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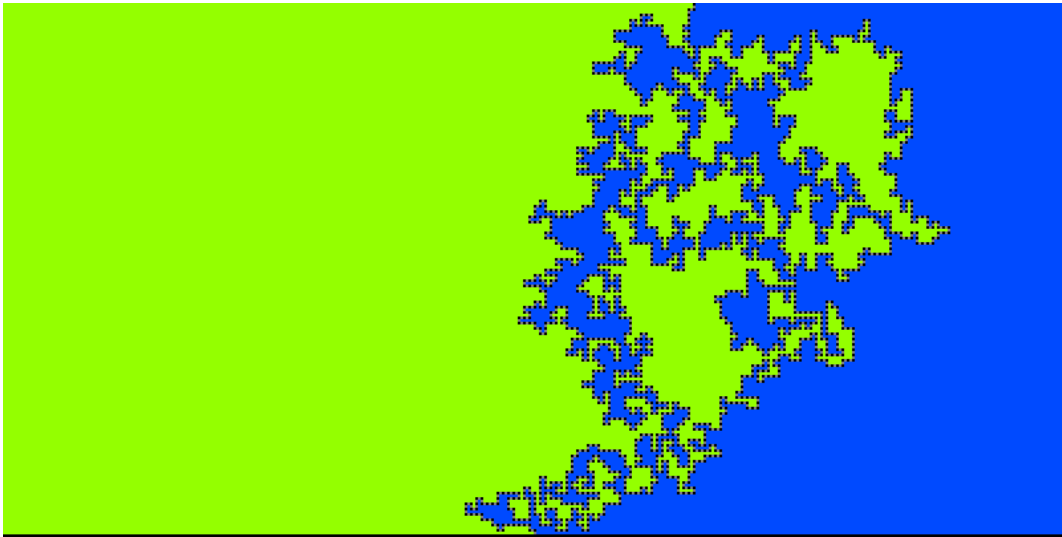
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# TREES OF SELF-AVOIDING WALKS

VINCENT BEFFARA AND CONG BANG HUYNH

ABSTRACT. We consider the biased random walk on a tree constructed from the set of finite self-avoiding walks on a lattice, and use it to construct probability measures on infinite self-avoiding walks. The limit measure (if it exists) obtained when the bias converges to its critical value is conjectured to coincide with the weak limit of the uniform SAW. Along the way, we obtain a criterion for the continuity of the escape probability of a biased random walk on a tree as a function of the bias, and show that the collection of escape probability functions for spherically symmetric trees of bounded degree is stable under uniform convergence.



*A realization of the limit walk in the upper-half plane, with bias  $\lambda = 1$ .*

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*Key words and phrases.* Self-avoiding walk, effective conductance, random walk on tree.

## 1. INTRODUCTION

An  $n$ -step self-avoiding walk (SAW) (or a self-avoiding walk of length  $n$ ) in a regular lattice  $\mathbb{L}$  (such as the integer lattice  $\mathbb{Z}^2$ , triangular lattice  $\mathbb{T}$ , hexagonal lattice, etc) is a nearest neighbor path  $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_n)$  that visits no vertex more than once. Self-avoiding walks were first introduced as a lattice model for polymer chains (see [5]); while they are very easy to define, they are extremely difficult to analyze rigorously and there are still many basic open questions about them (see [16], Chapter 1).

Let  $c_n$  be the number of SAWs of length  $n$  starting at the origin. The *connective constant* of  $\mathbb{L}$ , which we will denote by  $\mu$ , is defined by

$$c_n = \mu^{n+o(n)} \quad \text{when } n \rightarrow \infty.$$

The existence of the connective constant is easy to establish from the submultiplicativity relation  $c_{n+m} \leq c_n c_m$ , from which one can also deduce that  $c_n \geq \mu^n$  for all  $n$ ; the existence of  $\mu$  was first observed by Hammersley and Morton [7]. Nienhuis [18] gave a prediction that for all regular planar lattices,  $c_n = \mu^n n^{\alpha+o(1)}$  where  $\alpha = \frac{11}{32}$ , and this prediction is known to hold under the assumption of the existence of a conformally invariant scaling limit, see *e.g.* [13].

We are interested in defining a natural probability measure on the set  $SAW_\infty$  of *infinite* self-avoiding walks (*i.e.*, nearest-neighbors paths  $(\gamma_k)_{k \geq 0}$  visiting no vertex more than once, see the sections 5.2 and 6). Such a measure was constructed before in the half-plane case as the weak limit of the uniform measures on finite self-avoiding walks, relying on results by Kesten (see [16, 10]), and it is part of our goal to investigate whether that measure and our construction are related.

**1.1. The model.** In this paper, we consider a one-parameter family of probability measures on  $SAW_\infty$ , denoted by  $(\mathbb{P}_\lambda)_{\lambda > \lambda_c}$ , defined informally as follows (see Section 2.3 for a formal definition). Let  $\mathcal{T}_{\mathbb{Z}^2}$  be the tree whose vertices are the finite self-avoiding walks in the plane starting at the origin  $o := (0, 0)$ , where two such vertices are adjacent when one walk is a one-step extension of the other. We will call this tree the *self-avoiding tree* on  $\mathbb{Z}^2$ . Denoting by  $\mathbb{H}$  the upper-half plane in  $\mathbb{Z}^2$  and by  $\mathbb{Q}$  the first quadrant, defined as

$$\mathbb{H} := \{(x, y) \in \mathbb{Z}^2 : y \geq 0\} \quad \text{and} \quad \mathbb{Q} := \{(x, y) \in \mathbb{Z}^2 : x \geq 0 \text{ and } y \geq 0\},$$

one can construct the self-avoiding trees  $\mathcal{T}_{\mathbb{H}}$  and  $\mathcal{T}_{\mathbb{Q}}$  accordingly, and all the constructions below can be extended to these cases in a natural fashion which we will not make explicit in this introduction.

Then, consider the continuous-time biased random walk of parameter  $\lambda > 0$  on  $\mathcal{T}_{\mathbb{Z}^2}$ , which from a given location jumps towards the root with rate 1 and towards each of its children vertices with rate  $\lambda$ . If  $\lambda$  is such that the walk is

transient, its path determines an infinite branch in  $\mathcal{T}_{\mathbb{Z}^2}$  which can be seen as a random infinite self-avoiding walk  $\omega_\lambda^\infty$ ; we will denote by  $\mathbb{P}_\lambda^{\mathbb{Z}^2}$  the law of  $\omega_\lambda^\infty$ , omitting the mention of  $\mathbb{Z}^2$  in the notation when it is clear from the context, and call it the *limit walk* with parameter  $\lambda$ .

The idea of seeing the self-avoiding walk as a dynamical object is very natural, and not new; it seems that the biased walk on the “self-avoiding tree” was first considered, mostly for  $\lambda < \lambda_c$ , by Berretti and Sokal ([2], see also [20, 19]) as a Monte-Carlo method to estimate connective constants and sample finite-size self-avoiding paths uniformly. The model was discussed informally by one of the authors of the present paper (VB) with S. Sidoravicius and W. Werner a number of years ago, as a failed attempt to understand conformal invariance of the SAW model in the scaling limit, and in particular a proof of Theorem 1 was obtained at that time but never written down; one of our informal goals here is to revive this line of thought: even though the question of SAW proper still seems out of reach, the link with critical percolation (cf. Section 6.2) could be a promising direction for further research.

**1.2. Main results.** It is well-known that there exists a critical value  $\lambda_c = \lambda_c(\mathcal{T}_{\mathbb{Z}^2})$  such that if  $\lambda > \lambda_c$  the biased random walk is transient and if  $\lambda < \lambda_c$  it is recurrent (see Lyons [14]). In the general case of biased random walk on a tree, the recurrence or transience of the random walk at the critical point depends in subtle ways on the structure of the tree. The value of  $\lambda_c$  on the other hand is easier to determine: indeed, Lyons [14] proved that it coincides with the reciprocal of the branching number of the tree (for background on branching numbers and trees in general, see *e.g.* [15]). The following proposition gives the critical value for self-avoiding trees.

**Theorem 1.** *Let  $\mathcal{T}_{\mathbb{Z}^2}, \mathcal{T}_{\mathbb{H}}, \mathcal{T}_{\mathbb{Q}}$  be the self-avoiding trees defined as above, respectively in the plan, half-plane and first quadrant. Then,*

$$\lambda_c(\mathcal{T}_{\mathbb{Z}^2}) = \lambda_c(\mathcal{T}_{\mathbb{H}}) = \lambda_c(\mathcal{T}_{\mathbb{Q}}) = \frac{1}{\mu},$$

where  $\mu$  is the connective constant of lattice  $\mathbb{Z}^2$  as defined above.

This is a direct consequence of Proposition 49 below. Notice that it is clear from the definition that  $\mu$  is the growth rate of  $\mathcal{T}_{\mathbb{Z}^2}$ ; there are rather large classes of trees, including  $\mathcal{T}_{\mathbb{Z}^2}$ , for which the branching and growth coincide (for instance, this holds for sub- or super-periodic trees, cf. below, or for typical supercritical Galton-Watson trees), but none of the classical results seem to apply to  $\mathcal{T}_{\mathbb{H}}$  or  $\mathcal{T}_{\mathbb{Q}}$ .

The geometry of the limit walk is our main object of interest. As a first property of it, we obtain the following (see Section 6.3):

**Theorem 2.** *For all  $\lambda > \lambda_c$ , under the measures  $\mathbb{P}_\lambda^{\mathbb{Z}^2}$  and  $\mathbb{P}_\lambda^{\mathbb{H}}$ , the limit walk almost surely visits the line  $\mathbb{Z} \times \{0\}$  infinitely many times.*

A useful tool in our proofs is the *effective conductance* of the biased random walk on a tree  $\mathcal{T}$ , defined as the probability of never returning to the root  $o$  of  $\mathcal{T}$  and denoted by  $\mathcal{C}(\lambda, \mathcal{T})$  — see [15]. Along the way, we will be interested in several properties of it as a function of  $\lambda$ . Most important for us will be the question of continuity: in a general tree, the effective conductance is not necessarily a continuous function of  $\lambda$ . We will derive criteria for continuity, which are forms of *uniform transience* of the random walk, and apply them to prove that the effective conductance of self-avoiding trees is a continuous function (see Section 5.4):

**Theorem 3.** *The effective conductances  $\mathcal{C}(\lambda, \mathcal{T}_{\mathbb{Q}})$ ,  $\mathcal{C}(\lambda, \mathcal{T}_{\mathbb{H}})$  and  $\mathcal{C}(\lambda, \mathcal{T}_{\mathbb{Z}^2})$  are continuous functions of  $\lambda$  on the interval  $(\lambda_c, +\infty)$ .*

A related question is that of the convergence of effective conductance along a sequence of trees. More precisely, let  $(\mathcal{C}_n)_n$  denote the effective conductances for a sequence  $(\mathcal{T}_n)$  of infinite trees, again seen as functions of the bias parameter  $\lambda$ , and assume that  $(\mathcal{C}_n)_n$  converges uniformly towards a function  $\mathcal{C}$  that is not identically 0. The question is: is  $\mathcal{C}$  the effective conductance of a certain tree? We study this question on the class of spherically symmetric trees (a tree  $\mathcal{T}$  is said to be spherically symmetric if for every vertex  $\nu$ ,  $\deg \nu$  depends only on  $|\nu|$ , where  $|\nu|$  denote its distance from the root and  $\deg \nu$  is its number of neighbors). If  $\mathbb{S}$  denotes the set of spherically symmetric trees and  $m \in \mathbb{N}^*$  is fixed, define

$$A_m := \{\mathcal{T} \in \mathbb{S}; \forall \nu \in \mathcal{T}, \deg \nu \leq m\} \text{ and} \\ \mathbb{F}_m := \{f \in C^0([0, 1]) : \exists \mathcal{T} \in A_m, \forall \lambda > 0, \mathcal{C}(\lambda, \mathcal{T}) = f(\lambda)\}.$$

Then (see Section 4.2):

**Theorem 4.** *Let  $(f_n)_n$  be a sequence of functions in  $\mathbb{F}_m$ . Assume that  $f_n$  converges uniformly towards  $f \neq 0$ . Then  $f \in \mathbb{F}_m$ .*

**1.3. Open questions.** One natural probability measure on the set of infinite self-avoiding walks is the limit of  $\mathbb{P}_\lambda^{\mathbb{H}}$  as  $\lambda \rightarrow \lambda_c$ , assuming that this limit exists. We were not able to show convergence, but obtained partial results in this direction by restricting the set of allowed paths. Our conjecture is that the limit exists and has to do with Kesten's measure, *i.e.* the weak limit of uniform finite self-avoiding walks in the half-plane, in a way similar to the fact that the two definitions of the incipient infinite cluster for percolation (seen as a limit as  $p \rightarrow p_c$  or as a limit of conditioned critical percolation) coincide, see [12].

This is motivated by a few observations. First, the model for  $\lambda < \lambda_c$  gives rise to a recurrent random walk on  $\mathcal{T}_{\mathbb{H}}$  for which the invariant measure  $\mu_\lambda$  is

rather explicit (by reversibility, the mass of a vertex  $\nu$  is proportional to  $\lambda^{|\nu|}$ ), in particular it depends only on the distance to the root, and on the other hand it tends to be concentrated on longer and longer walks as  $\lambda \uparrow \lambda_c$ . This means that the initial segment of a walk distributed as the stationary measure can be seen as the initial segment of a uniform self-avoiding walk with random total length, and we get convergence to Kesten's measure as soon as we can show that for all  $\nu$ ,  $\mu_\lambda(\{\nu\}) \rightarrow 0$  as  $\lambda \uparrow \lambda_c$ . On the other hand, the behavior of the biased walk in a fixed neighborhood of the origin changes very little when  $\lambda$  is close to  $\lambda_c$ , so for  $\lambda$  slightly larger than  $\lambda_c$  it seems reasonable to predict that the walk will spend a long time close to the origin, following an occupation measure close to  $\mu_{\lambda_c^-}$ , before escaping to infinity. Unfortunately we were unable to formalize this intuition.

Another observation is that convergence of the law of the limit walk holds within the class of paths for which the bridge decomposition involves only bridges of height less than some fixed bound  $m > 0$ . More precisely: for fixed  $m$ , the critical parameter is  $\lambda_{c,m} \geq \lambda_c$ , and the limit  $\lambda \downarrow \lambda_{c,m}$  followed by  $m \rightarrow \infty$  leads to Kesten's measure, while the limit  $m \rightarrow \infty$  for fixed  $\lambda$  coincides with the limit walk on  $\mathcal{T}_{\mathbb{H}}$  with parameter  $\lambda$  — see Theorem 67 for more detail. Exchanging the limits would lead to the claim. Unfortunately, it is not true that this can be done in the general setting of biased walks on trees, due to phenomena similar to those described in Section 3, so it seems that a deeper understanding of the structure of  $\mathcal{T}_{\mathbb{H}}$  would be necessary to conclude.

**1.4. Organization of the paper.** The paper is structured as follows. In Section 2, we review some basic definitions on graphs, trees, branching number and growth rate of a tree, as well as a few classical results about random walks on trees. Section 3 gathers some relevant examples and counter-examples exhibiting some similarities to the self-avoiding trees while being treatable explicitly. The criterion for the continuity of the effective conductance is given in Section 4. Then Section 5 provides some background on self-avoiding walks and the self-avoiding trees, and some properties of the limit walks are obtained in Section 6. Finally, we state a few conjectures and conditional results in Section 7.

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## 2. NOTATION AND BASIC DEFINITIONS

**2.1. Graphs and trees.** In this section, we review some basic definitions; we refer the reader to the book [15] for a more developed treatment. A *graph* is a pair  $\mathcal{G} = (V, E)$  where  $V$  is a set of *vertices* and  $E$  is a symmetric subset

of  $V \times V$  (i.e if  $(\nu, \mu) \in E$  then  $(\mu, \nu) \in E$ ), called the *edge set*, containing no element of the form  $(\nu, \nu)$ . If  $(\nu, \mu) \in E$ , then we call  $\nu$  and  $\mu$  *adjacent* or *neighbors* and we write  $\nu \sim \mu$ . For any vertex  $\nu \in V$ , denote by  $\deg \nu$  its number of neighbors. A *path* in a graph is a sequence of vertices, any two consecutive of which are adjacent. A *self-avoiding path* is a path which does not pass through any vertex more than once. For any  $(\nu, \mu) \in V \times V$ , the distance between  $\nu$  and  $\mu$  is the minimum number of edges among all paths joining  $\nu$  and  $\mu$ , denoted  $d(\nu, \mu)$ . A graph is *connected* if, for each pair  $(\nu, \mu) \in V \times V$ , there exist a path starting at  $\nu$  and ending at  $\mu$ . A connected graph with no cycles is called a *tree*. A *morphism* from a graph  $\mathcal{G}_1$  to a graph  $\mathcal{G}_2$  is a mapping  $\phi$  from  $V(\mathcal{G}_1)$  to  $V(\mathcal{G}_2)$  such that the image of any edge of  $\mathcal{G}_1$  is an edge of  $\mathcal{G}_2$ . We will always consider trees to be *rooted* by the choice of a vertex  $o$ , called the *root*.

Let  $\mathcal{T} = (V, E)$  be an infinite, locally finite, rooted tree with set of vertices  $V$  and set of edges  $E$ . Let  $o$  be the root of  $\mathcal{T}$ . For any vertex  $\nu \in V \setminus \{o\}$ , denote by  $\nu^{-1}$  its *parent* (we also say that  $\nu$  is a *child* of  $\nu^{-1}$ ), i.e. the neighbor of  $\nu$  with shortest distance from  $o$ . For any  $\nu \in V$ , let  $|\nu|$  be the number of edges in the unique self-avoiding path connecting  $\nu$  to  $o$  and call  $|\nu|$  the *generation* of  $\nu$ . In particular, we have  $|o| = 0$ . Denote by  $\mathcal{T}_n$  the set of all vertices of  $\mathcal{T}$  that are at generation  $n$ .

If a vertex has no child, it is called a *leaf*. For any edge  $e \in E$  denote by  $e^-$  and  $e^+$  its endpoints with  $|e^+| = |e^-| + 1$ , and define the generation of an edge as  $|e| = |e^+|$ . We define an order on  $V(\mathcal{T})$  as follows: if  $\nu, \mu \in V(\mathcal{T})$ , we say that  $\nu \leq \mu$  if the simple path joining  $o$  to  $\mu$  passes through  $\nu$ . For each  $\nu \in V(\mathcal{T})$ , we define the *subtree* of  $\mathcal{T}$  rooted at  $\nu$ , denoted by  $\mathcal{T}^\nu$ , where  $V(\mathcal{T}^\nu) := \{\mu \in V(\mathcal{T}) : \nu \leq \mu\}$  and  $E(\mathcal{T}^\nu) = E(\mathcal{T})|_{V(\mathcal{T}^\nu) \times V(\mathcal{T}^\nu)}$ .

An infinite simple path starting at  $o$  is called a *ray*. The set of all rays, denoted by  $\partial\mathcal{T}$ , is called the *boundary* of  $\mathcal{T}$ . The set  $\mathcal{T} \cup \partial\mathcal{T}$  can be equipped with a metric that makes it a compact space, see [15].

The remaining part of this paper, we consider only infinite, locally finite and rooted trees with the root  $o$ .

## 2.2. Branching and growth.

**Definition 5.** *Let  $\mathcal{T}$  be an infinite, locally finite and rooted tree. A  $E$ -cutset (resp.  $V$ -cutset) in  $\mathcal{T}$  is a set  $\pi$  of edges (resp. vertices) such that, for any infinite self-avoiding path  $(\nu_i)_{i \geq 0}$  started at the root, there exists a  $i \geq 0$  such that  $[\nu_{i-1}, \nu_i] \in \pi$  (resp.  $\nu_i \in \pi$ ). In other words, a  $E$ -cutset (resp.  $V$ -cutset) is a set of edges (resp. vertices) separating the root from infinity. We use  $\Pi$  to denote the set of  $E$ -cutsets.*

**Definition 6.** *Let  $\mathcal{T}$  be an infinite, locally finite and rooted tree.*

- The branching number of  $\mathcal{T}$  is defined by:

$$br(\mathcal{T}) = \sup \left\{ \lambda \geq 1 : \inf_{\pi \in \Pi} \sum_{e \in \pi} \lambda^{-|e|} > 0 \right\}$$

- We define also

$$\overline{gr}(\mathcal{T}) = \limsup |\mathcal{T}_n|^{1/n} \quad \text{and} \quad \underline{gr}(\mathcal{T}) = \liminf |\mathcal{T}_n|^{1/n}.$$

In the case  $\overline{gr}(\mathcal{T}) = \underline{gr}(\mathcal{T})$ , the growth rate of  $\mathcal{T}$  is defined by their common value and denoted by  $gr(\mathcal{T})$ .

**Remark 7.** It follows immediately from the definition of branching number that if  $\mathcal{T}'$  is a subtree of  $\mathcal{T}$ , then  $br(\mathcal{T}') \leq br(\mathcal{T})$ .

**Proposition 8** ([15]). Let  $\mathcal{T}$  be a tree, then  $br(\mathcal{T}) \leq \underline{gr}(\mathcal{T})$ .

In general, the inequality in Proposition 8 may be strict: The 1–3 tree (see [15], page 4) is an example for which the branching number is 1 and the growth rate is 2. There are classes of trees however where branching and growth match.

**Definition 9.** The tree  $\mathcal{T}$  is said to be spherically symmetric if  $\deg \nu$  depends only on  $|\nu|$ .

**Theorem 10** ([15] page 83). For every spherically symmetric tree  $\mathcal{T}$ ,  $br(\mathcal{T}) = \underline{gr}(\mathcal{T})$ .

**Definition 11.** Let  $N \geq 0$ : an infinite, locally finite and rooted tree  $\mathcal{T}$  with the root  $o$ , is said to be

- $N$ -sub-periodic if for every  $\nu \in V(\mathcal{T})$ , there exists an injective morphism  $f : \mathcal{T}^\nu \rightarrow \mathcal{T}^{f(\nu)}$  with  $|f(\nu)| \leq N$ .
- $N$ -super-periodic if for every  $\nu \in V(\mathcal{T})$ , there exists an injective morphism  $f : \mathcal{T} \rightarrow \mathcal{T}^{f(o)}$  with  $f(o) \in \mathcal{T}^\nu$  and  $|f(o)| - |\nu| \leq N$ .

