



**HAL**  
open science

# A breakdown of injectivity for weighted ray transforms in multidimensions

Fedor O Goncharov, Roman Novikov

► **To cite this version:**

Fedor O Goncharov, Roman Novikov. A breakdown of injectivity for weighted ray transforms in multidimensions. 2017. hal-01635188v1

**HAL Id: hal-01635188**

**<https://hal.science/hal-01635188v1>**

Preprint submitted on 14 Nov 2017 (v1), last revised 25 Mar 2018 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# A breakdown of injectivity for weighted ray transforms in multidimensions

F.O. Goncharov\*      R. G. Novikov\*†

November 14, 2017

## Abstract

We consider weighted ray-transforms  $P_W$  (weighted Radon transforms along straight lines) in  $\mathbb{R}^d$ ,  $d \geq 2$ , with strictly positive weights  $W$ . We construct an example of such a transform with non-trivial kernel in the space of infinitely smooth compactly supported functions on  $\mathbb{R}^d$ . In addition, the constructed weight  $W$  is rotation-invariant continuous and is infinitely smooth almost everywhere on  $\mathbb{R}^d \times \mathbb{S}^{d-1}$ . In particular, by this construction we give counterexamples to some well-known injectivity results for weighted ray transforms for the case when the regularity of  $W$  is slightly relaxed.

**Keywords:** Radon transforms, ray transforms, integral geometry, injectivity, non-injectivity

**AMS Mathematics Subject Classification:** 44A12, 53C65, 65R32

## 1 Introduction

We consider the weighted ray transforms  $P_W$  defined by

$$P_W f(x, \theta) = \int_{\mathbb{R}} W(x + t\theta, \theta) f(x + t\theta) dt, \quad (x, \theta) \in T\mathbb{S}^{d-1}, \quad d \geq 2, \quad (1.1)$$

$$T\mathbb{S}^{d-1} = \{(x, \theta) \in \mathbb{R}^d \times \mathbb{S}^{d-1} : x\theta = 0\}, \quad (1.2)$$

where  $f = f(x)$ ,  $W = W(x, \theta)$ ,  $x \in \mathbb{R}^d$ ,  $\theta \in \mathbb{S}^{d-1}$ . Here,  $W$  is the weight,  $f$  is a test function on  $\mathbb{R}^d$ . In addition, we interpret  $T\mathbb{S}^{d-1}$  as the set of all rays in  $\mathbb{R}^d$ . As a ray  $\gamma$  we understand a straight line with fixed orientation. If  $\gamma = \gamma(x, \theta)$ ,  $(x, \theta) \in T\mathbb{S}^{d-1}$ , then

$$\gamma(x, \theta) = \{y \in \mathbb{R}^d : y = x + t\theta, t \in \mathbb{R}\} \text{ (up to orientation),} \quad (1.3)$$

where  $\theta$  gives the orientation of  $\gamma$ .

We assume that

$$W = \overline{W} \geq c > 0, \quad W \in L^\infty(\mathbb{R}^d \times \mathbb{S}^{d-1}), \quad (1.4)$$

where  $\overline{W}$  denotes the complex conjugate of  $W$ ,  $c$  is a constant.

The aforementioned transforms  $P_W$  arise in various domains of pure and applied mathematics; see, e.g., [Bey84], [Bom93], [BQ87], [Fi86], [JoIlm16], [Kun92], [LB73], [MQ85], [Natt01], [Nov02], [Nov14], [Qui83] and references therein.

In particular, the related results are the most developed for the case when  $W \equiv 1$ . In this case  $P_W$  is reduced to the classical ray-transform  $P$  (Radon transform along straight lines). The transform  $P$  arises, in particular, in the X-ray transmission tomography. We refer to [GGV64], [Natt01] and references therein in connection with basic results for this classical case.

At present, many important results on transforms  $P_W$  with other weights  $W$  satisfying (1.4) are also known; see the publications mentioned above with non-constant  $W$  and references therein.

In particular, assuming (1.4) we have the following injectivity results.

**Injectivity 1** (see [Fi86]). Suppose that  $d \geq 3$  and  $W \in C^2(\mathbb{R}^d \times \mathbb{S}^{d-1})$ . Then  $P_W$  is injective on  $L_0^p(\mathbb{R}^d)$  for  $p > 2$ , where  $L_0^p$  denotes compactly supported functions from  $L^p$ .

---

\*CMAP, Ecole Polytechnique, CNRS, Université Paris-Saclay, 91128, Palaiseau, France;

email: fedor.goncharov.ol@gmail.com

†IEPT RAS, 117997 Moscow, Russia;

email: roman.novikov@polytechnique.edu

**Injectivity 2** (see [MQ85]). Suppose that  $d = 2$ ,  $W \in C^2(\mathbb{R}^2 \times \mathbb{S}^1)$  and

$$0 < c_0 \leq W, \|W\|_{C^2(\mathbb{R}^2 \times \mathbb{S}^1)} \leq N, \quad (1.5)$$

for some constants  $c_0$  and  $N$ . Then, for any  $p > 2$ , there is  $\delta = \delta(c_0, N, p) > 0$  such that  $P_W$  is injective on  $L^p(B(x_0, \delta))$  for any  $x_0 \in \mathbb{R}^2$ , where

$$\begin{aligned} L^p(B(x_0, \delta)) &= \{f \in L^p(\mathbb{R}^2) : \text{supp } f \subset \overline{B}(x_0, \delta)\}, \\ \overline{B}(x_0, \delta) &= \{x \in \mathbb{R}^2 : |x - x_0| \leq \delta\}. \end{aligned}$$

**Injectivity 3** (see [Qui83]). Suppose that  $d = 2$ ,  $W \in C^1(\mathbb{R}^2 \times \mathbb{S}^1)$  and  $W$  is rotation invariant (see formula (1.6) below). Then  $P_W$  is injective on  $L_0^p(\mathbb{R}^2)$  for  $p \geq 2$ .

In a similar way with [Qui83] we say that  $W$  is rotation invariant if and only if

$$W(x, \theta) = U(x\theta, |x - (x\theta)\theta|), \quad x \in \mathbb{R}^d, \theta \in \mathbb{S}^{d-1}, \quad (1.6)$$

for some positive and continuous  $U$  such that

$$U(r, s) = U(-r, s) = U(r, -s), \quad r \in \mathbb{R}, s \in \mathbb{R}. \quad (1.7)$$

Injectivity 1 is a global injectivity for  $d \geq 3$ . Injectivity 2 is a local injectivity for  $d = 2$ . Injectivity 3 is a global injectivity for  $d = 2$  for the rotation invariant case.

The results of Injectivity 1 and Injectivity 2 remain valid with  $C^\alpha$ ,  $\alpha > 1$ , in place of  $C^2$  in the assumptions on  $W$ ; see [Jollm16].

Injectivity 1 follows from Injectivity 2 in the framework of the layer-by-layer reconstruction approach. See [Fi86], [Novi02], [Jollm16] and references therein in connection with the layer-by-layer reconstruction approach for weighted and non-abelian ray transforms in dimension  $d \geq 3$ .

In the present work we obtain counterexamples to Injectivity 1, Injectivity 2 and Injectivity 3 for the case when the regularity of  $W$  is slightly relaxed. In particular, by these counterexamples we continue related studies of [MQ85], [Bom93] and [GN17].

More precisely, in the present work we construct  $W$  and  $f$  such that

$$P_W f \equiv 0 \text{ on } T\mathbb{S}^{d-1}, \quad d \geq 2, \quad (1.8)$$

where

$W$  satisfies (1.4),  $W$  is rotation-invariant (i.e., satisfies (1.6)),

$W$  is infinitely smooth almost everywhere on  $\mathbb{R}^d \times \mathbb{S}^{d-1}$  and (1.9)

$W \in C^\alpha(\mathbb{R}^d \times \mathbb{S}^{d-1})$ , at least, for any  $\alpha \in (0, \alpha_0)$ , where  $\alpha_0 = 1/16$ ;

$f$  is a non-zero spherically symmetric infinitely smooth compactly supported function on  $\mathbb{R}^d$ ; (1.10)

see Theorem 1 of Section 3.

These  $W$  and  $f$  directly give the aforementioned counterexamples to Injectivity 1 and Injectivity 3.

Our counterexample to Injectivity 1 is of particular interest (and is rather surprising) in view of the fact that the problem of finding  $f$  on  $\mathbb{R}^d$  from  $P_W f$  on  $T\mathbb{S}^{d-1}$  is strongly overdetermined for  $d \geq 3$ . Indeed,

$$\dim \mathbb{R}^d = d, \quad \dim T\mathbb{S}^{d-1} = 2d - 2,$$

$$d < 2d - 2 \text{ for } d \geq 3.$$

Our counterexample to Injectivity 3 is considerably stronger than the preceding counterexample of [MQ85], where  $W$  is not yet continuous and is not yet strictly positive (i.e., is not yet separated from zero by a positive constant).

Using our  $W$  and  $f$  of (1.9), (1.10) for  $d = 3$  we also obtain the aforementioned counterexample to Injectivity 2; see Corollary 1 of Section 3.

In the present work we adopt and develop considerations of the famous work [Bom93] and of our very recent work [GN17].

In Section 2 we give some preliminaries and notations.

Main results are presented in detail in Sections 3.

Related proofs are given in Sections 4, 5, 6, 7, 8.

## 2 Some preliminaries

**Notations.** Let

$$\Omega = \mathbb{R}^d \times \mathbb{S}^{d-1}, \quad (2.1)$$

$$r(x, \theta) = |x - (x\theta)\theta|, \quad (x, \theta) \in \Omega, \quad (2.2)$$

$$\Omega_0(\delta) = \{(x, \theta) \in \Omega : r(x, \theta) > \delta\}, \quad (2.3)$$

$$\Omega_1(\delta) = \Omega \setminus \Omega_0(\delta) = \{(x, \theta) \in \Omega : r(x, \theta) \leq \delta\}, \quad \delta > 0, \quad (2.4)$$

$$\Omega(\Lambda) = \{(x, \theta) \in \mathbb{R}^d \times \mathbb{S}^{d-1} : r(x, \theta) \in \Lambda\}, \quad \Lambda \subset [0, +\infty), \quad (2.5)$$

$$T_0(\delta) = \{(x, \theta) \in T\mathbb{S}^{d-1} : |x| > \delta\}, \quad (2.6)$$

$$T_1(\delta) = \{(x, \theta) \in T\mathbb{S}^{d-1} : |x| \leq \delta\}, \quad \delta > 0, \quad (2.7)$$

$$T(\Lambda) = \{(x, \theta) \in T\mathbb{S}^{d-1} : |x| \in \Lambda\}, \quad \Lambda \subset [0, +\infty), \quad (2.8)$$

$$\mathcal{J}_{r,\varepsilon} = (r - \varepsilon, r + \varepsilon) \cap [0, +\infty), \quad r \in [0, +\infty), \quad \varepsilon > 0. \quad (2.9)$$

The set  $T_0(\delta)$  in (2.6) is considered as the set of all rays in  $\mathbb{R}^d$  which are located at distance greater than  $\delta$  from the origin.

The set  $T_1(\delta)$  in (2.7) is considered as the set of all rays in  $\mathbb{R}^d$  which are located at distance less or equal than  $\delta$ .

We also consider the projection

$$\pi : \Omega \rightarrow T\mathbb{S}^{d-1}, \quad (2.10)$$

$$\pi(x, \theta) = (\pi_\theta x, \theta), \quad (x, \theta) \in \Omega, \quad (2.11)$$

$$\pi_\theta x = x - (x\theta)\theta. \quad (2.12)$$

In addition,  $r(x, \theta)$  of (2.2) is the distance from the origin  $\{0\} \in \mathbb{R}^d$  to the ray  $\gamma = \gamma(\pi(x, \theta))$  (i.e.,  $r(x, \theta) = |\pi_\theta x|$ ). The rays will be also denoted by

$$\gamma = \gamma(x, \theta) \stackrel{def}{=} \gamma(\pi(x, \theta)), \quad (x, \theta) \in \Omega. \quad (2.13)$$

We also consider

$$P_W f(x, \theta) = P_W f(\pi(x, \theta)) \text{ for } (x, \theta) \in \Omega. \quad (2.14)$$

We also define

$$B(x_0, \delta) = \{x \in \mathbb{R}^d : |x - x_0| < \delta\}, \quad \overline{B}(x_0, \delta) = \{x \in \mathbb{R}^d : |x - x_0| \leq \delta\}, \quad x_0 \in \mathbb{R}^d, \quad \delta > 0, \quad (2.15)$$

$$B = B(0, 1), \quad \overline{B} = \overline{B}(0, 1). \quad (2.16)$$

For a function  $f$  on  $\mathbb{R}^d$  we denote its restriction to a subset  $\Sigma \subset \mathbb{R}^d$  by  $f|_\Sigma$ .

