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► **To cite this version:**

Bahloul Rachid, Bahaj Mohamed, Sidki Omar. Periodic solutions of differential equations with three variable in vector-valued space. 2017. hal-01635197

HAL Id: hal-01635197

<https://hal.science/hal-01635197>

Preprint submitted on 14 Nov 2017

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Periodic solutions of differential equations with three variable in vector-valued space

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ABSTRACT

The aim of this work is to study the existence and uniqueness of a periodic solutions of differential partial functional differential equations with three variable:

$\frac{\partial}{\partial \mathbf{x}} \frac{\partial}{\partial \mathbf{y}} \frac{\partial}{\partial \mathbf{w}} \mathbf{u}(\mathbf{x}, \mathbf{y}, \mathbf{w}) + \mathbf{A}\mathbf{u}(\mathbf{x}, \mathbf{y}, \mathbf{w}) = \mathbf{f}(\mathbf{x}, \mathbf{y}, \mathbf{w}); \mathbf{x}, \mathbf{y}, \mathbf{w} \in [0, 2\pi]$. Our approach is based on the L^p -multipliers of linear operators.

Keywords: Differential equations; L^p -multipliers; vector-valued function space.

1 Introduction

Motivated by the fact that neutral functional integro-differential equations (abbreviated, NFDE) with finite delay arise in many areas of applied mathematics, this type of equations has received much attention in recent years. In particular, the problem of existence of periodic solutions, has been considered by several authors.

In this work, we study the existence and uniqueness of periodic solutions for the following integro-differential equation with two variable of the following form

$$\frac{\partial}{\partial \mathbf{x}} \frac{\partial}{\partial \mathbf{y}} \frac{\partial}{\partial \mathbf{w}} \mathbf{u}(\mathbf{x}, \mathbf{y}, \mathbf{w}) + \mathbf{A}\mathbf{u}(\mathbf{x}, \mathbf{y}, \mathbf{w}) = \mathbf{f}(\mathbf{x}, \mathbf{y}, \mathbf{w}) \text{ for } x, y, w \in \mathbb{R}, \quad (1.1)$$

where $(A, D(A))$ is a closed linear operator on a Banach space X , and $f : \mathbb{R}^3 \rightarrow X$ is a locally p -integrable and 2π -periodic function for $1 \leq p < \infty$.

In [1] and [2], Arendt gave necessary and sufficient conditions for the existence of periodic solutions of the following evolution equation in vecteur valued and Besov space.

In [3], Bahloul et al established results on the existence for some degenerate differential equation.

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In [4], Bahloul established results on the existence of periodic solution of Eq. (1.1) when $a = 0$, namely, for the following partial functional differential equation

$$\frac{\partial}{\partial x} \frac{\partial}{\partial y} u(x, y) + Au(x, y) = f(x, y)$$

where $(A, D(A))$ is a linear operator. Similar characterizations were then obtained for the scale of degenerate integro-differential and second differential equations [5], [6] and [7].

In [8], Hernan et al, studied the existence of periodic solution for the class of linear abstract neutral functional differential equation described in the following form:

$$\frac{d}{dt}[x(t) - Bx(t-r)] = Ax(t) + G(x_t) + f(t) \quad \text{for } t \in \mathbb{R}$$

where $A : D(A) \rightarrow X$ and $B : D(B) \rightarrow X$ are closed linear operator such that $D(A) \subset D(B)$ and $G \in B(L^p([-2\pi, 0], X); X)$. We refer the readers to papers [9], [10], [11] and [12] and the references listed therein for information on this subject.

In [13], Lizama gave necessary and sufficient conditions for the existence of periodic solutions of the following equation.

$$\frac{d}{dt}x(t) = Ax(t) + L(x_t) + f(t) \quad \text{for } t \in \mathbb{R},$$

where A is a closed linear operator on an UMD-space Y . We note that, starting with the work [13], the problem of characterization of maximal regularity for avolution equations with periodic initial conditions have been studied intensively in the last years. See e.g [14], [15] and [16], see also [17] and [18] and references therien.

the main novelty in this paper relies in the presence of two variable. This work can be considered as a progress on the treatment of such kind of problems.

