



HAL
open science

The critical threshold for Bargmann-Fock percolation

Alejandro Rivera, Hugo Vanneuille

► **To cite this version:**

Alejandro Rivera, Hugo Vanneuille. The critical threshold for Bargmann-Fock percolation. 2017.
hal-01634286v1

HAL Id: hal-01634286

<https://hal.science/hal-01634286v1>

Preprint submitted on 13 Nov 2017 (v1), last revised 18 Dec 2017 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

The critical threshold for Bargmann-Fock percolation

Alejandro Rivera*

Hugo Vanneuville†

Abstract

In this article, we study the excursion sets $\mathcal{D}_p = f^{-1}([-p, +\infty[)$ where f is a natural real-analytic planar Gaussian field called the Bargmann-Fock field. More precisely, f is the centered Gaussian field on \mathbb{R}^2 with covariance $(x, y) \mapsto \exp(-\frac{1}{2}|x - y|^2)$. In [BG16], Beffara and Gayet prove that, if $p \leq 0$, then a.s. \mathcal{D}_p has no unbounded component. We show that conversely, if $p > 0$, then a.s. \mathcal{D}_p has a unique unbounded component. As a result, the critical level of this percolation model is 0. We also prove exponential decay of crossing probabilities under the critical level. To show these results, we develop several tools including a KKL-type result for biased Gaussian vectors (based on the analogous result for product Gaussian vectors by Keller, Mossel and Sen in [KMS12]) and a sprinkling inspired discretization procedure. These intermediate results hold for more general Gaussian fields, for which we prove a discrete version of our main result.

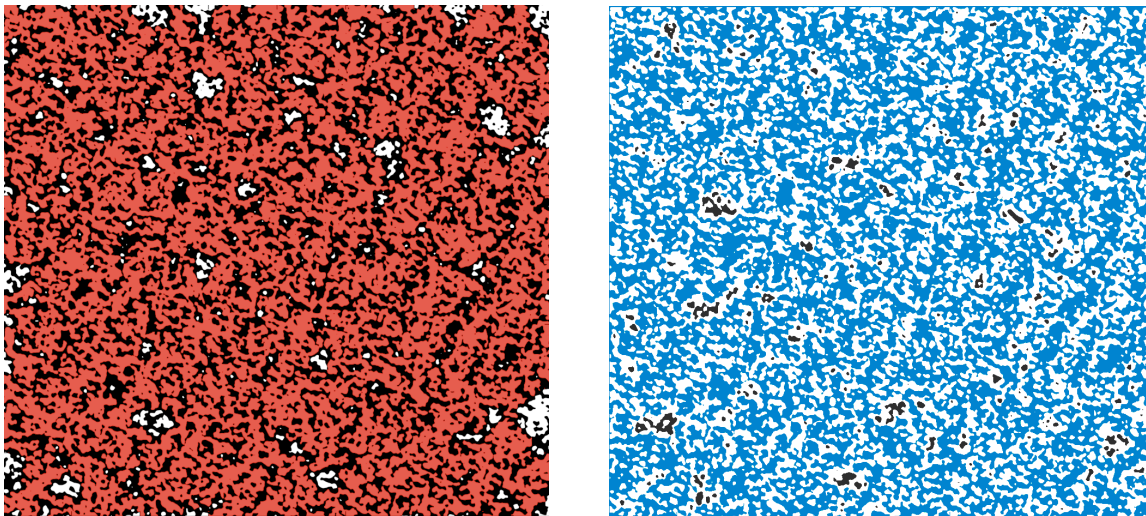


Figure 1: Percolation for an approximation of the Bargmann-Fock field f . On the left hand side, the region $f \geq 0.1$ is colored in black, the small components of the region $f < 0.1$ are colored in white, and the giant component of the region $f < 0.1$ is colored in red. On the right hand side, the small components of the region $f \geq -0.1$ are colored in black, the giant component of the region $f \geq -0.1$ is colored in blue, and the region $f < -0.1$ is colored in white. The two pictures correspond to the same sample of f .

*Univ. Grenoble Alpes, UMP5582, Institut Fourier, 38000 Grenoble, France, supported by the ERC grant Liko No 676999

†Univ. Lyon 1, Institut Camille Jordan, 69100 Villeurbanne, France, supported by the ERC grant Liko No 676999

Contents

1	Main results	2
2	Proof strategy and intermediate results	7
2.1	Some ingredients for the proof of Kesten’s theorem	7
2.2	Conditions on the Gaussian fields	8
2.3	A box-crossing criterion	10
2.4	A discretization procedure and a discrete phase transition theorem for more general fields	10
2.5	Sharp threshold at the discrete level	14
2.6	From discrete to continuous	19
3	Proofs of the phase transition theorems and of the exponential decay	20
3.1	Proof of the phase transition theorems	20
3.2	Proof of the exponential decay	26
4	Percolation arguments	27
4.1	Proof of Lemma 2.9	27
4.2	Proof of Proposition 2.20	29
5	A KKL theorem for biased Gaussian vectors: the proof of Theorem 2.19	30
5.1	Sub-linearity of influences implies Theorem 2.19	31
5.2	Sub-linearity of influences for monotonic events	33
6	An estimate on the infinite operator norm of square root of infinite matrices: the proof of Proposition 2.21	35
7	Sprinkling discretization scheme	40
A	An additional sub-linearity result for influences	41

1 Main results

In this article, we study the geometry of excursion sets of a planar centered Gaussian field f . The **covariance function** of f is the function $K : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by:

$$\forall x, y \in \mathbb{R}^2, K(x, y) = \mathbb{E}[f(x)f(y)].$$

We assume that f is normalized so that for each $x \in \mathbb{R}^2$, $K(x, x) = \text{Var}(f(x)) = 1$, that it is non-degenerate (i.e. for any pairwise distinct $x_1, \dots, x_k \in \mathbb{R}^2$, $(f(x_1), \dots, f(x_k))$ is non-degenerate), and that it is a.s. continuous and stationary. In particular, there exists a strictly positive definite continuous function $\kappa : \mathbb{R}^2 \rightarrow [-1, 1]$ such that $\kappa(0) = 1$ and, for each $x, y \in \mathbb{R}^2$, $K(x, y) = \kappa(x - y)$. For our main results (though not for all the intermediate results), we will also assume that f is positively correlated, which means that κ takes only non-negative values. We will also refer to κ as covariance function when there is no possible ambiguity. For each $p \in \mathbb{R}$ we call **level set** of f the random set $\mathcal{N}_p := f^{-1}(-p)$ and **excursion set** of f the random set $\mathcal{D}_p := f^{-1}([-p, +\infty])$.¹

¹This convention, while it may seem counterintuitive, is convenient because it makes \mathcal{D}_p increasing both in f and in p . See Section 2 for more details.

These sets have been studied through their connections to percolation theory (see [MS83a, MS83b, MS86], [Ale96], [BS07], [BG16], [BM17], [BMW17], [RV17]). In this theory, one wishes to determine whether or not there exist unbounded connected components of certain random sets. So far, we know that level sets have a.s. only bounded components for a very large family of (non-necessarily Gaussian) fields f with a positive association property (which is satisfied by positively correlated Gaussian fields, see Theorem 2.15 in the appendix). This is a result by Alexander (see Theorem 2.2 of [Ale96]). More recently, Beffara and Gayet [BG16] have initiated the study of connectivity properties for level and excursion sets of positively correlated Gaussian fields. In particular, they have obtained that, for a large family of positively correlated stationary Gaussian fields such that in particular $\kappa(x) = O(|x|^{-\alpha})$ for some α sufficiently large, a.s. all the connected components of \mathcal{D}_0 (and therefore of \mathcal{D}_p for any $p \leq 0$) are bounded. In [RV17], the authors of the present paper have revisited the results by [BG16] and weaken the assumptions on α . More precisely, we have the following:

Theorem 1.1 ([BG16] for α sufficiently large, [RV17]). ² *Assume that f is a non-degenerate, centered, normalized, continuous, stationary, positively correlated planar Gaussian field that satisfies the symmetry assumption Condition 2.2 below. Assume also that κ satisfies the differentiability assumption Condition 2.4 below and that $\kappa(x) \leq C|x|^\alpha$ for some $C < +\infty$ and $\alpha > 4$. Then, a.s. all the connected components of \mathcal{D}_0 are bounded.*

Both in [BG16] and in [RV17], this result heavily relies on a box-crossing estimate, which is an analog of the classical Russo-Seymour-Welsh theorem for planar percolation. This box-crossing estimate was proved in [BG16] for a large family of positively correlated stationary Gaussian fields such that $\kappa(x) = O(|x|^{-325})$. In [BM17], Beliaev and Muirhead have lowered the exponent $\alpha = 325$ to any $\alpha > 16$. In [RV17], we have obtained that such a result holds with any exponent $\alpha > 4$. More precisely, we have the following:

Theorem 1.2 ([BG16] for $\alpha > 325$, [BM17] for $\alpha > 16$, [RV17]). ³ *With the same hypotheses as Theorem 1.1, for every $\rho \in]0, +\infty[$ there exists $c = c(\rho, \kappa) > 0$ such that for each $R \in]0, +\infty[$, the probability that there is a continuous path in $\mathcal{D}_0 \cap [0, \rho R] \times [0, R]$ joining the left side of $[0, \rho R] \times [0, R]$ to its right side is at least c . Moreover, there exists $R_0 = R_0(\kappa) < +\infty$ such that the same result holds for \mathcal{N}_0 as long as $R \geq R_0$.*

A remaining natural question is whether or not, for $p > 0$, the excursion set \mathcal{D}_p has an unbounded component. Our main result provides an answer to this question for a specific, natural choice of f , see our Theorem 1.3. We obtain a positive answer for a particular model, arising naturally from real algebraic geometry: the Bargmann-Fock model that we now introduce.

The planar Bargmann-Fock field is defined as follows. Let $(a_{i,j})_{i,j \in \mathbb{N}}$ be a family of independent centered Gaussian random variables of variance 1. For each $x = (x_1, x_2) \in \mathbb{R}^2$, the Bargmann-Fock field at x is:

$$f(x) = e^{-\frac{1}{2}|x|^2} \sum_{i,j \in \mathbb{N}} a_{i,j} \frac{x_1^i x_2^j}{\sqrt{i!j!}}.$$

²More precisely, this is Proposition 4.6 of [RV17]. Moreover, this is Theorem 5.7 of [BG16] for α sufficiently large and with slightly different assumptions on the differentiability and on the non-degeneracy of κ .

³More precisely, this is Theorem 1.1 of [RV17]. Moreover, this is Theorem 4.9 of [BG16] (resp. Theorem 1.7 of [BM17]) for $\alpha \geq 325$ (resp. $\alpha > 16$) and with slightly different assumptions on the differentiability and on the non-degeneracy of κ .

The sum converges a.s. in $C^\infty(\mathbb{R}^2)$ to a random analytic function. Moreover, for each $x, y \in \mathbb{R}^2$:

$$K(x, y) = \mathbb{E}[f(x)f(y)] = \exp\left(-\frac{1}{2}|x - y|^2\right).$$

For a discussion of the relation of the Bargmann-Fock field with algebraic geometry, we refer the reader to the introduction of [BG16]. Theorem 1.1 above applies to the Bargmann-Fock model (see Subsection 2.2 for the non-degeneracy condition). In this paper, we prove that on the contrary if $p > 0$ then a.s. \mathcal{D}_p has a unique unbounded component, thus obtaining the following result:

Theorem 1.3. *Let f be the planar Bargmann-Fock field. Then, the probability that \mathcal{D}_p has an unbounded connected component is 1 if $p > 0$, and 0 otherwise. Moreover, if it exists, such a component is a.s. unique.*

In Section 2, we expose the general strategy of the proof of Theorem 1.3. This proof can be summed up as follows: i) we discretize our model as was done in [BG16], ii) we prove that there is a sharp threshold phenomenon at $p = 0$ in the discrete model, iii) we return to the continuum. The results at the heart of our proof of a sharp threshold phenomenon for the discrete model are on the one hand a discrete version of the box-crossing estimate Theorem 1.2 which goes back to [BG16] (see Theorem 2.13 below) and on the other hand a Kahn-Kalai-Linial (KKL)-type estimate for biased Gaussian vectors (see Theorem 2.19) that we show by using the analogous estimate for product Gaussian vectors proved by Keller, Mossel and Sen in [KMS12]. To go back to the continuum, we apply a sprinkling argument to a discretization procedure tailor-made for our setting (see Proposition 2.22). This step is especially delicate since Theorem 2.19 gives no relevant information when the discretization mesh is too fine (see Subsection 2.4 for more details).

Most of the intermediate results that we will prove work in a much wider setting, see in particular Proposition 3.5 where we explain how, for a large family of Gaussian fields f , the proof of an estimate on the correlation function would imply that Theorem 1.3 also holds for f . See also Theorem 2.13 which is a discrete analog of Theorem 1.3 for more general Gaussian fields.

As in [BG16], we are inspired by tools from percolation theory. Before going any further, let us make a short detour to present the results of planar percolation we used to guide our research. It will be helpful to have this analogy in mind to appreciate our results.

Planar Bernoulli percolation is a statistical mechanics model defined on a planar lattice, say \mathbb{Z}^2 , depending on a parameter $p \in [0, 1]$. Consider a family of independent Bernoulli random variables $(\omega_e)_e$ of parameter p indexed by the edges of the graph \mathbb{Z}^2 . We say that an edge e is black if the corresponding random variable ω_e equals 1 and white otherwise. The analogy with our model becomes apparent when one introduces the following classical coupling of the $(\omega_e)_e$ for various values of p . Consider a family $(U_e)_e$ of independent uniform random variables in $[0, 1]$ indexed by the set of edges of \mathbb{Z}^2 . For each $p \in [0, 1]$, let $\omega_e^p = \mathbb{1}_{\{U_e \geq 1-p\}}$. Then, the family $((\omega_e^p)_e)_p$ forms a coupling of Bernoulli percolation with parameters in $[0, 1]$. In this coupling, black edges are seen as excursion sets of the random field $(U_e)_e$. Theorem 1.2 is the analog of the Russo-Seymour-Welsh (RSW) estimates first proved for planar Bernoulli percolation in [Rus78, SW78] (see also Lemma 4 of Chapter 3 of [BR06b], Theorem 11.70 and Equation 11.72 of [Gri99] or Theorem 5.31 of [Gri10]). We now state the main result of percolation theory on \mathbb{Z}^2 , a celebrated theorem due to Kesten. Chapter 3 of [BR06b], Chapter 11 of [Gri99] and Chapter 5 of [Gri10] present different approaches to prove this result.

Theorem 1.4 (Kesten’s Theorem, [Kes80]). *Consider planar Bernoulli percolation of parameter $p \in [0, 1]$ on \mathbb{Z}^2 . If $p > 1/2$, then a.s. there exists an unbounded connected component made of black edges. On the other hand, if $p \leq 1/2$ then a.s. there is no unbounded connected component made of black edges.*

The parameter $p_c = 1/2$ is said to be critical for planar Bernoulli percolation on \mathbb{Z}^2 . It is also known that, if such an unbounded connected component exists, it is a.s. unique. In Theorem 1.4, the case where $p = 1/2$ goes back to Harris [Har60] and also follows easily from the RSW estimate. Next, the coupling constructed above yields the result for $p < 1/2$. Kesten’s theorem is closely linked with another, more quantitative result:

Theorem 1.5 (Exponential decay, [Kes80]). *Consider planar Bernoulli percolation with parameter $p > 1/2$ on \mathbb{Z}^2 . Then, for each $\rho > 0$, there exists a constant $c = c(p, \rho) > 0$ such that for each $R > 0$, the probability that there is a continuous path made of black edges in $[0, \rho R] \times [0, R]$ joining the left side of $[0, \rho R] \times [0, R]$ to its right side is at least $1 - e^{-cR}$.*

The value $p = 1/2$ is significant because with this choice of parameter, the induced percolation model on the dual graph of \mathbb{Z}^2 has the same law as the initial one (for more details see Lemma 1 of Chapter 3 of [BR06b] and the preceding discussion, or Chapter 11 of [Gri99]). For this reason, $1/2$ is called the self-dual point. In the case of our planar Gaussian model, self-duality arises at the parameter $p = 0$ (see the duality properties used in [BG16, RV17], for instance Remark A.12 of [RV17]). The results on Bernoulli percolation lead us to the following conjecture:

Conjecture 1.6. *For centered, normalized, non-degenerate, sufficiently smooth, stationary, isotropic, and positively correlated random fields on \mathbb{R}^2 with sufficient correlation decay, the probability that \mathcal{D}_p has an unbounded connected component is 1 if $p > 0$, and 0 otherwise.*

Of course, our Theorem 1.3 is an answer to the above conjecture for a particular model. We also have an analog for Theorem 1.5 that we prove in Subsection 3.2. Contrary to the other results of this paper, the fact that the correlation function of the Bargmann-Fock field decreases more than exponentially fast is crucial to prove Theorem 1.7.

Theorem 1.7. *Consider the Bargmann-Fock field and let $p > 0$. Then, for each $\rho > 0$ there exists a constant $c = c(p, \rho) > 0$ such that for each $R > 0$, the probability that there is a continuous path in $\mathcal{D}_p \cap [0, \rho R] \times [0, R]$ joining the left side of $[0, \rho R] \times [0, R]$ to its right side is at least $1 - e^{-cR}$.*

Organization of the paper. The paper is organized as follows:

- In Section 2, we explain the general strategy of the proof of our phase transition result Theorem 1.3. In particular, we explain the discretization we use and we state the main intermediate results including the KKL-type inequality for biased Gaussian vectors mentioned above.
- In Section 3, we combine all these intermediate results to prove Theorems 1.3 and 2.13.
- In Sections 4 to 7, we prove these intermediate results.

Related works. As explained above, the present article is in the continuity of [BG16] where the authors somewhat initiate the study of a rigorous connection between percolation theory and the behaviour of nodal lines. In [BM17], the authors optimize the results from [BG16] and the authors of the present paper optimize them further in [RV17]. See also [BMW17] where the

authors prove a box-crossing estimate for Gaussian fields on the sphere or the torus by adapting the strategy of [BG16]. In [BG16, BM17, BMW17, RV17] (while the approaches differ in some key places), the initial idea is the same, namely to use Tassion’s general method to prove box-crossing estimates, which goes back to [Tas16]. To apply such a method, we need to have in particular a positive association property and a quasi-independence property. In [RV17], we have proved such a quasi-independence property for planar Gaussian fields that we will also use in the present paper, see Claim 3.7. We will also rely to other results from [RV17], in particular a discrete RSW estimate. As we will explain in Subsection 2.4, we could have rather referred to the slightly weaker analogous results from [BG16], which would have been sufficient in order to prove our main result Theorem 1.3.

The use of a KKL theorem to prove our main result Theorem 1.3 shows that our work falls within the approach of recent proofs of phase transition that mix tools from percolation theory and tools from the theory of Boolean functions. See [GS14] for a book about how these theory can combine. Below, we list some related works in this spirit.

- During the elaboration of the present work, Duminil-Copin, Raoufi and Tassion have developed novel techniques based on the theory of randomized algorithms and have proved new sharp threshold results, see [DCRT17b, DCRT17a]. This method has proved robust enough to work in a variety of settings: discrete and continuous (the Ising model and Voronoi percolation), with dependent models (such as FK-percolation) and in any dimension. It seems worthwhile to note that the present model resists direct application. At present, we see at least two obstacles: first of all the influences that arise in our setting are not the same as those of [DCRT17b, DCRT17a] (more precisely, the influences studied by Duminil-Copin, Raoufi and Tassion can be expressed as covariances while ours cannot exactly, see Remark 2.18). Secondly, right now it is not obvious for us whether or not our measures are monotonic.
- Another related work whose strategy is closer to the present paper is [Rod15], where the author studies similar questions for the d -dimensional discrete (massive) Gaussian free field. Some elements of said work apply to general Gaussian fields. More precisely, following the proof of Proposition 2.2 of [Rod15], one can express the derivative probability with respect to the threshold p as a sum of covariances, which seems promising, especially in view of [DCRT17b, DCRT17a]. However, each covariance is weighted with a sum of coefficients of the inverse covariance matrix of the discretized field and at present we do not know how to deal with these sums. In [Rod15], these coefficients are very simple because we are dealing with the Gaussian free field.
- For other works where the authors compute the critical point of a planar percolation model by using a KKL inequality, see [BR06d, BR06c, BR06a, BDC12, DCRT16]. In [BDC12, Rod15, DCRT16], the authors use a KKL inequality for monotonic measures proved by Graham and Grimmett [GG06, GG11]. The same obstacles as in Item 1 above prevented us to use this KKL inequality.

A note on vocabulary. We end the first section by a remark on vocabulary on positive definite matrices and functions.

Remark 1.8. In all the paper, we are going to deal with positive definite functions and matrices. The convention seems to be that, on the one hand positive definite matrices are invertible whereas semi-positive definite matrices are not necessarily. On the other hand, a function g

is said strictly positive definite if for any $n \in \mathbb{Z}_{>0}$ and any x_1, \dots, x_n , $(g(x_i - x_j))_{1 \leq i, j \leq n}$ is a positive definite matrix while g is said positive definite if the matrices $(g(x_i - x_j))_{1 \leq i, j \leq n}$ are semi-positive definite. We will follow these conventions and hope this remark will clear up any ambiguities.

Acknowledgments:

We are grateful to Christophe Garban and Damien Gayet for many fruitful discussions and for their advice in the organization of the manuscript. We would also like to thank Vincent Beffara for his helpful comments. Finally, we are thankful to Thomas Letendre for pointing out useful references about Gaussian fields and for his help regarding Fourier techniques.

2 Proof strategy and intermediate results

In this section, we explain the global strategy of the proof of our main result Theorem 1.3. Since the case $p \leq 0$ is already known (see the beginning of Section 1), we focus to the case $p > 0$. We first discuss briefly some aspects of the analogous result for Bernoulli percolation: Kesten's theorem. Next, we give an informal explanation of our proof and state the main intermediate results. More precisely, we explain the discretization procedure used in our proof in Subsection 2.4. Then, we state the main intermediate results at the discrete level in Subsection 2.5, and in Subsection 2.6 we explain how to go from the discrete to the continuum.

2.1 Some ingredients for the proof of Kesten's theorem

Several proofs of Kesten's theorem (Theorem 1.4) are known (see [Gri10, BR06b]). Let us fix a parameter $p \in [0, 1]$ and consider Bernoulli percolation on \mathbb{Z}^2 with parameter p . As explained before, we are mainly interested in the proof of existence of an unbounded component, i.e. when $p > 1/2$. One possible proof of Kesten's theorem uses the following ingredients, that we will try to adapt to our setting.

- *A box-crossing criterion:* For all $\rho_1, \rho_2 > 0$, let $\text{Cross}_p^{\text{perco}}(\rho_1, \rho_2)$ denote the event that there is a continuous path of black edges that crosses the rectangle $[0, \rho_1] \times [0, \rho_2]$ from left to right, and assume that:

$$\sum_{k \in \mathbb{N}} \mathbb{P} \left[\neg \text{Cross}_p^{\text{perco}}(2^{k+1}, 2^k) \right] < +\infty.$$

Then, a.s. there exists an infinite black component.

