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# GLOBAL EXISTENCE AND BLOWUP FOR A CLASS OF THE FOCUSING NONLINEAR SCHRÖDINGER EQUATION WITH INVERSE-SQUARE POTENTIAL

VAN DUONG DINH

Abstract. We consider a class of the focusing nonlinear Schrödinger equation with inverse-square potential

$$i\partial_t u + \Delta u - c|x|^{-2}u = -|u|^\alpha u, \quad u(0) = u_0 \in H^1, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^d,$$

where $d \geq 3$, $\frac{4}{d} \leq \alpha \leq \frac{4}{d-2}$ and $c \neq 0$ satisfies $c > -\lambda(d) := -\left(\frac{d-2}{2}\right)^2$. In the mass-critical case $\alpha = \frac{4}{d}$, we prove the global existence and blowup below ground states for the equation with $d \geq 3$ and $c > -\lambda(d)$. In the mass and energy intercritical case $\frac{4}{d} < \alpha < \frac{4}{d-2}$, we prove the global existence and blowup below the ground state threshold for the equation. This extends similar results of [17] and [21] to any dimensions $d \geq 3$ and a full range $c > -\lambda(d)$. We finally prove the blowup below ground states for the equation in the energy-critical case $\alpha = \frac{4}{d-2}$ with $d \geq 3$ and $c > -\frac{d^2+4d}{(d+2)^2}\lambda(d)$.

## 1. Introduction

Consider the Cauchy problem for the focusing nonlinear Schrödinger equation with inverse-square potential

$$\begin{cases} i\partial_t u - P_c u &= -|u|^\alpha u, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^d, \\ u(0) &= u_0, \end{cases} \tag{NLS$_c$}$$

where $u : \mathbb{R}\times\mathbb{R}^d \to \mathbb{C}, u_0 : \mathbb{R}^d \to \mathbb{C}, d \geq 3, \alpha > 0$ and $P_c = -\Delta + c|x|^{-2}$ with $c \neq 0$ satisfies $c > -\lambda(d) := -\left(\frac{d-2}{2}\right)^2$. The case $c = 0$ is the well-known nonlinear Schrödinger equation which has been studied extensively over the last three decades. The nonlinear Schrödinger equation with inverse-square potential (NLS$_c$) appears in a variety of physical settings and is of interest in quantum mechanics (see e.g. [13] and references therein). The study of the (NLS$_c$) has attracted a lot of interest in the past several years (see e.g. [4, 25, 26, 28, 29, 18, 19, 17, 33, 21]).

The operator $P_c$ is the self-adjoint extension of $-\Delta + c|x|^{-2}$. It is well-known that in the range $-\lambda(d) < c < 1 - \lambda(d)$, the extension is not unique (see e.g. [13]). In this case, we do make a choice among possible extensions, such as Friedrichs extension. The restriction on $c$ comes from the sharp Hardy inequality, namely

$$\lambda(d)\int |x|^{-2}|u(x)|^2 dx \leq \int |\nabla u(x)|^2 dx, \quad \forall u \in H^1, \tag{1.1}$$

which ensures that $P_c$ is a positive operator.

Throughout this paper, we denote for $\gamma \in \mathbb{R}$ and $q \in [1,\infty]$ the usual homogeneous and inhomogeneous Sobolev spaces associated to the Laplacian $-\Delta$ by $\dot{W}^{\gamma,q}$ and $W^{\gamma,q}$ respectively. We

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also use $\dot{H}^\gamma := \dot{W}^{\gamma,2}$ and $H^\gamma := W^{\gamma,2}$. Similarly, we define the homogeneous Sobolev space $\dot{W}_c^{\gamma,q}$ associated to $P_c$ by the closure of $C_0^\infty(\mathbb{R}^d\backslash\{0\})$ under the norm

$$\|u\|_{\dot{W}_c^{\gamma,q}} := \|\sqrt{P_c}^\gamma u\|_{L^q}.$$

The inhomogeneous Sobolev space associated to $P_c$ is defined by the closure of $C_0^\infty(\mathbb{R}^d)$ under the norm

$$\|u\|_{W_c^{\gamma,q}} := \|\langle P_c\rangle^\gamma u\|_{L^q},$$

where $\langle\cdot\rangle$ is the Japanese bracket. We abbreviate $\dot{H}_c^\gamma := \dot{W}_c^{\gamma,2}$ and $H_c^\gamma := W_c^{\gamma,2}$. Note that by definition, we have

$$\|u\|_{\dot{H}_c^1}^2 = \int |\nabla u(x)|^2 + c|x|^{-2}|u(x)|^2 dx. \tag{1.2}$$

By the sharp Hardy inequality, we see that for $c > -\lambda(d)$,

$$\|u\|_{\dot{H}_c^1} \sim \|u\|_{\dot{H}^1}.$$

Before stating our results, let us recall some facts for the (NLS$_c$). We firstly note that the (NLS$_c$) is invariant under the scaling,

$$u_\lambda(t,x) := \lambda^{\frac{2}{\alpha}} u(\lambda^2 t, \lambda x), \quad \lambda > 0.$$

An easy computation shows

$$\|u_\lambda(0)\|_{\dot{H}^\gamma} = \lambda^{\gamma+\frac{2}{\alpha}-\frac{d}{2}}\|u_0\|_{\dot{H}^\gamma}.$$

Thus, the critical Sobolev exponent is given by

$$\gamma_{\mathrm{c}} := \frac{d}{2} - \frac{2}{\alpha}. \tag{1.3}$$

Moreover, the (NLS$_c$) has the following conserved quantities:

$$M(u(t)) := \int |u(t,x)|^2 dx = M(u_0), \tag{1.4}$$

$$E_c(u(t)) := \int \frac{1}{2}|\nabla u(t,x)|^2 + \frac{c}{2}|x|^{-2}|u(t,x)|^2 - \frac{1}{\alpha+2}|u(t,x)|^{\alpha+2} dx = E_c(u_0). \tag{1.5}$$

It is convenient to introduce the following numbers:

$$\alpha_\star := \frac{4}{d}, \quad \alpha^\star := \begin{cases} \frac{4}{d-2} & \text{if } d \geq 3, \\ \infty & \text{if } d = 1,2. \end{cases} \tag{1.6}$$

The main purpose of this paper is to study the global existence and blowup for the (NLS$_c$) in the mass-critical (i.e. $\alpha = \alpha_\star$), intercritical (mass-supercritical and energy-subcritical, i.e. $\alpha_\star < \alpha < \alpha^\star$) and energy-critical (i.e. $\alpha = \alpha^\star$) cases.