By  $C_0, C_0^\infty$  we denote continuous compactly supported and infinitely smooth compactly supported functions, respectively.

By  $C^\alpha(Y)$ ,  $\alpha \in (0, 1)$ , we denote the space of  $\alpha$ -Hölder functions on  $Y$  with the norm:

$$\begin{aligned} \|u\|_{C^\alpha(Y)} &= \|u\|_{C(Y)} + \|u\|'_{C^\alpha(Y)}, \\ \|u\|'_{C^\alpha(Y)} &= \sup_{\substack{y_1, y_2 \in Y \\ |y_1 - y_2| \leq 1}} \frac{|u(y_1) - u(y_2)|}{|y_1 - y_2|^\alpha}, \end{aligned} \quad (2.17)$$

where  $\|u\|_{C(Y)}$  denotes the maximum of  $|u|$  on  $Y$ .

**Rotation invariancy.** Symmetry (1.6) of  $W$  can be also written as

$$W(x, \theta) = \tilde{U}(|x|, x\theta), \quad (x, \theta) \in \Omega, \quad (2.18)$$

$$\tilde{U}(r, s) = \tilde{U}(-r, s) = \tilde{U}(r, -s), \quad r \in \mathbb{R}, \quad s \in \mathbb{R}. \quad (2.19)$$

where  $\tilde{U}$  is positive and continuous on  $\mathbb{R} \times \mathbb{R}$ .

Using the formula  $|x|^2 = |x\theta|^2 + r^2(x, \theta)$ , one can see that symmetries (1.6), (1.7) and symmetries (2.18), (2.19) of  $W$  are equivalent.

**Partition of unity.** We recall the following classical result (see, e.g., Theorem 5.6 in [MCar92]):  
*Let  $\mathcal{M}$  be a  $C^\infty$ -manifold, which is Hausdorff and has a countable base. Let also  $\{U_i\}_{i=1}^\infty$  be an open locally finite cover of  $\mathcal{M}$ .*

*Then there exists a  $C^\infty$ -smooth locally-finite partition of unity  $\{\psi_i\}_{i=1}^\infty$  on  $\mathcal{M}$ , such that*

$$\text{supp } \psi_i \subset U_i. \quad (2.20)$$

In particular, any open interval  $(a, b) \subset \mathbb{R}$  and  $\Omega$  satisfy the conditions for  $\mathcal{M}$  of this statement. It will be used in Subsection 3.1.

### 3 Main results

**Theorem 1.** *There exist a weight  $W$  satisfying (1.4) and a non-zero function  $f \in C_0^\infty(\mathbb{R}^d)$ ,  $d \geq 2$ , such that*

$$P_W f \equiv 0 \text{ on } T\mathbb{S}^{d-1}, \quad (3.1)$$

where  $P_W$  is defined in (1.1). In addition,  $W$  is rotation invariant, i.e., satisfies (1.6), and  $f$  is spherically symmetric with  $\text{supp } f \subseteq \overline{B}$ . Moreover,

$$W \in C^\infty(\Omega \setminus \Omega(1)), \quad (3.2)$$

$$W \in C^\alpha(\mathbb{R}^d \times \mathbb{S}^{d-1}) \text{ for any } \alpha \in (0, \alpha_0), \alpha_0 = 1/16, \quad (3.3)$$

$$W \geq 1/2 \text{ on } \Omega \text{ and } W \equiv 1 \text{ on } \Omega([1, +\infty)), \quad (3.4)$$

$$W(x, \theta) \equiv 1 \text{ for } |x| \geq R > 1, \theta \in \mathbb{S}^{d-1}, \quad (3.5)$$

where  $\Omega$ ,  $\Omega(1)$ ,  $\Omega([1, +\infty))$  are defined by (2.1), (2.5),  $R$  is a constant.

The construction of  $W$  and  $f$  proving Theorem 1 is presented below in Subsections 3.1, 3.2. In addition, this construction consists of its version in dimension  $d = 2$  (see Subsection 3.1) and a subsequent extension of  $W$  and  $f$  to the case of  $d \geq 3$  (see Subsection 3.2).

Theorem 1 directly gives counterexamples to Injectivity 1 and Injectivity 3 of Introduction. Theorem 1 also implies the following counterexample to Injectivity 2 of Introduction:

**Corollary 1.** *For any  $N > 0$ ,  $\delta > 0$  and  $\alpha \in (0, 1/16)$  there are  $W_\delta$ ,  $f_\delta$  such that*

$$W_\delta \text{ satisfies (1.4), } d = 2, \quad (3.6)$$

$$W_\delta \in C^\alpha(\mathbb{R}^2 \times \mathbb{S}^1), \|W_\delta\|_{C^\alpha(\mathbb{R}^2 \times \mathbb{S}^1)} \leq N, \quad (3.7)$$

$$f_\delta \in C^\infty(\mathbb{R}^2), f_\delta \not\equiv 0, \text{supp } f_\delta \subseteq \overline{B}(0, \delta), d = 2, \quad (3.8)$$

$$P_{W_\delta} f_\delta \equiv 0 \text{ on } T\mathbb{S}^1. \quad (3.9)$$

The construction of  $W_\delta$ ,  $f_\delta$  proving Corollary 1 is presented in Subsection 4.1.

#### 3.1 Construction of $f$ and $W$ for $d = 2$

In dimension  $d = 2$ , the construction of  $f$  and  $W$  adopts and develops considerations of [Bom93] and [GN17]. In particular, we construct  $f$ , first, and then  $W$  (in this construction we use notations of Section 2 for  $d = 2$ ).

**Construction of  $f$ .** The function  $f$  is constructed as follows:

$$f = \sum_{k=1}^{\infty} \frac{f_k}{k!}, \quad (3.10)$$

$$f_k(x) = \tilde{f}_k(|x|) = \Phi(2^k(1 - |x|)) \cos(8^k|x|^2), x \in \mathbb{R}^2, k \in \mathbb{N}, \quad (3.11)$$

for arbitrary  $\Phi \in C^\infty(\mathbb{R})$  such that

$$\text{supp } \Phi = [4/5, 6/5], \quad (3.12)$$

$$0 < \Phi(t) \leq 1 \text{ for } t \in (4/5, 6/5), \quad (3.13)$$

$$\Phi(t) = 1, \text{ for } t \in [9/10, 11/10], \quad (3.14)$$

$$\begin{aligned} &\Phi \text{ monotonously increases on } [4/5, 9/10] \\ &\text{and monotonously decreases on } [11/10, 6/5]. \end{aligned} \quad (3.15)$$

Properties (3.12), (3.13) imply that functions  $\tilde{f}_k$  (and functions  $f_k$ ) in (3.11) have disjoint supports:

$$\begin{aligned} &\text{supp } \tilde{f}_i \cap \text{supp } \tilde{f}_j = \emptyset \text{ if } i \neq j, \\ &\text{supp } \tilde{f}_k = [1 - 2^{-k} \left(\frac{6}{5}\right), 1 - 2^{-k} \left(\frac{4}{5}\right)], i, j, k \in \mathbb{N}. \end{aligned} \quad (3.16)$$

This implies the convergence of series in (3.10) for every fixed  $x \in \mathbb{R}^2$ .

**Lemma 1.** *Let  $f$  be defined by (3.10)-(3.14). Then  $f$  is spherically symmetric,  $f \in C_0^\infty(\mathbb{R}^2)$  and  $\text{supp } f \subseteq \overline{B}$ . In addition, if  $\gamma \in T\mathbb{S}^1$ ,  $\gamma \cap B \neq \emptyset$ , then  $f|_\gamma \not\equiv 0$  and  $f|_\gamma$  has non-constant sign.*

Lemma 1 is similar to Lemma 1 of [GN17] and it is, actually, proved in Section 4.1 of [GN17].

**Construction of  $W$ .** In our example  $W$  is of the following form:

$$\begin{aligned} W(x, \theta) &= \phi_1(x) \left( \sum_{i=0}^N \xi_i(r(x, \theta)) W_i(x, \theta) \right) + \phi_2(x) \\ &= \phi_1(x) \left( \xi_0(r(x, \theta)) W_0(x, \theta) + \sum_{i=1}^N \xi_i(r(x, \theta)) W_i(x, \theta) \right) + \phi_2(x), \quad (x, \theta) \in \Omega, \end{aligned} \quad (3.17)$$

where

$$\begin{aligned} \phi_1, \phi_2 &\text{ is a } C^\infty\text{-smooth partition of unity on } \mathbb{R}^2 \text{ such that,} \\ \phi_1 &\equiv 0 \text{ for } |x| \geq R > 1, \quad \phi_1 \equiv 1 \text{ for } |x| \leq 1, \end{aligned} \quad (3.18)$$

$$\begin{aligned} \phi_2 &\equiv 0 \text{ for } |x| \leq 1, \\ \{\xi_i(s), s \in \mathbb{R}\}_{i=0}^N &\text{ is a } C^\infty\text{- smooth partition of unity on } \mathbb{R}, \end{aligned} \quad (3.19)$$

$$\xi_i(s) = \xi_i(-s), \quad s \in \mathbb{R}, \quad i = \overline{0, N}, \quad (3.20)$$

$$\begin{aligned} W_i(x, \theta) &\text{ are bounded, continuous, strictly positive and rotation invariant} \\ &\text{(according to (1.6)), (2.19) on the open vicinities of } \text{supp } \xi_i(r(x, \theta)), \quad i = \overline{0, N}, \text{ respectively.} \end{aligned} \quad (3.21)$$

From the result of Lemma 1 and from (3.18) it follows that

$$P_W f(x, \theta) = \xi_0(|x|) P_{W_0} f(x, \theta) + \sum_{i=1}^N \xi_i(|x|) P_{W_i} f(x, \theta), \quad (x, \theta) \in TS^1, \quad (3.22)$$

where  $W$  is given by (3.17).

From (3.17)-(3.21) it follows that  $W$  of (3.17) satisfies the conditions (1.4), (2.18), (2.19).

The weight  $W_0$  is constructed in next paragraph and has the following properties:

$$W_0 \text{ is bounded, continuous and rotation invariant on } \Omega(1/2, +\infty), \quad (3.23)$$

$$W_0 \in C^\infty(\Omega((1/2, 1) \cup (1, +\infty))) \text{ and } W_0 \in C^\alpha(\Omega(1/2, +\infty)) \text{ for } \alpha \in (0, 1/16), \quad (3.24)$$

there exists  $\delta_0 \in (1/2, 1)$  such that:

$$W_0(x, \theta) \geq 1/2 \text{ if } r(x, \theta) > \delta_0, \quad (3.25)$$

$$W_0(x, \theta) = 1 \quad \text{if } r(x, \theta) \geq 1,$$

$$P_{W_0} f(x, \theta) = 0 \text{ on } \Omega((1/2, +\infty)), \quad (3.26)$$

where  $P_{W_0}$  is defined according to (1.1) for  $W = W_0$ ,  $f$  is given by (3.10), (3.11).

In addition,

$$\text{supp } \xi_0 \subset (-\infty, -\delta_0) \cup (\delta_0, +\infty), \quad (3.27)$$

$$\xi_0(s) = 1 \text{ for } |s| \geq 1, \quad (3.28)$$

where  $\delta_0$  is the number of (3.25).