**Theorem 12** (see [6, 15]). Let  $\mathcal{T}$  be an infinite, locally finite and rooted tree that is either  $N$ -sub-periodic, or  $N$ -super-periodic with  $\overline{gr}(\mathcal{T}) < \infty$ . Then the growth rate of  $\mathcal{T}$  exists and  $gr(\mathcal{T}) = br(\mathcal{T})$ .

**2.3. Random walks on trees.** Let  $\mathcal{T}$  be a tree, we now define the discrete-time biased random walk on  $\mathcal{T}$ . Working in discrete time will make some of the arguments below a little simpler, at the cost of a slightly heavier definition here — notice though that the definition of the measure  $\mathbb{P}_\lambda$  and the main results of the paper are not at all affected by this choice.

Let  $\lambda > 0$ : the biased walk  $RW_\lambda$  with bias  $\lambda$  on  $\mathcal{T}$  is the discrete-time Markov chain on the vertex set of  $\mathcal{T}$  with transition probabilities given, at a



vertex  $x \neq o$  with  $k$  children, by

$$p_\lambda(x, y) := \begin{cases} \frac{1}{1+k\lambda} & \text{if } y \text{ is the father of } x, \\ \frac{\lambda}{1+k\lambda} & \text{if } y \text{ is a child of } x, \\ 0 & \text{otherwise.} \end{cases}$$

If the root has  $k > 0$  children, then  $p_\lambda(o, x)$  is  $1/k$  if  $x$  is a child of  $o$  and 0 otherwise. The degenerate case  $\mathcal{T} = \{o\}$  where the root has no child will not occur in our context, so we will silently ignore it. We also allow ourselves to consider the cases  $\lambda \in \{0, \infty\}$ , with the natural convention that  $RW_0$  remains stuck at the root and that  $RW_\infty$  always moves away from the root, getting stuck whenever it reaches a leaf.

**Definition 13.** Let  $\mathcal{G} = (V, E)$  be a graph, and  $c : E \rightarrow \mathbb{R}_+^*$  be labels on the edges, referred to as conductances. Equivalently, one can fix resistances by letting  $r(e) := 1/c(e)$ . The pair  $(\mathcal{G}, c)$  is called a network. Given a subset  $K$  of  $V$ , the restriction of  $c$  to the edges joining vertices in  $K$  defines the induced sub-network  $\mathcal{G}|_K$ . The random walk on the network  $(\mathcal{G}, c)$  is the discrete-time Markov chain on  $V$  with transition probabilities proportional to the conductances.

Given a network  $(\mathcal{T}, c)$  on a tree, let  $\pi(o)$  be the sum of the conductances of the edges incident to the root, and denote by  $T(o)$  the first return time to the origin by the walk. Following [15] (page 25), we can define the *effective conductance* of the network by

$$(2.1) \quad \mathcal{C}_c(\mathcal{T}) := \pi(o)\tilde{\mathcal{C}}_c(\mathcal{T}),$$

where  $\tilde{\mathcal{C}}_c(\mathcal{T}) := \mathbb{P}[T(o) = +\infty]$ . The reciprocal  $\mathcal{R}_c(\mathcal{T})$  of the effective conductance is called the *effective resistance*.

We will be particularly interested in the case where the conductances are chosen exponentially in the distance to the root, more precisely if for every edge  $e = (x, y)$  where  $x$  is the parent of  $y$  we let  $c(e) = \lambda^{|x|}$ , because in that case the random walk on the network is exactly the same process as the random walk  $RW_\lambda$  defined earlier. We will use the following notation many times in what follows:

**Notation 14.** For every parameter  $\lambda > 0$ , assigning to every edge  $e = (x, y)$  conductance  $\lambda^{|x|}$ , we denote by  $\mathcal{C}(\lambda, \mathcal{T})$  (resp.  $\mathcal{R}(\lambda, \mathcal{T})$ ) the effective conductance (resp. resistance) of the associated network. Moreover, if  $\nu$  is a child of the root  $o$  of  $\mathcal{T}$ , we write  $\tilde{\mathcal{C}}(\lambda, \mathcal{T}, \nu)$  for the probability of the event that the random walk  $RW_\lambda$  on  $\mathcal{T}$ , started at the root (i.e  $X_0 = o$ ), visits  $\nu$  at its first step (i.e  $X_1 = \nu$ ) and never returns to the root.

**Theorem 15** (Rayleigh's monotonicity principle [15]). *Let  $\mathcal{T}$  be an infinite tree with two assignments,  $c$  and  $c'$ , of conductances on  $\mathcal{T}$  with  $c \leq c'$  (everywhere). Then the effective conductances are ordered in the same way:  $\mathcal{C}_c(\mathcal{T}) \leq \mathcal{C}_{c'}(\mathcal{T})$ .*

**Corollary 16.** *Let  $\mathcal{T}, \mathcal{T}'$  be two infinite trees; we say that  $\mathcal{T} \subset \mathcal{T}'$  if there exists an injective morphism  $f : \mathcal{T} \rightarrow \mathcal{T}'$ . If this holds, then for every  $\lambda > 0$ ,  $\mathcal{C}(\lambda, \mathcal{T}') \leq \mathcal{C}(\lambda, \mathcal{T})$ .*

In the case of spherically symmetric trees, the equivalent resistance is explicit:

**Proposition 17** (see [15]). *Let  $\mathcal{T}$  be spherically symmetric and  $(c(e))$  be conductances that are themselves constant on the levels of  $\mathcal{T}$ . Then  $\mathcal{R}_c(\mathcal{T}) = \sum_{n \geq 1} \frac{1}{c_n |\mathcal{T}_n|}$ , where  $c_n$  is the conductance of the edges going from level  $n - 1$  to level  $n$ .*

The following corollaries are the consequences of Proposition 17:

**Corollary 18.** *Let  $\mathcal{T}$  be a spherically symmetric tree. The effective conductance  $\mathcal{C}(\lambda, \mathcal{T})$  is a continuous function on  $(\lambda_c, +\infty)$ .*

**Corollary 19.** *Let  $\mathcal{T}$  be a spherically symmetric tree. Then  $RW_\lambda$  is transient if and only if  $\sum_n \frac{1}{\lambda^n |\mathcal{T}_n|} < \infty$ .*

**Theorem 20** (Nash-Williams criterion, see [17]). *If  $(\pi_n, n \geq 0)$  is a sequence of pairwise disjoint finite  $E$ -cutsets in a locally finite network  $\mathcal{G}$ , then*

$$\mathcal{R}_c(\mathcal{T}) \geq \sum_n \left( \sum_{e \in \pi_n} c(e) \right)^{-1}.$$

*In particular, if  $\sum_n \left( \sum_{e \in \pi_n} c(e) \right)^{-1} = +\infty$ , then the random walk associated to this family of conductances  $(c(e), e \in E(\mathcal{T}))$  is recurrent.*

We end this subsection by stating a classical theorem relating the recurrence or transience of  $RW_\lambda$  to the branching of the underlying tree:

**Theorem 21** (see [14]). *Let  $\mathcal{T}$  be an infinite, locally finite and rooted tree. If  $\lambda < \frac{1}{br(\mathcal{T})}$  then  $RW_\lambda$  is recurrent, whereas if  $\lambda > \frac{1}{br(\mathcal{T})}$ , then  $RW_\lambda$  is transient. The critical value of biased random walk on  $\mathcal{T}$  is therefore  $\lambda_c(\mathcal{T}) := \frac{1}{br(\mathcal{T})}$ .*

**2.4. The law of the first  $k$  steps of the limit walk.** Let  $\mathcal{T}$  be a tree and  $(c(e))$  be conductances on the edges of  $\mathcal{T}$  such that the associated random walk  $(X_n)$  is transient. For every  $k \geq 0$ , the walk visits  $\mathcal{T}_k$  finitely many times: we can define an infinite path  $\omega^\infty$  on  $\mathcal{T}$  by letting  $\omega^\infty(k)$  be the last vertex of  $\mathcal{T}_k$  visited by the walk. Equivalently:

$$(2.2) \quad \omega^\infty(k) = \nu \iff \nu \in \mathcal{T}_k \text{ and } \exists n_0, \forall n > n_0 : X_n \in \mathcal{T}^\nu.$$

Let  $k \in \mathbb{N}^*$  and  $\nu_0 = o, \nu_1, \nu_2, \dots, \nu_k$  be  $k$  elements of  $V(\mathcal{T})$  such that the path  $(\nu_0, \nu_1, \nu_2, \dots, \nu_k)$  is a possible prefix of  $\omega^\infty$ : we can then define

$$(2.3) \quad \varphi_c(\nu_0, \nu_1, \nu_2, \dots, \nu_k) := \mathbb{P}(\omega^\infty(0) = \nu_0, \omega^\infty(1) = \nu_1, \dots, \omega^\infty(k) = \nu_k).$$

We will refer to this function as the *law of first  $k$  steps of limit walk*. In the case of the biased walk  $RW_\lambda$ , we will denote the function by  $\varphi^{\lambda, k}$ ; this will not lead to ambiguities. We finish this section with the following lemma.

**Lemma 22.** *The value of  $\varphi_c(\nu_0, \dots, \nu_k)$  depends continuously on any finite collection of the conductances in the network. More precisely, given a finite set  $U = \{e_1, \dots, e_\ell\}$  of edges and a collection  $(c(e))$  of conductances, let  $\tilde{c}(u_1, \dots, u_\ell)$  be the family of conductances that coincides with  $c$  outside  $U$  and takes value  $u_i$  at  $e_i$ : then the map*

$$\psi_{U, c} : (u_1, \dots, u_\ell) \mapsto \varphi_{\tilde{c}(u_1, \dots, u_\ell)}(\nu_0, \dots, \nu_k)$$

is continuous on  $(\mathbb{R}_+^*)^\ell$ .

*Proof.* The proof is simple, therefore it is omitted.  $\square$

### 3. A FEW EXAMPLES

The self-avoiding tree in the plane, which we alluded to in the introduction and will formally introduce in the next section, is sub-periodic but quite inhomogeneous, and the self-avoiding tree in the half-plane sits in none of the classes of trees defined above. To get an intuition of the kind of behavior we should expect or rule out, we gather here a few examples of trees with some atypical features.

**3.1. Trees with discontinuous conductance.** Let  $0 < \lambda_0 \leq 1$ . In the first part of this section, we construct two trees  $\mathcal{T}, \bar{\mathcal{T}}$  with  $\lambda_c(\mathcal{T}) = \lambda_c(\bar{\mathcal{T}}) = \lambda_0$ , such that the effective conductances  $C(\lambda, \mathcal{T})$  and  $C(\lambda, \bar{\mathcal{T}})$  of the biased random walk  $RW_\lambda$  on  $\mathcal{T}$  and  $\bar{\mathcal{T}}$  satisfy  $\mathcal{C}(\lambda_c(\mathcal{T}), \mathcal{T}) = 0$  but  $\mathcal{C}(\lambda_c(\bar{\mathcal{T}}), \bar{\mathcal{T}}) > 0$ . In the second part, we construct a tree  $\mathcal{T}$  such that  $C(\lambda, \mathcal{T})$  is not continuous on  $(\lambda_c, 1)$ .

**Proposition 23.** *For every  $x \geq 1$ , there exist two trees  $\mathcal{T}$  and  $\bar{\mathcal{T}}$  such that:*

- $br(\mathcal{T}) = br(\bar{\mathcal{T}}) = x$ ;
- $RW_{1/x}$  is recurrent on  $\mathcal{T}$  and transient on  $\bar{\mathcal{T}}$ .

*Proof.* We will construct spherically symmetric trees satisfying both conditions. Denote by  $\lfloor y \rfloor$  the integer part of  $y$ . We construct the sequence  $(\ell_i)_{i \in \mathbb{N}^*}$  inductively as follows:

$$\ell_1 = \lfloor x \rfloor, \quad \ell_2 = \left\lfloor \frac{x^2}{\ell_1} \right\rfloor, \quad \ell_3 = \left\lfloor \frac{x^3}{\ell_1 \ell_2} \right\rfloor, \quad \dots, \quad \ell_n = \left\lfloor \frac{x^n}{\prod_{i=1}^{n-1} \ell_i} \right\rfloor, \quad \dots$$

and let  $\mathcal{T}$  be the tree where vertices at distance  $i - 1$  from  $o$  have  $\ell_i$  children, so that the sizes of the levels of  $\mathcal{T}$  are given by  $|\mathcal{T}_n| = \prod_{i=1}^n \ell_i$ . We construct the tree  $\overline{\mathcal{T}}$  from the degree sequence  $(\ell'_i)_{i \in \mathbb{N}}$  by posing  $\ell'_i = 2\ell_i$  if  $i$  can be written under the form  $i = k^2$ , and  $\ell'_i = \ell_i$  otherwise. Notice that  $|\overline{\mathcal{T}}_n| = 2^{\lfloor \sqrt{n} \rfloor} |\mathcal{T}_n|$ .

We first show that both trees have branching number  $x$ . Since they are spherically symmetric, it is enough to check that their growth rate is  $x$ ; the case  $x = 1$  is trivial, so assume  $x > 1$ . From the definition,

$$x^n - \prod_{i=1}^{n-1} \ell_i \leq \prod_{i=1}^n \ell_i \leq x^n \quad \text{hence} \quad x^n - x^{n-1} \leq |\mathcal{T}_n| \leq x^n$$

so  $gr(\mathcal{T}) = x$ ; the case of  $\overline{\mathcal{T}}$  follows directly.

The recurrence or transience of the critical random walks can be determined using Lemma 19:

$$\sum \frac{1}{\lambda_c^n |\mathcal{T}_n|} \geq \sum \frac{1}{\lambda_c^n x^n} = +\infty$$

so the critical walk on  $\mathcal{T}(x)$  is recurrent, while for  $x > 1$ ,

$$\sum \frac{1}{\lambda_c^n |\overline{\mathcal{T}}_n|} \leq \sum \frac{1}{\lambda_c^n (x^n - x^{n-1}) 2^{\lfloor \sqrt{n} \rfloor}} = \frac{x}{x-1} \sum \frac{1}{2^{\lfloor \sqrt{n} \rfloor}} < \infty$$

so the critical walk on  $\overline{\mathcal{T}}(x)$  is transient. In the case  $x = 1$  one gets  $\sum 2^{-\lfloor \sqrt{n} \rfloor} < \infty$  instead, and the conclusion is the same.  $\square$

**Proposition 24.** *For every  $k \in \mathbb{N}^*$  and  $\lambda_c \in (0, 1)$ , there exists a tree  $\mathcal{T}$  with critical drift  $\lambda_c(\mathcal{T}) = \lambda_c$  such that the ratio  $\mathcal{C}(\lambda, \mathcal{T})/(\lambda - \lambda_c)^k$  remains bounded away from 0 as  $\lambda \rightarrow \lambda_c^+$ .*

*Proof.* We construct a spherically symmetric tree  $\mathcal{T}$  which satisfies the conditions of this proposition in a similar way as before. Letting  $x = 1/\lambda_c > 1$ , define inductively:

$$\ell_1 = \lfloor x \rfloor, \quad \ell_2 = \left\lfloor \frac{x^2}{2^k \ell_1} \right\rfloor, \quad \dots, \quad \ell_n = \left\lfloor \frac{x^n}{n^k \prod_{i=1}^{n-1} \ell_i} \right\rfloor, \quad \dots$$

Let  $\mathcal{T}$  be the spherically symmetric tree with degree sequence  $(\ell_i)$ . It is easy to check that  $br(\mathcal{T}) = x$  like in the previous proposition; in a similar way,

$$x^n - n^k \prod_{i=1}^{n-1} \ell_i \leq n^k \prod_{i=1}^n \ell_i \leq x^n \quad \text{hence} \quad \frac{x^n}{n^k} - \frac{x^{n-1}}{(n-1)^k} \leq |\mathcal{T}_n| \leq \frac{x^n}{n^k}.$$

Recall that  $x = 1/\lambda_c$  and by using Proposition 17, the effective resistance at parameter  $\lambda > \lambda_c$  is given by

$$\mathcal{R}(\lambda, \mathcal{T}) = \sum \frac{1}{\lambda^n |\mathcal{T}_n|} \geq \sum \frac{n^k}{(\lambda x)^n} = \sum n^k \left( \frac{\lambda_c}{\lambda} \right)^n.$$

By an easy computation, there exists a constant  $C_k > 0$  such that

$$\sum n^k \left( \frac{\lambda_c}{\lambda} \right)^n \sim_{\lambda \rightarrow \lambda_c^+} \frac{C_k}{\left(1 - \frac{\lambda_c}{\lambda}\right)^{k+1}} \sim_{\lambda \rightarrow \lambda_c^+} \frac{\lambda^{k+1} C_k}{(\lambda - \lambda_c)^{k+1}},$$

implying that there exists a constant  $D_k > 0$ , uniform in  $\lambda$  close to  $\lambda_c$ , such that  $\mathcal{R}(\lambda, \mathcal{T}) \geq \frac{D_k}{(\lambda - \lambda_c)^{k+1}}$ . An upper bound of the same order can be obtained in a very similar fashion, leading to the conclusion.  $\square$

We end this subsection with the following proposition, showing that discontinuities can occur elsewhere than at  $\lambda_c$ :

**Proposition 25.** *There exists a tree  $\mathcal{T}$  such that the function  $\mathcal{C}(\lambda, \mathcal{T})$  is not continuous on  $(\lambda_c, 1)$ , i.e it will be discontinuous at a certain  $\lambda' \in (\lambda_c, 1)$ .*

*Proof.* Let  $0 < \lambda_1 < \lambda_2 < 1$ . By Proposition 23, there exist two trees  $\mathcal{H}$  and  $\mathcal{G}$  such that  $\lambda_c(\mathcal{H}) = \lambda_1$ ,  $\lambda_c(\mathcal{G}) = \lambda_2$  and

$$(3.1) \quad \mathcal{C}(\lambda_1, \mathcal{H}) = 0, \quad \mathcal{C}(\lambda_2, \mathcal{G}) > 0.$$

We construct a tree  $\mathcal{T}$  rooted at  $o$  as follows:

$$\mathcal{T}_1 = \{\nu_1, \nu_2\}, \quad \mathcal{T}^{\nu_1} = \mathcal{H} \quad \text{and} \quad \mathcal{T}^{\nu_2} = \mathcal{G}.$$

Hence,

$$\lambda_c(\mathcal{T}) = \lambda_1.$$

Denote by  $\deg \nu_1$  (resp.  $\deg \nu_2$ ) the degree of  $\nu_1$  (resp.  $\nu_2$ ) in the tree  $\mathcal{T}$ . By an easy computation, for any  $\lambda \in (\lambda_1, 1)$ , we obtain:

$$(3.2) \quad \mathcal{C}(\lambda, \mathcal{T}) = \frac{1}{2} \times \frac{\lambda \mathcal{C}(\lambda, \mathcal{H}) \deg \nu_1}{1 + \lambda \mathcal{C}(\lambda, \mathcal{H}) \deg \nu_1} + \frac{1}{2} \times \frac{\lambda \mathcal{C}(\lambda, \mathcal{G}) \deg \nu_2}{1 + \lambda \mathcal{C}(\lambda, \mathcal{G}) \deg \nu_2}.$$

By Corollary 18, the function  $\mathcal{C}(\lambda, \mathcal{H})$  is continuous on  $(\lambda_1, 1)$  and since  $\mathcal{C}(\lambda, \mathcal{G}) = 0$  for any  $\lambda \in (\lambda_1, \lambda_2)$ , therefore:

$$(3.3) \quad \lim_{\lambda \rightarrow \lambda_2^-} \mathcal{C}(\lambda, \mathcal{T}) = \frac{1}{2} \times \frac{\lambda_2 \mathcal{C}(\lambda_2, \mathcal{H}) \deg \nu_1}{1 + \lambda_2 \mathcal{C}(\lambda_2, \mathcal{H}) \deg \nu_1}.$$

By Equations (3.1), (3.2) and (3.3), we obtain:

$$\lim_{\lambda \rightarrow \lambda_2^-} \mathcal{C}(\lambda, \mathcal{T}) < \mathcal{C}(\lambda_2, \mathcal{T}).$$

The latter inequality implies that the function  $\mathcal{C}(\lambda, \mathcal{T})$  is discontinuous at  $\lambda_2$ .  $\square$

Note that continuity properties at  $\lambda \geq 1$  are actually easier to obtain, and we will investigate them further below.

**3.2. The convergence of the law of the first  $k$  steps.**

If  $\lim_{\lambda \rightarrow \lambda_c, \lambda > \lambda_c} \mathcal{C}(\lambda, \mathcal{T}) > 0$ , by Lemma 64 the limit of  $\varphi^{\lambda, k}(y_1, \dots, y_k)$  when  $\lambda$  decreases to  $\lambda_c$  exists. If one has  $\lim_{\lambda \downarrow \lambda_c} \mathcal{C}(\lambda, \mathcal{T}) = 0$ , the situation is more delicate and we cannot yet conclude on the limit of the function  $\varphi^{\lambda, k}(\nu_0, \dots, \nu_k)$  when  $\lambda$  decreases to  $\lambda_c$ . Indeed, convergence does not always hold, as we will see in a counterexample. The idea of what follows is easy to describe: we are going to construct a very inhomogeneous tree with various subtrees of higher and higher branching numbers, at locations alternating between two halves of the whole tree; a biased random walk will wander until it finds the first such subtree inside which it is transient, and escape to infinity within this subtree with high probability.

**Proposition 26.** *There exists a tree  $\mathcal{T}$  such that the function  $\varphi^{\lambda, 1}(y_0, y_1)$  does not converge as  $\lambda \rightarrow \lambda_c$ .*

**Notation 27.** *Let  $\mathcal{T}, \mathcal{T}'$  be two trees and  $A \subset V(\mathcal{T})$ . We can construct a new tree by grafting a copy of  $\mathcal{T}'$  at all the vertices of  $A$ ; we will denote by  $\mathcal{T} \overset{A}{\oplus} \mathcal{T}'$  this new tree. Note that for all  $x \in A$ ,  $(\mathcal{T} \overset{A}{\oplus} \mathcal{T}')^x \simeq \mathcal{T}'$ . In the case  $A = \{x\}$ , we will use the simpler notation  $\mathcal{T} \overset{x}{\oplus} \mathcal{T}'$  for  $\mathcal{T} \overset{\{x\}}{\oplus} \mathcal{T}'$ .*

*Proof.* Fix  $\varepsilon > 0$  small enough. By Proposition 23, for all  $0 < a \leq 1$ , there exists a tree, denoted by  $\mathcal{T}(a)$ , such that its branching number is  $\frac{1}{a}$  and  $\mathcal{C}(a, \mathcal{T}(a)) = 0$ . Let  $\mathcal{H} = \mathbb{Z}$ , seen as a tree rooted at 0, so that the integers are the vertices of  $\mathcal{H}$  (see the Figure 1). We are going to construct a tree inductively.

