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# A breakdown of injectivity for weighted ray transforms in multidimensions

F.O. Goncharov\*      R. G. Novikov\*†

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## Abstract

We consider weighted ray-transforms  $P_W$  (weighted Radon transforms along straight lines) in  $\mathbb{R}^d$ ,  $d \geq 2$ , with strictly positive weights  $W$ . We construct an example of such a transform with non-trivial kernel in the space of infinitely smooth compactly supported functions on  $\mathbb{R}^d$ . In addition, the constructed weight  $W$  is rotation-invariant continuous and is infinitely smooth almost everywhere on  $\mathbb{R}^d \times \mathbb{S}^{d-1}$ . In particular, by this construction we give counterexamples to some well-known injectivity results for weighted ray transforms for the case when the regularity of  $W$  is slightly relaxed.

**Keywords:** Radon transforms, ray transforms, integral geometry, injectivity, non-injectivity

**AMS Mathematics Subject Classification:** 44A12, 53C65, 65R32

## 1 Introduction

We consider the weighted ray transforms  $P_W$  defined by

$$P_W f(x, \theta) = \int_{\mathbb{R}} W(x + t\theta, \theta) f(x + t\theta) dt, \quad (x, \theta) \in T\mathbb{S}^{d-1}, \quad d \geq 2, \quad (1.1)$$

$$T\mathbb{S}^{d-1} = \{(x, \theta) \in \mathbb{R}^d \times \mathbb{S}^{d-1} : x\theta = 0\}, \quad (1.2)$$

where  $f = f(x)$ ,  $W = W(x, \theta)$ ,  $x \in \mathbb{R}^d$ ,  $\theta \in \mathbb{S}^{d-1}$ . Here,  $W$  is the weight,  $f$  is a test function on  $\mathbb{R}^d$ . In addition, we interpret  $T\mathbb{S}^{d-1}$  as the set of all rays in  $\mathbb{R}^d$ . As a ray  $\gamma$  we understand a straight line with fixed orientation. If  $\gamma = \gamma(x, \theta)$ ,  $(x, \theta) \in T\mathbb{S}^{d-1}$ , then

$$\gamma(x, \theta) = \{y \in \mathbb{R}^d : y = x + t\theta, t \in \mathbb{R}\} \text{ (up to orientation),} \quad (1.3)$$

where  $\theta$  gives the orientation of  $\gamma$ .

We assume that

$$W = \overline{W} \geq c > 0, \quad W \in L^\infty(\mathbb{R}^d \times \mathbb{S}^{d-1}), \quad (1.4)$$

where  $\overline{W}$  denotes the complex conjugate of  $W$ ,  $c$  is a constant.

The aforementioned transforms  $P_W$  arise in various domains of pure and applied mathematics; see, e.g., [Bey84], [Bom93], [BQ87], [Fi86], [JoIlm16], [Kun92], [LB73], [MQ85], [Natt01], [Nov02], [Nov14], [Qui83] and references therein.

In particular, the related results are the most developed for the case when  $W \equiv 1$ . In this case  $P_W$  is reduced to the classical ray-transform  $P$  (Radon transform along straight lines). The transform  $P$  arises, in particular, in the X-ray transmission tomography. We refer to [GGV64], [Natt01] and references therein in connection with basic results for this classical case.

At present, many important results on transforms  $P_W$  with other weights  $W$  satisfying (1.4) are also known; see the publications mentioned above with non-constant  $W$  and references therein.

In particular, assuming (1.4) we have the following injectivity results.

**Injectivity 1** (see [Fi86]). Suppose that  $d \geq 3$  and  $W \in C^2(\mathbb{R}^d \times \mathbb{S}^{d-1})$ . Then  $P_W$  is injective on  $L_0^p(\mathbb{R}^d)$  for  $p > 2$ , where  $L_0^p$  denotes compactly supported functions from  $L^p$ .

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\*CMAP, Ecole Polytechnique, CNRS, Université Paris-Saclay, 91128, Palaiseau, France;

email: fedor.goncharov.ol@gmail.com

†IEPT RAS, 117997 Moscow, Russia;

email: roman.novikov@polytechnique.edu

**Injectivity 2** (see [MQ85]). Suppose that  $d = 2$ ,  $W \in C^2(\mathbb{R}^2 \times \mathbb{S}^1)$  and

$$0 < c_0 \leq W, \|W\|_{C^2(\mathbb{R}^2 \times \mathbb{S}^1)} \leq N, \quad (1.5)$$

for some constants  $c_0$  and  $N$ . Then, for any  $p > 2$ , there is  $\delta = \delta(c_0, N, p) > 0$  such that  $P_W$  is injective on  $L^p(B(x_0, \delta))$  for any  $x_0 \in \mathbb{R}^2$ , where

$$\begin{aligned} L^p(B(x_0, \delta)) &= \{f \in L^p(\mathbb{R}^2) : \text{supp } f \subset \overline{B}(x_0, \delta)\}, \\ \overline{B}(x_0, \delta) &= \{x \in \mathbb{R}^2 : |x - x_0| \leq \delta\}. \end{aligned}$$

**Injectivity 3** (see [Qui83]). Suppose that  $d = 2$ ,  $W \in C^1(\mathbb{R}^2 \times \mathbb{S}^1)$  and  $W$  is rotation invariant (see formula (1.6) below). Then  $P_W$  is injective on  $L_0^p(\mathbb{R}^2)$  for  $p \geq 2$ .

In a similar way with [Qui83] we say that  $W$  is rotation invariant if and only if

$$W(x, \theta) = U(x\theta, |x - (x\theta)\theta|), \quad x \in \mathbb{R}^d, \theta \in \mathbb{S}^{d-1}, \quad (1.6)$$

for some positive and continuous  $U$  such that

$$U(r, s) = U(-r, s) = U(r, -s), \quad r \in \mathbb{R}, s \in \mathbb{R}. \quad (1.7)$$

Injectivity 1 is a global injectivity for  $d \geq 3$ . Injectivity 2 is a local injectivity for  $d = 2$ . Injectivity 3 is a global injectivity for  $d = 2$  for the rotation invariant case.

The results of Injectivity 1 and Injectivity 2 remain valid with  $C^\alpha$ ,  $\alpha > 1$ , in place of  $C^2$  in the assumptions on  $W$ ; see [Jollm16].

Injectivity 1 follows from Injectivity 2 in the framework of the layer-by-layer reconstruction approach. See [Fi86], [Novi02], [Jollm16] and references therein in connection with the layer-by-layer reconstruction approach for weighted and non-abelian ray transforms in dimension  $d \geq 3$ .

In the present work we obtain counterexamples to Injectivity 1, Injectivity 2 and Injectivity 3 for the case when the regularity of  $W$  is slightly relaxed. In particular, by these counterexamples we continue related studies of [MQ85], [Bom93] and [GN17].

More precisely, in the present work we construct  $W$  and  $f$  such that

$$P_W f \equiv 0 \text{ on } T\mathbb{S}^{d-1}, \quad d \geq 2, \quad (1.8)$$

where

$$\begin{aligned} W &\text{ satisfies (1.4), } W \text{ is rotation-invariant (i.e., satisfies (1.6)),} \\ W &\text{ is infinitely smooth almost everywhere on } \mathbb{R}^d \times \mathbb{S}^{d-1} \text{ and} \end{aligned} \quad (1.9)$$

$$W \in C^\alpha(\mathbb{R}^d \times \mathbb{S}^{d-1}), \text{ at least, for any } \alpha \in (0, \alpha_0), \text{ where } \alpha_0 = 1/16;$$

$$f \text{ is a non-zero spherically symmetric infinitely smooth compactly supported function on } \mathbb{R}^d; \quad (1.10)$$

see Theorem 1 of Section 3.

These  $W$  and  $f$  directly give the aforementioned counterexamples to Injectivity 1 and Injectivity 3.

Our counterexample to Injectivity 1 is of particular interest (and is rather surprising) in view of the fact that the problem of finding  $f$  on  $\mathbb{R}^d$  from  $P_W f$  on  $T\mathbb{S}^{d-1}$  is strongly overdetermined for  $d \geq 3$ . Indeed,

$$\begin{aligned} \dim \mathbb{R}^d &= d, \dim T\mathbb{S}^{d-1} = 2d - 2, \\ d &< 2d - 2 \text{ for } d \geq 3. \end{aligned}$$

Our counterexample to Injectivity 3 is considerably stronger than the preceding counterexample of [MQ85], where  $W$  is not yet continuous and is not yet strictly positive (i.e., is not yet separated from zero by a positive constant).

Using our  $W$  and  $f$  of (1.9), (1.10) for  $d = 3$  we also obtain the aforementioned counterexample to Injectivity 2; see Corollary 1 of Section 3.

In the present work we adopt and develop considerations of the famous work [Bom93] and of our very recent work [GN17].

In Section 2 we give some preliminaries and notations.

Main results are presented in detail in Sections 3.

Related proofs are given in Sections 4, 5, 6, 7, 8.

## 2 Some preliminaries

**Notations.** Let

$$\Omega = \mathbb{R}^d \times \mathbb{S}^{d-1}, \quad (2.1)$$

$$r(x, \theta) = |x - (x\theta)\theta|, \quad (x, \theta) \in \Omega, \quad (2.2)$$

$$\Omega_0(\delta) = \{(x, \theta) \in \Omega : r(x, \theta) > \delta\}, \quad (2.3)$$

$$\Omega_1(\delta) = \Omega \setminus \Omega_0(\delta) = \{(x, \theta) \in \Omega : r(x, \theta) \leq \delta\}, \quad \delta > 0, \quad (2.4)$$

$$\Omega(\Lambda) = \{(x, \theta) \in \mathbb{R}^d \times \mathbb{S}^{d-1} : r(x, \theta) \in \Lambda\}, \quad \Lambda \subset [0, +\infty), \quad (2.5)$$

$$T_0(\delta) = \{(x, \theta) \in T\mathbb{S}^{d-1} : |x| > \delta\}, \quad (2.6)$$

$$T_1(\delta) = \{(x, \theta) \in T\mathbb{S}^{d-1} : |x| \leq \delta\}, \quad \delta > 0, \quad (2.7)$$

$$T(\Lambda) = \{(x, \theta) \in T\mathbb{S}^{d-1} : |x| \in \Lambda\}, \quad \Lambda \subset [0, +\infty), \quad (2.8)$$

$$\mathcal{J}_{r,\varepsilon} = (r - \varepsilon, r + \varepsilon) \cap [0, +\infty), \quad r \in [0, +\infty), \quad \varepsilon > 0. \quad (2.9)$$

The set  $T_0(\delta)$  in (2.6) is considered as the set of all rays in  $\mathbb{R}^d$  which are located at distance greater than  $\delta$  from the origin.

The set  $T_1(\delta)$  in (2.7) is considered as the set of all rays in  $\mathbb{R}^d$  which are located at distance less or equal than  $\delta$ .