1.1. **Mass-critical case.** Let us firstly recall known results for the focusing mass-critical nonlinear Schrödinger equation, i.e. $c = 0$ and $\alpha = \alpha_\star$ in (NLS$_c$). One has the following (see e.g. [5, Chapter 6] for more details):

**Theorem 1.1.** *Let $u_0 \in H^1$ and $u$ be the corresponding solution to the mass-critical* (NLS$_0$) *(i.e. $c = 0$ and $\alpha = \alpha_\star$ in* (NLS$_c$)*).*

1. *Global existence* [31]*: If $d \geq 1$ and $\|u_0\|_{L^2} < \|Q_0\|_{L^2}$, where $Q_0$ is the unique positive radial solution to the elliptic equation*

$$\Delta Q_0 - Q_0 + Q_0^{\alpha_\star+1} = 0,$$

*then the solution $u$ exists globally in time and $\sup_{t\in\mathbb{R}}\|u(t)\|_{\dot{H}^1} < \infty$.*

2. *Blowup* [23, 24]*: The solution $u$ blows up in finite time if one of the following conditions holds true:*
   - $d \geq 1, E_0(u_0) < 0$ *and* $xu_0 \in L^2$,
   - $d \geq 2, E_0(u_0) < 0$ *and* $u_0$ *is radial,*
   - $d = 1$ *and* $E_0(u_0) < 0$.

**Remark 1.2.**
1. By the sharp Gagliardo-Nirenberg inequality, the condition $\|u_0\|_{L^2} < \|Q_0\|_{L^2}$ implies $E_0(u_0) > 0$.
2. The condition $\|u_0\|_{L^2} < \|Q_0\|_{L^2}$ is sharp for the global existence in the sense that for any $M_0 > \|Q_0\|_{L^2}$ (even for $M_0 = \|Q_0\|_{L^2}$, see Item 4 below), there exists $u_0 \in H^1$ satisfying $\|u_0\|_{L^2} = M_0$ and the corresponding solution $u$ blows up in finite time.
3. The assumption $E_0(u_0) < 0$ is a sufficient condition for finite time blowup but it is not necessary. One can show that for any $E_0 > 0$, there exists $u_0 \in H^1$ satisfying $E_0(u_0) = E_0$ and the corresponding solution blows up in finite time.
4. It is well-known (see e.g. [32] or [5, Remark 6.7.3]) that there exists a blowup solution to the mass-critical ($\mathrm{NLS}_0$) with $\|u_0\|_{L^2} = \|Q_0\|_{L^2}$ by using the speudo-conformal transformation.
5. Note also that Merle in [22] proved the following classification of miminal mass blowup solutions for the mass-critical ($\mathrm{NLS}_0$): Let $u_0 \in H^1$ be such that $\|u_0\|_{L^2} = \|Q_0\|_{L^2}$. If the corresponding solution blows up in finite time $0 < T < +\infty$, then up to symmetries of the equation, $u(t,x) = S(t-T,x)$, where
$$S(t,x) := \frac{1}{|t|^{\frac{d}{2}}} e^{-i\frac{|x|^2}{4t} + \frac{i}{t}} Q\left(\frac{x}{t}\right). \tag{1.7}$$

Now let us consider $c \neq 0$ satisfy $c > -\lambda(d)$. Let $C_{\mathrm{GN}}(c)$ be the sharp constant to the Gagliardo-Nirenberg inequality associated to the mass-critical ($\mathrm{NLS}_c$), namely,
$$C_{\mathrm{GN}}(c) := \sup\left\{\|f\|_{L^{\alpha_\star+2}}^{\alpha_\star+2} \div \left[\|f\|_{L^2}^{\alpha_\star}\|f\|_{\dot{H}_c^1}^2\right] \;\middle|\; f \in H_c^1\backslash\{0\}\right\}.$$
We will see in Theorem 4.1 that:

1. When $-\lambda(d) < c < 0$, the sharp constant $C_{\mathrm{GN}}(c)$ is attained by a non-negative radial solution to the elliptic equation
$$-P_c Q_c - Q_c + Q_c^{\alpha_\star+1} = 0.$$
2. When $c > 0$, $C_{\mathrm{GN}}(c) = C_{\mathrm{GN}}(0)$, where $C_{\mathrm{GN}}(0)$ is the sharp constant to the standard Gagliardo-Nirenberg inequality
$$\|f\|_{L^{\alpha_\star+2}}^{\alpha_\star+2} \leq C_{\mathrm{GN}}(0)\|f\|_{L^2}^{\alpha_\star}\|f\|_{\dot{H}^1}^2.$$
However, $C_{\mathrm{GN}}(c)$ is never attained. Moreover, if we restrict attention to the Gagliardo-Nirenberg inequality for radial functions, then the sharp constant for the radial Gagliardo-Nirenberg inequality associated to the mass-critical ($\mathrm{NLS}_c$), namely,
$$C_{\mathrm{GN}}(c,\mathrm{rad}) := \sup\left\{\|f\|_{L^{\alpha_\star+2}}^{\alpha_\star+2} \div \left[\|f\|_{L^2}^{\alpha_\star}\|f\|_{\dot{H}_c^1}^2\right] \;\middle|\; f \in H_c^1\backslash\{0\}, f \text{ radial}\right\}$$
is attended by a radial solution $Q_{c,\mathrm{rad}}$ to the elliptic equation
$$-P_c Q_{c,\mathrm{rad}} - Q_{c,\mathrm{rad}} + Q_{c,\mathrm{rad}}^{\alpha_\star+1} = 0.$$
Since $C_{\mathrm{GN}}(c)$ is never attained, the constant $C_{\mathrm{GN}}(c,\mathrm{rad})$ is strictly smaller than $C_{\mathrm{GN}}(c)$.

We will also see in Remark 4.2 that for $c > -\lambda(d)$,
$$C_{\mathrm{GN}}(c) = \frac{\alpha_\star + 2}{2\|Q_{\bar{c}}\|_{L^2}^{\alpha_\star}}, \tag{1.8}$$

where $\overline{c} := \min\{c, 0\}$. Moreover, for $c > 0$,

$$C_{\mathrm{GN}}(c, \mathrm{rad}) = \frac{\alpha_\star + 2}{2\|Q_{c,\mathrm{rad}}\|_{L^2}^{\alpha_\star}}, \tag{1.9}$$

Our first result is the following global existence and blowup for the mass-critical ($\mathrm{NLS}_c$).