Let  $(a_i)_{i \geq 1}$  be a decreasing sequence such that  $a_1 < 1$ . Set  $a_c := \lim a_i$  and assume that  $a_c > 0$ . Choose a sequence  $(b_i)_{i \geq 1}$  such that  $b_i \in (a_{i+1}, a_i)$  for all  $i$ . First, set  $\mathcal{H}^0 := (\mathcal{H} \overset{-2}{\oplus} \mathcal{T}(a_1)) \overset{2}{\oplus} \mathcal{T}(a_2)$ . We consider the biased random walk  $RW_{b_1}$ , then it is recurrent on  $\mathcal{T}(a_1)$  and transient on  $\mathcal{T}(a_2)$ . On  $\mathcal{H}^0$ , the biased random walk  $RW_{b_1}$  is transient, and in addition we know that it stays eventually within the copy of  $\mathcal{T}(a_2)$ . There exists then  $N_1 > 2$  such that the probability that the limit walk remains in that copy after time  $N_1 - 1$  is greater than  $1 - \varepsilon$ .

Then we set  $\mathcal{H}^1 = (\mathcal{H}^0 \overset{-N_1}{\oplus} \mathcal{T}(a_3))$ . On  $\mathcal{H}^1$ , the walk of bias  $b_1$  is still transient and still has probability at least  $1 - \varepsilon$  to escape through the copy of  $\mathcal{T}(a_2)$ , because  $\mathcal{T}(a_3)$  is grafted too far to be relevant. On the other hand, consider the biased random walk  $RW_{b_2}$ : it is still transient on  $\mathcal{H}^1$  but only through the new copy of  $\mathcal{T}(a_3)$ . There exists then  $N_2 > 2$  such that the probability that the limit walk remains in that copy after time  $N_2 - 1$  is greater than  $1 - \varepsilon$ .

We can set  $\mathcal{H}^2 := (\mathcal{H}^1 \overset{N_2}{\oplus} \mathcal{T}(a_4))$  and continue this procedure to graft all the trees  $\mathcal{T}(a_i)$ , further and further from the origin and alternatively on the left and on the right; denote by  $\mathcal{H}^\infty$  the union of all the  $\mathcal{H}^k$ .

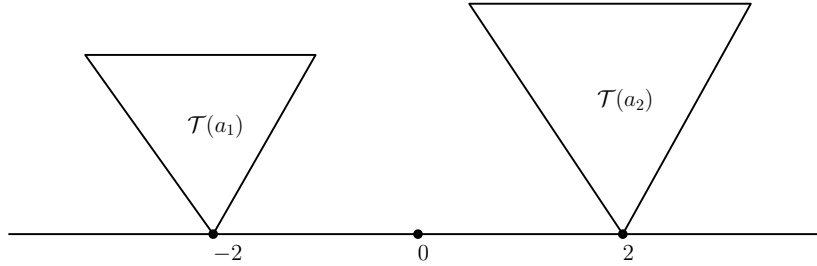


FIGURE 1. Tree  $H^0$

It remains to show that the function  $\varphi^{\lambda,1}(0,1)$  for the biased random walk on the tree  $\mathcal{H}^\infty$  does not converge. We have  $br(\mathcal{H}^\infty) = \sup_i br(\mathcal{T}(a_i)) = \frac{1}{a_c}$  and  $\varphi^{b_i,1}(0,1) \geq 1 - \varepsilon$  if  $i$  is odd while  $\varphi^{b_i,1}(0,-1) \geq 1 - \varepsilon$  if  $i$  is even. Then,

$$\forall k \geq 0, \begin{cases} \varphi^{b_i,1}(0,1) \geq 1 - \varepsilon & \text{if } i = 2k + 1 \\ \varphi^{b_i,1}(0,1) \leq \varepsilon & \text{if } i = 2k + 2 \end{cases}$$

This implies that the function  $\varphi^{\lambda,1}(0,1)$  does not converge when  $\lambda$  go to  $a_c$ .  $\square$

The tree we just constructed is tailored to be extremely inhomogeneous. At the other end of the spectrum, some trees have enough structure for all the functions we are considering to be essentially explicit:

**Definition 28.** A tree  $\mathcal{T}$  is called periodic (or finite type) if, for all  $v \in V(\mathcal{T}) \setminus \{o\}$ , there is a bijective morphism  $f : \mathcal{T}^v \rightarrow \mathcal{T}^{f(v)}$  with  $f(v)$  in a fixed, finite neighborhood of the root of  $\mathcal{T}$ .

**Definition 29.** Let  $\mathcal{T}$  be a finite tree and  $\mathcal{L}(\mathcal{T})$  be the set of leafs of  $\mathcal{T}$ . We set  $\mathcal{T}^1 = \mathcal{T} \overset{\mathcal{L}(\mathcal{T})}{\oplus} \mathcal{T}$ ,  $\mathcal{T}^2 = \mathcal{T}^1 \overset{\mathcal{L}(\mathcal{T}^1)}{\oplus} \mathcal{T}$ ,  $\dots$ ,  $\mathcal{T}^n = \mathcal{T}^{n-1} \overset{\mathcal{L}(\mathcal{T}^{n-1})}{\oplus} \mathcal{T}$  for every  $n \geq 2$ . We continue this procedure an infinite number of times to obtain an infinite tree  $\mathcal{T}^{\infty, \mathcal{T}}$ . Note that  $\mathcal{T}^{\infty, \mathcal{T}}$  is also a periodic tree.

**Fact 30** (see Lyons [14], theorem 5.1). Let  $\mathcal{T}$  be a periodic tree and  $(\nu_0 = o, \nu_1, \nu_2, \dots, \nu_k)$  be a simple path on  $\mathcal{T}$ . Then  $\varphi^{\lambda,k}(\nu_0, \nu_1, \dots, \nu_k)$  converges when  $\lambda$  decreases to  $\lambda_c(\mathcal{T})$ .

In the rest of this section we provide a new proof of a particular case (the case of  $\mathcal{T}^{\infty, \mathcal{T}}$ ) of fact 30:

**Proposition 31.** *Let  $\mathcal{T}$  be a finite tree and  $(y_0 = o, y_1, \dots, y_k)$  be a simple path on  $\mathcal{T}^{\infty, \mathcal{T}}$ . Then the function  $\varphi^{\lambda, k}(y_0, y_1, \dots, y_k)$  of  $\mathcal{T}^{\infty, \mathcal{T}}$  converges when  $\lambda$  decreases to  $\lambda_c(\mathcal{T}^{\infty, \mathcal{T}})$ .*

Before showing Proposition 31, we need to show the following lemma:

**Lemma 32.** *Let  $\mathcal{T}$  be a tree rooted at  $o$  such that  $\deg o = d_0$  and*

$$\left\{ \begin{array}{l} \mathcal{T}_1 = \{\nu_1, \nu_2, \dots, \nu_{d_0}\} \\ \forall i \in \{1, 2, \dots, d_0\}, \lambda_c(\mathcal{T}) = \lambda_c(\mathcal{T}^{\nu_i}) = \lambda_c \text{ and } \mathcal{C}(\lambda_c, \mathcal{T}) = \mathcal{C}(\lambda_c, \mathcal{T}^{\nu_i}) = 0 \end{array} \right.$$

*Then for all  $i$ , we have  $\tilde{\mathcal{C}}(\lambda, \mathcal{T}, \nu_i) = \frac{(d_{\nu_i}-1)\lambda\tilde{\mathcal{C}}(\lambda, \mathcal{T}^{\nu_i})}{d_0(1+(d_{\nu_i}-1)\lambda\tilde{\mathcal{C}}(\lambda, \mathcal{T}^{\nu_i}))}$ , where  $d_{\nu_i} = \deg \nu_i$ .*

*Proof.* Recall that  $\tilde{\mathcal{C}}(\lambda, \mathcal{T}, \nu_i) = \mathbb{P}(\mathcal{A})$ , where  $\mathcal{A}$  is the event that the random walk  $RW_\lambda$  on  $\mathcal{T}$ , started at the root (i.e  $X_0 = o$ ), never returns to it and reached  $\nu_i$  at the first step (i.e  $X_1 = \nu_i$ ). We can write

$$\mathcal{A} = \bigcup_{k \geq 0} \mathcal{A}_k$$

where

$$\mathcal{A}_k := \{\#\{j > 0 : X_j = o\} = 0\} \cap \{X_1 = \nu_i\} \cap \{\#\{j > 1 : X_j = \nu_i\} = k\}.$$

Let  $m = \frac{(d_{\nu_i}-1)\lambda}{1+(d_{\nu_i}-1)\lambda}$  and  $c = \tilde{\mathcal{C}}(\lambda, \mathcal{T}^{\nu_i})$ . Note that the sequence  $(\mathcal{A}_k, k \geq 0)$  are pairwise disjoint and  $\mathbb{P}(\mathcal{A}_k) = \frac{mc(m(1-c))^k}{d_0}$ , therefore we obtain:

$$\tilde{\mathcal{C}}(\lambda, \mathcal{T}, \nu_i) = \frac{mc}{d_0} \sum_{k=0}^{\infty} (m(1-c))^k = \frac{(d_{\nu_i}-1)\lambda\tilde{\mathcal{C}}(\lambda, \mathcal{T}^{\nu_i})}{d_0(1+(d_{\nu_i}-1)\lambda\tilde{\mathcal{C}}(\lambda, \mathcal{T}^{\nu_i}))}. \quad \square$$

**Proof of Proposition 31.** First, since  $\mathcal{T}^{\infty, \mathcal{T}}$  is a periodic tree, the biased random walk  $RW_{\lambda_c}$  on  $\mathcal{T}^{\infty, \mathcal{T}}$  is recurrent (see [14]). Recall that  $L(\mathcal{T})$  is the set of all leafs of the finite tree  $\mathcal{T}$ ; let  $S^i$  be the set of all finite paths starting at the origin, ending at one element of  $L(\mathcal{T})$  and passing through  $\nu_i$ . For all  $\nu \in L(\mathcal{T})$ , we have  $(\mathcal{T}^{\infty, \mathcal{T}})^\nu = \mathcal{T}^{\infty, \mathcal{T}}$  and we can apply Lemma 32 several times to obtain:

$$\tilde{\mathcal{C}}(\lambda, \mathcal{T}^{\infty, \mathcal{T}}, \nu_i) = \sum_{\gamma \in S^i} f_1^\gamma(\lambda) f_2^\gamma(\lambda) \cdots f_{|\gamma|}^\gamma(\lambda) \tilde{\mathcal{C}}(\lambda, (\mathcal{T}^{\infty, \mathcal{T}})^{\gamma_{|\gamma|}}),$$

where  $f_j^\gamma(\lambda) = \frac{m_{\gamma_j} \lambda}{m_{\gamma_{j-1}}(1+m_{\gamma_j} \lambda \mathcal{C}(\lambda, (\mathcal{T}^{\infty, \mathcal{T}})^{\gamma_j}))}$  and  $m_{\gamma_j} = d_{\gamma_j} - 1$  if  $j > 1$  and  $m_{\gamma_0} = d_0$ . Moreover, we have

$$\tilde{\mathcal{C}}(\lambda, (\mathcal{T}^{\infty, \mathcal{T}})^{\gamma_{|\gamma|}}) = \tilde{\mathcal{C}}(\lambda, \mathcal{T}^{\infty, \mathcal{T}})$$



then

$$\tilde{\mathcal{C}}(\lambda, \mathcal{T}^{\infty, \mathcal{T}}, \nu_i) = \sum_{\gamma \in S^i} f_1^\gamma(\lambda) f_2^\gamma(\lambda) \cdots f_{|\gamma|}^\gamma(\lambda) \tilde{\mathcal{C}}(\lambda, \mathcal{T}^{\infty, \mathcal{T}}).$$

By Lemma 64, we obtain

$$\varphi^{\lambda, 1}(o, \nu_i) = \frac{\tilde{\mathcal{C}}(\lambda, \mathcal{T}^{\infty, \mathcal{T}}, \nu_i)}{\tilde{\mathcal{C}}(\lambda, \mathcal{T}^{\infty, \mathcal{T}})} = \sum_{\gamma \in S^i} f_1^\gamma(\lambda) f_2^\gamma(\lambda) \cdots f_{|\gamma|}^\gamma(\lambda).$$

Note that for all  $\gamma \in S^i$  we have  $m_{\gamma_0} = m(\gamma_{|\gamma|})$ . Moreover, since the biased random walk  $RW_{\lambda_c}$  on  $\mathcal{T}^{\infty, \mathcal{T}}$  is recurrent, for all  $0 \leq j \leq |\gamma|$ , we have:

$$\mathcal{C}(\lambda_c(\mathcal{T}^{\infty, \mathcal{T}}), (\mathcal{T}^{\infty, \mathcal{T}})^{\gamma_j}) = 0.$$

Hence,  $\varphi^{\lambda, 1}(o, \nu_i)$  converges when  $\lambda$  decreases towards  $\lambda_c(\mathcal{T}^{\infty, \mathcal{T}})$  and

$$(3.4) \quad \lim_{\lambda \rightarrow \lambda_c(\mathcal{T}^{\infty, \mathcal{T}})} \varphi^{\lambda, 1}(o, \nu_i) = \sum_{\gamma \in S^i} \lambda_c^{|\gamma|}.$$

This proves the statement of the proposition in the case  $k = 1$ ; general cases are handled in a very similar fashion, enumerating the vertices at distance  $k$  from the root rather than children of  $o$ .  $\square$

**Remark 33.** *The equation (3.4) gives us a way to calculate the critical value of  $RW_\lambda$  on  $\mathcal{T}^{\infty, \mathcal{T}}$ , as the solution of the following equation:*

$$(3.5) \quad \sum_{i=1}^{m_o} \sum_{\gamma \in S^i} x^{|\gamma|} = 1.$$

#### 4. THE CONTINUITY OF EFFECTIVE CONDUCTANCE

We end the first half of the paper with a few results on the conductance functions of trees, namely we prove a criterion for the continuity of  $\mathcal{C}(\lambda, \mathcal{T})$  in  $\lambda$  (see Theorems 37 and 38 below) and study the set of conductance functions of spherically symmetric trees of bounded degree (see Theorem 4).

##### 4.1. Left- and right-continuity of $\mathcal{C}(\mathcal{T}, \lambda)$ .

**Lemma 34.** *Let  $\mathcal{T}$  be an infinite, locally finite and rooted tree. Then  $\mathcal{C}(\lambda, \mathcal{T})$  is right continuous on  $(0, +\infty)$ .*

*Proof.* Let  $(X_n, n \geq 0)$  be the biased random walk with parameter  $\lambda$  on  $\mathcal{T}$ . We define  $S_0 := \inf \{k > 0 : X_k = o\}$  and for any  $n > 0$ ,

$$S_n := \inf \{k > 0 : d(o, X_k) = n\}.$$

Recall that the random walk on a network  $(\mathcal{T}, c)$ , where  $c(e) = \lambda^{|e|}$  is exactly the same process as the biased random walk with parameter  $\lambda$ . We use Equation (2.1) to obtain

$$\mathcal{C}(\lambda, \mathcal{T}) = \pi(o) \lim_{n \rightarrow +\infty} \mathbb{P}(S_n < S_0).$$

We set  $\mathcal{C}(\lambda, \mathcal{T}, n) := \pi(o) \mathbb{P}(S_n < S_0)$ . It is easy to see that  $\mathcal{C}(\lambda, \mathcal{T}, n) \geq \mathcal{C}(\lambda, \mathcal{T}, n+1)$ . On the other hand, by Lemma 22, we obtain  $\mathcal{C}(\lambda, \mathcal{T}, n)$  is a continuous function. Hence,  $\mathcal{C}(\lambda, \mathcal{T}, n)$  is a continuous increasing function for each  $n$ . It implies that  $\mathcal{C}(\lambda, \mathcal{T})$  is the decreasing limit of increasing functions. Therefore  $\mathcal{C}(\lambda, \mathcal{T})$  is right continuous.  $\square$

**Definition 35.** Let  $\mathcal{T}$  be a tree. For each  $\nu \in \mathcal{T}$ , denote by  $X_n^\nu$  the biased random walk on the subtree  $\mathcal{T}^\nu$  (i.e.  $X_0^\nu = \nu$  and  $\forall n > 0, X_n^\nu \in \mathcal{T}^\nu$ ). We say that  $\mathcal{T}$  is uniformly transient if

$$\forall \lambda > \lambda_c, \exists \alpha_\lambda > 0, \forall \nu \in \mathcal{T}, \mathbb{P}(\forall n > 0, X_n^\nu \neq \nu) \geq \alpha_\lambda.$$

It is called weakly uniformly transient if there exists a sequence of finite pairwise disjoint  $V$ -cutsets  $(\pi_n, n \geq 1)$ , such that

$$\forall \lambda > \lambda_c, \exists \alpha_\lambda > 0, \forall \nu \in \bigcup_{k=1}^{+\infty} \pi_k, \mathbb{P}(\forall n > 0, X_n^\nu \neq \nu) \geq \alpha_\lambda.$$

**Remark 36.** It is easy to see that if  $\lambda_c(\mathcal{T}) = 1$ , then  $\mathcal{T}$  is uniformly transient: indeed, on every infinite subtree and for every  $\lambda > 1$ , escape probabilities are bounded below by the escape probability in  $\mathbb{Z}_+$  which is itself strictly positive for  $\lambda > 1$ .

**Theorem 37.** Let  $\mathcal{T}$  be a uniformly transient tree. Then  $\mathcal{C}(\lambda, \mathcal{T})$  is left continuous on  $(\lambda_c, +\infty)$ .

*Proof.* Fix  $\lambda_1 > \lambda_c$ , we will prove that  $\mathcal{C}(\lambda, \mathcal{T})$  is left continuous at  $\lambda_1$ . Choose  $\lambda_0 \in (\lambda_c, \lambda_1)$ . By Theorem 15, we can find a constant  $\alpha > 0$  (does not depend on  $\lambda \in [\lambda_0, \lambda_1]$ ) such that

$$\forall \lambda \in [\lambda_0, \lambda_1], \forall \nu \in V(\mathcal{T}), \mathbb{P}(\forall n > 0, X_n^\nu \neq \nu) \geq \alpha.$$

Given a family of conductances  $c = c(e)_{e \in E(\mathcal{T})} \in (0, +\infty)^E$ , let  $Y_n$  be the associated random walk. Let  $A \subset (0, +\infty)^E$  be the subset of elements of  $(0, +\infty)^E$  such that  $Y_n$  is transient for those choices of conductances. Then we define the function  $\psi : A \rightarrow \mathbb{R}_+^*$  as

$$\psi(c) := \mathcal{C}_c(\mathcal{T}).$$

Recall that  $\mathcal{T}_k$  is the collection of all the vertices at distance  $k$  from the root: then we have

$$\mathcal{C}(\lambda, \mathcal{T}) = \psi(\underbrace{\lambda, \lambda, \dots, \lambda}_{|\mathcal{T}_1|}, \underbrace{\lambda^2, \lambda^2, \dots, \lambda^2}_{|\mathcal{T}_2|}, \dots).$$

We will abuse notation until the end of the argument, writing

$$\psi(\lambda_1, \lambda_2^2, \lambda_3^3, \dots) \quad \text{for} \quad \psi(\underbrace{\lambda_1, \lambda_1, \dots, \lambda_1}_{|\mathcal{T}_1|}, \underbrace{\lambda_2^2, \lambda_2^2, \dots, \lambda_2^2}_{|\mathcal{T}_2|}, \dots)$$

so that in particular  $\mathcal{C}(\lambda, \mathcal{T}) = \psi(\lambda, \lambda^2, \lambda^3, \dots)$ .

Let  $\varepsilon > 0$ , we choose  $L \in \mathbb{N}$  such that  $(1 - \alpha)^L < \varepsilon$ . For  $\lambda \in (\lambda_0, \lambda_1)$  we have  $|\mathcal{C}(\lambda_1, \mathcal{T}) - \mathcal{C}(\lambda, \mathcal{T})| = |\psi(\lambda_1, \lambda_1^2, \lambda_1^3, \dots) - \psi(\lambda, \lambda^2, \lambda^3, \dots)|$  and by the triangular inequality, we get

$$(4.1) \quad \begin{aligned} |\mathcal{C}(\lambda_1, \mathcal{T}) - \mathcal{C}(\lambda, \mathcal{T})| &\leq |\psi(\lambda_1, \dots, \lambda_1^L, b_1) - \psi(\lambda, \dots, \lambda^L, b_1)| \\ &\quad + |\psi(\lambda, \dots, \lambda^L, b_1) - \psi(\lambda, \dots, \lambda^L, b)| \end{aligned}$$

where  $b := (\lambda^{L+k})_{k \geq 1}$  and  $b_1 := (\lambda_1^{L+k})_{k \geq 1}$ .