We also consider the projection

$$\pi : \Omega \rightarrow T\mathbb{S}^{d-1}, \quad (2.10)$$

$$\pi(x, \theta) = (\pi_\theta x, \theta), \quad (x, \theta) \in \Omega, \quad (2.11)$$

$$\pi_\theta x = x - (x\theta)\theta. \quad (2.12)$$

In addition,  $r(x, \theta)$  of (2.2) is the distance from the origin  $\{0\} \in \mathbb{R}^d$  to the ray  $\gamma = \gamma(\pi(x, \theta))$  (i.e.,  $r(x, \theta) = |\pi_\theta x|$ ). The rays will be also denoted by

$$\gamma = \gamma(x, \theta) \stackrel{def}{=} \gamma(\pi(x, \theta)), \quad (x, \theta) \in \Omega. \quad (2.13)$$

We also consider

$$P_W f(x, \theta) = P_W f(\pi(x, \theta)) \text{ for } (x, \theta) \in \Omega. \quad (2.14)$$

We also define

$$B(x_0, \delta) = \{x \in \mathbb{R}^d : |x - x_0| < \delta\}, \quad \overline{B}(x_0, \delta) = \{x \in \mathbb{R}^d : |x - x_0| \leq \delta\}, \quad x_0 \in \mathbb{R}^d, \quad \delta > 0, \quad (2.15)$$

$$B = B(0, 1), \quad \overline{B} = \overline{B}(0, 1). \quad (2.16)$$

For a function  $f$  on  $\mathbb{R}^d$  we denote its restriction to a subset  $\Sigma \subset \mathbb{R}^d$  by  $f|_\Sigma$ .

By  $C_0, C_0^\infty$  we denote continuous compactly supported and infinitely smooth compactly supported functions, respectively.

By  $C^\alpha(Y)$ ,  $\alpha \in (0, 1)$ , we denote the space of  $\alpha$ -Hölder functions on  $Y$  with the norm:

$$\begin{aligned} \|u\|_{C^\alpha(Y)} &= \|u\|_{C(Y)} + \|u\|'_{C^\alpha(Y)}, \\ \|u\|'_{C^\alpha(Y)} &= \sup_{\substack{y_1, y_2 \in Y \\ |y_1 - y_2| \leq 1}} \frac{|u(y_1) - u(y_2)|}{|y_1 - y_2|^\alpha}, \end{aligned} \quad (2.17)$$

where  $\|u\|_{C(Y)}$  denotes the maximum of  $|u|$  on  $Y$ .

**Rotation invariancy.** Symmetry (1.6) of  $W$  can be also written as

$$W(x, \theta) = \tilde{U}(|x|, x\theta), \quad (x, \theta) \in \Omega, \quad (2.18)$$

$$\tilde{U}(r, s) = \tilde{U}(-r, s) = \tilde{U}(r, -s), \quad r \in \mathbb{R}, \quad s \in \mathbb{R}. \quad (2.19)$$

where  $\tilde{U}$  is positive and continuous on  $\mathbb{R} \times \mathbb{R}$ .

Using the formula  $|x|^2 = |x\theta|^2 + r^2(x, \theta)$ , one can see that symmetries (1.6), (1.7) and symmetries (2.18), (2.19) of  $W$  are equivalent.

**Partition of unity.** We recall the following classical result (see, e.g., Theorem 5.6 in [MCar92]):  
*Let  $\mathcal{M}$  be a  $C^\infty$ -manifold, which is Hausdorff and has a countable base. Let also  $\{U_i\}_{i=1}^\infty$  be an open locally finite cover of  $\mathcal{M}$ .*

*Then there exists a  $C^\infty$ -smooth locally-finite partition of unity  $\{\psi_i\}_{i=1}^\infty$  on  $\mathcal{M}$ , such that*

$$\text{supp } \psi_i \subset U_i. \quad (2.20)$$

In particular, any open interval  $(a, b) \subset \mathbb{R}$  and  $\Omega$  satisfy the conditions for  $\mathcal{M}$  of this statement. It will be used in Subsection 3.1.

### 3 Main results

**Theorem 1.** *There exist a weight  $W$  satisfying (1.4) and a non-zero function  $f \in C_0^\infty(\mathbb{R}^d)$ ,  $d \geq 2$ , such that*

$$P_W f \equiv 0 \text{ on } T\mathbb{S}^{d-1}, \quad (3.1)$$

where  $P_W$  is defined in (1.1). In addition,  $W$  is rotation invariant, i.e., satisfies (1.6), and  $f$  is spherically symmetric with  $\text{supp } f \subseteq \overline{B}$ . Moreover,

$$W \in C^\infty(\Omega \setminus \Omega(1)), \quad (3.2)$$

$$W \in C^\alpha(\mathbb{R}^d \times \mathbb{S}^{d-1}) \text{ for any } \alpha \in (0, \alpha_0), \alpha_0 = 1/16, \quad (3.3)$$

$$W \geq 1/2 \text{ on } \Omega \text{ and } W \equiv 1 \text{ on } \Omega([1, +\infty)), \quad (3.4)$$

$$W(x, \theta) \equiv 1 \text{ for } |x| \geq R > 1, \theta \in \mathbb{S}^{d-1}, \quad (3.5)$$

where  $\Omega, \Omega(1), \Omega([1, +\infty))$  are defined by (2.1), (2.5),  $R$  is a constant.

The construction of  $W$  and  $f$  proving Theorem 1 is presented below in Subsections 3.1, 3.2. In addition, this construction consists of its version in dimension  $d = 2$  (see Subsection 3.1) and a subsequent extension of  $W$  and  $f$  to the case of  $d \geq 3$  (see Subsection 3.2).

Theorem 1 directly gives counterexamples to Injectivity 1 and Injectivity 3 of Introduction. Theorem 1 also implies the following counterexample to Injectivity 2 of Introduction:

**Corollary 1.** *For any  $N > 0, \delta > 0$  and  $\alpha \in (0, 1/16)$  there are  $W_\delta, f_\delta$  such that*

$$W_\delta \text{ satisfies (1.4), } d = 2, \quad (3.6)$$

$$W_\delta \in C^\alpha(\mathbb{R}^2 \times \mathbb{S}^1), \|W_\delta\|_{C^\alpha(\mathbb{R}^2 \times \mathbb{S}^1)} \leq N, \quad (3.7)$$

$$f_\delta \in C^\infty(\mathbb{R}^2), f_\delta \not\equiv 0, \text{supp } f_\delta \subseteq \overline{B}(0, \delta), d = 2, \quad (3.8)$$

$$P_{W_\delta} f_\delta \equiv 0 \text{ on } T\mathbb{S}^1. \quad (3.9)$$

The construction of  $W_\delta, f_\delta$  proving Corollary 1 is presented in Subsection 4.1.

#### 3.1 Construction of $f$ and $W$ for $d = 2$

In dimension  $d = 2$ , the construction of  $f$  and  $W$  adopts and develops considerations of [Bom93] and [GN17]. In particular, we construct  $f$ , first, and then  $W$  (in this construction we use notations of Section 2 for  $d = 2$ ).

**Construction of  $f$ .** The function  $f$  is constructed as follows:

$$f = \sum_{k=1}^{\infty} \frac{f_k}{k!}, \quad (3.10)$$

$$f_k(x) = \tilde{f}_k(|x|) = \Phi(2^k(1 - |x|)) \cos(8^k|x|^2), x \in \mathbb{R}^2, k \in \mathbb{N}, \quad (3.11)$$

for arbitrary  $\Phi \in C^\infty(\mathbb{R})$  such that

$$\text{supp } \Phi = [4/5, 6/5], \quad (3.12)$$

$$0 < \Phi(t) \leq 1 \text{ for } t \in (4/5, 6/5), \quad (3.13)$$

$$\Phi(t) = 1, \text{ for } t \in [9/10, 11/10], \quad (3.14)$$

$$\begin{aligned} &\Phi \text{ monotonously increases on } [4/5, 9/10] \\ &\text{and monotonously decreases on } [11/10, 6/5]. \end{aligned} \quad (3.15)$$

Properties (3.12), (3.13) imply that functions  $\tilde{f}_k$  (and functions  $f_k$ ) in (3.11) have disjoint supports:

$$\begin{aligned} &\text{supp } \tilde{f}_i \cap \text{supp } \tilde{f}_j = \emptyset \text{ if } i \neq j, \\ &\text{supp } \tilde{f}_k = [1 - 2^{-k} \left(\frac{6}{5}\right), 1 - 2^{-k} \left(\frac{4}{5}\right)], i, j, k \in \mathbb{N}. \end{aligned} \quad (3.16)$$

This implies the convergence of series in (3.10) for every fixed  $x \in \mathbb{R}^2$ .

**Lemma 1.** *Let  $f$  be defined by (3.10)-(3.14). Then  $f$  is spherically symmetric,  $f \in C_0^\infty(\mathbb{R}^2)$  and  $\text{supp } f \subseteq \overline{B}$ . In addition, if  $\gamma \in T\mathbb{S}^1, \gamma \cap B \neq \emptyset$ , then  $f|_\gamma \not\equiv 0$  and  $f|_\gamma$  has non-constant sign.*

Lemma 1 is similar to Lemma 1 of [GN17] and it is, actually, proved in Section 4.1 of [GN17].

**Construction of  $W$ .** In our example  $W$  is of the following form:

$$\begin{aligned} W(x, \theta) &= \phi_1(x) \left( \sum_{i=0}^N \xi_i(r(x, \theta)) W_i(x, \theta) \right) + \phi_2(x) \\ &= \phi_1(x) \left( \xi_0(r(x, \theta)) W_0(x, \theta) + \sum_{i=1}^N \xi_i(r(x, \theta)) W_i(x, \theta) \right) + \phi_2(x), \quad (x, \theta) \in \Omega, \end{aligned} \quad (3.17)$$

where

$$\begin{aligned} \phi_1, \phi_2 &\text{ is a } C^\infty\text{-smooth partition of unity on } \mathbb{R}^2 \text{ such that,} \\ \phi_1 &\equiv 0 \text{ for } |x| \geq R > 1, \quad \phi_1 \equiv 1 \text{ for } |x| \leq 1, \end{aligned} \quad (3.18)$$

$$\begin{aligned} \phi_2 &\equiv 0 \text{ for } |x| \leq 1, \\ \{\xi_i(s), s \in \mathbb{R}\}_{i=0}^N &\text{ is a } C^\infty\text{- smooth partition of unity on } \mathbb{R}, \end{aligned} \quad (3.19)$$

$$\xi_i(s) = \xi_i(-s), \quad s \in \mathbb{R}, \quad i = \overline{0, N}, \quad (3.20)$$

$$\begin{aligned} W_i(x, \theta) &\text{ are bounded, continuous, strictly positive and rotation invariant} \\ &\text{(according to (1.6)), (2.19) on the open vicinities of } \text{supp } \xi_i(r(x, \theta)), \quad i = \overline{0, N}, \text{ respectively.} \end{aligned} \quad (3.21)$$

From the result of Lemma 1 and from (3.18) it follows that

$$P_W f(x, \theta) = \xi_0(|x|) P_{W_0} f(x, \theta) + \sum_{i=1}^N \xi_i(|x|) P_{W_i} f(x, \theta), \quad (x, \theta) \in TS^1, \quad (3.22)$$

where  $W$  is given by (3.17).

From (3.17)-(3.21) it follows that  $W$  of (3.17) satisfies the conditions (1.4), (2.18), (2.19).

The weight  $W_0$  is constructed in next paragraph and has the following properties:

$$W_0 \text{ is bounded, continuous and rotation invariant on } \Omega(1/2, +\infty), \quad (3.23)$$

$$W_0 \in C^\infty(\Omega((1/2, 1) \cup (1, +\infty))) \text{ and } W_0 \in C^\alpha(\Omega(1/2, +\infty)) \text{ for } \alpha \in (0, 1/16), \quad (3.24)$$

there exists  $\delta_0 \in (1/2, 1)$  such that:

$$W_0(x, \theta) \geq 1/2 \text{ if } r(x, \theta) > \delta_0, \quad (3.25)$$

$$W_0(x, \theta) = 1 \quad \text{if } r(x, \theta) \geq 1,$$

$$P_{W_0} f(x, \theta) = 0 \text{ on } \Omega((1/2, +\infty)), \quad (3.26)$$

where  $P_{W_0}$  is defined according to (1.1) for  $W = W_0$ ,  $f$  is given by (3.10), (3.11).