**Theorem 1.3.** *Let $d \geq 3$ and $c \neq 0$ be such that $c > -\lambda(d)$. Let $u_0 \in H^1$ and $u$ be the corresponding solution to the mass-critical* ($\mathrm{NLS}_c$) *(i.e. $\alpha = \alpha_\star$).*

1. *If $\|u_0\|_{L^2} < \|Q_{\overline{c}}\|_{L^2}$, then the solution $u$ exists globally and $\sup_{t\in\mathbb{R}} \|u(t)\|_{\dot{H}^1_c} < \infty$.*
2. *If $E_c(u_0) < 0$ and either $xu_0 \in L^2$ or $u_0$ is radial, then the solution $u$ blows up in finite time.*

**Remark 1.4.**

1. In [8], the authors proved the global existence for the mass-critical ($\mathrm{NLS}_c$) with $-\lambda(d) < c < 0$ under the assumption $\|u_0\|_{L^2} < \|Q_c\|_{L^2}$. Here we extend their result to any $c \neq 0$ and $c > -\lambda(d)$.
2. By the sharp Gagliardo-Nirenberg inequality associated to ($\mathrm{NLS}_c$), we see that the condition $\|u_0\|_{L^2} < \|Q_{\overline{c}}\|_{L^2}$ implies that $E_c(u_0) > 0$. Indeed, applying the sharp Gagliardo-Nirenberg inequality and (1.8),
$$\begin{aligned} E_c(u_0) &= \frac{1}{2}\|u_0\|_{\dot{H}^1_c}^2 - \frac{1}{\alpha_\star + 2}\|u_0\|_{L^{\alpha_\star+2}}^{\alpha_\star+2} \\ &\geq \frac{1}{2}\|u_0\|_{\dot{H}^1_c}^2 - \frac{1}{\alpha_\star + 2} C_{\mathrm{GN}}(c)\|u_0\|_{L^2}^{\alpha_\star}\|u_0\|_{\dot{H}^1_c}^2 \\ &\geq \frac{1}{2}\|u_0\|_{\dot{H}^1_c}^2 \Big[1 - \Big(\frac{\|u_0\|_{L^2}}{\|Q_{\overline{c}}\|_{L^2}}\Big)^{\alpha_\star}\Big] > 0. \end{aligned}$$
3. When $-\lambda(d) < c < 0$, the condition $\|u_0\|_{L^2} < \|Q_{\overline{c}}\|_{L^2} = \|Q_c\|_{L^2}$ is sharp for the global existence. In fact, for any $M_c > \|Q_c\|_{L^2}$ (even for $M_c = \|Q_c\|_{L^2}$, see Item 5 below), we can show (see Remark 6.1) that there exists $u_0 \in H^1$ satisfying $\|u_0\|_{L^2} = M_c$ and the corresponding solution $u$ to the mass-critical ($\mathrm{NLS}_c$) blows up in finite time. When $c > 0$, the condition $\|u_0\|_{L^2} < \|Q_0\|_{L^2}$ is not sharp. Indeed, if $u_0$ is radial and satisfies $\|u_0\|_{L^2} < \|Q_{c,\mathrm{rad}}\|_{L^2}$, then the corresponding solution exists globally. Note that $\|Q_{c,\mathrm{rad}}\|_{L^2} > \|Q_0\|_{L^2}$. Moreover, for any $M_c > \|Q_{c,\mathrm{rad}}\|_{L^2}$ (even for $M_c = \|Q_{c,\mathrm{rad}}\|_{L^2}$, see again Item 5 below), we can show (see again Remark 6.1) that there exists $u_0 \in H^1$ radial satisfying $\|u_0\|_{L^2} = M_c$ and the corresponding solution blows up in finite time.
4. The condition $E_c(u_0) < 0$ is a sufficient condition for finite time blowup, but it is not necessary. We will see in Remark 7.1 that for any $E_c > 0$, there exists $u_0 \in H^1$ satisfying $E_c(u_0) = E_c$ and the corresponding solution blows up in finite time.
5. Recently, Csobo-Genoud in [8, Lemma 1] made use of the speudo-conformal transformation to show that for $-\lambda(d) < c < 0$, there exists a blowup solution to the mass-critical ($\mathrm{NLS}_c$) with $\|u_0\|_{L^2} = \|Q_c\|_{L^2}$. By a similar argument, we can show (see Remark 6.2) that for $c > 0$, there exists a radial blowup solution to the mass-critical ($\mathrm{NLS}_c$) with $\|u_0\|_{L^2} = \|Q_{c,\mathrm{rad}}\|_{L^2}$.
6. In [8], the authors also proved the classification of miminal mass blowup solutions for the mass-critical ($\mathrm{NLS}_c$) with $-\lambda(d) < c < 0$. Their result is as follows: Let $u_0 \in H^1$ be such that $\|u_0\|_{L^2} = \|Q_c\|_{L^2}$. If the corresponding solution blows up in finite time $0 < T < +\infty$, then up to symmetries of the equation [1], $u(t,x) = S(t-T, x)$, where $S$ is as in (1.7). We expect that a similar result should hold for radial blowup solutions with $c > 0$.

[1]The ($\mathrm{NLS}_c$) does not enjoy the space translation invariance and the Galilean invariance.

1.2. **Intercritical case.** We next consider the intercritical (i.e. mass-supercritical and energy-subcritical) case. Let us recall known results for the focusing intercritical nonlinear Schrödinger equation, i.e. $c = 0$ and $\alpha_\star < \alpha < \alpha^\star$ in ($\mathrm{NLS}_c$). The global existence, scattering and blowup were studied in [12, 9, 10]. In order to state these results, let us define the following quantities:

$$H(0) := E_0(Q_0)M(Q_0)^\sigma, \quad K(0) := \|Q_0\|_{\dot{H}^1}\|Q_0\|_{L^2}^\sigma,$$

where

$$\sigma := \frac{1-\gamma_c}{\gamma_c} = \frac{4-(d-2)\alpha}{d\alpha-4}. \tag{1.10}$$

and $Q_0$ is the unique positive radial solution to the elliptic equation

$$\Delta Q_0 - Q_0 + Q_0^{\alpha+1} = 0.$$

**Theorem 1.5** ([12, 9, 10])**.** *Let $d \geq 1, u_0 \in H^1$ and $u$ be the corresponding solution to the intercritical* ($\mathrm{NLS}_0$) *(i.e. $c = 0$ and $\alpha_\star < \alpha < \alpha^\star$ in* ($\mathrm{NLS}_c$)*). Suppose that $E_0(u_0)M(u_0)^\sigma < H(0)$.*

1. *If $\|u_0\|_{\dot{H}^1}\|u_0\|_{L^2}^\sigma < K(0)$, then the solution $u$ exists globally in time and*
$$\|u(t)\|_{\dot{H}^1}\|u(t)\|_{L^2}^\sigma < K(0),$$
*for any $t \in \mathbb{R}$. Moreover, the solution $u$ scatters in $H^1$.*
2. *If $\|u_0\|_{\dot{H}^1}\|u_0\|_{L^2}^\sigma > K(0)$ and either*
   - *$xu_0 \in L^2$,*
   - *or $d \geq 3$, $u_0$ is radial,*
   - *or $d = 2$, $u_0$ is radial and $\alpha_\star < \alpha < 4$,*

   *then the solution $u$ blows up in finite time and*
$$\|u(t)\|_{\dot{H}^1}\|u(t)\|_{L^2}^\sigma > K(0),$$
*for any $t$ in the existence time.*