This work is organized as follows : After a creterion in the second section, we are able to characterize in section 3 the existence and uniqueness of strong L^p -solution for the problem (1.1) solely in terms of a property of L^p -multiplier for the sequence of operators $-iklz(-iklzI + A)^{-1}$. We optain that the following assertion are equivalent in general Banach spaces:

- (1) $(-iklzI + A)$ is invertible and $\{-iklz(-iklzI + A)^{-1}, (k, l, k) \in \mathbb{Z}^3\}$ is an L^p -muliplier.
- (2) For every $f \in L^p(\mathbb{T}^3; X)$ there exist a unique function $u \in H^{1,p}(\mathbb{T}^3; X)$ such that $u \in D(A)$ and equation (1.1) holds for a.e $(x, y, w) \in [0, 2\pi]^3$.

In section 4, we give the main abstract result (theorem (4.2)) of this work.

In section 5, we propose an application to some integro-differential equation. In section 6, we give the conclusion.

2 A creterion for periodic solutions

Let X be a Banach Space. Firstly, we denote By \mathbb{T} the group defined as the quotient $\mathbb{R}/2\pi\mathbb{Z}$. There is an identification between functions on \mathbb{T} and 2π -periodic functions on \mathbb{R} .

We consider the interval $[0; 2\pi)$ as a model for \mathbb{T} . Given $1 \leq p < \infty$, we denote by $L^p(\mathbb{T}^3; X)$ the space of 2π -periodic locally p -integrable functions from \mathbb{R}^3 into X , with the norm:

$$\|f\|_p := \left(\int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \|f(x, y, w)\|^p dx dy dw \right)^{1/p}$$

For $f \in L^p(\mathbb{T}^3; X)$, we denote by $\hat{f}(k, l, z)$, $(k, l, z) \in \mathbb{Z}$ the (k, l, z) -th Fourier coefficient of f that is defined by:

$$\hat{f}^3(k, l, z) = \left(\frac{1}{2\pi}\right)^3 \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} e^{-i(kx+ly+zw)} f(x, y, w) dx dy dw$$

for $(k, l, z) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ and $(x, y, w) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$.

Notation 2.1. : Let $(k, l, z) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ and $1 \leq p < \infty$. Denote by

$$H^{1,p}(\mathbb{T}^3; X) = \{u \in L^p(\mathbb{T}^3; X) : \exists v \in L^p(\mathbb{T}^3; X), \hat{v}^3(k, z) = -iklz \hat{u}^3(k, z)\}$$

Remark 2.1. [9]

Let $f \in L^1(\mathbb{T}; X)$. If $g(t) = \int_0^t f(s) ds$ and $k \in \mathbb{Z}, k \neq 0$, then

$$\hat{g}(k) = \frac{i}{k} \hat{f}(0) - \frac{i}{k} \hat{f}(k)$$

such that: $\hat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikx} dx$.

Lemma 2.2. [4]

Let $f \in L^1(\mathbb{T} \times \mathbb{T}; X)$.

If $w(x, t) = \int_0^x \int_0^t f(s, \xi) d\xi ds$ and $(k, z) \in \mathbb{Z} \times \mathbb{Z} - \{(0, 0)\}$ then

$$\hat{w}(k, z) = -\frac{1}{kz} \hat{f}(0, 0) + \frac{1}{kz} \hat{f}(0, z) + \frac{1}{kz} \hat{f}(k, 0) - \frac{1}{kz} \hat{f}(k, z)$$

such that: $\hat{f}(k, z) = \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi} \int_0^{2\pi} e^{-i(kx+zy)} dx dy$.