In addition,

$$\text{supp } \xi_0 \subset (-\infty, -\delta_0) \cup (\delta_0, +\infty), \quad (3.27)$$

$$\xi_0(s) = 1 \text{ for } |s| \geq 1, \quad (3.28)$$

where  $\delta_0$  is the number of (3.25).

In particular, from (3.25), (3.27) it follows that

$$W_0(x, \theta) \xi_0(r(x, \theta)) > 0 \text{ if } \xi_0(r(x, \theta)) > 0. \quad (3.29)$$

In addition,

$$\xi_i(r(x, \theta)) W_i(x, \theta) \text{ are bounded, rotation invariant and } C^\infty \text{ on } \Omega, \quad (3.30)$$

$$W_i(x, \theta) \geq 1/2 \text{ if } \xi_i(r(x, \theta)) \neq 0, \quad (3.31)$$

$$P_{W_i} f(x, \theta) = 0 \text{ on } (x, \theta) \in TS^1, \text{ such that } \xi_i(r(x, \theta)) \neq 0, \quad (3.32)$$

$$i = \overline{1, N}, \quad (x, \theta) \in \Omega.$$

Weights  $W_1, \dots, W_N$  of (3.17) and  $\{\xi_i\}_{i=0}^N$  are constructed in Subsection 3.1.

Theorem 1 for  $d = 2$  follows from Lemma 1 and formulas (3.17)-(3.26), (3.29)-(3.32).

We point out that the construction of  $W_0$  of (3.17) is substantially different from the construction of  $W_1, \dots, W_N$ . The weight  $W_0$  is defined for the rays  $\gamma \in TS^1$  which can be close to the boundary  $\partial B$  of  $B$  which results in restrictions on global smoothness of  $W_0$ .

**Construction of  $W_0$ .** Let

$$\{\psi_k\}_{k=1}^{\infty} \text{ be a } C^{\infty} \text{ partition of unity on } (1/2, 1), \text{ such that } \text{supp } \psi_k \subset (1 - 2^{-k+1}, 1 - 2^{-k-1}), k \in \mathbb{N}, \quad (3.33)$$

$$\text{first derivatives } \psi'_k \text{ satisfy the bounds: } \sup |\psi'_k| \leq C2^k, \quad (3.34)$$

where  $C$  is a positive constant. Actually, functions  $\{\psi_k\}_{k=1}^{\infty}$  satisfying (3.33), (3.34) were used in considerations of [Bom93].

Note that

$$1 - 2^{-(k-2)-1} < 1 - 2^{-k}(6/5), k \geq 3. \quad (3.35)$$

Therefore,

$$\forall s_0, t_0 \in \mathbb{R} : s_0 \in \text{supp } \psi_{k-2}, t_0 \in \text{supp } \Phi(2^k(1-t)) \Rightarrow s_0 < t_0, k \geq 3. \quad (3.36)$$

Weight  $W_0$  is defined by the following formulas

$$W_0(x, \theta) = \begin{cases} 1 - G(x, \theta) \sum_{k=3}^{\infty} k! f_k(x) \frac{\psi_{k-2}(r(x, \theta))}{H_k(x, \theta)}, & 1/2 < r(x, \theta) < 1, \\ 1, & r(x, \theta) \geq 1 \end{cases}, \quad (3.37)$$

$$G(x, \theta) = \int_{\gamma(x, \theta)} f(y) dy, H_k(x, \theta) = \int_{\gamma(x, \theta)} f_k^2(y) dy, x \in \mathbb{R}^2, \theta \in \mathbb{S}^1, \quad (3.38)$$

where  $f, f_k$  are defined in (3.10), (3.11), respectively, rays  $\gamma(x, \theta)$  are given by (2.13).

Formula (3.37) implies that  $W_0$  is defined on  $\Omega_0(1/2) \subset \Omega$ .

Due to (3.11)-(3.14), (3.33), (3.36), in (3.38) we have that

$$H_k(x, \theta) \neq 0 \text{ if } \psi_{k-2}(r(x, \theta)) \neq 0, (x, \theta) \in \Omega, \quad (3.39)$$

$$\frac{\psi_{k-2}(r(x, \theta))}{H_k(x, \theta)} \in C^{\infty}(\Omega(1/2, 1)), \quad (3.40)$$

where  $r(x, \theta)$  is defined in (2.2),  $\Omega, \Omega(\cdot)$  are defined in (2.1), (2.5),  $d = 2$ .

Also, for any fixed  $(x, \theta) \in \Omega, 1/2 < r(x, \theta)$ , the series in the right hand-side of (3.37) has only a finite number of non-zero terms (in fact, no more than two) and, hence, the weight  $W_0$  is well-defined.

By the spherical symmetry of  $f$ , functions  $G, H_k$  in (3.37) are of the type (1.6) (and (2.18)). Therefore,  $W_0$  is rotation invariant (in the sense of (1.6) and (2.18)).

Actually, formula (3.26) follows from (3.10), (3.11), (3.37), (3.38) (see Subsection 5.2 for details).

Using the construction of  $W_0$  and the assumption that  $r(x, \theta) > 1/2$  one can see that  $W_0$  is  $C^{\infty}$  on its domain of definition, possibly, except points with  $r(x, \theta) = 1$ .

Note also that due to (3.10), (3.11), the functions  $f_k, G, H_k$ , used in (3.37), (3.38) can be considered as functions of one-dimensional arguments.

Formulas (3.23)-(3.25) are proved in Subsection 5.1.

## Construction of $W_1, \dots, W_N$ and $\xi_0, \dots, \xi_N$

**Lemma 2.** *Let  $f \in C_0^{\infty}(\mathbb{R}^2)$  be spherically symmetric,  $(x_0, \theta_0) \in T\mathbb{S}^1, f|_{\gamma(x_0, \theta_0)} \neq 0$  and  $f|_{\gamma(x_0, \theta_0)}$  changes the sign. Then there exist  $\varepsilon_0 > 0$  and weight  $W_{(x_0, \theta_0), \varepsilon_0}$  such that*

$$P_{W_{(x_0, \theta_0), \varepsilon_0}} f = 0 \text{ on } \Omega(\mathcal{J}_{r(x_0, \theta_0), \varepsilon_0}), \quad (3.41)$$

$$\begin{aligned} W_{(x_0, \theta_0), \varepsilon_0} & \text{ is bounded, infinitely smooth,} \\ & \text{strictly positive and rotation invariant on } \Omega(\mathcal{J}_{r(x_0, \theta_0), \varepsilon_0}), \end{aligned} \quad (3.42)$$

where  $\Omega(\mathcal{J}_{r, \varepsilon_0}), \mathcal{J}_{r, \varepsilon_0}$  are defined in (2.5) and (2.9), respectively.

Lemma 2 is proved in Section 6.

Let  $f$  be the function of (3.10), (3.11). Then, using Lemmas 1, 2 one can see that

$$\begin{aligned} \forall \delta \in (0, 1) \text{ there exist } \{J_i = \mathcal{J}_{r_i, \varepsilon_i}, W_i = W_{(x_i, \theta_i), \varepsilon_i}\}_{i=1}^N \\ \text{such that } J_i, i = \overline{1, N}, \text{ is an open cover of } [0, \delta] \\ \text{and } W_i \text{ satisfy (3.41) and (3.42) on } \Omega(J_i), \text{ respectively.} \end{aligned} \quad (3.43)$$

Actually, we consider (3.43) for the case of  $\delta = \delta_0$  of (3.25).

Note that in this case  $\{\Omega(J_i)\}_{i=1}^N$  for  $J_i$  of (3.43) is the open cover of  $\Omega_1(\delta_0)$ .

To the set  $\Omega_0(\delta_0)$  we associate the open set

$$J_0 = (-\infty, \delta_0) \cup (\delta_0, +\infty) \subset \mathbb{R}. \quad (3.44)$$

Therefore, the collection of intervals  $\{\pm J_i, i = \overline{0, N}\}$  is an open cover of  $\mathbb{R}$ , where  $-J_i$  is the symmetrical reflection of  $J_i$  with respect to  $\{0\} \in \mathbb{R}$ .

We construct the partition of unity  $\{\xi_i\}_{i=0}^N$  as follows:

$$\xi_i(s) = \xi_i(|s|) = \frac{1}{2}(\tilde{\xi}_i(s) + \tilde{\xi}_i(-s)), \quad s \in \mathbb{R}, \quad (3.45)$$

$$\text{supp } \xi_i \subset J_i \cup (-J_i), \quad i = \overline{0, N}, \quad (3.46)$$

where  $\{\tilde{\xi}_i\}_{i=0}^N$  is a partition of unity for the open cover  $\{J_i \cup (-J_i)\}_{i=0}^N$  (see Section 2, Partition of unity, for  $U_i = J_i$ ).

Properties (3.27), (3.46) follow from (2.20) for  $\{\tilde{\xi}_i\}_{i=0}^N$  with  $U_i = J_i \cup (-J_i)$ , the symmetry of  $J_i \cup (-J_i)$ ,  $i = \overline{1, N}$ , choice of  $J_0$  in (3.44) and from (3.45).

In turn, (3.28) follows from (3.44) and the construction of  $J_i$ ,  $i = \overline{1, N}$ , from (3.43) (see the proof of Lemma 2 and properties (3.43) in Section 6 for details).

Properties (3.30)-(3.32) follow from (3.43) for  $\delta = \delta_0$  and from (3.44)-(3.46).

This completes the description of  $W_1, \dots, W_N$  and  $\{\xi_i\}_{i=0}^N$ .

### 3.2 Construction of $W$ and $f$ for $d \geq 3$

Consider  $f$  and  $W$  of Theorem 1, for  $d = 2$ , constructed in Subsection 3.1. For these  $f$  and  $W$  consider  $\tilde{f}$  and  $\tilde{U}$  such that

$$f(x) = \tilde{f}(|x|), \quad W(x, \theta) = \tilde{U}(|x|, |x\theta|), \quad x \in \mathbb{R}^2, \quad \theta \in \mathbb{S}^1. \quad (3.47)$$

**Proposition 1.** *Let  $W$  and  $f$ , for  $d \geq 3$ , be defined as*

$$W(x, \theta) = \tilde{U}(|x|, |x\theta|), \quad (x, \theta) \in \mathbb{R}^d \times \mathbb{S}^{d-1}, \quad (3.48)$$

$$f(x) = \tilde{f}(|x|), \quad x \in \mathbb{R}^d, \quad (3.49)$$

where  $\tilde{U}, \tilde{f}$  are the functions of (3.47). Then

$$P_W f \equiv 0 \text{ on } T\mathbb{S}^{d-1}. \quad (3.50)$$

In addition, weight  $W$  satisfies properties (3.2)-(3.5),  $f$  is spherically symmetric infinitely smooth and compactly supported on  $\mathbb{R}^d$ ,  $f \neq 0$ .

Proposition 1 is proved in Subsection 4.2.

This completes the proof of Theorem 1.

## 4 Proofs of Corollary 1 and Proposition 1

### 4.1 Proof of Corollary 1

Let

$$X_r = \{x_1 e_1 + x_2 e_2 + r e_3 : (x_1, x_2) \in \mathbb{R}^2\}, \quad 0 \leq r < 1, \quad (4.1)$$

$$S = X_0 \cap \mathbb{S}^2 = \{(\cos \phi, \sin \phi, 0) \in \mathbb{R}^3 : \phi \in [0, 2\pi)\} \simeq \mathbb{S}^1. \quad (4.2)$$

where  $(e_1, e_2, e_3)$  is the standard orthonormal basis in  $\mathbb{R}^3$ .