- *The RSW (Russo-Seymour-Welsh) theorem* (see Lemma 4 of Chapter 3 of [BR06b], Theorem 11.70 and Equation 11.72 of [Gri99] or Theorem 5.31 of [Gri10]) which implies that, for every $\rho > 0$, there exists a constant $c = c(\rho) > 0$ such that, for every $R > 0$, $\mathbb{P} \left[\text{Cross}_{1/2}^{\text{perco}}(\rho R, R) \right] \geq c(\rho)$.
- *The FKG-Harris inequality* (see [Gri99, BR06b]) that says that increasing events are positively correlated.
- *Russo's differential formula* ([Gri99, BR06b]): Let $n \in \mathbb{N}_+$, $\mathbb{P}_p^n := (p\delta_1 + (1-p)\delta_0)^{\otimes n}$ and $A \subseteq \{0, 1\}^n$. For every $i \in \{1, \dots, n\}$, let $\text{Inf}_i^p(A)$ denote the influence of i on A at the

parameter p , which is the probability that changing the value of the coordinate i modifies $\mathbb{1}_A$. If A is increasing, then we have the following differential formula:

$$\frac{d}{dp} \mathbb{P}_p^n[A] = \sum_{i=1}^n \text{Inf}_i^p(A).$$

- *A KKL (Kahn-Kalai-Linial) theorem* (see Theorem 4.29 of [Gri10], Theorem 12 of Chapter 2 of [BR06b] or Theorems 1.16 and 3.4 of [GS14]): The sum of influences can be estimated thanks to the celebrated KKL theorem. Here, we present the version of the KKL theorem that implies that, if all the influences are small, then the sum of the influences is large. It was proved in [KKL88] for $p = 1/2$ and generalized in [BKK⁺92] to every p . Let \mathbb{P}_p^n be as above. There exists an absolute constant $c > 0$ such that, for every $p \in [0, 1]$, every $n \in \mathbb{N}_+$ and every $A \subseteq \{0, 1\}^n$, we have:

$$\sum_{i=1}^n \text{Inf}_i^p(A) \geq c \mathbb{P}_p^n[A] \cdot (1 - \mathbb{P}_p^n[A]) \cdot \log \left(\frac{1}{\max_i \text{Inf}_i^p(A)} \right).$$

These ingredients can be used as follows to prove that there exists an infinite black component when $p > 1/2$ (for more details, see for instance Section 3.4 of [BR06b], Section 5.8 of [Gri99] or Section 3.4 of [GS14]): The RSW theorem and the FKG-Harris inequality can be used to prove that the influences of the crossing events decrease polynomially fast in R . Thanks to this polynomial decay, Russo's formula, the RSW theorem and the KKL theorem, one can prove that, for every $p > 1/2$, $\mathbb{P}[\text{Cross}_p^{\text{perco}}(2R, R)] \geq 1 - R^{-a}$ for some $a = a(p) > 0$ (actually, with a bit more work, one can deduce exponential decay, see for instance Section 3.5 of [BR06b]). Thus, the box-crossing criterion is satisfied and we are done.

Our global strategy to prove Theorem 1.3 will be based on similar ingredients and on a discretization procedure used in [BG16, BM17] (together with a sprinkling argument). Some of these ingredients are already known, the others will be proved in our paper. We list them in the remaining subsections of Section 2, and in Section 3.1 we will explain how we can combine all these ingredients to prove Theorem 1.3.

Since most of our intermediate results work in a much wider setting, we first state the conditions on the planar Gaussian field f under which we work.

2.2 Conditions on the Gaussian fields

First, we state Condition 2.1 that we will assume during all the work:

Condition 2.1. The field f is non-degenerate (i.e. for any pairwise distinct $x_1, \dots, x_k \in \mathbb{R}^2$, $(f(x_1), \dots, f(x_k))$ is non-degenerate), centered, normalized, continuous, and stationary. In particular, there exists a strictly positive definite continuous function $\kappa : \mathbb{R}^2 \rightarrow [-1, 1]$ such that $K(x, y) := \mathbb{E}[f(x)f(y)] = \kappa(y - x)$ and $\kappa(0) = 1$.

Depending on the intermediate results we prove, we will also need to assume some of the following additional conditions:

Condition 2.2 (Useful to apply percolation arguments.). The field f is positively correlated, invariant by $\frac{\pi}{2}$ -rotation, and reflection through the horizontal axis.

Condition 2.3 (Useful to have quasi-independence.). Depends on a parameter $\alpha > 0$.] There exists $C < +\infty$ such that for each $x \in \mathbb{R}^2$, $|\kappa(x)| \leq C|x|^{-\alpha}$.

Condition 2.4 (Technical conditions to have quasi-independence, see [RV17]). The function κ is C^8 and for each $\beta \in \mathbb{N}^2$ with $\beta_1 + \beta_2 \leq 2$, $\partial^\beta \kappa(x) \xrightarrow{|x| \rightarrow +\infty} 0$.

Condition 2.5 (Useful to do Fourier calculations on the correlation function. Depends on a parameter $\alpha > 0$). The Fourier transform of κ takes only positive values. Moreover, κ is C^3 and there exists $C < +\infty$ such that for every $\beta \in \mathbb{N}^2$ with $\beta_1 + \beta_2 \leq 3$, we have:

$$|\partial^\beta \kappa(x)| \leq C|x|^{-\alpha}.$$

We will often suppose regularity conditions on f and κ . It will be interesting to have the following in mind (see for instance Appendices A.3 and A.9 of [NS15]):

Lemma 2.6. *Assume that f satisfies Condition 2.1. Let $k \in \mathbb{N}$. If κ is $C^{2(k+1)}$, then a.s. f is C^k . Conversely, if a.s. f is C^k , then κ is C^{2k} and for every multi-indices $\beta, \gamma \in \mathbb{N}^2$ such that $\beta_1 + \beta_2 \leq k$ and $\gamma_1 + \gamma_2 \leq k$, we have:*

$$\text{Cov} \left(\partial^\beta f(x), \partial^\gamma f(y) \right) = \mathbb{E} \left[\partial^\beta f(x) \partial^\gamma f(y) \right] = (-1)^{|\gamma|} \partial^{\beta+\gamma} \kappa(x-y).$$

It is easy to check that the conditions are all satisfied by the Bargmann-Fock field (see Lemma 2.7 for the non-degeneracy condition). In particular, Conditions 2.4 and 2.5 hold for any $\alpha > 0$. Also, the Bargmann-Fock field is a.s. analytic and its covariance is analytic.

An other example of Gaussian fields in the plane that satisfy the above conditions (with the parameter α which depends on the parameter n) is the field with correlation function:

$$\kappa : x = (x_1, x_2) \in \mathbb{R}^2 \mapsto \left(\frac{1}{(1+x_1^2)(1+x_2^2)} \right)^n, \quad (2.1)$$

where $n \in \mathbb{Z}_{>0}$. This is indeed a strictly positive definite function by the following lemma.

Lemma 2.7. *The Fourier transform of a continuous and integrable function $\mathbb{R}^2 \rightarrow \mathbb{R}_+$ which is not 0 is strictly positive definite. In particular, the Gaussian function $x \mapsto \exp(-\frac{1}{2}|x|^2)$ and the function (2.1) (for any $n \in \mathbb{Z}_{>0}$) are strictly positive definite.*

Proof. This is a direct consequence of Theorem 3 of Chapter 13 of [CL09] (which is the strictly positive definite version of the easy part of Bochner theorem). We can apply this to the Bargmann-Fock field and to (2.1) since the Fourier transform of a Gaussian function is still a Gaussian function and since the Fourier transform of $x \mapsto \frac{1}{(1+x_1^2)(1+x_2^2)}$ is $\xi \mapsto cst \exp(-(|\xi_1| + |\xi_2|))$, hence $x \in \mathbb{R}^2 \mapsto \left(\frac{1}{(1+x_1^2)(1+x_2^2)} \right)^n$ is the Fourier transform of the function $\xi \mapsto cst \exp(-(|\xi_1| + |\xi_2|))$ convoluted n times, see Paragraph 1.2.3 of [Rud]. \square

If one wants to consider a large family of examples of planar Gaussian fields, one can consider a function $\rho : \mathbb{R}^2 \rightarrow \mathbb{R}_+$ sufficiently smooth, that is not 0 and such that ρ and its derivatives decay sufficiently fast. One can note that $\kappa = \rho * \rho$ has the same properties and is strictly positive definite by Lemma 2.7 since $\hat{\kappa} = (\hat{\rho})^2$. Moreover, if ρ has sufficiently many symmetries, then the Gaussian field with covariance κ above satisfies Conditions from 2.1 to 2.5. Now, if W

is a two-dimensional white noise, that is, the free field associated to the Hilbert space $L^2(\mathbb{R}^2)$ (see Definition 2.5 of [She07]), then (if ρ is even)

$$f = \rho * W$$

defines a Gaussian field on \mathbb{R}^2 with covariance κ .

We now list the main intermediate results of our proof.

2.3 A box-crossing criterion

As in the strategy of the proof of Kesten's theorem presented in Subsection 2.1, our goal will be to prove a box-crossing criterion. We start by introducing the following notation.

Notation 2.8. For each $\rho_1, \rho_2 > 0$ and each $p \in \mathbb{R}$, we write $\text{Cross}_p(\rho_1, \rho_2)$ for the event that there is a continuous path in $\mathcal{D}_p \cap [0, \rho_1] \times [0, \rho_2]$ joining the left side of $[0, \rho_1] \times [0, \rho_2]$ to its right side.

In Subsection 4.1, we will prove the following proposition.

Lemma 2.9. *Assume that f satisfies Condition 2.1 and that κ is invariant by $\frac{\pi}{2}$ -rotations. Let $p > 0$ and assume that:*

$$\sum_{k \in \mathbb{N}} \mathbb{P} \left[\neg \text{Cross}_p(2^{k+1}, 2^k) \right] < +\infty. \quad (2.2)$$

Then, a.s. there exists a unique unbounded component in \mathcal{D}_p .

Thus, our goal turns to prove that, if $p > 0$, then $\mathbb{P}[\text{Cross}_p(2R, R)]$ goes to 1 as R goes to $+\infty$, sufficiently fast so that the above sum is finite. In order to prove such a result, we will show a Russo-type formula and a KKL-type theorem for discrete Gaussian fields. To apply such result, we first need to discretize our model, as it was done in [BG16].

2.4 A discretization procedure and a discrete phase transition theorem for more general fields

We consider the following discrete percolation model: let $\mathcal{T} = (\mathcal{V}, \mathcal{E})$ be the face-centered square lattice defined in Figure 2. Note that the exact choice of lattice is not essential in most of our arguments. We mostly use the fact that it is a periodic triangulation with nice symmetries. However, we do need a little more information to apply Theorem 2.13 (see the beginning of Section 3 of [RV17]) and in Section 6 of the present work we use the fact that the sites of \mathcal{T} are the vertices of a rotated and rescaled \mathbb{Z}^2 -lattice. We will denote by $\mathcal{T}^\varepsilon = (\mathcal{V}^\varepsilon, \mathcal{E}^\varepsilon)$ this lattice scaled by a factor ε . Given a realization of our Gaussian field f and some $\varepsilon > 0$, we color the plane as follows: For each $x \in \mathbb{R}^2$, if $x \in \mathcal{V}^\varepsilon$ and $f(x) \geq -p$ or if x belongs to an edge of \mathcal{E}^ε whose two extremities y_1, y_2 satisfy $f(y_1) \geq -p$ and $f(y_2) \geq -p$, then x is colored black. Otherwise, x is colored white. In other words, we study a **correlated site percolation model** on \mathcal{T}^ε .

We will use the following notation analogous to Notation 2.8.

Notation 2.10. Let $\varepsilon > 0$, $p \in \mathbb{R}$, and consider the above discrete percolation model. For every $\rho_1, \rho_2 > 0$, write $\text{Cross}_p^\varepsilon(\rho_1, \rho_2)$ for the event that there is a continuous black path in $[0, \rho_1] \times [0, \rho_2]$ joining the left side of $[0, \rho_1] \times [0, \rho_2]$ to its right side.

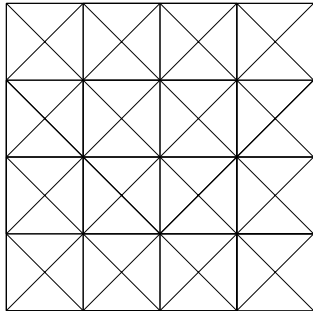


Figure 2: The face-centered square lattice (the vertices are the points of \mathbb{Z}^2 and the centers of the squares of the \mathbb{Z}^2 -lattice).

As explained in Subsection 2.3, we want to estimate the quantities $\mathbb{P}[\text{Cross}_p(2R, R)]$ for $p > 0$. However, the tools that we develop in our paper are suitable to study the quantities $\mathbb{P}[\text{Cross}_p^\varepsilon(2R, R)]$. We will thus have to choose ε so that, on the one hand, we can find nice estimates at the discrete level and, on the other hand, the discrete field is a good approximation of the continuous field. As a reading guide for the global strategy, we list below what are the constraints on ε for our intermediate results to hold. See also Figure 3. We write them for the Bargmann-Fock field. Actually, most of the intermediate results including Item 2 below hold for more general fields, see the following subsections for more details.

1. We want to establish a lower bound for $\mathbb{P}[\text{Cross}_p^\varepsilon(2R, R)]$ that goes (sufficiently fast for us) to 1 as R goes to $+\infty$. Our techniques work as long as the number of vertices is not too large (see in particular Subsection 2.5.5). They yield such a bound provided that there exists $\delta > 0$ such that (R, ε) satisfy the condition:

$$\varepsilon \geq \log^{-(1/2-\delta)}(R). \quad (2.3)$$

See Subsubsection 2.5.6.

2. Then, we will have to estimate how much $\text{Cross}_p^\varepsilon(2R, R)$ approximates well $\text{Cross}_p(2R, R)$. At this point, it seems natural to use the quantitative approximation results from [BG16, BM17]. In particular, as explained briefly in Subsection 2.6, it seems that results from [BM17] (based on discretization schemes that generalize the methods in [BG16]) imply that the event $\text{Cross}_p^\varepsilon(2R, R)$ approximates $\text{Cross}_p(2R, R)$ well if:

$$\varepsilon \leq R^{-(1+\delta)}, \quad (2.4)$$

for some $\delta > 0$. Unfortunately, this constraint is not compatible with the constraint (2.3). For this reason, we will instead use a sprinkling discretization procedure. More precisely, we will see in Proposition 2.22 that $\text{Cross}_{p/2}^\varepsilon(2R, R)$ implies $\text{Cross}_p(2R, R)$ with high probability if (R, ε) satisfy the following condition:

$$\varepsilon \geq \log^{-(1/4+\delta)}(R) \quad (2.5)$$

for some $\delta > 0$. This time, the constraint combines very well with the constraint (2.3) and we will be able to conclude.

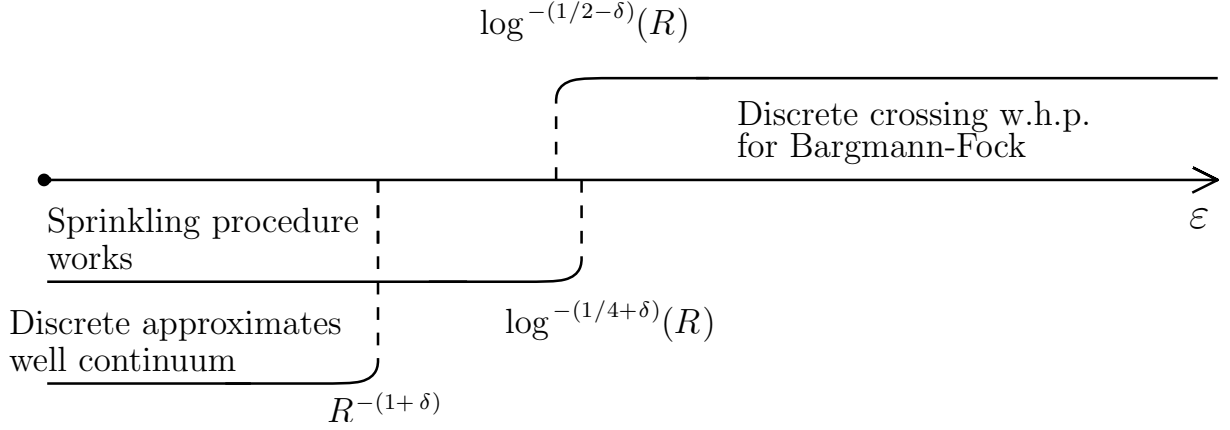


Figure 3: The different constraints on (R, ε) .

If we work with the Bargmann-Fock field and if we choose for instance $\varepsilon = \varepsilon(R) = \log^{-1/3}(R)$ then, as shown in Figure 3, we obtain that $\text{Cross}_p^\varepsilon(2R, R)$ holds with high probability and that the sprinkling discretization procedure works. As explained in Section 3.1, we will thus obtain that $\sum_{k \in \mathbb{N}} \mathbb{P}[-\text{Cross}_p(2^{k+1}, 2^k)] < +\infty$ and conclude thanks to Lemma 2.9. See also Figure 4.

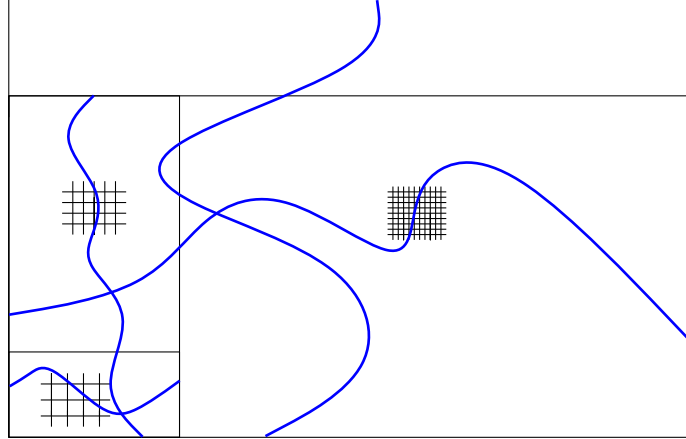


Figure 4: A right choice of ε is $\varepsilon = \log^{-1/3}(R)$ with whom we will be able to prove that $\mathbb{P}[\text{Cross}(2R, R)]$ goes sufficiently fast to 1 so that the box-crossing criterion of Lemma 2.9 is satisfied. As we will see in the proof of this lemma, this criterion implies that, for a well-chosen of rectangles $(Q_k)_k$ of typical length R_k , a.s. all the Q_k 's for k sufficiently large are crossed, which implies that there exists an infinite path. The idea is that, for each k , we study the probability that it is crossed by comparing the continuous event with the discrete event with mesh $\varepsilon = \log^{-1/3}(R_k)$.

Note that Item 1 above implies in particular that, if we work with the Bargmann-Fock field and if we fix $\varepsilon \in]0, 1]$,⁴ then for $p > 0$, $\mathbb{P}[\text{Cross}_p^\varepsilon(2R, R)]$ goes to 1 (sufficiently fast for us) as R goes to $+\infty$. We will actually obtain such a result for more general fields, which will enable us to

⁴In this paper, we only look at the case $\varepsilon \in]0, 1]$ though a lot of results could probably be extended to any $\varepsilon > 0$.

prove that a discrete box-crossing criterion analogous to Lemma 2.9 is satisfied and deduce the following in Section 3.1:

Theorem 2.11. *Suppose that f satisfies Conditions 2.1 and 2.2 and Condition 2.5 for some $\alpha > 5$ (thus in particular Condition 2.3 is satisfied for some $\alpha > 5$). Then, for each $\varepsilon \in]0, 1]$, the critical threshold of the discrete percolation model on \mathcal{T}^ε defined in the present subsection is 0. More precisely: the probability that there is an unbounded black connected component is 1 if $p > 0$, and 0 otherwise. Moreover, if it exists, such a component is a.s. unique. In particular, this result holds when f is the Bargmann-Fock field or the centered Gaussian field with covariance given by (2.1) with $n \geq 3$.*

As in the continuous setting, the case $p \leq 0$ of Theorem 2.11 goes back to [BG16], at least for α large enough. In [RV17], we have optimized this result and obtain the following:

Theorem 2.12 ([BG16] for α sufficiently large, [RV17]). ⁵ *Assume that f satisfies Conditions 2.1, 2.2, 2.4 as well as Condition 2.3 for some $\alpha > 4$. Let $\varepsilon \in]0, 1]$ and consider the discrete percolation model on \mathcal{T}^ε defined in the present subsection with parameter $p = 0$. Then, a.s. there is no infinite connected black component.*

Again as in the continuous case, this result heavily relies on a RSW estimate, which we state below (see Theorem 2.13). This estimate is a uniform lower bound on the crossing probability of a quad scaled by R on a lattice with mesh ε . The proof of such an estimate goes back to [BG16] (for $\alpha > 16$) and was later optimized to $\alpha > 8$ in [BM17]. The key property in this estimate is that the lower bound of the crossing properties does not depend on the choice of the mesh ε . To obtain such a result, the authors of [BG16] had to impose some conditions on (R, ε) . For instance, if we consider the Bargmann-Fock field, the constraint was $\varepsilon \geq C \exp(-cR^2)$ where $C < +\infty$ and $c > 0$ are fixed. Actually, it seems likely that, one could deduce a discrete RSW estimate with no constraint on (R, ε) by using their quantitative approximation results. In [RV17], we prove such a discrete RSW estimate with no constraint on (R, ε) and without using quantitative discretization estimates, but rather by using new quasi-independence results. Note that to prove our main result Theorem 1.3, it would not have been a problem for us to rather use the result by Beffara and Gayet with constraints on (R, ε) since, as explained above, we will use results from the discrete model with $\varepsilon = \varepsilon(R) = \log^{-1/3}(R)$ - which of course satisfies the condition $\varepsilon \geq C \exp(-cR^2)$. We have the following:

Theorem 2.13 ([BG16] for $\alpha > 16$, [BM17] for $\alpha > 8$, [RV17]). ⁶ *Assume that f satisfies Conditions 2.1, 2.2, 2.4 as well as Condition 2.3 for some $\alpha > 4$. For every $\rho \in]0, +\infty[$, there exists a constant $c = c(\kappa, \rho) > 0$ such that the following holds: let $\varepsilon \in]0, 1]$, and consider the discrete percolation model on \mathcal{T}^ε defined in the present subsection with parameter $p = 0$. Then, for every $R \in]0, +\infty[$ we have:*

$$\mathbb{P}[\text{Cross}_0^\varepsilon(\rho R, R)] \geq c.$$

What remains to prove in order to show Theorem 2.11 is that, if $p > 0$, a.s. there is a unique infinite black component. Exactly as in the continuum, our goal will be to show that a box-crossing criterion is satisfied. The following lemma is proved in Subsection 4.1.

⁵More precisely, this is Proposition B.7 of [RV17]. Moreover, this can be extracted from the proof of Theorem 5.7 of [BG16] for α sufficiently large and with slightly different assumptions on the differentiability and the non-degeneracy of κ .

⁶More precisely, this is Proposition B.2 of [RV17]. Moreover, this is Theorem 2.2 from [BG16] combined with the results of their Section 4 (resp. this is Appendix C of [BM17]) for $\alpha > 16$ (resp. $\alpha > 8$), with some constraints on (R, ε) , and with slightly different assumptions on the differentiability and the non-degeneracy of κ .

Lemma 2.14. *Assume that f satisfies Condition 2.1 and that κ is invariant by $\frac{\pi}{2}$ -rotations. Let $\varepsilon > 0$, let $p \in \mathbb{R}$ and suppose that:*

$$\sum_{k \in \mathbb{N}} \mathbb{P} \left[-\text{Cross}_p^\varepsilon(2^{k+1}, 2^k) \right] < +\infty. \quad (2.6)$$

Then, a.s. there exists a unique unbounded black component in the discrete percolation model defined in the present subsection.

2.5 Sharp threshold at the discrete level

We list the intermediate results at the discrete level. Among all the following results (and actually among all the intermediate results of the paper) the only result specific to the Bargmann-Fock field is the second result of Proposition 2.21. All the others work in a quite general setting.

2.5.1 The FKG inequality for Gaussian vectors

The FKG inequality is a crucial tool to apply percolation arguments. We say that a Borel subset $A \subseteq \mathbb{R}^n$ is **increasing** if for each $x \in A$ and $y \in \mathbb{R}^n$ such that $y_i \geq x_i$ for every $i \in \{1, \dots, n\}$, we have $y \in A$. We say that A is decreasing if the complement of A is increasing. The FKG inequality for Gaussian vectors was proved by Pitt and can be stated as follows.