In particular, from (3.25), (3.27) it follows that

$$W_0(x, \theta) \xi_0(r(x, \theta)) > 0 \text{ if } \xi_0(r(x, \theta)) > 0. \quad (3.29)$$

In addition,

$$\xi_i(r(x, \theta)) W_i(x, \theta) \text{ are bounded, rotation invariant and } C^\infty \text{ on } \Omega, \quad (3.30)$$

$$W_i(x, \theta) \geq 1/2 \text{ if } \xi_i(r(x, \theta)) \neq 0, \quad (3.31)$$

$$P_{W_i} f(x, \theta) = 0 \text{ on } (x, \theta) \in TS^1, \text{ such that } \xi_i(r(x, \theta)) \neq 0, \quad (3.32)$$

$$i = \overline{1, N}, \quad (x, \theta) \in \Omega.$$

Weights  $W_1, \dots, W_N$  of (3.17) and  $\{\xi_i\}_{i=0}^N$  are constructed in Subsection 3.1.

Theorem 1 for  $d = 2$  follows from Lemma 1 and formulas (3.17)-(3.26), (3.29)-(3.32).

We point out that the construction of  $W_0$  of (3.17) is substantially different from the construction of  $W_1, \dots, W_N$ . The weight  $W_0$  is defined for the rays  $\gamma \in TS^1$  which can be close to the boundary  $\partial B$  of  $B$  which results in restrictions on global smoothness of  $W_0$ .

**Construction of  $W_0$ .** Let

$$\{\psi_k\}_{k=1}^{\infty} \text{ be a } C^{\infty} \text{ partition of unity on } (1/2, 1), \text{ such that } \text{supp } \psi_k \subset (1 - 2^{-k+1}, 1 - 2^{-k-1}), k \in \mathbb{N}, \quad (3.33)$$

$$\text{first derivatives } \psi'_k \text{ satisfy the bounds: } \sup |\psi'_k| \leq C2^k, \quad (3.34)$$

where  $C$  is a positive constant. Actually, functions  $\{\psi_k\}_{k=1}^{\infty}$  satisfying (3.33), (3.34) were used in considerations of [Bom93].

Note that

$$1 - 2^{-(k-2)-1} < 1 - 2^{-k}(6/5), k \geq 3. \quad (3.35)$$

Therefore,

$$\forall s_0, t_0 \in \mathbb{R} : s_0 \in \text{supp } \psi_{k-2}, t_0 \in \text{supp } \Phi(2^k(1-t)) \Rightarrow s_0 < t_0, k \geq 3. \quad (3.36)$$

Weight  $W_0$  is defined by the following formulas

$$W_0(x, \theta) = \begin{cases} 1 - G(x, \theta) \sum_{k=3}^{\infty} k! f_k(x) \frac{\psi_{k-2}(r(x, \theta))}{H_k(x, \theta)}, & 1/2 < r(x, \theta) < 1, \\ 1, & r(x, \theta) \geq 1 \end{cases}, \quad (3.37)$$

$$G(x, \theta) = \int_{\gamma(x, \theta)} f(y) dy, H_k(x, \theta) = \int_{\gamma(x, \theta)} f_k^2(y) dy, x \in \mathbb{R}^2, \theta \in \mathbb{S}^1, \quad (3.38)$$

where  $f, f_k$  are defined in (3.10), (3.11), respectively, rays  $\gamma(x, \theta)$  are given by (2.13).

Formula (3.37) implies that  $W_0$  is defined on  $\Omega_0(1/2) \subset \Omega$ .

Due to (3.11)-(3.14), (3.33), (3.36), in (3.38) we have that

$$H_k(x, \theta) \neq 0 \text{ if } \psi_{k-2}(r(x, \theta)) \neq 0, (x, \theta) \in \Omega, \quad (3.39)$$

$$\frac{\psi_{k-2}(r(x, \theta))}{H_k(x, \theta)} \in C^{\infty}(\Omega(1/2, 1)), \quad (3.40)$$

where  $r(x, \theta)$  is defined in (2.2),  $\Omega, \Omega(\cdot)$  are defined in (2.1), (2.5),  $d = 2$ .

Also, for any fixed  $(x, \theta) \in \Omega, 1/2 < r(x, \theta)$ , the series in the right hand-side of (3.37) has only a finite number of non-zero terms (in fact, no more than two) and, hence, the weight  $W_0$  is well-defined.

By the spherical symmetry of  $f$ , functions  $G, H_k$  in (3.37) are of the type (1.6) (and (2.18)). Therefore,  $W_0$  is rotation invariant (in the sense of (1.6) and (2.18)).

Actually, formula (3.26) follows from (3.10), (3.11), (3.37), (3.38) (see Subsection 5.2 for details).

Using the construction of  $W_0$  and the assumption that  $r(x, \theta) > 1/2$  one can see that  $W_0$  is  $C^{\infty}$  on its domain of definition, possibly, except points with  $r(x, \theta) = 1$ .

Note also that due to (3.10), (3.11), the functions  $f_k, G, H_k$ , used in (3.37), (3.38) can be considered as functions of one-dimensional arguments.

Formulas (3.23)-(3.25) are proved in Subsection 5.1.

## Construction of $W_1, \dots, W_N$ and $\xi_0, \dots, \xi_N$

**Lemma 2.** *Let  $f \in C_0^{\infty}(\mathbb{R}^2)$  be spherically symmetric,  $(x_0, \theta_0) \in T\mathbb{S}^1, f|_{\gamma(x_0, \theta_0)} \neq 0$  and  $f|_{\gamma(x_0, \theta_0)}$  changes the sign. Then there exist  $\varepsilon_0 > 0$  and weight  $W_{(x_0, \theta_0), \varepsilon_0}$  such that*

$$P_{W_{(x_0, \theta_0), \varepsilon_0}} f = 0 \text{ on } \Omega(\mathcal{J}_{r(x_0, \theta_0), \varepsilon_0}), \quad (3.41)$$

$$W_{(x_0, \theta_0), \varepsilon_0} \text{ is bounded, infinitely smooth,} \quad (3.42)$$

$$\text{strictly positive and rotation invariant on } \Omega(\mathcal{J}_{r(x_0, \theta_0), \varepsilon_0}),$$

where  $\Omega(\mathcal{J}_{r, \varepsilon_0}), \mathcal{J}_{r, \varepsilon_0}$  are defined in (2.5) and (2.9), respectively.

Lemma 2 is proved in Section 6.

Let  $f$  be the function of (3.10), (3.11). Then, using Lemmas 1, 2 one can see that

$$\forall \delta \in (0, 1) \text{ there exist } \{J_i = \mathcal{J}_{r_i, \varepsilon_i}, W_i = W_{(x_i, \theta_i), \varepsilon_i}\}_{i=1}^N$$

$$\text{such that } J_i, i = \overline{1, N}, \text{ is an open cover of } [0, \delta] \quad (3.43)$$

$$\text{and } W_i \text{ satisfy (3.41) and (3.42) on } \Omega(J_i), \text{ respectively.}$$

Actually, we consider (3.43) for the case of  $\delta = \delta_0$  of (3.25).

Note that in this case  $\{\Omega(J_i)\}_{i=1}^N$  for  $J_i$  of (3.43) is the open cover of  $\Omega_1(\delta_0)$ .

To the set  $\Omega_0(\delta_0)$  we associate the open set

$$J_0 = (-\infty, \delta_0) \cup (\delta_0, +\infty) \subset \mathbb{R}. \quad (3.44)$$

Therefore, the collection of intervals  $\{\pm J_i, i = \overline{0, N}\}$  is an open cover of  $\mathbb{R}$ , where  $-J_i$  is the symmetrical reflection of  $J_i$  with respect to  $\{0\} \in \mathbb{R}$ .

We construct the partition of unity  $\{\xi_i\}_{i=0}^N$  as follows:

$$\xi_i(s) = \xi_i(|s|) = \frac{1}{2}(\tilde{\xi}_i(s) + \tilde{\xi}_i(-s)), \quad s \in \mathbb{R}, \quad (3.45)$$

$$\text{supp } \xi_i \subset J_i \cup (-J_i), \quad i = \overline{0, N}, \quad (3.46)$$

where  $\{\tilde{\xi}_i\}_{i=0}^N$  is a partition of unity for the open cover  $\{J_i \cup (-J_i)\}_{i=0}^N$  (see Section 2, Partition of unity, for  $U_i = J_i$ ).

Properties (3.27), (3.46) follow from (2.20) for  $\{\tilde{\xi}_i\}_{i=0}^N$  with  $U_i = J_i \cup (-J_i)$ , the symmetry of  $J_i \cup (-J_i)$ ,  $i = \overline{1, N}$ , choice of  $J_0$  in (3.44) and from (3.45).

In turn, (3.28) follows from (3.44) and the construction of  $J_i$ ,  $i = \overline{1, N}$ , from (3.43) (see the proof of Lemma 2 and properties (3.43) in Section 6 for details).

Properties (3.30)-(3.32) follow from (3.43) for  $\delta = \delta_0$  and from (3.44)-(3.46).

This completes the description of  $W_1, \dots, W_N$  and  $\{\xi_i\}_{i=0}^N$ .

### 3.2 Construction of $W$ and $f$ for $d \geq 3$

Consider  $f$  and  $W$  of Theorem 1, for  $d = 2$ , constructed in Subsection 3.1. For these  $f$  and  $W$  consider  $\tilde{f}$  and  $\tilde{U}$  such that

$$f(x) = \tilde{f}(|x|), \quad W(x, \theta) = \tilde{U}(|x|, |x\theta|), \quad x \in \mathbb{R}^2, \quad \theta \in \mathbb{S}^1. \quad (3.47)$$

**Proposition 1.** *Let  $W$  and  $f$ , for  $d \geq 3$ , be defined as*

$$W(x, \theta) = \tilde{U}(|x|, |x\theta|), \quad (x, \theta) \in \mathbb{R}^d \times \mathbb{S}^{d-1}, \quad (3.48)$$

$$f(x) = \tilde{f}(|x|), \quad x \in \mathbb{R}^d, \quad (3.49)$$

where  $\tilde{U}, \tilde{f}$  are the functions of (3.47). Then

$$P_W f \equiv 0 \text{ on } T\mathbb{S}^{d-1}. \quad (3.50)$$

In addition, weight  $W$  satisfies properties (3.2)-(3.5),  $f$  is spherically symmetric infinitely smooth and compactly supported on  $\mathbb{R}^d$ ,  $f \neq 0$ .

Proposition 1 is proved in Subsection 4.2.

This completes the proof of Theorem 1.

## 4 Proofs of Corollary 1 and Proposition 1

### 4.1 Proof of Corollary 1

Let

$$X_r = \{x_1 e_1 + x_2 e_2 + r e_3 : (x_1, x_2) \in \mathbb{R}^2\}, \quad 0 \leq r < 1, \quad (4.1)$$

$$S = X_0 \cap \mathbb{S}^2 = \{(\cos \phi, \sin \phi, 0) \in \mathbb{R}^3 : \phi \in [0, 2\pi)\} \simeq \mathbb{S}^1. \quad (4.2)$$

where  $(e_1, e_2, e_3)$  is the standard orthonormal basis in  $\mathbb{R}^3$ .

Without loss of generality we assume that  $0 < \delta < 1$ . Choosing  $r$  so that  $\sqrt{1 - \delta^2} \leq r < 1$ , we have that the intersection of the three dimensional ball  $B(0, 1)$  with  $X_r$  is the two-dimensional disk  $B(0, \delta')$ ,  $\delta' \leq \delta$  (with respect to the coordinates  $(x_1, x_2)$  induced by basis  $(e_1, e_2)$  on  $X_r$ ).

We define  $W_\delta$  on  $\mathbb{R}^2 \times \mathbb{S}^1$  and  $f_\delta$  on  $\mathbb{R}^2$  as follows:

$$W_\delta := \frac{N}{\|W\|_{C^\alpha(\mathbb{R}^3 \times \mathbb{S}^2)}} W|_{X_r \times S}, \quad (4.3)$$

$$f_\delta := f|_{X_r}, \quad (4.4)$$

$$\text{for } r = \sqrt{1 - \delta^2},$$

where  $W$  and  $f$  are the functions of Theorem 1 for  $d = 3$ .