Now let $c \neq 0$ be such that $c > -\lambda(d)$, and let $C_{\mathrm{GN}}(c)$ be the sharp constant in the Gagliardo-Nirenberg inequality associated to the intercritical ($\mathrm{NLS}_c$), namely,

$$C_{\mathrm{GN}}(c) := \sup\left\{\|f\|_{L^{\alpha+2}}^{\alpha+2} \div \left[\|f\|^{\frac{4-(d-2)\alpha}{2}}\|f\|_{\dot{H}_c^1}^{\frac{d\alpha}{2}}\right] \;\middle|\; f \in H_c^1\backslash\{0\}\right\}.$$

We will see in Theorem 4.1 that:

1. When $-\lambda(d) < c < 0$, the sharp constant $C_{\mathrm{GN}}(c)$ is attained by a solution $Q_c$ to the elliptic equation
$$-P_cQ_c - Q_c + Q_c^{\alpha+1} = 0.$$
2. When $c > 0$, $C_{\mathrm{GN}}(c) = C_{\mathrm{GN}}(0)$, where $C_{\mathrm{GN}}(0)$ is again the sharp constant to the standard Gagliardo-Nirenberg inequality
$$\|f\|_{L^{\alpha+2}}^{\alpha+2} \leq C_{\mathrm{GN}}(0)\|f\|_{L^2}^{\frac{4-(d-2)\alpha}{2}}\|f\|_{\dot{H}^1}^{\frac{d\alpha}{2}}.$$
Moreover, $C_{\mathrm{GN}}(c)$ is never attained. However, if we restrict attention to the Gagliardo-Nirenberg inequality for radial functions, then the sharp constant for the radial Gagliardo-Nirenberg inequality associated to the intercritical ($\mathrm{NLS}_c$), namely,
$$C_{\mathrm{GN}}(c,\mathrm{rad}) := \sup\left\{\|f\|_{L^{\alpha+2}}^{\alpha+2} \div \left[\|f\|^{\frac{4-(d-2)\alpha}{2}}\|f\|_{\dot{H}_c^1}^{\frac{d\alpha}{2}}\right] \;\middle|\; f \in H_c^1\backslash\{0\}, f \text{ radial}\right\}$$
is attended by a radial solution $Q_{c,\mathrm{rad}}$ to the elliptic equation
$$-P_cQ_{c,\mathrm{rad}} - Q_{c,\mathrm{rad}} + Q_{c,\mathrm{rad}}^{\alpha+1} = 0.$$
Since $C_{\mathrm{GN}}(c)$ is never attained, the constant $C_{\mathrm{GN}}(c,\mathrm{rad})$ is strictly smaller than $C_{\mathrm{GN}}(c)$.

We define the following quantities:

$$H(c) := E_{\overline{c}}(Q_{\overline{c}})M(Q_{\overline{c}})^{\sigma}, \quad K(c) := \|Q_{\overline{c}}\|_{\dot{H}^1_{\overline{c}}}\|Q_{\overline{c}}\|^{\sigma}_{L^2}, \tag{1.11}$$

where $\overline{c} = \min\{c, 0\}$. Our next result is the following global existence and blowup for the intercritical ($\mathrm{NLS}_c$).

**Theorem 1.6.** *Let $d \geq 3, \alpha_\star < \alpha < \alpha^\star$ and $c \neq 0$ be such that $c > -\lambda(d)$. Let $u_0 \in H^1$ and $u$ be the corresponding solution of the intercritical ($\mathrm{NLS}_c$) (i.e. $\alpha_\star < \alpha < \alpha^\star$). Suppose that*

$$E_c(u_0)M(u_0)^{\sigma} < H(c). \tag{1.12}$$

1. *Global existence: If*

$$\|u_0\|_{\dot{H}^1_c}\|u_0\|^{\sigma}_{L^2} < K(c), \tag{1.13}$$

*then the solution $u$ exists globally in time and*

$$\|u(t)\|_{\dot{H}^1_c}\|u(t)\|^{\sigma}_{L^2} < K(c). \tag{1.14}$$

*for any $t \in \mathbb{R}$.*

2. *Blowup: If*

$$\|u_0\|_{\dot{H}^1_c}\|u_0\|^{\sigma}_{L^2} > K(c), \tag{1.15}$$

*and either $xu_0 \in L^2$ or $u_0$ is radial, then the solution $u$ blows up in finite time and*

$$\|u(t)\|_{\dot{H}^1_c}\|u(t)\|^{\sigma}_{L^2} > K(c), \tag{1.16}$$

*for any $t$ in the existence time.*

**Remark 1.7.**

1. In [17], the authors considered the cubic ($\mathrm{NLS}_c$) in $3D$ (i.e. $\alpha = 2$ and $c > -\frac{1}{4}$) and proved that the global existence as well as scattering hold true under the assumptions (1.12), (1.13) and the blowup holds true under the assumptions (1.12), (1.15). Recently, Lu-Miao-Murphy in [21] proved a similar result as in [17] for the intercritical ($\mathrm{NLS}_c$) with

$$\begin{cases} c > -\frac{1}{4} & \text{if } d = 3, \quad \frac{4}{3} < \alpha \leq 2 \\ c > -\lambda(d) + \left(\frac{d-2}{2} - \frac{1}{\alpha}\right)^2 & \text{if } 3 \leq d \leq 6, \quad \max\left\{\frac{2}{d-2}, \frac{4}{d}\right\} < \alpha < \frac{4}{d-2}. \end{cases}$$

Here we extend the global existence and blowup results of [17, 21] to any dimensions $d \geq 3$ and the full range $c > -\lambda(d)$. We expect that the global solution in Theorem 1.6 scatters in $H^1$ under a certain restriction on $c$. Note that the scattering of global solutions depends heavily on Strichartz estimates which were proved in [4, 2]. In order to successfully apply Strichartz estimates, we need the equivalence of Sobolev norms between the ones associated to $P_c$ and those associated to $-\Delta$ (see Subsection 2.2 for more details). This will lead to a restriction on the validity of $c$.

2. Theorem 1.6 says that the condition (1.13) is sharp for the global existence except for the threshold level

$$\|u_0\|_{\dot{H}^1_c}\|u_0\|^{\sigma}_{L^2} = K(c).$$

It is an interesting open problem to show that there exists blowup solutions to the intercritical ($\mathrm{NLS}_0$) and ($\mathrm{NLS}_c$) equations at this threshold.

3. It is worth mentioning that if the energy of the initial data is negative, then (1.12) is always satisfied. Indeed, we will see in (4.9) that

$$E(Q_{\overline{c}}) = \frac{d\alpha - 4}{2(4 - (d-2)\alpha)}\|Q_{\overline{c}}\|^2_{L^2} = \frac{d\alpha - 4}{2d\alpha}\|Q_{\overline{c}}\|^2_{\dot{H}^1_{\overline{c}}},$$

hence $H(c)$ is always non-negative.

In the case $c > 0$, we have the following improved result for radial solutions.