Let  $\lambda' \in [\lambda_0, \lambda_1]$  and denote by  $S_n^{\lambda'}$  the first hitting point of  $\mathcal{T}_n$  by the random walk with conductances

$$\underbrace{(\lambda, \dots, \lambda)}_{|\mathcal{T}_1|}, \underbrace{\lambda^2, \dots, \lambda^2}_{|\mathcal{T}_2|}, \dots, \underbrace{\lambda^L, \dots, \lambda^L}_{|\mathcal{T}_L|}, \underbrace{(\lambda')^{L+1}, \dots, (\lambda')^{L+1}}_{|\mathcal{T}_{L+1}|}, \dots$$

We can see that the law of  $S_L^{\lambda_1}$  and the law of  $S_L^\lambda$  are identical. Since  $\mathcal{T}$  is uniformly transient, then when the random walk reaches  $\mathcal{T}_L$ , it returns to  $o$  with a probability strictly smaller than  $(1 - \alpha)^L$ . It implies that

$$(4.2) \quad |\psi(\lambda, \dots, \lambda^L, b_1) - \psi(\lambda, \dots, \lambda^L, b)| \leq 2(1 - \alpha)^L \leq 2\varepsilon.$$

It remains to estimate  $|\psi(\lambda_1, \dots, \lambda_1^L, b_1) - \psi(\lambda, \dots, \lambda^L, b_1)|$ . By Theorem 15, we have

$$\psi(\lambda_1, \dots, \lambda_1^L, b_1) \geq \mathcal{C}(\lambda_0, \mathcal{T}) > 0 \quad \text{and} \quad \psi(\lambda, \dots, \lambda^L, b) \geq \mathcal{C}(\lambda_0, \mathcal{T}) > 0.$$

We apply the Lemma 22 to obtain

$$(4.3) \quad \exists \delta > 0, \forall \lambda \in [\lambda_1 - \delta, \lambda_1], |\psi(\lambda_1, \dots, \lambda_1^L, b_1) - \psi(\lambda, \dots, \lambda^L, b_1)| < \varepsilon.$$

We combine (4.1), (4.2) and (4.3) to get

$$\exists \delta > 0, \forall \lambda \in [\lambda_0, \lambda_1] \text{ such that } \lambda_1 - \lambda < \delta : |\mathcal{C}(\lambda_1, \mathcal{T}) - \mathcal{C}(\lambda, \mathcal{T})| \leq 3\varepsilon.$$

This implies that  $\mathcal{C}(\lambda, \mathcal{T})$  is left continuous at  $\lambda_1$ .  $\square$

Using the same method as in the proof of Theorem 37, we can prove the slightly stronger result (the proof of which we omit):

**Theorem 38.** *Let  $\mathcal{T}$  be a weakly uniformly transient tree: then the effective conductance  $\mathcal{C}(\lambda, \mathcal{T})$  is left continuous on  $(\lambda_c, 1]$ .*

#### 4.2. Proof of Theorem 4.

**Definition 39.** Let  $(\mathcal{T}^n, n \geq 1)$  be a sequence of infinite, locally finite and rooted trees. We say that  $\mathcal{T}^n$  converges locally towards  $\mathcal{T}^\infty$  if  $\forall k, \exists n_0, \forall n \geq n_0, \mathcal{T}_{\leq k}^n = \mathcal{T}_{\leq k}^\infty$ , where  $\mathcal{T}_{\leq k}$  is a finite tree defined by:

$$\begin{cases} V(\mathcal{T}_{\leq k}) := \{\nu \in V(\mathcal{T}), d(o, \nu) \leq k\} \\ E(\mathcal{T}_{\leq k}) = E|_{V(\mathcal{T}_{\leq k}) \times V(\mathcal{T}_{\leq k})} \end{cases}$$

Recall from the introduction that  $\mathbb{F}_m$  denotes the collection of all effective conductance functions for spherically symmetric trees with degree uniformly bounded by  $m$ .

**Lemma 40.** Let  $(f_n, n \geq 1)$  be a sequence of functions in  $\mathbb{F}_m$ . Assume that  $f_n$  converges towards some function  $f$ . Then, there exists a function  $g \in \mathbb{F}_m$  such that, for any  $\lambda > 0$ ,

$$f(\lambda) \leq g(\lambda).$$

*Proof.* Recall from the introduction that  $A_m$  denotes the collection of all spherically symmetric trees with maximal degree at most  $m$ ; let  $(\mathcal{T}^n, n \geq 1)$  be a sequence of elements of  $A_m$  such that, for every  $n > 0$ ,

$$f_n(\lambda) = \mathcal{C}(\lambda, \mathcal{T}^n).$$

Since the degree of vertices of  $\mathcal{T}^n$  are bounded by  $m$ , we can apply the diagonal extraction argument. After renumbering indices, there exists a subsequence of  $(\mathcal{T}^n, n \geq 1)$ , denoted also by  $(\mathcal{T}^n, n \geq 1)$  below, which converges locally towards some tree  $\mathcal{T}^\infty \in A_m$ . Moreover, we can assume that for any  $n > 0$ ,

$$(4.4) \quad \mathcal{T}_{\leq n}^n = \mathcal{T}_{\leq n}^\infty.$$

Set  $g(\lambda) = \mathcal{C}(\lambda, \mathcal{T}^\infty)$ , it remains to show that for every  $\lambda > 0$ ,

$$f(\lambda) \leq g(\lambda).$$

Assume that there exists  $\lambda_0$  such that  $f(\lambda_0) > g(\lambda_0)$  and let  $c := f(\lambda_0) - g(\lambda_0) > 0$ . Since the sequence  $(f_n(\lambda_0), n \geq 1)$  converges towards  $f(\lambda_0)$ ,

$$(4.5) \quad \exists \ell_1 > 0, \forall n \geq \ell_1, f_n(\lambda_0) > f(\lambda_0) - \frac{c}{4}.$$

Recall the definition of the function  $\mathcal{C}(\lambda_0, \mathcal{T}, n)$  in the proof of Lemma 34, the sequence  $(\mathcal{C}(\lambda_0, \mathcal{T}^\infty, n), n \geq 1)$  decreases towards  $g(\lambda_0)$ , implying that

$$(4.6) \quad \exists \ell_2 > 0, \forall n \geq \ell_2, \mathcal{C}(\lambda_0, \mathcal{T}^\infty, n) < g(\lambda_0) + \frac{c}{4}.$$

Letting  $\ell := \ell_1 \vee \ell_2$ , combine (4.5) and (4.6) to obtain:

$$(4.7) \quad f_\ell(\lambda_0) > f(\lambda_0) - \frac{c}{4} \quad \text{and} \quad \mathcal{C}(\lambda_0, \mathcal{T}^\infty, \ell) < g(\lambda_0) + \frac{c}{4}.$$

On the other hand  $\mathcal{C}(\lambda_0, \mathcal{T}^\ell, \ell) = \mathcal{C}(\lambda_0, \mathcal{T}^\infty, \ell)$  and by (4.7) we obtain:

$$(4.8) \quad f_\ell(\lambda_0) > f(\lambda_0) - \frac{c}{4} \quad \text{and} \quad \mathcal{C}(\lambda_0, \mathcal{T}^\ell, \ell) < g(\lambda_0) + \frac{c}{4}.$$

The sequence  $(\mathcal{C}(\lambda_0, \mathcal{T}^\ell, k), k \geq 1)$  decreases towards  $f_\ell(\lambda_0)$  when  $k$  goes to  $+\infty$ . Hence,

$$(4.9) \quad f_\ell(\lambda_0) \leq \mathcal{C}(\lambda_0, \mathcal{T}^\ell, \ell) < g(\lambda_0) + \frac{c}{4}.$$

From (4.8) and (4.9) we obtain

$$f(\lambda_0) - \frac{c}{4} < f_\ell(\lambda_0) < g(\lambda_0) + \frac{c}{4},$$

hence  $c = f(\lambda_0) - g(\lambda_0) < \frac{c}{2}$ , leading to a contradiction.  $\square$

*Proof of Theorem 4.* Let  $(\mathcal{T}^n, n \geq 1)$  be a sequence of elements of  $A_m$  such that, for any  $n > 0$ ,

$$f_n(\lambda) = \mathcal{C}(\lambda, \mathcal{T}^n).$$

Fix a sub-sequence of  $(\mathcal{T}^n, n \geq 1)$  which converges locally towards  $\mathcal{T}^\infty$  and such that (4.4) holds as in the proof of the Lemma 40. We set  $g(\lambda) = \mathcal{C}(\lambda, \mathcal{T}^\infty)$  and we need to prove that  $f = g$ .

By Lemma 40, we have  $f(\lambda) \leq g(\lambda)$ . Assume that there exists  $\lambda_0$  such that  $0 < f(\lambda_0) < g(\lambda_0)$ . We prove that for any  $\lambda < \lambda_0$ , we have  $f(\lambda) = 0$ .

Use Proposition 17 to obtain

$$(4.10) \quad \begin{cases} \forall n > 0, \mathcal{R}(\lambda_0, \mathcal{T}^n) &= \sum_{k=1}^{+\infty} \frac{1}{\lambda_0^k |\mathcal{T}_k^n|} \\ \mathcal{R}(\lambda_0, \mathcal{T}^\infty) &= \sum_{k=1}^{\infty} \frac{1}{\lambda_0^k |\mathcal{T}_k^\infty|} \end{cases}$$

We write

$$\mathcal{R}(\lambda_0, \mathcal{T}^n) = \sum_{k=1}^{+\infty} \frac{1}{\lambda_0^k |\mathcal{T}_k^n|} = \sum_{k \leq n} \frac{1}{\lambda_0^k |\mathcal{T}_k^n|} + \sum_{k > n} \frac{1}{\lambda_0^k |\mathcal{T}_k^n|}.$$

On the other hand, for any  $k \leq n$  we have  $|\mathcal{T}_k^n| = |\mathcal{T}_k^\infty|$ , hence

$$(4.11) \quad \mathcal{R}(\lambda_0, \mathcal{T}^n) = \sum_{k \leq n} \frac{1}{\lambda_0^k |\mathcal{T}_k^\infty|} + \sum_{k > n} \frac{1}{\lambda_0^k |\mathcal{T}_k^n|}.$$

Since  $f_n$  converges to  $f$ , then

$$(4.12) \quad \begin{cases} \lim_{n \rightarrow \infty} \mathcal{R}(\lambda_0, \mathcal{T}^n) &= \frac{1}{f(\lambda_0)} < \infty \\ \mathcal{R}(\lambda_0, \mathcal{T}^\infty) &= \frac{1}{g(\lambda_0)} < \frac{1}{f(\lambda_0)} \end{cases}$$

By using (4.11) and (4.12), we obtain

$$(4.13) \quad \lim_{n \rightarrow +\infty} \sum_{k > n} \frac{1}{\lambda_0^k |\mathcal{T}_k^n|} = \frac{1}{f(\lambda_0)} - \frac{1}{g(\lambda_0)} > 0.$$

Now we take  $\lambda < \lambda_0$  and we apply the Proposition 17 in order to get

$$(4.14) \quad \mathcal{R}(\lambda, \mathcal{T}^n) = \sum_{k=0}^{+\infty} \frac{1}{\lambda^k |\mathcal{T}_k^n|} > \sum_{k>n} \frac{1}{\lambda^k |\mathcal{T}_k^n|} \geq \left(\frac{\lambda_0}{\lambda}\right)^n \sum_{k>n} \frac{1}{\lambda_0^k |\mathcal{T}_k^n|}.$$

We combine (4.13) and (4.14) to obtain:

$$(4.15) \quad \lim_{n \rightarrow \infty} \mathcal{R}(\lambda, \mathcal{T}^n) = \infty$$

This implies that  $f(\lambda) = \lim_{n \rightarrow \infty} f_n(\lambda) = \lim_{n \rightarrow \infty} \frac{1}{\mathcal{R}(\lambda, \mathcal{T}^n)} = 0$ . Therefore, we proved that:

$$\forall \lambda < \lambda_0, f(\lambda) = 0.$$

Since  $f \neq 0$ , we can define  $\lambda_c := \inf \{0 \leq \lambda \leq 1 : f(\lambda) > 0\}$ : we proved that

$$(4.16) \quad \forall \lambda > \lambda_c, f(\lambda) = g(\lambda).$$

As the sequence  $(f_n)_n$  converges uniformly to  $f$ , then  $f$  is continuous, and hence  $f(\lambda_c) = 0$ . By Lemma 34,  $g$  is right continuous, so we get

$$(4.17) \quad f(\lambda_c) = \lim_{\lambda \rightarrow \lambda_c^+} f(\lambda) = \lim_{\lambda \rightarrow \lambda_c^+} g(\lambda) = g(\lambda_c) = 0.$$

On the other hand, by Theorem 15,  $g$  is a non-decreasing function, hence:

$$(4.18) \quad \forall \lambda < \lambda_c, g(\lambda) = 0 = f(\lambda).$$

Combine (4.16), (4.17) and (4.18) to obtain the identity  $f = g$ , thus concluding the proof.  $\square$

## 5. SELF-AVOIDING WALKS

The main goal of this section is to prove Theorem 1 (Section 5.3) and Theorem 3 (Section 5.4).

**5.1. Walks and bridges.** In this section, we review some definitions on the self-avoiding walk, bridges and connective constant (see [16]). Denote by  $c_n$  the number of self-avoiding walks of length  $n$ , starting at the origin on the considered graph. If  $\mathcal{G}$  is transitive, the sequence  $c_n^{1/n}$  converges to a constant when  $n$  goes to infinity. This constant is called the connective constant of  $\mathcal{G}$ .

**Definition 41.** *An  $n$ -step bridge in the plane  $\mathbb{Z}^2$  (or upper half-plane  $\mathbb{H}$ ) is an  $n$ -step self-avoiding walk (SAW)  $\gamma$  such that*

$$\forall i = 1, 2, \dots, n, \quad \gamma_1(0) < \gamma_1(i) \leq \gamma_1(n)$$

where  $\gamma_1(i)$  is the first coordinate of  $\gamma(i)$ . Denote by  $b_n$  the number of all  $n$ -step bridges with  $\gamma(0) = o$ . By convention, set  $b_0 = 1$ .

We have  $b_{m+n} \geq b_m \cdot b_n$ , hence we can define

$$\mu_b = \lim_{n \rightarrow +\infty} b_n^{\frac{1}{n}} = \sup_n b_n^{\frac{1}{n}}.$$

Moreover,  $b_n \leq \mu_b^n \leq \mu^n$ .

**Definition 42.** *Given a bridge  $\gamma$  of length  $n$ ,  $\gamma$  is called an irreducible bridge if it cannot be decomposed into two bridges of length strictly smaller than  $n$ . It means, we cannot find  $i \in [1, n-1]$  such that  $\gamma|_{[0,i]}$ ,  $\gamma|_{[i,n]}$  are two bridges. The set of all irreducible bridges is denoted by  $iSAW$ .*

**5.2. Kesten's measure.** For this section, we refer the reader to ([10],[4]) for a more precise description. Denote by  $SAW_\infty$  the set of all self-avoiding walks on the plane  $\mathbb{Z}^2$  or half-plane  $\mathbb{H}$ . In this section, we review the Kesten measure, which is a probability measure on the set of infinite self-avoiding paths in the half-plane constructed from finite bridges. Denote by  $\mathbb{B}$  (resp.  $\mathbb{I}$ ) the set of bridges (resp. irreducible bridges) starting at the origin. Denote by  $p_n$  the number of irreducible bridges starting at the origin, of length  $n$ .

We define a notion of concatenation of paths. If  $\gamma^1 = [\gamma^1(0), \dots, \gamma^1(m)]$  and  $\gamma^2 = [\gamma^2(0), \dots, \gamma^2(n)]$  are two SAWs, we define  $\gamma^1 \oplus \gamma^2$  to be the  $(m+n)$ -step walk (which is not necessarily self-avoiding) as

$$\begin{aligned} \gamma^1 \oplus \gamma^2 &:= [0, \gamma^1(1), \dots, \gamma^1(m), \\ &\quad \gamma^1(m) + \gamma^2(1) - \gamma^2(0), \dots, \gamma^1(m) + \gamma^2(n) - \gamma^2(0)]. \end{aligned}$$

Similarly, we can define  $\gamma^1 \oplus \gamma^2 \oplus \dots \oplus \gamma^k$ . We begin with the following equality:

**Fact 43** (Kesten [10, Theorem 5]). *We have*

$$\sum_{n=1}^{+\infty} \frac{p_n}{\mu^n} = 1.$$

**Remark 44.** *An obvious consequence of this equality and the existence of arbitrarily large bridges is that  $\sum_{\omega \in \mathbb{I}} \beta^{|\omega|} < \infty$  is finite for  $\beta < 1/\mu$  and infinite for  $\beta > 1/\mu$ .*

Let us now define the Kesten measure on the  $SAW_\infty$  in the half-plane. We fix  $\beta \leq \frac{1}{\mu}$  and denote by  $Q^\beta$  the probability measure on  $\mathbb{I}$  defined by

$$Q^\beta(\omega) = \frac{\beta^{|\omega|}}{Z_\beta}, \omega \in \mathbb{I}$$

where  $Z_\beta = \sum_{\omega \in \mathbb{I}} \beta^{|\omega|}$ . By Fact 43 and Remark 44,  $Z_\beta$  is finite and thus  $Q^\beta$  is a probability measure on  $\mathbb{I}$ .

Let  $k \geq 1$ , we consider the product space  $\mathbb{I}^k$  and define the product probability measure  $Q_k^\beta$  on  $k$ -tuples of bridges; we also write  $Q_k^\beta$  for the measure on

finite self-avoiding paths defined as

$$Q_k^\beta(\gamma) = \begin{cases} Q_k^\beta(\omega_1, \dots, \omega_k) & \text{if } \gamma = \omega^1 \oplus \omega^2 \oplus \dots \oplus \omega^k, (\omega_i)_{i \leq k} \in \mathbb{I}^k \\ 0 & \text{otherwise.} \end{cases}$$

The measure  $Q_\infty^\beta$  on the set of infinite self-avoiding paths is defined in the same way as the product measure on the bridge decomposition, or equivalently as the projective limit of the  $Q_k^\beta$ . From the definition, we directly obtain the following property:

**Fact 45.** *Under the  $\beta$ -Kesten measure, the infinite self-avoiding walk, denoted by  $\omega_K^{\infty, \beta}$ , almost surely does not reach the line  $\mathbb{Z} \times \{0\}$ .*

### 5.3. Proof of Theorem 1.

**Notation 46.** *Consider the self-avoiding walks in the lattice  $\mathbb{Z}^2$  starting at the origin. We construct a tree  $\mathcal{T}_{\mathbb{Z}^2}$ , which is called self-avoiding tree, from these self-avoiding walks: The vertices of  $\mathcal{T}_{\mathbb{Z}^2}$  are the finite self-avoiding walks and two such vertices joined when one path is an extension by one step of the other. Formally, denote by  $\Omega_n$  the set of self-avoiding walks of length  $n$  starting at the origin and  $V := \bigcup_{n=0}^{+\infty} \Omega_n$ . Two elements  $x, y \in V$  are adjacent if one path is an extension by one step of the other. We then define  $\mathcal{T}_{\mathbb{Z}^2} = (V, E)$ . In the same way, we can define other self-avoiding trees  $\mathcal{T}_{\mathbb{H}}, \mathcal{T}_{\mathbb{Q}}$ , where  $\mathbb{H}$  is a half-plane and  $\mathbb{Q}$  is a quarter-plane.*

**Remark 47.** *Note that each vertex (resp. a ray) of  $\mathcal{T}_{\mathbb{Z}^2}$  (or  $\mathcal{T}_{\mathbb{H}}, \mathcal{T}_{\mathbb{Q}}$ ) is a finite self-avoiding path (resp. an infinite self-avoiding path). Moreover, it is easy to see that the number of vertices at generation  $n$  of  $\mathcal{T}_{\mathbb{Z}^2}$  (or  $\mathcal{T}_{\mathbb{H}}, \mathcal{T}_{\mathbb{Q}}$ ) is the number of self-avoiding walks of length  $n$  in  $\mathbb{Z}^2$  (resp.  $\mathbb{H}, \mathbb{Q}$ ).*

**Notation 48.** *In [10], Kesten proved that all bridges in a half-plane can be decomposed into a sequence of irreducible bridges in a unique way. An infinite self-avoiding path starting at the origin is called “ $m$ -good” if it possesses a decomposition into irreducible bridges of length at most  $m$ . Denote by  $G_m$  the set of infinite self-avoiding paths which are  $m$ -good, and let  $\mathcal{T}^m$  be the subtree of  $\mathcal{T}_{\mathbb{Z}^2}$ , which we will refer to as the  $m$ -good tree, spanning the vertex set*

$$V(\mathcal{T}^m) := \{\omega \in V(\mathcal{T}_{\mathbb{Z}^2}) : \text{there exists } \gamma \in G_m \text{ such that } \gamma|_{[0, |\omega|]} = \omega\}.$$

**Proposition 49.** *Let  $\mathcal{T}_{\mathbb{H}}$  and  $\mathcal{T}_{\mathbb{Q}}$  be defined as above. Then,*

$$gr(\mathcal{T}_{\mathbb{Z}^2}) = br(\mathcal{T}_{\mathbb{Z}^2}) = gr(\mathcal{T}_{\mathbb{H}}) = br(\mathcal{T}_{\mathbb{H}}) = gr(\mathcal{T}_{\mathbb{Q}}) = br(\mathcal{T}_{\mathbb{Q}}) = \mu,$$

where  $\mu$  is the connective constant of the lattice  $\mathbb{Z}^2$ .

*Proof.* As explained in the introduction, there are rather large classes of trees, including  $\mathcal{T}_{\mathbb{Z}^2}$ , for which the branching and growth coincide (for instance, this holds for sub- or super-periodic trees, cf. below, or for typical supercritical



Galton-Watson trees), but none of the classical results seem to apply to  $\mathcal{T}_{\mathbb{H}}$  or  $\mathcal{T}_{\mathbb{Q}}$ .