Theorem 2.15 ([Pit82]). *Let $X = (X_1, \dots, X_n)$ be a n -dimensional Gaussian vector whose correlation matrix has non-negative entries. Then, for every increasing Borel subsets $A, B \subseteq \mathbb{R}^n$, we have:*

$$\mathbb{P}[X \in A \cap B] \geq \mathbb{P}[X \in A] \cdot \mathbb{P}[X \in B].$$

This result is the reason why we work with positively correlated Gaussian fields. Indeed, the FKG inequality is a crucial ingredient in the proof of RSW-type results. Note however that the very recent [BG17] proves a box-crossing property without this inequality, albeit only in a discrete setting.

2.5.2 A differential formula

As in the case of Bernoulli percolation, we need to introduce a notion of influence. This notion of influence is inspired by the **geometric influences** studied by Keller, Mossel and Sen in the case of product spaces [KMS12, KMS14]. (For more about the relations between the geometric influences and the influences of Definition 2.16 below - which are roughly the same in the case of product measures - see Subsection 5.1.)

Definition 2.16. Let μ be a finite Borel measure on \mathbb{R}^n , let $v \in \mathbb{R}^n$, and let A be a Borel subset of \mathbb{R}^n . The influence of v on A under μ is:

$$I_{v,\mu}(A) := \liminf_{r \downarrow 0} \frac{\mu(A + [-r, r]v) - \mu(A)}{r} \in [0, +\infty].$$

Write (e_1, \dots, e_n) for the canonical basis of \mathbb{R}^n . We will use the following simplified notations:

$$I_{i,\mu}(A) := I_{e_i,\mu}(A).$$

The events we are interested in are “threshold events” and the measures we are interested in are Gaussian distributions: Let $X = (X_1, \dots, X_n)$ be a n -dimensional non-degenerate centered Gaussian vector, write μ_X for the law of X and, for every $\vec{p} = (p_1, \dots, p_n) \in \mathbb{R}^n$ and every $i \in \{1, \dots, n\}$, write:

$$\omega_i^{\vec{p}} := \mathbb{1}_{\{X_i \geq -p_i\}}.$$

This defines a random variable $\omega^{\vec{p}}$ with values in $\{0, 1\}^n$. If $B \subseteq \{0, 1\}^n$, we call the event $\{\omega^{\vec{p}} \in B\}$ a **threshold event**. For every $i \in \{1, \dots, n\}$, we let $\text{Piv}_i^{\vec{p}}(B)$ denote the event that changing the value of the bit i in $\omega^{\vec{p}}$ modifies $\mathbb{1}_B(\omega^{\vec{p}})$. In other words,

$$\text{Piv}_i^{\vec{p}}(B) = \{(\omega_1^{\vec{p}}, \dots, \omega_{i-1}^{\vec{p}}, 0, \omega_{i+1}^{\vec{p}}, \dots, \omega_n^{\vec{p}}) \in B\} \triangle \{(\omega_1^{\vec{p}}, \dots, \omega_{i-1}^{\vec{p}}, 1, \omega_{i+1}^{\vec{p}}, \dots, \omega_n^{\vec{p}}) \in B\},$$

where $E \triangle F := (E \setminus F) \cup (F \setminus E)$. Such an event is called a **pivotal event**. We say that a subset $B \subseteq \{0, 1\}^n$ is **increasing** if for every $\omega \in B$ and $\omega' \in \{0, 1\}^n$, the fact that $\omega'_i \geq \omega_i$ for every i implies that $\omega' \in B$. Moreover, if $p \in \mathbb{R}$, we write $\mathbf{p} = (p, \dots, p) \in \mathbb{R}^n$ and we use the following notations:

$$\omega^p := \omega^{\mathbf{p}} \text{ and } \text{Piv}_i^p(B) := \text{Piv}_i^{\mathbf{p}}(B).$$

Proposition 2.17. *Assume that X is a n -dimensional non-degenerate centered Gaussian vector and let Σ be its covariance matrix. Let B be an increasing subset of $\{0, 1\}^n$. Then:*

$$\frac{\partial \mathbb{P}[\omega^{\vec{p}} \in B]}{\partial p_i} = I_{i, \mu_X}(\omega^{\vec{p}} \in B) = \mathbb{P}[\text{Piv}_i^{\vec{p}}(B) \mid X_i = -p_i] \frac{1}{\sqrt{2\pi\Sigma_{i,i}}} \exp\left(-\frac{1}{2\Sigma_{i,i}}p_i^2\right).$$

In particular, if $p \in \mathbb{R}$, then:

$$\frac{d\mathbb{P}[\omega^p \in B]}{dp} = \sum_{i=1}^n I_{i, \mu_X}(\omega^p \in B) = \sum_{i=1}^n \mathbb{P}[\text{Piv}_i^p(B) \mid X_i = -p] \frac{1}{\sqrt{2\pi\Sigma_{i,i}}} \exp\left(-\frac{1}{2\Sigma_{i,i}}p^2\right).$$

Remark 2.18. With the same hypotheses as in Proposition 2.17: For $\epsilon \in \{0, 1\}$, let $B_\epsilon^i = \{\omega \in \{0, 1\}^n : (\omega_1, \dots, \omega_{i-1}, \epsilon, \omega_{i+1}, \dots, \omega_n) \in B\}$. Note that we have:

$$\mathbb{P}[\text{Piv}_i^{\vec{p}}(B) \mid X_i = -p_i] = \mathbb{P}[\omega^{\vec{p}} \in B_1^i \mid X_i = -p_i] - \mathbb{P}[\omega^{\vec{p}} \in B_0^i \mid X_i = -p_i].$$

Hence the form of our influences is close to the covariance between $\mathbb{1}_{\{\omega_i^{\vec{p}}=1\}}$ and $\mathbb{1}_{\{\omega^{\vec{p}} \in B\}}$ but is not exactly a covariance. This is important to have in mind if one wants to try to apply techniques from [DCRT17b, DCRT17a] in future.

Since the proof of Proposition 2.17 is rather short, we include this here. The reader essentially interested in the strategy of proof can skip this in a first reading.

Proof of Proposition 2.17. Without loss of generality, we assume that $i = n$. For any $C \subseteq \{0, 1\}^n$, let $C^{\vec{p}} \subseteq \mathbb{R}^n$ be the preimage of C by the map $x \in \mathbb{R}^n \mapsto (\mathbb{1}_{x_i \geq -p_i})_{i \in \{1, \dots, n\}} \in \{0, 1\}^n$. Let $e_n = (0, \dots, 0, 1) \in \mathbb{R}^n$, $h > 0$, let $B \subseteq \{0, 1\}^n$ be an increasing event, and let $\tilde{X} = (X_i)_{1 \leq i \leq n-1}$. Then:

$$\mathbb{P}[\omega^{\vec{p}+he_n} \in B] = \mathbb{P}[X \in B^{\vec{p}+he_n}] = \mathbb{P}[X + he_n \in B^{\vec{p}}] = \mathbb{P}[(\tilde{X}, X_n + h) \in B^{\vec{p}}],$$

$$\text{and } \mathbb{P}[\omega^{\vec{p}} \in B] = \mathbb{P}[(\tilde{X}, X_n) \in B^{\vec{p}}].$$

Also:

$$\begin{aligned} \mathbb{P}\left[(\tilde{X}, X_n + h) \in B^{\vec{p}}\right] - \mathbb{P}\left[(\tilde{X}, X_n) \in B^{\vec{p}}\right] &= \mathbb{P}\left[(\tilde{X}, X_n + h) \in B^{\vec{p}}, (\tilde{X}, X_n) \notin B^{\vec{p}}\right] \\ &= \mathbb{P}\left[\text{Piv}_n^{\vec{p}}(B), X_n \in [-p_n - h, -p_n[\right]. \end{aligned}$$

Since Σ is positive definite, the Gaussian measure has smooth density with respect to the Lebesgue measure. Taking the difference of the two probabilities and letting $h \downarrow 0$, we get:

$$\begin{aligned} &h^{-1} \left(\mathbb{P}\left[\omega^{\vec{p} + h e_n} \in B\right] - \mathbb{P}\left[\omega^{\vec{p}} \in B\right] \right) \\ &= h^{-1} \mathbb{P}\left[\text{Piv}_n^{\vec{p}}(B), X_n \in [-p_n - h, -p_n[\right] \\ &= \frac{1}{h \sqrt{2\pi \Sigma_{n,n}}} \int_{-p_n - h}^{-p_n} \mathbb{P}\left[\text{Piv}_n^{\vec{p}}(B) \mid X_n = t\right] \exp\left(-\frac{1}{2\Sigma_{n,n}} t^2\right) dt \\ &\xrightarrow{h \downarrow 0} \mathbb{P}\left[\text{Piv}_n^{\vec{p}}(B) \mid X_n = -p_n\right] \frac{1}{\sqrt{2\pi \Sigma_{n,n}}} \exp\left(-\frac{1}{2\Sigma_{n,n}} p_n^2\right). \end{aligned}$$

In the last step we use the continuity of the conditional probability of a threshold event with respect to the conditioning value. This is an easy consequence of Proposition 1.2 of [AW09]. The calculation for $h < 0$ is analogous, hence we are done. \square

2.5.3 A KKL-KMS (Kahn-Kalai-Linial - Keller-Mossel-Sen) theorem for non-product Gaussian vectors

One of the contributions of this paper is the derivation of a KKL theorem for biased Gaussian vectors, namely Theorem 2.19, based on a similar result for product Gaussian vectors by Keller, Mossel and Sen, [KMS12]. (This similar result by [KMS12] is stated in Theorem 5.1 of our paper⁷.) Actually, with our techniques of Section 5, most of the results from [KMS12] could be extended to non-product Gaussian vectors (and to monotonic or semi-algebraic events).

Theorem 2.19. *There exists an absolute constant $c > 0$ such that the following holds: Let $n \in \mathbb{Z}_{>0}$, let Σ be a $n \times n$ symmetric positive definite matrix⁸ and let $\mu = \mathcal{N}(0, \Sigma)$. Also, let $\sqrt{\Sigma}$ be a symmetric square root of Σ and write $\|\sqrt{\Sigma}\|_{\infty, op}$ for the operator norm of $\sqrt{\Sigma}$ for the infinite norm⁹ on \mathbb{R}^n . For every A either monotonic or semi-algebraic¹⁰ Borel subset of \mathbb{R}^n we have:*

$$\sum_{i=1}^n I_{i, \mu}(A) \geq c \|\sqrt{\Sigma}\|_{\infty, op}^{-1} \mu(A) (1 - \mu(A)) \sqrt{\log_+ \left(\frac{1}{\|\sqrt{\Sigma}\|_{\infty, op} \cdot \max_{i \in \{1, \dots, n\}} I_{i, \mu}(A)} \right)}.$$

This theorem probably holds for a wider class of sets. Since we need it only for monotonic events, we did not try to identify the weakest possible assumptions for the property to hold. In particular, we have not found any examples of Borel sets for which the theorem does not hold.

A way to understand the constant $\|\sqrt{\Sigma}\|_{\infty, op}$ is to see what happens in the extreme cases:

⁷As we will explain, Theorem 5.1 a simple consequence of Item (1) of Theorem 1.5 of [KMS12].

⁸Remember Remark 1.8: this means in particular that Σ is non-degenerate.

⁹I.e. $\|\sqrt{\Sigma}\|_{\infty, op} = \sup_{x \in \mathbb{R}^n \setminus \{0\}} \frac{|\sqrt{\Sigma}(x)|_{\infty}}{|x|_{\infty}}$. Equivalently, $\|\sqrt{\Sigma}\|_{\infty, op} = \max_{i \in \{1, \dots, n\}} \sum_{j=1}^n |\sqrt{\Sigma}(i, j)|$.

¹⁰We say that a set $A \subset \mathbb{R}^n$ is semi-algebraic if it belongs to the Boolean algebra generated by sets of the form $P^{-1}([0, +\infty[)$ where $P \in \mathbb{R}[X_1, \dots, X_n]$.

- If Σ is the identity matrix, then $\|\sqrt{\Sigma}\|_{\infty,op} = 1$ and the above result corresponds to the product case from [KMS12].
- If $\Sigma_{i,j} = 1$ for every i, j (which corresponds to the fact that, if $X \sim \Sigma$, then $X_i = X_j$ for every i, j) then $\|\sqrt{\Sigma}\|_{\infty,op} = n$ which is coherent since there is no threshold phenomenon for a single variable.

The proof of Theorem 2.19 is organized as follows: In Subsection 5.1, we explain how to deduce Theorem 2.19 from a sub-linear property of the influences (see Proposition 5.3) and from the results of [KMS12]. In Subsection 5.2 (respectively in Appendix A), we prove the sub-linearity property for monotonic (respectively semi-algebraic) subsets. The reason why we postpone the proof in the semi-algebraic case to the appendix is that the events that will interest us (namely, the events $\text{Cross}_p^\varepsilon(2R, R)$) are both monotonic and semi-algebraic (indeed, any threshold event is semi-algebraic). Hence it is enough for our purpose to prove Theorem 2.19 only for A monotonic or A semi-algebraic (and the monotonic case is easier).

Remember that we want to prove that $\mathbb{P}[\text{Cross}_p(2R, R)]$ is close to 1 and that our first step is to prove that it is the case for $\mathbb{P}[\text{Cross}_p^\varepsilon(2R, R)]$. To do so, we use that for $p = 0$ this probability is bounded from below (by Theorem 2.13) and we differentiate the probability with respect to p using Proposition 2.17. We then apply Theorem 2.19 to the right-hand side so that it is sufficient to prove that the maximum of the corresponding influences is small and that the operator norm for the infinite norm of the correlation matrix of our model is not too large. In the two following subsections, we state results in this spirit.

2.5.4 Polynomial decay of influences

Thanks to the RSW estimate and quasi-independence results, one can obtain that the probability of “discrete arm events” decay polynomially fast, see Subsection 5.3 of [BG16] and Proposition B.6 of [RV17]. Such a result together with monotonic arguments imply that we have a polynomial bound on the influences for crossing events. We state this bound here and we prove it in Subsection 4.2. We first need some notations. Let $\mathcal{V}_R^\varepsilon$ denote the set of vertices $x \in \mathcal{V}^\varepsilon$ that belong to an edge that intersects $[0, 2R] \times [0, R]$, and let X be our Gaussian field f restricted to $\mathcal{V}_R^\varepsilon$. Thus, X is a finite dimensional Gaussian field. For every $p \in \mathbb{R}$, $x \in \mathcal{V}_R^\varepsilon$ and $B \subseteq \mathbb{R}^{\mathcal{V}_R^\varepsilon}$, we use the notations ω_x^p and $\text{Piv}_x^p(B)$ from Subsubsection 2.5.2. In particular, for every $x \in \mathcal{V}_R^\varepsilon$ and every $p \in \mathbb{R}$ we write:

$$\omega_x^p = \mathbb{1}_{\{X_x \geq -p\}} = \mathbb{1}_{\{f(x) \geq -p\}}.$$

Also, we write $\text{Cross}^\varepsilon(2R, R)$ for the subset of $\{0, 1\}^{\mathcal{V}_R^\varepsilon}$ that corresponds to the crossing from left to right of $[0, 2R] \times [0, R]$. Note that $\text{Cross}_p^\varepsilon(2R, R) = \{\omega^p \in \text{Cross}^\varepsilon(2R, R)\}$. We have the following result (see Proposition 2.17 for its link with the influences):

Proposition 2.20. *Assume that f satisfies Conditions 2.1, 2.2, 2.4 as well as Condition 2.3 for some $\alpha > 4$. Then, there exist constants $\eta = \eta(\kappa) > 0$ and $C = C(\kappa) < +\infty$ such that, for every $\varepsilon \in]0, 1]$, every $R \in]0, +\infty[$, every $x \in \mathcal{V}_R^\varepsilon$ and every $p \in [-1, 1]$ we have:*

$$\mathbb{P} \left[\text{Piv}_x^p(\text{Cross}^\varepsilon(2R, R)) \mid f(x) = -p \right] \leq C R^{-\eta}.$$

Note that the constants in Proposition 2.20 do not depend on p (as long p belongs to a bounded interval, in our proof we have chosen $[-1, 1]$).

2.5.5 A bound on the infinite norm of the square root matrix

In order to use Theorem 2.19, it is important to control the quantity $\|\sqrt{\Sigma}\|_{\infty,op}$. We are interested in the case $\Sigma = K_R^\varepsilon :=$ the correlation matrix K restricted to $\mathcal{V}_R^\varepsilon$ (see Subsubsection 2.5.4 the notation $\mathcal{V}_R^\varepsilon$). However, we were not able to estimate $\|\sqrt{K_R^\varepsilon}\|_{\infty,op}$. Instead, we obtained estimates on $\|\sqrt{K^\varepsilon}\|_{\infty,op}$ where K^ε is K restricted to \mathcal{V}^ε , by using Fourier techniques (which work only in a translation invariant setting, hence not for K_R^ε). Here, K^ε is an infinite matrix, so let us be more precise about what we mean by square root of K^ε : To define the product of two infinite matrices, we ask the infinite sums that arise to be absolutely convergent. A square root of K^ε is simply an infinite matrix $\sqrt{K^\varepsilon}$ such that in this sense we have $\sqrt{K^\varepsilon}^2 = K^\varepsilon$. It may be that classical spectral theory arguments imply that such a square root exists. However, we will not need such arguments since we will have to construct quite explicitly $\sqrt{K^\varepsilon}$ in order to estimate its infinite operator norm (where as in the finite dimensional case we let $\|\sqrt{K^\varepsilon}\|_{\infty,op} := \max_i \sum_j |\sqrt{K^\varepsilon}(i,j)|$, one difference being that this sum might be infinite). Let us also note that the matrices $\sqrt{K^\varepsilon}$ that we will construct will be positive definite in the sense that their restriction to finite sets of indices is positive definite. By spectral theory arguments it may be that only one such matrix exists. However, we do not need such a result either.

Note that, since Theorem 2.19 is stated for finite dimensional Gaussian fields, it would have been better to have an estimate on $\|\sqrt{K_R^\varepsilon}\|_{\infty,op}$. See Lemma 3.1 for more details on this issue.

We will prove the following in Section 6. The reason why our main result Theorem 1.3 is only proved for the Bargmann-Fock model is that we were unable to find a general estimate on $\|\sqrt{K^\varepsilon}\|_{\infty,op}$ with explicit dependence on ε as $\varepsilon \downarrow 0$ (this is the only place where we prove a result only for the Bargmann-Fock field). If one could prove a bound by $C\varepsilon^{-2+\delta}$ for some $\delta > 0$ and $C < +\infty$ for another process (which also satisfies some of the conditions of Subsection 2.2), the rest of the proof of Theorem 1.3 would piece together and yield the same result for this process. See Proposition 3.5.

Proposition 2.21. *Assume that f satisfies Condition 2.1 and Condition 2.5 for some $\alpha > 5$. Then, for all $\varepsilon > 0$, there exists a symmetric square root $\sqrt{K^\varepsilon}$ of K^ε such that:*

$$\|\sqrt{K^\varepsilon}\|_{\infty,op} < +\infty.$$

Assume further that $\kappa(x) = \exp(-\frac{1}{2}|x|^2)$ (which is the covariance kernel function of the Bargmann-Fock field). Then, there exist $\varepsilon_0 > 0$ and $C < +\infty$ such that, for all $\varepsilon \in]0, \varepsilon_0]$, there exists a symmetric square root $\sqrt{K^\varepsilon}$ of K^ε such that:

$$\|\sqrt{K^\varepsilon}\|_{\infty,op} \leq C \frac{1}{\varepsilon} \log\left(\frac{1}{\varepsilon}\right).$$

2.5.6 Combining all the above results

The following Equation (2.7) is not used anywhere in the paper though it appears implicitly in Lemma 3.1. However, since it sums up the crux of the proof, we state it here as a guide to the reader. Its proof could easily be extracted from the arguments below (see Remark 3.4).

As we will explain in Section 3.1, by combining the results of Subsections 2.5.2, 2.5.3, 2.5.4 and 2.5.5, we can obtain the following: Consider the Bargmann-Fock field and let $p \in]0, 1]$.

There exists $C < +\infty$ such that, if $\varepsilon \in]0, 1]$ and $R \in]0, +\infty[$, then:

$$\mathbb{P} [\text{Cross}_p^\varepsilon(2R, R)] \geq 1 - C_1 \exp \left(-p c_2 \frac{\varepsilon}{\log(\frac{1}{\varepsilon})} \sqrt{\log_+ \left(C_1 R^{-\eta} \frac{\log(\frac{1}{\varepsilon})}{\varepsilon} \right)} \right), \quad (2.7)$$

where $C_1 < +\infty$ and $c_2 > 0$ are some absolute constants. One can deduce from (2.7) that, if $\varepsilon = \varepsilon(R) \geq \log^{-(1/2-\delta)}(R)$, then $\mathbb{P} [\text{Cross}_p^\varepsilon(2R, R)]$ goes to 1 as R goes to $+\infty$ (sufficiently fast so that $\sum_k \mathbb{P} [\text{Cross}_p^{\varepsilon(2^k)}(2^{k+1}, 2^k)] < +\infty$ as required, see (2.3)).

We will also obtain results analogous to Equation (2.7) for more general fields and with $\|\sqrt{K^\varepsilon}\|_{\infty, op}$ instead of $\frac{1}{\varepsilon} \log(\frac{1}{\varepsilon})$, which will enable us to prove (2.6) and thus obtain Theorem 2.11.

2.6 From discrete to continuous

It is helpful to keep in mind in this subsection that, as explained above, in the case of the Bargmann-Fock field, the quantity $\mathbb{P} [\text{Cross}_p^\varepsilon(2R, R)]$ goes to 1 as $R \rightarrow +\infty$ if $\varepsilon = \varepsilon(R) \geq \log^{-(1/2-\delta)}(R)$. In order to obtain an analogous result for the continuum model, we need to measure the extent to which $\mathbb{P} [\text{Cross}_p^\varepsilon(2R, R)]$ approximates $\mathbb{P} [\text{Cross}_p(2R, R)]$. To this purpose, it seems natural to use the approximation estimates based on the discretization schemes of [BG16] which have been generalized in [BM17]. More precisely, we think that the following can easily be extracted from [BM17] (however, we do not write a formal proof since we will not need such a result):¹¹ Assume that f satisfies Condition 2.1 and that κ is C^6 . Then, for every $p \in \mathbb{R}$ and $\delta > 0$, there exists $C = C(\kappa, p, \delta) < +\infty$ such that for each $\varepsilon \in]0, 1]$ and each $R \in \varepsilon\mathbb{Z}_{>0}$ we have:

$$\mathbb{P} [\text{Cross}_p^\varepsilon(2R, R) \Delta \text{Cross}_p(2R, R)] \leq C \varepsilon^{2-\delta} R^2. \quad (2.8)$$

This would imply in particular that, for every sequence $(\varepsilon(R))_{R>0}$ such that: (i) for each $R > 0$, $R \in \varepsilon(R)\mathbb{Z}_{>0}$ and: (ii) $\varepsilon(R) = O(R^{-(1+\delta)})$ for some $\delta > 0$, we have:

$$\mathbb{P} [\text{Cross}_p^{\varepsilon(R)}(2R, R) \Delta \text{Cross}_p(2R, R)] \xrightarrow{R \rightarrow +\infty} 0.$$

Unfortunately, the constraint $\varepsilon \leq R^{-(1+\delta)}$ is incompatible with $\varepsilon \geq \log^{-(1/2-\delta)}(R)$. To solve this quandary, we will replace the discretization result with a sprinkling argument.

Let $p > 0$. The idea is to compare the probability of a continuum crossing at parameter p to the probability of a discrete crossing at parameter $p/2$. Instead of using the discretization results of [BG16, BM17], we will use the following result, which we prove in Subsection 7. For references about the strategy of sprinkling in the context of Bernoulli percolation, see the notes on Section 2.6 of [Gri99].

