some Schauder like controls, which are, because of our methodology that requires pointwise controls of the derivatives, more adapted than the standard Sobolev estimates (see e.g. Pardoux and Veretennikov [PV01]).

There are two kinds of assumptions that guarantee the solution φ of (6.1) enjoys the required smoothness of Theorems 3 and 4.

(a) If b, σ and f are *smooth*, under suitable confluence like conditions stated below in **(D)**, the probabilistic representation $\varphi(x) = -\int_0^{+\infty} \mathbb{E}\big[\big(f(Y_t^{0,x}) - \nu(f)\big)\big]dt$ of the solution of (6.1) can be differentiated using iterated tangent flows to establish that $\varphi \in \mathcal{C}^{3,\beta}(\mathbb{R}^d, \mathbb{R})$, for some $\beta \in (0, 1]$. It suffices for that to have $f \in C^{3,\beta}(\mathbb{R}^d, \mathbb{R})$ and $b \in C^{3,\beta}(\mathbb{R}^d, \mathbb{R}^d), \sigma \in C_b^{3,\beta}(\mathbb{R}^d, \mathbb{R}^d \otimes \mathbb{R}^r)$. We refer to Section 5.3 of [HMP17] for additional details. Importantly, under such assumptions, specific non-degeneracy conditions are not needed.

(b) If we do not assume such an *a priori* smoothness on f, b, σ, we need to make an *extra* non-degeneracy assumption which will allow some regularity gain (elliptic bootstrap). Precisely, assuming that the bounded diffusion coefficient Σ is also uniformly elliptic, we exploit the results of Krylov and Priola [KP10] (Theorems 2.4-2.6) to derive that, up to an additional technical condition on Σ when $d > 1$, we actually have $\varphi \in \mathcal{C}^{3,\beta}(\mathbb{R}^d, \mathbb{R})$ for some $\beta \in (0, 1)$, as soon as $f \in C^{1,\beta}(\mathbb{R}^d, \mathbb{R})$ and $b \in C^{1,\beta}(\mathbb{R}^d, \mathbb{R}^d), \sigma \in C_b^{1,\beta}(\mathbb{R}^d, \mathbb{R}^d \otimes \mathbb{R}^d)$.

We point out that the second set of assumptions is very important in order to go towards the *usual* setting of functional/transport inequalities which typically involves Lipschitz continuous test functions. This is for instance the case for the controls of the Wasserstein distances between the law of Y_t in (1.1) and the invariant measure ν (see e.g. [BGL14]). We are thus able, in the non-degenerate framework (b), to consider directly sources $f \in \mathcal{C}^{1,\beta}(\mathbb{R}^d, \mathbb{R})$. Observe that, when $\beta \to 0$, we are *almost* Lipschitz. Actually, we can, in this setting, handle functions $f \in \mathcal{C}^{0,1}(\mathbb{R}^d, \mathbb{R})$ up to a spatial regularization which leads to a constraint on the steps (see Theorem 6 below).

6.1. Assumptions and Regularity Results.

The regularity results stated here can be found in Section 5 of [HMP17]. Let us now recall the useful assumptions needed.

(UE) Uniform ellipticity. We assume that $r \geq d$ in (1.1) and that there is $\underline{\sigma} > 0$ such that

$$\forall \xi \in \mathbb{R}^d,\ \langle \sigma\sigma^*(x)\xi, \xi\rangle \geq \underline{\sigma}|\xi|^2.$$

For $\beta \in (0, 1)$, we introduce the following condition.

($\mathbf{R}_{1,\beta}$) Regularity Condition. From equation (1.1), we suppose $b \in \mathcal{C}^{1,\beta}(\mathbb{R}^d, \mathbb{R}^d), \sigma \in \mathcal{C}_b^{1,\beta}(\mathbb{R}^d, \mathbb{R}^d)$.

(D) Confluence Conditions. We assume that there exists $\alpha > 0$ and $p \in [1, 2)$ such that for all $x \in \mathbb{R}^d$, $\xi \in \mathbb{R}^d$

$$\Big\langle \frac{Db(x) + Db(x)^*}{2}\xi, \xi\Big\rangle + \frac{1}{2}\sum_{j=1}^{r}\Big((p-2)\frac{|\langle D\sigma_{\cdot j}(x)\xi, \xi\rangle|^2}{|\xi|^2} + |D\sigma_{\cdot j}\xi|^2\Big) \leq -\alpha|\xi|^2, \tag{6.2}$$

where Db stands here for the Jacobian of b, $\sigma_{\cdot j}$ stands for the jth column of the diffusion matrix σ and $D\sigma_{\cdot j}$ for its Jacobian matrix.

There are others assumptions than **(D)** which yield, in the non-degenerate setting, Schauder estimates. This is the case for the so-called Bakry and Émery curvature criterion ([BE85, BGL14]) which is however pretty hard to check for general multidimensional diffusion coefficients.

We eventually introduce, as in [HMP17], a technical condition on the diffusion coefficient σ. It allows to prove that each partial derivative $\partial_{x_i}\varphi$ of the solution of (6.1) satisfies an autonomous scalar Poisson problem. We suppose:

(Σ) for every $(i,j) \in [\![1,d]\!]^2$ and $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, $\Sigma_{i,j}(x) = \Sigma_{i,j}(x_{i\wedge j}, \cdots, x_d)$.

We say that assumption $(\mathbf{P}_\beta)$ is satisfied if **(UE)**, **(D)**, $(\mathbf{R}_{1,\beta})$, and **(Σ)** are in force. From Section 5.3 of [HMP17] we have the following result.

Theorem 5 (Elliptic Bootstrap in a non-degenerate setting)**.** *Assume* $(\mathbf{P}_\beta)$ *holds for some* $\beta \in (0,1)$ *and that* $f \in \mathcal{C}^{1,\beta}(\mathbb{R}^d, \mathbb{R})$*. Then, there is a unique* $\varphi \in \mathcal{C}^{3,\beta}(\mathbb{R}^d, \mathbb{R})$ *solving* (6.1)*.*

Note as well from Remark 2, that there exists $C > 0$, $|D^2\varphi(x)| \le C(1+|x|)^{-1}$. In other words, the solution φ of (6.1) satisfies $(\mathbf{T}_\beta)$.

Remark 11 (On Schauder estimates for $\beta = 1$)**.** *We insist on the fact that* $\beta \in (0,1)$ *in the above theorem. Indeed, it is well known that the Hölder exponent* β *cannot go to* 1 *in the Schauder estimates. Note that, for the particular case* $f \in \mathcal{C}^{1,1}(\mathbb{R}^d, \mathbb{R})$*, we also have* $f \in \mathcal{C}^{1,\beta}(\mathbb{R}^d, \mathbb{R})$ *for all* $\beta \in (0,1)$*. This means that, for such* f*, the elliptic bootstrap works up to an arbitrarily small correction.*

From Theorem 5 we readily have:

Corollary 1 (Smoothness for the Poisson problem with *Carré du champ* source)**.** *Assume* $(\mathbf{P}_\beta)$ *holds for some* $\beta \in (0,1)$ *and that* $f \in \mathcal{C}^{1,\beta}(\mathbb{R}^d, \mathbb{R})$*. Then, there is a unique* $\vartheta \in \mathcal{C}^{3,\beta}(\mathbb{R}^d, \mathbb{R})$ *solving*

$$\mathcal{A}\vartheta = |\sigma^*\nabla\varphi|^2 - \nu(|\sigma^*\nabla\varphi|^2)$$

and satisfying $(\mathbf{T}_\beta)$*.*

Indeed, it suffices to observe from Theorem 5 and the assumption $(\mathbf{R}_{1,\beta})$ in $(\mathbf{P}_\beta)$ that $\tilde{f} := |\sigma^*\nabla\varphi|^2 \in \mathcal{C}^{1,\beta}(\mathbb{R}^d, \mathbb{R})$ and to apply again Theorem 5 for this source.

In relation with our previous Slutsky like Theorem 4, we also have from Theorem 5, taking $\bar{f} := f^2$ as source, the following corollary.

Corollary 2 (Smoothness for the Poisson problem for the square of the initial source)**.** *Assume* $(\mathbf{P}_\beta)$ *holds for some* $\beta \in (0,1)$ *and that* $f^2 \in \mathcal{C}^{1,\beta}(\mathbb{R}^d, \mathbb{R})$*. Then, there is a unique* $\tilde{\varphi} \in \mathcal{C}^{3,\beta}(\mathbb{R}^d, \mathbb{R})$ *solving*

$$\mathcal{A}\tilde{\varphi} = f^2 - \nu(f^2)$$

and satisfying $(\mathbf{T}_\beta)$*.*

6.2. **Concentration bounds for a Lipschitz source in a non-degenerate setting.** As indicated in the introduction of the Section, we aim at controlling deviations for Lipschitz sources. In the current Lipschitz framework we aim to address, we need a slightly different set of assumptions. Namely, we will assume that **(C1)**, **(GC)**, **(C2)**, $(\mathcal{L}_\mathbf{V})$, **(U)**, **(S)** and $(\mathbf{P}_\beta)$ are in force and we will say that $(\mathbf{L}_\beta)$ holds. Under this new assumption, we have the following result.