Note that  $\mathcal{T}_{\mathbb{Z}^2}$  is a sub-periodic tree. By Theorem 12 and the definition of connective constant, we have

$$(5.1) \quad gr(\mathcal{T}_{\mathbb{Z}^2}) = br(\mathcal{T}_{\mathbb{Z}^2}) = \mu.$$

We also know (see [1, 8]) that there exists a constant  $B$  and  $n_0 \in \mathbb{N}$  such that for any  $n > n_0$ ,  $c_n \leq b_n e^{B\sqrt{n}}$  from which we obtain

$$(5.2) \quad \mu \leq \lim_{n \rightarrow \infty} (b_n)^{\frac{1}{n}} \leq gr(\mathcal{T}_{\mathbb{H}}) \leq gr(\mathcal{T}_{\mathbb{Z}^2}) = \mu.$$

Hence,  $gr(\mathcal{T}_{\mathbb{H}}) = \mu$  (as already mentioned in [10]) and, by Proposition 8,

$$(5.3) \quad br(\mathcal{T}_{\mathbb{H}}) \leq \mu.$$

Let  $b_n^{(m)}$  be the number of bridges of length  $n$  which possess a decomposition into irreducible bridges of length at most  $m$ . Recall that  $(\mathcal{T}^m)_n$  is the number of vertices of  $\mathcal{T}^m$  at generation  $n$ . Then for any  $n > 0$ , we have

$$(5.4) \quad |(\mathcal{T}^m)_n| \geq b_n^{(m)}.$$

Note that  $\mathcal{T}^m$  is also a subtree of  $\mathcal{T}_{\mathbb{H}}$ , so that by Remark 7 we have

$$(5.5) \quad br(\mathcal{T}^m) \leq br(\mathcal{T}_{\mathbb{H}}).$$

On the other hand,  $\mathcal{T}^m$  is  $m$ -super-periodic (because from any of its vertices, one can complete the current irreducible bridge in at most  $m$  steps after which every self-avoiding path in  $\mathcal{T}^m$  provides a possible continuation), so we can apply Theorem 12 to obtain the existence of  $gr(\mathcal{T}^m)$  and the equality

$$(5.6) \quad br(\mathcal{T}^m) = gr(\mathcal{T}^m).$$

We use (5.5) and (5.6) to obtain, for any  $m > 0$ ,

$$(5.7) \quad br(\mathcal{T}_{\mathbb{H}}) \geq gr(\mathcal{T}^m).$$

It remains to prove that  $\lim_{m \rightarrow \infty} gr(\mathcal{T}^m) = \mu$ . Noting that the concatenation of two bridges is itself a bridge, we see that the sequence  $(b_n)$  is super-multiplicative: for any  $m, n$ ,

$$(5.8) \quad b_{m+n} \geq b_m b_n \quad \text{and} \quad b_{n_1+n_2}^{(m)} \geq b_{n_1}^{(m)} b_{n_2}^{(m)},$$

implying the existence of

$$(5.9) \quad \mu_m := \lim_{n \rightarrow \infty} (b_n^{(m)})^{1/n} = \sup_{n \rightarrow \infty} (b_n^{(m)})^{1/n}.$$

Fix  $\varepsilon > 0$ : by (5.8) there exists  $m_0$  such that for all  $m \geq m_0$ ,

$$(5.10) \quad \left| \mu - (b_m)^{\frac{1}{m}} \right| \leq \varepsilon.$$

As we already mentioned (see Notation 48), all bridges in a half-plane can be decomposed into a sequence of irreducible bridges in a unique way. Therefore each bridge of length  $m$  possesses a decomposition into irreducible bridges of length at most  $m$ . Hence, for any  $m > m_0$ ,

$$(5.11) \quad b_m = b_m^{(m)}.$$

Combining (5.8), (5.9), (5.10) and (5.11), we obtain that for any  $m > m_0$ ,

$$(5.12) \quad \mu_m \geq (b_{km}^{(m)})^{\frac{1}{km}} \geq \left( (b_m^{(m)})^k \right)^{\frac{1}{km}} = (b_m^{(m)})^{\frac{1}{m}} = (b_m)^{\frac{1}{m}} \geq \mu - \varepsilon.$$

By (5.9), the sequence  $(b_\ell^{(m)})^{1/\ell}$  converges to  $\mu_m$ , hence  $\lim_{k \rightarrow \infty} (b_{km}^{(m)})^{1/km} = \mu_m$ . Using (5.4) and (5.12), for any  $m > m_0$ , we have  $\mu \geq gr(\mathcal{T}^m) \geq \mu_m \geq \mu - \varepsilon$  and then,

$$(5.13) \quad \lim_{m \rightarrow \infty} gr(\mathcal{T}^m) = \mu.$$

Combining (5.3), (5.7) and (5.13) leads to  $br(\mathcal{T}_{\mathbb{H}}) = \mu$ . By following a similar strategy in  $\mathbb{Q}$ , we obtain  $gr(\mathcal{T}_{\mathbb{Q}}) = br(\mathcal{T}_{\mathbb{Q}}) = \mu$ .  $\square$

Theorem 1 is a consequence of Theorem 21 and Proposition 49.

**5.4. Proof of Theorem 3.** Now, we apply the results in Section 4.1 for the self-avoiding trees  $\mathcal{T}_{\mathbb{Q}}$ ,  $\mathcal{T}_{\mathbb{H}}$  and  $\mathcal{T}_{\mathbb{Z}^2}$ .

**Notation.** For any  $n \in \mathbb{N}$ , let  $\Lambda_n := \llbracket -n, n \rrbracket^2$  be a subdomain of  $\mathbb{Z}^2$ . Denote by  $\partial\Lambda_n$  the boundary of  $\Lambda_n$ , i.e,

$$\partial\Lambda_n := \{(a, b) \in \Lambda_n : |a| = n \text{ or } |b| = n\}.$$

We write  $\overset{\circ}{\Lambda}_n := \Lambda_n \setminus \partial\Lambda_n$  for the interior of  $\Lambda_n$ . Let  $\gamma$  be a finite self-avoiding walk: we say that  $\gamma$  is a *self-avoiding walk in the domain  $\Lambda_n$*  if for any  $0 \leq k \leq |\gamma|$ , we have  $\gamma(k) \in \Lambda_n$ . Denote by  $\Omega(\Lambda_n)$  the set of self-avoiding in  $\Lambda_n$  starting from the origin  $o = (0, 0)$ .

**Lemma 50.** *The functions  $\mathcal{C}(\lambda, \mathcal{T}_{\mathbb{Q}})$ ,  $\mathcal{C}(\lambda, \mathcal{T}_{\mathbb{H}})$  and  $\mathcal{C}(\lambda, \mathcal{T}_{\mathbb{Z}^2})$  are right continuous on  $(\lambda_c, +\infty)$ .*

*Proof.* It follows immediately from Lemma 34.  $\square$

**Lemma 51.** *The functions  $\mathcal{C}(\lambda, \mathcal{T}_{\mathbb{Q}})$ ,  $\mathcal{C}(\lambda, \mathcal{T}_{\mathbb{H}})$  and  $\mathcal{C}(\lambda, \mathcal{T}_{\mathbb{Z}^2})$  are left continuous on  $(\lambda_c, +\infty)$ .*

*Proof.* We prove this Lemma for the case  $\mathcal{T}_{\mathbb{H}}$  and we use the same argument for other cases ( $\mathcal{T}_{\mathbb{Q}}$  and  $\mathcal{T}_{\mathbb{Z}^2}$ ). Note that  $\mathcal{T}_{\mathbb{H}}$  is not uniformly transient, therefore we cannot use Theorem 37. Fortunately, we can prove that  $\mathcal{T}_{\mathbb{H}}$  is weakly uniformly transient. For this purpose, we define a sequence of cutsets  $(\pi_n, n \geq 1)$  as

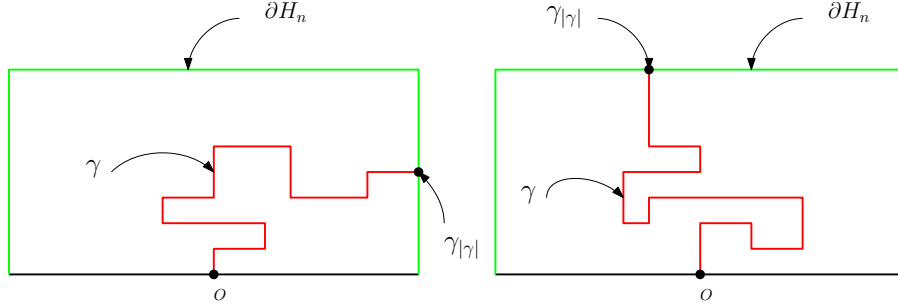


FIGURE 2. The boundary of  $\mathbb{H}_n$  is green and the self-avoiding walk  $\gamma$  is red. Recall that  $\gamma$  is a vertex of the tree  $\mathcal{T}_{\mathbb{H}}$ . On the left (resp. right), we can add a new quadrant  $\mathbb{Q}$  (resp. new half-plane  $\mathbb{H}$ ) rooted at  $\gamma_{|\gamma|}$ . Hence, on the left (resp. on the right) the subtree  $(\mathcal{T}_{\mathbb{H}})^\gamma$  contains the tree  $\mathcal{T}_{\mathbb{Q}}$  (resp.  $\mathcal{T}_{\mathbb{H}}$ ).

follows. Set  $\mathbb{H}_n := \Lambda_n \cap \mathbb{H}$  and  $\partial\mathbb{H}_n := (\partial\Lambda_n) \cap \mathbb{H}$  (see Figure 2). Recall that  $\Omega(\mathbb{H}_n)$  is the set of self-avoiding walks of domain  $\mathbb{H}_n$ . For any  $n \geq 1$ ,

$$\pi_n := \left\{ \gamma \in \Omega(\mathbb{H}_n) : \text{for any } 0 \leq k < |\gamma|, \gamma(k) \in \overset{\circ}{\mathbb{H}}_n \text{ and } \gamma_{|\gamma|} \in \partial(\mathbb{H}_n) \right\}$$

Since  $\mathbb{H}_n$  is a finite domain of  $\mathbb{H}$ , therefore any infinite self-avoiding walk starting at the origin of  $\mathbb{H}$ , must touch the boundary of  $\mathbb{H}_n$ . Hence, for any  $n \geq 1$ , we have  $\pi_n$  is a V-cutset of  $\mathcal{T}_{\mathbb{H}}$ . Let  $\Gamma := \bigcup_{n \geq 1} \pi_n$ : it remains to verify that

$$(5.14) \quad \forall \lambda > \lambda_c (= \frac{1}{\mu}), \exists \alpha_\lambda > 0, \forall \gamma \in \Gamma, \mathbb{P}(\forall n > 0, X_n^\gamma \neq \gamma) \geq \alpha_\lambda.$$

Note that for any  $\gamma \in \Gamma$ , the subtree  $(\mathcal{T}_{\mathbb{H}})^\gamma$  contains the tree  $\mathcal{T}_{\mathbb{H}}$  or  $\mathcal{T}_{\mathbb{Q}}$  (see Figure 2). Hence, (5.14) is a consequence of Theorem 1 and Theorem 15. We use Theorem 38 to complete the proof of Lemma.  $\square$

Theorem 4 is a consequence of Lemmas 50 and 51.

## 6. THE BIASED WALK ON THE SELF-AVOIDING TREE

We now begin the study of our main object of interest, which is the biased random walk on the self-avoiding tree. We will use the results that were obtained in the previous section to prove the properties of the limit walk. In the next section, we will gather a few natural conjectures.

**6.1. The limit walk.** Let  $\lambda \in [0, +\infty]$  and consider the biased random walk  $RW_\lambda$  on  $\mathcal{T}$  where  $\mathcal{T} = \mathcal{T}_{\mathbb{H}}$  or  $\mathcal{T} = \mathcal{T}_{\mathbb{Z}^2}$ . For  $\lambda > \lambda_c$ , the biased random walk is transient so almost surely, the random walk does not visit  $\mathcal{T}_k$  anymore after a

sufficiently large time. We can then define the limit walk, as denoted by  $\omega_\lambda^\infty$  in the following way:

$$\omega_\lambda^\infty(i) = x_i \iff \left\{ \begin{array}{l} x_i \in \mathcal{T}_i \\ \exists n_0, \forall n > n_0 : X_n \in \mathcal{T}^{x_i} \end{array} \right\}.$$

$\omega_\lambda^\infty$  is a random ray. Denote by  $\mathbb{P}_\lambda^{\mathbb{H}}$  the law of  $\omega_\lambda^\infty$  in the half-plane  $\mathbb{H}$  and  $\mathbb{P}_\lambda^{\mathbb{Z}^2}$ , the law of  $\omega_\lambda^\infty$  in the plane  $\mathbb{Z}^2$ . We can see  $\mathbb{P}_\lambda^{\mathbb{H}}$  (respectively  $\mathbb{P}_\lambda^{\mathbb{Z}^2}$ ) as a probability measure on  $SAW_\infty$  in the half-plane (respectively the plane).

For what follows, it will be useful to have the following definition: removing all the finite branches of  $\mathcal{T}_R$  (where  $R$  is a regular lattice), leads to a new tree without leaf, which we will denote by  $\tilde{\mathcal{T}}_R$ .

**6.2. The case  $\lambda = +\infty$  and percolation.** First, we review some definitions of percolation theory. Percolation was introduced by Broadbent and Hammersley in 1957 (see [3]). For  $p \in [0, 1]$ , we consider the triangular lattice  $\mathbb{T}$ , a site of  $\mathbb{T}$  is open with probability  $p$  or closed with probability  $1 - p$ , independently of the others. This can also be seen as a random colouring (in black or white) of the faces of hexagonal lattice  $\mathbb{T}^*$  dual of  $\mathbb{T}$ .

We define the exploration curve as follows (see [21], section 6.1.2 for more details). Let  $\Omega$  be a simply connected subgraph of the triangular lattice and  $A, B$  be two points on its boundary. We can then divide the hexagonal cells of  $\partial\Omega$  into two arcs, going from  $A$  to  $B$  in two directions (clockwise and counter-clockwise). These arcs will be denoted by  $\mathbb{B}$  and  $\mathbb{W}$  such that  $A, \mathbb{B}, B, \mathbb{W}$  is in the clockwise direction. Assume that all of the hexagons in  $\mathbb{B}$  are colored in black and that all of the hexagons in  $\mathbb{W}$  are colored in white. The color of the hexagonal faces in  $\Omega$  is chosen at random (black with probability  $p$  and white with probability  $1 - p$ ), independently of the others. We define the *exploration curve*  $\gamma$  starting at  $A$  and ending at  $B$  which separates the black component containing  $\mathbb{B}$  from the white component containing  $\mathbb{W}$ .

Then the exploration curve  $\gamma$  is a self-avoiding walk using the vertices and edges of hexagonal lattice  $\mathbb{T}^*$ . We can define this interface  $\gamma$  in an equivalent, dynamical way, informally described as follows. At each step,  $\gamma$  looks at its three neighbors on the hexagonal lattice, one of which is occupied by the previous step of  $\gamma$ . For the next step,  $\gamma$  randomly chooses one of these neighbors that has not yet occupied by  $\gamma$ . If there is just one neighbor that has not yet been occupied, then we choose this neighbor and if there are two neighbors, then we choose the right neighbor with probability  $p$  and the left neighbor with probability  $1 - p$ .

We know that there exists  $p_c \in [0, 1]$  such that for  $p < p_c$  there is almost surely no infinite cluster, while for  $p > p_c$  there is almost surely an infinite cluster. This parameter is called *critical point*. It is known that the critical point of site-percolation on the triangular lattice equals  $\frac{1}{2}$ . The lower bound

of critical point was proven by Harris in [9]. A similar theorem in the case of bond percolation on square lattice was given by Kesten in [11], and the result on the triangular lattice is obtained in a similar fashion.

Now, take  $\Omega = \mathbb{T}_+^*$ , the half-plane of hexagonal lattice. The hexagons on the boundary of  $\Omega$  ( $\partial\Omega$ ) and on the right of origin (denoted by  $\partial^+\Omega$ ) are colored in black and the hexagons on  $\partial\Omega$  and on the left of origin ( $\partial^-\Omega$ ) are colored in white. In this case, the exploration curve is an (random) infinite self-avoiding walk. Denote by  $\mathcal{T}_{\mathbb{T}_+^*}$  the self-avoiding tree constructed from the self-avoiding walks in  $\mathbb{T}_+^*$ .

In the case  $\lambda = +\infty$ , one can reinterpret the second construction of the exploration curve as the limit walk  $\omega^\infty$  on  $\tilde{\mathcal{T}}_{\mathbb{T}_+^*}$ . This is very useful because every feature of the curve  $\gamma$  is also one for  $\omega^\infty$  and can therefore be restated in terms of the biased walk on the self-avoiding tree. One of these properties is that  $\gamma$  almost surely reaches the boundary of  $\Omega$  infinitely many times, which follows from Russo-Seymour-Welsh type arguments. As we will see below, this property is still valid in the case  $RW_\lambda$ , for all  $\lambda > \lambda_c$  (see Theorem 2).

**6.3. Proof of Theorem 2.** In this section, for any  $z \in \mathbb{Z}^2$ , we write  $\Re z$  (resp.  $\Im z$ ) for the real part (resp. imaginary part) of  $z$ . Before going into the proof, we introduce a map from the set of all finite self-avoiding walks on a graph to the set of vertices of that graph, which to each walk associates the location of its last vertex (the “head of the snake”): formally, if  $\omega = (\omega_0, \dots, \omega_{|\omega|})$  is a self-avoiding path, its head is the vertex  $\omega_{|\omega|}$ . In our setup, the underlying graph is either  $\mathbb{H}$  or  $\mathbb{Z}^2$ , so the walk itself is a vertex of the corresponding tree ( $\mathcal{T} = \mathcal{T}_{\mathbb{H}}$  or  $\mathcal{T} = \mathcal{T}_{\mathbb{Z}^2}$  respectively), and we define  $p : V(\mathcal{T}) \rightarrow \mathbb{Z}^2$  by letting

$$p(\omega) := \omega_{|\omega|}.$$

The image by  $p$  of a random walk on  $\mathcal{T}$  is thus a random process on  $\mathbb{Z}^2$ , which we are going to investigate.

The proof of Theorem 2 has several steps. In the first step, we study the trajectory of the biased random walk  $X_n$  on  $\mathcal{T}$ . We prove that, under the measures  $\mathbb{P}_\lambda^{\mathbb{H}}$  and  $\mathbb{P}_\lambda^{\mathbb{Z}^2}$ ,  $p(X_n)$  almost surely reaches the line  $\mathbb{Z} \times \{0\}$ . In the second step, we prove furthermore that it almost surely reaches the line  $\mathbb{Z} \times \{0\}$  an infinite number of times. In the third step, we prove that under  $\mathbb{P}_\lambda^{\mathbb{Z}^2}$ , the limit walk (*i.e.*, the infinite self-avoiding path obtained as the limit of  $(X_n)$  as  $n \rightarrow \infty$ ) almost surely touches the line  $\mathbb{Z} \times \{0\}$  an infinite number of times — note that this is not a deterministic consequence of the previous step, since there are trajectories for  $(X_n)$  which do touch the line infinitely many times but whose limit path does not. In the last step, we prove the corresponding result for the half-plane, *i.e.* we show that under  $\mathbb{P}_\lambda^{\mathbb{H}}$ , the limit walk almost surely touches the line  $\mathbb{Z} \times \{0\}$  an infinite number of times. For simplicity, we will write  $Y_n$  for  $p(X_n)$ .

6.3.1. **The first step.** In this step, we study the trajectory of  $RW_\lambda$ . We begin with the following simple lemma:

**Lemma 52.** *Let  $\lambda > \lambda_c$  and consider the biased random walk  $(X_n)$  of law  $RW_\lambda$  on  $\mathcal{T}_{\mathbb{Z}^2}$  or  $\mathcal{T}_{\mathbb{H}}$ . Then almost surely  $\limsup |\Re(p(X_n))| = +\infty$ .*

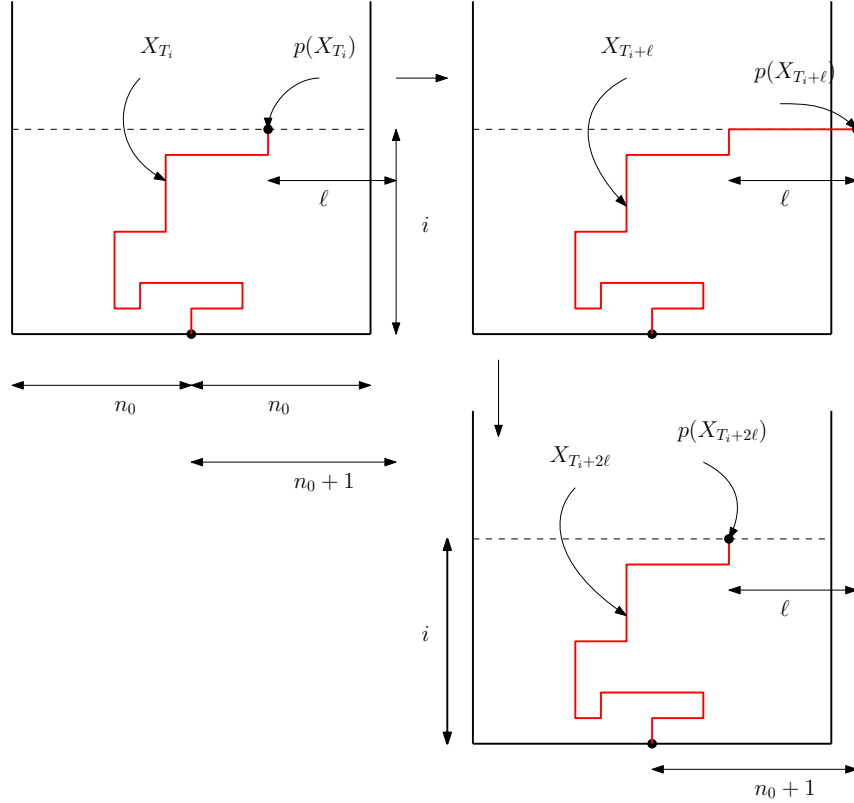


FIGURE 3. Illustration of the proof of Lemma 52

*Proof.* Recall that we defined  $Y_n := p(X_n)$ . We prove the lemma in the case of  $\mathcal{T}_{\mathbb{H}}$ ; the result for  $\mathcal{T}_{\mathbb{Z}^2}$  can be obtained in a similar way. The idea of the argument is straightforward: if the real part of  $Y_n$  is constrained, then its imaginary part has to take large values and every time it visits a new height, the real part has a chance of becoming large. What follows is a formalization of this idea. Assume that  $\alpha := \mathbb{P}(\limsup |\Re(Y_n)| < +\infty) > 0$ , then there exists a constant  $n_0 > 0$  such that,

$$(6.1) \quad \beta := \mathbb{P} \{ \forall n > 0, -n_0 \leq \Re(Y_n) \leq n_0 \} > 0.$$

For any  $i \geq 0$ , define

$$(6.2) \quad T(i) := \inf \{ n \geq 0 : \Im(Y_n) = i \}.$$

Note that  $T(i) < +\infty$  on the event {for all  $n > 0 : -n_0 \leq \Re(Y_n) \leq n_0$ }. We remark that, at time  $T(i)$ ,  $X$  can always be extended towards the left or the right. For any  $i \geq 0$ , define

$$S_i := \{\exists! k : |\Re(Y_k)| = n_0 + 1, \Im(Y_k) = i \text{ and } \forall n \neq k : -n_0 \leq \Re(Y_n) \leq n_0\}.$$

If the walk is at time  $T(i)$ , then we go towards the left or the right to reach the domain

$$\{\Re z = n_0 + 1\} \cup \{\Re z = -n_0 - 1\},$$

and after, we go back to  $X_{T(i)}$  (see Figure 3). We need at most  $2n_0$  steps to do this. Then, there exist a constant  $c > 0$  such that for any  $i > 0$ ,

$$(6.3) \quad \mathbb{P}(S_i) \geq c\beta.$$

On the other hand, we have

$$(6.4) \quad \bigcup_{i=0}^{+\infty} S_i \subset \{\forall n \geq 0, -n_0 - 1 \leq \Re(Y_n) \leq n_0 + 1\}.$$

Since these  $S_i$  are pairwise disjoint, by using (6.3) and (6.4) we obtain:

$$\mathbb{P}(\forall n \geq 0, -n_0 - 1 \leq \Re(Y_n) \leq n_0 + 1) \geq \sum_{i=0}^{\infty} \mathbb{P}(S_i) \geq \sum_{i=0}^{\infty} c\beta = +\infty.$$

This is a contradiction and therefore almost surely  $\limsup |\Re(Y_n)| = +\infty$ .  $\square$

**Lemma 53.** *Let  $\lambda > \lambda_c$  and consider the biased random walk  $(X_n)$  of law  $RW_\lambda$  on  $\mathcal{T}_{\mathbb{Z}^2}$  or  $\mathcal{T}_{\mathbb{H}}$ . Then  $\#\{n > 0 : \Im(p(X_n)) = 0\} \geq 1$  almost surely.*

*Proof.* Recall again that  $Y_n = p(X_n)$  is the location of the head of the walk at time  $n$ . The idea of the proof is somehow similar to the previous one: given that  $|\Re(Y_n)|$  reaches arbitrarily large values, we will look at the path at record times of  $|\Re(Y_n)|$  and argue from the previous lemma that from each such record time, the walk touches the real axis with positive probability. We again write the proof in  $\mathbb{Z}^2$  in detail, since the half-plane case is almost identical.