Due to (3.2), (3.3), (3.4), (4.3) we have that

$$W_\delta \geq \frac{N}{2\|W\|_{C^\alpha(\mathbb{R}^2 \times \mathbb{S}^1)}}, \quad \|W_\delta\|_{C^\alpha(\mathbb{R}^2 \times \mathbb{S}^1)} \leq N. \quad (4.5)$$

Properties (4.5) imply (3.6), (3.7).

In view of Lemma 1 for the function  $f$  of Theorem 1, we have that  $f_\delta$  is spherically symmetric,  $f_\delta \in C_0^\infty(B(0, \delta'))$ ,  $f_\delta \neq 0$ .

Using (3.1), (4.3), (4.4) one can see that (3.9) holds.

This completes the proof of Corollary 1.



## 4.2 Proof of Proposition 1

Let

$$I(r) = \int_{\gamma_r} \tilde{U}(|y|, r) \tilde{f}(|y|) dy, \quad r \geq 0, \quad \gamma_r = \gamma(re_2, e_1), \quad (4.6)$$

where  $\gamma(x, \theta)$  is defined by (1.3),  $(e_1, \dots, e_d)$  is the standard basis in  $\mathbb{R}^d$ .

Due to formula (3.1) of Theorem 1 for  $d = 2$  and formulas (3.47), (4.6) we have that

$$I(r) = P_W f(re_2, e_1) = 0 \text{ for } r \geq 0. \quad (4.7)$$

Next, using (1.1), (3.47), (4.7) we have also that

$$P_W f(x, \theta) = \int_{\gamma(x, \theta)} \tilde{U}(|y|, |y - (y\theta)\theta|) \tilde{f}(|y|) dy = I(|x|) = 0 \text{ for } (x, \theta) \in T\mathbb{S}^{d-1}, \quad (4.8)$$

where  $\gamma(x, \theta)$  is defined in (1.3).

Formula (4.8) implies (3.50). Properties of  $W$  and  $f$  mentioned in Proposition 1 follow from properties (3.2)-(3.5) of  $W$  and of  $f$  of Theorem 1 for  $d = 2$ .

This completes the proof of Proposition 1.

## 5 Proofs of formulas (3.23)-(3.26)

### 5.1 Proof of formulas (3.23)-(3.25)

**Lemma 3.** *Let  $W_0$  be defined by (3.37), (3.38). Then  $W_0$  admits the following representation:*

$$W_0(x, \theta) = U_0(x\theta, |x - (x\theta)\theta|), \quad (x, \theta) \in \Omega((1/2, +\infty)), \quad (5.1)$$

$$U_0(s, r) = \begin{cases} 1 - \tilde{G}(r) \sum_{k=3}^{\infty} k! \tilde{f}_k((s^2 + r^2)^{1/2}) \frac{\psi_{k-2}(r)}{\tilde{H}_k(r)}, & 1/2 < r < 1, \\ 1, & r \geq 1 \end{cases}, \quad (5.2)$$

$$\tilde{G}(r) \stackrel{def}{=} \int_{\gamma_r} \tilde{f}(|y|) dy, \quad \tilde{H}_k(r) \stackrel{def}{=} \int_{\gamma_r} \tilde{f}_k^2(|y|) dy, \quad \tilde{f} = \sum_{k=1}^{\infty} \frac{\tilde{f}_k}{k!}, \quad (5.3)$$

$$s \in \mathbb{R}, \quad x \in \mathbb{R}^2, \quad \gamma_r \text{ is an arbitrary ray in } T(r), \quad r > 1/2,$$

where  $T(r)$  is defined by (2.8),  $d = 2$ ,  $\tilde{f}_k$  are defined by (3.11). In addition:

$$U_0 \text{ is infinitely smooth on } \mathbb{R} \times \{(1/2, 1) \cup (1, +\infty)\}, \quad (5.4)$$

$$U_0(s, r) \rightarrow 1 \text{ as } r \rightarrow 1 \text{ (uniformly in } s \in \mathbb{R}), \quad (5.5)$$

$$U_0(s, r) = 1 \text{ if } s^2 + r^2 \geq 1, \quad (5.6)$$

$$|1 - U_0(s, r)| \leq C_0(1 - r)^{1/2} \log_2^4 \left( \frac{1}{1 - r} \right), \quad (5.7)$$

$$\text{for } s \in \mathbb{R}, \quad 1/2 < r < 1,$$

$$|U_0(s, r) - U_0(s', r')| \leq C_1 |s - s'|^{1/\alpha} + C_1 |r - r'|^{1/\alpha}, \quad (5.8)$$

$$\text{for } \alpha \in (0, 1/16), \quad s, s' \in \mathbb{R}, \quad r, r' > 1/2,$$

where  $C_0, C_1$  are positive constants depending on  $\Phi$  of (3.12)-(3.14).

Lemma 3 is proved Section 7.

Lemma 3 implies (3.23)-(3.25) as follows.

The continuity and rotation invariancy of  $W_0$  in (3.23) follow from (1.6), (1.7), (5.1), (5.8).

Due to (3.33), (5.1), (5.2), (5.3) we have also that

$$U_0 \text{ admits a continuous extension to } \mathbb{R} \times [1/2, +\infty). \quad (5.9)$$

Properties (5.6), (5.9) imply the boundedness of  $W_0$  on  $\Omega_0(1/2)$ , where  $\Omega_0(\cdot)$  is defined in (2.3),  $d = 2$ . This completes the proof of (3.23).

Formula (3.24) follows from (5.1), (5.4), (5.8) and from the fact that  $x\theta, |x - (x\theta)\theta|$  are infinitely smooth functions on  $\Omega_0(1/2)$  and are Lipschitz in  $(x, \theta)$  for  $x \in \overline{B}(0, R)$ ,  $R > 1$ .

Formula (3.25) follows from (3.23), (5.1), (5.2), (5.5), (5.6).

This completes the proof of (3.23)-(3.25).

## 5.2 Proof of formula (3.26)

From (1.1), (3.10)-(3.13), (3.33), (3.37), (3.38) it follows that:

$$P_{W_0} f(x, \theta) = \int_{\gamma(x, \theta)} f(y) dy - G(x, \theta) \sum_{k=3}^{\infty} k! \psi_{k-2}(r(x, \theta)) \frac{\int_{\gamma(x, \theta)} f(y) f_k(y) dy}{H_k(x, \theta)} \quad (5.10)$$

$$= \int_{\gamma(x, \theta)} f(y) dy - \int_{\gamma(x, \theta)} f(y) dy \sum_{k=3}^{\infty} \psi_{k-2}(r(x, \theta)) \frac{\int_{\gamma(x, \theta)} f_k^2(y) dy}{\int_{\gamma(x, \theta)} f_k^2(y) dy} \quad (5.11)$$

$$= \int_{\gamma(x, \theta)} f(y) dy - \int_{\gamma(x, \theta)} f(y) dy \sum_{k=3}^{\infty} \psi_{k-2}(r(x, \theta)) = 0 \text{ for } (x, \theta) \in \Omega_0(1/2), \quad (5.12)$$

where  $\gamma(x, \theta)$  is defined in (1.3),  $\Omega_0(\cdot)$  is defined in (2.3),  $d = 2$ .

Formula (3.26) is proved.

## 6 Proof of Lemma 2

By  $u \in \mathbb{R}$  we denote the coordinates on a fixed ray  $\gamma(x, \theta)$ ,  $(x, \theta) \in \Omega$ ,  $d = 2$ , taking into account the orientation, where  $u = 0$  at the point  $x - (x\theta)\theta \in \gamma(x, \theta)$ ; see notation (2.13).

Using Lemma 1, one can see that

$$f|_{\gamma(x, \theta)} \in C_0^\infty(\mathbb{R}), f|_{\gamma(x, \theta)}(u) = f|_{\gamma(x, \theta)}(|u|), u \in \mathbb{R}. \quad (6.1)$$

Using (6.1) and the assumption that  $f|_{\gamma(x_0, \theta_0)}(u)$  changes the sign, one can see that there exists  $\psi_{(x_0, \theta_0)}$  such that

$$\psi_{(x_0, \theta_0)} \in C_0^\infty(\mathbb{R}), \psi_{(x_0, \theta_0)} \geq 0, \psi_{(x_0, \theta_0)}(u) = \psi_{(x_0, \theta_0)}(|u|), u \in \mathbb{R}, \quad (6.2)$$

$$\int_{\gamma(x_0, \theta_0)} f \psi_{(x_0, \theta_0)} d\sigma \neq 0, \quad (6.3)$$

and if

$$\int_{\gamma(x_0, \theta_0)} f d\sigma \neq 0 \quad (6.4)$$

then also

$$\text{sgn}\left(\int_{\gamma(x_0, \theta_0)} f d\sigma\right) \text{sgn}\left(\int_{\gamma(x_0, \theta_0)} f \psi_{(x_0, \theta_0)} d\sigma\right) = -1, \quad (6.5)$$

where  $d\sigma = du$  (i.e.,  $\sigma$  is the standard Euclidean measure on  $\gamma(x, \theta)$ ).

Let

$$W_{(x_0, \theta_0)}(x, \theta) = 1 - \psi_{(x_0, \theta_0)}(x\theta) \frac{\int_{\gamma(x, \theta)} f d\sigma}{\int_{\gamma(x, \theta)} f \psi_{(x_0, \theta_0)} d\sigma}, x \in \mathbb{R}^2, \theta \in \mathbb{S}^1, \quad (6.6)$$

where  $d\sigma = du$ , where  $u$  is the coordinate on  $\gamma(x, \theta)$ .

Lemma 1 and property (6.2) imply that

$$\int_{\gamma(x, \theta)} f d\sigma \text{ and } \int_{\gamma(x, \theta)} f \psi_{(x_0, \theta_0)} d\sigma \text{ depend only on } r(x, \theta), \text{ where } (x, \theta) \in \Omega, \quad (6.7)$$

where  $r(x, \theta)$  is defined in (2.2),  $\Omega$  is defined in (2.1),  $d = 2$ .

From (6.2), (6.6), (6.7) it follows that  $W_{(x_0, \theta_0)}$  is rotation-invariant in the sense (1.6).

Formulas (6.3), (6.6), (6.7), properties of  $f$  of Lemma 1 and properties of  $\psi_{(x_0, \theta_0)}$  of (6.2) imply that

$$\exists \varepsilon_1 > 0 : \int_{\gamma(x, \theta)} f \psi_{(x_0, \theta_0)} d\sigma \neq 0 \text{ for } (x, \theta) \in \Omega(\mathcal{J}_{r(x_0, \theta_0), \varepsilon_1}), \quad (6.8)$$

where sets  $\Omega(\mathcal{J}_{s, \varepsilon})$ ,  $\mathcal{J}_{s, \varepsilon}$  are defined in (2.5), (2.9), respectively.

In addition, using properties of  $f$  of Lemma 1 and also using (3.10), (3.16), (6.2), (6.6), (6.8), one can see that

$$W_{(x_0, \theta_0)} \in C^\infty(\Omega(\mathcal{J}_{r(x_0, \theta_0), \varepsilon_1})). \quad (6.9)$$

In addition, from (6.1)-(6.7) it follows that

$$\begin{aligned} \text{if } r(x, \theta) = r(x_0, \theta_0) \text{ then } W_{(x_0, \theta_0)}(x, \theta) &= 1 - \psi_{(x_0, \theta_0)}(x\theta) \frac{\int_{\gamma(x_0, \theta_0)} f d\sigma}{\int_{\gamma(x_0, \theta_0)} f \psi_{(x_0, \theta_0)} d\sigma} \\ &= 1 - \psi_{(x_0, \theta_0)}(x\theta) \frac{\int_{\gamma(x_0, \theta_0)} f d\sigma}{\int_{\gamma(x_0, \theta_0)} f \psi_{(x_0, \theta_0)} d\sigma} \geq 1, \end{aligned} \quad (6.10)$$

where  $r(x, \theta)$  is defined in (2.2),  $d = 2$ .