Theorem 6 (Non-asymptotic concentration bounds for Lipschitz continuous source)**.** *Assume that* $(\mathbf{L}_\beta)$ *is in force. Let* f *be a Lipschitz continuous function. For a time step sequence* $(\gamma_k)_{k\ge1}$ *of the form* $\gamma_k \asymp k^{-\theta}$*,* $\theta \in (1/2, 1]$*, we have that, there exist two explicit monotonic sequences* $c_n \le 1 \le C_n$*,* $n \ge 1$*, with* $\lim_n C_n = \lim_n c_n = 1$ *such that for all* $n \ge 1$ *and for every* $a > 0$*:*

$$\mathbb{P}\big[|\sqrt{\Gamma_n}\big(\nu_n(f) - \nu(f)\big)| \ge a\big] \le 2C_n \exp\big(-c_n \frac{a^2}{2\nu(|\sigma^*\nabla\varphi|^2)}\big), \tag{6.3}$$

where $\varphi \in \mathcal{C}^{0,1}(\mathbb{R}^d, \mathbb{R}) \cap W^2_{2,loc}(\mathbb{R}^d, \mathbb{R})$ *is a weak solution of the Poisson equation* $\mathcal{A}\varphi = f - \nu(f)$*.*

Sketch of the proof. To prove the above result, the starting point consists in regularizing the source f by mollification. Namely, we consider $f_\delta = f \star \eta_\delta$, where $\star$ denotes the usual convolution, for a suitable mollifier $\eta_\delta(\cdot) := \frac{1}{\delta^d}\eta(\frac{\cdot}{\delta})$, $\delta > 0$, where η is a compactly supported non-negative function s.t. $\int_{\mathbb{R}^d}\eta(x)dx = 1$. We then write $\nu_n(f) - \nu(f) = \nu_n(f_\delta) - \nu(f_\delta) + (\nu_n - \nu)(f - f_\delta) =: \nu_n\big(f_\delta - \nu(f_\delta)\big) + R_{n,\delta}$. We aim at letting δ go to 0 so that $R_{n,\delta}$ can be viewed as a remainder. On the other hand, we will apply the same strategy as in the proof of Theorem 3 to analyze the deviations of $\nu_n\big(f_\delta - \nu(f_\delta)\big) = \nu_n(\mathcal{A}\varphi_\delta)$. Precisely, reproducing the arguments of Section 5.4 of [HMP17] to equilibrate the explosions of the derivatives of φ_δ in the proof of Theorem 3 yields that there exists two explicit monotonic sequences $\tilde{c}_n \le 1 \le C_n,\ n \ge 1$, with $\lim_n C_n = \lim_n \tilde{c}_n = 1$ s.t.

$$\mathbb{P}\big[|\sqrt{\Gamma_n}\big(\nu_n(f) - \nu(f)\big)| \ge a\big] \le 2C_n \exp\big(-\tilde{c}_n \frac{a^2}{2\nu(|\sigma^*\nabla\varphi_\delta|^2)}\big). \tag{6.4}$$

From the previous Schauder estimates, we know that $\varphi_\delta \in \mathcal{C}^{3,\beta}(\mathbb{R}^d, \mathbb{R})$ for all $\delta > 0$ with explosive $\mathcal{C}^{3,\beta}$ norm in δ but with bounded gradient. Recall indeed that, for all $\beta \in (0,1)$,

$$\|f_\delta\|_{\mathcal{C}^{1,\beta}} \le C\delta^{-\beta},\ |\nabla\varphi_\delta| \le \frac{[f_\delta]_1}{\alpha} = \frac{[f]_1}{\alpha}. \tag{6.5}$$

We again refer to Lemma 6 and Section 5.4 in [HMP17] for details. On the other hand, it is well known, see e.g. [PV01], that $\varphi_\delta(x) = -\int_0^{+\infty}\mathbb{E}[f_\delta(Y_t^{0,x}) - \nu(f_\delta)]dt$. From their Proposition 1, we have in our case that, denoting by $\nu_{Y_t^{0,x}}$ the law of $Y_t^{0,x}$ we have the following control for the total variation between $\nu_{Y_t^{0,x}}$ and ν. There exists constants $(C,c) := (C,c)((\mathbf{L}_\beta))$ s.t.

$$\|\nu_{Y_t^{0,x}} - \nu\|_{T.V.} \le C\exp(c|x|)\exp(-\alpha t). \tag{6.6}$$

Introducing now $\bar{f}_\delta = f_\delta - f - \nu\big(f_\delta - f\big)$, we rewrite that, for all $x \in \mathbb{R}^d$:

$$\begin{aligned}
\big(\varphi_\delta - \varphi\big)(x) &= -\int_0^{+\infty}\mathbb{E}[\bar{f}_\delta(Y_t^{0,x})]dt,\\
|\big(\varphi_\delta - \varphi\big)(x)| &\le \int_0^{+\infty}\Big(\int_{\mathbb{R}^d}|\bar{f}_\delta(y)|^2\big(\nu_{Y_t^{0,x}} + \nu\big)(dy)\Big)^{1/2}\|\nu_{Y_t^{0,x}} - \nu\|_{T.V.}^{1/2}dt\\
&\underset{(6.6)}{\le} C^{1/2}\exp(\frac{c}{2}|x|)\|\bar{f}_\delta\|_\infty\int_0^{+\infty}\exp\Big(-\frac{\alpha}{2}t\Big)dt.
\end{aligned}$$

Recalling that f is Lipschitz and that $(f - f_\delta)(x) = \int_{\mathbb{R}^d}\big(f(x-y) - f(x)\big)\eta_\delta(y)dy$, we actually have $\|\bar{f}_\delta\|_\infty \le C[f]_1\delta$, which establishes the pointwise convergence $\varphi_\delta(x) \underset{\delta\to 0}{\longrightarrow} -\int_0^{+\infty}\mathbb{E}[f(Y_t^{0,x}) - \nu(f)]dt =: \varphi(x)$ which is the only weak solution of (6.1) in $W^2_{p,loc}(\mathbb{R}^d, \mathbb{R}), p > 1$ (see Theorem 1 in [PV01]).

Let us now prove that

$$\lim_{\delta\to 0}\nu(|\sigma^*\nabla\varphi_\delta|^2) = \nu(|\sigma^*\nabla\varphi|^2). \tag{6.7}$$

For all $\varepsilon > 0$, there is a compact $K := K(\varepsilon)$ such that, denoting by $K^c := \mathbb{R}^d\backslash K$, we have:

$$\int_{\mathbb{R}^d}\big|\nabla\varphi_\delta - \nabla\varphi\big|^2(x)\mathbb{1}_{K^c}(x)\nu(dx) \le 4\|\nabla\varphi\|_\infty^2\int_{\mathbb{R}^d}\mathbb{1}_{K^c}(x)\nu(dx) \le \frac{\varepsilon}{2}.$$

Also, since $\varphi \in W^2_{2,loc}(\mathbb{R}^d,\mathbb{R})$, we write:

$$\begin{aligned}
&\int_{\mathbb{R}^d} \big|\nabla\varphi_\delta - \nabla\varphi\big|^2(x)\mathbb{1}_K(x)\nu(dx) \le \int_{\mathbb{R}^d}\Big|\int_{\mathbb{R}^d}\big(\nabla\varphi(x-z)-\nabla\varphi(x)\big)\rho_\delta(z)dz\Big|^2\mathbb{1}_K(x)\nu(dx)\\
&= \int_{\mathbb{R}^d}\Big|\int_{\mathbb{R}^d}\big(\int_0^1 D^2\varphi(x-\lambda z)zd\lambda\big)\rho_\delta(z)dz\Big|^2\mathbb{1}_K(x)\nu(dx)\\
&\le \int_0^1 d\lambda\int_{\mathbb{R}^d}\Big|\int_{\mathbb{R}^d} D^2\varphi(x-\lambda z)z\rho_\delta(z)dz\Big|^2\mathbb{1}_K(x)\nu(dx)\\
&\le \int_0^1 d\lambda\Big(\int_{\mathbb{R}^d} dz|z|^2\rho_\delta(z)\int_{\mathbb{R}^d}|D^2\varphi(x-\lambda z)|^2\mathbb{1}_K(x)\nu(dx)\Big) \le C\delta^2\|\varphi\|^2_{W^2_2(\bar{K},\mathbb{R})} < \frac{\varepsilon}{2},
\end{aligned}$$

for δ small enough, using the Cauchy-Schwarz inequality for the penultimate control and denoting by $\bar{K}$ a compact set such that for all $z \in B(0, C\delta) \supset \mathrm{supp}(\eta_\delta)$, $x \in K$, $x - z \in \bar{K}$. This in particular gives (6.7). Hence, setting

$$c_n := \tilde{c}_n \frac{\nu(|\sigma^*\nabla\varphi|^2)}{\nu(|\sigma^*\nabla\varphi_\delta|^2)} \to_n 1,$$

and recalling from Section 5.4. in [HMP17] that $\delta := \delta(n) \to_n 0$ [4], we derive that (6.3) follows from (6.4) up to a modification of $\tilde{c}_n$.