Lemma 2.3. [4]

Let $1 \leq p < \infty$ and let $f, g \in L^1(\mathbb{T} \times \mathbb{T}; X)$. The following are equivalent:

(1) $\hat{v}^3(k, l, z) = 0$ if $k = 0, l = 0$ or $k = 0$ and there exists $a \in X$ such that

$$u(x, y, w) = a + \int_0^x \int_0^y \int_0^w v(s, \tau, \xi) d\xi d\tau ds, \quad (x, y, w) \in \mathbb{T}^3$$

(2) $\hat{v}^3(k, l, z) = -iklz \hat{u}^3(k, l, z), (k, l, z) \in \mathbb{Z}^3$

Proof. Its proof is inspired by that of [4, Lemma 2.3]. □

Remark 2.2. By Lemma (4.1) we have $u \in H^{1,p}(\mathbb{T}^3; X) \Leftrightarrow$ there exists $v \in L^p(\mathbb{T}^3; X)$ such that $\hat{v}^3(k, l, z) = 0$ if $k = 0, l = 0$ or $k = 0$ and there exists $a \in X$ such that $u(x, y, w) = a + \int_0^x \int_0^y \int_0^w v(s, \tau, \xi) d\xi d\tau ds, (x, y, w) \in \mathbb{T}^3$.

Now we can formulate the following multiplier definitions.

Definition 2.1. For $1 \leq p < \infty$, a sequence $\{M_{k,l,z}\}_{(k,l,z) \in \mathbb{Z}^3}$ is said to be an L^p -multiplier (or (L^p, L^p) -multiplier) if for each $f \in L^p(\mathbb{T}^3, X)$, there exists $u \in L^p(\mathbb{T}^3, Y)$ such that $\widehat{g}^3(k, l, z) = M_{k,l,z} \widehat{f}^3(k, l, z)$ for all $(k, l, z) \in \mathbb{Z}^3$.

Definition 2.2. : For $1 \leq p < \infty$, a sequence $\{M_{k,l,z}\}_{(k,l,z) \in \mathbb{Z} \times \mathbb{Z}}$ is said to be an $(L^p, H^{1,p})$ -multiplier if for each $f \in L^p(\mathbb{T}^3, X)$, there exists $u \in H^{1,p}(\mathbb{T}^3, Y)$ such that $\widehat{g}^3(k, l, z) = M_{k,l,z} \widehat{f}^3(k, l, z)$ for all $(k, l, z) \in \mathbb{Z}^3$.

We recall the following results

Lemma 2.4. [4] Let $f, g \in L^p([0, 2\pi]^3; X)$, where $1 \leq p < \infty$. Then the following are equivalent.

- (i) $f(x, y, w) \in D(A)$ and $Af(x, y, w) = g(x, y, w)$,
- (ii) $\widehat{f}^3(k, l, z) \in D(A)$ and $A\widehat{f}^3(k, l, z) = \widehat{g}^3(k, l, z)$ for all $(k, l, z) \in \mathbb{Z}^3$

3 A characterization of strong L^p -solution

Definition 3.1. : Let $f \in L^p(\mathbb{T}^3; X)$. A function $u \in H^{1,p}(\mathbb{T}^3; X)$ is said to be a 2π -periodic strong L^p -solution of Eq. (1.1) if $u \in D(A)$ and Eq.(1.1) holds almost every where.

Proposition 3.1. Let X be a Banach space. Suppose that for every $f \in L^p(\mathbb{T}^3; X)$ there exists a unique strong solution of Eq. (1.1) for $1 \leq p < \infty$. Then

- (1) for every $(k, l, z) \in \mathbb{Z}^3$ the operator $(-iklzI + A)$ has invertible.
- (2) $\{-iklz(-iklzI + A)^{-1}\}_{(k,l,z) \in \mathbb{Z} \times \mathbb{Z}}$ is an L^p -multiplier.

Proof. 1. Let $k, l, z \in \mathbb{Z}$ and $v \in X$. Then for $f(x, y, w) = e^{i(kx+ly+zw)}v$, there exists $u \in H^{1,p}(\mathbb{T}^3; X)$ such that:

$$\frac{\partial}{\partial x} \frac{\partial}{\partial y} \frac{\partial}{\partial w} u(x, y, w) + Au(x, y, w) = +f(x, y, w)$$

Taking Fourier transform. Then we obtain

$$(-iklzI + A)\widehat{u}^3(k, l, z) = \widehat{f}^3(k, l, z) = v \Rightarrow (-iklzI + A) \text{ is surjective.}$$

if $(-iklzI + A)v = 0$, then, $u(x, y, w) = e^{i(kx+ly+zw)}v$ is a 2π -periodic strong L^p -solution of Eq. (1.1) corresponding to the function $f = 0$ Hence $u(x, y, w) = 0$ and $v = 0$ then $(-iklzI + A)$ is injective.