**Theorem 1.8.** *Let $d \geq 3, \alpha_\star < \alpha < \alpha^\star$ and $c > 0$. Let $u_0 \in H^1$ be radial and $u$ the corresponding solution of the intercritical* (NLS$_c$) *(i.e. $\alpha_\star < \alpha < \alpha^\star$). Suppose that*

$$E_c(u_0)M(u_0)^\sigma < H(c, \mathrm{rad}) =: E_c(Q_{c,\mathrm{rad}})M(Q_{c,\mathrm{rad}})^\sigma. \tag{1.17}$$

1. *Global existence: If*

$$\|u_0\|_{\dot{H}^1_c}\|u_0\|_{L^2}^\sigma < K(c, \mathrm{rad}) =: \|Q_{c,\mathrm{rad}}\|_{\dot{H}^1_c}\|Q_{c,\mathrm{rad}}\|_{L^2}^\sigma, \tag{1.18}$$

*then the solution $u$ exists globally in time and*

$$\|u(t)\|_{\dot{H}^1_c}\|u(t)\|_{L^2}^\sigma < K(c, \mathrm{rad}). \tag{1.19}$$

*for any $t \in \mathbb{R}$.*

2. *Blowup: If*

$$\|u_0\|_{\dot{H}^1_c}\|u_0\|_{L^2}^\sigma > K(c, \mathrm{rad}), \tag{1.20}$$

*then the solution $u$ blows up in finite time and*

$$\|u(t)\|_{\dot{H}^1_c}\|u(t)\|_{L^2}^\sigma > K(c, \mathrm{rad}), \tag{1.21}$$

*for any $t$ in the existence time.*

Since $C_{\mathrm{GN}}(c, \mathrm{rad}) < C_{\mathrm{GN}}(c)$, we will see in Remark 4.2 that $H(c) < H(c, \mathrm{rad})$ and $K(c) < K(c, \mathrm{rad})$. This shows that the class of radial solutions enjoys strictly larger thresholds for the global existence and the blowup.

1.3. **Energy-critical case.** We finally consider the energy-critical case. As above, we recall known results for the focusing energy-critical nonlinear Schrödinger equation, i.e. $c = 0$ and $\alpha = \alpha^\star$ in (NLS$_c$). The global existence, scattering and blowup for the energy-critical (NLS$_0$) were first studied in [14] where the authors proved the global existence, scattering and blowup for the equation under the radial assumption of initial data in dimensions $d = 3, 4, 5$. This was extended to dimensions $d \geq 3$ in [15]. Later, Killip-Visan in [16] proved the global existence and scattering for the equation with general (non-radial) data in dimensions five and higher. They also proved the existence of blowup solutions in dimensions $d \geq 3$. The global existence and scattering for the energy-critical (NLS$_0$) for general data still remain open for $d = 3, 4$. To state their results, we recall the following facts. Let

$$W_0(x) := \Big(1 + \frac{|x|^2}{d(d-2)}\Big)^{-\frac{d-2}{2}}. \tag{1.22}$$

It is well-known that $W$ solves the elliptic equation

$$\Delta W_0 + |W_0|^{\alpha^\star} W_0 = 0.$$

In particular, $W_0$ is a stationary solution to the energy-critical (NLS$_0$). Note that $W_0 \in \dot{H}^1$ but it need not belong to $L^2$.

**Theorem 1.9** ([14])**.** *Let $d = 3, 4, 5$. Let $u_0 \in \dot{H}^1$ be radial and $u$ be the corresponding solution to the energy-critical* (NLS$_0$) *(i.e. $c = 0$ and $\alpha = \alpha^\star$ in* (NLS$_c$)*). Suppose that $E_0(u_0) < E_0(W_0)$.*

1. *If $\|u_0\|_{\dot{H}^1} < \|W_0\|_{\dot{H}^1}$, then the solution $u$ exists globally and scatters in $\dot{H}^1$.*
2. *If $\|u_0\|_{\dot{H}^1} > \|W_0\|_{\dot{H}^1}$ and either $xu_0 \in L^2$ or $u_0 \in H^1$ is radial, then the solution $u$ blows up in finite time.*

**Theorem 1.10** ([16])**.** *Let $u_0 \in \dot{H}^1$ and $u$ be the corresponding solution to the energy-critical* (NLS$_0$)*. Suppose that $E_0(u_0) < E_0(W_0)$.*

1. *If $d \geq 5$ and $\|u_0\|_{\dot{H}^1} < \|W_0\|_{\dot{H}^1}$, then the solution $u$ exists globally and scatters in $\dot{H}^1$.*
2. *If $d \geq 3, \|u_0\|_{\dot{H}^1} > \|W_0\|_{\dot{H}^1}$ and either $xu_0 \in L^2$ or $u_0 \in H^1$ is radial, then the solution $u$ blows up in finite time.*

**Remark 1.11.** Note that the conditions $E_0(u_0) < E_0(W_0)$ and $\|u_0\|_{\dot{H}^1} = \|W_0\|_{\dot{H}^1}$ are incompatible.

Now let $c \neq 0$ satisfy $c > -\lambda(d)$, and let $C_{\mathrm{SE}}(c)$ be the sharp constant in the Sobolev embedding inequality associated to the energy-critical ($\mathrm{NLS}_c$), namely,

$$C_{\mathrm{SE}}(c) := \sup\left\{\|f\|_{L^{\alpha^\star+2}} \div \|f\|_{\dot{H}^1_c} \mid f \in \dot{H}^1_c \backslash \{0\}\right\}.$$

We will see in Theorem 4.3 that:

1. When $-\lambda(d) < c < 0$, the sharp constant $C_{\mathrm{SE}}(c)$ is attained by functions $f(x)$ of the form $\lambda W_c(\mu x)$ for some $\lambda \in \mathbb{C}$ and $\mu > 0$, where
$$W_c(x) := [d(d-2)\beta^2]^{\frac{d-2}{4}} \left[\frac{|x|^{\beta-1}}{1+|x|^{2\beta}}\right]^{\frac{d-2}{2}}, \tag{1.23}$$
with $\beta = 1 - \frac{2\rho}{d-2}$ (see (2.3) for the definition of $\rho$).
2. When $c > 0$, $C_{\mathrm{SE}}(c) = C_{\mathrm{SE}}(0)$, where $C_{\mathrm{SE}}(0)$ is the sharp constant to the standard Sobolev embedding inequality
$$\|f\|_{L^{\alpha^\star+2}} \leq C_{\mathrm{SE}}(0)\|f\|_{\dot{H}^1}.$$
Moreover, $C_{\mathrm{SE}}(c)$ is never attained. Note that the constant $C_{\mathrm{SE}}(0)$ is attained by functions $f(x)$ of a form $\lambda W_0(\mu x + y)$ for some $\lambda \in \mathbb{C}, y \in \mathbb{R}^d$ and $\mu > 0$. However, if we restrict attention to radial functions, then the sharp constant for the radial Sobolev embedding associated to the energy-critical ($\mathrm{NLS}_c$), namely,
$$C_{\mathrm{SE}}(c, \mathrm{rad}) := \sup\left\{\|f\|_{L^{\alpha^\star+2}} \div \|f\|_{\dot{H}^1_c} \mid f \in \dot{H}^1_c \backslash \{0\}, f \text{ radial}\right\}$$
is attained by functions $f(x)$ of the form $\lambda W_c(\mu x)$ for some $\lambda \in \mathbb{C}$ and $\mu > 0$.