Furthermore, let us point out that, the result can alternatively be stated replacing the *carré du champ* in (6.3) by the variance of the Lipschitz source under the invariant law. In fact, we can write by the dominated convergence theorem:

$$\lim_{\delta\to 0}\nu(|\sigma^*\nabla\varphi_\delta|^2) = \lim_{\delta\to 0}\nu\big(\big(f_\delta - \nu(f_\delta)\big)^2\big) = \nu\big(\big(f-\nu(f)\big)^2\big) = \nu(|\sigma^*\nabla\varphi|^2). \tag{6.8}$$

□

Remark 12. *The new threshold $\theta > \frac{1}{2}$ comes from the specific Lipschitz regularity of the test function f. Intuitively, this threshold naturally appears when we consider $\beta \to 0$ in the previous condition $\theta \in (\frac{1}{2+\beta}, 1]$ induced by the regularity of $\varphi \in \mathcal{C}^{3,\beta}(\mathbb{R}^d,\mathbb{R})$ which holds, under* $(\mathbf{P}_\beta)$*, when $f \in \mathcal{C}^{1,\beta}(\mathbb{R}^d,\mathbb{R})$. We underline anyhow that, for $\beta = 0$, the Schauder estimates do not directly apply.*

7. Optimisation over ρ under *Gaussian* and *super Gaussian deviations*

In Lemma 4, we performed an asymptotic estimation of the upper-bound for *Gaussian deviations*. However, from a numerical point of view, it appears to be more significant to optimize over ρ in whole generality, i.e. not only when $\frac{a}{\sqrt{\Gamma_n}} \to 0$. This procedure conducts to deviations bounds that are much closer to the realizations.

In particular, in *super Gaussian deviations* framework (i.e. $\frac{a}{\sqrt{\Gamma_n}} \to 0$), we provide here a "weaker" concentration inequality than the Gaussian one, which precisely comes from the optimization over ρ for this regime. This loss of concentration, with the terminology of Remark 4, is intrinsic to our method, as it will be shown in the proofs of Theorem 7 and Lemma 7 below, see also Remark 18.

Theorem 7 (Deviations in the *super Gaussian* regime)**.** *Assume* **(A)** *is in force. If there exists $\vartheta \in \mathcal{C}^{3,\beta}(\mathbb{R}^d,\mathbb{R})$, $\beta \in (0,1]$ satisfying* $(\mathbf{T}_\beta)$ *s.t.*

$$\mathcal{A}\vartheta = |\sigma^*\nabla\varphi|^2 - \nu(|\sigma^*\nabla\varphi|^2), \tag{7.9}$$

4. which was anyhow constrained to go to 0 sufficiently *slowly* in order to balance the explosions in the derivatives coming from the Schauder estimates, see (6.5). It is specifically this feature that led to the condition $\gamma_n \asymp n^{-\theta}, \theta \in (1/2, 1]$.

then, for $\theta \in (\frac{1}{2+\beta}, 1]$ *, there exist explicit non-negative sequences* $(c_n)_{n\geq 1}$ *and* $(C_n)_{n\geq 1}$*, respectively increasing and decreasing for* n *large enough, with* $\lim_n C_n = \lim_n c_n = 1$ *s.t. for all* $n \geq 1$*,* $a > 0$*, the following bounds hold. When* $\frac{a}{\sqrt{\Gamma_n}} \to +\infty$ *(Super Gaussian deviations):*

$$\mathbb{P}\big[|\sqrt{\Gamma_n}\nu_n(\mathcal{A}\varphi)| \geq a\big] \leq 2\, C_n \exp\big(-\, c_n \frac{a^{4/3}\Gamma_n^{1/3}}{2\|\sigma\|_\infty^{2/3}[\vartheta]_1^{2/3}}\big).$$

Remark 13. *Observe from Corollary 1, that, the function* ϑ *enjoys the required smoothness as soon as assumption* $(\mathbf{P}_\beta)$ *introduced in Section 6 holds.*

Remark 14. *For super Gaussian deviations, we obtain a sharper bound than in Theorem 2. Nonetheless, asymptotically, this regime is less sharp than Theorem 2 in* [HMP17] *which provides a Gaussian bound with deteriorated constants (see also the User's guide to the proof in Section 3.1 below). Even if, from a numerical point of view, the deviation bounds in Theorem 8 below yield sharper controls with respect to simulated empirical measures (see Figure 1 in the numerical Section below).*

For "intermediate Gaussian deviations", i.e. for $\frac{a}{\sqrt{\Gamma_n}} \underset{n}{\to} C > 0$*, the constants in the Gaussian bound deteriorate. So, it seems reasonable to see this situation like for the first regime* $\frac{a}{\sqrt{\Gamma_n}} \underset{n}{\to} 0$*, namely where there are constants* $C_\infty > 1, c_\infty < 1$ *such that* $\lim_n C_n = C_\infty$*,* $\lim_n c_n = c_\infty$ *and*

$$\mathbb{P}\big[|\sqrt{\Gamma_n}\nu_n(\mathcal{A}\varphi)| \geq a\big] \leq 2\, C_n \exp\big(-\, c_n \frac{a^2}{2\nu(|\sigma^*\nabla\varphi|^2)}\big).$$

Observe that for such regimes, there is an equivalence, up to multiplicative constants, between the bounds in Theorem 3 and in Theorem 7.

The idea of the proof of Theorem 7 follows the same lines as for Theorem 3, except for the optimization over ρ which is more fussy, see Lemma 7 below.

We recall that the analysis in the proof of Theorem 3 leaving open a possible optimization over the parameter ρ which we now perform The next lemma indicates that this optimization implies a Gaussian regime for $\frac{a}{\sqrt{\Gamma_n}} \underset{n}{\to} 0$ and a super Gaussian one for $\frac{a}{\sqrt{\Gamma_n}} \underset{n}{\to} +\infty$.

Lemma 7 (Choice of ρ for the concentration regime)**.** *For* $P_{\min}(a, \Gamma_n, \rho)$ *as in* (3.31)*,*

(a) If $\frac{a}{\sqrt{\Gamma_n}} \underset{n}{\to} 0$*, taking* $\rho := \rho(a, n)$ *s.t.* $\rho - 1 = \frac{1}{2}\frac{\tilde{B}_n^{1/2}a}{\tilde{A}_n^{3/2}\sqrt{\Gamma_n}}(1 + o(1))$

$$P_{\min}(a, \Gamma_n, \rho) \underset{\frac{a}{\sqrt{\Gamma_n}} \underset{n}{\to} 0}{=} -\frac{a^2}{2\nu(|\sigma^*\nabla\varphi|^2)}(1 + o(1)).$$

(b) If $\frac{a}{\sqrt{\Gamma_n}} \underset{n}{\to} +\infty$*, taking* $\rho := \rho(a, n)$ *s.t.* $\rho - 1 = \frac{1}{2} + o(1)$

$$P_{\min}(a, \Gamma_n, \rho) \underset{\frac{a}{\sqrt{\Gamma_n}} \to +\infty}{=} -\frac{a^{4/3}\Gamma_n^{1/3}}{2\|\sigma\|_\infty^{2/3}[\vartheta]_1^{2/3}}(1 + o(1)).$$

Remark 15. *For Gaussian deviations,* $\rho - 1 = \frac{1}{2}\frac{\tilde{B}_n^{1/2}a}{\tilde{A}_n^{3/2}\sqrt{\Gamma_n}}(1 + o(1)) \asymp \frac{a}{\sqrt{\Gamma_n}}$ *which corresponds to our choice in Lemma 4. We then retrieve the Gaussian regime. In the super Gaussian deviations framework, the optimization over* ρ *leads to consider* $\rho - 1 = \frac{1}{2} + o(1)$ *which yields the loss in the concentration inequality.*

Let us first continue with the proof of Lemma 7 which is purely analytical and rather independent of our probabilistic setting.

Proof of Lemma 7 . We keep the notations of Lemma 4, that we bring to mind.

$$\begin{aligned}\Phi_n(a,\rho) &= \Big(\frac{a}{\sqrt{\Gamma_n}\tilde{B}_n}+\big(\frac{a^2}{\tilde{B}_n^2\Gamma_n}+(\rho-1)\big(\frac{2\tilde{A}_n}{3\tilde{B}_n}\big)^3\big)^{\frac{1}{2}}\Big)^{\frac{1}{3}}+\Big(\frac{a}{\sqrt{\Gamma_n}\tilde{B}_n}-\big(\frac{a^2}{\tilde{B}_n^2\Gamma_n}+(\rho-1)\big(\frac{2\tilde{A}_n}{3\tilde{B}_n}\big)^3\big)^{\frac{1}{2}}\Big)^{\frac{1}{3}}\\ &= \frac{a^{1/3}}{\tilde{B}_n^{1/3}\Gamma_n^{1/6}}\big((1+\sqrt{1+\xi})^{1/3}+(1-\sqrt{1+\xi})^{1/3}\big),\end{aligned}$$

for

$$\rho-1:=\xi\frac{27}{8}\frac{\tilde{B}_n a^2}{\tilde{A}_n^3\Gamma_n}, \tag{7.10}$$

where $\xi:=\xi(a,n)>0$ is a parameter that we are going to optimize. Furthermore:

$$P(\lambda_n)=-\frac{3^2}{2^4\tilde{A}_n}a^2 f_\Psi(\xi), \tag{7.11}$$

for

$$f_\Psi:\xi\in\mathbb{R}_+\longmapsto\frac{g(\xi)}{\Psi\xi+1}, \tag{7.12}$$

where

$$g:\xi\longmapsto\xi^{1/3}\big((1+\sqrt{1+\xi})^{1/3}+(1-\sqrt{1+\xi})^{1/3}\big)\Big(1-\frac{\xi^{1/3}}{2}\big((1+\sqrt{1+\xi})^{1/3}+(1-\sqrt{1+\xi})^{1/3}\big)\Big),$$

and

$$\Psi:=\frac{27}{8}\frac{\tilde{B}_n a^2}{\tilde{A}_n^3\Gamma_n}\big(\overset{(4.10)}{=}\frac{\rho-1}{\xi}\big). \tag{7.13}$$