Say that a path  $\omega$  is *good* if its head is further to the left or right than its other vertices, namely if

$$|\Re p(\omega)| > \max_{k < |\omega|} |\Re(\omega_k)|.$$

A consequence of the previous lemma is that almost surely,  $X_n$  is good infinitely many times: indeed, it is automatically good whenever  $|\Re(Y_n)|$  reaches a record value. Now define inductively the following stopping times:  $T_0 = S_0 = 0$ ,

$$T_{k+1} = \inf\{n > S_k : X_n \text{ is good}\} \text{ and}$$

$$S_{k+1} = \inf\{n > T_{k+1} : \Im(Y_n) = 0 \text{ or } \Im(Y_n) = 2\Im(Y_{T_{k+1}})\}.$$

All these stopping times are a.s. finite, the  $T_k$  as mentioned above and the  $S_k$  by the same argument as in the proof of the previous lemma, which excluded  $(Y_n)$  from remaining within a strip.

Assume by contradiction that  $\alpha := \mathbb{P}(\forall n > 0, \mathfrak{S}Y_n > 0)$  is positive. The key observation, which we will use several times in various forms in what follows, is that the behavior of the walk after time  $T_k$ , and until its first visit to the parent of  $X_{T_k}$  in  $\mathcal{T}$ , matches the similar process defined in  $\mathbb{Z}^2 \setminus \{X_{T_k}(i) : 0 \leq i < |X_{T_k}|\}$ . Here, this domain contains the half-plane

$$\mathbb{H}_k := \{z \in \mathbb{Z}^2 : |\Re(z)| \geq |\Re(Y_{T_k})| \text{ and } \Re(z)\Re(Y_{T_k}) \geq 0\}.$$

With conditional probability  $\lambda/(1+3\lambda)$ , the first step of  $Y$  after time  $T_k$  will be horizontal and into  $\mathbb{H}_k$ , after which  $Y$  will remain within  $\mathbb{H}_k$  with conditional probability at least  $\alpha$  by our running hypothesis. If such is the case, then by symmetry,  $\mathfrak{S}(Y_{S_k}) = 0$  with conditional probability  $1/2$ . To summarize,

$$\mathbb{P}(\mathfrak{S}(Y_{S_k}) = 0 | \mathcal{F}_{T_k}) \geq \frac{\alpha\lambda}{2(1+3\lambda)} > 0,$$

where as is customary  $\mathcal{F}_{T_k}$  denotes the  $\sigma$ -field generated by the process  $(X_n)$  up to the stopping time  $T_k$ . Since this inequality holds for all  $k$ , there a.s. exists some  $k_0$  such that  $\mathfrak{S}(Y_{S_{k_0}}) = 0$ , which contradicts our assumption.  $\square$

As a shortcut in later arguments, we will refer to the kind of construction in the proof above as *considering a new half-plane with origin  $Y_{T_k}$* .

**6.3.2. The second step.** The goal of this step is to prove the following lemma:

**Lemma 54.** *Let  $\lambda > \lambda_c$  and consider the biased random walk  $RW_\lambda$  on  $\mathcal{T}_{\mathbb{Z}^2}$  or  $\mathcal{T}_{\mathbb{H}}$ . Then almost surely  $\#\{n > 0 : \mathfrak{S}(Y_n) = 0\} = +\infty$ .*

*Proof.* We again need to deal separately with the two cases.

**Case I: the tree  $\mathcal{T}_{\mathbb{H}}$ .** Denote by  $A$  the event that  $(Y_n)$  touches the real line infinitely often, namely

$$A := \{\#\{n > 0 : \mathfrak{S}Y_n = 0\} = +\infty\} = \{\forall k, \exists n > k : \mathfrak{S}Y_n = 0\}.$$

Assume by contradiction that  $\mathbb{P}(A^c) > 0$ . Then, there exists  $n_0 > 0$  such that

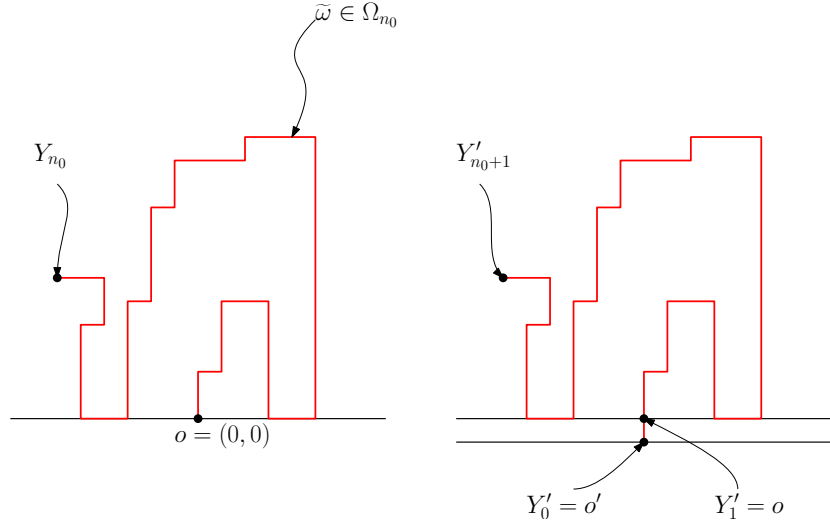
$$(6.5) \quad \mathbb{P}(\forall n \geq n_0 : \mathfrak{S}Y_n > 0) > 0.$$

Now, consider the random walk until time  $n_0$  and denote by  $\Omega_{n_0}$  the set of all possible trajectories of  $(X_n)$  in the tree  $\mathcal{T}_{\mathbb{H}}$  up to that time. For each  $\omega \in \Omega_{n_0}$ , define the events

$$(6.6) \quad A_\omega := \{(X_n)_{n \leq n_0} = \omega\} \quad \text{and} \quad B_\omega := A_\omega \cap \{\forall n \geq n_0, \mathfrak{S}(Y_n) > 0\}.$$

Since the cardinality of  $\Omega_{n_0}$  is finite, there exists  $\tilde{\omega} \in \Omega_{n_0}$  such that  $\mathbb{P}(B_{\tilde{\omega}}) > 0$ ; fix such a trajectory  $\tilde{\omega} = (\tilde{\omega}_0, \dots, \tilde{\omega}_{n_0})$  from now on.



FIGURE 4. Illustration of the proof of Lemma 54, case  $\mathcal{T}_{\mathbb{H}}$ 

We add a new line under the line  $\mathbb{Z} \times \{0\}$  and consider a new half-plane  $\mathbb{H}'$  with origin  $O' = (0, -1)$  (see the Figure 4 and the discussion in the proof of Lemma 53). Observe an independent biased random walk  $X'_n$  with parameter  $\lambda$  on  $\mathcal{T}_{\mathbb{H}'}$  and denote by  $Y'_n$  its head,  $Y'_n := p(X'_n)$ . Let

$$A'_{\tilde{\omega}} := \{\forall n \leq n_0, X'_{1+n} = O' \oplus \tilde{\omega}_n\}$$

where as defined earlier,  $O' \oplus \tilde{\omega}_n$  is the self-avoiding path in  $\mathbb{H}'$  obtained by prepending the vertex  $O'$  to the path  $\tilde{\omega}_n$ ; obviously  $P(A'_{\tilde{\omega}}) > 0$ . Now, conditionally on the events  $A_{\tilde{\omega}}$  and  $A'_{\tilde{\omega}}$ , the walks  $(X_n)$  and  $(X'_n)$  can be coupled after time  $n_0$  in such a way that they agree on the event  $B_{\tilde{\omega}}$ ; and whenever this is the case, the imaginary part of  $Y'_n$  remains positive for all times  $n > 0$ , thus showing that this happens with positive probability and contradicting Lemma 53 applied to  $X'$ .

**Case II: the tree  $\mathcal{T}_{\mathbb{Z}^2}$ .** Assume again that with positive probability, the process  $(Y_n)$  reaches the line  $\mathbb{Z} \times \{0\}$  finitely many times. By the same argument as in the case of  $\mathcal{T}_{\mathbb{H}}$ , there exists a positive number  $n_0$  and a trajectory  $\tilde{\omega}$  in the tree  $\mathcal{T}_{\mathbb{Z}^2}$  such that the events

$$A_{\tilde{\omega}} := \{(X_n)_{n \leq n_0} = \tilde{\omega}\} \quad \text{and} \quad B_{\tilde{\omega}} := A_{\tilde{\omega}} \cap \{\forall n \geq n_0, \Im Y_n < 0\}$$

both have positive probability. The key step of the proof is to show the existence of an integer  $h > 0$  and of a self-avoiding path  $\eta$  in the half-plane below  $\mathbb{Z} \times \{h\}$ , starting at  $(0, h)$ , which has the same head as  $\tilde{\omega}_{n_0}$  and whose complement in the lower half-plane has the same unbounded connected component as that of  $\tilde{\omega}_{n_0}$  (meaning that its intersection with the lower half-plane is the



with positive probability, the process  $RW_\lambda$  in the half-plane below  $\mathbb{Z} \times \{h\}$  first grows along  $\eta$  and then does not visit  $\mathbb{Z} \times \{0\}$  afterwards, which contradicts the half-plane case of the present lemma.

It remains to construct the path  $\eta$ ; here is an informal description of one possible construction: consider all the excursions of  $\tilde{\omega}_{n_0}$  below the line  $\mathbb{Z} \times \{0\}$ , keep the maximal ones (*i.e.*, those not separated from infinity by any other) and “collect” them linearly along the axis, first from the left up to the exit point of  $\tilde{\omega}_{n_0}$ , then from the right — see Figure 6.

By an *excursion* of  $\tilde{\omega}_{n_0}$ , we will always mean a sub-path of  $\tilde{\omega}_{n_0}$  of the form  $(\tilde{\omega}_{n_0}(s), \tilde{\omega}_{n_0}(s+1), \dots, \tilde{\omega}_{n_0}(t))$  where  $\Im \tilde{\omega}_{n_0}(s) = \tilde{\omega}_{n_0}(t) = 0$  and  $\Im \tilde{\omega}_{n_0}(u) < 0$  whenever  $s < u < t$ . Notice first that the head  $Y_{n_0} = p(\tilde{\omega}_{n_0})$  is not surrounded by any of the excursions of  $\tilde{\omega}_{n_0}$ , since that would contradict the assumption that  $P(B_{\tilde{\omega}}) > 0$ . Let  $(a, 0)$  be the last point of  $\mathbb{Z} \times \{0\}$  that  $\tilde{\omega}_{n_0}$  visits,  $\tilde{\omega}$  be the path followed by  $\tilde{\omega}_{n_0}$  after visiting  $(a, 0)$ , and define  $\Delta_-$  (resp.  $\Delta_+$ ) as the set of all integers  $x < a$  (resp.  $x > a$ ) such that the vertex  $(x, 0)$  is visited by  $\tilde{\omega}_{n_0}$ . If  $(x, 0)$  is one endpoint of an excursion of  $\tilde{\omega}_{n_0}$ , denote its other endpoint by  $(\varphi(x), 0)$  and let  $\delta(x)$  be sub-path of  $\tilde{\omega}_{n_0}$  between  $(x, 0)$  and  $(\varphi(x), 0)$ , parameterized to start at  $(x, 0)$  (walking backwards if needed); otherwise, set  $\varphi(x) = x$  and  $\delta(x) = \{x\}$ .

Assume first that  $\Delta_- \neq \emptyset$ : then let  $a_1 := \min \Delta_-$ ,  $b_1 := \varphi(a_1)$ , and inductively let  $a_{i+1} := \min\{x > b_i : x \in \Delta_-\}$  and  $b_{i+1} = \varphi(a_{i+1})$ , as long as the set in the definition of  $b_{i+1}$  is nonempty. This leads to a finite sequence  $a_1 \leq b_1 < a_2 \leq b_2 < \dots < a_{i_0} \leq b_{i_0} < a$ . Define  $\eta_-$  as the path obtained by concatenating the  $\delta(a_i)$  and the interval of the form  $[(b_i, 0), (a_{i+1}, 0)]$ :

$$\eta_- := \delta(a_1) \oplus [(b_1, 0), (a_2, 0)] \oplus \delta(a_2) \oplus \dots \oplus \delta(a_{i_0}).$$

If  $\Delta_- = \emptyset$ , let  $\eta_- := \{(-1, 0)\}$ . Similarly, collect the excursions intersecting with  $\Delta_+$  starting from the rightmost one, leading to a non-increasing sequence  $c_1 \geq d_1 > c_2 \geq d_2 > \dots > c_{j_0} \geq d_{j_0} > 0$ , and concatenate them to form

$$\eta_+ := \delta(c_1) \oplus [(d_1, 0), (c_1, 0)] \oplus \delta(c_2) \oplus \dots \oplus \delta(c_{j_0}),$$

again setting  $\eta_+ := \{(1, 0)\}$  if  $\Delta_+ = \emptyset$ . Denote by  $A_- = (a_-, 0)$  and  $B_- = (b_-, 0)$ , resp.  $A_+ = (a_+, 0)$  and  $B_+ = (b_+, 0)$ , the first and last points of  $\eta_-$ , resp.  $\eta_+$ . Last, define

$$\begin{aligned} \eta := & ((0, 2), (-1, 2), \dots, (a_-, 2), (a_-, 1), (a_-, 0)) \oplus \eta_- \\ & \oplus ((b_-, 0), (b_-, 1), (b_- + 1, 1), \dots, (a_+, 1), (a_+, 0)) \oplus \eta_+ \\ & \oplus ((b_+, 0), (b_+ - 1, 0), \dots, (a, 0)) \oplus \tilde{\omega}. \end{aligned}$$

It is easy now to see that  $\eta$  satisfies our requirements, thus concluding the proof of the lemma.  $\square$

**Remark 55.** Recall that if  $\mathcal{T}$  is a tree, we denote by  $\tilde{\mathcal{T}}$  the subtree obtained from  $\mathcal{T}$  by recursively erasing all its leaves; in terms of our dynamical self-avoiding walk model, this corresponds to preventing the path from entering traps. The reader can easily check that the previous arguments still apply in the cases of  $\tilde{\mathcal{T}}_{\mathbb{H}}$  and  $\tilde{\mathcal{T}}_{\mathbb{Z}^2}$ . Note that since the limit walk is the same on these trees without leaves as in the original ones, it is sufficient to prove Theorem 2 in the case of  $\tilde{\mathcal{T}}_{\mathbb{H}}$  and  $\tilde{\mathcal{T}}_{\mathbb{Z}^2}$ .

**6.3.3. The third step.** In this step, we give a proof of Theorem 2 in the case of  $\mathbb{P}_{\lambda}^{\mathbb{Z}^2}$ . We start with a definition:

**Definition 56.** Let  $A$  be a finite subset of  $\mathbb{Z}^2$ , its boundary is the set

$$\partial A := \{x \notin A : \exists z \in A, d(x, z) \leq \sqrt{2}\}$$

(where here  $d$  denotes the euclidean distance on  $\mathbb{Z}^2$ ) and its outer boundary is the set  $\partial_e A$  of all vertices in  $\partial A$  from which there exists an infinite self-avoiding path that does not intersect  $A$ .

A crucial fact, which we will use in several instances below, is that if  $A$  is connected (seen as a sub-graph of  $\mathbb{Z}^2$ ), then  $\partial_e A$  is connected as well. This is intuitively clear: informally, one can simply walk around  $A$  while remaining in  $\partial_e A$  (see figure 7). A formal proof is easy but tedious to write, and is therefore omitted here.

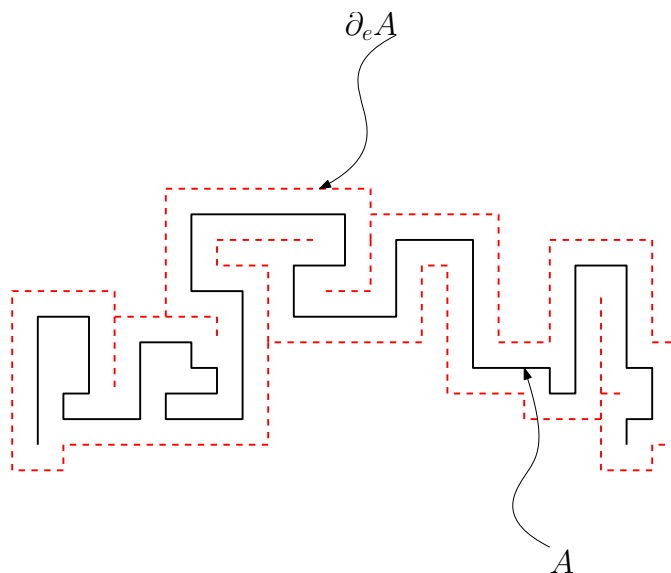


FIGURE 7. The outer boundary of a sub-graph  $A$  of  $\mathbb{Z}^2$

**Proof of Theorem 2 in the case of  $\mathbb{P}_\lambda^{\mathbb{Z}^2}$ .** Denote by  $\mathcal{A}$  the following event:

$$\mathcal{A} := \{\#\{n > 0 : \mathfrak{S}\omega_\lambda^\infty(n) = 0\} = \infty\}.$$

We want to show that  $P[\mathcal{A}] = 1$ , and to do that we are going to apply a strategy which is similar to the one used in step 2. We will need some additional notation: for every finite self-avoiding path  $\omega$ , let  $\mathcal{B}_\omega$  denote the event

$$\mathcal{B}_\omega := \begin{cases} \omega_\lambda^\infty(0) = \omega(0), \omega_\lambda^\infty(1) = \omega(1), \dots, \omega_\lambda^\infty(|\omega|) = \omega(|\omega|); \\ \forall n > |\omega| : \mathfrak{S}\omega_\lambda^\infty(n) < 0. \end{cases}$$

It is enough to show that  $P[\mathcal{B}_\omega] = 0$  for every  $\omega$ , so we fix a finite self-avoiding path  $\omega$  for the rest of the proof. Let  $n_0 := |\omega|$  be its length; without loss of generality we can always assume that  $\mathfrak{S}\omega(n_0) < 0$  and  $P[\mathcal{B}_\omega] > 0$ . Define

$$D_\omega := \left\{ (x, y) \in \mathbb{Z}^2 : y \geq 0 \text{ and } x \notin \{\mathfrak{R}\omega(i) : 0 \leq i \leq n_0\} \right\}$$

and let  $V_\omega$  be the set of all points in  $(\mathbb{Z} \times \{0\}) \setminus D_\omega$  from which there exists an infinite self-avoiding path in the lower half-plane which does not intersect  $\omega$  (see Figure 8).

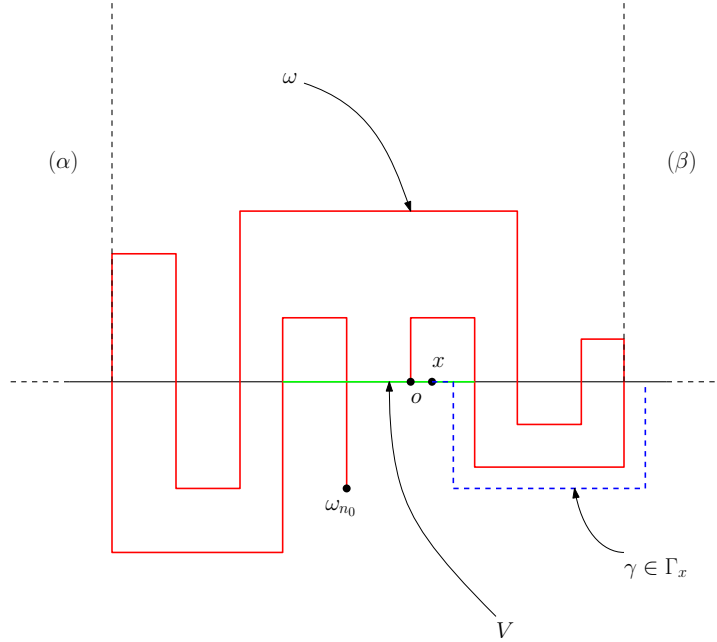


FIGURE 8. The self-avoiding walk  $\omega$  is colored by red; the domain  $D$  is the union of two quadrants  $\alpha$  and  $\beta$  and the set  $V$  is colored by green.