Proposition 2.22. *Assume that f satisfies Conditions 2.1 and that it is a.s. C^2 . Let $p > 0$. Then, there exist constants $C_1 < +\infty$ and $c_2 = c_2(\kappa, p) > 0$ as well as $\varepsilon_0 = \varepsilon_0(\kappa, p) > 0$ such that for each $\varepsilon \in]0, \varepsilon_0]$ and $R > 1$:*

$$\mathbb{P} [\text{Cross}_{p/2}^\varepsilon(2R, R) \setminus \text{Cross}_p(2R, R)] \leq C_1 R^2 \varepsilon^{-2} \exp(-c_2 \varepsilon^{-4}).$$

¹¹Let us be a little more precise here: one can use for instance Propositions 6.1 and 6.3 of [BM17] and (more classical) results about the regularity of \mathcal{D}_p as for instance Lemma A.10 of [RV17].

In particular, for every sequence $(\varepsilon(R))_{R>0}$ such that $\varepsilon(R) \geq \log^{-(1/4+\delta)}(R)$ for some $\delta > 0$, we have:

$$\mathbb{P} \left[\text{Cross}_{p/2}^\varepsilon(2R, R) \setminus \text{Cross}_p(2R, R) \right] \xrightarrow{R \rightarrow +\infty} 0.$$

Remark 2.23. While this estimate may seem stronger than (2.8), we wish to emphasize that it only shows that discretization is legitimate in one direction and also involves a change in threshold from $-p/2$ to $-p$.

If we choose for instance $\varepsilon = \varepsilon(R) = \log^{-1/3}(R)$, then:

$$\log^{-(1/2-\delta)}(R) \ll \varepsilon(R) \ll \log^{-(1/4+\delta)}(R)$$

for any $\delta > 0$ small enough. Hence, with this choice of ε and for the Bargmann-Fock field, on the one hand for any $p > 0$ we have a lower bound on $\mathbb{P} \left[\text{Cross}_{p/2}^\varepsilon(2R, R) \right]$ which is close to 1 (see Subsubsection 2.5.6), and on the other hand we can use Proposition 2.22 to go back to the continuum and obtain a lower bound on $\mathbb{P} [\text{Cross}_p(2R, R)]$ which is close to 1. More precisely, we will see in Section 3.1 that we can obtain:

$$\mathbb{P} [\text{Cross}_p(2R, R)] \geq 1 - \exp \left(-c \log^{1/7}(R) \right),$$

for some $c = c(p) > 0$, which implies that the box-crossing criterion of Lemma 2.9 is satisfied, and ends the proof of our main result Theorem 1.3.

3 Proofs of the phase transition theorems and of the exponential decay

3.1 Proof of the phase transition theorems

In this subsection, we combine the results stated in Section 2 in order to prove Theorem 1.3. We also prove Theorem 2.11. Most of the proof of the former carries over to that of the latter. The first half of the proof is the following lemma, in which we work with the following function:

$$g_R^\varepsilon : p \mapsto \log \left(\frac{\mathbb{P} [\text{Cross}_p^\varepsilon(2R, R)]}{1 - \mathbb{P} [\text{Cross}_p^\varepsilon(2R, R)]} \right). \quad (3.1)$$

Lemma 3.1. *There exists an absolute constant $c > 0$ such that the following holds: Assume that f satisfies Condition 2.1, and fix $\varepsilon > 0$ and $R > 0$. Moreover, let μ_R^ε be the law of f restricted to $\mathcal{V}_R^\varepsilon$ (defined in Subsubsection 2.5.4) and let K^ε be K restricted to \mathcal{V}^ε . Assume that there exists a symmetric square root $\sqrt{K^\varepsilon}$ of K^ε . Then:*

$$\forall p \in [0, 1], \quad \frac{d}{dp} g_R^\varepsilon(p) \geq c \|\sqrt{K^\varepsilon}\|_{\infty, op}^{-1} \sqrt{\log_+ \left(\frac{1}{\|\sqrt{K^\varepsilon}\|_{\infty, op} \cdot \max_{x \in \mathcal{V}_R^\varepsilon} I_{x, \mu_R^\varepsilon}(\text{Cross}_p^\varepsilon(2R, R))} \right)}. \quad (3.2)$$

Here, square root and infinite norm of an infinite matrix have the same meaning as in Subsubsection 2.5.5.

This lemma is mostly a consequence of Theorem 5.1. The only difficulty comes from the fact that, while Theorem 2.19 deals with finite dimensional Gaussian fields, the correlation matrix

on the right hand side of Equation (3.2) is the infinite matrix K^ε (which we need to apply Proposition 2.21 later in this section). To overcome such issues, we proceed by approximation and, instead of dealing with a μ_R^ε variable directly, we apply Theorem 2.19 to a Gaussian vector defined using $\sqrt{K^\varepsilon}$ (namely, the Gaussian vector $Y^{\varepsilon,\rho}$ below).

Proof of Lemma 3.1. Fix $\varepsilon > 0$ and $R > 0$. For each $\rho > 0$, let $H^{\varepsilon,\rho}$ be $\sqrt{K^\varepsilon}$ restricted to $\mathcal{V}^\varepsilon \cap [-\rho, \rho]^2$, let $K^{\varepsilon,\rho} = (H^{\varepsilon,\rho})^2$, and let $Y^{\varepsilon,\rho} \sim \mathcal{N}(0, K^{\varepsilon,\rho}) =: \mu^{\varepsilon,\rho}$. A simple computation shows that since K^ε is non-degenerate,¹² so is its square root. Restriction preserves non-degeneracy so $Y^{\varepsilon,\rho}$ is indeed non-degenerate and we can apply Theorem 2.19 to $\mu^{\varepsilon,\rho}$ and the (increasing) event $\text{Cross}_p^\varepsilon(2R, R)$. We obtain that, if ρ is sufficiently large:

$$\sum_{i \in \mathcal{V}_R^\varepsilon} I_{i, \mu^{\varepsilon,\rho}}(\text{Cross}_p^\varepsilon(2R, R)) \geq c \|H^{\varepsilon,\rho}\|_{\infty, op}^{-1} \mu^{\varepsilon,\rho}(\text{Cross}_p^\varepsilon(2R, R)) (1 - \mu^{\varepsilon,\rho}(\text{Cross}_p^\varepsilon(2R, R))) \times \sqrt{\log_+ \left(\frac{1}{\|H^{\varepsilon,\rho}\|_{\infty, op} \max_{i \in \mathcal{V}_R^\varepsilon} I_{i, \mu^{\varepsilon,\rho}}(\text{Cross}_p^\varepsilon(2R, R))} \right)}. \quad (3.3)$$

Here we use the fact that, since $\text{Cross}_p^\varepsilon(2R, R)$ depends only on the sites of $\mathcal{V}_R^\varepsilon$, the influences on this event of all the sites outside of this box vanish. Now, observe that:

$$\frac{d}{dp} g_R^\varepsilon(p) = \frac{d}{dp} \mathbb{P}[\text{Cross}_p^\varepsilon(2R, R)] \cdot \frac{1}{\mathbb{P}[\text{Cross}_p^\varepsilon(2R, R)] (1 - \mathbb{P}[\text{Cross}_p^\varepsilon(2R, R)])}.$$

By Proposition 2.17,

$$\frac{d}{dp} g_R^\varepsilon(p) = \frac{\exp(-p^2/2)}{\sqrt{2\pi} \mathbb{P}[\text{Cross}_p^\varepsilon(2R, R)] (1 - \mathbb{P}[\text{Cross}_p^\varepsilon(2R, R)])} \sum_{i \in \mathcal{V}_R^\varepsilon} I_{i, \mu_R^\varepsilon}(\text{Cross}_p^\varepsilon(2R, R)). \quad (3.4)$$

Also, by definition of $\|\cdot\|_\infty$, for every ρ , we have:

$$\|H^{\varepsilon,\rho}\|_{\infty, op} \leq \|\sqrt{K^\varepsilon}\|_{\infty, op}. \quad (3.5)$$

In view of Equations (3.3), (3.4) and (3.5), we are done as long as we prove that:

1. $\mu^{\varepsilon,\rho}(\text{Cross}_p^\varepsilon(2R, R)) \xrightarrow{\rho \rightarrow +\infty} \mu_R^\varepsilon(\text{Cross}_p^\varepsilon(2R, R)) (= \mathbb{P}[\text{Cross}_p^\varepsilon(2R, R)])$ and:
2. $\forall i \in \mathcal{V}_R^\varepsilon, I_{i, \mu^{\varepsilon,\rho}}(\text{Cross}_p^\varepsilon(2R, R)) \xrightarrow{\rho \rightarrow +\infty} I_{i, \mu_R^\varepsilon}(\text{Cross}_p^\varepsilon(2R, R))$.

To this purpose, we need the following elementary lemma:

Lemma 3.2. *Let $(Y_\rho)_{\rho>0}$ be a sequence of non-degenerate Gaussian vectors in \mathbb{R}^n with covariance Σ_ρ and mean m_ρ , assume that Σ_ρ converges to some invertible matrix Σ as ρ goes to $+\infty$, and that m_ρ converges to some $m \in \mathbb{R}^n$ as ρ goes to $+\infty$. Let $Y \sim \mathcal{N}(m, \Sigma)$. Then, for any Borel subset $A \subset \mathbb{R}^n$, any $i \in \{1, \dots, n\}$ and any $q \in \mathbb{R}$, we have:*

$$\mathbb{P}[Y_\rho \in A] \xrightarrow{\rho \rightarrow +\infty} \mathbb{P}[Y \in A],$$

$$\mathbb{P}[Y_\rho \in A \mid Y_\rho(i) = q] \xrightarrow{\rho \rightarrow +\infty} \mathbb{P}[Y \in A \mid Y(i) = q].$$

¹²Meaning that for any non-zero finitely supported $v \in \mathbb{R}^{\mathcal{V}^\varepsilon}$, $K^\varepsilon v \neq 0$.

Proof. For the first statement, just notice that the density function of a non-degenerate Gaussian vector is a continuous function of its covariance and of its mean, and apply dominated convergence. By Proposition 1.2 of [AW09], the law of Y_ρ conditioned on the value of $Y_\rho(i)$ is that of a Gaussian vector whose mean and covariance depend continuously on Σ_ρ and m_ρ . Note also that the law $(Y_\rho(j))_{j \neq i}$ when we condition on $\{Y_\rho(i) = q\}$ (respectively the law $(Y(j))_{j \neq i}$ when we condition on $\{Y(i) = q\}$) is still non-degenerate.¹³ We conclude by applying the first statement. \square

Item 1 above is a direct consequence of the first part of Lemma 3.2. Regarding Item 2, note that, since $\text{Cross}_p^\varepsilon(2R, R)$ is an increasing threshold event, by Proposition 2.17, we have the following equalities:

$$\begin{aligned} I_{i, \mu^{\varepsilon, \rho}}(\text{Cross}_p^\varepsilon(2R, R)) \\ = \mathbb{P} \left[Y^{\varepsilon, \rho} \in \text{Piv}_i(\text{Cross}_p^\varepsilon(2R, R)) \mid Y^{\varepsilon, \rho}(i) = -p \right] \frac{1}{\sqrt{2\pi K^{\varepsilon, \rho}(i, i)}} e^{-p^2/(2K^{\varepsilon, \rho}(i, i))}, \end{aligned}$$

and:

$$\begin{aligned} I_{i, \mu_R^\varepsilon}(\text{Cross}_p^\varepsilon(2R, R)) \\ = \mathbb{P} \left[Y_R^\varepsilon \in \text{Piv}_i(\text{Cross}_p^\varepsilon(2R, R)) \mid Y_R^\varepsilon(i) = -p \right] \frac{1}{\sqrt{2\pi K_{\rho, R}^\varepsilon(i, i)}} e^{-p^2/(2K_{\rho, R}^\varepsilon(i, i))}, \end{aligned}$$

where $Y_R^\varepsilon \sim \mathcal{N}(0, K_R^\varepsilon)$. We conclude by applying the second part of Lemma 3.2. \square

We are now ready to prove our main result.

Proof of Theorem 1.3. We consider the Bargmann-Fock field f . By Theorem 1.1, for every $p \leq 0$, a.s. there is no unbounded connected component in \mathcal{D}_p . We henceforth consider some parameter $p_0 \in]0, 1]$. By Lemma 2.9, it is enough to prove that f satisfies criterion (2.2) (note that if we prove that this criterion is satisfied for every $p_0 \in]0, 1]$ then we obtain the result for every p_0 since the quantities $\mathbb{P}[-\text{Cross}_{p_0}(2R, R)]$ are non-increasing in p_0). To this end we first fix $R_0 < +\infty$ to be determined later, we let $R > R_0$, and we use the discretization procedure introduced in Subsection 2.4 with:

$$\varepsilon = \varepsilon(R) = \log^{-1/3}(R).$$

Let $g_R := g_R^{\varepsilon(R)}$ be as in (3.1). We are going to apply Lemma 3.1. Let us estimate the quantities that appear in this lemma. By Proposition 2.21, there exists a constant $C < +\infty$ independent of R such that:

$$\|\sqrt{K^{\varepsilon(R)}}\|_{\infty, op} \leq C \frac{1}{\varepsilon(R)} \log \left(\frac{1}{\varepsilon(R)} \right).$$

Moreover by Propositions 2.17 and 2.20, if R_0 is large enough, there exists $\eta > 0$ independent of $R > R_0$ and of $p \in [-1, 1]$ such that:

$$\forall x \in \mathcal{V}_R^{\varepsilon(R)}, I_{x, \mu_R^{\varepsilon(R)}}(\text{Cross}_p^{\varepsilon(R)}(2R, R)) \leq R^{-\eta}. \quad (3.6)$$

¹³To see this, complete the vector e_i into an orthogonal basis for Σ_ρ (respectively Σ) and express Y_ρ (respectively Y) in this basis.

In particular, $\|\sqrt{K^{\varepsilon(R)}}\|_{\infty,op} \cdot \max_{x \in \mathcal{V}_R^{\varepsilon(R)}} I_{x, \mu_R^{\varepsilon(R)}} \left(\text{Cross}_p^{\varepsilon(R)}(2R, R) \right) < 1$ if R_0 is large enough, and by applying Lemma 3.1 we obtain:

$$\begin{aligned} \frac{d}{dp} g_R(p) &\geq \frac{c}{C} \varepsilon(R) \log^{-1} \left(\frac{1}{\varepsilon(R)} \right) \sqrt{\eta \log(R) - \log \left(\frac{1}{C \varepsilon(R)} \right) - \log \left(\log \left(\frac{1}{\varepsilon(R)} \right) \right)} \\ &\geq \frac{c}{3C} \log^{-1/3}(R) \log^{-1}(\log(R)) \sqrt{\eta \log(R) - \frac{1}{3} \log \left(\frac{\log(R)}{C} \right) - \log(3 \log(\log(R)))} \\ &\geq \frac{c}{3C} \sqrt{\eta/2} \log^{1/6}(R) \log^{-1}(\log(R)) \\ &\geq \frac{c}{C} \sqrt{\eta/2} \log^{1/7}(R). \end{aligned}$$

By integrating from 0 to $p_0/2$, we get:

$$\begin{aligned} \log \left(\frac{1}{1 - \mathbb{P} \left[\text{Cross}_{p_0/2}^{\varepsilon(R)}(2R, R) \right]} \right) &\geq \log \left(\frac{\mathbb{P} \left[\text{Cross}_{p_0/2}^{\varepsilon(R)}(2R, R) \right]}{1 - \mathbb{P} \left[\text{Cross}_{p_0/2}^{\varepsilon(R)}(2R, R) \right]} \right) \\ &\geq \frac{p_0}{2} \frac{c}{C} \sqrt{\eta/2} \log^{1/7}(R) + \log \left(\frac{\mathbb{P} \left[\text{Cross}_0^{\varepsilon(R)}(2R, R) \right]}{1 - \mathbb{P} \left[\text{Cross}_0^{\varepsilon(R)}(2R, R) \right]} \right). \end{aligned}$$

By Theorem 2.13, if R_0 is large enough, there exists $C' < +\infty$ independent of $R > R_0$ such that:

$$\log \left(\frac{\mathbb{P} \left[\text{Cross}_0^{\varepsilon(R)}(2R, R) \right]}{1 - \mathbb{P} \left[\text{Cross}_0^{\varepsilon(R)}(2R, R) \right]} \right) \geq -C'.$$

Therefore:

$$\mathbb{P} \left[\text{Cross}_{p_0/2}^{\varepsilon(R)}(2R, R) \right] \geq 1 - \exp \left(-\frac{c}{2C} \sqrt{\eta/2} p_0 \log^{1/7}(R) + C' \right). \quad (3.7)$$

Now:

$$\mathbb{P} \left[\text{Cross}_{p_0}(2R, R) \right] \geq \mathbb{P} \left[\text{Cross}_{p_0/2}^{\varepsilon(R)}(2R, R) \right] - \mathbb{P} \left[\text{Cross}_{p_0/2}^{\varepsilon(R)}(2R, R) \setminus [\text{Cross}_{p_0}(2R, R)] \right],$$

and by Proposition 2.22, if R_0 is large enough, there exist constants $c_1 > 0$ and $C_2 = C_2(p_0) < +\infty$ independent of $R > R_0$ such that:

$$\begin{aligned} \mathbb{P} \left[\text{Cross}_{p_0/2}^{\varepsilon(R)}(2R, R) \setminus [\text{Cross}_{p_0}(2R, R)] \right] &\leq C_2 R^2 \varepsilon(R)^{-2} \exp(-c_1 \varepsilon(R)^{-4}) \\ &\leq C_2 \exp \left(2 \log(R) + (2/3) \log(\log(R)) - c_1 \log^{4/3}(R) \right) \\ &\leq C_2 \exp \left(-\frac{c_1}{2} \log^{4/3}(R) \right). \end{aligned}$$

Combining this estimate with Equation (3.7) we get:

$$\mathbb{P} \left[\text{Cross}_{p_0}(2R, R) \right] \geq 1 - \exp \left(-\frac{c}{2C} \sqrt{\eta/2} p_0 \log^{1/7}(R) + C' \right) - C_2 \exp \left(-\frac{c_1}{2} \log^{4/3}(R) \right).$$

All in all, for a large enough choice of R_0 , there exists $c_3 = c_3(p_0) > 0$ such that for each $R > R_0$,

$$\mathbb{P} \left[\text{Cross}_{p_0}(2R, R) \right] \geq 1 - \exp \left(-c_3 \log^{1/7}(R) \right).$$

Hence, criterion 2.2 is satisfied and we are done. \square

Remark 3.3. Note that this proof also implies that:

$$\forall p > 0, \lim_{R \rightarrow +\infty} \text{Cross}_p(2R, R) = 1, \quad (3.8)$$

which will be useful for the proof of Theorem 1.7.

Remark 3.4. Note that if we follow the above proof without taking $\varepsilon = \log^{-1/3}(R)$ but by taking any $\varepsilon \in]0, 1]$ and R larger than some constant that does not depend on ε , we obtain (2.7).

The only result that we have used in the proof of Theorem 1.3 and that we have managed to prove only for the Bargmann-Fock field is the $\frac{1}{\varepsilon} \log\left(\frac{1}{\varepsilon}\right)$ estimate on the infinite operator norm of $\sqrt{K^\varepsilon}$. We have the following more general result:

Proposition 3.5. *Assume that f satisfies Conditions 2.1, 2.2, 2.4 as well as Condition 2.3 for some $\alpha > 4$. Assume also that there exist $\delta > 0$ and $C < +\infty$ such that for every ε sufficiently small K^ε admits a symmetric square root and:*

$$\|\sqrt{K^\varepsilon}\|_{\infty, op} \leq C \frac{1}{\varepsilon^{2-\delta}}. \quad (3.9)$$

Then the critical threshold for the continuous percolation model induced by f is 0. More precisely, the probability that \mathcal{D}_p has an unbounded connected component is 1 if $p > 0$, and 0 if $p \leq 0$. Moreover, in the case where $p > 0$, if it exists, such a component is a.s. unique.

Remark 3.6. A way to prove that (3.9) is to use our results of Section 6 (where we assume that f satisfies Condition 2.1 and Condition 2.5 for some $\alpha > 5$): Define the quantity $\Upsilon(\varepsilon)$ as in Lemma 6.3. If one proves that there exists $\delta > 0$ and $C_1 < +\infty$ such that, for every ε sufficiently small, we have $\Upsilon(\varepsilon) \leq C_1 \varepsilon^{-1+\delta}$, then one obtains that there exists $C_2 < +\infty$ such that, for every ε sufficiently small:

$$\|\sqrt{K^\varepsilon}\|_{\infty, op} \leq C_2 \frac{1}{\varepsilon^{2-\delta}} \log\left(\frac{1}{\varepsilon}\right).$$

Proof of Proposition 3.5. The proof is almost identical to that of Theorem 1.3 so we will not detail elementary computations. First fix some $h > 0$ such that:

$$(-1/4 - h)(2 - \delta) + 1/2 > 0. \quad (3.10)$$

By Theorem 1.1, for every $p \leq 0$, a.s. there is no unbounded connected component in \mathcal{D}_p . We henceforth consider some parameter $p_0 \in]0, 1]$. Fix $R_0 < +\infty$ to be determined later, and let:

$$\varepsilon = \varepsilon(R) = \log^{-1/4-h}(R).$$

By Propositions 2.17 and 2.20 if R_0 is large enough, there exists $\eta > 0$ independent of $R > R_0$ and of $p \in [-1, 1]$ such that:

$$\forall x \in \mathcal{V}_R^{\varepsilon(R)}, I_{x, \mu_R^{\varepsilon(R)}}\left(\text{Cross}_p^{\varepsilon(R)}(2R, R)\right) \leq R^{-\eta}.$$

Hence, by Lemma 3.1, if R_0 is large enough:

$$\frac{d}{dp} g_R(p) \geq \frac{c}{C} \log^{(-1/4-h)(2-\delta)+1/2}(R) \sqrt{\eta/2} \log^{1/2}(R).$$

By Theorem 2.13, if R_0 is large enough, there exists $C' < +\infty$ independent of $R > R_0$ such that:

$$\log \left(\frac{\mathbb{P} [\text{Cross}_0^{\varepsilon(R)}(2R, R)]}{1 - \mathbb{P} [\text{Cross}_0^{\varepsilon(R)}(2R, R)]} \right) \geq -C' .$$

Integrating from 0 to $p_0/2$ we get:

$$\mathbb{P} [\text{Cross}_{p_0/2}^{\varepsilon(R)}(2R, R)] \geq 1 - \exp \left(-\frac{c}{2C} \frac{p_0}{2} \log^{(-1/4-h)(2-\delta)+1/2}(R) \sqrt{\eta/2} + C' \right) .$$

Now, if we apply Proposition 2.22, we obtain that, if R_0 is large enough, there exist constants $c_1 = c_1(\kappa) > 0$ and $C_2 = C_2(\kappa, p_0) < +\infty$ independent of $R > R_0$ such that:

$$\begin{aligned} & \mathbb{P} [\text{Cross}_{p_0}(2R, R)] \\ & \geq \mathbb{P} [\text{Cross}_{p_0/2}^{\varepsilon(R)}(2R, R)] - C_2 \exp \left(-c_1 \log^{1+4h}(R) \right) \\ & \geq 1 - \exp \left(-\frac{c}{2C} p_0 \log^{(-1/4-h)(2-\delta)+1/2}(R) \sqrt{\eta/2} + C' \right) - C_2 \exp \left(-c_1 \log^{1+4h}(R) \right) . \end{aligned}$$

Since (by (3.10)) we have $(-1/4 - h)(2 - \delta) + 1/2 > 0$, the criterion of Lemma 2.9 is satisfied and we obtain that a.s. \mathcal{D}_{p_0} has a unique unbounded component. \square

The following proof is also almost identical to that of Theorem 1.3 except that we do not have to use discretization estimates to conclude.