• Let us first focus on case *(a)*, "Gaussian deviations" ($\frac{a}{\sqrt{\Gamma_n}}\underset{n}{\to}0$). We, now anyhow, want to maximize Λ in ξ to obtain the best possible concentration bound. Let $\mathscr{A}:=\{\xi\in[0,+\infty]:f_\Psi(\xi)=\|f_\Psi\|_\infty\}$ be the set of points where f_Ψ reaches its maximum. Observe that for a fixed Ψ, $f_\Psi(\xi)\underset{\xi\to\infty}{\to}0$. Thus, $+\infty\notin\mathscr{A}$. Let now ξ_* be an arbitrary point in $\mathscr{A}$. From the smoothness of f_Ψ, the optimality condition writes:

$$f'_\Psi(\xi_*)=\frac{g'(\xi_*)}{(\Psi\xi_*+1)}-\frac{\Psi g(\xi_*)}{(\Psi\xi_*+1)^2}=0\ \Leftrightarrow\frac{g'(\xi_*)}{(\Psi\xi_*+1)}=\frac{\Psi f_\Psi(\xi_*)}{(\Psi\xi_*+1)}\ \Leftrightarrow f_\Psi(\xi_*)=\frac{g'(\xi_*)}{\Psi}. \tag{7.14}$$

Recall now that we want to maximize over the ξ s.t. $\xi\to+\infty$, $\xi\Psi\underset{\frac{a}{\sqrt{\Gamma_n}}\to0}{\to}0$. Indeed, from the proof of Lemma 4, we saw that for such a choice, we obtain the expected Gaussian concentration, namely $P(\lambda_n)\underset{\frac{a}{\sqrt{\Gamma_n}}\underset{n}{\to}0}{\sim}-\frac{a^2}{4\tilde{A}_n}$.

From the computations of Lemma 8 in Appendix A, we have:

$$g'(\xi)\underset{\xi\to+\infty}{=}\frac{8}{3^5\xi^2}+o(\frac{1}{\xi^2}). \tag{7.15}$$

So,

$$\Lambda(\xi_*)\underset{\xi_*\to+\infty}{=}\frac{1}{2\cdot3^3\xi_*^2\Psi}(1+o(1))\underset{\frac{a}{\sqrt{\Gamma_n}}\to0}{\to}\frac{1}{4}, \tag{7.16}$$

where $o(1)$ denotes here a quantity going to 0 as $\xi_*\to+\infty$ and $\xi_*\Psi\underset{\frac{a}{\sqrt{\Gamma_n}}\to0}{\to}0$. Inspired by the identity (7.16), and motivated the numerical simulations (see Section 8), we set

$$\bar{\xi}_*:=\frac{2^{1/2}}{3^{3/2}\sqrt{\Psi}}=\frac{2^2\tilde{A}_n^{3/2}\sqrt{\Gamma_n}}{3^9\tilde{B}_n^{1/2}a}, \tag{7.17}$$

which indeed satisfies (4.16), so that

$$\min_{\rho>1} P(\lambda_n) \leq -\frac{a^2}{\tilde{A}_n}\Lambda(\bar{\xi}_*)(1+o(1)) = \frac{a^2}{4\tilde{A}_n}(1+o(1)) \underset{n\to+\infty}{\sim} -\frac{a^2}{2\nu(|\sigma^*\nabla\varphi|^2)}.$$

Observe as well that this choice yields:

$$\rho - 1 = \frac{2^{1/2}\Psi}{3^{3/2}\sqrt{\Psi}} = \frac{1}{2}\frac{\tilde{B}_n^{1/2}a}{\tilde{A}_n^{3/2}\sqrt{\Gamma_n}}. \tag{7.18}$$

• Now we will study the the case *(b)* "super Gaussian deviations" for which $\frac{a}{\sqrt{\Gamma_n}} \underset{n}{\to} +\infty$.

Note that we cannot expect a Gaussian regime in this case. In fact, for Ψ going to infinity (see definition (7.13)), by (7.14), to get a Gaussian regime at a maximizer ξ_* of Λ, we have from (7.11) that $f_\Psi(\xi_*) = \frac{g'(\xi_*)}{\Psi}$ has to remain separated from 0 when $\frac{a}{\sqrt{\Gamma_n}} \to +\infty$. Since, in this case $\Psi \underset{n}{\to} +\infty$, this imposes to consider points $\xi \to 0$ in order to exploit the asymptotic behaviour (see again Lemma 8 for more details):

$$g'(\xi) \underset{\xi\to 0}{=} \frac{2^{1/3}}{3\xi^{\frac{2}{3}}}\big(1+o(1)\big). \tag{7.19}$$

In other words, from (7.19), we expect that there is a constant $K_* > 0$ s.t.

$$f_\Psi(\xi_*) \underset{\xi^*\to 0}{\sim} \frac{2^{1/3}}{3\xi_*^{2/3}\Psi} \geq K_*.$$

So $\xi_* \leq \frac{C}{\Psi^{3/2}} \underset{\Psi\to+\infty}{\to} 0$. Now, $|f_\Psi(\xi_*)| = |\frac{g(\xi_*)}{\Psi\xi+1}| \leq |g(\xi_*)| \underset{\xi_*\to 0}{\to} 0$. This means that it is impossible to stay in a Gaussian regime.

We now still look at the optimal $\xi_* \to 0$ which allows to stay at "the biggest possible regime". Thenceforth, we will estimate ξ_* directly from the map f_Ψ defined in (7.12):

$$f_\Psi(\xi) \underset{\xi\to 0}{=} \frac{2^{1/3}\xi^{1/3}}{\Psi\xi+1}(1+o(1)) =: f_{\Psi,0}(\xi)(1+o(1)). \tag{7.20}$$

It can be directly checked that $\arg\max_{\xi\in\mathbb{R}_+} f_{\Psi,0}(\xi) = \frac{1}{2\Psi}$. We therefore get:

$$\xi_* \underset{\Psi\to+\infty}{=} \frac{1}{2\Psi} + o(\frac{1}{\Psi}) \underset{\frac{a}{\sqrt{\Gamma_n}}\to+\infty}{=} \frac{4\tilde{A}_n^3\Gamma_n}{27\tilde{B}_n a^2} + o(\frac{\Gamma_n}{a^2}) \to 0. \tag{7.21}$$

From (7.11) and (7.20), we get:

$$\begin{aligned}
\min_{\rho>1} P(\lambda_n) &\underset{\frac{a}{\sqrt{\Gamma_n}}\underset{n}{\to}+\infty}{\leq} -\frac{3^2}{2^4\tilde{A}_n}a^2\frac{2^{1/3}\xi_*^{1/3}}{\Psi\xi_*+1}(1+o(1)) \underset{(7.21)}{=} -\frac{3^2}{2^4}\frac{a^2}{\tilde{A}_n(\frac{1}{2}+1)}\Psi^{-1/3}(1+o(1)) \\
&\underset{(4.16)}{\leq} -\frac{3^2}{2^4}\frac{a^2}{\tilde{A}_n(\frac{1}{2}+1)}(\frac{2^3\tilde{A}_n^3\Gamma_n}{3^3\tilde{B}_n a^2})^{1/3}(1+o(1)) \\
&\underset{\frac{a}{\sqrt{\Gamma_n}}\underset{n}{\to}+\infty}{=} -\frac{a^{4/3}\Gamma_n^{1/3}}{2^2\tilde{B}_n^{1/3}}(1+o(1)) \underset{\frac{a}{\sqrt{\Gamma_n}}\underset{n}{\to}+\infty}{\overset{(3.28)}{=}} -\frac{a^{4/3}\Gamma_n^{1/3}}{2\|\sigma\|_\infty^{2/3}[\vartheta]_1^{2/3}}(1+o(1)).
\end{aligned}$$

From equations (4.10) and (7.13), the choice (7.21) yields:

$$\rho - 1 = \xi\Psi = \frac{1}{2} + o(1). \tag{7.22}$$

□

Remark 16 (Controls of the optimized parameters λ and ρ for *super Gaussian deviations*)**.** *We give here some useful estimates to control the remainder terms in the truncation procedure associated with unbounded innovations, see proof of Lemma 6. They specify the behaviour of the quantity* $\frac{\rho\lambda_n}{\Gamma_n}$.