For each  $x \in V_\omega$ , denote by  $\Gamma_x$  the collection of all (finite) self-avoiding paths from  $x$  to a point in  $D_\omega$  that are not intersecting  $\omega$  and are contained in

$\partial\omega \cup (\mathbb{Z} \times \{0\})$ . The set  $\Gamma_x$  is finite and it is not empty by the previously noted connectivity fact about set boundaries. We then set  $p_\omega := \max_{x \in V_\omega} \max_{\gamma \in \Gamma_x} |\gamma|$ .

We first make the following remark: let  $\tau$  be a stopping time such that on the event  $\{\tau < \infty\}$ ,  $\omega$  is a prefix of  $X_\tau$  and  $X_\tau \cap (\mathbb{Z} \times \{0\}) = \omega \cap (\mathbb{Z} \times \{0\})$ . On this event, defining

$$\begin{cases} \sigma_1 := \inf\{t > \tau : p(X_t) \in \mathbb{Z} \times \{0\} \text{ and} \\ \sigma_2 := \inf\{t > \tau : \omega \text{ not prefix of } X_t\}, \end{cases}$$

it is always the case that  $\min(\sigma_1, \sigma_2) < \infty$ . Indeed, if neither of these occur, then  $(X_n)$  in particular does not touch the axis infinitely many times, thus contradicting Lemma 54 under our assumption that  $P[\mathcal{B}_\omega] > 0$ . On the other hand, on the event  $\mathcal{B}_\omega$ , the case where  $\min(\sigma_1, \sigma_2) = \sigma_2 < \infty$  can only occur finitely many times, by definition of the limit walk. Thus we only need to rule out the scenario where the case  $\min(\sigma_1, \sigma_2) = \sigma_1 < \infty$  occurs infinitely many times.

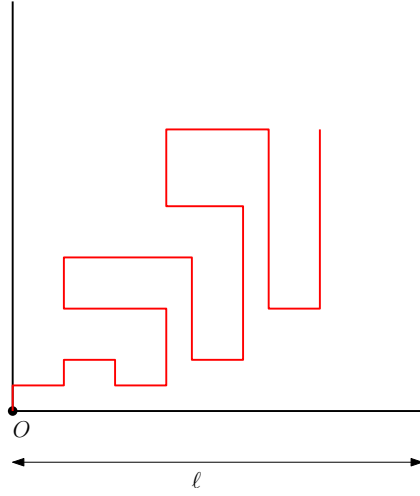
To do this, consider a realization of the process where  $\sigma_1 < \sigma_2$ . Let  $x := p(X_{\sigma_1})$ . We claim that at least one element  $\gamma$  of  $\Gamma_x$  intersects  $X_{\sigma_1}$  only at  $x$ : indeed, otherwise the path  $X_{\sigma_1}$  would have to form a loop around  $x$ , closed strictly before time  $\sigma_1$ , and this is impossible for the process constructed on the leafless tree that we are considering. Thus, with conditional probability at least  $(\lambda/(1+3\lambda))^{p_\omega}$ , the trajectory of the process  $(X_n)$  right after time  $\sigma_1$  will follow the path  $\gamma$  until reaching  $D_\omega$  at some point  $y$ . When this occurs, the process now sees an empty quadrant, and by the positivity of effective conductance  $C(\lambda, \mathcal{T}_\mathbb{Q})$  of the tree constructed on such a quadrant, with uniformly positive probability it will never backtrack further than  $y$ , in which case the event  $\mathcal{B}_\omega$  will not be realized. This concludes the proof.  $\square$

**6.3.4. The last step.** In this section, we give a proof of Theorem 2 in the case  $\mathbb{P}_\lambda^{\mathbb{H}}$ . Before beginning the proof, we will need some combinatorial information about the connective constants of strips.

**Definition 57.** A strip of width  $\ell$ , denoted by  $B_\ell$ , is a sub-domain of  $\mathbb{Z}^2$  which is limited by two lines  $\{\Im z = a\}$  and  $\{\Im z = b\}$  (for a horizontal strip) or  $\{\Re z = a\}$  and  $\{\Re z = b\}$  (for a vertical strip), such that  $|a - b| = \ell$ .

Fix an origin  $O \in \{\Im z = a\} \cup \{\Im z = b\}$  (or  $\{\Re z = a\} \cup \{\Re z = b\}$ ) of  $B_\ell$  and let  $\gamma$  be a finite self-avoiding path starting at  $O$ . We say that  $\gamma$  is a self-avoiding path in the strip  $B_\ell$  if for any  $0 \leq k \leq |\gamma|$ , we have  $\gamma(k) \in B_\ell$ ; we define the self-avoiding tree  $\mathcal{T}_{B_\ell}$  from the self-avoiding paths from  $O$  in  $B_\ell$  as in Notation 46.

A bridge (resp. irreducible bridge) in  $B_\ell$  is defined in the same way as in the half-plane; see Figure 9.

FIGURE 9. A bridge in a strip  $B_\ell$ 

**Lemma 58** (The subadditivity property). *For  $\ell, n$  positive natural numbers, denote by  $p_n^{(\ell)}$  the number of bridges of length  $n$  starting at the origin in the strip  $B_\ell$ . Then, for any  $\ell, n, m, k \in \mathbb{N}^*$ ,*

$$p_{n+m}^{(2\ell)} \geq p_m^{(\ell)} p_n^{(\ell)} \quad \text{and} \quad p_{kn}^{(2\ell)} \geq (p_n^{(\ell)})^k.$$

*Proof.* Divide the (vertical) strip  $B_{2\ell}$  into two small strips  $B_\ell^1, B_\ell^2$  of width  $\ell$  (see Figure 10). Consider  $\gamma_1, \gamma_2$  two bridges in the strip  $B_\ell^1$ , of length  $m$  and  $n$  respectively, and concatenate  $\gamma_1$  and  $\gamma_2$  to obtain a new bridge  $\gamma_{12} := \gamma_1 \oplus \gamma_2$  of length  $m+n$  in the strip  $B_{2\ell}$  (see Figure 10 again). This is an injection, and hence for any  $\ell, n, m \in \mathbb{N}^*$ ,

$$p_{n+m}^{(2\ell)} \geq p_m^{(\ell)} p_n^{(\ell)}$$

which is the first claim of the Lemma.

Given a third bridge  $\gamma_3$  in  $B_\ell^1$  of length  $q$ , we build a bridge  $\gamma_{123}$  of length  $m+n+q$  in  $B_{2\ell}$  as follows (see Figure 10):

$$\begin{cases} \gamma_{123} = \gamma_{12} \oplus \gamma_3 & \text{if } \gamma_{12}(|\gamma_{12}|) \in B_\ell^1 \\ \gamma_{123} = \gamma_{12} \oplus S(\gamma_3) & \text{if } \gamma_{12}(|\gamma_{12}|) \in B_\ell^2, \end{cases}$$

where  $S$  denotes the reflection across the vertical line going through  $(0, 0)$ . This is again an injection, so for any  $\ell, n, m, q \in \mathbb{N}^*$ ,

$$p_{n+m+q}^{(2\ell)} \geq p_m^{(\ell)} p_n^{(\ell)} p_q^{(\ell)}.$$

Iterating the same construction leads to the second claim of the Lemma, and thus finishes the proof.  $\square$

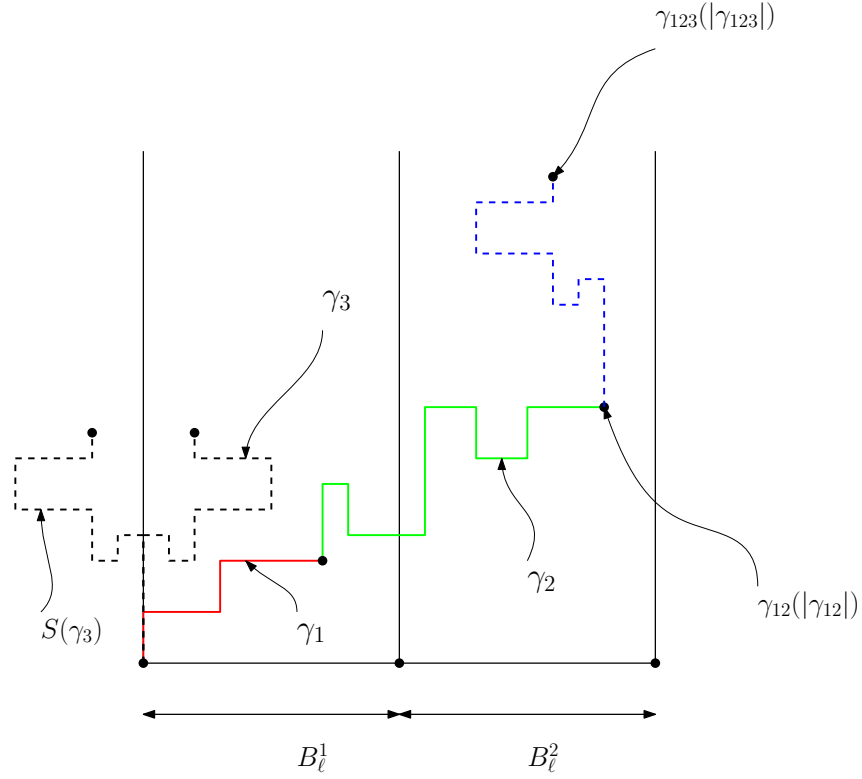


FIGURE 10. A concatenation of 3 bridges in  $B_L^1$ .

**Lemma 59.** Denote by  $\mu(\ell)$  the connective constant of the strip  $B_\ell$ , which exists from the previous Lemma. Then,

$$\lim_{\ell \rightarrow \infty} \mu(\ell) = \mu,$$

where  $\mu$  is the connective constant of  $\mathbb{Z}^2$ .

*Proof.* Denote by  $b_n^{\mathbb{Q}}$  the number of bridges of length  $n$  in  $\mathbb{Q}$ , starting at the origin. Note that for any  $\ell$ , we have:

$$(6.7) \quad \lim_{n \rightarrow \infty} (p_n^{(\ell)})^{\frac{1}{n}} = \mu(\ell) \quad \text{and} \quad p_\ell^{(\ell)} = b_\ell^{\mathbb{Q}}.$$

Moreover, we also have:

$$(6.8) \quad \lim_{n \rightarrow \infty} (b_n^{\mathbb{Q}})^{\frac{1}{n}} = \mu.$$

By using Lemma 58, for any  $\ell, n, k$ :

$$(6.9) \quad p_{kn}^{(2\ell)} \geq (p_n^{(\ell)})^k.$$



Fix  $\varepsilon > 0$  and by (6.8), there exists  $n_0$  such that for any  $n > n_0$ , we have

$$(6.10) \quad \left| (b_n^{\mathbb{Q}})^{\frac{1}{n}} - \mu \right| \leq \varepsilon.$$

Let  $\ell > n_0$  and  $k > 0$ . By (6.7), (6.9) and (6.10), we have:

$$(6.11) \quad \left( p_{k\ell}^{(2\ell)} \right)^{\frac{1}{k\ell}} \geq \left( p_{\ell}^{(\ell)} \right)^{\frac{1}{\ell}} = (b_{\ell}^{\mathbb{Q}})^{\frac{1}{\ell}} \geq \mu - \varepsilon.$$

Since the sequence  $(p_{k\ell}^{(2\ell)})^{\frac{1}{k\ell}}$  converges towards  $\mu(2\ell)$  when  $k$  goes to infinity, we use (6.11) to obtain:

$$(6.12) \quad \mu \geq \mu_{2\ell} \geq \mu - \varepsilon,$$

where inequality  $\mu \geq \mu_{2\ell}$  is obvious. Hence, the sequence  $(\mu(\ell), \ell \geq 1)$  converges towards  $\mu$  when  $\ell$  goes to  $+\infty$ .  $\square$

**Proposition 60.** *Denote by  $br(\mathcal{T}_{B_{\ell}})$  the branching number of  $\mathcal{T}_{B_{\ell}}$ . Then,*

$$\lim_{\ell \rightarrow \infty} br(\mathcal{T}_{B_{\ell}}) = \mu,$$

where  $\mu$  is again the connective constant of  $\mathbb{Z}^2$ .

*Proof.* The following argument is very close in spirit to the proof of Proposition 49. Recall that an infinite self-avoiding path starting at the origin in  $B_{\ell}$  is *m-good* if it possesses a decomposition into irreducible bridges of length at most  $m$ . Denote by  $G_m(B_{\ell})$  the set of infinite, *m-good* self-avoiding paths in  $B_{\ell}$ . Let  $\mathcal{T}_{B_{\ell}}^{(m)}$  be the subtree of  $\mathcal{T}_{B_{\ell}}$ , which we will refer to as the *m-good tree*, spanning

$$V(\mathcal{T}_{B_{\ell}}^{(m)}) := \{\omega \in V(\mathcal{T}_{B_{\ell}}) : \text{there exists } \gamma \in G_m(B_{\ell}) \text{ such that } \gamma_{[0, |\omega|]} = \omega\}$$

(i.e., the tree formed of all finite self-avoiding paths in  $B_{\ell}$  that can be extended into an infinite, *m-good* path).

Denote by  $p_n^{(\ell, m)}$  be the number of bridges of length  $n$  in  $B_{\ell}$ , starting at the origin, which possess a decomposition into bridges of length at most  $m$ . Recall that  $p_n^{(\ell)}$  is the number of bridges of length  $n$  in  $B_{\ell}$  starting at origin, and that  $(\mathcal{T}_{B_{\ell}}^{(m)})_n$  denotes the number of vertices of  $\mathcal{T}_{B_{\ell}}^{(m)}$  at generation  $n$ . Then for any  $n > 0$ , we have

$$(6.13) \quad \left| (\mathcal{T}_{B_{\ell}}^{(m)})_n \right| \geq p_n^{(m)}.$$

By using Lemma 58, for any  $\ell, m, n, k$  we have:

$$(6.14) \quad p_{nk}^{(2\ell)} \geq (p_n^{(\ell)})^k \quad \text{and} \quad p_{nk}^{(2\ell, m)} \geq (p_n^{(\ell, m)})^k.$$

We know (see Notation 48) that all bridges in a half-plane can be decomposed into a sequence of irreducible bridges in a unique way; therefore each bridge

in  $B_\ell$  of length  $m$  possesses a unique decomposition into irreducible bridges of length at most  $m$ . Hence, for any  $m, \ell > 0$ ,

$$(6.15) \quad p_m^{(\ell)} = p_m^{(\ell, m)}.$$

Fix  $\varepsilon > 0$ , by Lemma 59, there exists  $\ell_0$  such that for any  $\ell > \ell_0$ ,

$$(6.16) \quad \mu \geq \mu(2\ell) > \mu - \varepsilon.$$

Moreover, since  $\mu(2\ell) = \lim_{n \rightarrow \infty} (p_n^{(2\ell)})^{\frac{1}{n}}$ , then there exists  $n_0$  such that for any  $n > n_0$ :

$$(6.17) \quad (p_n^{(2\ell)})^{\frac{1}{n}} > \mu(2\ell) - \varepsilon.$$

Hence by (6.15), (6.14), (6.16) and (6.17),

$$(6.18) \quad (p_{kn}^{(4\ell, n)})^{\frac{1}{kn}} \geq (p_n^{(2\ell, n)})^{\frac{1}{n}} = (p_n^{(2\ell)})^{\frac{1}{n}} \geq \mu(2\ell) - \varepsilon \geq \mu - 2\varepsilon.$$

Therefore for  $\ell > \ell_0$  and  $m > m_0(\ell)$ , we have

$$(6.19) \quad \overline{gr(\mathcal{T}_{B_{4\ell}}^{(m)})} \geq \mu - 2\varepsilon.$$

On the other hand, noting that by construction,  $\mathcal{T}_{B_{4\ell}}^{(m)}$  is  $(m + 4\ell)$ -super-periodic and has finite growth, we can use Theorem 12 to get:

$$(6.20) \quad gr(\mathcal{T}_{B_{4\ell}}^{(m)}) \text{ exists and } gr(\mathcal{T}_{B_{4\ell}}^{(m)}) = br(\mathcal{T}_{B_{4\ell}}^{(m)}).$$

Since  $\mathcal{T}_{B_{4\ell}}^{(m)} \subset \mathcal{T}_{B_{4\ell}}$ , by using (6.19), (6.20) and Proposition 8 we obtain for any  $\ell > \ell_0$ :

$$(6.21) \quad \mu \geq br(\mathcal{T}_{B_{4\ell}}) \geq \mu - 2\varepsilon,$$

where we used  $\mathcal{T}_{B_{4\ell}} \subset \mathcal{T}_{\mathbb{H}}$  for the first inequality. Therefore, the sequence  $(br(\mathcal{T}_{B_\ell}))_{\ell \geq 1}$  converges towards  $\mu$  when  $\ell$  goes to infinity.  $\square$

We can now turn to the study of the limit self-avoiding path in the half-plane.

**Proposition 61.** *Consider the biased random walk  $RW_\lambda$  on  $\tilde{\mathcal{T}}_{\mathbb{H}}$ . Let  $(B_\ell)_{\ell \geq 1}$  be the sequence of (horizontal) strips in  $\mathbb{H}$  where  $B_\ell$  is the strip between the lines  $\{\Im z = 0\}$  and  $\{\Im z = \ell\}$ . Suppose that  $\lambda > \frac{1}{\mu}$ , where  $\mu$  is the connective constant of  $\mathbb{H}$ . Then, whenever  $\ell > 0$  is large enough that  $\mu(B_\ell) > 1/\lambda$  (which holds for  $\ell$  large enough by the previous Lemma), the limit walk  $\omega_\lambda^\infty$  in  $\mathbb{H}$  almost surely touches the strip  $B_\ell$  infinitely often.*

*Proof.* Fix  $\ell$  such that  $\mu(B_\ell) > 1/\lambda$ . We use again the same argument as in the previous steps: assuming that  $\omega_\lambda$  touches  $B_\ell$  finitely many times with positive

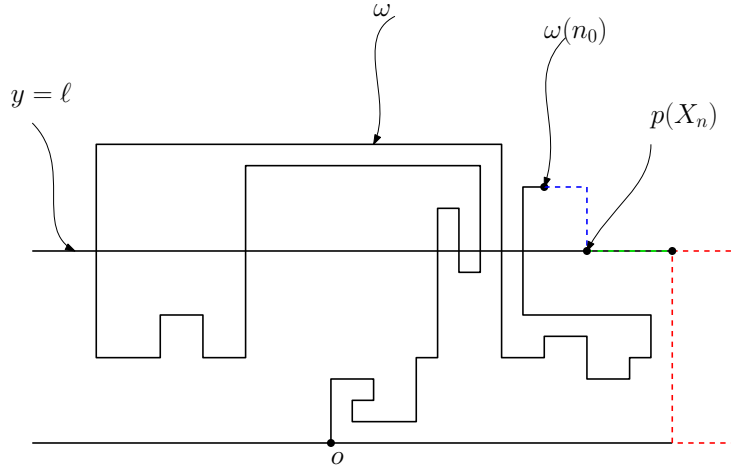


FIGURE 11. Illustration of the proof of Proposition 61

probability, there exists  $n_0 > 0$  and a self-avoiding path  $\omega = [\omega(0), \dots, \omega(n_0)]$  such that the following event has strictly positive probability:

$$B := \begin{cases} \omega_\lambda^\infty(0) = \omega(0), \omega_\lambda^\infty(1) = \omega(1), \dots, \omega_\lambda^\infty(n_0) = \omega(n_0) \\ \forall n > n_0 : \mathfrak{S}\omega_\lambda^\infty(n) > \ell. \end{cases}$$

By Lemma 54, we know that the head  $p(X_n)$  of the dynamical path almost surely reaches the line  $\mathbb{Z} \times \{0\}$  an infinite number of times, therefore on the event  $B$  it must be the case that  $\mathfrak{S}p(X_n) = \ell$  infinitely many times after time  $n_0$ . Let  $n$  be the first time when this occurs: at time  $n$ , the process sees a strip of width  $\ell$  that is empty except for the trace of  $\omega$ ; the corresponding subtree has positive conductance (with a lower bound depending only on  $\lambda$ ,  $\ell$  and  $\omega$ ), and therefore  $X$  has positive probability of never backtracking through  $X_n$ , in which case the second condition defining the event  $B$  does not hold. Iterating the same argument at each successive visit of  $B_\ell$  separated by backtracking, this leads to a contradiction. See Figure 11 for illustration.  $\square$

**Proof of Theorem 2 in the case of  $\mathbb{P}_\lambda^{\mathbb{H}}$ .** By Proposition 61, we can fix a number  $\ell > 0$  such that the limit walk almost surely reaches the strip  $B_\ell$  infinitely often. We just need to prove that it therefore visits the line  $\mathbb{Z} \times \{0\}$ .

The argument is again similar to the previous ones, so we do not flesh it out in detail. We know that  $\omega_\lambda$  visits  $B_\ell$  infinitely many times, therefore it occurs infinitely many times that

$$p(X_n) \in B_\ell \text{ and } |p(X_n)| > \max\{|p(X_m)| : m < n \text{ and } p(X_m) \in B_\ell\}.$$

Whenever this holds, the process has positive probability to reach the line  $\mathbb{Z} \times \{0\}$  by going down vertically, and once it reaches it, seeing an empty half-strip of width  $\ell$ , to never backtrack through that point.  $\square$

**Remark 62.** *We applied a similar reasoning several times above; informally, the general principle is that if there is a region  $V \subset \mathbb{Z}^2$  that the head  $p(X_n)$  visits infinitely often with probability 1, and which is such that the effective conductance of  $\mathcal{T}_{V \setminus K}$  can be bounded below for every finite  $K$  uniformly in the choice of the root, then the limit walk has infinitely many points in  $V$ . Writing a formal statement with this flavor would involve technicalities that are not needed in the setups that we were in (where  $V$  is either a half-plane or a strip).*

**6.4. The law of first steps of the limit walk.** We consider the biased random walk  $RW_\lambda$  on  $\mathcal{T}_{\mathbb{Z}^2}$  (resp.  $\mathcal{T}_{\mathbb{H}}, \mathcal{T}_{\mathbb{Q}}$  — the argument below is exactly the same all three cases). Recall that  $\omega_\lambda^\infty$  is the associated limit walk and  $\mathbb{P}_\lambda^{\mathbb{H}}$  denotes its law.