Proof of Theorem 2.11. Fix $\varepsilon \in]0, 1]$. By Theorem 2.12, for every $p \leq 0$, a.s., there is no unbounded black component. From now on, we fix some $p_0 \in]0, 1]$. By Lemma 2.14, it is enough to prove that f satisfies criterion (2.6). To this end we first fix $R_0 > 0$ to be determined later and we let $R > R_0$. By Proposition 2.21, there exists $C = C(\kappa, \varepsilon) < +\infty$ such that:

$$\|\sqrt{K^\varepsilon}\|_{\infty, op} \leq C .$$

Moreover by Propositions 2.17 and 2.20 if R_0 is large enough, there exists $\eta > 0$ independent of $R > R_0$ and $p \in [-1, 1]$ such that:

$$\forall x \in \mathcal{V}_R^\varepsilon, I_{x, \mu_R^\varepsilon} (\text{Cross}_p^\varepsilon(2R, R)) \leq R^{-\eta} .$$

Hence, by Lemma 3.1, if R_0 is large enough:

$$\frac{d}{dp} g_R(p) \geq \frac{c}{C} \sqrt{\frac{\eta}{2} \log(R)} .$$

By Theorem 2.13, if R_0 is large enough, there exists $C' < +\infty$ independent of $R > R_0$ such that:

$$\log \left(\frac{\mathbb{P} [\text{Cross}_0^\varepsilon(2R, R)]}{1 - \mathbb{P} [\text{Cross}_0^\varepsilon(2R, R)]} \right) \geq -C' .$$

Integrating from 0 to p_0 we get:

$$\mathbb{P} [\text{Cross}_{p_0}^\varepsilon(2R, R)] \geq 1 - \exp \left(-\frac{c}{C} p_0 \sqrt{\frac{\eta}{2} \log(R)} + C' \right) .$$

Hence, f satisfies criterion 2.6 and we are done. \square

3.2 Proof of the exponential decay

In this subsection, we prove Theorem 1.7 by following classical arguments of planar percolation that involve a recursion on crossing probabilities. In order to start the induction, we use the following property: According to Equation (3.8), if f is the Bargmann-Fock field, then:

$$\forall p > 0, \mathbb{P}[\text{Cross}_p(2R, R)] \xrightarrow{R \rightarrow +\infty} 1. \quad (3.11)$$

We can easily deduce from (3.11) that the same result holds with the factor 2 replaced by any $\rho > 0$ (indeed, for every $\rho_1 > 1$ and $\rho_2 > 0$, there exists $N = N(\rho_1, \rho_2) \in \mathbb{Z}_{>0}$ such that, for every $R > 0$, there exist N $\rho_1 R \times R$ rectangles such that if these rectangles are crossed lengthwise then $[0, \rho_2 R] \times [0, R]$ is also crossed lengthwise):

$$\forall \rho > 0, \forall p > 0, \mathbb{P}[\text{Cross}_p(\rho R, R)] \xrightarrow{R \rightarrow +\infty} 1. \quad (3.12)$$

Proof of Theorem 1.7. Let f be the Bargmann-Fock field. First observe that, by the argument above, it is enough to prove that, for every $p > 0$, there exists $c = c(p) > 0$ such that for every $R > 0$ we have:

$$\mathbb{P}[\text{Cross}_p(3R, R)] \geq 1 - \exp(-cR). \quad (3.13)$$

In order to prove (3.13), we follow classical ideas in planar percolation theory, see for instance the second version of the proof of Theorem 10 in [BR06b] or the proof of Proposition 3.1 in [ATT16]. Write¹⁴ $a_R = 1 - \mathbb{P}[\text{Cross}_p(3R, R)] + \exp(-R)$. We will show that there exists $R_1 = R_1(p) < +\infty$ such that for any $R \geq R_1$:

$$a_{3R} \leq 49a_R^2. \quad (3.14)$$

By Equation (3.12), $\lim_{R \rightarrow +\infty} a_R = 0$. This observation combined with Equation (3.14) yields (3.13) by an elementary induction argument.

In order to prove Equation (3.14) we first introduce two events, see Figure 5. First, the event $\text{FiveCross}_p(R)$ is the event that:

- the $3R \times R$ rectangles $Q_i = [(2i - 2)R, (2i + 1)R] \times [0, R]$ for $i = 1, 2, 3, 4$ are crossed from left to right by a continuous path in \mathcal{D}_p ;
- the $R \times R$ squares $Q_{4+j} = [2jR, (2j + 1)R] \times [0, R]$ for $j = 1, 2, 3$ are crossed from top to bottom by a continuous path in \mathcal{D}_p .

Second, the event $\text{FiveCross}'_p(R)$ is the event $\text{FiveCross}_p(R)$ translated by $(0, 2R)$. Also, we write Q'_i for the rectangle obtained by translating Q_i by $(0, 2R)$. Note that $\text{FiveCross}_p(R) \cup \text{FiveCross}'_p(R) \subseteq \text{Cross}_p(9R, 3R)$. Hence:

$$\mathbb{P}[\text{Cross}_p(9R, 3R)] \geq 1 - \mathbb{P}[\neg \text{FiveCross}_p(R) \cap \neg \text{FiveCross}'_p(R)]. \quad (3.15)$$

We claim that the events $\text{FiveCross}_p(R)$ and $\text{FiveCross}'_p(R)$ are asymptotically independent. More precisely, the following is a direct consequence of Theorem 1.12 from [RV17].

¹⁴Note that if we were working with Bernoulli edge-percolation on \mathbb{Z}^2 , say, the first two terms in the following definition would be enough. However, dependencies create small errors when estimating the probability term. Adding the last term, $\exp(-R)$, allows us to preserve the inductive inequality (3.14) while taking into account these errors.

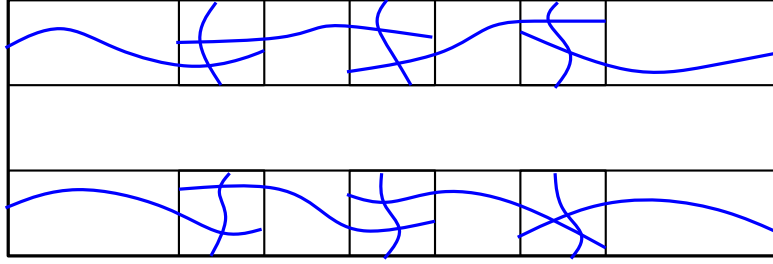


Figure 5: The events $\text{FiveCross}_p(R)$ and $\text{FiveCross}'_p(R)$.

Claim 3.7. *Let $p \in \mathbb{R}$. There exists $R_2 = R_2(p) < +\infty$ such that, for every $R \geq R_2$:*

$$|\mathbb{P}[\neg \text{FiveCross}_p(R) \cap \neg \text{FiveCross}'_p(R)] - \mathbb{P}[\neg \text{FiveCross}_p(R)] \mathbb{P}[\neg \text{FiveCross}'_p(R)]| \leq e^{-\frac{R^2}{4}}.$$

Let $R \geq R_2$. Applying Claim 3.7 in Equation (3.15), we get:

$$\begin{aligned} \mathbb{P}[\text{Cross}_p(9R, 3R)] &\geq 1 - \mathbb{P}[\neg \text{FiveCross}_p(R)] \cdot \mathbb{P}[\neg \text{FiveCross}'_p(R)] - e^{-\frac{R^2}{4}} \\ &= 1 - \mathbb{P}[\neg \text{FiveCross}_p(R)]^2 - e^{-\frac{R^2}{4}} \text{ by stationarity.} \end{aligned}$$

By a union bound we have:

$$\mathbb{P}[\neg \text{FiveCross}_p(R)] \leq 4(1 - \mathbb{P}[\text{Cross}_p(3R, R)]) + 3(1 - \mathbb{P}[\text{Cross}_p(R, R)]) \leq 7(1 - \mathbb{P}[\text{Cross}_p(3R, R)]).$$

Thus:

$$1 - \mathbb{P}[\text{Cross}_p(9R, 3R)] \leq (7(1 - \mathbb{P}[\text{Cross}_p(3R, R)]))^2 + e^{-\frac{R^2}{4}},$$

and:

$$\begin{aligned} a_{3R} &\leq (7(1 - \mathbb{P}[\text{Cross}_p(3R, R)]))^2 + e^{-\frac{R^2}{4}} + \exp(-3R) \\ &\leq (7a_R)^2 \text{ if } R \text{ is sufficiently large.} \end{aligned}$$

This is exactly Equation (3.14). All that remains is to prove the claim. \square

4 Percolation arguments

In this section we prove Lemma 2.9 and Proposition 2.20. In doing so we also obtain Lemma 2.14.

For every $x \in \mathbb{R}^2$ and every $\rho > 0$, we set $B(x, \rho) = x + [-\rho, \rho]^2$. For every $x \in \mathbb{R}^2$ $0 < \rho_1 \leq \rho_2$, we set:

$$\text{Ann}(\rho_1, \rho_2) = [-\rho_2, \rho_2]^2 \setminus]\rho_1, \rho_2[^2,$$

and $\text{Ann}(x, \rho_1, \rho_2) = x + \text{Ann}(\rho_1, \rho_2)$.

4.1 Proof of Lemma 2.9

In this subsection, we prove Lemma 2.9. The proof of Lemma 2.9 also works for Lemma 2.14 with only a few obvious changes. For the proofs of these results, we never use the fact that our Gaussian field is non-degenerate.

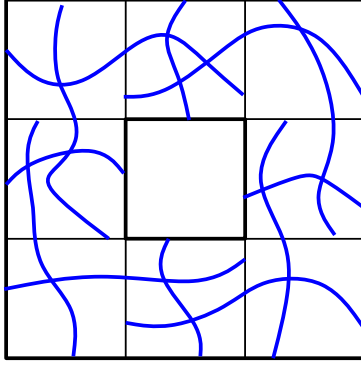


Figure 6: If eight $2^{k+1} \times 2^k$ well chosen rectangles and four well chosen $2^k \times 2^k$ squares are crossed, then there is a circuit in $\text{Ann}(2^k, 2^{k+1})$.

Proof of Lemma 2.9. In this proof, we write $\text{Cross}_p(k) = \text{Cross}_p(2^{k+1}, 2^k)$ and we write $\text{Cross}'_p(k)$ for the event that there is a continuous path joining the bottom side of the rectangle $[0, 2^k] \times [0, 2^{k+1}]$ to its top side in $\mathcal{D}_p \cap [0, 2^k] \times [0, 2^{k+1}]$. By translation invariance and $\frac{\pi}{2}$ -rotation invariance, $\mathbb{P}[\text{Cross}'_p(k)] = \mathbb{P}[\text{Cross}_p(k)]$ for every k . Thus:

$$\sum_{k \in \mathbb{N}} \mathbb{P}[\neg(\text{Cross}_p(k) \cap \text{Cross}'_p(k))] \leq 2 \sum_{k \in \mathbb{N}} \mathbb{P}[\neg \text{Cross}_p(k)] < +\infty \text{ by assumption.}$$

Together with Borel-Cantelli lemma, it implies that a.s. there exists $k_0 \in \mathbb{N}$ such that, for every $k \geq k_0$, $\text{Cross}_p(k) \cap \text{Cross}'_p(k)$ is satisfied. Note that any crossing from left to right of $[0, 2^{k+1}] \times [0, 2^k]$ and any crossing from top to bottom of $[0, 2^{k+1}] \times [0, 2^{k+2}]$ must intersect. Similarly, any crossing from top to bottom of $[0, 2^k] \times [0, 2^{k+1}]$ and any crossing from left to right of $[0, 2^{k+2}] \times [0, 2^{k+1}]$ must intersect. All these intersecting crossings then form an unbounded connected component in \mathcal{D}_p .

Let us end the proof by showing that this unbounded component is a.s. unique. The proof follows the same overall structure: First, write $\text{Circ}_p(k)$ for the event that there is a circuit in $\mathcal{D}_p \cap \text{Ann}(2^k, 2^{k+1})$ surrounding the square $[-2^k, 2^k]^2$. Note that, if eight well-chosen $2^{k+1} \times 2^k$ rectangles are crossed lengthwise, two well-chosen $2^k \times 2^k$ squares are crossed from left to right, and two well-chosen $2^k \times 2^k$ squares are crossed from top to bottom, then $\text{Circ}_p(k)$ holds (see Figure 6 where we already used such a property). Since the probability that a $2^k \times 2^k$ square is crossed is at least equal to the probability that a $2^{k+1} \times 2^k$ rectangle is crossed lengthwise, we get:

$$\mathbb{P}[\neg \text{Circ}_p(k)] \leq 12\mathbb{P}[\neg \text{Cross}_p(k)].$$

Thus, as before,

$$\sum_{k \in \mathbb{N}} \mathbb{P}[\neg \text{Circ}_p(k)] < +\infty,$$

and Borel-Cantelli lemma implies that a.s. there exists k_0 such that, for every $k \geq k_0$, $\text{Circ}_p(k)$ holds. Now, note that, for any unbounded connected component, there exists k_1 such that this component crosses the annuli $\text{Ann}(2^k, 2^{k+1})$ for every $k \geq k_1$. Thus, this component contains any circuit of this annulus for every $k \geq k_0 \vee k_1$. In particular, it must be unique. \square

4.2 Proof of Proposition 2.20

The goal of this subsection is to prove Proposition 2.20. To this purpose, we use the following result from [RV17] which is a consequence of the discrete RSW estimate and quasi-independence. Such a result goes back to [BG16] for $\alpha > 16$. Let $\varepsilon > 0$ and consider the discrete percolation model defined in Subsection 2.4. Let $\text{Arm}_p^\varepsilon(R)$ (respectively $\text{Arm}_p^{\varepsilon,*}(R)$) be the event that in this model there is a black (respectively white) path in the annulus $\text{Ann}(1, R)$ from the inner boundary of this annulus to its outer boundary. We have the following:

Proposition 4.1 ([BG16] for $\alpha > 16$, [RV17]).¹⁵ *Assume that f satisfies Conditions 2.1, 2.2, 2.4 as well as Condition 2.3 for some $\alpha > 4$. Then, there exists $C = C(\kappa) < +\infty$ and $\eta = \eta(\kappa) > 0$ such that, for each $\varepsilon \in]0, 1]$, for each $R \in]0, +\infty[$:*

$$\mathbb{P}[\text{Arm}_0^\varepsilon(R)], \mathbb{P}[\text{Arm}_0^{*,\varepsilon}(R)] \leq C R^{-\eta}.$$

To deduce Proposition 2.20 from Proposition 4.1, we need the following lemma and more particularly its consequence Corollary 4.3.

Lemma 4.2. *Let $\Sigma = (\Sigma(i, j))_{0 \leq i, j \leq n}$ be a $(n+1) \times (n+1)$ symmetric positive definite matrix, let $(m_0, m) = (m_0, m_1, \dots, m_n) \in \mathbb{R}^{n+1}$ and let $(X_0, X) = (X_0, X_1, \dots, X_n) \sim \mathcal{N}((m_0, m), \Sigma)$. Assume that $\Sigma(0, i) \geq 0$ for every $i \in \{1, \dots, n\}$. Then, for every non-decreasing¹⁶ function $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$, the following quantity is non-decreasing in q :*

$$\mathbb{E}[\varphi(X) \mid X_0 = q].$$

Similarly, if φ is non-increasing, then this quantity is non-increasing in q .

Proof. This is a direct consequence of the following result (see for instance Proposition 1.2 of [AW09]): Under the probability measure $\mathbb{P}[\cdot \mid X_0 = q]$, X is a Gaussian vector whose covariance matrix does not depend on q and whose mean is:

$$m + (q - m_0)v,$$

where $v = \left(\frac{\Sigma(0, i)}{\Sigma(0, 0)} \right)_{1 \leq i \leq n}$ (which has only non-negative entries). □

From this lemma we deduce the following:

Corollary 4.3. *Let (X_0, X) be as in Lemma 4.2 and let $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ be a non-decreasing function. Then, for every $p \in \mathbb{R}$ we have:*

$$\mathbb{E}[\varphi(X) \mid X_0 = -p] \leq \mathbb{E}[\varphi(X) \mid X_0 \geq -p].$$

Similarly, if φ is non-increasing, then:

$$\mathbb{E}[\varphi(X) \mid X_0 = -p] \leq \mathbb{E}[\varphi(X) \mid X_0 \leq -p].$$

¹⁵More precisely, this is Proposition B.6 of [RV17]. Moreover, this can be extracted from the proof of Theorem 5.7 of [BG16] for $\alpha > 16$ and with slightly different assumptions on the differentiability and the non-degeneracy of κ .

¹⁶For the partial order $(x_1, \dots, x_n) \leq (y_1, \dots, y_n)$ if $x_i \leq y_i$ for every $i \in \{1, \dots, n\}$.

Proof. Assume that φ is non-decreasing and let γ_0 be the density of X_0 , which exists because Σ is positive definite. We have :

$$\begin{aligned} \mathbb{E} \left[\varphi(X) \mid X_0 \geq -p \right] &= \frac{1}{\mathbb{P}[X_0 \geq -p]} \int_{-p}^{+\infty} \mathbb{E} \left[\varphi(X) \mid X_0 = t \right] \gamma_0(t) dt \\ &\geq \frac{1}{\mathbb{P}[X_0 \geq -p]} \mathbb{E} \left[\varphi(X) \mid X_0 = -p \right] \int_{-p}^{+\infty} \gamma_0(t) dt \\ &= \mathbb{E} \left[\varphi(X) \mid X_0 = -p \right] \end{aligned}$$

where the inequality comes from Lemma 4.2 applied with $q = -p$. \square

We are now ready to prove Proposition 2.20. As usual in percolation theory, we are going to control our p -dependent probabilities by probabilities at the self-dual point $p = 0$.

Proof of Proposition 2.20. Fix $R \geq 2$ and $\varepsilon \in]0, 1]$. Let us prove the proposition in the case $p \geq 0$. Let $x \in \mathcal{V}_R^\varepsilon$ and let $\text{Arm}_p^{\varepsilon,*}(x, R/2)$ be the event that there is a white path in $\text{Ann}(x, 1, R/2)$ from $\partial B(x, 1)$ to $\partial B(x, R/2)$. We claim that $\text{Piv}_x^p(\text{Cross}^\varepsilon(2R, R)) \subseteq \text{Arm}_p^{\varepsilon,*}(x, R/2)$. Indeed, $\text{Piv}_x^p(\text{Cross}^\varepsilon(2R, R))$ is the event that there are two white paths from the top and bottom sides of the rectangle $[0, 2R] \times [0, R]$ to two neighbors of x and two black paths from the left and right sides of $[0, 2R] \times [0, R]$ to two other neighbors of x (with an exception when x does not belong to $[0, 2R] \times [0, R]$ but is a neighbor of a point $y \in [0, 2R] \times [0, R]$ such that the edge between x and y crosses the left or right side of $[0, 2R] \times [0, R]$; in this case $\text{Piv}_x^p(\text{Cross}^\varepsilon(R))$ is the event that there is a white path from top to bottom that goes through y and a black path from y to the opposite side). In particular, at least one black path reaches a point at distance at least $R/2$ of x .

Since $\text{Arm}_p^{\varepsilon,*}(x, R/2)$ is decreasing and does not depend on $f(x)$, then (by Corollary 4.3 and with $a := \mathbb{P}[Z \leq -1]$ where $Z \sim \mathcal{N}(0, 1)$):

$$\begin{aligned} \mathbb{P} \left[\text{Arm}_p^{\varepsilon,*}(x, R/2) \mid f(x) = -p \right] &\leq \mathbb{P} \left[\text{Arm}_p^{\varepsilon,*}(x, R/2) \mid f(x) \leq -p \right] \\ &\leq \frac{\mathbb{P} \left[\text{Arm}_p^{\varepsilon,*}(x, R/2) \right]}{\mathbb{P} \left[f(x) \leq -p \right]} \\ &\leq a^{-1} \mathbb{P} \left[\text{Arm}_p^{\varepsilon,*}(x, R/2) \right] \text{ since } p \leq 1 \\ &\leq a^{-1} \mathbb{P} \left[\text{Arm}_0^{\varepsilon,*}(x, R/2) \right] \text{ since } p \geq 0. \end{aligned}$$

The result follows from Proposition 4.1 (and stationarity). The case $p \leq 0$ is treated similarly (by noting that $\text{Piv}_x^p(\text{Cross}^\varepsilon(2R, R)) \subseteq \text{Arm}_p^\varepsilon(x, R/2)$ and by studying this increasing event exactly as above). \square

5 A KKL theorem for biased Gaussian vectors: the proof of Theorem 2.19

In this section we prove Theorem 2.19. The proofs presented here do not rely on any results of the other sections. Recall that we use the following definition for influence. Given a vector $v \in \mathbb{R}^n$ and a Borel probability measure μ on \mathbb{R}^n the influence of v on A under μ is

$$I_{v,\mu}(A) = \liminf_{r \downarrow 0} \frac{\mu(A + [-r, r]v) - \mu(A)}{r} \in [0, +\infty].$$

5.1 Sub-linearity of influences implies Theorem 2.19

The aim of this subsection is to prove Theorem 2.19. That this theorem holds for product Gaussian measures is a result by Keller, Mossel and Sen, [KMS12] (see Corollary 5.2 below). In order to extend this result to general Gaussian measures, we use a sub-linearity property for influences. In the present subsection, we state this property and use it to derive Theorem 2.19 from the product case. In Subsection 5.2 (respectively in Appendix A), we prove the sub-linearity property for monotonic (respectively semi-algebraic) sets.

A KKL theorem for product Gaussian vectors The authors of [KMS12] introduce and study the notion of *geometric influences* which are defined as follows: Let ν be a probability measure on \mathbb{R} and let A be a Borel subset of \mathbb{R}^n . If $i \in \{1, \dots, n\}$, let:

$$A_i^x := \{y \in \mathbb{R} : (x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_n) \in A\}.$$

The geometric influence of i on A under the measure $\nu^{\otimes n}$ is:

$$I_{i,\nu}^{\mathcal{G}}(A) := \mathbb{E}_{x \sim \nu^{\otimes n}} [\nu^+(A_i^x)] \in [0, +\infty],$$

where ν^+ is the lower Minkowski content, defined as follows: for all $B \subseteq \mathbb{R}$ Borel,

$$\nu^+(B) := \liminf_{r \downarrow 0} \frac{\nu(B + [-r, r]) - \nu(B)}{r} \in [0, +\infty].$$

In the case where $\mu = \nu^{\otimes n}$, $I_{i,\nu}^{\mathcal{G}}$ and $I_{i,\mu}$ are closely related. Indeed, firstly, by Fubini's Theorem and Fatou's lemma, for each Borel subset $A \subset \mathbb{R}^n$ and each $i \in \{1, \dots, n\}$,

$$I_{i,\nu^{\otimes n}}(A) = \liminf_{r \downarrow 0} \mathbb{E}_{x \sim \nu^{\otimes n}} \left[\frac{\nu(A_i^x + [-r, r]) - \nu(A)}{r} \right] \geq I_{i,\nu}^{\mathcal{G}}. \quad (5.1)$$

While the reverse inequality seems not true in general, we expect it to hold for a wide class of events. In particular, our Lemma 5.5 and (2.6) from [KMS12] imply that this is the case for monotonic events. Since it is not useful to us, we do not investigate this matter any further.

We will need the following result, which is a direct consequence of Item (1) of Theorem 1.5 of [KMS12]. Several results of this type can also be found in the more recent [CEL12] (see for instance the paragraph above Corollary 7 therein).