In the regime of super Gaussian deviations, identities (4.11), (7.13) *and the choice* (7.21) *(i.e.* $\xi_* = \frac{1}{2\Psi}$*) yields:*

$$\frac{\rho^2\lambda_n^2}{\Gamma_n^2} \asymp C\Psi \underbrace{\xi_*^{2/3}\big((1+\sqrt{1+\xi_*})^{1/3} + (1-\sqrt{1+\xi_*})^{1/3}\big)^2}_{\asymp \frac{2^{1/3}}{\Psi^{2/3}}} \underset{\frac{a}{\sqrt{\Gamma_n}}\to+\infty}{\to} +\infty.$$

Remark 17. *Theorems 3 and 7 are actually a consequence of the more general following result which has a real importance for numerical applications. Indeed, for a given* $n \in \mathbb{N}$*, we have to control the non-asymptotic error in Theorems 3 and 7 . Furthermore, in Section 8 (see Remark 19) we will see that for* $\theta \in (\frac{1}{3}, 1]$ *, for "reasonable"* n *(e.g.* $n = 5\cdot 10^4$ *in the following Section 8) and* a (≈ 1) *we are already "out" of the Gaussian deviations regime, namely* $\frac{a}{\sqrt{\Gamma_n}} \asymp 1$*. This illustration justifies the interest of optimizing over* ρ *for Gaussian deviations and super Gaussian deviations.*

Theorem 8. *Let the assumptions of Theorem 3 be in force. For* $\theta \in (\frac{1}{3}, 1]$*, there exist explicit non-negative sequences* $(c_n)_{n\geq 1}$ *and* $(C_n)_{n\geq 1}$*, respectively increasing and decreasing for* n *large enough, with* $\lim_n C_n = \lim_n c_n = 1$ *s.t. for all* $n \geq 1$ *for all* $a > 0$*,*

$$\mathbb{P}\big[|\sqrt{\Gamma_n}\nu_n(\mathcal{A}\varphi)| \geq a\big] \leq 2\, C_n \exp\big(c_n P_{\min}(a, \Gamma_n, \rho)\big),$$

where $\rho > 1$ *and*

$$P_{\min}\big(a, \Gamma_n, \rho\big) = -\frac{(\rho-1)^{1/3}}{\rho}\frac{\sqrt{\Gamma_n}\Phi_n(a,\rho)}{8}(3a - \sqrt{\Gamma_n}(\rho-1)^{1/3}\tilde{A}_n\Phi_n(a,\rho)),$$

with

(7.23)

$$\Phi_n(a,\rho) := \Big(\frac{a}{\sqrt{\Gamma_n}\tilde{B}_n} + \big(\frac{a^2}{\tilde{B}_n^2\Gamma_n} + (\rho-1)\big(\frac{2\tilde{A}_n}{3\tilde{B}_n}\big)^3\big)^{\frac{1}{2}}\Big)^{\frac{1}{3}} + \Big(\frac{a}{\sqrt{\Gamma_n}\tilde{B}_n} - \big(\frac{a^2}{\tilde{B}_n^2\Gamma_n} + (\rho-1)\big(\frac{2\tilde{A}_n}{3\tilde{B}_n}\big)^3\big)^{\frac{1}{2}}\Big)^{\frac{1}{3}}$$

$$\widetilde{A}_n := \frac{q\nu(|\sigma^*\nabla\varphi|^2)}{2} + e_n \quad \text{and} \quad \widetilde{B}_n = \frac{q^3}{4}\Big(\frac{\bar{q}\|\sigma\|_\infty^2\|\nabla\vartheta\|_\infty^2}{2} + e_n\Big), \tag{7.24}$$

where $q := q(n) > 1$*,* $\bar{q} := \bar{q}(n) > 1$ *with* $q \underset{n}{\to} 1$*,* $\bar{q} \underset{n}{\to} 1$*, and* e_n *is an explicit sequence going to* 0*.*

The results of Theorem 3 explicitly follow taking (see for more details the proof of Lemma 7):

$$\begin{aligned}\rho - 1 &= \frac{1}{2}\frac{\tilde{B}_n^{1/2}a}{\tilde{A}_n^{3/2}\sqrt{\Gamma_n}}, \text{ for } \frac{a}{\sqrt{\Gamma_n}} \underset{n}{\to} 0,\\ \rho - 1 &= \frac{1}{2}, \text{ for } \frac{a}{\sqrt{\Gamma_n}} \to +\infty.\end{aligned}$$

Remark 18. *It is natural to wonder if it is possible to get a sharper variance for super Gaussian deviations. Thereby it would be tempting to bootstrap Lemma 3. Such an iteration would lead to optimize polynomials of higher degrees. Recall from Lemma 4, that we already have to handle a polynomial of order 4 in the current setting. This illustrates that for very large deviations the highest term dominates and deteriorates the concentration. This is intrinsic to our approach. Such a phenomenon would even more pregnant when iterating the procedure the polynomial of higher degree yields, for a certain very large deviation regime, concentration bounds that become closer and closer to the exponential. However, bootstrapping might allow to improve the constants in the successive deteriorated concentration regimes. As indicated in Remark 17, this could be useful for some numerical purposes.*

8. Numerical Results

8.1. **Degenerate diffusion.** Here, we have chosen to highlight the possible absence of non-degeneracy assumption for our results. To oversimplify, simulations are done with $r = d = 1$, X_0 and U_1 follow the standard normal distribution. Naturally, for a better convergence speed, we take $\theta \approx \frac{1}{3}$, precisely $\theta = \frac{1}{3} + \frac{1}{1000}$.

For this first example, we choose φ (solution of the Poisson equation). We take $\varphi = \sigma = \cos$ and for all $x \in \mathbb{R}$, $b(x) = -\frac{x}{2}$. By this pick, we compute numerically $\nu(|\sigma^*\nabla\varphi|^2) \approx 0.1515$ and $\nu(|\sigma|^2) \approx 0.4171$ (that we provide here for comparison with the previous results in [HMP17]), with the same parameters ($\theta = \frac{1}{3} + \frac{1}{1000}$, $n = 5 \cdot 10^4$ and $MC = 10^4$).

Heed, for a non trivial test function φ, with our method, we cannot choose functions b and $\sigma \neq 0$ canceling at the same point (0 here). Otherwise, the Poisson equation associated with the *carré du champ* source, $\mathcal{A}\vartheta = |\sigma^*\nabla\varphi|^2 - \nu(|\sigma^*\nabla\varphi|^2)$, would imply that $-\nu(|\sigma^*\nabla\varphi|^2) = 0$, then $\nabla\varphi = 0$, ν almost surely.

Let us now check that the Confluence Conditions **(D)** are satisfied. For $p \in [1, 2)$, we have for all $x \in \mathbb{R}^d$, $\xi \in \mathbb{R}^d$

$$\begin{aligned}&\Big\langle \frac{Db(x) + Db(x)^*}{2}\xi, \xi\Big\rangle + \frac{1}{2}\sum_{j=1}^{r}\Big((p-2)\frac{|\langle D\sigma_{\cdot j}(x)\xi, \xi\rangle|^2}{|\xi|^2} + |D\sigma_{\cdot j}\xi|^2\Big)\\ &= -\frac{1}{2}\xi^2 + \frac{1}{2}\sin^2(x)\xi^2(p-1).\end{aligned} \tag{8.1}$$

So, for $p = \frac{3}{2}$, we directly obtain:

$$\begin{aligned}&\Big\langle \frac{Db(x) + Db(x)^*}{2}\xi, \xi\Big\rangle + \frac{1}{2}\sum_{j=1}^{r}\Big((\frac{3}{2}-2)\frac{|\langle D\sigma_{\cdot j}(x)\xi, \xi\rangle|^2}{|\xi|^2} + |D\sigma_{\cdot j}\xi|^2\Big)\\ &= -\frac{1}{2}\xi^2 + \frac{1}{4}\sin^2(x)\xi^2 \leq -\frac{1}{4}\xi^2 =: -\alpha\xi^2.\end{aligned} \tag{8.2}$$

Note that, we have chosen a diffusion coefficient σ which degenerates on $\{k\pi, k \in \mathbb{Z}\}$. However, thanks to the smoothness of the diffusion parameters, we can still here apply Lemma 6 in [HMP17] which gives us a pointwise gradient bound of the solution of the Poisson problem in the current degenerate context. In other words:

$$[\vartheta]_1 \leq \frac{[|\sigma^*\nabla\varphi|^2]_1}{\alpha} = 4[\cos^2\sin^2]_1 = 4\sup_{x\in\mathbb{R}}(\cos(2x)\sin(2x)) = 2.$$

Hence, this inequality leads us to approximate $[\vartheta]_1$ by 2. Pay attention that the control of the Lipschitz constant $[\vartheta]_1$ is important for the *super Gaussian deviations*. Like illustrated in Remarks 17 and 19, this regime appears "sooner" than we might expect.

From Theorem 3, the function $a \mapsto g_n(a) := \ln(\mathbb{P}[\sqrt{\Gamma_n}|\nu_n(\mathcal{A}\varphi)| \geq a])$ is s.t. for $a > 0$:

$$g_n(a) \leq -c_n\frac{a^2}{2\nu(|\sigma^*\nabla\varphi|^2)} + \ln(2C_n),$$

where $(c_n)_{n\geq1}$ and $(C_n)_{n\geq1}$ are sequences respectively increasing and decreasing for n large enough, with $\lim_n C_n = \lim_n c_n = 1$.