Let  $k \in \mathbb{N}^*$  and  $y_1, y_2, \dots, y_k$  be  $k$  elements of  $V(\mathcal{T}_{\mathbb{H}})$  such that the path  $(o, y_1, y_2, \dots, y_k)$  in  $\mathcal{T}_{\mathbb{H}}$  is simple. For each  $\lambda > \lambda_c$ , recall that the law of first  $k$  steps of  $\omega_\lambda$  is defined by:

$$(6.22) \quad \varphi^{\lambda,k}(y_1, y_2, \dots, y_k) = \mathbb{P}_\lambda^{\mathbb{H}}(\omega_\lambda^\infty(1) = y_1, \omega_\lambda^\infty(2) = y_2, \dots, \omega_\lambda^\infty(k) = y_k).$$

We prove the continuity of this function.

**Theorem 63.** *For every  $k \in \mathbb{N}^*$  and  $(y_1, y_2, \dots, y_k) \in V^k$ , the function  $\varphi^{\lambda,k}$  depends continuously on  $\lambda$  on the whole interval  $(\lambda_c, +\infty)$ .*

Let  $\mathcal{T}$  be an infinite, locally finite and rooted tree and  $\nu$  is a child of the root. Recall the definition of  $\tilde{\mathcal{C}}(\lambda, \mathcal{T})$  and  $\tilde{\mathcal{C}}(\lambda, \mathcal{T}, \nu)$  in Section 2.3. To prove Theorem 63, we need the following lemma:

**Lemma 64.** *We have*

$$\varphi^{\lambda,k}(y_1, y_2, \dots, y_k) = \frac{\tilde{\mathcal{C}}(\lambda, \mathcal{T}, y_1)}{\tilde{\mathcal{C}}(\lambda, \mathcal{T})} \times \frac{\tilde{\mathcal{C}}(\lambda, \mathcal{T}^{y_1}, y_2)}{\tilde{\mathcal{C}}(\lambda, \mathcal{T}^{y_1})} \times \dots \times \frac{\tilde{\mathcal{C}}(\lambda, \mathcal{T}^{y_{k-1}}, y_k)}{\tilde{\mathcal{C}}(\lambda, \mathcal{T}^{y_{k-1}})}.$$

*Proof.* We prove this lemma in the case  $k = 1$ , and leave the (slightly more complicated, but following the same lines) cases  $k \geq 2$  to the reader. Denote by  $\tilde{\mathcal{C}}_i(\lambda, \mathcal{T})$  the probability that the biased random walk on  $\mathcal{T}$  returns to origin exactly  $i$  times before going to infinity. Define the following events:

$$\begin{cases} \mathcal{A} := & \{\omega_\lambda^\infty(1) = y_1\}, \\ \mathcal{A}_i := & \{\omega_\lambda^\infty(1) = y_1 \text{ and } \#\{n > 0 : X_n = o\} = i\}. \end{cases}$$

The events  $\mathcal{A}_i$  are disjoint, and by transience,  $\mathcal{A} = \bigcup \mathcal{A}_i$ . On the other hand, by the Markov property, for any  $i \geq 0$ , we have

$$\mathbb{P}(\mathcal{A}_i) = \tilde{\mathcal{C}}(\lambda, \mathcal{T}, y_1) \left(1 - \tilde{\mathcal{C}}(\lambda, \mathcal{T})\right)^i.$$

Summing this identity over  $i \geq 0$  leads to

$$\mathbb{P}(\mathcal{A}) = \sum_{i=0}^{+\infty} \mathbb{P}(\mathcal{A}_i) = \frac{\tilde{\mathcal{C}}(\lambda, \mathcal{T}, y_1)}{\tilde{\mathcal{C}}(\lambda, \mathcal{T})},$$

which is indeed the claim of the Lemma for the case  $k = 1$ .  $\square$

*Proof of Theorem 63.* By Lemma 64 applied to the case  $\mathcal{T} = \mathcal{T}_{\mathbb{H}}$ , we have

$$\varphi^{\lambda, k}(y_1, y_2, \dots, y_k) = \frac{\tilde{\mathcal{C}}(\lambda, \mathcal{T}, y_1)}{\tilde{\mathcal{C}}(\lambda, \mathcal{T})} \times \frac{\tilde{\mathcal{C}}(\lambda, \mathcal{T}^{y_1}, y_2)}{\tilde{\mathcal{C}}(\lambda, \mathcal{T}^{y_1})} \times \dots \times \frac{\tilde{\mathcal{C}}(\lambda, \mathcal{T}^{y_{k-1}}, y_k)}{\tilde{\mathcal{C}}(\lambda, \mathcal{T}^{y_{k-1}})}.$$

It is enough to prove that the functions  $\tilde{\mathcal{C}}(\lambda, \mathcal{T}^{y_i}, y_{i+1})$  and  $\tilde{\mathcal{C}}(\lambda, \mathcal{T}^{y_i})$  are continuous. This follows from previous arguments, which we are going to adapt.

The continuity of  $\tilde{\mathcal{C}}(\lambda, \mathcal{T}^{y_i})$  is a consequence of Theorem 38. All that is needed is to show that the trees  $\mathcal{T}^{y_i}$  are all weakly uniformly transient, and this is true because they are finite modifications of  $\mathcal{T}_{\mathbb{H}}$ , which is itself weakly uniformly transient.

For the continuity of  $\tilde{\mathcal{C}}(\lambda, \mathcal{T}^{y_i}, y_{i+1})$ , this function can be written in terms of effective conductances: denoting by  $\hat{\mathcal{T}}^{y_i}$  the tree obtained from  $\mathcal{T}^{y_i}$  by adding one extra vertex that is a parent of the root, using the Markov property at time 1 we get

$$\tilde{\mathcal{C}}(\lambda, \mathcal{T}^{y_i}, y_{i+1}) = \frac{1}{d(y_i)} \cdot \mathcal{C}(\hat{\mathcal{T}}^{y_i})$$

which is therefore continuous by another application of Theorem 38.  $\square$

## 7. THE CRITICAL PROBABILITY MEASURE VIA BIASED RANDOM WALK

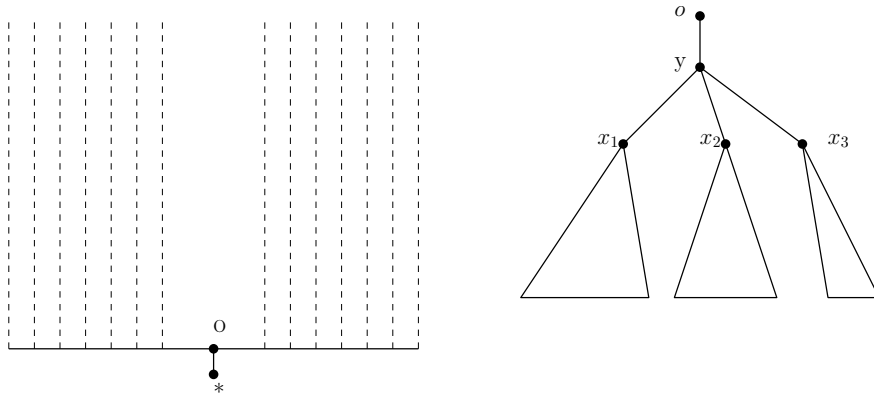


FIGURE 12. The upper-half plane on the left and the tree  $\mathcal{T}_{\mathbb{H}}$  on the right.

**7.1. The critical probability measure.** In this section, we only work on the half-plane  $\mathbb{H}$  and the associated self-avoiding tree  $\mathcal{T}_{\mathbb{H}}$ . It will be useful below to extend the half-plane by adding one more vertex below the origin: namely,

$$\bar{\mathbb{H}} := \mathbb{H} \cup \{*\} \quad \text{with} \quad * := (0, -1)$$

and to consider the biased walk on the corresponding tree  $\mathcal{T}_{\bar{\mathbb{H}}}$ ; notice that the limit walk in  $\bar{\mathbb{H}}$  has the same law as the concatenation  $[*, o] \oplus \gamma$  where  $\gamma$  is the limit walk on  $\mathbb{H}$ .

We aim to construct a critical probability measure through the biased random walk on the self-avoiding tree by letting  $\lambda$  decrease to its critical value. First, we review the construction of Madras and Slade (see [16] for detail). Let  $\mathcal{B}_n$  denote the set of all self-avoiding bridges of length  $n$  in  $\bar{\mathbb{H}}$  starting at  $*$ , and let  $\beta_n := |\mathcal{B}_n|$  be the number of such bridges. Given  $n \geq m$  and an  $m$ -step self-avoiding walk  $\gamma$  in  $\bar{\mathbb{H}}$ , let  $\mathbb{P}_{m,n}^{\mathcal{B}}(\gamma)$  denote the fraction of  $n$ -step bridges in  $\mathcal{B}_n$  that extend  $\gamma$ , *i.e.*

$$(7.1) \quad \mathbb{P}_{m,n}^{\mathcal{B}}(\gamma) = \frac{|\mathcal{F}_n(\gamma) \cap \mathcal{B}_n|}{\beta_n} = \frac{|\mathcal{F}_n(\gamma)|}{\beta_n},$$

where  $\mathcal{F}_n(\gamma)$  is the set of all  $n$ -step bridges which extend  $\gamma$ . One can think of  $\mathbb{P}_{m,n}^{\mathcal{B}}(\gamma)$  as the probability that a bridge uniformly chosen from among all  $n$ -step bridges is an extension of  $\gamma$ , it defines a probability measure on  $\mathcal{T}_{\bar{\mathbb{H}}}^m$ . Theorem 8.3.1 in [16] states that the limit

$$(7.2) \quad \mathbb{P}_m^{\mathcal{B}}(\gamma) := \lim_{n \rightarrow \infty} \mathbb{P}_{m,n}^{\mathcal{B}}(\gamma)$$

exists. By the Kolmogorov theorem, this allows for the definition of a measure  $\mathbb{P}_{\infty}^{\mathcal{B}}$  on the set  $\text{SAW}_{\infty}$  of infinite self-avoiding paths in  $\bar{\mathbb{H}}$  by imposing, for every  $\gamma \in \text{SAW}_{\infty}$ ,  $m \geq 0$  and  $\gamma'$  of length  $m$ ,

$$\mathbb{P}_{\infty}^{\mathcal{B}}(\gamma[0, m] = \gamma') = \mathbb{P}_m^{\mathcal{B}}(\gamma').$$

**Theorem 65** ([16], Theorem 8.3.2). *The probability measure  $\mathbb{P}_{\infty}^{\mathcal{B}}$  coincides with the Kesten measure at parameter  $\mu^{-1}$ , where  $\mu$  is the connective constant of the square lattice.*

Recall that for all  $m \geq 1$ ,  $\mathcal{T}_{\bar{\mathbb{H}}}^{(m)}$  is the  $m$ -good tree built from self-avoiding paths having only irreducible bridges of length at most  $m$  in their decomposition (see Notation 48). Fix  $k > 0$  and a self-avoiding path  $\gamma$  of length  $k$  started at  $*$ ; define  $\varphi^{\lambda}(\gamma)$  and  $\varphi^{\lambda, m}(\gamma)$  as the probability that  $\gamma$  is the prefix of length  $m$  of the limit walk with parameter  $\lambda$  respectively on  $\mathcal{T}_{\bar{\mathbb{H}}}$  and  $\mathcal{T}_{\bar{\mathbb{H}}}^{(m)}$ . Note that  $\varphi^{\lambda}$  is defined for all  $\lambda > \mu^{-1}$  and  $\varphi^{\lambda, m}$  for all  $\lambda > \lambda_m := \lambda_c(\mathcal{T}_{\bar{\mathbb{H}}}^{(m)})$ .

**Lemma 66.** *As  $m \rightarrow \infty$ , we have  $\lambda_m \rightarrow \lambda_c(\mathcal{T}_{\bar{\mathbb{H}}})$ .*

*Proof.* This follows from the same proof as that of Proposition 60: in the last paragraph, instead of embedding the tree  $\mathcal{T}_{B_{4\ell}}^{(m)}$  into  $\mathcal{T}_{B_{4\ell}}$ , one can embed it into  $\mathcal{T}_{\mathbb{H}}^{(m)}$  to derive the lower bound  $br(\mathcal{T}_{\mathbb{H}}^{(m)}) \geq \mu - 2\varepsilon$  and conclude in a similar fashion.  $\square$

**Theorem 67.** *For every  $k > 0$  and  $\gamma$  of length  $k$ , the following hold:*

- (1)  $\varphi^{\lambda, m}(\gamma)$  converges as  $\lambda$  decreases to  $\lambda_m$ : let  $\varphi^{(m)}(\gamma) := \lim_{\lambda \searrow \lambda_m} \varphi^{\lambda, m}(\gamma)$ ;
- (2)  $\varphi^{(m)}(\gamma)$  converges as  $m \rightarrow \infty$ : let  $\varphi^{(\infty)}(\gamma) := \lim_{m \rightarrow \infty} \varphi^{(m)}(\gamma)$ ;
- (3) Moreover, we have the following diagram:

$$\begin{array}{ccc} \varphi^{m, \lambda}(\gamma) & \xrightarrow[\lambda > \mu^{-1}]{m \rightarrow \infty} & \varphi^{\lambda}(\gamma) \\ \lambda \rightarrow \lambda_m \downarrow & & \\ \varphi^{(m)}(\gamma) & \xrightarrow{m \rightarrow \infty} & \varphi^{(\infty)}(\gamma) \end{array}$$

*Proof.* As was the case for Theorem 63, all the ideas of the argument are already present in the case  $k = 2$ , so we focus on that case below and omit the details of the extension to general  $k \geq 3$ .

**Item (1):** The tree  $\mathcal{T}_{\mathbb{H}}^m$  is periodic, so we are in the framework of Proposition 31: as  $\lambda \searrow \lambda_m$ , convergence will follow from that proposition and formula (3.4) will give the limit. However, a little care has to be given because of the fact that the second step of  $\gamma$  is not necessarily within the first irreducible bridge in the decomposition. Let  $x_1 := (-1, 0)$ ,  $x_2 := (0, 1)$  and  $x_3 := (1, 0)$  be the three neighbors of  $o$  in  $\mathbb{H}$ . Moreover, let  $S_i^m$  denote the set of all irreducible bridges of length at most  $m$  having  $[\ast, o, x_i]$  as a prefix.

Formula (3.4) applies directly to the cases of  $x_1$  and  $x_3$ : as  $\lambda \searrow \lambda_m$ ,

$$\varphi^{m, \lambda}([\ast, o, x_1]) = \varphi^{m, \lambda}([\ast, o, x_1]) \rightarrow \sum_{\gamma \in S_1^m} \lambda^{|\gamma|}.$$

The case of  $x_2$  has two sub-cases, since either the first step  $[\ast, o]$  is an irreducible bridge in the decomposition by itself (in which case passing through  $x_2$  afterwards is automatic), or it is the first step of a bridge going through  $x_2$ : therefore, applying (3.4) to both sub-cases, we obtain that as  $\lambda \searrow \lambda_m$ ,

$$\varphi^{m, \lambda}([\ast, o, x_2]) \rightarrow \lambda_m + \sum_{\gamma \in S_2^m} \lambda^{|\gamma|}.$$

Notice that relation (3.5) implies that the sum of the three limits is indeed equal to 1, since the set of relevant irreducible bridges is  $[\ast, o] \cup S_1^m \cup S_2^m \cup S_3^m$ . This gives both the convergence and the value of  $\varphi^{(m)}$ . To summarize, letting  $p_{i, n}$  be the number of irreducible bridges of length  $n \geq 2$  which pass through

$x_i$ , we can rewrite the limit as

$$(7.3) \quad \varphi^{(m)}([*, o, x_i]) = \delta_i^2 \lambda_m + \sum_{n=2}^m p_{i,n} \lambda_m^n.$$

**Item (2):** For all  $m$  we have  $\lambda_m \geq \lambda_c$  because  $\mathcal{T}^m \subset \mathcal{T}_{\mathbb{H}}$ , hence

$$(7.4) \quad \varphi^{(m)}([*, o, x_i]) \geq \delta_i^2 \lambda_c + \sum_{n=2}^m p_{i,n} \lambda_c^n.$$

As  $m \rightarrow \infty$ , the right-hand term above increases to

$$\tilde{\varphi}([*, o, x_i]) := \delta_i^2 \lambda_c + \sum_{n=2}^{\infty} p_{i,n} \lambda_c^n$$

which coincides with the law of the first two steps in the Kesten measure discussed earlier; in particular, Kesten's identity implies that the sum of the  $\tilde{\varphi}([*, o, x_i])$  is equal to 1. We claim that  $\varphi^{(m)}([*, o, x_i]) \rightarrow \tilde{\varphi}([*, o, x_i])$  (so that in the limit  $\varphi^{(\infty)} = \tilde{\varphi}$ ); this directly follows from an elementary result:

**Lemma 68.** *Let  $(x_m), (y_m), (z_m), (\alpha_m), (\beta_m), (\gamma_m)$  be sequences taking values in  $[0, 1]$  and satisfying the assumptions*

- (1) *For all  $m$ ,  $x_m \geq \alpha_m$ ,  $y_m \geq \beta_m$  and  $z_m \geq \gamma_m$ ;*
- (2) *For all  $m$ ,  $x_m + y_m + z_m = 1$ ;*
- (3) *As  $m \rightarrow \infty$ ,  $\alpha_m \rightarrow \alpha$ ,  $\beta_m \rightarrow \beta$  and  $\gamma_m \rightarrow \gamma$  with  $\alpha + \beta + \gamma = 1$ .*

*Then,  $x_m \rightarrow \alpha$ ,  $y_m \rightarrow \beta$  and  $z_m \rightarrow \gamma$ .*

We only need to prove the lemma. To do that, let  $(\xi, \eta, \zeta)$  be any subsequential limit of  $(x_m, y_m, z_m)$ : assumption (1) implies that  $\xi \geq \alpha$ ,  $\eta \geq \beta$  and  $\zeta \geq \gamma$ ; assumption (2) shows that  $\xi + \eta + \zeta = 1$ ; and these readily imply that  $\xi = \alpha$ ,  $\eta = \beta$  and  $\zeta = \gamma$ , thus showing that the only possible subsequential limit of  $(x_m, y_m, z_m)$  is  $(\alpha, \beta, \gamma)$ .

**Item (3):** It remains to prove that for every  $\lambda > \lambda_c$  and every finite  $\gamma$ ,  $\varphi^{m,\lambda}(\gamma) \rightarrow \varphi^\lambda(\gamma)$  as  $m \rightarrow \infty$ . We begin with a combinatorial remark. Let  $\ell > 1$ ; if  $\gamma$  is a self-avoiding path of length  $\ell$  in  $\mathbb{H}$  which can be extended to an infinite self-avoiding path (*i.e.*,  $\gamma$  is a vertex at height  $\ell$  in the leaf-erased version of  $\mathcal{T}_{\mathbb{H}}$ ), let  $m(\gamma) < \infty$  be the smallest  $m$  such that  $\gamma \in \mathcal{T}_{\mathbb{H}}^m$ , and let  $M_\ell < \infty$  be the largest such  $m(\gamma)$ . By definition, the leaf-erased trees  $\tilde{\mathcal{T}}_{\mathbb{H}}$  and  $\tilde{\mathcal{T}}_{\mathbb{H}}^{M_\ell}$  coincide within distance  $\ell$  of their roots.

Now fix  $\lambda > \lambda_c$  and  $m_0$  such that  $\lambda_{m_0} < \lambda$ ; also fix a finite self-avoiding path  $\gamma$  and some  $\varepsilon > 0$ . By uniform transience, there exists  $L < \infty$  such that the probability that the biased walk in either  $\tilde{\mathcal{T}}_{\mathbb{H}}$  or  $\tilde{\mathcal{T}}_{\mathbb{H}}^m$  (for any  $m \geq m_0$ ) has probability at most  $\varepsilon$  of revisiting level  $|\gamma|$  after visiting level  $L$ .



For any  $m \geq \max(m_0, M_L)$ , the biased walks in  $\tilde{\mathcal{T}}_{\mathbb{H}}$  and  $\tilde{\mathcal{T}}_{\mathbb{H}}^m$  coincide until they reach level  $L$  (because the trees coincide below that level), after which they revisit level  $|\gamma|$  with probability at most  $\varepsilon$  (because of the definition of  $L$ ), thus  $|\varphi^{\lambda, m}(\gamma) - \varphi^\lambda(\gamma)| \leq \varepsilon$ . This concludes the proof of the Theorem.  $\square$

**7.2. Conjectures.** If we take a sequence of cutsets  $\pi_n := \mathcal{T}_n$  and we set  $c(e) = \mu^{-|e|}$ , then

$$\sum_n \left( \sum_{e \in \pi_n} c(e) \right)^{-1} = \sum_{n=1}^{+\infty} \frac{\mu^n}{c_n}.$$

If the prediction of Nienhuis [18] were true, we would obtain

$$\sum_{n=1}^{+\infty} \frac{\mu^n}{c_n} \geq c \sum_{n=1}^{+\infty} \frac{1}{n^{\frac{11}{32}}} = +\infty$$

and by Theorem 20, this would imply the recurrence of the critical biased random walk on the self-avoiding tree. We believe that recurrence does hold and that it might be provable without computing the critical exponent above, so we leave it as a conjecture:

**Conjecture 69.** *The biased random walk  $RW_{\lambda_c}$  on  $T_{\mathbb{H}}$  (or  $T_{\mathbb{Z}^2}$ ) is recurrent.*

Finally, we believe that one can take the limit  $\lambda \rightarrow \lambda_c$  without restricting the lengths of the irreducible bridges in the decomposition:

**Conjecture 70.** *The following convergence diagram holds*

$$\begin{array}{ccc} \varphi^{m, \lambda}(\gamma) & \xrightarrow[\lambda > \mu^{-1}]{m \rightarrow \infty} & \varphi^\lambda(\gamma) \\ \lambda \rightarrow \lambda_m \downarrow & & \downarrow \lambda \rightarrow \lambda_c \\ \varphi^{(m)}(\gamma) & \xrightarrow{m \rightarrow \infty} & \varphi^{(\infty)}(\gamma) \end{array}$$

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