Our last result concerns with the blowup for the energy-critical ($\mathrm{NLS}_c$).

**Theorem 1.12.** *Let $d \geq 3$ and $c \neq 0$ be such that $c > -\frac{d^2+4d}{(d+2)^2}\lambda(d)$. Let $u_0 \in \dot{H}^1$ and $u$ be the corresponding solution to the energy-critical* ($\mathrm{NLS}_c$) *(i.e. $\alpha = \alpha^\star$). Suppose that $E_c(u_0) < E_{\overline{c}}(W_{\overline{c}})$ and $\|u_0\|_{\dot{H}^1_c} > \|W_{\overline{c}}\|_{\dot{H}^1_{\overline{c}}}$, where $\overline{c} = \min\{c, 0\}$. If $xu_0 \in L^2$ or $u_0$ is radial, then the solution $u$ blows up in finite time.*

**Remark 1.13.**
1. As in Remark 1.11, the conditions $E_c(u_0) < E_{\overline{c}}(W_{\overline{c}})$ and $\|u_0\|_{\dot{H}^1_c} = \|W_{\overline{c}}\|_{\dot{H}^1_{\overline{c}}}$ are incompatible.
2. Theorem 1.12 was stated in [19] without proof. In this paper, we give a proof for this result. The restriction of $c$ comes from the local theory via Strichartz estimates (see Proposition 3.3).
3. We expect that the global existence as well as scattering for the energy-critical ($\mathrm{NLS}_c$) hold true for $u_0 \in \dot{H}^1$ satisfying $E_c(u_0) < E_{\overline{c}}(W_{\overline{c}})$ and $\|u_0\|_{\dot{H}^1_c} < \|W_{\overline{c}}\|_{\dot{H}^1_{\overline{c}}}$. It is a delicate open problem.

In the case $c > 0$, we have the following blowup result for radial solutions.

**Theorem 1.14.** *Let $d \geq 3$ and $c > 0$. Let $u_0 \in \dot{H}^1$ radial and $u$ be the corresponding solution to the energy-critical* ($\mathrm{NLS}_c$) *(i.e. $\alpha = \alpha^\star$). Suppose that $E_c(u_0) < E_c(W_c)$ and $\|u_0\|_{\dot{H}^1_c} > \|W_c\|_{\dot{H}^1_c}$. Then the solution $u$ blows up in finite time.*

Since $C_{\mathrm{GN}}(c) > C_{\mathrm{GN}}(c, \mathrm{rad})$, we have from (4.19) and (4.22) that $E_0(W_0) < E_c(W_c)$. This shows that the blowup threshold for radial solutions is strictly larger than the one for non-radial solutions.

The paper is organized as follows. In Section 2, we recall some preliminary results related to the $(\mathrm{NLS}_c)$. In Section 3, we recall the local well-posedness for the $(\mathrm{NLS}_c)$ in the energy-subcritical and energy-critical cases. In Section 4, we recall the sharp Gagliardo-Nirenberg inequality and the sharp Sobolev embedding inequality for the $(\mathrm{NLS}_c)$ by using the variational analysis. We next derive the standard virial identity as well as the localized virial estimate in Section 5. Section 6 is devoted to the proofs of global existence results. Finally, we give the proofs of blowup results in Section 7.

## 2. Preliminaries

In the sequel, the notation $A \lesssim B$ denotes an estimate of the form $A \leq CB$ for some constant $C > 0$. The notation $A \sim B$ means $A \lesssim B$ and $B \lesssim A$. The various constant $C$ may change from line to line.

### 2.1. Strichartz estimates.

Let $J \subset \mathbb{R}$ and $p, q \in [1, \infty]$. We define the mixed norm

$$\|u\|_{L^p(J,L^q)} := \Big( \int_J \Big( \int_{\mathbb{R}^d} |u(t,x)|^q dx \Big)^{\frac{1}{q}} \Big)^{\frac{1}{p}}$$

with a usual modification when either $p$ or $q$ are infinity.

**Definition 2.1.** A pair $(p, q)$ is said to be **Schrödinger admissible**, for short $(p, q) \in S$, if

$$(p,q) \in [2,\infty]^2, \quad (p,q,d) \neq (2,\infty,2), \quad \frac{2}{p} + \frac{d}{q} = \frac{d}{2}.$$

We recall Strichartz estimates for the inhomogeneous Schrödinger equation with inverse-square potential.

**Proposition 2.2** (Strichartz estimates [4, 2])**.** *Let $d \geq 3$ and $c > -\lambda(d)$. Let $u$ be a solution to the inhomogeneous Schrödinger equation with inverse-square potential, namely*

$$u(t) = e^{itP_c} u_0 + \int_0^t e^{i(t-s)P_c} F(s) ds,$$

*for some data $u_0, F$. Then, for any $(p, q), (a, b) \in S$,*

$$\|u\|_{L^p(\mathbb{R},L^q)} \lesssim \|u_0\|_{L^2} + \|F\|_{L^{a'}(\mathbb{R},L^{b'})}. \tag{2.1}$$

*Moreover, for any $\gamma \in \mathbb{R}$, $(p, q), (a, b) \in S$,*

$$\|u\|_{L^p(\mathbb{R},\dot{W}_c^{\gamma,q})} \lesssim \|u_0\|_{\dot{H}_c^\gamma} + \|F\|_{L_t^{a'}(\mathbb{R},\dot{W}_c^{\gamma,b'})}. \tag{2.2}$$

*Here $(a, a')$ and $(b, b')$ are conjugate pairs.*

Note that Strichartz estimates for the homogeneous nonlinear Schrödinger equation with inverse-square potential were first proved by Burq-Planchon-Stalker-Zadeh in [4] except the endpoint $(p, q) = (2, \frac{2d}{d-2})$. Recently, Bouclet-Mizutani in [2] proved Strichartz estimates with the full set of Schrödinger admissible pairs for the homogeneous and inhomogeneous nonlinear Schrödinger equation with critical potentials including the inverse-square potential. We refer the reader to [4, 2] for more details.