2) Let $f \in L^p(\mathbb{T}^3; X)$. By hypothesis, there exists a unique $u \in H^{1,p}(\mathbb{T}^3, X)$ such that the Eq.(1.1) is valid. Taking Fourier transforms, we deduce that $\widehat{u}^3(k, l, z) = (-iklzI + A)^{-1} \widehat{f}^3(k, l, z)$ for all $k, l, z \in \mathbb{Z}$, then $\{(-iklzI + A)^{-1}\}_{(k,l,z) \in \mathbb{Z}^3}$ is an L^p -multiplier. Hence $-iklz\widehat{u}^3(k, l, z) = -iklz(-iklzI + A)^{-1} \widehat{f}^3(k, l, z)$ for all $k, l, z \in \mathbb{Z}$. On the other hand, since $u \in H^{1,p}(\mathbb{T}^3; X)$, there exists $w \in L^p(\mathbb{T}^3; X)$ such that $\widehat{w}^3(k, l, z) = -iklz\widehat{u}^3(k, l, z) = -iklz(-iklzI + A)^{-1} \widehat{f}^3(k, l, z)$ i.e $\{-iklz(-iklzI + A)^{-1}\}_{(k,l,z) \in \mathbb{Z}^3}$ is an L^p -multiplier. \square

4 Main result

The next Theorem is the main result of this section.

Lemma 4.1. *Let $f \in L^1(\mathbb{T}^3; X)$.*

If $w(x, y, w) = \int_0^x \int_0^y \int_0^w f(s, \tau, \xi) d\xi d\tau ds$ and $(k, z) \in \mathbb{Z}^3 - \{(0, 0, 0)\}$ then

$$\widehat{w}^3(k, l, z) = \frac{i^3}{klz} [\widehat{f}^3(0, 0, 0) - \widehat{f}^3(k, 0, 0) - \widehat{f}^3(0, l, 0) - \widehat{f}^3(0, 0, z) + \widehat{f}^3(k, l, 0) + \widehat{f}^3(0, l, z) + \widehat{f}^3(k, 0, z) - \widehat{f}^3(k, l, z)]$$

Proof.

$$\begin{aligned} \widehat{w}^3(k, l, z) &= \frac{i}{z} \left[\int_0^x \int_0^y \widehat{f}(s, \tau, 0) d\tau ds \right] - \frac{i}{z} \left[\int_0^x \int_0^y \widehat{f}(s, \tau, z) d\tau ds \right] \\ &= \frac{i}{z} \left[\frac{i}{l} \int_0^x \widehat{f}(s, 0, 0) ds - \frac{i}{l} \int_0^x \widehat{f}(s, l, 0) ds \right] - \frac{i}{z} \left[\frac{i}{l} \int_0^x \widehat{f}(s, 0, z) ds - \frac{i}{l} \int_0^x \widehat{f}(s, l, z) ds \right] \\ &= \frac{i^3}{zl} \left[\int_0^x \widehat{f}(s, 0, 0) ds - \int_0^x \widehat{f}(s, l, 0) ds - \int_0^x \widehat{f}(s, 0, z) ds + \int_0^x \widehat{f}(s, l, z) ds \right] \\ &= \frac{i^3}{zl} \left[\left(\frac{i}{k} \widehat{f}^3(0, 0, 0) - \frac{i}{k} \widehat{f}^3(k, 0, 0) \right) - \left(\frac{i}{k} \widehat{f}^3(0, l, 0) - \frac{i}{k} \widehat{f}^3(k, l, 0) \right) - \left(\frac{i}{k} \widehat{f}^3(0, 0, z) - \frac{i}{k} \widehat{f}^3(k, 0, z) \right) \right. \\ &\quad \left. + \left(\frac{i}{k} \widehat{f}^3(0, l, 0) - \frac{i}{k} \widehat{f}^3(k, l, 0) \right) \right] \\ &= \frac{i^3}{klz} [\widehat{f}^3(0, 0, 0) - \widehat{f}^3(k, 0, 0) - \widehat{f}^3(0, l, 0) - \widehat{f}^3(0, 0, z) + \widehat{f}^3(k, l, 0) + \widehat{f}^3(0, l, z) + \widehat{f}^3(k, 0, z) - \widehat{f}^3(k, l, z)]. \end{aligned}$$