Without loss of generality we assume that  $0 < \delta < 1$ . Choosing  $r$  so that  $\sqrt{1 - \delta^2} \leq r < 1$ , we have that the intersection of the three dimensional ball  $B(0, 1)$  with  $X_r$  is the two-dimensional disk  $B(0, \delta')$ ,  $\delta' \leq \delta$  (with respect to the coordinates  $(x_1, x_2)$  induced by basis  $(e_1, e_2)$  on  $X_r$ ).

We define  $W_\delta$  on  $\mathbb{R}^2 \times \mathbb{S}^1$  and  $f_\delta$  on  $\mathbb{R}^2$  as follows:

$$W_\delta := \frac{N}{\|W\|_{C^\alpha(\mathbb{R}^3 \times \mathbb{S}^2)}} W|_{X_r \times S}, \quad (4.3)$$

$$f_\delta := f|_{X_r}, \quad (4.4)$$

$$\text{for } r = \sqrt{1 - \delta^2},$$

where  $W$  and  $f$  are the functions of Theorem 1 for  $d = 3$ .

Due to (3.2), (3.3), (3.4), (4.3) we have that

$$W_\delta \geq \frac{N}{2\|W\|_{C^\alpha(\mathbb{R}^2 \times \mathbb{S}^1)}}, \quad \|W_\delta\|_{C^\alpha(\mathbb{R}^2 \times \mathbb{S}^1)} \leq N. \quad (4.5)$$

Properties (4.5) imply (3.6), (3.7).

In view of Lemma 1 for the function  $f$  of Theorem 1, we have that  $f_\delta$  is spherically symmetric,  $f_\delta \in C_0^\infty(B(0, \delta'))$ ,  $f_\delta \neq 0$ .

Using (3.1), (4.3), (4.4) one can see that (3.9) holds.

This completes the proof of Corollary 1.



## 4.2 Proof of Proposition 1

Let

$$I(r) = \int_{\gamma_r} \tilde{U}(|y|, r) \tilde{f}(|y|) dy, \quad r \geq 0, \quad \gamma_r = \gamma(re_2, e_1), \quad (4.6)$$

where  $\gamma(x, \theta)$  is defined by (1.3),  $(e_1, \dots, e_d)$  is the standard basis in  $\mathbb{R}^d$ .

Due to formula (3.1) of Theorem 1 for  $d = 2$  and formulas (3.47), (4.6) we have that

$$I(r) = P_W f(re_2, e_1) = 0 \text{ for } r \geq 0. \quad (4.7)$$

Next, using (1.1), (3.47), (4.7) we have also that

$$P_W f(x, \theta) = \int_{\gamma(x, \theta)} \tilde{U}(|y|, |y - (y\theta)\theta|) \tilde{f}(|y|) dy = I(|x|) = 0 \text{ for } (x, \theta) \in T\mathbb{S}^{d-1}, \quad (4.8)$$

where  $\gamma(x, \theta)$  is defined in (1.3).

Formula (4.8) implies (3.50). Properties of  $W$  and  $f$  mentioned in Proposition 1 follow from properties (3.2)-(3.5) of  $W$  and of  $f$  of Theorem 1 for  $d = 2$ .

This completes the proof of Proposition 1.

## 5 Proofs of formulas (3.23)-(3.26)

### 5.1 Proof of formulas (3.23)-(3.25)

**Lemma 3.** *Let  $W_0$  be defined by (3.37), (3.38). Then  $W_0$  admits the following representation:*

$$W_0(x, \theta) = U_0(x\theta, |x - (x\theta)\theta|), \quad (x, \theta) \in \Omega((1/2, +\infty)), \quad (5.1)$$

$$U_0(s, r) = \begin{cases} 1 - \tilde{G}(r) \sum_{k=3}^{\infty} k! \tilde{f}_k((s^2 + r^2)^{1/2}) \frac{\psi_{k-2}(r)}{\tilde{H}_k(r)}, & 1/2 < r < 1, \\ 1, & r \geq 1 \end{cases}, \quad (5.2)$$

$$\tilde{G}(r) \stackrel{def}{=} \int_{\gamma_r} \tilde{f}(|y|) dy, \quad \tilde{H}_k(r) \stackrel{def}{=} \int_{\gamma_r} \tilde{f}_k^2(|y|) dy, \quad \tilde{f} = \sum_{k=1}^{\infty} \frac{\tilde{f}_k}{k!}, \quad (5.3)$$

$$s \in \mathbb{R}, \quad x \in \mathbb{R}^2, \quad \gamma_r \text{ is an arbitrary ray in } T(r), \quad r > 1/2,$$

where  $T(r)$  is defined by (2.8),  $d = 2$ ,  $\tilde{f}_k$  are defined by (3.11). In addition:

$$U_0 \text{ is infinitely smooth on } \mathbb{R} \times \{(1/2, 1) \cup (1, +\infty)\}, \quad (5.4)$$

$$U_0(s, r) \rightarrow 1 \text{ as } r \rightarrow 1 \text{ (uniformly in } s \in \mathbb{R}), \quad (5.5)$$

$$U_0(s, r) = 1 \text{ if } s^2 + r^2 \geq 1, \quad (5.6)$$

$$|1 - U_0(s, r)| \leq C_0(1 - r)^{1/2} \log_2^4 \left( \frac{1}{1 - r} \right), \quad (5.7)$$

$$\text{for } s \in \mathbb{R}, \quad 1/2 < r < 1,$$

$$|U_0(s, r) - U_0(s', r')| \leq C_1 |s - s'|^{1/\alpha} + C_1 |r - r'|^{1/\alpha}, \quad (5.8)$$

$$\text{for } \alpha \in (0, 1/16), \quad s, s' \in \mathbb{R}, \quad r, r' > 1/2,$$

where  $C_0, C_1$  are positive constants depending on  $\Phi$  of (3.12)-(3.14).

Lemma 3 is proved Section 7.

Lemma 3 implies (3.23)-(3.25) as follows.

The continuity and rotation invariancy of  $W_0$  in (3.23) follow from (1.6), (1.7), (5.1), (5.8).

Due to (3.33), (5.1), (5.2), (5.3) we have also that

$$U_0 \text{ admits a continuous extension to } \mathbb{R} \times [1/2, +\infty). \quad (5.9)$$

Properties (5.6), (5.9) imply the boundedness of  $W_0$  on  $\Omega_0(1/2)$ , where  $\Omega_0(\cdot)$  is defined in (2.3),  $d = 2$ . This completes the proof of (3.23).

Formula (3.24) follows from (5.1), (5.4), (5.8) and from the fact that  $x\theta$ ,  $|x - (x\theta)\theta|$  are infinitely smooth functions on  $\Omega_0(1/2)$  and are Lipschitz in  $(x, \theta)$  for  $x \in \overline{B}(0, R)$ ,  $R > 1$ .

Formula (3.25) follows from (3.23), (5.1), (5.2), (5.5), (5.6).

This completes the proof of (3.23)-(3.25).

## 5.2 Proof of formula (3.26)

From (1.1), (3.10)-(3.13), (3.33), (3.37), (3.38) it follows that:

$$P_{W_0} f(x, \theta) = \int_{\gamma(x, \theta)} f(y) dy - G(x, \theta) \sum_{k=3}^{\infty} k! \psi_{k-2}(r(x, \theta)) \frac{\int_{\gamma(x, \theta)} f(y) f_k(y) dy}{H_k(x, \theta)} \quad (5.10)$$

$$= \int_{\gamma(x, \theta)} f(y) dy - \int_{\gamma(x, \theta)} f(y) dy \sum_{k=3}^{\infty} \psi_{k-2}(r(x, \theta)) \frac{\int_{\gamma(x, \theta)} f_k^2(y) dy}{\int_{\gamma(x, \theta)} f_k^2(y) dy} \quad (5.11)$$

$$= \int_{\gamma(x, \theta)} f(y) dy - \int_{\gamma(x, \theta)} f(y) dy \sum_{k=3}^{\infty} \psi_{k-2}(r(x, \theta)) = 0 \text{ for } (x, \theta) \in \Omega_0(1/2), \quad (5.12)$$

where  $\gamma(x, \theta)$  is defined in (1.3),  $\Omega_0(\cdot)$  is defined in (2.3),  $d = 2$ .

Formula (3.26) is proved.

## 6 Proof of Lemma 2

By  $u \in \mathbb{R}$  we denote the coordinates on a fixed ray  $\gamma(x, \theta)$ ,  $(x, \theta) \in \Omega$ ,  $d = 2$ , taking into account the orientation, where  $u = 0$  at the point  $x - (x\theta)\theta \in \gamma(x, \theta)$ ; see notation (2.13).

Using Lemma 1, one can see that

$$f|_{\gamma(x, \theta)} \in C_0^\infty(\mathbb{R}), f|_{\gamma(x, \theta)}(u) = f|_{\gamma(x, \theta)}(|u|), u \in \mathbb{R}. \quad (6.1)$$

Using (6.1) and the assumption that  $f|_{\gamma(x_0, \theta_0)}(u)$  changes the sign, one can see that there exists  $\psi_{(x_0, \theta_0)}$  such that

$$\psi_{(x_0, \theta_0)} \in C_0^\infty(\mathbb{R}), \psi_{(x_0, \theta_0)} \geq 0, \psi_{(x_0, \theta_0)}(u) = \psi_{(x_0, \theta_0)}(|u|), u \in \mathbb{R}, \quad (6.2)$$

$$\int_{\gamma(x_0, \theta_0)} f \psi_{(x_0, \theta_0)} d\sigma \neq 0, \quad (6.3)$$

and if

$$\int_{\gamma(x_0, \theta_0)} f d\sigma \neq 0 \quad (6.4)$$

then also

$$\text{sgn}\left(\int_{\gamma(x_0, \theta_0)} f d\sigma\right) \text{sgn}\left(\int_{\gamma(x_0, \theta_0)} f \psi_{(x_0, \theta_0)} d\sigma\right) = -1, \quad (6.5)$$

where  $d\sigma = du$  (i.e.,  $\sigma$  is the standard Euclidean measure on  $\gamma(x, \theta)$ ).

Let

$$W_{(x_0, \theta_0)}(x, \theta) = 1 - \psi_{(x_0, \theta_0)}(x\theta) \frac{\int_{\gamma(x, \theta)} f d\sigma}{\int_{\gamma(x, \theta)} f \psi_{(x_0, \theta_0)} d\sigma}, x \in \mathbb{R}^2, \theta \in \mathbb{S}^1, \quad (6.6)$$

where  $d\sigma = du$ , where  $u$  is the coordinate on  $\gamma(x, \theta)$ .

Lemma 1 and property (6.2) imply that

$$\int_{\gamma(x, \theta)} f d\sigma \text{ and } \int_{\gamma(x, \theta)} f \psi_{(x_0, \theta_0)} d\sigma \text{ depend only on } r(x, \theta), \text{ where } (x, \theta) \in \Omega, \quad (6.7)$$

where  $r(x, \theta)$  is defined in (2.2),  $\Omega$  is defined in (2.1),  $d = 2$ .

From (6.2), (6.6), (6.7) it follows that  $W_{(x_0, \theta_0)}$  is rotation-invariant in the sense (1.6).