For Figure 1, the simulations have been performed for $n = 5 \cdot 10^4$ and the probability estimated by Monte Carlo simulation for $MC = 10^4$ realizations of the random variable $\sqrt{\Gamma_n}|\nu_n(\mathcal{A}\varphi)|$.

The corresponding 95% confidence intervals have size at most of order 0.0016. We introduce the functions:

$$S(a) := -\frac{a^2}{2\nu(|\sigma^*\nabla\varphi|^2)},\ S_{\sup}(a) := -\frac{a^2}{2\|\sigma\|_\infty^2\|\nabla\varphi\|_\infty^2}.$$

Like in Theorem 8, we take

$$P_{\min}(a,\Gamma_n,\rho) = -\frac{(\rho-1)^{1/3}}{\rho}\frac{\sqrt{\Gamma_n}\Phi_n(a,\rho)}{8}(3a-(\rho-1)^{1/3}\tilde{A}_n\Phi_n(a,\rho)\sqrt{\Gamma_n}),$$

where $\tilde{A}_n, \tilde{B}_n$ and $\Phi_n(a,\rho)$ are defined in (7.24). Through our numerical results, we take $e_n = 0$.

We set also:

$$\rho_0 := 1 + \frac{1}{2}\frac{\bar{B}_n^{1/2}a}{\bar{A}_n^{3/2}\sqrt{\Gamma_n}},\ \rho_\infty := \frac{3}{2}.$$

We recall here that ρ and ρ_∞ respectivly correspond to the optimal values of ρ in the *Gaussian deviations* and *super Gaussian deviations* (see Lemma 7). Eventually, we introduce:

$$\begin{aligned} P_{n,0,\infty}(a) &:= \min\Big(P_{\min}(a,\Gamma_n,\rho_0), P_{\min}(a,\Gamma_n,\rho_\infty)\Big), \\ P_n(a) &:= \min_{\rho>1} P_{\min}(a,\Gamma_n,\rho). \end{aligned}$$

Note that, the function $P_{n,0,\infty}$ takes into account the multi-regime competition. We have estimated $P_n(a)$ by a mesh method for $\rho \in (1,2)$ and for a grid with $5\cdot 10^5$ steps.

From the above notations, we add the subscript σ to mean that we change $\nu(|\sigma^*\nabla\varphi|^2)$ into $\|\nabla\varphi\|_\infty^2\nu(\|\sigma\|^2)$, i.e.

$$S_\sigma(a) = -\frac{a^2}{2\|\nabla\varphi\|_\infty^2\nu(\|\sigma\|^2)},\ P_{n,\sigma}(a) := \min_{\rho>1} P_{\min,\sigma}(a,\Gamma_n,\rho),$$

and we have changed $\tilde{A}_n$ into

$$\tilde{A}_{\sigma,n} := \frac{\|\nabla\varphi\|_\infty^2\nu(\|\sigma\|^2)}{2}.$$

The quantities with subscript σ are those associated with the results in [HMP17], recalled in the previous Theorem 2, where the variance is less sharp than the constants appearing in Theorems 3, 7 and 8. Thus, we can compare our main results with Remark 10 of [HMP17] which is a weakened form of Theorem 8 where the *carré du champ* is changed into $\|\nabla\varphi\|_\infty^2\nu(\|\sigma\|^2)$ like in Theorem 2.

Figure 1 reveals that the asymptotic curve S is much less sharp with respect to the realizations g_n than our main estimations P_n and $P_{n,0,\infty}$. In fact, these latter are very close to the realization g_n. This claim enhances the significance of controlling finely, non-asymptotically, the deviation of the empirical measure.

In this plot, we can see that our pick of ρ for $P_{n,0,\infty}$, set in Lemma 7, is very close to the numerical optimization of P_n over ρ. Nevertheless, observe that for $a > 0.5$, $P_{n,0,\infty}(a)$ and $P_n(a)$ slightly differ. It means that progressively the regime goes from *Gaussian deviations* (i.e. $\frac{a}{\sqrt{\Gamma_n}} \to 0$) to *intermediate Gaussian deviations* (i.e. $\frac{a}{\sqrt{\Gamma_n}} = O(1)$). Hence, the importance of optimizing globally the function $\rho \mapsto P_{\min}(a,\Gamma_n,\rho)$ (appearing in (3.32)) in all regimes.

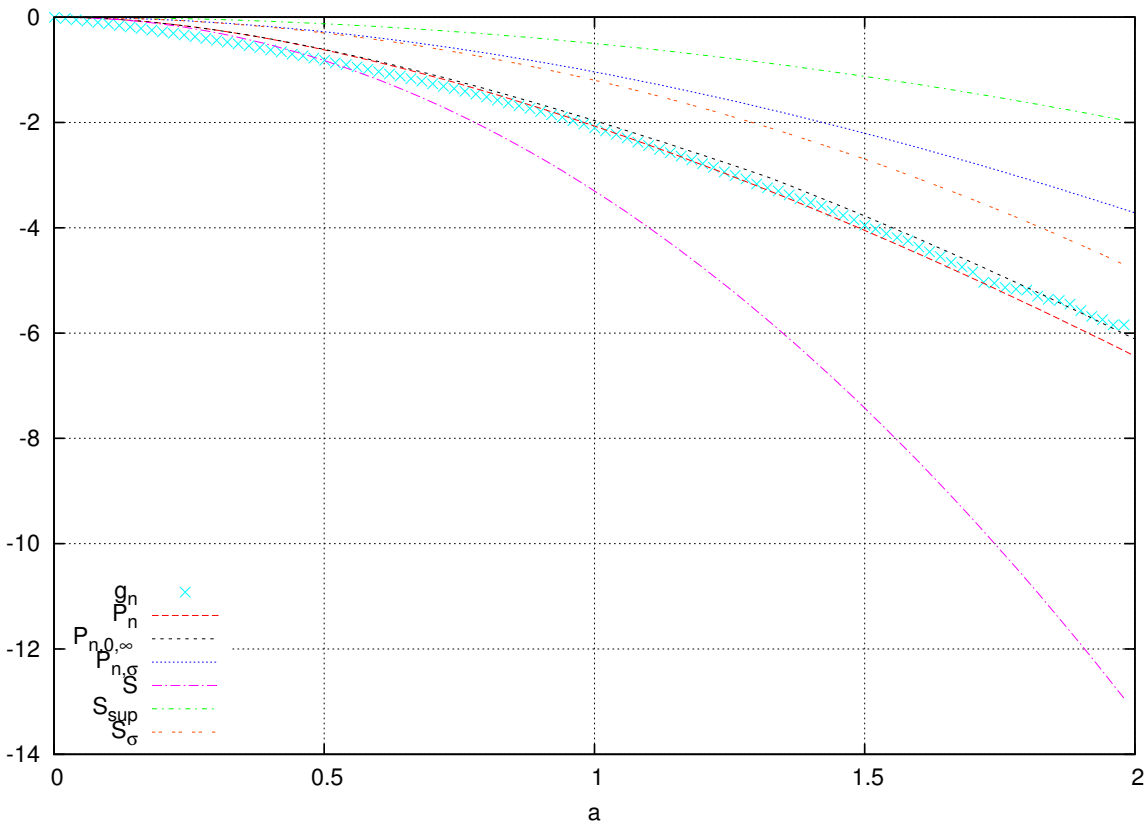

FIGURE 1. Plot of $a \mapsto g_n(a)$ with $\varphi(x) = \sigma(x) = \cos(x)$.

8.2. **Slutsky like property.** In this subsection, we display the deviation result in Theorem 4, for a multidimensional process, precisely for $r = d = 2$. The innovations $(U_k)_{k\geq 1}$ are Gaussian random variables. We take $\theta = \frac{1}{2+\beta} + \frac{1}{1000}$, for $\beta = 0.5$, to get a fast convergence without any bias. We use a "toy model" for which assumption $\mathbf{(P_\beta)}$ introduced in Section 6 holds. Namely, we pick for all $x = (x_1, x_2) \in \mathbb{R}^2$:

$$f(x) = \sin\Big(\frac{|x|^{1+\beta}}{1+|x|^\beta}\Big), \beta = 0.5,\ b(x) = \begin{pmatrix} -4x_1 + 6x_2 \\ -5x_1 - 5x_2 \end{pmatrix},\ \sigma\sigma^*(x) = \begin{pmatrix} \frac{\cos(x_1+x_2)}{2} + 1 & \frac{\sin(x_1)\sin(x_2)}{4} \\ \frac{\sin(x_1)\sin(x_2)}{4} & 1 - \frac{\sin(x_2)}{2}. \end{pmatrix}.$$

Note, that f is in $\mathcal{C}_b^{1,\beta}(\mathbb{R}^d, \mathbb{R})$, which corresponds to $\eta = \frac{1}{2}$ in statement of Theorem 4. The non-degeneracy condition **(UE)** is fulfilled by $\Sigma = \sigma\sigma^*$, and also the assumption (Σ), namely $\Sigma_{i,j}(x) = \Sigma_{i,j}(x_i, \ldots, x_d)$, for all $1 \leq i \leq j \leq d$. By the Cholesky decomposition, we get:

$$\sigma(x) = \begin{pmatrix} \sqrt{\frac{\cos(x_1+x_2)}{2} + 1} & 0 \\ \frac{\sin(x_1)\sin(x_2)}{4\sqrt{\frac{\cos(x_1+x_2)}{2}+1}} & \sqrt{-\frac{\sin(x_1)^2\sin(x_2)^2}{16(\frac{\cos(x_1+x_2)}{2}+1)} + 1 - \frac{\sin(x_2)}{2}} \end{pmatrix}.$$