Theorem 5.1. *There exists an absolute constant $c > 0$ such that the following holds: Let $\nu = \mathcal{N}(0, 1)$ and let A be a Borel measurable subset of \mathbb{R}^n . Then:*

$$\sum_{i=1}^n I_{i,\nu}^{\mathcal{G}}(A) \geq c \nu^{\otimes n}(A) (1 - \nu^{\otimes n}(A)) \sqrt{\log_+ \left(\frac{1}{\max_{i \in \{1, \dots, n\}} I_{i,\nu}^{\mathcal{G}}(A)} \right)}.$$

Corollary 5.2. *Let ν and A be as in Theorem 5.1. Then:*

$$\sum_{i=1}^n I_{i,\nu^{\otimes n}}(A) \geq c \nu^{\otimes n}(A) (1 - \nu^{\otimes n}(A)) \sqrt{\log_+ \left(\frac{1}{\max_{i \in \{1, \dots, n\}} I_{i,\nu^{\otimes n}}(A)} \right)}.$$

Proof. This is a direct consequence of Theorem 5.1 and Equation (5.1). □

Now, let us state a sub-linearity property for the influences that we will be proved in Subsection 5.2 and Appendix A.

Proposition 5.3. *Let Σ be a $n \times n$ symmetric positive definite matrix, let $\mu \sim \mathcal{N}(0, \Sigma)$ and let (e_1, \dots, e_n) be the canonical basis of \mathbb{R}^n . Also, let $v = \sum_{j=1}^n v_j e_j \in \mathbb{R}^n$. For every $i \in \{1, \dots, n\}$ and every A Borel subset of \mathbb{R}^n which is monotonic or semi-algebraic, we have:*

$$I_{v, \mu}(A) \leq \sum_{i=1}^n |v_i| I_{i, \mu}(A). \quad (5.2)$$

We expect the above result to hold for a larger class of Borel subsets A (in particular, we have not found any examples of Borel sets for which this does not hold) and not only for the decomposition on the canonical basis (see in particular Proposition A.1).

Let us now derive Theorem 2.19 from Corollary 5.2 using Proposition 5.3.

Proof of Theorem 2.19. Let $\sqrt{\Sigma}$ be a symmetric square root of Σ and let $\nu = \mathcal{N}(0, 1)$. Then, $\mu = \mathcal{N}(0, \Sigma)$ is the pushforward measure of $\nu^{\otimes n}$ by $\sqrt{\Sigma}$. Thanks to Corollary 5.2 (applied to the event $\sqrt{\Sigma}^{-1}(A)$), it is sufficient to prove the following claim. \square

Claim 5.4. *Let $\nu = \mathcal{N}(0, 1)$. We have:*

$$\max_{j \in \{1, \dots, n\}} I_{j, \nu^{\otimes n}}(\sqrt{\Sigma}^{-1}(A)) \leq \|\sqrt{\Sigma}\|_{\infty, op} \cdot \max_{i \in \{1, \dots, n\}} I_{i, \mu}(A). \quad (5.3)$$

Moreover:

$$\sum_{j=1}^n I_{j, \nu^{\otimes n}}(\sqrt{\Sigma}^{-1}(A)) \leq \|\sqrt{\Sigma}\|_{\infty, op} \cdot \sum_{i=1}^n I_{i, \mu}(A). \quad (5.4)$$

Proof. For each $j \in \{1, \dots, n\}$:

$$\begin{aligned} I_{j, \nu^{\otimes n}}(\sqrt{\Sigma}^{-1}(A)) &= \liminf_{r \downarrow 0} \frac{\nu^{\otimes n}(\sqrt{\Sigma}^{-1}(A) + [-r, r] e_j) - \nu^{\otimes n}(\sqrt{\Sigma}^{-1}(A))}{r} \\ &= \liminf_{r \downarrow 0} \frac{\mu(A + [-r, r] \sqrt{\Sigma} \cdot e_j) - \mu(A)}{r} \\ &= I_{\sqrt{\Sigma} \cdot e_j, \mu}(A). \end{aligned}$$

By using Proposition 5.3, we obtain that:

$$I_{j, \nu^{\otimes n}}(\sqrt{\Sigma}^{-1}(A)) \leq \sum_{i=1}^n |\sqrt{\Sigma}(i, j)| I_{i, \mu}(A). \quad (5.5)$$

Hence:

$$I_{j, \nu^{\otimes n}}(\sqrt{\Sigma}^{-1}(A)) \leq \sum_{i=1}^n |\sqrt{\Sigma}(i, j)| \max_{i \in \{1, \dots, n\}} I_{i, \mu}(A) \leq \|\sqrt{\Sigma}\|_{\infty, op} \max_{i \in \{1, \dots, n\}} I_{i, \mu}(A).$$

Since $\sqrt{\Sigma}$ is symmetric, we obtain (5.3) by taking the supremum over j . Inequality (5.5) also implies that:

$$\begin{aligned} \sum_{j=1}^n I_{\sqrt{\Sigma} \cdot e_j, \mu}(A) &\leq \sum_{j=1}^n \sum_{i=1}^n |\sqrt{\Sigma}(i, j)| I_{i, \mu}(A) \\ &= \sum_{i=1}^n I_{i, \mu}(A) \sum_{j=1}^n |\sqrt{\Sigma}(i, j)| \\ &\leq \|\sqrt{\Sigma}\|_{\infty, op} \cdot \sum_{i=1}^n I_{i, \mu}(A), \end{aligned}$$

which is (5.4). □

5.2 Sub-linearity of influences for monotonic events

The proof of Proposition 5.3 for monotonic events relies on the following lemma.

Lemma 5.5. *Let Σ be a $n \times n$ symmetric positive definite matrix and let $\mu = \mathcal{N}(0, \Sigma)$. Moreover, let $v \in \mathbb{R}^n$. For every A monotonic Borel subset of \mathbb{R}^n and every $i \in \{1, \dots, n\}$, we have:*

$$\exists \lim_{r \downarrow 0} \frac{\mu(A + [-r, r]e_i + [-r, r]v) - \mu(A + [-r, r]v)}{r} = I_{i, \mu}(A).$$

We first prove Proposition 5.3 using Lemma 5.5.

Proof of Proposition 5.3. Let A be a monotonic Borel subset of \mathbb{R}^n and fix $v \in \mathbb{R}^n$. Let $v^i = \sum_{k=i}^n v_k e_k$. We will prove that, for every $i \in \{1, \dots, n-1\}$:

$$I_{v^i, \mu}(A) \leq |v_i| I_{i, \mu}(A) + I_{v^{i+1}, \mu}(A). \quad (5.6)$$

The result will then follow directly by induction. For any $w_1, w_2 \subseteq \mathbb{R}^n$, we have $[-r, r](w_1 + w_2) \subseteq [-r, r]w_1 + [-r, r]w_2$. Hence:

$$\begin{aligned} \mu(A + [-r, r]v^i) &= \mu(A + [-r, r](v_i e_i + v^{i+1})) \\ &\leq \mu(A + [-|v_i|r, |v_i|r]e_i + [-r, r]v^i) \\ &= \mu(A + [-|v_i|r, |v_i|r]e_i + [-r, r]v^{i+1}) \\ &\quad - \mu(A + [-r, r]v^{i+1}) + \mu(A + [-r, r]v^{i+1}). \end{aligned}$$

By Lemma 5.5 we have:

$$\frac{\mu(A + [-|v_i|r, |v_i|r]e_i + [-r, r]v^{i+1}) - \mu(A + [-r, r]v^i)}{r} \xrightarrow{r \downarrow 0} |v_i| I_{i, \mu}(A).$$

Equation (5.6) follows. □

Proof of Lemma 5.5. We are inspired by the proof of Proposition 1.3 in [KMS12]. We write the proof for A decreasing since the proof for A increasing is identical. Also, we prove the result in the case $i = n$. We write \tilde{x} for the first $(n-1)$ coordinates of any $x \in \mathbb{R}^n$. Let $A_r = A + [-r, r]v$, note that A_r is decreasing, and write for any $\tilde{x} \in \mathbb{R}^{n-1}$:

$$\begin{aligned} s(\tilde{x}) &:= \sup\{x_n \in \mathbb{R} : (\tilde{x}, x_n) \in A\} \in [-\infty, +\infty], \\ s_r(\tilde{x}) &:= \sup\{x_n \in \mathbb{R} : (\tilde{x}, x_n) \in A_r\} \in [-\infty, +\infty]. \end{aligned}$$

Let λ be the density function of μ . Since A and A_r are decreasing, we have:

$$\begin{aligned}\frac{\mu(A + [-r, r]e_n) - \mu(A)}{r} &= \frac{1}{r} \int_{\mathbb{R}^{n-1}} \left(\int_{s(\tilde{x})}^{s(\tilde{x})+r} \lambda(\tilde{x}, x_n) dx_n \right) d\tilde{x}, \\ \frac{\mu(A_r + [-r, r]e_n) - \mu(A_r)}{r} &= \frac{1}{r} \int_{\mathbb{R}^{n-1}} \left(\int_{s_r(\tilde{x})}^{s_r(\tilde{x})+r} \lambda(\tilde{x}, x_n) dx_n \right) d\tilde{x},\end{aligned}\quad (5.7)$$

where by convention $\int_{-\infty}^{-\infty+r} = \int_{+\infty}^{+\infty+r} = 0$. For each $\tilde{x} \in \mathbb{R}^{n-1}$ let:

$$g_1(\tilde{x}) = \sup_{x_n \in \mathbb{R}} \lambda(\tilde{x}, x_n); \quad g_2(\tilde{x}) = \sup_{x_n \in \mathbb{R}} \left| \frac{\partial \lambda}{\partial x_n}(\tilde{x}, x_n) \right|.$$

Direct computation shows that $g_1, g_2 \in L^1(\mathbb{R}^{n-1})$. By the mean value inequality, for each $\tilde{x} \in \mathbb{R}^{n-1}$:

$$\left| \frac{1}{r} \left(\int_{s(\tilde{x})}^{s(\tilde{x})+r} \lambda(\tilde{x}, x_n) dx_n \right) - \lambda(\tilde{x}, s(\tilde{x})) \right| + \left| \frac{1}{r} \left(\int_{s_r(\tilde{x})}^{s_r(\tilde{x})+r} \lambda(\tilde{x}, x_n) dx_n \right) - \lambda(\tilde{x}, s_r(\tilde{x})) \right|$$

is no greater than $2r g_2(\tilde{x})$. Combining this with Equation (5.7) we get:

$$\begin{aligned}\left| \frac{\mu(A + [-r, r]e_n) - \mu(A)}{r} - \frac{\mu(A_r + [-r, r]e_n) - \mu(A_r)}{r} \right| \\ \leq \left| \int_{\mathbb{R}^{n-1}} \lambda(\tilde{x}, s(\tilde{x})) - \lambda(\tilde{x}, s_r(\tilde{x})) d\tilde{x} \right| + 2r \int_{\mathbb{R}^{n-1}} g_2(\tilde{x}) d\tilde{x}.\end{aligned}$$

Since $g_2 \in L^1(\mathbb{R}^{n-1})$, the second integral in the last inequality is finite and independent of r . Moreover:

$$\int_{\mathbb{R}^{n-1}} \lambda(\tilde{x}, s_r(\tilde{x})) - \lambda(\tilde{x}, s(\tilde{x})) d\tilde{x} = \int_{\mathbb{R}^{n-1}} \int_{s(\tilde{x})}^{s_r(\tilde{x})} \frac{\partial \lambda}{\partial x_n}(\tilde{x}, x_n) dx_n d\tilde{x}.$$

Since $\frac{\partial \lambda}{\partial x_n} \in L^1(\mathbb{R}^n)$, by dominated convergence, all that remains is to show that for a.e. $\tilde{x} \in \mathbb{R}^{n-1}$: $s_r(\tilde{x}) \xrightarrow[r \downarrow 0]{} s(\tilde{x})$. Since for each \tilde{x} the sequence $s_r(\tilde{x})$ is decreasing, it converges to some $s_\infty(\tilde{x}) \geq s(\tilde{x})$. Let us prove that, for a.e. $\tilde{x} \in \mathbb{R}^{n-1}$, $s_\infty(\tilde{x}) = s(\tilde{x})$. To do so, first note that, since A is decreasing, we have:

$$0 \leq \mu(A_r) - \mu(A) \leq \mathbb{P} \left[X - \sum_{i=1}^n r |v_i| e_i \in A \right] - \mathbb{P} [X \in A],$$

where $X \sim \mathcal{N}(0, \Sigma)$. By dominated convergence, the right hand side tends to 0 when $r \rightarrow 0$. Now, note that:

$$\mu(A_r) - \mu(A) = \int_{\mathbb{R}^{n-1}} \int_{s(\tilde{x})}^{s_r(\tilde{x})} \lambda(\tilde{x}, x_n) dx_n d\tilde{x}.$$

Hence, by Fatou's lemma:

$$0 = \int_{\mathbb{R}^{n-1}} \int_{s(\tilde{x})}^{s_\infty(\tilde{x})} \lambda(\tilde{x}, x_n) dx_n d\tilde{x}.$$

Since the λ takes only positive values, this implies that for a.e. $\tilde{x} \in \mathbb{R}^{n-1}$, $s_\infty(\tilde{x}) = s(\tilde{x})$. \square

6 An estimate on the infinite operator norm of square root of infinite matrices: the proof of Proposition 2.21

The goal of this section is to prove Proposition 2.21. We assume that f satisfies Condition 2.1 and Condition 2.5 for some $\alpha > 5$. In particular, the Fourier transform of κ takes only positive values. Moreover, we assume that κ is C^3 and there exists $C < +\infty$ such that for every $\beta \in \mathbb{N}^2$ such that $\beta_1 + \beta_2 \leq 3$, we have:

$$|\partial^\beta \kappa(x)| \leq C|x|^{-\alpha}, \quad (6.1)$$

for some $\alpha > 5$. In this subsection, we never use the fact that our Gaussian field is non-degenerate.

Recall that for each $\varepsilon > 0$, \mathcal{T}^ε is the lattice \mathcal{T} scaled by a factor ε , that \mathcal{V}^ε is the set of vertices of \mathcal{T}^ε , and that K^ε is the restriction of K to \mathcal{V}^ε . We begin by observing that the face-centered square lattice \mathcal{T} , when rotated by $\frac{\pi}{4}$ and rescaled by $\sqrt{2}$, has the same vertices as \mathbb{Z}^2 . Since by Condition 2.5 our field is invariant by $\frac{\pi}{4}$ -rotation, **we will simply replace \mathcal{T} by \mathbb{Z}^2 throughout the rest of this section.**

Let \mathbb{T}^2 be the flat 2D torus corresponding to the circle of length 2π . Throughout this section, we will identify $\lambda\mathbb{Z}^2$ -periodic functions on \mathbb{R}^2 (for some $\lambda > 0$) with functions on $\lambda\mathbb{T}^2$ and their integrals over the box $[-\lambda\pi, \lambda\pi]^2$ with integrals over $\lambda\mathbb{T}^2$. We will use the following convention for the Fourier transform:

$$\forall \xi \in \mathbb{R}^2, \hat{\kappa}(\xi) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} e^{-i\langle \xi, x \rangle} \kappa(x) dx.$$

Let us begin with a sketch of the construction of the square root $\sqrt{K^\varepsilon}$:

1. First note that if we find some symmetric function $\eta_\varepsilon : \mathbb{Z}^2 \rightarrow \mathbb{R}^2$ such that:

$$\forall m \in \mathbb{Z}^2, \eta_\varepsilon * \eta_\varepsilon(m) := \sum_{m' \in \varepsilon\mathbb{Z}^2} \eta_\varepsilon(m') \eta_\varepsilon(m - m') = \kappa(\varepsilon m),$$

then $(\varepsilon m_1, \varepsilon m_2) \in \varepsilon\mathbb{Z}^2 \mapsto \eta_\varepsilon(m_1 - m_2)$ is a symmetric square root of K^ε .

2. Let κ_ε be κ restricted to $\varepsilon\mathbb{Z}^2$ and let us try to construct η_ε above. The first idea is that, if $\eta_\varepsilon * \eta_\varepsilon = \kappa_\varepsilon$, then the Fourier transform of η_ε should be the square root of the Fourier transform of κ_ε . In other words:

$$F(\eta_\varepsilon) = \sqrt{F(\kappa_\varepsilon)},$$

where $F(\eta_\varepsilon)(\xi) = \sum_{m \in \mathbb{Z}^2} \eta_\varepsilon(m) e^{-i\langle \xi, m \rangle}$ and similarly for $F(\kappa_\varepsilon)$.

3. Thus:

$$\begin{aligned} \eta_\varepsilon(m) &= F^{-1} \left(\sqrt{F(\kappa_\varepsilon)} \right) \\ &= \frac{1}{4\pi^2} \int_{\mathbb{T}^2} \sqrt{F(\kappa_\varepsilon)}(\xi) d\xi. \end{aligned}$$

4. In the expression above, it seems difficult to deal with the term $F(\kappa_\varepsilon)$. To simplify the expression, we can use the Poisson summation formula and deduce that:

$$F(\kappa_\varepsilon)(\xi) = 4\pi^2 \varepsilon^{-2} \sum_{m \in \mathbb{Z}^2} \hat{\kappa}(\varepsilon^{-1}(2\pi m - \xi)).$$

For the Bargmann-Fock process, $\hat{\kappa}$ is well known since κ is simply the Gaussian function.

There will be two main steps in the proof of Proposition 2.21. First, we will make the above construction of η_ε rigorous by considering the four items above in the reverse order. More precisely, we will first set λ_ε as in Lemma 6.1 below, then we will apply the Poisson summation formula to prove that $2\pi\varepsilon^{-1}\lambda_\varepsilon = \sqrt{F(\kappa_\varepsilon)}$. Next, we will define η_ε as in Lemma 6.1, we will show that $F(\eta_\varepsilon) = \sqrt{F(\kappa_\varepsilon)}$, and we will conclude that η_ε is a convolution square root of κ_ε . All of this will be done in the proof Lemma 6.1. Secondly, we will prove estimates on $\sum_{m \in \mathbb{Z}^2} |\eta_\varepsilon(m)|$ when $\kappa(x) = e^{-\frac{1}{2}|x|^2}$. This will be the purpose of Lemma 6.2 (and most of the proof of this lemma will be written in a more general setting than $\kappa(x) = e^{-\frac{1}{2}|x|^2}$).

In the proofs, we will use a subscript ε to denote functions $:\mathbb{Z}^2 \rightarrow \mathbb{R}$ or to denote functions defined on \mathbb{T}^2 (or equivalently functions $2\pi\mathbb{Z}^2$ -periodic). On the other hand, we will use a superscript ε to denote functions $:\varepsilon^{-1}\mathbb{Z}^2 \rightarrow \mathbb{R}$ or to denote functions defined on $\varepsilon^{-1}\mathbb{T}^2$ (or equivalently functions $2\pi\varepsilon^{-1}\mathbb{Z}^2$ -periodic).

Proposition 2.21 is a direct consequence of the following Lemmas 6.1 and 6.2.

Lemma 6.1. *Assume that f satisfies Condition 2.1 as well as Condition 2.2 for $\alpha > 5$. Fix $\varepsilon > 0$ and let:*

$$\lambda_\varepsilon : \xi \in \mathbb{R}^2 \mapsto \sqrt{\sum_{m \in \mathbb{Z}^2} \hat{\kappa}(\varepsilon^{-1}(\xi - 2\pi m))}.$$

Then, λ_ε is a C^3 , positive, even and $2\pi\varepsilon^{-1}\mathbb{Z}^2$ -periodic function. Next, define:

$$\eta_\varepsilon : m \in \mathbb{Z}^2 \mapsto \frac{1}{\varepsilon(2\pi)} \int_{\mathbb{T}^2} e^{i\langle m, \xi \rangle} \lambda_\varepsilon(\xi) d\xi.$$

Then, $(\varepsilon m_1, \varepsilon m_2) \in \varepsilon\mathbb{Z}^2 \mapsto \eta_\varepsilon(m_1 - m_2)$ is a symmetric square root of K^ε and we have:

$$\sum_{m \in \mathbb{Z}^2} |\eta_\varepsilon(m)| < +\infty.$$

Lemma 6.2. *Assume that $\kappa(x) = e^{-\frac{1}{2}|x|^2}$. Then, there exist constants $C_0 < +\infty$ and $\varepsilon_0 > 0$ such that for $\varepsilon \in]0, \varepsilon_0]$:*

$$\sum_{m \in \mathbb{Z}^2} |\eta_\varepsilon(m)| \leq C_0 \frac{1}{\varepsilon} \log\left(\frac{1}{\varepsilon}\right).$$

Proof of Lemma 6.1. First note that (6.1) implies that $\hat{\kappa}$ is C^3 and that for every $\beta \in \mathbb{N}^2$ such that $\beta_1 + \beta_2 \leq 3$, we have:

$$|\partial^\beta \hat{\kappa}(\xi)| \leq C' |\xi|^{-3}, \tag{6.2}$$

for some $C' = C'(\kappa) < +\infty$. This implies that the series under the square root of the definition of λ_ε converges in C^3 -norm towards a C^3 function. Moreover, this series is clearly $2\pi\mathbb{Z}^2$ -periodic, even, and positive (since $\hat{\kappa}$ takes only positive values). Thus, λ_ε is well defined and also satisfies these properties.

Let us now prove the second part of the lemma. For each $\varepsilon > 0$ let $\kappa_\varepsilon : \mathbb{Z}^2 \rightarrow \mathbb{R}$ be defined as $\kappa_\varepsilon(m) = \kappa(\varepsilon m)$. The discrete Fourier transform of κ_ε is:

$$F(\kappa_\varepsilon) : \xi \in \mathbb{T}^2 \mapsto \sum_{m \in \mathbb{Z}^2} \kappa_\varepsilon(m) e^{-i\langle \xi, m \rangle}$$

(since κ and its derivatives of order up to 3 decay polynomially fast with an exponent larger than 5, the above series converges in C^3 -norm). Now, since κ and $\hat{\kappa}$ decay polynomially fast with an exponent larger than 2, we can apply the Poisson summation formula (see¹⁷ Theorem 3.1.17 of [Gra09]) which implies that:

$$\forall \xi \in \mathbb{T}^2, F(\kappa_\varepsilon)(\xi) = 4\pi^2 \varepsilon^{-2} \sum_{m \in \mathbb{Z}^2} \hat{\kappa}(\varepsilon^{-1}(2\pi m - \xi)) = 4\pi^2 \varepsilon^{-2} \lambda_\varepsilon(\xi)^2.$$

As a result:

$$\eta_\varepsilon(m) = \frac{1}{4\pi^2} \int_{\mathbb{T}^2} \sqrt{F(\kappa_\varepsilon)}(\xi) e^{i\langle m, \xi \rangle} d\xi.$$

In other words, the $\eta_\varepsilon(m)$'s are the Fourier coefficients of the C^3 , positive and $2\pi\mathbb{Z}^2$ -periodic function $\sqrt{F(\kappa_\varepsilon)}$, which implies in particular that $|\eta_\varepsilon(m)| \leq C''|m|^{-3}$ for some $C'' = C''(\kappa, \varepsilon) < +\infty$. As a result:

$$\sum_{m \in \mathbb{Z}^2} |\eta_\varepsilon(m)| < +\infty. \quad (6.3)$$

Thanks to (6.3), we can apply the Fourier inversion formula (see for instance Proposition 3.1.14 of [Gra09]) which implies that:

$$F(\eta_\varepsilon) = \sqrt{F(\kappa_\varepsilon)}. \quad (6.4)$$

Now, let us use the convolution formula (see for instance Paragraph 1.3.3 of [Rud]). Since $\sum_{m \in \mathbb{Z}^2} |\eta_\varepsilon(m)| < +\infty$, we have:

$$\sum_{m \in \mathbb{Z}^2} \sum_{m' \in \mathbb{Z}^2} |\eta_\varepsilon(m') \eta_\varepsilon(m - m')| < +\infty, \quad (6.5)$$

and:

$$F(\eta_\varepsilon * \eta_\varepsilon) = F(\eta_\varepsilon)^2, \quad (6.6)$$

where $\eta_\varepsilon * \eta_\varepsilon : m \in \mathbb{Z}^2 \mapsto \sum_{m' \in \mathbb{Z}^2} \eta_\varepsilon(m') \eta_\varepsilon(m - m')$.