Formulas (6.3), (6.6), (6.7), properties of  $f$  of Lemma 1 and properties of  $\psi_{(x_0, \theta_0)}$  of (6.2) imply that

$$\exists \varepsilon_1 > 0 : \int_{\gamma(x, \theta)} f \psi_{(x_0, \theta_0)} d\sigma \neq 0 \text{ for } (x, \theta) \in \Omega(\mathcal{J}_{r(x_0, \theta_0), \varepsilon_1}), \quad (6.8)$$

where sets  $\Omega(\mathcal{J}_{s, \varepsilon})$ ,  $\mathcal{J}_{s, \varepsilon}$  are defined in (2.5), (2.9), respectively.

In addition, using properties of  $f$  of Lemma 1 and also using (3.10), (3.16), (6.2), (6.6), (6.8), one can see that

$$W_{(x_0, \theta_0)} \in C^\infty(\Omega(\mathcal{J}_{r(x_0, \theta_0), \varepsilon_1})). \quad (6.9)$$

In addition, from (6.1)-(6.7) it follows that

$$\begin{aligned} \text{if } r(x, \theta) = r(x_0, \theta_0) \text{ then } W_{(x_0, \theta_0)}(x, \theta) &= 1 - \psi_{(x_0, \theta_0)}(x\theta) \frac{\int_{\gamma(x_0, \theta_0)} f d\sigma}{\int_{\gamma(x_0, \theta_0)} f \psi_{(x_0, \theta_0)} d\sigma} \\ &= 1 - \psi_{(x_0, \theta_0)}(x\theta) \frac{\int_{\gamma(x_0, \theta_0)} f d\sigma}{\int_{\gamma(x_0, \theta_0)} f \psi_{(x_0, \theta_0)} d\sigma} \geq 1, \end{aligned} \quad (6.10)$$

where  $r(x, \theta)$  is defined in (2.2),  $d = 2$ .

From properties of  $f$  of Lemma 1, properties of  $\psi_{(x_0, \theta_0)}$  of (6.2) and from formulas (6.6), (6.8), (6.9), (6.10) it follows that

$$\exists \varepsilon_0 > 0 (\varepsilon_0 < \varepsilon_1) : W_{(x_0, \theta_0)}(x, \theta) \geq 1/2 \text{ for } (x, \theta) \in \Omega(\mathcal{J}_{r(x_0, \theta_0), \varepsilon_0}). \quad (6.11)$$

Let

$$W_{(x_0, \theta_0), \varepsilon_0} := W_{(x_0, \theta_0)} \text{ for } (x, \theta) \in \Omega(\mathcal{J}_{r(x_0, \theta_0), \varepsilon_0}), \quad (6.12)$$

where  $W_{(x_0, \theta_0)}$  is defined in (6.6).

Properties (6.7), (6.9), (6.11) imply (3.42) for  $W_{(x_0, \theta_0), \varepsilon_0}$  of (6.12).

Using (1.1), (6.6), (6.8), (6.12) one can see that

$$\begin{aligned} P_{W_{(x_0, \theta_0), \varepsilon_0}} f(x, \theta) &= \int_{\gamma(x, \theta)} W_{(x_0, \theta_0)}(\cdot, \theta) f d\sigma \\ &= \int_{\gamma(x, \theta)} f d\sigma - \frac{\int_{\gamma(x, \theta)} f d\sigma}{\int_{\gamma(x, \theta)} f \psi_{(x_0, \theta_0)} d\sigma} \int_{\gamma(x, \theta)} f \psi_{(x_0, \theta_0)} d\sigma = 0 \text{ for } (x, \theta) \in \Omega(\mathcal{J}_{r(x_0, \theta_0), \varepsilon_0}), \end{aligned} \quad (6.13)$$

where  $\Omega(\cdot)$  is defined in (2.5),  $d = 2$ ,  $\mathcal{J}_{r, \varepsilon}$  is defined in (2.9). Formula (3.41) follows from (6.13).

Lemma 2 is proved.

## 7 Proof of Lemma 3

**Proof of (5.1)-(5.3).** Using (2.2), (3.10), (3.11), (3.38), (5.3) we obtain

$$G(x, \theta) = \tilde{G}(r(x, \theta)) = \int_{\gamma(x, \theta)} f(x) dx, \quad (7.1)$$

$$H_k(x, \theta) = \tilde{H}_k(r(x, \theta)) = \int_{\gamma(x, \theta)} f_k^2(x) dx, \quad (7.2)$$

$$\tilde{f}_k(|x|) = \tilde{f}_k((|x\theta|^2 + |x - (x\theta)\theta|^2)^{1/2}), (x, \theta) \in \Omega_0(1/2), \quad (7.3)$$

where  $\Omega_0(\cdot)$  is defined in (2.3),  $d = 2$ ,  $\gamma(x, \theta)$  is defined as in (2.13).

Formulas (3.37), (3.38), (7.1)-(7.3) imply (5.1)-(5.3).

**Proof of (5.4).** Let

$$\Lambda_k = (1 - 2^{-k+3}, 1 - 2^{-k+1}), k \in \mathbb{N}, k \geq 4. \quad (7.4)$$

From (3.33) it follows that, for  $k \geq 4$ :

$$\text{supp } \psi_{k-1} \subset (1 - 2^{-k+2}, 1 - 2^{-k}), \quad (7.5)$$

$$\text{supp } \psi_{k-2} \subset (1 - 2^{-k+3}, 1 - 2^{-k+1}) = \Lambda_k, \quad (7.6)$$

$$\text{supp } \psi_{k-3} \subset (1 - 2^{-k+4}, 1 - 2^{-k+2}). \quad (7.7)$$

Due to (5.2), (5.3), (7.5)-(7.7), we have the following formula for  $U_0$ :

$$\begin{aligned} U_0(s, r) &= 1 - \tilde{G}(r) \left( (k-1)! \tilde{f}_{k-1}((s^2 + r^2)^{1/2}) \frac{\psi_{k-3}(r)}{\tilde{H}_{k-1}(r)} \right. \\ &\quad + k! \tilde{f}_k((s^2 + r^2)^{1/2}) \frac{\psi_{k-2}(r)}{\tilde{H}_k(r)} \\ &\quad \left. + (k+1)! \tilde{f}_{k+1}((s^2 + r^2)^{1/2}) \frac{\psi_{k-1}(r)}{\tilde{H}_{k+1}(r)} \right) \text{ for } r \in \Lambda_k, s \in \mathbb{R}, k \geq 4. \end{aligned} \quad (7.8)$$

From (5.3), (7.8) it follows that

$$\begin{aligned} \frac{\partial^n U_0}{\partial s^n}(s, r) = & -\tilde{G}(r) \left( (k-1)! \frac{\partial^n \tilde{f}_{k-1}((s^2+r^2)^{1/2})}{\partial s^n} \frac{\psi_{k-3}(r)}{\tilde{H}_{k-1}(r)} \right. \\ & + k! \frac{\partial^n \tilde{f}_k((s^2+r^2)^{1/2})}{\partial s^n} \frac{\psi_{k-2}(r)}{\tilde{H}_k(r)} \\ & \left. + (k+1)! \frac{\partial^n \tilde{f}_{k+1}((s^2+r^2)^{1/2})}{\partial s^n} \frac{\psi_{k-1}(r)}{\tilde{H}_{k+1}(r)} \right), \end{aligned} \quad (7.9)$$

$$\frac{\partial^n \tilde{G}}{\partial r^n}(r) = \int_{-\infty}^{+\infty} \frac{\partial^n}{\partial r^n} \tilde{f}((s^2+r^2)^{1/2}) ds, \quad \frac{\partial^n \tilde{H}_m}{\partial r^n}(r) = \int_{-\infty}^{+\infty} \frac{\partial^n}{\partial r^n} \tilde{f}_m^2((s^2+r^2)^{1/2}) ds, \quad (7.10)$$

$$r \in \Lambda_k, s \in \mathbb{R}, m \geq 1, n \geq 0, k \geq 4,$$

where  $\tilde{G}, \tilde{H}_m$  are defined in (5.3).

Using Lemma 1 and formulas (3.10), (3.11), (3.33)-(3.40), (5.3) one can see that:

$$\tilde{f}, \tilde{f}_{m-2}, \tilde{G}, \tilde{H}_m \text{ belong to } C_0^\infty(\mathbb{R}), \frac{\psi_{m-2}}{\tilde{H}_m} \text{ belongs to } C_0^\infty((1/2, 1)) \text{ for any } m \geq 3. \quad (7.11)$$

From (7.9)-(7.11) it follows that  $U_0(s, r)$  has continuous partial derivatives of all orders with respect to  $r \in \Lambda_k, s \in \mathbb{R}$ . It implies that  $U_0 \in C^\infty(\mathbb{R} \times \Lambda_k)$ . From the fact that  $\Lambda_k, k \geq 4$ , is an open cover of  $(1/2, 1)$  and from definition (5.2) of  $U_0$ , it follows that  $U_0 \in C^\infty(\mathbb{R} \times \{(1/2, 1) \cup (1, +\infty)\})$ .

This completes the proof of (5.4).

**Proof of (5.6).** From (3.11)-(3.14) it follows that

$$\tilde{f}_k(|x|) = 0 \text{ if } |x| \geq 1 \text{ for } k \in \mathbb{N}. \quad (7.12)$$

Formula  $|x|^2 = |x\theta|^2 + |x - (x\theta)\theta|^2, x \in \mathbb{R}^2, \theta \in \mathbb{S}^1$ , and formulas (5.2), (7.12) imply (5.6).

**Proofs of (5.7)-(5.8).**

**Lemma 4.** *There are positive constants  $c, k_1$  depending on  $\Phi$  of (3.12)-(3.14), such that*

(i) *for all  $k \in \mathbb{N}$  the following estimates hold:*

$$|\tilde{f}_k| \leq 1, \quad (7.13)$$

$$|\tilde{f}'_k| \leq c8^k, \quad (7.14)$$

where  $\tilde{f}'_k$  denotes the derivative of  $\tilde{f}_k$  defined in (5.3).

(ii) *for  $k \geq k_1$  and  $1/2 < r \leq 1$  the following estimates hold:*

$$\left| \frac{\psi_{k-2}(r)}{\tilde{H}_k(r)} \right| \leq c2^k, \quad (7.15)$$

$$\left| \frac{d}{dr} \left( \frac{\psi_{k-2}(r)}{\tilde{H}_k(r)} \right) \right| \leq c2^{5k}, \quad (7.16)$$

where  $\psi_k$  are defined in (3.33),  $\tilde{H}_k$  is defined in (5.3).

(iii) *for  $k \geq 3$  and  $r \geq 1 - 2^{-k}$  the following estimates hold:*

$$|\tilde{G}(r)| \leq c \frac{(2\sqrt{2})^{-k}}{k!}, \quad (7.17)$$

$$\left| \frac{d\tilde{G}}{dr}(r) \right| \leq c \frac{8^k}{k!}, \quad (7.18)$$

where  $\tilde{G}$  is defined in (5.3).

**Lemma 5.** *Let  $U_0$  be defined by (5.2)-(5.3). Then the following estimates are valid:*

$$\left| \frac{\partial U_0}{\partial s}(s, r) \right| \leq \frac{C}{(1-r)^3}, \quad \left| \frac{\partial U_0}{\partial r}(s, r) \right| \leq \frac{C}{(1-r)^5} \text{ for } s \in \mathbb{R}, r \in (1/2, 1), \quad (7.19)$$

where  $C$  is a constant depending only on  $\Phi$  of (3.12)-(3.14).

Lemmas 4, 5 are proved in Subsections 8.1, 8.2, respectively.

**Proof of (5.7).** From (7.15), (7.17) it follows that

$$|\tilde{G}(r)| \leq c(2\sqrt{2})^{-k+3}/(k-3)!, \quad (7.20)$$

$$\left| \frac{\psi_{k-2}(r)}{\tilde{H}_k(r)} \right| \leq c2^k, \quad (7.21)$$

for  $r \in \Lambda_k$ ,  $k \geq \max(4, k_1)$ ,

where  $\Lambda_k$  is defined in (7.4).