From properties of  $f$  of Lemma 1, properties of  $\psi_{(x_0, \theta_0)}$  of (6.2) and from formulas (6.6), (6.8), (6.9), (6.10) it follows that

$$\exists \varepsilon_0 > 0 (\varepsilon_0 < \varepsilon_1) : W_{(x_0, \theta_0)}(x, \theta) \geq 1/2 \text{ for } (x, \theta) \in \Omega(\mathcal{J}_{r(x_0, \theta_0), \varepsilon_0}). \quad (6.11)$$

Let

$$W_{(x_0, \theta_0), \varepsilon_0} := W_{(x_0, \theta_0)} \text{ for } (x, \theta) \in \Omega(\mathcal{J}_{r(x_0, \theta_0), \varepsilon_0}), \quad (6.12)$$

where  $W_{(x_0, \theta_0)}$  is defined in (6.6).

Properties (6.7), (6.9), (6.11) imply (3.42) for  $W_{(x_0, \theta_0), \varepsilon_0}$  of (6.12).

Using (1.1), (6.6), (6.8), (6.12) one can see that

$$\begin{aligned} P_{W_{(x_0, \theta_0), \varepsilon_0}} f(x, \theta) &= \int_{\gamma(x, \theta)} W_{(x_0, \theta_0)}(\cdot, \theta) f d\sigma \\ &= \int_{\gamma(x, \theta)} f d\sigma - \frac{\int_{\gamma(x, \theta)} f d\sigma}{\int_{\gamma(x, \theta)} f \psi_{(x_0, \theta_0)} d\sigma} \int_{\gamma(x, \theta)} f \psi_{(x_0, \theta_0)} d\sigma = 0 \text{ for } (x, \theta) \in \Omega(\mathcal{J}_{r(x_0, \theta_0), \varepsilon_0}), \end{aligned} \quad (6.13)$$

where  $\Omega(\cdot)$  is defined in (2.5),  $d = 2$ ,  $\mathcal{J}_{r, \varepsilon}$  is defined in (2.9). Formula (3.41) follows from (6.13).

Lemma 2 is proved.

## 7 Proof of Lemma 3

**Proof of (5.1)-(5.3).** Using (2.2), (3.10), (3.11), (3.38), (5.3) we obtain

$$G(x, \theta) = \tilde{G}(r(x, \theta)) = \int_{\gamma(x, \theta)} f(x) dx, \quad (7.1)$$

$$H_k(x, \theta) = \tilde{H}_k(r(x, \theta)) = \int_{\gamma(x, \theta)} f_k^2(x) dx, \quad (7.2)$$

$$\tilde{f}_k(|x|) = \tilde{f}_k((|x\theta|^2 + |x - (x\theta)\theta|^2)^{1/2}), (x, \theta) \in \Omega_0(1/2), \quad (7.3)$$

where  $\Omega_0(\cdot)$  is defined in (2.3),  $d = 2$ ,  $\gamma(x, \theta)$  is defined as in (2.13).

Formulas (3.37), (3.38), (7.1)-(7.3) imply (5.1)-(5.3).

**Proof of (5.4).** Let

$$\Lambda_k = (1 - 2^{-k+3}, 1 - 2^{-k+1}), k \in \mathbb{N}, k \geq 4. \quad (7.4)$$

From (3.33) it follows that, for  $k \geq 4$ :

$$\text{supp } \psi_{k-1} \subset (1 - 2^{-k+2}, 1 - 2^{-k}), \quad (7.5)$$

$$\text{supp } \psi_{k-2} \subset (1 - 2^{-k+3}, 1 - 2^{-k+1}) = \Lambda_k, \quad (7.6)$$

$$\text{supp } \psi_{k-3} \subset (1 - 2^{-k+4}, 1 - 2^{-k+2}). \quad (7.7)$$

Due to (5.2), (5.3), (7.5)-(7.7), we have the following formula for  $U_0$ :

$$\begin{aligned} U_0(s, r) &= 1 - \tilde{G}(r) \left( (k-1)! \tilde{f}_{k-1}((s^2 + r^2)^{1/2}) \frac{\psi_{k-3}(r)}{\tilde{H}_{k-1}(r)} \right. \\ &\quad + k! \tilde{f}_k((s^2 + r^2)^{1/2}) \frac{\psi_{k-2}(r)}{\tilde{H}_k(r)} \\ &\quad \left. + (k+1)! \tilde{f}_{k+1}((s^2 + r^2)^{1/2}) \frac{\psi_{k-1}(r)}{\tilde{H}_{k+1}(r)} \right) \text{ for } r \in \Lambda_k, s \in \mathbb{R}, k \geq 4. \end{aligned} \quad (7.8)$$

From (5.3), (7.8) it follows that

$$\begin{aligned} \frac{\partial^n U_0}{\partial s^n}(s, r) = & -\tilde{G}(r) \left( (k-1)! \frac{\partial^n \tilde{f}_{k-1}((s^2+r^2)^{1/2})}{\partial s^n} \frac{\psi_{k-3}(r)}{\tilde{H}_{k-1}(r)} \right. \\ & + k! \frac{\partial^n \tilde{f}_k((s^2+r^2)^{1/2})}{\partial s^n} \frac{\psi_{k-2}(r)}{\tilde{H}_k(r)} \\ & \left. + (k+1)! \frac{\partial^n \tilde{f}_{k+1}((s^2+r^2)^{1/2})}{\partial s^n} \frac{\psi_{k-1}(r)}{\tilde{H}_{k+1}(r)} \right), \end{aligned} \quad (7.9)$$

$$\frac{\partial^n \tilde{G}}{\partial r^n}(r) = \int_{-\infty}^{+\infty} \frac{\partial^n}{\partial r^n} \tilde{f}((s^2+r^2)^{1/2}) ds, \quad \frac{\partial^n \tilde{H}_m}{\partial r^n}(r) = \int_{-\infty}^{+\infty} \frac{\partial^n}{\partial r^n} \tilde{f}_m^2((s^2+r^2)^{1/2}) ds, \quad (7.10)$$

$$r \in \Lambda_k, s \in \mathbb{R}, m \geq 1, n \geq 0, k \geq 4,$$

where  $\tilde{G}, \tilde{H}_m$  are defined in (5.3).

Using Lemma 1 and formulas (3.10), (3.11), (3.33)-(3.40), (5.3) one can see that:

$$\tilde{f}, \tilde{f}_{m-2}, \tilde{G}, \tilde{H}_m \text{ belong to } C_0^\infty(\mathbb{R}), \frac{\psi_{m-2}}{\tilde{H}_m} \text{ belongs to } C_0^\infty((1/2, 1)) \text{ for any } m \geq 3. \quad (7.11)$$

From (7.9)-(7.11) it follows that  $U_0(s, r)$  has continuous partial derivatives of all orders with respect to  $r \in \Lambda_k, s \in \mathbb{R}$ . It implies that  $U_0 \in C^\infty(\mathbb{R} \times \Lambda_k)$ . From the fact that  $\Lambda_k, k \geq 4$ , is an open cover of  $(1/2, 1)$  and from definition (5.2) of  $U_0$ , it follows that  $U_0 \in C^\infty(\mathbb{R} \times \{(1/2, 1) \cup (1, +\infty)\})$ .

This completes the proof of (5.4).

**Proof of (5.6).** From (3.11)-(3.14) it follows that

$$\tilde{f}_k(|x|) = 0 \text{ if } |x| \geq 1 \text{ for } k \in \mathbb{N}. \quad (7.12)$$

Formula  $|x|^2 = |x\theta|^2 + |x - (x\theta)\theta|^2$ ,  $x \in \mathbb{R}^2, \theta \in \mathbb{S}^1$ , and formulas (5.2), (7.12) imply (5.6).

**Proofs of (5.7)-(5.8).**

**Lemma 4.** *There are positive constants  $c, k_1$  depending on  $\Phi$  of (3.12)-(3.14), such that*

(i) *for all  $k \in \mathbb{N}$  the following estimates hold:*

$$|\tilde{f}_k| \leq 1, \quad (7.13)$$

$$|\tilde{f}'_k| \leq c8^k, \quad (7.14)$$

where  $\tilde{f}'_k$  denotes the derivative of  $\tilde{f}_k$  defined in (5.3).

(ii) *for  $k \geq k_1$  and  $1/2 < r \leq 1$  the following estimates hold:*

$$\left| \frac{\psi_{k-2}(r)}{\tilde{H}_k(r)} \right| \leq c2^k, \quad (7.15)$$

$$\left| \frac{d}{dr} \left( \frac{\psi_{k-2}(r)}{\tilde{H}_k(r)} \right) \right| \leq c2^{5k}, \quad (7.16)$$

where  $\psi_k$  are defined in (3.33),  $\tilde{H}_k$  is defined in (5.3).

(iii) *for  $k \geq 3$  and  $r \geq 1 - 2^{-k}$  the following estimates hold:*

$$|\tilde{G}(r)| \leq c \frac{(2\sqrt{2})^{-k}}{k!}, \quad (7.17)$$

$$\left| \frac{d\tilde{G}}{dr}(r) \right| \leq c \frac{8^k}{k!}, \quad (7.18)$$

where  $\tilde{G}$  is defined in (5.3).

**Lemma 5.** *Let  $U_0$  be defined by (5.2)-(5.3). Then the following estimates are valid:*

$$\left| \frac{\partial U_0}{\partial s}(s, r) \right| \leq \frac{C}{(1-r)^3}, \quad \left| \frac{\partial U_0}{\partial r}(s, r) \right| \leq \frac{C}{(1-r)^5} \text{ for } s \in \mathbb{R}, r \in (1/2, 1), \quad (7.19)$$

where  $C$  is a constant depending only on  $\Phi$  of (3.12)-(3.14).

Lemmas 4, 5 are proved in Subsections 8.1, 8.2, respectively.

**Proof of (5.7).** From (7.15), (7.17) it follows that

$$|\tilde{G}(r)| \leq c(2\sqrt{2})^{-k+3}/(k-3)!, \quad (7.20)$$

$$\left| \frac{\psi_{k-2}(r)}{\tilde{H}_k(r)} \right| \leq c2^k, \quad (7.21)$$

for  $r \in \Lambda_k$ ,  $k \geq \max(4, k_1)$ ,

where  $\Lambda_k$  is defined in (7.4).

Properties (7.5)-(7.7) and estimate (7.15) imply that

$$\begin{cases} \psi_{k-1}(r) = 0, \\ \left| \frac{\psi_{k-3}(r)}{\tilde{H}_{k-1}(r)} \right| \leq c2^{k-1} \end{cases} \quad \text{if } r \in (1 - 2^{-k+3}, 1 - 2^{-k+2}), \quad (7.22)$$

$$\begin{cases} \psi_{k-2}(r) = 0, \\ \left| \frac{\psi_{k-1}(r)}{\tilde{H}_{k+1}(r)} \right| \leq c2^{k+1} \end{cases} \quad \text{if } r \in (1 - 2^{-k+2}, 1 - 2^{-k+1}), \quad (7.23)$$

$$\begin{cases} \psi_{k-1}(r) = 0, \\ \psi_{k-3}(r) = 0 \end{cases} \quad \text{if } r = 1 - 2^{-k+2}, \quad (7.24)$$

for  $k \geq \max(4, k_1)$ .

Note that the assumption that  $r \in \Lambda_k$  is splitted into the assumptions on  $r$  of (7.22), (7.23), (7.24).