We deduce from (6.4) and (6.6) that $F(\eta_\varepsilon * \eta_\varepsilon) = F(\kappa_\varepsilon)$. Since, by the dominated convergence theorem, the Fourier coefficients of $F(\eta_\varepsilon * \eta_\varepsilon)$ are the $\eta_\varepsilon * \eta_\varepsilon(m)$'s and the Fourier coefficients of $F(\kappa_\varepsilon)$ are the $\kappa_\varepsilon(m)$'s, we obtain that:

$$\eta_\varepsilon * \eta_\varepsilon = \kappa_\varepsilon.$$

This is equivalent to saying that $(\varepsilon m_1, \varepsilon m_2) \in \varepsilon\mathbb{Z}^2 \mapsto \eta_\varepsilon(m_1 - m_2)$ is a symmetric square root of K^ε . \square

The proof of Lemma 6.2 is split into two sub-lemmas:

Lemma 6.3. *Assume that f satisfies Condition 2.1 and Condition 2.2 for $\alpha > 5$. In this lemma, we work with the function:*

$$\rho^\varepsilon : \xi \in \mathbb{R}^2 \mapsto \lambda(\varepsilon\xi) = \sqrt{\sum_{m \in \mathbb{Z}^2} \hat{\kappa}(\xi - 2\pi\varepsilon^{-1}m)}.$$

¹⁷Be aware that the conventions used in [Gra09] are different from ours.

For each $\varepsilon > 0$, let:

$$\Upsilon(\varepsilon) = \max \left(\int_{\varepsilon^{-1}\mathbb{T}^2} |\rho^\varepsilon(\xi)| d\xi, \int_{\varepsilon^{-1}\mathbb{T}^2} |\Delta\rho^\varepsilon(\xi)| d\xi, \int_{\varepsilon^{-1}\mathbb{T}^2} |\partial_1\Delta\rho^\varepsilon(\xi)| + |\partial_2\Delta\rho^\varepsilon(\xi)| d\xi \right).$$

Then, for every $\varepsilon > 0$, $\Upsilon(\varepsilon) < +\infty$. Moreover, there exist an absolute constants $C_1 < +\infty$ and a constant $\varepsilon_1 = \varepsilon_1(\kappa) > 0$ such that for every $\varepsilon \in]0, \varepsilon_1]$ we have:

$$\sum_{m \in \mathbb{Z}^2} |\eta_\varepsilon(m)| \leq C_1 \Upsilon(\varepsilon) \frac{1}{\varepsilon} \log \left(\frac{1}{\varepsilon} \right).$$

Lemma 6.4. Assume that $\kappa(x) = e^{-\frac{1}{2}|x|^2}$. Then, there exists $\varepsilon_2 > 0$ such that $\sup_{\varepsilon \in]0, \varepsilon_2]} \Upsilon(\varepsilon) < +\infty$.

Lemma 6.3 follows easily from appropriate integration by parts. The proof of Lemma 6.4 is just straightforward elementary computation and uses very crude estimates throughout.

Proof of Lemma 6.3. First note that $\Upsilon(\varepsilon) < +\infty$ comes from the fact that (by Lemma 6.1) $\lambda_\varepsilon \in C^3(\mathbb{T}^2)$, hence $\rho^\varepsilon \in C^3(\varepsilon^{-1}\mathbb{T}^2)$. Next, note that by an obvious change of variable, we have:

$$\eta_\varepsilon(m) = \frac{\varepsilon}{2\pi} \int_{\varepsilon^{-1}\mathbb{T}^2} e^{i\langle \varepsilon m, \xi \rangle} \rho^\varepsilon(\xi) d\xi.$$

Hence, $\eta_\varepsilon(0) \leq \frac{\varepsilon}{2\pi} \int_{\varepsilon^{-1}\mathbb{T}^2} |\mu_\varepsilon(\xi)| d\xi \leq \frac{\varepsilon}{2\pi} \Upsilon(\varepsilon)$. Now, let $m \neq 0$. By integration by parts, we have:

$$\eta_\varepsilon(m) = \frac{1}{2\pi\varepsilon|m|^2} \int_{\varepsilon^{-1}\mathbb{T}^2} \Delta\rho^\varepsilon(\xi) e^{i\langle \varepsilon m, \xi \rangle} d\xi, \quad (6.7)$$

and:

$$\eta^\varepsilon(m) = \frac{1}{i2\pi\varepsilon^2|m|^2(m_1 \pm m_2)} \int_{\varepsilon^{-1}\mathbb{T}^2} (\partial_1 \pm \partial_2) \Delta\rho^\varepsilon(\xi) e^{i\langle \varepsilon m, \xi \rangle} d\xi. \quad (6.8)$$

In (6.8), since $m \neq 0$, at least one of the two expressions is well defined. Thus, for each $m \in \mathbb{Z}^2$:

$$|\eta_\varepsilon(m)| \leq \frac{1}{\varepsilon} \frac{1}{2\pi} \Upsilon(\varepsilon) \min \left(\varepsilon^2, \frac{1}{|m|^2}, \frac{1}{\varepsilon|m|^2(|m_1| + |m_2|)} \right).$$

This implies that:

$$\begin{aligned} \sum_{m \in \mathbb{Z}^2} |\eta_\varepsilon(m)| &= |\eta_\varepsilon(0)| + \sum_{m \in \mathbb{Z}^2, 0 < |m| \leq \varepsilon^{-1}} |\eta_\varepsilon(m)| + \sum_{m \in \mathbb{Z}^2, |m| > \varepsilon^{-1}} |\eta_\varepsilon(m)| \\ &\leq \frac{1}{\varepsilon} \frac{1}{2\pi} \Upsilon(\varepsilon) \left(\varepsilon + \frac{1}{\varepsilon} \sum_{0 < |m| \leq \varepsilon^{-1}} \frac{1}{|m|^2} + \frac{1}{\varepsilon^2} \sum_{|m| > \varepsilon^{-1}} \frac{1}{|m|^2(|m_1| + |m_2|)} \right) \\ &\leq C'' \Upsilon(\varepsilon) \left(\varepsilon + \frac{1}{\varepsilon} \log \left(\frac{1}{\varepsilon} \right) + \frac{1}{\varepsilon} \right), \end{aligned}$$

for some $C'' < +\infty$, all this being valid for small enough values of $\varepsilon > 0$. □

Proof of Lemma 6.4. Since $\kappa(x) = e^{-\frac{1}{2}|x|^2}$ the Fourier transform of κ is

$$\hat{\kappa} : \xi \in \mathbb{R}^2 \mapsto \frac{1}{4\pi^2} \int_{\mathbb{R}^2} e^{-i\langle \xi, x \rangle} \kappa(x) dx = \frac{1}{2\pi} e^{-\frac{1}{2}|\xi|^2},$$

so that

$$\rho^\varepsilon(\xi) = \sqrt{\sum_{m \in \mathbb{Z}^2} e^{-\frac{1}{2}|\xi - 2\pi\varepsilon^{-1}m|^2}}.$$

Let $P_1 = Id$, $P_2 = \Delta$ and $P_3 = 2\partial_1\Delta$. Then:

$$\Upsilon(\varepsilon) = \max\left(\int_{\varepsilon^{-1}\mathbb{T}^2} |P_1\rho^\varepsilon(\xi)|d\xi, \int_{\varepsilon^{-1}\mathbb{T}^2} |P_2\rho^\varepsilon(\xi)|d\xi, \int_{\varepsilon^{-1}\mathbb{T}^2} |P_3\rho^\varepsilon(\xi)|d\xi\right).$$

For the last argument of the max we use the fact that ρ^ε remains unchanged when switching the two coordinates of \mathbb{R}^2 . We begin by justifying the following claim:

Claim 6.5. *There exists $C' < +\infty$ such that for each $\xi \in \varepsilon^{-1}\mathbb{T}^2$ and each $j \in \{1, 2, 3\}$,*

$$|P_j\rho^\varepsilon(\xi)| \leq C' \left(\sum_{m \in \mathbb{Z}^2} e^{-\frac{1}{13}|\xi - 2\pi\varepsilon^{-1}m|^2} \right)^3.$$

Proof. Elementary algebra shows that for each $j \in \{1, 2, 3\}$, there exists a polynomial function $Q_j : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ of degree at most j such that for each $\xi \in \varepsilon^{-1}\mathbb{T}^2$, $\rho^\varepsilon(\xi)^5 P_j\rho^\varepsilon(\xi)$ equals:

$$\sum_{m_1, m_2, m_3 \in \mathbb{Z}^2} Q_j(\xi - 2\pi\varepsilon^{-1}m_1, \xi - 2\pi\varepsilon^{-1}m_2, \xi - 2\pi\varepsilon^{-1}m_3) \prod_{i=1}^3 e^{-\frac{1}{2}|\xi - 2\pi\varepsilon^{-1}m_i|^2}. \quad (6.9)$$

By using the very crude bound $\forall m \in \mathbb{Z}^2, \rho^\varepsilon(\xi) \geq e^{-\frac{1}{4}|\xi - 2\pi\varepsilon^{-1}m|^2}$, we obtain that for each $\xi \in \varepsilon^{-1}\mathbb{T}^2$, $|P_j\rho^\varepsilon(\xi)|$ is at most:

$$\begin{aligned} & \sum_{m_1, m_2, m_3 \in \mathbb{Z}^2} |Q_j(\xi - 2\pi\varepsilon^{-1}m_1, \xi - 2\pi\varepsilon^{-1}m_2, \xi - 2\pi\varepsilon^{-1}m_3)| \prod_{i=1}^3 e^{-\left(\frac{1}{2} - \frac{5}{3} \times \frac{1}{4}\right)|\xi - 2\pi\varepsilon^{-1}m_i|^2} \\ &= \sum_{m_1, m_2, m_3 \in \mathbb{Z}^2} |Q_j(\xi - 2\pi\varepsilon^{-1}m_1, \xi - 2\pi\varepsilon^{-1}m_2, \xi - 2\pi\varepsilon^{-1}m_3)| \prod_{i=1}^3 e^{-\frac{1}{12}|\xi - 2\pi\varepsilon^{-1}m_i|^2}. \end{aligned}$$

Now, note that there exists a constant $C' < +\infty$ such that, for each $j \in \{1, 2, 3\}$, for each $m_1, m_2, m_3 \in \mathbb{Z}^2$, and for each $\xi \in \mathbb{R}^2$:

$$\left| Q_j(\xi - 2\pi\varepsilon^{-1}m_1, \xi - 2\pi\varepsilon^{-1}m_2, \xi - 2\pi\varepsilon^{-1}m_3) \prod_{i=1}^3 e^{-\frac{1}{12}|\xi - 2\pi\varepsilon^{-1}m_i|^2} \right| \leq C' \prod_{i=1}^3 e^{-\frac{1}{13}|\xi - 2\pi\varepsilon^{-1}m_i|^2},$$

and we are done. \square

Let us use the claim to conclude. Let $m \in \mathbb{Z}^2$ be such that $|m| \geq 2$ and $\xi \in [-\varepsilon^{-1}\pi, \varepsilon^{-1}\pi]^2$. Then, $|\xi - 2\pi\varepsilon^{-1}m| \geq \pi\varepsilon^{-1}|m|$. Therefore:

$$\begin{aligned} \sum_{m \in \mathbb{Z}^2, |m| \geq 2} e^{-\frac{1}{13}|\xi - 2\pi\varepsilon^{-1}m|^2} &\leq \sum_{m \in \mathbb{Z}^2, |m| \geq 2} e^{-\frac{\pi}{13}\varepsilon^{-2}|m|^2} \\ &\leq C'' e^{-\frac{4\pi}{13}\varepsilon^{-2}}, \end{aligned}$$

for some $C'' < +\infty$ and if ε is sufficiently small. Moreover, $\sum_{m \in \mathbb{Z}^2, |m| \leq 1} e^{-\frac{1}{13}|\xi - 2\pi\varepsilon^{-1}m|^2} \leq 5$, hence by expanding the cubed sum in Claim 6.5, we obtain that if ε is sufficiently small, then:

$$|P_j\rho^\varepsilon(\xi)| \leq C''' \left(e^{-\frac{4\pi}{13}\varepsilon^{-2}} + \sum_{m \in \mathbb{Z}^2, |m| \leq 1} e^{-\frac{1}{13}|\xi - 2\pi\varepsilon^{-1}m|^2} \right),$$

for some $C''' < +\infty$. Finally:

$$\begin{aligned} \Upsilon(\varepsilon) &\leq C''' \int_{\varepsilon^{-1}\mathbb{T}^2} \left(e^{-\frac{4\pi}{13}\varepsilon^{-2}} + \sum_{m \in \mathbb{Z}^2, |m| \leq 1} e^{-\frac{1}{13}|\xi - 2\pi\varepsilon^{-1}m|^2} \right) d\xi \\ &\leq C'''(2\pi\varepsilon)^2 e^{-\frac{4\pi}{13}\varepsilon^{-2}} + 5C''' \int_{\mathbb{R}^2} e^{-\frac{1}{13}|\xi|^2} d\xi, \end{aligned}$$

which is less than some absolute constant since we consider only ε small. \square

7 Sprinkling discretization scheme

In this section, we prove Proposition 2.22. We do not rely on arguments from other sections. Recall that $\mathcal{T} = (\mathcal{V}, \mathcal{E})$ is the square face-centered lattice and that $\mathcal{T}^\varepsilon = (\mathcal{V}^\varepsilon, \mathcal{E}^\varepsilon)$ denotes \mathcal{T} scaled by ε . Given an edge $e = (x, y)$, we take the liberty of writing “ $z \in e$ ” as a shorthand for “ $\exists t \in [0, 1]$ such that $z = ty + (1-t)x$ ”. For each $R > 0$ and $\varepsilon > 0$, let $\mathcal{T}_R^\varepsilon = (\mathcal{E}_R^\varepsilon, \mathcal{V}_R^\varepsilon)$ denote the sublattice of \mathcal{T}^ε generated by the edges e that intersect $[0, 2R] \times [0, R]$.

In this section, we never use the fact that our Gaussian field is non-degenerate. As we shall see at the end of this section, Proposition 2.22 is an easy consequence of the following approximation estimate.

Proposition 7.1. *Assume that f satisfies Condition 2.2 and that it is a.s. C^2 . Let $p > 0$. Given $\varepsilon > 0$ and $e = (x, y) \in \mathcal{E}^\varepsilon$, we call $\text{Fold}(e)$ the event that there exists $z \in e$ such that $f(x) \geq -\frac{p}{2}$, $f(y) \geq -\frac{p}{2}$, and $f(z) < -p$. There exist constants $c_2 = c_2(\kappa, p) > 0$ and $\varepsilon_0 = \varepsilon_0(\kappa, p) > 0$ such that for each $\varepsilon \in]0, \varepsilon_0]$ we have:*

$$\forall e \in \mathcal{E}^\varepsilon, \mathbb{P}[\text{Fold}(e)] \leq C_1 \exp(-c_2\varepsilon^{-4}).$$

A key ingredient in proving this inequality will be the Borell-Tsirelson-Ibragimov-Sudakov (or BTIS) inequality (see Theorem 2.9 of [AW09]).

Proof of Proposition 7.1. Let us fix $e = (x, y) \in \mathcal{E}^\varepsilon$ and consider the vector v defined by $\varepsilon v = y - x$. On the event $\text{Fold}(e, \varepsilon)$, by Taylor’s inequality applied to f between points x, z and y , there exist $w_1, w_2 \in e$ such that $\partial_v f(w_1) > \frac{p|v|}{2\varepsilon}$ and $\partial_v f(w_2) < -\frac{p|v|}{2\varepsilon}$. Applying Taylor’s estimate to $\partial_v f$ between w_1 and w_2 we conclude that there exists $w_3 \in e$ such that $|\partial_{v,v}^2 f(w_3)| > \frac{p|v|}{\varepsilon^2}$. Hence:

$$\mathbb{P}[\text{Fold}(e)] \leq \mathbb{P}\left[\sup_{w \in e} |\partial_{v,v}^2 f(w)| > \frac{p|v|}{\varepsilon^2}\right].$$

Let $x_t = \varepsilon tv$ and $g_\varepsilon^v(t) = \partial_{v,v}^2 f(x_t)$. By translation invariance of f , we have:

$$\mathbb{P}\left[\sup_{w \in e} |f_{vv}(w)| > \frac{p|v|}{\varepsilon^2}\right] = \mathbb{P}\left[\sup_{t \in [0,1]} |g_\varepsilon^v(t)| > \frac{p|v|}{\varepsilon^2}\right].$$

The strategy is to apply the BTIS inequality to g_ε^v . Note that g_ε^v is a centered Gaussian field on $[0, 1]$ which is a.s. bounded. Hence, by Theorem 2.9 of [AW09], $\mathbb{E}\left[\sup_{t \in [0,1]} |g_\varepsilon^v(t)|\right] < +\infty$. Note that $\mathbb{E}\left[\sup_{t \in [0,1]} |g_\varepsilon^v(t)|\right]$ is non-decreasing in ε , let $C_4 = \max_{v' \in \Gamma} \mathbb{E}\left[\sup_{t \in [0,1]} |g_1^{v'}(t)|\right]$,

and choose $\varepsilon_0 \in]0, 1]$ sufficiently small so that $\min_{v' \in \Gamma} \frac{p|v'|}{\varepsilon_0^2} > 2C_4$. Note that, by translation invariance:

$$\sigma^2 := \sup_{t \in [0,1]} \text{Var}(g_\varepsilon^v(t)) = \text{Var}(g_\varepsilon^v(0)) = \partial_{v,v,v,v}^4 \kappa(0).$$

If $\partial_{v,v,v,v}^4 \kappa(0) = 0$, then $g_\varepsilon^v \equiv 0$ a.s. and we are done. Assume now that $\partial_{v,v,v,v}^4 \kappa(0) > 0$. Then, by the BTIS inequality (see Theorem 2.9 in [AW09]), for each $\varepsilon \in]0, \varepsilon_0]$:

$$\begin{aligned} \mathbb{P} \left[\sup_{t \in [0,1]} |g_\varepsilon^v(t)| > \frac{p|v|}{\varepsilon^2} \right] &\leq 2 \mathbb{P} \left[\sup_{t \in [0,1]} g_\varepsilon^v(t) > \frac{p|v|}{\varepsilon^2} \right] \\ &\leq 2 \mathbb{P} \left[\left| \sup_{t \in [0,1]} g_\varepsilon^v(t) - \mathbb{E} \left[\sup_{t \in [0,1]} g_\varepsilon^v(t) \right] \right| > \frac{p|v|}{2\varepsilon^2} \right] \\ &\leq 4 \exp \left(-\frac{1}{2\sigma^2} \left(\frac{p|v|}{2\varepsilon^2} \right)^2 \right). \end{aligned}$$

Since v can take only finitely many values, we have obtain what we want. \square

Proof of Proposition 2.22. By Proposition 7.1 (and by a simple union bound), it is enough to prove that $\text{Cross}_{p/2}^\varepsilon(2R, R) \cap (\cup_e \text{Fold}(e))^c \subseteq \text{Cross}_p(2R, R)$, where the union is over each $e \in \mathcal{E}_R^\varepsilon$. Assume that for every $e \in \mathcal{E} \text{Fold}(e)$ is not satisfied. Then, for each edge $e = (x, y) \in \mathcal{E}_R^\varepsilon$ which is colored black in the discrete model of parameter $p/2$, and for each $z \in e$, we have $f(z) \geq -p$. In other words, each black edge is contained in \mathcal{D}_p . If in addition $\text{Cross}_{p/2}^\varepsilon(2R, R)$ is satisfied, then there exists a crossing of $[0, 2R] \times [0, R]$ from left to right made up of black edges. This crossing belongs to \mathcal{D}_p so that $\text{Cross}_p(2R, R)$ is satisfied. \square

A An additional sub-linearity result for influences

In this section, we prove that geometric influences are sub-linear for semi-algebraic sets. The following results are not useful for the proof of the main results of the article. However, they are useful for the proof of Theorem 2.19 in the case of semi-algebraic sets. This case is included in the theorem in order to provide a more complete picture of the range of application of Theorem 2.19.

Let us first recall their definition. Given $A \subseteq \mathbb{R}^n$, we denote by ∂A its topological boundary. Moreover, for each $k \in \mathbb{N}$, we denote by \mathcal{H}^k the k -dimensional Hausdorff measure. We say that A is semi-algebraic if it belongs to the Boolean algebra generated by sets B of the form $P^{-1}(]0, +\infty[)$ where $P \in \mathbb{R}[X_1, \dots, X_n]$.

We say that A has piecewise C^1 -boundary if there exists $E \subseteq \partial A$ satisfying $\mathcal{H}^{n-1}(E) = 0$ such that for each $x \in \partial A \setminus E$ there exists an open neighborhood $U \subseteq \mathbb{R}^n$ of x and a C^1 -diffeomorphism from U to an open subset $V \subseteq \mathbb{R}^n$ sending $A \cap U$ to the intersection of V and the closed upper-half plane.

The goal of this subsection is to prove the following proposition.

Proposition A.1. *Let Σ be a symmetric positive definite $n \times n$ matrix. Let $\mu = \mathcal{N}(0, \Sigma)$ and let λ be density of μ . Let $A \subseteq \mathbb{R}^n$ be a semi-algebraic set. Then, for each $v \in \mathbb{R}^n$:*

$$\exists \lim_{r \downarrow 0} \frac{1}{r} \int_{A+[-r,r]v \setminus A} \lambda(x) dx = I_{v,\mu}(A),$$

and we have the following properties.¹⁸

$$\begin{aligned}\forall r \in \mathbb{R}, v \in \mathbb{R}^n, I_{rv, \mu}(A) &= |r| I_{v, \mu}(A), \\ \forall u, v \in \mathbb{R}^n, I_{u+v, \mu}(A) &\leq I_{u, \mu}(A) + I_{v, \mu}(A).\end{aligned}$$

The regular case of Proposition 5.3 follows directly from this result. We prove Proposition A.1 at the end of this subsection.

Remark A.2. As is apparent in its proof, the assumptions made in Proposition A.1 can be substantially weakened. For the sake of simplicity, we did not attempt to prove it in full generality.

Definition A.3. Let $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ be a measurable function. For any $\delta > 0$, we denote by $\bar{\varphi}_\delta$ the function defined by $\forall x \in \mathbb{R}^n, \bar{\varphi}_\delta(x) = \sup_{|y-x| < \delta} |\varphi(y)|$. A set $A \subseteq \mathbb{R}^n$ will be called **φ -tame** if A has piecewise C^1 -boundary and if there exists $\delta > 0$ such that:

$$\int_{\partial A} \bar{\varphi}_\delta(x) d\mathcal{H}^{n-1}(x) < +\infty.$$

Here, the integral coincides with the usual surface-area integral $\int \bar{\varphi}_\delta d\sigma_S$ on S the C^1 part of ∂A .

A key step in the proof of Proposition A.1 is the following result.

Proposition A.4. Fix $\varphi \in C(\mathbb{R}^n, \mathbb{R})$. Let $A \subseteq \mathbb{R}^n$ be a φ -tame set. Let ν be the outward normal vector of A defined \mathcal{H}^{n-1} -almost everywhere on ∂A . Then, for each $v \in \mathbb{R}^n$:

$$J_{v, \varphi}(A) := \lim_{r \downarrow 0} r^{-1} \int_{(A+[-r, r]v) \setminus A} \varphi(x) dx = \int_{\partial A} |v \cdot \nu(x)| \varphi(x) d\mathcal{H}^{n-1}(x).$$

The following result is an immediate consequence of the proposition.