Properties (7.5)-(7.7) and estimate (7.15) imply that

$$\begin{cases} \psi_{k-1}(r) = 0, \\ \left| \frac{\psi_{k-3}(r)}{\tilde{H}_{k-1}(r)} \right| \leq c2^{k-1} \end{cases} \quad \text{if } r \in (1 - 2^{-k+3}, 1 - 2^{-k+2}), \quad (7.22)$$

$$\begin{cases} \psi_{k-2}(r) = 0, \\ \left| \frac{\psi_{k-1}(r)}{\tilde{H}_{k+1}(r)} \right| \leq c2^{k+1} \end{cases} \quad \text{if } r \in (1 - 2^{-k+2}, 1 - 2^{-k+1}), \quad (7.23)$$

$$\begin{cases} \psi_{k-1}(r) = 0, \\ \psi_{k-3}(r) = 0 \end{cases} \quad \text{if } r = 1 - 2^{-k+2}, \quad (7.24)$$

for  $k \geq \max(4, k_1)$ .

Note that the assumption that  $r \in \Lambda_k$  is splitted into the assumptions on  $r$  of (7.22), (7.23), (7.24).

Using formulas (7.8), (7.20)-(7.24), we obtain the following estimates:

$$\begin{aligned} |1 - U_0(s, r)| &= |\tilde{G}(r)| \left| (k-1)! \tilde{f}_{k-1}((s^2 + r^2)^{1/2}) \frac{\psi_{k-3}(r)}{\tilde{H}_{k-1}(r)} + k! \tilde{f}_k((s^2 + r^2)^{1/2}) \frac{\psi_{k-2}(r)}{\tilde{H}_k(r)} \right| \\ &\leq c(2\sqrt{2})^{-k+3} (c(k-2)(k-1)2^{k-1} + c(k-2)(k-1)k2^k) \\ &\leq 2^5 \sqrt{2} c^2 2^{-k/2} k^3 \quad \text{if } r \in (1 - 2^{-k+3}, 1 - 2^{-k+2}), \end{aligned} \quad (7.25)$$

$$\begin{aligned} |1 - U_0(s, r)| &= |\tilde{G}(r)| \left| k! \tilde{f}_k((s^2 + r^2)^{1/2}) \frac{\psi_{k-2}(r)}{\tilde{H}_k(r)} + (k+1)! \tilde{f}_{k+1}((s^2 + r^2)^{1/2}) \frac{\psi_{k-1}(r)}{\tilde{H}_{k+1}(r)} \right| \\ &\leq c(2\sqrt{2})^{-k+3} (c2^k(k-2)(k-1)k + c2^{k+1}(k-2)(k-1)k(k+1)) \\ &\leq 2^{10} \sqrt{2} c^2 2^{-k/2} k^4 \quad \text{if } r \in (1 - 2^{-k+2}, 1 - 2^{-k+1}), \end{aligned} \quad (7.26)$$

$$\begin{aligned} |1 - U_0(s, r)| &= |\tilde{G}(r)| \left| k! \tilde{f}_k((s^2 + r^2)^{1/2}) \frac{\psi_{k-2}(r)}{\tilde{H}_k(r)} \right| \\ &\leq 2^4 \sqrt{2} c^2 2^{-k/2} k^3 \quad \text{if } r = 1 - 2^{-k+2}, \end{aligned} \quad (7.27)$$

for  $s \in \mathbb{R}$ ,  $k \geq \max(4, k_1)$ . Estimates (7.25)-(7.27) imply that

$$|1 - U_0(s, r)| \leq C 2^{-k/2} k^4, \quad r \in \Lambda_k, \quad s \in \mathbb{R}, \quad k \geq \max(4, k_1), \quad (7.28)$$

where  $C$  is a positive constant depending on  $c$  of Lemma 4.

In addition, for  $r \in \Lambda_k$  we have that  $2^{-k+1} < (1-r) < 2^{-k+3}$ , which together with (7.28) imply (5.7).

This completes the proof of (5.7).

**Proof of (5.8).** We consider the following cases of  $s, s', r, r'$  in (5.8):

1. Let

$$s, s' \in \mathbb{R} \text{ and } r, r' \geq 1. \quad (7.29)$$

Due to (5.2) we have that

$$U_0(s, r) = 1, \quad U_0(s', r') = 1. \quad (7.30)$$

Identities in (7.30) and assumption (7.29) imply (5.8) for this case.

2. Let

$$s, s' \in \mathbb{R}, \quad 1/2 < r < 1 \text{ and } r' \geq 1. \quad (7.31)$$

Then, due to (5.2), (5.7) we have that

$$|1 - U_0(s, r)| \leq C(1-r)^{1/3}, \quad (7.32)$$

$$U_0(s', r') = 1, \quad (7.33)$$

where  $s, s', r, r'$  satisfy assumption (7.31),  $C$  is a constant depending only on  $\Phi$ . In particular, inequality (7.32) follows from (5.7) due to the following simple property of the logarithm:

$$\log_2^a \left( \frac{1}{1-r} \right) \leq C(a, \varepsilon)(1-r)^{-\varepsilon} \text{ for any } \varepsilon > 0, r \in [0, 1), a > 0, \quad (7.34)$$

where  $C(a, \varepsilon)$  is some positive constant depending only on  $a$  and  $\varepsilon$ .

Due to (7.31), (7.32), (7.33) we have that

$$\begin{aligned} |U_0(s', r') - U_0(s, r)| &= |1 - U_0(s, r)| \leq C(1-r)^{1/3} \\ &\leq C|r - r'|^{1/3} \leq C(|r - r'|^{1/3} + |s - s'|^{1/3}), \end{aligned} \quad (7.35)$$

where  $C$  is a constant depending only on  $\Phi$ .

Estimate (7.35) and assumptions (7.31) imply (5.8) for this case.

3. Let

$$s, s' \in \mathbb{R} \text{ and } r, r' \in (1/2, 1). \quad (7.36)$$

In addition, without loss of generality we assume that  $r > r'$ .

Next, using (5.4) one can see that

$$\begin{aligned} |U_0(s, r) - U_0(s', r')| &= |U_0(s, r) - U_0(s', r) + U_0(s', r) - U_0(s', r')| \\ &\leq |U_0(s, r) - U_0(s', r)| + |U_0(s', r) - U_0(s', r')| \\ &\leq \left| \frac{\partial U_0}{\partial s}(\hat{s}, r) \right| |s - s'| + \left| \frac{\partial U_0}{\partial r}(s', \hat{r}) \right| |r - r'|, \end{aligned} \quad (7.37)$$

for  $s, s' \in \mathbb{R}, r, r' > 1/2$ , and for appropriate  $\hat{s}, \hat{r}$ .

Note that  $\hat{s}, \hat{r}$  belong to open intervals  $(s, s'), (r', r)$ , respectively.

Using (5.7), (7.19), (7.32), (7.37) and the property that  $1/2 < r' < \hat{r} < r < 1$  we obtain

$$|U_0(s, r) - U_0(s', r')| \leq C((1-r)^{1/3} + (1-r')^{1/3}), \quad (7.38)$$

$$|U_0(s, r) - U_0(s', r')| \leq \frac{C}{(1-r)^5} (|s - s'| + |r - r'|), \quad (7.39)$$

where  $C$  is a constant depending only on  $\Phi$ .

We have that

$$\begin{aligned} (1-r)^{1/3} + (1-r')^{1/3} &= (1-r)^{1/3} + ((1-r) + (r-r'))^{1/3} \\ &\leq 2(1-r)^{1/3} + |r-r'|^{1/3} \\ &\leq \begin{cases} 3|r-r'|^{1/3} & \text{if } 1-r \leq |r-r'|, \\ 3(1-r)^{1/3} & \text{if } 1-r > |r-r'|, \end{cases} \end{aligned} \quad (7.40)$$

where  $r, r'$  satisfy (7.36). Note that in (7.40) we used the following inequality:

$$(a+b)^{1/m} \leq a^{1/m} + b^{1/m} \text{ for } a \geq 0, b \geq 0, m \in \mathbb{N}. \quad (7.41)$$

In particular, using (7.38), (7.40) we have that

$$|U_0(s, r) - U_0(s', r')|^{15} \leq 3^{15} C^{15} (1-r)^5 \text{ if } 1-r > |r-r'|, \quad (7.42)$$

where  $s, s', r, r'$  satisfy assumption (7.36),  $C$  is a constant of (7.38), (7.39).

Multiplying the left and the right hand-sides of (7.39), (7.42) we obtain

$$|U_0(s, r) - U_0(s', r')|^{16} \leq 3^{15} C^{16} (|s - s'| + |r - r'|), \text{ if } 1-r > |r - r'|. \quad (7.43)$$

Using (7.38), (7.40) we obtain

$$|U_0(s, r) - U_0(s', r')| \leq 3C|r - r'|^{1/3}, \text{ if } 1-r \leq |r - r'|, \quad (7.44)$$

where  $C$  is a constant of (7.38), (7.39) depending only on  $\Phi$ . Using (7.43) and (7.41) for  $m = 16, a = |s - s'|, b = |r - r'|$ , we have that

$$|U_0(s, r) - U_0(s', r')| \leq 3C(|s - s'|^{1/16} + |r - r'|^{1/16}), \text{ if } 1-r > |r - r'|, \quad (7.45)$$

where  $s, s', r, r'$  satisfy assumption (7.36),  $C$  is a constant of (7.38), (7.39) which depends only on  $\Phi$ .

Formulas (7.44), (7.45) imply (5.8) for this case.

Note that assumptions (7.29), (7.31), (7.36) for cases 1, 2, 3, respectively, cover all possible choices of  $s, s', r, r'$  in (5.8).

This completes the proof of (5.8).

This completes the proof of Lemma 3.

## 8 Proofs of Lemmas 4, 5

### 8.1 Proof of Lemma 4

**Proof of (7.13), (7.14).** Estimates (7.13), (7.14) follow directly from (3.11)-(3.14).

**Proof of (7.17).** We will use the following parametrization of the points  $y$  on  $\gamma(x, \theta) \in TS^1$ ,  $(x, \theta) \in \Omega$ ,  $r(x, \theta) \neq 0$  (see notations (2.1), (2.2), (2.13) for  $d = 2$ ):

$$y(\beta) = x - (x\theta)\theta + \tan(\beta)r(x, \theta)\theta, \quad \beta \in (-\pi/2, \pi/2), \quad (8.1)$$

where  $\beta$  is the parameter.

We have that:

$$d\sigma(\beta) = r d(\tan(\beta)) = \frac{r d\beta}{\cos^2 \beta}, \quad r = r(x, \theta), \quad (8.2)$$

where  $\sigma$  is the standard Lebesgue measure on  $\gamma(x, \theta)$ .

From definitions (3.10), (5.3) it follows that

$$\tilde{G}(r) = \sum_{k=1}^{\infty} \frac{\tilde{G}_k(r)}{k!}, \quad (8.3)$$

$$\tilde{G}_k(r) = \int_{\gamma_r} \tilde{f}_k(|y|) dy, \quad \gamma_r \in T(r), \quad r > 1/2, \quad (8.4)$$

where  $T(r)$  is defined by (2.8).