Using formulas (7.8), (7.20)-(7.24), we obtain the following estimates:

$$\begin{aligned} |1 - U_0(s, r)| &= |\tilde{G}(r)| \left| (k-1)! \tilde{f}_{k-1}((s^2 + r^2)^{1/2}) \frac{\psi_{k-3}(r)}{\tilde{H}_{k-1}(r)} + k! \tilde{f}_k((s^2 + r^2)^{1/2}) \frac{\psi_{k-2}(r)}{\tilde{H}_k(r)} \right| \\ &\leq c(2\sqrt{2})^{-k+3} (c(k-2)(k-1)2^{k-1} + c(k-2)(k-1)k2^k) \\ &\leq 2^5 \sqrt{2} c^2 2^{-k/2} k^3 \quad \text{if } r \in (1 - 2^{-k+3}, 1 - 2^{-k+2}), \end{aligned} \quad (7.25)$$

$$\begin{aligned} |1 - U_0(s, r)| &= |\tilde{G}(r)| \left| k! \tilde{f}_k((s^2 + r^2)^{1/2}) \frac{\psi_{k-2}(r)}{\tilde{H}_k(r)} + (k+1)! \tilde{f}_{k+1}((s^2 + r^2)^{1/2}) \frac{\psi_{k-1}(r)}{\tilde{H}_{k+1}(r)} \right| \\ &\leq c(2\sqrt{2})^{-k+3} (c2^k(k-2)(k-1)k + c2^{k+1}(k-2)(k-1)k(k+1)) \\ &\leq 2^{10} \sqrt{2} c^2 2^{-k/2} k^4 \quad \text{if } r \in (1 - 2^{-k+2}, 1 - 2^{-k+1}), \end{aligned} \quad (7.26)$$

$$\begin{aligned} |1 - U_0(s, r)| &= |\tilde{G}(r)| \left| k! \tilde{f}_k((s^2 + r^2)^{1/2}) \frac{\psi_{k-2}(r)}{\tilde{H}_k(r)} \right| \\ &\leq 2^4 \sqrt{2} c^2 2^{-k/2} k^3 \quad \text{if } r = 1 - 2^{-k+2}, \end{aligned} \quad (7.27)$$

for  $s \in \mathbb{R}$ ,  $k \geq \max(4, k_1)$ . Estimates (7.25)-(7.27) imply that

$$|1 - U_0(s, r)| \leq C 2^{-k/2} k^4, \quad r \in \Lambda_k, \quad s \in \mathbb{R}, \quad k \geq \max(4, k_1), \quad (7.28)$$

where  $C$  is a positive constant depending on  $c$  of Lemma 4.

In addition, for  $r \in \Lambda_k$  we have that  $2^{-k+1} < (1-r) < 2^{-k+3}$ , which together with (7.28) imply (5.7).

This completes the proof of (5.7).

**Proof of (5.8).** We consider the following cases of  $s, s', r, r'$  in (5.8):

1. Let

$$s, s' \in \mathbb{R} \text{ and } r, r' \geq 1. \quad (7.29)$$

Due to (5.2) we have that

$$U_0(s, r) = 1, \quad U_0(s', r') = 1. \quad (7.30)$$

Identities in (7.30) and assumption (7.29) imply (5.8) for this case.

2. Let

$$s, s' \in \mathbb{R}, \quad 1/2 < r < 1 \text{ and } r' \geq 1. \quad (7.31)$$

Then, due to (5.2), (5.7) we have that

$$|1 - U_0(s, r)| \leq C(1-r)^{1/3}, \quad (7.32)$$

$$U_0(s', r') = 1, \quad (7.33)$$

where  $s, s', r, r'$  satisfy assumption (7.31),  $C$  is a constant depending only on  $\Phi$ . In particular, inequality (7.32) follows from (5.7) due to the following simple property of the logarithm:

$$\log_2^a \left( \frac{1}{1-r} \right) \leq C(a, \varepsilon)(1-r)^{-\varepsilon} \text{ for any } \varepsilon > 0, r \in [0, 1), a > 0, \quad (7.34)$$

where  $C(a, \varepsilon)$  is some positive constant depending only on  $a$  and  $\varepsilon$ .

Due to (7.31), (7.32), (7.33) we have that

$$\begin{aligned} |U_0(s', r') - U_0(s, r)| &= |1 - U_0(s, r)| \leq C(1-r)^{1/3} \\ &\leq C|r - r'|^{1/3} \leq C(|r - r'|^{1/3} + |s - s'|^{1/3}), \end{aligned} \quad (7.35)$$

where  $C$  is a constant depending only on  $\Phi$ .

Estimate (7.35) and assumptions (7.31) imply (5.8) for this case.

3. Let

$$s, s' \in \mathbb{R} \text{ and } r, r' \in (1/2, 1). \quad (7.36)$$

In addition, without loss of generality we assume that  $r > r'$ .

Next, using (5.4) one can see that

$$\begin{aligned} |U_0(s, r) - U_0(s', r')| &= |U_0(s, r) - U_0(s', r) + U_0(s', r) - U_0(s', r')| \\ &\leq |U_0(s, r) - U_0(s', r)| + |U_0(s', r) - U_0(s', r')| \\ &\leq \left| \frac{\partial U_0}{\partial s}(\hat{s}, r) \right| |s - s'| + \left| \frac{\partial U_0}{\partial r}(s', \hat{r}) \right| |r - r'|, \end{aligned} \quad (7.37)$$

for  $s, s' \in \mathbb{R}, r, r' > 1/2$ , and for appropriate  $\hat{s}, \hat{r}$ .

Note that  $\hat{s}, \hat{r}$  belong to open intervals  $(s, s'), (r', r)$ , respectively.

Using (5.7), (7.19), (7.32), (7.37) and the property that  $1/2 < r' < \hat{r} < r < 1$  we obtain

$$|U_0(s, r) - U_0(s', r')| \leq C((1-r)^{1/3} + (1-r')^{1/3}), \quad (7.38)$$

$$|U_0(s, r) - U_0(s', r')| \leq \frac{C}{(1-r)^5} (|s - s'| + |r - r'|), \quad (7.39)$$

where  $C$  is a constant depending only on  $\Phi$ .

We have that

$$\begin{aligned} (1-r)^{1/3} + (1-r')^{1/3} &= (1-r)^{1/3} + ((1-r) + (r-r'))^{1/3} \\ &\leq 2(1-r)^{1/3} + |r-r'|^{1/3} \\ &\leq \begin{cases} 3|r-r'|^{1/3} & \text{if } 1-r \leq |r-r'|, \\ 3(1-r)^{1/3} & \text{if } 1-r > |r-r'|, \end{cases} \end{aligned} \quad (7.40)$$

where  $r, r'$  satisfy (7.36). Note that in (7.40) we used the following inequality:

$$(a+b)^{1/m} \leq a^{1/m} + b^{1/m} \text{ for } a \geq 0, b \geq 0, m \in \mathbb{N}. \quad (7.41)$$

In particular, using (7.38), (7.40) we have that

$$|U_0(s, r) - U_0(s', r')|^{15} \leq 3^{15} C^{15} (1-r)^5 \text{ if } 1-r > |r-r'|, \quad (7.42)$$

where  $s, s', r, r'$  satisfy assumption (7.36),  $C$  is a constant of (7.38), (7.39).

Multiplying the left and the right hand-sides of (7.39), (7.42) we obtain

$$|U_0(s, r) - U_0(s', r')|^{16} \leq 3^{15} C^{16} (|s - s'| + |r - r'|), \text{ if } 1-r > |r - r'|. \quad (7.43)$$

Using (7.38), (7.40) we obtain

$$|U_0(s, r) - U_0(s', r')| \leq 3C|r - r'|^{1/3}, \text{ if } 1-r \leq |r - r'|, \quad (7.44)$$

where  $C$  is a constant of (7.38), (7.39) depending only on  $\Phi$ . Using (7.43) and (7.41) for  $m = 16, a = |s - s'|, b = |r - r'|$ , we have that

$$|U_0(s, r) - U_0(s', r')| \leq 3C(|s - s'|^{1/16} + |r - r'|^{1/16}), \text{ if } 1-r > |r - r'|, \quad (7.45)$$

where  $s, s', r, r'$  satisfy assumption (7.36),  $C$  is a constant of (7.38), (7.39) which depends only on  $\Phi$ .

Formulas (7.44), (7.45) imply (5.8) for this case.

Note that assumptions (7.29), (7.31), (7.36) for cases 1, 2, 3, respectively, cover all possible choices of  $s, s', r, r'$  in (5.8).

This completes the proof of (5.8).

This completes the proof of Lemma 3.

## 8 Proofs of Lemmas 4, 5

### 8.1 Proof of Lemma 4

**Proof of (7.13), (7.14).** Estimates (7.13), (7.14) follow directly from (3.11)-(3.14).

**Proof of (7.17).** We will use the following parametrization of the points  $y$  on  $\gamma(x, \theta) \in TS^1$ ,  $(x, \theta) \in \Omega$ ,  $r(x, \theta) \neq 0$  (see notations (2.1), (2.2), (2.13) for  $d = 2$ ):

$$y(\beta) = x - (x\theta)\theta + \tan(\beta)r(x, \theta)\theta, \quad \beta \in (-\pi/2, \pi/2), \quad (8.1)$$

where  $\beta$  is the parameter.

We have that:

$$d\sigma(\beta) = r d(\tan(\beta)) = \frac{r d\beta}{\cos^2 \beta}, \quad r = r(x, \theta), \quad (8.2)$$

where  $\sigma$  is the standard Lebesgue measure on  $\gamma(x, \theta)$ .

From definitions (3.10), (5.3) it follows that

$$\tilde{G}(r) = \sum_{k=1}^{\infty} \frac{\tilde{G}_k(r)}{k!}, \quad (8.3)$$

$$\tilde{G}_k(r) = \int_{\gamma_r} \tilde{f}_k(|y|) dy, \quad \gamma_r \in T(r), \quad r > 1/2, \quad (8.4)$$

where  $T(r)$  is defined by (2.8).

Using (3.11), (8.1), (8.2), (8.4) we obtain the following formula for  $\tilde{G}_k$ :

$$\begin{aligned} \tilde{G}_k(r) &= r \int_{-\pi/2}^{\pi/2} \Phi \left( 2^k \left( 1 - \frac{r}{\cos \beta} \right) \right) \cos \left( 8^k \frac{r^2}{\cos^2 \beta} \right) \frac{d\beta}{\cos^2 \beta} \\ &= \{u = \tan(\beta)\} = 2r \int_0^{+\infty} \Phi \left( 2^k \left( 1 - r\sqrt{u^2 + 1} \right) \right) \cos \left( 8^k r^2 (u^2 + 1) \right) du \\ &= \{t = u^2\} = r \int_0^{+\infty} \Phi \left( 2^k \left( 1 - r\sqrt{t+1} \right) \right) \cos \left( 8^k r^2 (t+1) \right) \frac{dt}{\sqrt{t}} \\ &= r \cos(8^k r^2) \int_0^{+\infty} \Phi(2^k(1 - r\sqrt{t+1})) \frac{\cos(8^k r^2 t)}{\sqrt{t}} dt \\ &\quad - r \sin(8^k r^2) \int_0^{+\infty} \Phi(2^k(1 - r\sqrt{t+1})) \frac{\sin(8^k r^2 t)}{\sqrt{t}} dt \\ &= 8^{-k/2} r^{-1} \cos(8^k r^2) \int_0^{+\infty} \Phi_k(t, r) \frac{\cos(t)}{\sqrt{t}} dt \\ &\quad - 8^{-k/2} r^{-1} \sin(8^k r^2) \int_0^{+\infty} \Phi_k(t, r) \frac{\sin(t)}{\sqrt{t}} dt, \quad r > 1/2, \end{aligned} \quad (8.5)$$

where

$$\Phi_k(t, r) = \Phi(2^k(1 - r\sqrt{8^{-k}r^{-2}t + 1})), \quad t \geq 0, \quad r > 1/2, \quad k \in \mathbb{N}. \quad (8.6)$$

For integrals arising in (8.5) the following estimates hold:

$$\left| \int_0^{+\infty} \Phi_k(t, r) \frac{\sin(t)}{\sqrt{t}} dt \right| \leq C_1 < +\infty, \quad (8.7)$$

$$\left| \int_0^{+\infty} \Phi_k(t, r) \frac{\cos(t)}{\sqrt{t}} dt \right| \leq C_2 < +\infty, \quad (8.8)$$

for  $1/2 < r < 1$ ,  $k \geq 1$ .

where  $\Phi_k$  is defined in (8.6),  $C_1, C_2$  are some positive constants depending only on  $\Phi$  and not depending on  $k$  and  $r$ .

Estimates (8.7), (8.8) are proved in Subsection 8.3.