2.2. **Equivalence of Sobolev norms.** In this subsection, we recall the equivalence between Sobolev norms defined by $P_c$ and the ones defined by the usual Laplacian $-\Delta$. In [4, Proposition 1], the authors proved the following:

$$\|u\|_{\dot{H}^\gamma_c} \sim \|u\|_{\dot{H}^\gamma}, \quad \forall \gamma \in [-1,1].$$

Later, Zhang-Zheng in [33] extended this result to homogeneous Sobolev spaces $\dot{W}^{\gamma,q}_c$ and $\dot{W}^{\gamma,q}$ for $0 \le \gamma \le 1$ and a certain range of $q$. Recently, Killip-Miao-Visan-Zhang-Zheng extended these results to a more general setting. To state their result, let us introduce

$$\rho := \frac{d-2}{2} - \sqrt{\left(\frac{d-2}{2}\right)^2 + c}. \tag{2.3}$$

**Proposition 2.3** (Equivalence of Sobolev norms [18])**.** *Let $d \ge 3, c \ge -\lambda(d), 0 < \gamma < 2$ and $\rho$ be as in (2.3).*

1. *If $1 < q < \infty$ satisfies $\frac{\gamma+\rho}{d} < \frac{1}{q} < \min\left\{1, \frac{d-\rho}{d}\right\}$, then*
$$\|f\|_{\dot{W}^{\gamma,q}} \lesssim \|f\|_{\dot{W}^{\gamma,q}_c},$$
*for all $f \in C_0^\infty(\mathbb{R}^d \backslash \{0\})$.*
2. *If $1 < q < \infty$ satisfies $\max\left\{\frac{\gamma}{d}, \frac{\rho}{d}\right\} < \frac{1}{q} < \min\left\{1, \frac{d-\rho}{d}\right\}$, then*
$$\|f\|_{\dot{W}^{\gamma,q}_c} \lesssim \|f\|_{\dot{W}^{\gamma,q}},$$
*for all $f \in C_0^\infty(\mathbb{R}^d \backslash \{0\})$.*

**Remark 2.4.**

1. When $c > 0$, we have $\rho < 0$. Therefore, $\|u\|_{\dot{W}^{\gamma,q}}$ is equivalent to $\|u\|_{\dot{W}^{\gamma,q}_c}$ provided that $0 < \gamma < 2$ and
$$\frac{\gamma}{d} < \frac{1}{q} < 1 \quad \text{or} \quad 1 < q < \frac{d}{\gamma}. \tag{2.4}$$
2. When $-\lambda(d) \le c < 0$, we have $0 < \rho < \frac{d-2}{2}$. Thus $\|u\|_{\dot{W}^{\gamma,q}} \sim \|u\|_{\dot{W}^{\gamma,q}_c}$ provided that $0 < \gamma < 2$ and
$$\frac{\gamma+\rho}{d} < \frac{1}{q} < \frac{d-\rho}{d} \quad \text{or} \quad \frac{d}{d-\rho} < q < \frac{d}{\gamma+\rho}. \tag{2.5}$$

We next recall the fractional derivative estimates due to Christ-Weinstein [7]. The equivalence of Sobolev spaces given in Proposition 2.3 allows us to use the same estimates for powers of $P_c$ with a certain set of exponents.

**Lemma 2.5** (Fractional derivative estimates)**.**

1. *Let $\gamma \ge 0, 1 < r < \infty$ and $1 < p_1, q_1, p_2, q_2 \le \infty$ satisfying $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$. Then*
$$\||\nabla|^\gamma (fg)\|_{L^r} \lesssim \|f\|_{L^{p_1}} \||\nabla|^\gamma g\|_{L^{q_1}} + \||\nabla|^\gamma f\|_{L^{p_2}} \|g\|_{L^{q_2}}.$$
2. *Let $G \in C^1(\mathbb{C}), \gamma \in (0,1], 1 < r, q < \infty$ and $1 < p \le \infty$ satisfying $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Then*
$$\||\nabla|^\gamma G(f)\|_{L^r} \lesssim \|G'(f)\|_{L^p} \||\nabla|^\gamma f\|_{L^q}.$$

2.3. **Convergences of operators.** In this subsection, we recall the convergence of operators of [19] arising from the fact that $P_c$ does not commute with translations.

**Definition 2.6.** Suppose $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}^d$. We define

$$P_c^n := -\Delta + \frac{c}{|x+x_n|^2}, \quad P_c^\infty := \begin{cases} -\Delta + \frac{c}{|x+x_\infty|^2} & \text{if } x_n \to x_\infty \in \mathbb{R}^d, \\ -\Delta & \text{if } |x_n| \to \infty. \end{cases} \tag{2.6}$$

By definition, we have $P_c[f(x-x_n)] = [P_c^n f](x-x_n)$. The operator $P_c^\infty$ appears as a limit of the operators $P_c^n$ in the following senses:

**Lemma 2.7** (Convergence of operators [19])**.** *Let $d \geq 3$ and $c \neq 0$ be such that $c > -\lambda(d)$. Suppose $(t_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ satisfies $t_n \to t_\infty \in \mathbb{R}$, and $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}^d$ satisfies $x_n \to x_\infty \in \mathbb{R}^d$ or $|x_n| \to \infty$. Then,*

$$\lim_{n\to\infty} \|P_c^n f - P_c^\infty f\|_{\dot{H}^{-1}} = 0, \quad \textit{for all} \quad f \in \dot{H}^1, \tag{2.7}$$

$$\lim_{n\to\infty} \|e^{-it_n P_c^n} f - e^{-it_\infty P_c^\infty} f\|_{\dot{H}^{-1}} = 0, \quad \textit{for all} \quad f \in \dot{H}^{-1}, \tag{2.8}$$

$$\lim_{n\to\infty} \|\sqrt{P_c^n} f - \sqrt{P_c^\infty} f\|_{L^2} = 0, \quad \textit{for all} \quad f \in \dot{H}^1. \tag{2.9}$$

*Furthermore, for any $(p,q) \in S$ with $p \neq 2$,*

$$\lim_{n\to\infty} \|e^{-itP_c^n} f - e^{-itP_c^\infty} f\|_{L^p(\mathbb{R},L^q)} = 0, \quad \textit{for all} \quad f \in L^2. \tag{2.10}$$

We refer the reader to [19, Lemma 3.3] for the proof of Lemma 2.7.

## 3. Local well-posedness

In this section, we study the local well-posedness for the (NLS$_c$) in the energy-subcritical and energy-critical cases. To our knowledge, there are two possible ways to show the local well-posedness in $H^1$ for the classical nonlinear Schrödinger equation (NLS$_0$): the Kato's method and the energy method. The Kato's method is based on the contraction mapping principle using Strichartz estimates. This method is very effective to study the (NLS$_0$) in general Sobolev spaces. The energy method, on the other hand, does not use Strichartz estimates and only allows to prove the existence of solutions in the energy space. But, on one hand, it provides a useful tool to study the (NLS$_0$) in a general domain $\Omega$ where Strichartz estimates are not available in general. We refer the reader to [5] for more details. In the presence of the singular potential $c|x|^{-2}$, even though Strichartz estimates are available (see [4, 2]), the Kato's method does not allow to study the (NLS$_c$) in the energy space with the full range $c > -\lambda(d)$. The reason for this is that the homogeneous Sobolev spaces $\dot{W}_c^{\gamma,q}$ and the usual ones $\dot{W}^{\gamma,q}$ are equivalent only in a certain range of $\gamma$ and $q$ (see Subsection 2.2). Moreover, Okazawa-Suzuki-Yokota in [26] pointed out that the energy method developed by Cazenave is not enough to study the (NLS$_c$) in the energy space. They thus formulated an improved energy method to treat the equation. More precisely, they proved the following:

**Theorem 3.1** ([26])**.** *Let $d \geq 3, c > -\lambda(d)$. Then the* (NLS$_c$) *is well posed in $H^1$:*

- *locally if $0 \leq \alpha < \alpha^\star$,*
- *globally if $0 \leq \alpha < \alpha_\star$.*

*Here $\alpha_\star, \alpha^\star$ are given in* (1.6)*.*

We refer the reader to [26, Theorem 5.1] for the proof of this result.