Using (3.11), (8.1), (8.2), (8.4) we obtain the following formula for  $\tilde{G}_k$ :

$$\begin{aligned} \tilde{G}_k(r) &= r \int_{-\pi/2}^{\pi/2} \Phi \left( 2^k \left( 1 - \frac{r}{\cos \beta} \right) \right) \cos \left( 8^k \frac{r^2}{\cos^2 \beta} \right) \frac{d\beta}{\cos^2 \beta} \\ &= \{u = \tan(\beta)\} = 2r \int_0^{+\infty} \Phi \left( 2^k \left( 1 - r\sqrt{u^2 + 1} \right) \right) \cos \left( 8^k r^2 (u^2 + 1) \right) du \\ &= \{t = u^2\} = r \int_0^{+\infty} \Phi \left( 2^k \left( 1 - r\sqrt{t+1} \right) \right) \cos \left( 8^k r^2 (t+1) \right) \frac{dt}{\sqrt{t}} \\ &= r \cos(8^k r^2) \int_0^{+\infty} \Phi(2^k(1 - r\sqrt{t+1})) \frac{\cos(8^k r^2 t)}{\sqrt{t}} dt \\ &\quad - r \sin(8^k r^2) \int_0^{+\infty} \Phi(2^k(1 - r\sqrt{t+1})) \frac{\sin(8^k r^2 t)}{\sqrt{t}} dt \\ &= 8^{-k/2} r^{-1} \cos(8^k r^2) \int_0^{+\infty} \Phi_k(t, r) \frac{\cos(t)}{\sqrt{t}} dt \\ &\quad - 8^{-k/2} r^{-1} \sin(8^k r^2) \int_0^{+\infty} \Phi_k(t, r) \frac{\sin(t)}{\sqrt{t}} dt, \quad r > 1/2, \end{aligned} \quad (8.5)$$

where

$$\Phi_k(t, r) = \Phi(2^k(1 - r\sqrt{8^{-k}r^{-2}t + 1})), \quad t \geq 0, \quad r > 1/2, \quad k \in \mathbb{N}. \quad (8.6)$$

For integrals arising in (8.5) the following estimates hold:

$$\left| \int_0^{+\infty} \Phi_k(t, r) \frac{\sin(t)}{\sqrt{t}} dt \right| \leq C_1 < +\infty, \quad (8.7)$$

$$\left| \int_0^{+\infty} \Phi_k(t, r) \frac{\cos(t)}{\sqrt{t}} dt \right| \leq C_2 < +\infty, \quad (8.8)$$

for  $1/2 < r < 1$ ,  $k \geq 1$ .

where  $\Phi_k$  is defined in (8.6),  $C_1, C_2$  are some positive constants depending only on  $\Phi$  and not depending on  $k$  and  $r$ .

Estimates (8.7), (8.8) are proved in Subsection 8.3.

From (8.5)-(8.8) it follows that

$$|\tilde{G}_k(r)| \leq 2 \cdot 8^{-k/2} (C_1 + C_2) \text{ for } r > 1/2, k \in \mathbb{N}. \quad (8.9)$$

Note that for  $y \in \gamma_r$ , the following inequality holds:

$$2^k(1 - |y|) \leq 2^k(1 - r) \leq 2^{k-m} \leq 1/2 < 4/5 \text{ for } 1 - 2^{-m} \leq r < 1, k < m, m \geq 3, \quad (8.10)$$

where  $\gamma_r$  is a ray in  $T(r)$  (see notations of (2.8),  $d = 2$ ).

Formulas (3.11), (3.12), (5.3), (8.10) imply that

$$\gamma_r \cap \text{supp } f_k = \emptyset \text{ if } r \geq 1 - 2^{-m}, k < m, \quad (8.11)$$

In turn, (8.4), (8.11) imply that

$$\tilde{G}_k(r) = 0 \text{ for } r \geq 1 - 2^{-m}, k < m, m \geq 3. \quad (8.12)$$

Due to (8.3), (8.4), (8.9), (8.12) we have that:

$$\begin{aligned} |\tilde{G}(r)| &\leq \sum_{k=m}^{\infty} |\tilde{G}_k(r)|/k! \\ &\leq 2(C_1 + C_2) \frac{(2\sqrt{2})^{-m}}{m!} \sum_{k=0}^{\infty} (2\sqrt{2})^{-k} = c_1 \frac{(2\sqrt{2})^{-m}}{m!}, \quad c_1 = (C_1 + C_2) \frac{4\sqrt{2}}{2\sqrt{2} - 1}, \\ &\text{for } r \geq 1 - 2^{-m}, m \geq 3. \end{aligned} \quad (8.13)$$

This completes the proof of estimate (7.17).

**Proof of (7.18).** Using (8.3), (8.4) we have that:

$$\left| \frac{d\tilde{G}}{dr}(r) \right| \leq \sum_{k=1}^{\infty} \frac{1}{k!} \left| \frac{d\tilde{G}_k(r)}{dr} \right|. \quad (8.14)$$

Formulas (3.11), (7.10) for  $n = 1$ , (7.14), (8.4) imply that

$$\begin{aligned} \left| \frac{d\tilde{G}_k}{dr}(r) \right| &= \left| \int_{-\infty}^{+\infty} \frac{r \tilde{f}'_k((s^2 + r^2)^{1/2})}{\sqrt{r^2 + s^2}} ds \right| \\ &\leq \int_{-\infty}^{+\infty} |\tilde{f}'_k((s^2 + r^2)^{1/2})| ds = \int_{\gamma_r} |\tilde{f}'_k(|y|)| dy \leq c8^k \int_{\gamma_r \cap B(0,1)} dy \leq 2c8^k, \end{aligned} \quad (8.15)$$

where  $B(0, 1)$  is defined in (2.15),  $d = 2$ .

At the same time, formula (8.12) implies that

$$\frac{d\tilde{G}_k(r)}{dr} = 0 \text{ for } r \geq 1 - 2^{-m}, k < m, m \geq 3. \quad (8.16)$$

Formulas (8.14), (8.15), (8.16) imply the following sequence of inequalities:

$$\left| \frac{d\tilde{G}(r)}{dr} \right| \leq \sum_{k=m}^{\infty} \frac{1}{k!} \left| \frac{d\tilde{G}_k(r)}{dr} \right| \leq c \frac{8^m}{m!} \sum_{k=0}^{\infty} \frac{m!8^k}{(k+m)!}, \quad r \geq 1 - 2^{-m}, m \geq 3. \quad (8.17)$$

The series in the right hand-side in (8.17) admits the following estimate:

$$\sum_{k=0}^{\infty} \frac{m!8^k}{(k+m)!} \leq \sum_{k=0}^{\infty} \frac{8^k}{k!} = e^8 \text{ and the estimate does not depend on } m. \quad (8.18)$$

Formulas (8.17), (8.18) imply (7.18).



**Proof of (7.15).** For each  $\psi_k$  from (3.33) we have that

$$|\psi_k| \leq 1. \quad (8.19)$$

Therefore, it is sufficient to show that

$$\tilde{H}_k \geq C2^{-k} \text{ for } k \geq k_1, C = c^{-1}. \quad (8.20)$$

Proceeding from (5.3) and in a similar way with (8.5) we obtain the formulas

$$\tilde{H}_k(r) = r \int_0^{+\infty} \frac{\Phi^2(2^k(1-r\sqrt{t+1}))}{\sqrt{t}} \cos^2(8^k r^2(t+1)) dt = \tilde{H}_{k,1}(r) + \tilde{H}_{k,2}(r), \quad r > 1/2, \quad (8.21)$$

$$\tilde{H}_{k,1}(r) = \frac{r}{2} \int_0^{+\infty} \frac{\Phi^2(2^k(1-r\sqrt{t+1}))}{\sqrt{t}} dt, \quad (8.22)$$

$$\tilde{H}_{k,2}(r) = \frac{r}{2} \int_0^{+\infty} \frac{\Phi^2(2^k(1-r\sqrt{t+1}))}{\sqrt{t}} \cos(2 \cdot 8^k r^2(t+1)) dt. \quad (8.23)$$

In addition, we have that:

$$\text{supp}_t \Phi^2(2^k(1-r\sqrt{t+1})) \subset [0, 3] \text{ for } 1/2 < r \leq 1 - 2^{-k+1}, k \geq 3, \quad (8.24)$$

where  $\text{supp}_t$  denotes the support of the function in variable  $t$ . Property (8.24) is proved below in this paragraph (see formulas (8.26)-(8.29)).

Note that

$$2^k(1-r) \geq 2^k \cdot 2^{-k+1} \geq 2 > 6/5 \text{ for } 1/2 < r \leq 1 - 2^{-k+1}, k \geq 3. \quad (8.25)$$

From (3.12), (3.13) and from (8.25) we have that:

$$\text{supp}_t \Phi^2(2^k(1-r\sqrt{t+1})) \subset [0, +\infty) \text{ for } 1/2 < r \leq 1 - 2^{-k+1}, k \geq 3. \quad (8.26)$$

We have that

$$\exists t_1^{(k)} = t_1^{(k)}(r) \geq 0, t_2^{(k)} = t_2^{(k)}(r) \geq 0, t_2^{(k)} > t_1^{(k)}, \text{ such that } \begin{cases} 2^k(1-r\sqrt{t_1^{(k)}+1}) = 11/10, \\ 2^k(1-r\sqrt{t_2^{(k)}+1}) = 9/10, \end{cases} \quad (8.27)$$

$$|t_2^{(k)} - t_1^{(k)}| \geq \left( \sqrt{t_2^{(k)}+1} - \sqrt{t_1^{(k)}+1} \right) = \frac{2^{-k}}{5} r^{-1} \geq \frac{2^{-k}}{5}, \quad (8.28)$$

for  $1/2 < r \leq 1 - 2^{-k+1}, k \geq 3$ .

In addition, from (8.27) it follows that

$$t_1^{(k)} = \frac{(1 - 2^{-k} \frac{11}{10})^2}{r^2} - 1 \leq 4(1 - 2^{-k} \frac{11}{10})^2 - 1 \leq 3, \quad (8.29)$$

$$t_2^{(k)} = \frac{(1 - 2^{-k} \frac{9}{10})^2}{r^2} - 1 \leq 4(1 - 2^{-k} \frac{11}{10})^2 - 1 \leq 3,$$

for  $1/2 < r \leq 1 - 2^{-k+1}, k \geq 3$ .

Using (3.12)-(3.14), (8.22), (8.24), (8.27)-(8.29) we have that

$$\begin{aligned} \tilde{H}_{k,1}(r) &\geq \frac{r}{2} \int_{t_1^{(k)}}^{t_2^{(k)}} \frac{dt}{\sqrt{t}} \geq \frac{r}{2} \int_3^{3+|t_2^{(k)}-t_1^{(k)}|} \frac{dt}{\sqrt{t}} \\ &\geq \frac{r}{6} \int_3^{3+|t_2^{(k)}-t_1^{(k)}|} dt = \frac{r}{6} |t_2^{(k)} - t_1^{(k)}| \geq \frac{2^{-k}}{30} \text{ for } 1/2 < r \leq 1 - 2^{-k+1}, k \geq 3. \end{aligned} \quad (8.30)$$

On the other hand, proceeding from using (8.23) and, in a similar way with (8.5)-(8.9), we have

$$\begin{aligned}
|\tilde{H}_{k,2}(r)| &= \frac{r}{2} \left| \int_0^{+\infty} \frac{\Phi^2(2^k(1-r\sqrt{t+1}))}{\sqrt{t}} \cos(2 \cdot 8^k r^2(t+1)) dt \right| \\
&\leq \frac{r}{2} |\cos(2 \cdot 8^k r^2)| \left| \int_0^{+\infty} \Phi^2(2^k(1-r\sqrt{t+1})) \frac{\cos(2 \cdot 8^k r^2 t)}{\sqrt{t}} dt \right| \\
&\quad + \frac{r}{2} |\sin(2 \cdot 8^k r^2)| \left| \int_0^{+\infty} \Phi^2(2^k(1-r\sqrt{t+1})) \frac{\sin(2 \cdot 8^k r^2 t)}{\sqrt{t}} dt \right| \\
&\leq 8^{-k/2} \frac{r^{-1}}{2} \left| \int_0^{+\infty} \Phi_k^2(t, r) \frac{\cos(2t)}{\sqrt{t}} dt \right| + 8^{-k/2} \frac{r^{-1}}{2} \left| \int_0^{+\infty} \Phi_k^2(t, r) \frac{\sin(2t)}{\sqrt{t}} dt \right| \\
&\leq 8^{-k/2} C, \text{ for } 1/2 < r < 1 - 2^{-k+1}, k \geq 3,
\end{aligned} \tag{8.31}$$

where  $\Phi_k(t, r)$  is defined in (8.6),  $C$  is some constant depending only on  $\Phi$  and not depending on  $k, r$ . In (8.31) we have also used that  $\Phi^2(t)$  satisfies assumptions (3.12)-(3.14).