□

Theorem 4.2. *Let A be a closed linear operator and $1 \leq p < \infty$. If*

(1) *for every $(k, l, z) \in \mathbb{Z}^3$ the operator $M_{k,l,z} = (-iklzI + A)$ has invertible.*

(2) $\{-iklz(-iklzI + A)^{-1}\}_{(k,l,z) \in \mathbb{Z}^3}$ *is an L^p -multiplier.*

Then for every $f \in L^p(\mathbb{T}^3; X)$ there exists a unique 2π -periodic strong L^p -solution of Eq. (1.1).

Lemma 4.3. [4] *Let $1 \leq p < \infty$ and X be a Banach space. The following assertions are equivalent*

(1) $\{M_{k,l,z}\}_{(k,l,z) \in \mathbb{Z}^3}$ *is an (L^p, H^p) -multiplier.*

(2) $\{-iklzM_{k,l,z}\}_{(k,l,z) \in \mathbb{Z}^3}$ *is an L^p -multiplier.*

Proof. 1 \Rightarrow 2. Let $f \in L^p(\mathbb{T}^3, X)$, then there exists $g \in H^{1,p}(\mathbb{T}^3; X)$ (in particular $g \in L^p(\mathbb{T}^3; X)$) such that $\widehat{g}^3(k, l, z) = -iklzM_{k,l,z}\widehat{f}^3(k, l, z)$, then $\{-iklzM_{k,l,z}\}_{(k,l,z) \in \mathbb{Z}^3}$ is an L^p -multiplier.

2 \Rightarrow 1. Let $f \in L^p(\mathbb{T}^3; X)$, by hypothesis $\exists v \in L^p(\mathbb{T}^3; X)$ such that $\widehat{v}^3(k, l, z) = -iklzM_{k,l,z}\widehat{f}^3(k, l, z)$. In particular $\widehat{v}^3(k, l, z) = 0$ if $k = 0, l = 0$ or $z = 0$.

Let

$$w(x, y, w) = \int_0^x \int_0^y \int_0^w v(s, \tau, \xi) d\xi d\tau ds,$$

then by Lemma (4.1) we have,

$$\widehat{w}^3(k, l, z) = \frac{i^3}{klz} [\widehat{v}^3(0, 0, 0) - \widehat{v}^3(k, 0, 0) - \widehat{v}^3(0, l, 0) - \widehat{v}^3(0, 0, z) + \widehat{v}^3(k, l, 0) + \widehat{v}^3(0, l, z) + \widehat{v}^3(k, 0, z) - \widehat{v}^3(k, l, z)]$$

$$\widehat{w}^3(k, l, z) = \frac{-1}{iklz} \widehat{v}^3(k, l, z) = M_{k,l,z} \widehat{f}^3(k, l, z), \text{ for all } k \neq 0, l \neq 0 \text{ and } z \neq 0$$

Let $u(x, y, z) = w(x, y, z) + M_{0,0,0} \widehat{f}^3(0, 0, 0) - \widehat{w}^3(0, 0, 0)$ i.e

$u(x, y, z) = a + \int_0^x \int_0^y \int_0^w v(s, \tau, \xi) d\xi d\tau ds$, $a = M_{0,0,0} \widehat{f}^3(0, 0, 0) - \widehat{w}^3(0, 0, 0)$. Then by Remark (2.1) we have $u \in H^{1,p}(\mathbb{T}^3; X)$.