We have chosen exactly the same coefficients as in [HMP17], in the part 6.2.2 (except for the test function f), thereby the computations are identical. Furthermore, remark that hypothesis **(D)** is satisfied, $\frac{Db+Db^*}{2}$ is a constant matrix whose eigenvalues are $\{-\frac{\sqrt{2}+9}{2}, \frac{\sqrt{2}-9}{2}\}$, hence for all $x \in \mathbb{R}^d$,

and $\xi \in \mathbb{R}^d$:

$$\left\langle \frac{Db(x) + Db(x)^*}{2}\xi, \xi \right\rangle + \frac{1}{2}\sum_{j=1}^{r} |D\sigma_{\cdot j}(x)\xi|^2 \leq -3.085|\xi|^2.$$

It is obvious that $(\mathbf{R}_{1,\beta})$ is true for these parameters. Hence by Theorem 5 (see also, Theorem 6 in [HMP17]), under assumptions $(\mathbf{P}_\beta)$ and $(\mathbf{A})$, the Poisson equation $\mathcal{A}\varphi = f - \nu(f)$ admits a unique solution $\varphi \in \mathcal{C}^{3,\beta}(\mathbb{R}^d, \mathbb{R})$, and there exists $C_\varphi > 0$ s.t. for all $x \in \mathbb{R}^d$, $|D^2\varphi(x)| \leq \frac{C_\varphi}{1+|x|}$ [5].

To compute the deviation, we need firstly to estimate $\nu(f)$. We do it thanks to the ergodic theorem ($\nu_n(f) \xrightarrow{a.s.} \nu(f)$), and we take $\nu_{n^c}(f) \approx 0.5848$ for a "large" $n^c = 5 \cdot 10^5$. Running $MC = 10^2$ samples, we find that the size of the associated 95% confidence interval is to the order of 10^{-4}. Finally, the simulations are performed for $n = 5 \cdot 10^4$, and the probability is calculated by the Monte Carlo algorithm for $MC = 10^3$ realizations. Remark that the running time $\sqrt{\Gamma_{n^c}} \approx 51.2$ while the running time for the computation of g_n^ν is $\sqrt{\Gamma_n} \approx 25.7$. In other words, the empirical measure ν_n is compared to ν_{n^c} which is approximately twice as accurate (even if the mean reversion assumption, i.e. the linear growth of b, may lead to a faster convergence).

For this plot, we set

$$g_n^\nu(a) = \ln(\mathbb{P}[\sqrt{\Gamma_n}|\frac{\nu_n(f) - \nu(f)}{\nu_n(f^2) - \nu_n(f)^2}| \geq a),$$

estimated by the Monte Carlo method ($MC = 10^4$). Gathering Theorem 4 and Theorem 3, we obtain immediately, under the same assumptions, that, there exist two explicit monotonic sequences $c_n \leq 1 \leq C_n$, $n \geq 1$, with $\lim_n C_n = \lim_n c_n = 1$ such that for all $n \geq 1$, for all $0 < a = o(\Gamma_n^{1/4})$ $(= o(\Gamma_n^{\frac{\eta}{1+2\eta}})$, for $\eta = \frac{1}{2}$, as established in Theorem 4 for a bounded source f) , with the previously introduced notations:

$$g_n^\nu(a) \leq \ln(2C_n) - \frac{c_n a^2}{2}.$$

In the following graphic we also take

$$S_\nu(a) = -\frac{a^2}{2}.$$

The graphic 2 reveals that we can control properly the log probability of the discretized process $(X_n)_{n\geq 0}$ without any knowledge of the solution of the Poisson equation $\mathcal{A}\varphi = f - \nu(f)$.

Remark 19. *Remark that for the graphics 1 and 2, we chose $n = 5 \cdot 10^4$, but for $\theta \approx \frac{1}{3}$, $\sqrt{\Gamma_n} \approx 37$ and for $\theta \approx \frac{1}{2+0.5}$, $\sqrt{\Gamma_n} \approx 26$. In other words, for $a \approx 1$ we have intermediate Gaussian deviations as emphasized by the graphics. Hence the importance of the study of both regimes, Gaussian deviations ($\frac{a}{\sqrt{\Gamma_n}} \to 0$) and super Gaussian deviations ($\frac{a}{\sqrt{\Gamma_n}} \to +\infty$).*

5. Pay attention that, here, the explicit knowledge of φ is not needed. We only use some of its quantitative properties deriving from Theorem 5.

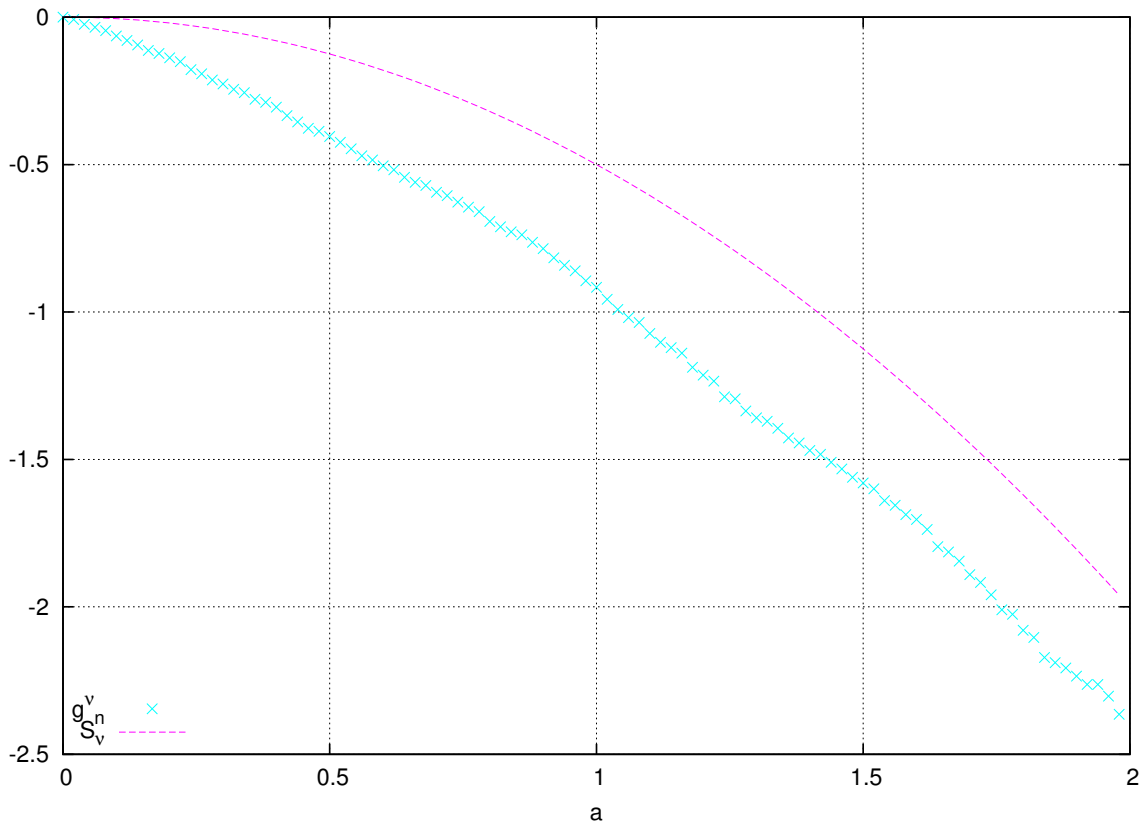

FIGURE 2. Plot of $a \mapsto g_n^\nu(a)$.

APPENDIX A. COMPUTATION OF ASYMPTOTIC ANALYSIS

In this section, we perform asymptotic analysis for the map g' defined in (7.12) in proof of Lemma 4. We recall that for all $\xi \in \mathbb{R}$:

$$g(\xi) = \xi^{1/3}\Big((1+\sqrt{1+\xi})^{1/3} + (1-\sqrt{1+\xi})^{1/3}\Big)\Big(1 - \tfrac{\xi^{1/3}}{2}\big((1+\sqrt{1+\xi})^{1/3} + (1-\sqrt{1+\xi})^{1/3}\big)\Big).$$

Lemma 8.

$$g'(\xi) \underset{\xi\to 0}{=} \frac{2^{1/3}}{3\xi^{\frac{2}{3}}}\big(1+o(1)\big), \quad g'(\xi) \underset{\xi\to+\infty}{=} \frac{8}{3^5\xi^2} + o(\frac{1}{\xi^2}).$$