Corollary A.5. Fix $\varphi \in C(\mathbb{R}^n, \mathbb{R}_+)$. Let $A \subseteq \mathbb{R}^n$ be a φ -tame set. Then, we have the following relations.

$$\begin{aligned}\forall \lambda \in \mathbb{R}, v \in \mathbb{R}^n, J_{\lambda v, \varphi}(A) &= |\lambda| J_{v, \varphi}(A), \\ \forall u, v \in \mathbb{R}^n, J_{u+v, \varphi}(A) &\leq J_{u, \varphi}(A) + J_{v, \varphi}(A).\end{aligned}$$

In order to prove Proposition A.4, we introduce the following notation. Let $\pi_1 : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ (respectively $\pi_2 : \mathbb{R}^n \rightarrow \mathbb{R}$) be the projection on the first $n-1$ coordinates (respectively last coordinate). Given $x \in \mathbb{R}^n$, we write $\tilde{x} := \pi_1(x)$. Given $A \subseteq \mathbb{R}^n$ and $\tilde{x} \in \mathbb{R}^{n-1}$ and $r > 0$, we write $A_r(\tilde{x}) = \pi_2(\pi_1^{-1}(\tilde{x}) \cap ((A + [-r, r]e_n) \setminus A)) = \{x_n \in \mathbb{R} : (\tilde{x}, x_n) \in (A + [-r, r]e_n) \setminus A\}$.

The proof of Proposition A.4 will rely on two lemmas which we state along the way and prove at the end of the subsection.

Proof of Proposition A.4. First note that it is enough to prove the result for φ non-negative and for $v \in \mathbb{S}^{n-1}$. Since the statement is invariant by rotations, it is enough to prove the result for $v = e_n := (0, \dots, 0, 1)$.

Next, we introduce some notation. From this point on, we assume that A has piecewise C^1 -boundary. Let $E \subseteq \partial A$ such that $\mathcal{H}^{n-1}(E) = 0$ and $S = \partial A \setminus E$ is a C^1 -hypersurface of \mathbb{R}^n .

¹⁸These properties amount to saying that $v \mapsto I_{v, \mu}(A)$ defines a seminorm on \mathbb{R}^n .

Since π_1 is a Lipschitz map, $\mathcal{H}^{n-1}(\pi_1(E)) = 0$. Let $\pi = \pi_1|_S$. This function is of class C^1 so by Sard's lemma the set $\text{Crit}_{\text{val}}(\pi)$ of its critical values has measure 0 in \mathbb{R}^{n-1} . For each $x \in S$, write $B_r(x) = \pi_2(\{\{x\} + [-r, r]e_n\} \setminus A)$. Finally, for any $r > 0$ let:

$$F_r : \mathbb{R}^{n-1} \rightarrow [0, +\infty]$$

$$\tilde{x} \mapsto r^{-1} \int_{A_r(\tilde{x})} \varphi(\tilde{x}, t) dt.$$

For any $r > 0$, by Fubini's theorem,

$$r^{-1} \int_{(A+[-r, r]e_n) \setminus A} \varphi(x) dx = \int_{\mathbb{R}^{n-1}} \left[r^{-1} \int_{A_r(\tilde{x})} \varphi(\tilde{x}, t) dt \right] d\tilde{x} = \int_{\mathbb{R}^{n-1}} F_r(\tilde{x}) d\tilde{x}. \quad (\text{A.1})$$

In order to prove Proposition A.4 we will apply the dominated convergence theorem to the sequence $(F_r)_{r>0}$ as $r \downarrow 0$. In order to do so, we begin by finding a uniform bound on the sequence $(F_r)_{r>0}$ by an integrable function. Let us make two observations:

- Each $\tilde{x} \in \mathbb{R}^{n-1} \setminus (\pi_1(E) \cup \text{Crit}_{\text{val}}(\pi))$, \tilde{x} is a regular value of π so: for each $x \in \pi^{-1}(\tilde{x})$, $e_n \notin T_x S$ and there exists $r' = r'(x) > 0$ such that for each $r \in]0, r'[$:

$$B_r(x) =]x_n, x_n + r] \text{ or } [x_n - r, x_n[\quad (\text{A.2})$$

and

$$\forall y \in \pi^{-1}(\tilde{x}) \setminus \{x\}, B_r(x) \cap B_r(y) = \emptyset.$$

(In particular, $\pi^{-1}(\tilde{x})$ is countable and has no accumulation point.)

- For each $\tilde{x} \in \mathbb{R}^{n-1} \setminus (\pi_1(E) \cup \text{Crit}_{\text{val}}(\pi))$ and each $r > 0$, since $\varphi \geq 0$ and $A_r(\tilde{x}) = \cup_{x \in \pi^{-1}(\tilde{x})} B_r(x) \subseteq \pi_2(\pi^{-1}(\tilde{x}) + [-r, r]e_n)$, we have:

$$F_r(\tilde{x}) = r^{-1} \int_{A_r(\tilde{x})} \varphi(\tilde{x}, t) dt \leq \sum_{x \in \pi^{-1}(\tilde{x})} r^{-1} \int_{B_r(x)} \varphi(\tilde{x}, t) dt \quad (\text{A.3})$$

$$\leq 2 \sum_{x \in \pi^{-1}(\tilde{x})} \bar{\varphi}_r(x) =: 2G_r(\tilde{x}). \quad (\text{A.4})$$

By construction, the sequence $(G_r)_{r>0}$ is pointwise non-increasing in r . We wish to prove that G_r is integrable for r small enough. To do so, we use the following lemma, which is a direct application of the coarea formula.

Lemma A.6. *For any non-negative measurable function $\psi : S \rightarrow \mathbb{R}$, the function $\tilde{x} \mapsto \sum_{x \in \pi^{-1}(\tilde{x})} \psi(x)$ is measurable on $\mathbb{R}^{n-1} \setminus \text{Crit}_{\text{val}}(\pi)$ with values in $[0, +\infty]$ and satisfies:*

$$\int_{\mathbb{R}^{n-1}} \sum_{x \in \pi^{-1}(\tilde{x})} \psi(x) d\tilde{x} = \int_S |\nu(x) \cdot e_n| \psi(x) d\sigma_S(x),$$

where $d\sigma_S$ is the standard surface-area measure on S .

By Lemma A.6, for each $r > 0$ and almost each $\tilde{x} \in \mathbb{R}^{n-1}$:

$$\int_{\mathbb{R}^{n-1}} G_r(\tilde{x}) d\tilde{x} = \int_S |\nu(x) \cdot e_n| \bar{\varphi}_r(x) d\mathcal{H}^{n-1}(x)$$

(remember Definition A.3: the integral above coincide with the integral under $d\sigma_S$). Moreover, since A is φ -tame, there exists $r_0 > 0$ such that:

$$\int_S |\nu(x) \cdot e_n| \bar{\varphi}_{r_0}(x) d\mathcal{H}^{n-1}(x) \leq \int_S \bar{\varphi}_{r_0}(x) |d\mathcal{H}^{n-1}(x)| < +\infty.$$

In summary, for each $r \in]0, r_0]$ and almost every $\tilde{x} \in \mathbb{R}^{n-1}$, by (A.4) we have:

$$F_r(\tilde{x}) \leq 2G_r(\tilde{x}) \leq 2G_{r_0}(\tilde{x}). \quad (\text{A.5})$$

Moreover, G_{r_0} is integrable. This will allow us to apply the dominated convergence theorem once we have established pointwise convergence, which is given by the following lemma:

Lemma A.7. *For a.e. $\tilde{x} \in \mathbb{R}^{n-1}$,*

$$F_r(\tilde{x}) \xrightarrow[r \downarrow 0]{} \sum_{x \in \pi^{-1}(\tilde{x})} \varphi(x).$$

By Lemma A.6, the limit $\sum_{x \in \pi^{-1}(\tilde{x})} \varphi(x)$ is a measurable function of \tilde{x} . By Equation (A.5), the family $(F_r)_{r>0}$ is pointwise bounded for $r \in]0, r_0]$ by an integrable function, namely G_{r_0} . By dominated convergence,

$$\int_{\mathbb{R}^{n-1}} F_r(\tilde{x}) d\tilde{x} \xrightarrow[r \downarrow 0]{} \int_{\mathbb{R}^{n-1}} \sum_{x \in \pi^{-1}(\tilde{x})} \varphi(x) d\tilde{x}.$$

We replace the left-hand side using Equation (A.1) and the right-hand side using Lemma A.6 to get:

$$\int_{A+[-r,r]e_n \setminus A} \varphi(x) dx \xrightarrow[r \downarrow 0]{} \int_S |\nu(x) \cdot e_n| \varphi(x) |dV_S|(x) = \int_{\partial A} |\nu(x) \cdot e_n| \varphi(x) d\mathcal{H}^{n-1}(x).$$

□

Proof of Lemma A.6. Let $\psi : S \rightarrow \mathbb{R}$ be a measurable non-negative function. As a C^1 -hypersurface of \mathbb{R}^n , S is naturally endowed with a Riemannian metric by restriction of the Euclidean scalar product. By the coarea formula (see Corollary 13.4.6 of [BZ88]) applied to π , the function

$$\begin{aligned} \mathbb{R}^{n-1} &\rightarrow \mathbb{R} \\ \tilde{x} &\mapsto \sum_{x \in \pi^{-1}(\tilde{x})} \psi(x) \end{aligned}$$

is measurable, and:

$$\int_{\mathbb{R}^{n-1}} \sum_{x \in \pi^{-1}(\tilde{x})} \psi(x) d\tilde{x} = \int_S |J_\pi(x)| \psi(x) d\mathcal{H}^{n-1}(x),$$

where $J_\pi(x)$ is the determinant of the matrix of $d_x\pi$ between orthonormal bases of T_xS and \mathbb{R}^{n-1} . The map $d_x\pi$ is the projection on the first $n-1$ coordinates. Based on this information, we claim that $J_\pi(x) = \pm\nu_n(x) = \pm\nu(x) \cdot e_n$. If $T_xS = \mathbb{R}^{n-1} \times \{0\}$ then $d_x\pi$ acts as the identity so $J_\pi(x) = 1$. On the other hand, if $T_xS \neq \mathbb{R}^{n-1} \times \{0\}$, then the intersection of these two hyperplanes is a linear subspace $H_x \subseteq \mathbb{R}^n$ of dimension $n-2$. Let $u(x)$ (respectively $v(x)$) be a unit vector spanning the orthogonal to H_x in T_xS (respectively $\mathbb{R}^{n-1} \times \{0\}$). Since $d_x\pi$ is an orthogonal projection,

it preserves H_x as well as its orthogonal, so $J_\pi(x) = \pm d_x \pi(u(x)) \cdot v(x) = \pm u(x) \cdot v(x)$. Note that the vectors $u(x), \nu(x), e_n, v(x)$ all belong to the plane orthogonal to H_x in \mathbb{R}^n . Since $v(x)$ is orthogonal to e_n and $u(x)$ is orthogonal to $\nu(x)$, $u(x) \cdot v(x) = \pm \nu(x) \cdot e_n$. Thus, $J_\pi(x) = \pm \nu(x) \cdot e_n$ as announced. Hence,

$$\int_{\mathbb{R}^{n-1}} \sum_{x \in \pi^{-1}(\tilde{x})} \psi(x) d\tilde{x} = \int_S |e_n \cdot \nu(x)| \psi(x) d\mathcal{H}^{n-1}(x).$$

□

Proof of Lemma A.7. First note that, by Equation (A.2) (and since φ is continuous), for each $\tilde{x} \in \mathbb{R}^{n-1} \setminus (\pi_1(E) \cup \text{Crit}_{\text{val}}(\pi))$:

$$r^{-1} \int_{B_r(x)} \varphi(\tilde{x}, t) dt \xrightarrow[r \downarrow 0]{} \varphi(x). \quad (\text{A.6})$$

Let us use (A.6) and (A.3) to prove that if for a.e. $\tilde{x} \in \mathbb{R}^{n-1} \setminus (\pi_1(E) \cup \text{Crit}_{\text{val}}(\pi))$, we have:

$$\limsup_{r \downarrow 0} r^{-1} \int_{A_r(\tilde{x})} \varphi(\tilde{x}, t) dt \leq \sum_{x \in \pi^{-1}(\tilde{x})} \varphi(x). \quad (\text{A.7})$$

To prove the above, one can use the dominated convergence theorem, which can be applied by noting that for every $r \in]0, r_0]$, $r^{-1} \int_{B_r(x)} \varphi(\tilde{x}, t) dt \leq \bar{\varphi}_{r_0}(x)$ and that $\sum_{x \in \pi^{-1}(\tilde{x})} \bar{\varphi}_{r_0}(x) = G_{r_0}(\tilde{x})$ is a.e. finite since it is integrable.

Let us now study the liminf. Equation (A.2) implies that, for each $\tilde{x} \in \mathbb{R}^{n-1} \setminus \text{Crit}_{\text{val}}(\pi)$ and each $y_1, \dots, y_k \in \pi^{-1}(x)$, there exists $r_1 > 0$ such that for each $r \in]0, r_1]$, the intervals $B_r(y_j)$ for $j = 1, \dots, k$ have length r and are pairwise disjoint. Hence, for each $r \in]0, r_1]$,

$$r^{-1} \int_{A_r(\tilde{x})} \varphi(\tilde{x}, t) dt \geq \sum_{j=1}^k r^{-1} \int_{B_r(y_j)} \varphi(\tilde{x}, t) dt.$$

Taking the liminf followed by the supremum over all such y_1, \dots, y_k we have, by Equation (A.6):

$$\liminf_{r \downarrow 0} r^{-1} \int_{A_r(\tilde{x})} \varphi(\tilde{x}, t) dt \geq \sum_{x \in \pi^{-1}(\tilde{x})} \varphi(x).$$

Together with Equation (A.7) this concludes the proof of the lemma. □

We are now ready to prove Proposition A.1

Proof of Proposition A.1. Let $P \in \mathbb{R}[X_1, \dots, X_n]$ and let $A = P^{-1}(]0, +\infty[)$. Then A has piecewise C^1 -boundary $\tilde{Y} := \partial A = P^{-1}(0)$, which is the image of a projective algebraic hypersurface $Y \subseteq \mathbb{R}P^n$ by a standard chart.¹⁹ Let $g \in C^\infty(\mathbb{R}^n)$ be the density of the Fubini-Study volume measure on $\mathbb{R}P^n$ when read in a canonical chart. Since \tilde{Y} is piecewise C^1 , we have:

$$\int_{\tilde{Y}} g(x) d\mathcal{H}^{n-1}(x) = \int_{\tilde{Y}} g(y) d\sigma(y) = \text{Vol}_{\mathbb{R}}(Y) < +\infty,$$

¹⁹Indeed, \tilde{Y} is a finite union of irreducible components each of whose singular set is a proper algebraic variety, and thus has (algebraic) dimension at most $n - 2$. Moreover the intersection of two such components has also at most dimension $n - 2$. By induction, each of these algebraic submanifolds is a finite union of smooth manifolds of dimension $\leq n - 2$ and so has Hausdorff dimension $\leq n - 2$.

where $\text{Vol}_{\mathbb{R}}(Y)$ is the volume of Y with respect to the Fubini-Study measure (which is finite: see for instance [Mon12], Corollary A.2) and $d\sigma$ is the usual area measure (see Theorem 3.2.3 of [Fed69]). For each $x \in \mathbb{R}^n$, $g(x) = \frac{4}{(1+|x|^2)^2}$. In particular, for each $\delta > 0$ there exists $C_\delta < +\infty$ such that $\bar{g}_\delta \leq C_\delta g$. This shows that A is g -tame. Moreover, note that there exists a constant $C < +\infty$ such that $0 \leq \lambda \leq Cg$, where λ is the density of our Gaussian measure μ . Thus, A is λ -tame. Since this is valid for any set of the form $P^{-1}([0, +\infty[)$ and since the set of tame sets is clearly stable by complement, finite union and finite intersection, any semi-algebraic set is λ -tame. We conclude by applying Corollary A.5 to $\varphi = \lambda$. \square

References

- [Ale96] Kenneth S. Alexander. Boundedness of level lines for two-dimensional random fields. *The Annals of Probability*, 24(4):1653–1674, 1996.
- [ATT16] Daniel Ahlberg, Vincent Tassion, and Augusto Teixeira. Sharpness of the phase transition for continuum percolation in \mathbb{R}^2 . *arXiv preprint arXiv:1605.05926*, 2016.
- [AW09] Jean-Marc Azaïs and Mario Wschebor. *Level sets and extrema of random processes and fields*. John Wiley & Sons, Inc., Hoboken, NJ, 2009.
- [BDC12] Vincent Beffara and Hugo Duminil-Copin. The self-dual point of the two-dimensional random-cluster model is critical for $q \geq 1$. *Probability Theory and Related Fields*, 153(3):511–542, 2012.
- [BG16] Vincent Beffara and Damien Gayet. Percolation of random nodal lines. *arXiv preprint arXiv:1605.08605*, to appear in *Inst. Hautes Etudes Sci. Publ. Math.*, 2016.
- [BG17] Vincent Beffara and Damien Gayet. Percolation without FKG. *arXiv preprint arXiv:1710.10644*, 2017.
- [BKK⁺92] Jean Bourgain, Jeff Kahn, Gil Kalai, Yitzhak Katznelson, and Nathan Linial. The influence of variables in product spaces. *Israel Journal of Mathematics*, 77(1-2):55–64, 1992.
- [BM17] Dmitry Beliaev and Stephen Muirhead. Discretisation schemes for level sets of planar Gaussian fields. *arXiv preprint arXiv:1702.02134*, 2017.
- [BMW17] Dmitry Beliaev, Stephen Muirhead, and Igor Wigman. Russo-Seymour-Welsh estimates for the kostlan ensemble of random polynomials. *arXiv preprint arXiv:1709.08961*, 2017.
- [BR06a] Béla Bollobás and Oliver Riordan. The critical probability for random Voronoi percolation in the plane is $1/2$. *Probability theory and related fields*, 136(3):417–468, 2006.
- [BR06b] Béla Bollobás and Oliver Riordan. *Percolation*. Cambridge University Press, 2006.
- [BR06c] Béla Bollobás and Oliver Riordan. Sharp thresholds and percolation in the plane. *Random Structures & Algorithms*, 29(4):524–548, 2006.
- [BR06d] Béla Bollobás and Oliver Riordan. A short proof of the Harris–Kesten theorem. *Bulletin of the London Mathematical Society*, 38(3):470–484, 2006.

- [BS07] Eugene Bogomolny and Charles Schmit. Random wavefunctions and percolation. *Journal of Physics A: Mathematical and Theoretical*, 40(47):14033, 2007.
- [BZ88] Yurii D. Burago and Viktor A. Zalgaller. *Geometric inequalities*. Grundlehren der mathematischen Wissenschaften, Vol. 285 v. 285. Springer, 1988.
- [CEL12] Dario Cordero-Erausquin and Michel Ledoux. Hypercontractive measures, tala-grand’s inequality, and influences. In *Geometric aspects of functional analysis*, pages 169–189. Springer, 2012.
- [CL09] Elliott Ward Cheney and William Allan Light. *A course in approximation theory*, volume 101. American Mathematical Soc., 2009.
- [DCRT16] Hugo Duminil-Copin, Aran Raoufi, and Vincent Tassion. A new computation of the critical point for the planar random cluster model with $q \geq 1$. *arXiv preprint arXiv:1604.03702*, 2016.
- [DCRT17a] Hugo Duminil-Copin, Aran Raoufi, and Vincent Tassion. Exponential decay of connection probabilities for subcritical Voronoi percolation in \mathbb{R}^d . *arXiv preprint arXiv:1705.07978*, 2017.
- [DCRT17b] Hugo Duminil-Copin, Aran Raoufi, and Vincent Tassion. Sharp phase transition for the random-cluster and Potts models via decision trees. *arXiv preprint arXiv:1705.03104*, 2017.
- [Fed69] Herbert Federer. *Geometric measure theory*. Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.
- [GG06] Benjamin T. Graham and Geoffrey R. Grimmett. Influence and sharp-threshold theorems for monotonic measures. *The Annals of Probability*, pages 1726–1745, 2006.
- [GG11] Benjamin T. Graham and Geoffrey R. Grimmett. Sharp thresholds for the random-cluster and Ising models. *The Annals of Applied Probability*, pages 240–265, 2011.
- [Gra09] Loukas Grafakos. *Classical Fourier analysis*. Graduate Texts in Mathematics 249. Springer, 2nd edition, 2009.
- [Gri99] Geoffrey R. Grimmett. *Percolation (Grundlehren der mathematischen Wissenschaften)*. Springer: Berlin, Germany, 1999.
- [Gri10] Geoffrey Grimmett. Probability on graphs. *Lecture Notes on Stochastic Processes on Graphs and Lattices*. Statistical Laboratory, University of Cambridge, 2010.
- [GS14] Christophe Garban and Jeffrey Steif. *Noise sensitivity of Boolean functions and percolation*. Cambridge University Press, 2014.
- [Har60] Theodore E. Harris. A lower bound for the critical probability in a certain percolation process. 56(01):13–20, 1960.
- [Kes80] Harris Kesten. The critical probability of bond percolation on the square lattice equals $1/2$. *Comm. Math. Phys.* 74, no. 1, 1980.

- [KKL88] Jeff Kahn, Gil Kalai, and Nathan Linial. The influence of variables on Boolean functions. In *Foundations of Computer Science, 1988., 29th Annual Symposium on*, pages 68–80. IEEE, 1988.
- [KMS12] Nathan Keller, Elchanan Mossel, and Arnab Sen. Geometric influences. *The Annals of Probability*, pages 1135–1166, 2012.
- [KMS14] Nathan Keller, Elchanan Mossel, and Arnab Sen. Geometric influences ii: Correlation inequalities and noise sensitivity. In *Annales de l’Institut Henri Poincaré, Probabilités et Statistiques*, volume 50, pages 1121–1139. Institut Henri Poincaré, 2014.
- [Mon12] Arnaud Moncet. Real versus complex volumes on real algebraic surfaces. *Int. Math. Res. Not.*, 2012(16):3723–3762, 2012.
- [MS83a] Stanislav A. Molchanov and A.K. Stepanov. Percolation in random fields. i. *Theoretical and Mathematical Physics*, 55(2):478–484, 1983.
- [MS83b] Stanislav A. Molchanov and A.K. Stepanov. Percolation in random fields. ii. *Theoretical and Mathematical Physics*, 55(3):592–599, 1983.
- [MS86] Stanislav A. Molchanov and A.K. Stepanov. Percolation in random fields. iii. *Theoretical and Mathematical Physics*, 67(2):434–439, 1986.
- [NS15] Fedor Nazarov and Mikhail Sodin. Asymptotic laws for the spatial distribution and the number of connected components of zero sets of Gaussian random functions. *arXiv preprint arXiv:1507.02017*, 2015.
- [Pit82] Loren D. Pitt. Positively correlated normal variables are associated. *The Annals of Probability*, pages 496–499, 1982.
- [Rod15] Pierre-François Rodriguez. A 0-1 law for the massive Gaussian free field. *Probability Theory and Related Fields*, pages 1–30, 2015.
- [Rud] Walter Rudin. *Fourier analysis on groups*. Interscience Tracts in Pure and Applied Mathematics. 12., Year = 1962,.
- [Rus78] Lucio Russo. A note on percolation. *Probability Theory and Related Fields*, 43(1):39–48, 1978.
- [RV17] Alejandro Rivera and Hugo Vanneuville. Quasi-independence for nodal lines. *arXiv preprint*, 2017.
- [She07] Scott Sheffield. Gaussian free fields for mathematicians. *Probab. Theory Related Fields*, 139(3-4):521–541, 2007.
- [SW78] Paul D. Seymour and Dominic J.A. Welsh. Percolation probabilities on the square lattice. *Annals of Discrete Mathematics*, 3:227–245, 1978.
- [Tas16] Vincent Tassion. Crossing probabilities for Voronoi percolation. *The Annals of Probability*, 44(5):3385–3398, 2016.

Alejandro Rivera

Univ. Grenoble Alpes

UMP5582, Institut Fourier, 38000 Grenoble, France

alejandro.rivera@univ-grenoble-alpes.fr

Supported by the ERC grant Liko No 676999

Hugo Vanneuille

Univ. Lyon 1

UMR5208, Institut Camille Jordan, 69100 Villeurbanne, France

vanneuille@math.univ-lyon1.fr

<http://math.univ-lyon1.fr/~vanneuille/>

Supported by the ERC grant Liko No 676999