$$\widehat{u}^3(k, l, z) = \widehat{w}^3(k, l, z) + 0 = \widehat{w}^3(k, l, z) = M_{k,l,z} \widehat{w}^3(k, l, z), \text{ for all } (k, l, z) \in \mathbb{Z}^3,$$

Proof of theorem (4.2). Existence: Let $f \in L^p(\mathbb{T}^3; X)$. Define $M_{k,z} = (-iklzI + A)^{-1}$ by theorem (4.2), the family $\{-iklz(-iklzI + A)^{-1}\}_{(k,l,z) \in \mathbb{Z}^3}$ is an L^p -multiplier and by Lemma (4.3), it is equivalent to the fact that the family $\{M_{k,l,z}\}_{(k,l,z) \in \mathbb{Z}^3}$ is an (L^p, H^p) -multiplier, i.e, there exists $u \in H^{1,p}(\mathbb{T}^3; X)$ such that

$$\widehat{u}^3(k, l, z) = M_{k,l,z} \widehat{f}^3(k, l, z) = (-iklzI + A)^{-1} \widehat{f}^3(k, l, z) \quad (4.1)$$

In particular, $u \in L^p(\mathbb{T}^3, X)$ and there exists $v \in L^p(\mathbb{T}^3; X)$ such that

$$\widehat{v}^3(k, l, z) = -iklz \widehat{u}^3(k, l, z) \quad (4.2)$$

Moreover, $\widehat{\frac{\partial}{\partial x} \frac{\partial}{\partial y} \frac{\partial}{\partial w} u}^3(k, l, z) = -iklz \widehat{v}^3(k, l, z)$.

By Fejer's theorem one has in $L^p(\mathbb{T}^3; X)$

$$\widehat{u}^3(k, l, z) = \lim_{n \rightarrow \infty} \lim_{n' \rightarrow \infty} \lim_{m \rightarrow \infty} \sigma_{n,n',m}(u)(x, y, w)$$

such that

$$\sigma_{n,n',m}(u)(x, y, w) = \frac{1}{n+1} \frac{1}{n'+1} \frac{1}{m+1} \sum_{p=0}^n \sum_{k=-p}^p \sum_{p'=0}^{n'} \sum_{l=-p'}^{p'} \sum_{p''=0}^m \sum_{z=-p''}^{p''} e^{i(kx+ly+zw)} \widehat{u}^3(k, l, z).$$

By (4.1) and (4.2) we have

$$-iklz \widehat{u}^3(k, l, z) + A \widehat{u}^3(k, l, z) = \widehat{f}^3(k, l, z)$$

For all $(k, l, z) \in \mathbb{Z}^3$. Then using that A is closed, we conclude that $u \in D(A)$ and

$$\frac{\partial}{\partial x} \frac{\partial}{\partial y} \frac{\partial}{\partial w} u(x, y, w) + Au(x, y, w) = f(x, y, w).$$

Uniqueness : Suppose that the Eq. (1.1) has two strong L^p -solution u_1 and u_2 , then $u = u_1 - u_2$ is a solution of Eq. (1.1) corresponding to the function $f = 0$. Taking Fourier transform on both sides of this equality we get $(-iklzI + A) \widehat{u}^3(k, l, z)$, which implies that $\widehat{u}^3(k, l, z) = 0$ for all $(k, l, z) \in \mathbb{Z}^3$, consequently, $u = 0$. Then $u_1 = u_2$. \square

5 Conclusion

For every function $f \in L^p(\mathbb{T}^3; X)$, this paper deals with the existence and uniqueness of periodic solutions of the differential equation with three variable $\frac{\partial}{\partial \mathbf{x}} \frac{\partial}{\partial \mathbf{y}} \frac{\partial}{\partial \mathbf{w}} \mathbf{u}(\mathbf{x}, \mathbf{y}, \mathbf{w}) + \mathbf{A}\mathbf{u}(\mathbf{x}, \mathbf{y}, \mathbf{w}) = \mathbf{f}(\mathbf{x}, \mathbf{y}, \mathbf{w})$. Using the method of Fourier transform of equation (1.1) and L^p -multipliers of linear operators. the results obtained in this paper generalizes the result of Bahloul [4].

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