Proof. Denote $h(\xi) := \xi^{1/3}\left((1+\sqrt{1+\xi})^{1/3} + (1-\sqrt{1+\xi})^{1/3}\right)$, so $g(\xi) = h(\xi)(1 - \frac{h(\xi)}{2})$. Differentiating, we get:

$$\begin{aligned} h'(\xi) &= \frac{(1-\sqrt{1+\xi})^{\frac{1}{3}} + (1+\sqrt{1+\xi})^{\frac{1}{3}}}{3\,\xi^{\frac{2}{3}}} \\ &\quad + \xi^{\frac{1}{3}}\,(\frac{1}{6\sqrt{1+\xi}\,(1+\sqrt{1+\xi})^{\frac{2}{3}}} - \frac{1}{6\sqrt{1+\xi}\,(1-\sqrt{1+\xi})^{\frac{2}{3}}}) \\ &= \frac{\big(1-\sqrt{1+\xi}\big)^{\frac{1}{3}} + \big(1+\sqrt{1+\xi}\big)^{\frac{1}{3}}}{3\,\xi^{\frac{2}{3}}} + \frac{1}{6\sqrt{1+\xi}}\frac{(1-\sqrt{1+\xi})^{2/3} - (1+\sqrt{1+\xi})^{2/3}}{\xi^{2/3}}. \end{aligned}$$

(a) For $\xi \to 0$,

$$h(\xi) \underset{\xi\to 0}{=} 2^{1/3}\xi^{1/3} + o(\xi^{1/3}),$$

and

$$h'(\xi) \underset{\xi\to 0}{=} \frac{2^{1/3}}{3\xi^{\frac{2}{3}}} + \xi^{\frac{1}{3}}\,(\frac{1}{6\times 2^{2/3}} - \frac{2^{2/3}}{\xi^{2/3}}) + o(\frac{1}{\xi^{2/3}}) = \frac{2^{1/3}}{3\xi^{\frac{2}{3}}} + o(\frac{1}{\xi^{2/3}}),$$

which yields that

$$g'(\xi) = h'(\xi)\big(1-h(\xi)\big) \underset{\xi\to 0}{=} \frac{2^{1/3}}{3\xi^{\frac{2}{3}}} - +o(\frac{1}{\xi^{2/3}}).$$

(b) For $\xi \to +\infty$,

In order to estimate g' we need to do a Taylor expansion up to the third order:

$$\begin{aligned}
h(\xi) \underset{\xi\to+\infty}{=} & \ \xi^{1/3}(1+\xi)^{1/6}\big((1+\frac{1}{\sqrt{1+\xi}})^{1/3} - (1-\frac{1}{\sqrt{1+\xi}})^{1/3}\big) \\
= & \ \xi^{1/3}(1+\xi)^{1/6}\Big(1+\frac{1}{3\sqrt{1+\xi}} - \frac{1}{3^2(1+\xi)} + \frac{2\times 5}{3^3 3!(1+\xi)^{3/2}} \\
& -(1-\frac{1}{3\sqrt{1+\xi}} - \frac{1}{3^2(1+\xi)} - \frac{2\times 5}{3^3 3!(1+\xi)^{3/2}}) + o(\frac{1}{\xi^{3/2}})\Big) \\
= & \ \xi^{1/2}\big(1+\frac{1}{6\xi}+o(\frac{1}{\xi})\big)\big(\frac{2}{3\sqrt{1+\xi}} + \frac{10}{3^4(1+\xi)^{3/2}} + o(\frac{1}{\xi^{3/2}})\big) \\
= & \ \xi^{1/2}\big(1+\frac{1}{6\xi}+o(\frac{1}{\xi})\big)\big(\frac{2}{3\sqrt{\xi}} - \frac{1}{3\xi^{3/2}} + \frac{10}{3^4(1+\xi)^{3/2}} + o(\frac{1}{\xi^{3/2}})\big) \\
= & \ \xi^{1/2}\big(\frac{2}{3\sqrt{\xi}} + \frac{1}{3^2\xi^{3/2}} - \frac{1}{3\xi^{3/2}} + \frac{10}{3^4(1+\xi)^{3/2}} + o(\frac{1}{\xi^{3/2}})\big) \\
= & \ \frac{2}{3} - \frac{8}{81\xi} + o(\frac{1}{\xi}).
\end{aligned}$$

Differentiating the above expression, we get:

$$h'(\xi) \underset{\xi\to+\infty}{=} \frac{8}{81\xi^2} + o(\frac{1}{\xi^2}),$$

which yields that

$$\begin{aligned}
g'(\xi) &= h'(\xi)\big(1-h(\xi)\big) \underset{\xi\to 0}{=} \big(\frac{8}{81\xi^2} + o(\frac{1}{\xi^2})\big)\big(1 - \frac{2}{3} + \frac{8}{81\xi} + o(\frac{1}{\xi})\big) \\
&= \frac{16}{3^5\xi^2} - \frac{8}{3^5\xi^2} + o(\frac{1}{\xi^2}) = \frac{8}{3^5\xi^2} + o(\frac{1}{\xi^2}).
\end{aligned}$$

□

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References

[BB06] G. Blower and F. Bolley. Concentration inequalities on product spaces with applications to Markov processes. *Studia Mathematica*, 175-1:47–72, 2006.

[BE85] D. Bakry and M. Émery. Diffusions hypercontractives. *Séminaire de probabilités*, XIX:177–206, 1985.

[BGL14] D. Bakry, I. Gentil, and M. Ledoux. *Analysis and geometry of Markov diffusion operators*, volume 348 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Cham, 2014.

[Bha82] R. N. Bhattacharya. On the functional central limit theorem and the law of the iterated logarithm for Markov processes. *Z. Wahrsch. Verw. Gebiete*, 60(2):185–201, 1982.

[BLG14] Emmanuel Boissard and Thibaut Le Gouic. On the mean speed of convergence of empirical and occupation measures in Wasserstein distance. *Ann. Inst. Henri Poincaré Probab. Stat.*, 50(2):539–563, 2014.

[Boi11] E. Boissard. Simple bounds for the convergence of empirical and occupation measures in 1-Wasserstein distance. *Electronic Journal of Probability*, 16, 2011.

[DG15] J. Dedecker and S. Gouëzel. Subgaussian concentration inequalities for geometrically ergodic markov chains. *Electronic Communications in Probability*, 20, Article 64:1–12, 2015.

[EK86] E. Ethier and T. Kurtz. *Markov Processes. Characterization and Convergence.* Wiley, 1986.

[FM12] N. Frikha and S. Menozzi. Concentration bounds for stochastic approximations. *Electron. Commun. Probab.*, 17:no. 47, 15, 2012.

[GL78] M.I. Gordin and B.A. Lifsic. On the central limit theorem for stationnary markov processes. *Soviet Math. Dokl.*, 19(2):392–394, 1978.

[HMP17] I Honoré, S Menozzi, and G Pagès. Non-Asymptotic Gaussian Estimates for the Recursive Approximation of the Invariant Measure of a Diffusion. working paper or preprint, July 2017.

[Hon18] I. Honoré. *PhD Thesis, Université d'Evry Val d'Essonne.* in progress, 2018.

[KM11] R. Khasminskii and G.N. Milstein. *Stochastic Stability of Differential Equations.* Stochastic Modelling and Applied Probability. Springer Berlin Heidelberg, 2011.

[KP10] N. V. Krylov and E. Priola. Elliptic and parabolic second-order PDEs with growing coefficients. *Comm. Partial Differential Equations*, 35(1):1–22, 2010.

[Led99] M. Ledoux. Concentration of measure and logarithmic Sobolev inequalities. In *Séminaire de Probabilités, XXXIII*, volume 1709 of *Lecture Notes in Math.*, pages 120–216. Springer, Berlin, 1999.

[Lem05] V. Lemaire. An adaptive scheme for the approximation of dissipative systems. February 2005.

[LP02] D. Lamberton and G. Pagès. Recursive computation of the invariant distribution of a diffusion. *Bernoulli*, 8–3:367–405, 2002.

[MT06] F. Malrieu and D. Talay. Concentration inequalities for Euler Schemes. In H. Niederreiter and D. Talay, editors, *Monte Carlo and Quasi-Monte Carlo Methods 2004*, pages 355–371. Springer Berlin Heidelberg, 2006.

[Pan08a] F. Panloup. Computation of the invariant measure of a levy driven SDE: Rate of convergence. *Stochastic processes and Applications*, 118–8:1351–1384, 2008.

[Pan08b] F. Panloup. Recursive computation of the invariant measure of a stochastic differential equation driven by a lévy process. *Ann. Appl. Probab.*, 18(2):379–426, 04 2008.

[PP12] G. Pagès and F. Panloup. Ergodic approximation of the distribution of a stationary diffusion: rate of convergence. *Ann. Appl. Probab.*, 22(3):1059–1100, 2012.

[PV01] E. Pardoux and A. Veretennikov. On the Poisson Equation and Diffusion Approximation. I. *Ann. Probab.*, 29–3:1061–1085, 2001.

[Ste70] E. M. Stein. *Singular integrals and differentiability properties of functions.* Princeton university press, 1970.

[Tal02] D. Talay. Stochastic Hamiltonian dissipative systems: exponential convergence to the invariant measure, and discretization by the implicit Euler scheme. *Markov Processes and Related Fields*, 8–2:163–198, 2002.

[TT90] D. Talay and L. Tubaro. Expansion of the global error for numerical schemes solving stochastic differential equations. *Stoch. Anal. and App.*, 8-4:94–120, 1990.

[Vil09] C. Villani. Hypocoercivity. *Mem. Amer. Math. Soc.*, 202(950):iv+141, 2009.

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