From (8.5)-(8.8) it follows that

$$|\tilde{G}_k(r)| \leq 2 \cdot 8^{-k/2} (C_1 + C_2) \text{ for } r > 1/2, k \in \mathbb{N}. \quad (8.9)$$

Note that for  $y \in \gamma_r$ , the following inequality holds:

$$2^k(1 - |y|) \leq 2^k(1 - r) \leq 2^{k-m} \leq 1/2 < 4/5 \text{ for } 1 - 2^{-m} \leq r < 1, k < m, m \geq 3, \quad (8.10)$$

where  $\gamma_r$  is a ray in  $T(r)$  (see notations of (2.8),  $d = 2$ ).

Formulas (3.11), (3.12), (5.3), (8.10) imply that

$$\gamma_r \cap \text{supp } f_k = \emptyset \text{ if } r \geq 1 - 2^{-m}, k < m, \quad (8.11)$$

In turn, (8.4), (8.11) imply that

$$\tilde{G}_k(r) = 0 \text{ for } r \geq 1 - 2^{-m}, k < m, m \geq 3. \quad (8.12)$$

Due to (8.3), (8.4), (8.9), (8.12) we have that:

$$\begin{aligned} |\tilde{G}(r)| &\leq \sum_{k=m}^{\infty} |\tilde{G}_k(r)|/k! \\ &\leq 2(C_1 + C_2) \frac{(2\sqrt{2})^{-m}}{m!} \sum_{k=0}^{\infty} (2\sqrt{2})^{-k} = c_1 \frac{(2\sqrt{2})^{-m}}{m!}, \quad c_1 = (C_1 + C_2) \frac{4\sqrt{2}}{2\sqrt{2} - 1}, \\ &\text{for } r \geq 1 - 2^{-m}, m \geq 3. \end{aligned} \quad (8.13)$$

This completes the proof of estimate (7.17).

**Proof of (7.18).** Using (8.3), (8.4) we have that:

$$\left| \frac{d\tilde{G}}{dr}(r) \right| \leq \sum_{k=1}^{\infty} \frac{1}{k!} \left| \frac{d\tilde{G}_k(r)}{dr} \right|. \quad (8.14)$$

Formulas (3.11), (7.10) for  $n = 1$ , (7.14), (8.4) imply that

$$\begin{aligned} \left| \frac{d\tilde{G}_k}{dr}(r) \right| &= \left| \int_{-\infty}^{+\infty} \frac{r \tilde{f}'_k((s^2 + r^2)^{1/2})}{\sqrt{r^2 + s^2}} ds \right| \\ &\leq \int_{-\infty}^{+\infty} |\tilde{f}'_k((s^2 + r^2)^{1/2})| ds = \int_{\gamma_r} |\tilde{f}'_k(|y|)| dy \leq c8^k \int_{\gamma_r \cap B(0,1)} dy \leq 2c8^k, \end{aligned} \quad (8.15)$$

where  $B(0, 1)$  is defined in (2.15),  $d = 2$ .

At the same time, formula (8.12) implies that

$$\frac{d\tilde{G}_k(r)}{dr} = 0 \text{ for } r \geq 1 - 2^{-m}, k < m, m \geq 3. \quad (8.16)$$

Formulas (8.14), (8.15), (8.16) imply the following sequence of inequalities:

$$\left| \frac{d\tilde{G}(r)}{dr} \right| \leq \sum_{k=m}^{\infty} \frac{1}{k!} \left| \frac{d\tilde{G}_k(r)}{dr} \right| \leq c \frac{8^m}{m!} \sum_{k=0}^{\infty} \frac{m!8^k}{(k+m)!}, \quad r \geq 1 - 2^{-m}, m \geq 3. \quad (8.17)$$

The series in the right hand-side in (8.17) admits the following estimate:

$$\sum_{k=0}^{\infty} \frac{m!8^k}{(k+m)!} \leq \sum_{k=0}^{\infty} \frac{8^k}{k!} = e^8 \text{ and the estimate does not depend on } m. \quad (8.18)$$

Formulas (8.17), (8.18) imply (7.18).



**Proof of (7.15).** For each  $\psi_k$  from (3.33) we have that

$$|\psi_k| \leq 1. \quad (8.19)$$

Therefore, it is sufficient to show that

$$\tilde{H}_k \geq C2^{-k} \text{ for } k \geq k_1, C = c^{-1}. \quad (8.20)$$

Proceeding from (5.3) and in a similar way with (8.5) we obtain the formulas

$$\tilde{H}_k(r) = r \int_0^{+\infty} \frac{\Phi^2(2^k(1-r\sqrt{t+1}))}{\sqrt{t}} \cos^2(8^k r^2(t+1)) dt = \tilde{H}_{k,1}(r) + \tilde{H}_{k,2}(r), \quad r > 1/2, \quad (8.21)$$

$$\tilde{H}_{k,1}(r) = \frac{r}{2} \int_0^{+\infty} \frac{\Phi^2(2^k(1-r\sqrt{t+1}))}{\sqrt{t}} dt, \quad (8.22)$$

$$\tilde{H}_{k,2}(r) = \frac{r}{2} \int_0^{+\infty} \frac{\Phi^2(2^k(1-r\sqrt{t+1}))}{\sqrt{t}} \cos(2 \cdot 8^k r^2(t+1)) dt. \quad (8.23)$$

In addition, we have that:

$$\text{supp}_t \Phi^2(2^k(1-r\sqrt{t+1})) \subset [0, 3] \text{ for } 1/2 < r \leq 1 - 2^{-k+1}, k \geq 3, \quad (8.24)$$

where  $\text{supp}_t$  denotes the support of the function in variable  $t$ . Property (8.24) is proved below in this paragraph (see formulas (8.26)-(8.29)).

Note that

$$2^k(1-r) \geq 2^k \cdot 2^{-k+1} \geq 2 > 6/5 \text{ for } 1/2 < r \leq 1 - 2^{-k+1}, k \geq 3. \quad (8.25)$$

From (3.12), (3.13) and from (8.25) we have that:

$$\text{supp}_t \Phi^2(2^k(1-r\sqrt{t+1})) \subset [0, +\infty) \text{ for } 1/2 < r \leq 1 - 2^{-k+1}, k \geq 3. \quad (8.26)$$

We have that

$$\exists t_1^{(k)} = t_1^{(k)}(r) \geq 0, t_2^{(k)} = t_2^{(k)}(r) \geq 0, t_2^{(k)} > t_1^{(k)}, \text{ such that } \begin{cases} 2^k(1-r\sqrt{t_1^{(k)}+1}) = 11/10, \\ 2^k(1-r\sqrt{t_2^{(k)}+1}) = 9/10, \end{cases} \quad (8.27)$$

$$|t_2^{(k)} - t_1^{(k)}| \geq \left( \sqrt{t_2^{(k)}+1} - \sqrt{t_1^{(k)}+1} \right) = \frac{2^{-k}}{5} r^{-1} \geq \frac{2^{-k}}{5}, \quad (8.28)$$

for  $1/2 < r \leq 1 - 2^{-k+1}, k \geq 3$ .

In addition, from (8.27) it follows that

$$t_1^{(k)} = \frac{(1 - 2^{-k} \frac{11}{10})^2}{r^2} - 1 \leq 4(1 - 2^{-k} \frac{11}{10})^2 - 1 \leq 3, \quad (8.29)$$

$$t_2^{(k)} = \frac{(1 - 2^{-k} \frac{9}{10})^2}{r^2} - 1 \leq 4(1 - 2^{-k} \frac{11}{10})^2 - 1 \leq 3,$$

for  $1/2 < r \leq 1 - 2^{-k+1}, k \geq 3$ .

Using (3.12)-(3.14), (8.22), (8.24), (8.27)-(8.29) we have that

$$\begin{aligned} \tilde{H}_{k,1}(r) &\geq \frac{r}{2} \int_{t_1^{(k)}}^{t_2^{(k)}} \frac{dt}{\sqrt{t}} \geq \frac{r}{2} \int_3^{3+|t_2^{(k)}-t_1^{(k)}|} \frac{dt}{\sqrt{t}} \\ &\geq \frac{r}{6} \int_3^{3+|t_2^{(k)}-t_1^{(k)}|} dt = \frac{r}{6} |t_2^{(k)} - t_1^{(k)}| \geq \frac{2^{-k}}{30} \text{ for } 1/2 < r \leq 1 - 2^{-k+1}, k \geq 3. \end{aligned} \quad (8.30)$$

On the other hand, proceeding from using (8.23) and, in a similar way with (8.5)-(8.9), we have

$$\begin{aligned}
|\tilde{H}_{k,2}(r)| &= \frac{r}{2} \left| \int_0^{+\infty} \frac{\Phi^2(2^k(1-r\sqrt{t+1}))}{\sqrt{t}} \cos(2 \cdot 8^k r^2(t+1)) dt \right| \\
&\leq \frac{r}{2} |\cos(2 \cdot 8^k r^2)| \left| \int_0^{+\infty} \Phi^2(2^k(1-r\sqrt{t+1})) \frac{\cos(2 \cdot 8^k r^2 t)}{\sqrt{t}} dt \right| \\
&\quad + \frac{r}{2} |\sin(2 \cdot 8^k r^2)| \left| \int_0^{+\infty} \Phi^2(2^k(1-r\sqrt{t+1})) \frac{\sin(2 \cdot 8^k r^2 t)}{\sqrt{t}} dt \right| \\
&\leq 8^{-k/2} \frac{r^{-1}}{2} \left| \int_0^{+\infty} \Phi_k^2(t, r) \frac{\cos(2t)}{\sqrt{t}} dt \right| + 8^{-k/2} \frac{r^{-1}}{2} \left| \int_0^{+\infty} \Phi_k^2(t, r) \frac{\sin(2t)}{\sqrt{t}} dt \right| \\
&\leq 8^{-k/2} C, \text{ for } 1/2 < r < 1 - 2^{-k+1}, k \geq 3,
\end{aligned} \tag{8.31}$$

where  $\Phi_k(t, r)$  is defined in (8.6),  $C$  is some constant depending only on  $\Phi$  and not depending on  $k, r$ . In (8.31) we have also used that  $\Phi^2(t)$  satisfies assumptions (3.12)-(3.14).

Note also that  $\Phi^2(t)$  satisfies assumptions (3.12)-(3.14) for  $\Phi(t)$ .

Using (8.21)-(8.23), (8.30), (8.31) we obtain

$$\begin{aligned}
|\tilde{H}_k(r)| &\geq |\tilde{H}_{k,1}(r)| - |\tilde{H}_{k,2}(r)| \\
&\geq \frac{2^{-k}}{30} - C' \cdot 8^{-k/2} \\
&\geq 2^{-k} \left( \frac{1}{30} - \frac{C'}{(\sqrt{2})^k} \right) \\
&\geq C 2^{-k} \text{ for } 1/2 < r < 1 - 2^{-k+1}, k \geq k_1 \geq 3, \\
C &= \frac{1}{30} - C'(\sqrt{2})^{-k_1},
\end{aligned} \tag{8.32}$$

where  $C'$  depends only on  $\Phi$ ,  $k_1$  is arbitrary constant such that  $k_1 \geq 3$  and  $C$  is positive.

Formulas (7.15) follows from (3.33), (8.32).

This completes the proof (7.15).

**Proof of (7.16).** The following formula holds:

$$\frac{d}{dr} \left( \frac{\psi_{k-2}(r)}{\tilde{H}_k(r)} \right) = - \frac{\tilde{H}'_k(r)\psi_{k-2}(r) - \tilde{H}_k(r)\psi'_{k-2}(r)}{\tilde{H}_k^2(r)}, \quad 1/2 < r < 1, \tag{8.33}$$

where  $\tilde{H}'_k, \psi'_{k-2}$  denote the derivatives of  $\tilde{H}_k, \psi_k$ , defined in (5.3), (3.33), respectively.