Note also that  $\Phi^2(t)$  satisfies assumptions (3.12)-(3.14) for  $\Phi(t)$ .

Using (8.21)-(8.23), (8.30), (8.31) we obtain

$$\begin{aligned}
|\tilde{H}_k(r)| &\geq |\tilde{H}_{k,1}(r)| - |\tilde{H}_{k,2}(r)| \\
&\geq \frac{2^{-k}}{30} - C' \cdot 8^{-k/2} \\
&\geq 2^{-k} \left( \frac{1}{30} - \frac{C'}{(\sqrt{2})^k} \right) \\
&\geq C 2^{-k} \text{ for } 1/2 < r < 1 - 2^{-k+1}, k \geq k_1 \geq 3, \\
C &= \frac{1}{30} - C'(\sqrt{2})^{-k_1},
\end{aligned} \tag{8.32}$$

where  $C'$  depends only on  $\Phi$ ,  $k_1$  is arbitrary constant such that  $k_1 \geq 3$  and  $C$  is positive.

Formulas (7.15) follows from (3.33), (8.32).

This completes the proof (7.15).

**Proof of (7.16).** The following formula holds:

$$\frac{d}{dr} \left( \frac{\psi_{k-2}(r)}{\tilde{H}_k(r)} \right) = - \frac{\tilde{H}'_k(r) \psi_{k-2}(r) - \tilde{H}_k(r) \psi'_{k-2}(r)}{\tilde{H}_k^2(r)}, \quad 1/2 < r < 1, \tag{8.33}$$

where  $\tilde{H}'_k, \psi'_{k-2}$  denote the derivatives of  $\tilde{H}_k, \psi_k$ , defined in (5.3), (3.33), respectively.

Using (3.11), (5.3), (7.10),  $n = 1$ , (7.13), (7.14) we have that

$$\begin{aligned}
|\tilde{H}'_k(r)| &= 2 \left| \int_{-\infty}^{+\infty} \frac{r}{\sqrt{r^2+s^2}} \tilde{f}_k(\sqrt{r^2+s^2}) \tilde{f}'_k(\sqrt{r^2+s^2}) ds \right| \\
&\leq 2 \int_{-\infty}^{+\infty} |\tilde{f}_k(\sqrt{r^2+s^2}) \tilde{f}'_k(\sqrt{r^2+s^2})| ds = 2 \int_{\gamma_r} |\tilde{f}_k(|y|) \tilde{f}'_k(|y|)| dy \\
&\leq 2c 8^k \int_{\gamma_r \cap B(0,1)} dy \leq 4c 8^k, \quad \gamma_r \in T(r), k \geq 3, r > 1/2,
\end{aligned} \tag{8.34}$$

where we use notations (2.8), (2.15),  $d=2$ .

Using (3.33), (3.34), (7.15), (8.32)-(8.34) we have that

$$\begin{aligned}
\left| \frac{d}{dr} \left( \frac{\psi_{k-2}(r)}{\tilde{H}_k(r)} \right) \right| &\leq C 2^{2k} (|\tilde{H}'_k(r)| + |\tilde{H}_k(r)| \cdot |\psi'_{k-2}(r)|) \leq C' 2^{5k}, \\
&\text{for } 1/2 < r < 1 - 2^{-k+1}, k \geq k_1 \geq 3,
\end{aligned} \tag{8.35}$$

where  $C'$  is a constant not depending on  $k$  and  $r$  and depending only on  $\Phi$ .

This completes the proof of Lemma 4.

## 8.2 Proof of Lemma 5

It is sufficient to show that

$$\left| \frac{\partial U_0(s, r)}{\partial s} \right| \leq \frac{C}{(1-r)^3}, \quad (8.36)$$

$$\left| \frac{\partial U_0(s, r)}{\partial r} \right| \leq \frac{C}{(1-r)^5}, \quad (8.37)$$

for  $s \in \mathbb{R}$ ,  $r \in \Lambda_k$ ,  $k \geq \max(4, k_1)$ ,

where  $C$  is a positive constant depending only on  $\Phi$  of (3.11),  $\Lambda_k$  is defined in (7.4),  $k_1$  is a constant arising in Lemma 4 and depending only on  $\Phi$ .

Indeed, estimates (7.19) follow from (5.4), (8.36), (8.37) and the fact that  $\Lambda_k$ ,  $k \geq 4$ , is an open cover of  $(1/2, 1)$ .

In turn, estimates (8.36), (8.37) follow from the estimates

$$\left| \frac{\partial U_0(s, r)}{\partial s} \right| \leq C \cdot 8^k, \quad (8.38)$$

$$\left| \frac{\partial U_0(s, r)}{\partial r} \right| \leq C \cdot (32)^k, \quad (8.39)$$

for  $s \in \mathbb{R}$ ,  $r \in \Lambda_k$ ,

and from the fact that  $2^{-k+1} < 1-r < 2^{-k+3}$ ,  $k \geq \max(4, k_1)$ , for  $r \in \Lambda_k$ , where  $C$  is a positive constant depending only on  $\Phi$ .

Estimate (8.38) follows from formula (7.9) for  $n = 1$  and estimates (7.14), (7.15), (7.20)-(7.24).

Estimate (8.39) follows from (7.8), (7.13)-(7.16), (7.20)-(7.24) and from the estimates:

$$\left| \frac{d}{dr} \left( \frac{\psi_{k-i}(r)}{\tilde{H}_{k-i+2}(r)} \right) \right| \leq c 2^{5(k+1)}, \quad (8.40)$$

$$\left| \frac{d\tilde{G}(r)}{dr} \right| \leq c \frac{8^{-k+3}}{(k-3)!}, \quad (8.41)$$

for  $r \in \Lambda_k$ ,  $i \in \{1, 2, 3\}$ ,

where  $c$  is a constant arising in Lemma 4.

Estimate (8.40) follows from (7.16) (used with  $k-1$ ,  $k$ ,  $k+1$  in place of  $k$ ). Estimate (8.41) follows from (7.18) (used with  $k-3$  in place of  $k$ ).

This completes the proof of Lemma 5.

## 8.3 Proof of estimates (8.7), (8.8)

We use the following Bonnet's integration formulas (see, e.g., [Fich59], Chapter 2):

$$\int_a^b f_1(t)h(t) dt = f_1(a) \int_a^{\xi_1} h(t) dt, \quad (8.42)$$

$$\int_a^b f_2(t)h(t) dt = f_2(b) \int_{\xi_2}^b h(t) dt, \quad (8.43)$$

for some appropriate  $\xi_1, \xi_2 \in [a, b]$ , where

$$\begin{aligned} f_1 &\text{ is monotonously decreasing on } [a, b], f_1 \geq 0, \\ f_2 &\text{ is monotonously increasing on } [a, b], f_2 \geq 0, \\ h(t) &\text{ is integrable on } [a, b]. \end{aligned} \quad (8.44)$$

Let

$$g_1(t) = \frac{\sin(t)}{\sqrt{t}}, \quad g_2(t) = \frac{\cos(t)}{\sqrt{t}}, \quad t > 0, \quad (8.45)$$

$$G_1(s) = \int_0^s \frac{\sin(t)}{\sqrt{t}} dt, \quad G_2(s) = \int_0^s \frac{\cos(t)}{\sqrt{t}} dt, \quad s \geq 0. \quad (8.46)$$

We recall that

$$\lim_{s \rightarrow +\infty} G_1(s) = \lim_{s \rightarrow +\infty} G_2(s) = \sqrt{\frac{\pi}{2}}. \quad (8.47)$$

From (8.45), (8.46), (8.47) it follows that

$$G_1, G_2 \text{ are continuous and bounded on } [0, +\infty). \quad (8.48)$$

Due to (3.12)-(3.15), (8.6) and monotonicity of the function  $2^k(1 - r\sqrt{8^{-k}r^{-2}t + 1})$  in  $t$  on  $[0, +\infty)$  it follows that

$$\Phi_k(t, r) \text{ is monotonously decreasing on } [0, +\infty), \text{ if } 2^k(1 - r) \leq 11/10, \quad (8.49)$$

$$\Phi_k(t, r) \text{ is monotonously increasing on } [0, t_0] \text{ for some } t_0 > 0 \quad (8.50)$$

$$\text{and is monotonously decreasing on } [t_0, +\infty), \text{ if } 2^k(1 - r) > 11/10.$$

for  $r > 1/2, k \in \mathbb{N}$ ,

Moreover, due to (3.12)-(3.14), (8.6), for  $T_k = 8^k, k \in \mathbb{N}$ , we have that

$$\Phi_k(T_k, r) = \Phi(2^k(1 - r\sqrt{r^{-2} + 1})) = \Phi(2^k(1 - \sqrt{1 + r^{-2}})) = 0, \quad (8.51)$$

$$\Phi_k(t, r) = 0 \text{ for } t \geq T_k, \quad (8.52)$$

$$|\Phi_k(t, r)| \leq 1 \text{ for } t \geq 0, \quad (8.53)$$

$r > 1/2, k \in \mathbb{N}$ .

Using (8.6), (8.45)-(8.50), (8.52) and (8.42)-(8.44) we obtain

$$\begin{aligned} \int_0^{+\infty} \Phi_k(t, r)g_i(t) dt &= \int_0^{T_k} \Phi_k(t, r)g_i(t) dt = \Phi_k(0, r) \int_0^\xi g_i(t) dt \\ &= \Phi_k(0, r)G_i(\xi) \text{ for appropriate } \xi \in [0, T_k], \text{ if } 2^k(1 - r) \leq 11/10, \end{aligned} \quad (8.54)$$

$$\begin{aligned} \int_0^\infty \Phi_k(t, r)g_i(t) dt &= \int_0^{T_k} \Phi_k(t, r)g_i(t) dt = \int_0^{t_0} \Phi_k(t, r)g_i(t) dt + \int_{t_0}^{T_k} \Phi_k(t, r)g_i(t) dt \\ &= \Phi_k(t_0, r) \int_{\xi'}^{t_0} g_i(t) dt + \Phi_k(t_0, r) \int_{t_0}^{\xi''} g_i(t) dt \\ &= \Phi_k(t_0, r)(G_i(\xi'') - G_i(\xi')) \text{ for appropriate } \xi' \in [0, t_0], \xi'' \in [t_0, T_k], \\ &\text{if } 2^k(1 - r) > 11/10, \end{aligned} \quad (8.55)$$

where  $i = \overline{1, 2}$ .

Estimates (8.7), (8.8) follow from (8.45), (8.46), (8.48), (8.53)-(8.55).

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