Using (3.11), (5.3), (7.10),  $n = 1$ , (7.13), (7.14) we have that

$$\begin{aligned}
|\tilde{H}'_k(r)| &= 2 \left| \int_{-\infty}^{+\infty} \frac{r}{\sqrt{r^2+s^2}} \tilde{f}_k(\sqrt{r^2+s^2}) \tilde{f}'_k(\sqrt{r^2+s^2}) ds \right| \\
&\leq 2 \int_{-\infty}^{+\infty} |\tilde{f}_k(\sqrt{r^2+s^2}) \tilde{f}'_k(\sqrt{r^2+s^2})| ds = 2 \int_{\gamma_r} |\tilde{f}_k(|y|) \tilde{f}'_k(|y|)| dy \\
&\leq 2c 8^k \int_{\gamma_r \cap B(0,1)} dy \leq 4c 8^k, \quad \gamma_r \in T(r), k \geq 3, r > 1/2,
\end{aligned} \tag{8.34}$$

where we use notations (2.8), (2.15),  $d=2$ .

Using (3.33), (3.34), (7.15), (8.32)-(8.34) we have that

$$\begin{aligned}
\left| \frac{d}{dr} \left( \frac{\psi_{k-2}(r)}{\tilde{H}_k(r)} \right) \right| &\leq C 2^{2k} (|\tilde{H}'_k(r)| + |\tilde{H}_k(r)| \cdot |\psi'_{k-2}(r)|) \leq C' 2^{5k}, \\
&\text{for } 1/2 < r < 1 - 2^{-k+1}, k \geq k_1 \geq 3,
\end{aligned} \tag{8.35}$$

where  $C'$  is a constant not depending on  $k$  and  $r$  and depending only on  $\Phi$ .

This completes the proof of Lemma 4.

## 8.2 Proof of Lemma 5

It is sufficient to show that

$$\left| \frac{\partial U_0(s, r)}{\partial s} \right| \leq \frac{C}{(1-r)^3}, \quad (8.36)$$

$$\left| \frac{\partial U_0(s, r)}{\partial r} \right| \leq \frac{C}{(1-r)^5}, \quad (8.37)$$

for  $s \in \mathbb{R}$ ,  $r \in \Lambda_k$ ,  $k \geq \max(4, k_1)$ ,

where  $C$  is a positive constant depending only on  $\Phi$  of (3.11),  $\Lambda_k$  is defined in (7.4),  $k_1$  is a constant arising in Lemma 4 and depending only on  $\Phi$ .

Indeed, estimates (7.19) follow from (5.4), (8.36), (8.37) and the fact that  $\Lambda_k$ ,  $k \geq 4$ , is an open cover of  $(1/2, 1)$ .

In turn, estimates (8.36), (8.37) follow from the estimates

$$\left| \frac{\partial U_0(s, r)}{\partial s} \right| \leq C \cdot 8^k, \quad (8.38)$$

$$\left| \frac{\partial U_0(s, r)}{\partial r} \right| \leq C \cdot (32)^k, \quad (8.39)$$

for  $s \in \mathbb{R}$ ,  $r \in \Lambda_k$ ,

and from the fact that  $2^{-k+1} < 1-r < 2^{-k+3}$ ,  $k \geq \max(4, k_1)$ , for  $r \in \Lambda_k$ , where  $C$  is a positive constant depending only on  $\Phi$ .

Estimate (8.38) follows from formula (7.9) for  $n = 1$  and estimates (7.14), (7.15), (7.20)-(7.24).

Estimate (8.39) follows from (7.8), (7.13)-(7.16), (7.20)-(7.24) and from the estimates:

$$\left| \frac{d}{dr} \left( \frac{\psi_{k-i}(r)}{\tilde{H}_{k-i+2}(r)} \right) \right| \leq c 2^{5(k+1)}, \quad (8.40)$$

$$\left| \frac{d\tilde{G}(r)}{dr} \right| \leq c \frac{8^{-k+3}}{(k-3)!}, \quad (8.41)$$

for  $r \in \Lambda_k$ ,  $i \in \{1, 2, 3\}$ ,

where  $c$  is a constant arising in Lemma 4.

Estimate (8.40) follows from (7.16) (used with  $k-1$ ,  $k$ ,  $k+1$  in place of  $k$ ). Estimate (8.41) follows from (7.18) (used with  $k-3$  in place of  $k$ ).

This completes the proof of Lemma 5.

## 8.3 Proof of estimates (8.7), (8.8)

We use the following Bonnet's integration formulas (see, e.g., [Fich59], Chapter 2):

$$\int_a^b f_1(t)h(t) dt = f_1(a) \int_a^{\xi_1} h(t) dt, \quad (8.42)$$

$$\int_a^b f_2(t)h(t) dt = f_2(b) \int_{\xi_2}^b h(t) dt, \quad (8.43)$$

for some appropriate  $\xi_1, \xi_2 \in [a, b]$ , where

$$\begin{aligned} f_1 &\text{ is monotonously decreasing on } [a, b], f_1 \geq 0, \\ f_2 &\text{ is monotonously increasing on } [a, b], f_2 \geq 0, \\ h(t) &\text{ is integrable on } [a, b]. \end{aligned} \quad (8.44)$$

Let

$$g_1(t) = \frac{\sin(t)}{\sqrt{t}}, \quad g_2(t) = \frac{\cos(t)}{\sqrt{t}}, \quad t > 0, \quad (8.45)$$

$$G_1(s) = \int_0^s \frac{\sin(t)}{\sqrt{t}} dt, \quad G_2(s) = \int_0^s \frac{\cos(t)}{\sqrt{t}} dt, \quad s \geq 0. \quad (8.46)$$

We recall that

$$\lim_{s \rightarrow +\infty} G_1(s) = \lim_{s \rightarrow +\infty} G_2(s) = \sqrt{\frac{\pi}{2}}. \quad (8.47)$$

From (8.45), (8.46), (8.47) it follows that

$$G_1, G_2 \text{ are continuous and bounded on } [0, +\infty). \quad (8.48)$$

Due to (3.12)-(3.15), (8.6) and monotonicity of the function  $2^k(1 - r\sqrt{8^{-k}r^{-2}t + 1})$  in  $t$  on  $[0, +\infty)$  it follows that

$$\Phi_k(t, r) \text{ is monotonously decreasing on } [0, +\infty), \text{ if } 2^k(1 - r) \leq 11/10, \quad (8.49)$$

$$\Phi_k(t, r) \text{ is monotonously increasing on } [0, t_0] \text{ for some } t_0 > 0 \quad (8.50)$$

$$\text{and is monotonously decreasing on } [t_0, +\infty), \text{ if } 2^k(1 - r) > 11/10.$$

$$\text{for } r > 1/2, k \in \mathbb{N},$$

Moreover, due to (3.12)-(3.14), (8.6), for  $T_k = 8^k$ ,  $k \in \mathbb{N}$ , we have that

$$\Phi_k(T_k, r) = \Phi(2^k(1 - r\sqrt{r^{-2} + 1})) = \Phi(2^k(1 - \sqrt{1 + r^{-2}})) = 0, \quad (8.51)$$

$$\Phi_k(t, r) = 0 \text{ for } t \geq T_k, \quad (8.52)$$

$$|\Phi_k(t, r)| \leq 1 \text{ for } t \geq 0, \quad (8.53)$$

$$r > 1/2, k \in \mathbb{N}.$$

Using (8.6), (8.45)-(8.50), (8.52) and (8.42)-(8.44) we obtain

$$\begin{aligned} \int_0^{+\infty} \Phi_k(t, r)g_i(t) dt &= \int_0^{T_k} \Phi_k(t, r)g_i(t) dt = \Phi_k(0, r) \int_0^\xi g_i(t) dt \\ &= \Phi_k(0, r)G_i(\xi) \text{ for appropriate } \xi \in [0, T_k], \text{ if } 2^k(1 - r) \leq 11/10, \end{aligned} \quad (8.54)$$

$$\begin{aligned} \int_0^\infty \Phi_k(t, r)g_i(t) dt &= \int_0^{T_k} \Phi_k(t, r)g_i(t) dt = \int_0^{t_0} \Phi_k(t, r)g_i(t) dt + \int_{t_0}^{T_k} \Phi_k(t, r)g_i(t) dt \\ &= \Phi_k(t_0, r) \int_{\xi'}^{t_0} g_i(t) dt + \Phi_k(t_0, r) \int_{t_0}^{\xi''} g_i(t) dt \\ &= \Phi_k(t_0, r)(G_i(\xi'') - G_i(\xi')) \text{ for appropriate } \xi' \in [0, t_0], \xi'' \in [t_0, T_k], \\ &\text{if } 2^k(1 - r) > 11/10, \end{aligned} \quad (8.55)$$

where  $i = \overline{1, 2}$ .

Estimates (8.7), (8.8) follow from (8.45), (8.46), (8.48), (8.53)-(8.55).

## References

- [Bey84] Beylkin, G., The inversion problem and applications of the generalized Radon transform. *Communications on pure and applied mathematics*, 37(5):579-599, 1984.
- [BQ87] Boman, J., Quinto, E.T., Support theorems for real-analytic Radon transforms. *Duke Mathematical J.*, 55(4):943-948, 1987.
- [Bom93] Boman, J., An example of non-uniqueness for a generalized Radon transform. *Journal d'Analyse Mathematique*, 61(1):395-401, 1993.
- [MCar92] Do Carmo, M. P., Riemannian Geometry. Birkhäuser Basel, 1992.
- [Fich59] Fichtenholz, G.M., A course of differential and integral calculus, volume II, Moscow, 1959.
- [Fi86] Finch, D., Uniqueness for the attenuated X-ray transform in the physical range. *Inverse Problems* 2(2), 1986.
- [GGV64] Gel'fand, I. M., Graev, M. I., Vilenkin, N. Ya., Generalized Functions, Volume 5: Integral Geometry and Representation Theory. AMS Chelsea Publishing, 1966.
- [GN16] Goncharov, F.O., Novikov, R.G., An analog of Chang inversion formula for weighted Radon transforms in multidimensions. *Eurasian Journal of Mathematical and Computer Applications*, 4(2):23-32, 2016.
- [Gon16] Goncharov, F.O., An iterative inversion of weighted Radon transforms along hyperplanes. *Inverse Problems* (<https://doi.org/10.1088/1361-6420/aa91a4>), 2017.
- [GN17] Goncharov F.O., Novikov, R.G. An example of non-uniqueness for Radon transforms with continuous positive rotation invariant weights. *hal-01593781v1*, 2017

- [GuNo14] Guillement J.-P., Novikov R. G., Inversion of weighted Radon transforms via finite Fourier series weight approximations. *Inverse Problems in Science and Engineering*, 22(5):787–802, 2014.
- [Jollm16] Ilmavirta, J., Coherent quantum tomography. *SIAM Journal on Mathematical Analysis*, 48(5):3039–3064, 2016.
- [Kun92] Kunyansky, L., Generalized and attenuated Radon transforms: restorative approach to the numerical inversion. *Inverse Problems*, 8(5):809-819, 1992.
- [LB73] Lavrent'ev, M. M., Bukhgeim, A. L., A class of operator equations of the first kind. *Functional Analysis and Its Applications*, 7(4):290-298, 1973.
- [MQ85] Markoe, A., Quinto, E.T. An elementary proof of local invertibility for generalized and attenuated Radon transforms. *SIAM Journal on Mathematical Analysis*, 16(5):1114–1119, 1985.
- [Natt01] Natterer, F., *The Mathematics of Computerized Tomography*. SIAM, 2001.
- [Novi02] Novikov, R.G., On determination of a gauge field on  $\mathbb{R}^d$  from its non-abelian Radon transform along oriented straight lines. *Journal of the Institute of Mathematics of Jussieu*, 1(4), 559-629., 2002.
- [Nov02] Novikov, R.G., An inversion formula for the attenuated X-ray transformation. *Arkiv för matematik*, 40(1):145-167, 2002.
- [Nov14] Novikov, R.G., Weighted Radon transforms and first order differential systems on the plane. *Moscow Mathematical Journal*, 14(4):807–823, 2014.
- [Qui83] Quinto, E.T., The invertibility of rotation invariant Radon transforms. *Journal of Mathematical Analysis and Applications*, 91(2):510–522, 1983.