**Remark 3.2.** 1. The energy method developed by Okazawa-Suzuki-Yokota is only available for the energy-subcritical case (i.e. $\alpha < \alpha^\star$) and not for the energy-critical case $\alpha = \alpha^\star$. The last case should rely on Kato's method (see Proposition 3.3 below).
2. Theorem 3.1 tells us that $H^1$ blowup solutions may occur only on $\alpha_\star \leq \alpha \leq \alpha^\star$.
3. The same well-posedness for the (NLS$_c$) as in Theorem 3.1 holds true when one replaces $\mathbb{R}^d$ by a bounded domain $\Omega$ (see again [26]). In this consideration, Suzuki in [29] proved a similar result for the (NLS$_c$) on $\Omega$ with $c = \lambda(d)$.

We now consider the energy-critical case $\alpha = \alpha^\star$.

**Proposition 3.3.** *Let $d \geq 3, c > -\frac{d^2+4d}{(d+2)^2}\lambda(d)$ and $\alpha = \alpha^\star$. Then for every $u_0 \in H^1$, there exist $T_*, T^* \in (0, \infty]$ and a unique strong $H^1$ solution to the* (NLS$_c$) *defined on the maximal interval $(-T_*, T^*)$. Moreover, if $\|u_0\|_{\dot H^1} < \epsilon$ for some $\epsilon > 0$ small enough, then $T_* = T^* = \infty$ and the solution is scattering in $H^1$, i.e. there exist $u_0^\pm \in H^1$ such that*

$$\lim_{t\to\pm\infty} \|u(t) - e^{itP_c}u_0^\pm\|_{H^1} = 0.$$

Before giving the proof of this result, let us introduce some notations. In this section, we denote

$$p = \frac{2(d+2)}{d-2}, \quad q = \frac{2d(d+2)}{d^2+4}.$$

It is easy to check that $(p, q)$ is a Schrödinger admissible pair and

$$\frac{1}{p} = \frac{1}{q} - \frac{1}{d}.$$

The last equality allows us to use the Sobolev embedding $\dot W^{1,q} \subset L^p$. Moreover, in the view of (2.4) and (2.5), it is easy to check that $\dot W_c^{1,q}$ is equivalent to $W^{1,q}$ provided that $c > -\frac{d^2+4d}{(d+2)^2}\lambda(d)$.
*Proof of Proposition 3.3.* We only consider the positive time, the negative time is similar. Let us define

$$X := \left\{ u \in C(I, H^1) \cap L^p(I, W^{1,q}) \;\middle|\; \|u\|_{L^p(I,\dot W^{1,q})} \leq M \right\}$$

equipped with the distance

$$d(u, v) := \|u - v\|_{L^p(I,L^q)},$$

where $I = [0, T]$ with $T, M > 0$ to be chosen later. By the Duhamel formula, it suffices to prove that the functional

$$\Phi(u)(t) = e^{-itP_c}u_0 + i\int_0^t e^{-i(t-s)P_c}|u(s)|^{\alpha^\star}u(s)ds =: u_{\text{hom}}(t) + u_{\text{inh}}(t)$$

is a contraction on $(X, d)$. Using Strichartz estimates and the fact $\|u\|_{\dot W_c^{1,q}} \sim \|u\|_{\dot W^{1,q}}$, we have

$$\|u_{\text{hom}}\|_{L^p(I,\dot W^{1,q})} \sim \|u_{\text{hom}}\|_{L^p(I,\dot W_c^{1,q})} \lesssim \|u_0\|_{\dot H_c^1} \sim \|u_0\|_{\dot H^1}.$$

This shows that $\|u_{\text{hom}}\|_{L^p(I,\dot W^{1,q})} \leq \epsilon$ for some $\epsilon > 0$ small enough provided that $T$ is small or $\|u_0\|_{\dot H^1}$ is small. By Strichartz estimates, the equivalence $\|u\|_{\dot W_c^{1,q}} \sim \|u\|_{\dot W^{1,q}}$, the fractional derivative estimates and the Sobolev embedding $\dot W^{1,q} \subset L^p$,

$$\begin{aligned}\|u_{\text{inh}}\|_{L^p(I,\dot W_c^{1,q})} \sim \|u_{\text{inh}}\|_{L^p(I,\dot W_c^{1,q})} &\lesssim \||u|^{\alpha^\star}u\|_{L^2(I,\dot W_c^{1,\frac{2d}{d+2}})} \sim \||u|^{\alpha^\star}u\|_{L^2(I,\dot W^{1,\frac{2d}{d+2}})} \\ &\lesssim \|u\|^{\alpha^\star}_{L^p(I,L^p)}\|u\|_{L^p(I,\dot W^{1,q})} \lesssim \|u\|^{\alpha^\star+1}_{L^p(I,\dot W^{1,q})}.\end{aligned}$$

Note that it is easy to check that $\dot{W}_c^{1,\frac{2d}{d+2}} \sim \dot{W}^{1,\frac{2d}{d+2}}$. Similarly,

$$\begin{aligned}\||u|^{\alpha^\star}u - |v|^{\alpha^\star}v\|_{L^2(I,L^{\frac{2d}{d+2}})} &\lesssim \Big(\|u\|_{L^p(I,L^p)}^{\alpha^\star} + \|v\|_{L^p(I,L^p)}^{\alpha^\star}\Big)\|u-v\|_{L^p(I,L^q)}\\ &\lesssim \Big(\|u\|_{L^p(I,\dot{W}^{1,q})}^{\alpha^\star} + \|v\|_{L^p(I,\dot{W}^{1,q})}^{\alpha^\star}\Big)\|u-v\|_{L^p(I,L^q)}.\end{aligned}$$

This implies that for any $u, v \in X$, there exists $C > 0$ independent of $T$ and $u_0 \in H^1$ such that

$$\begin{aligned}\|\Phi(u)\|_{L^p(I,\dot{W}_c^{1,q})} &\le \epsilon + CM^{\alpha^\star+1},\\ d(\Phi(u),\Phi(v)) &\le CM^{\alpha^\star} d(u,v).\end{aligned}$$

If we choose $\epsilon$ and $M$ small so that

$$CM^{\alpha^\star} \le \frac{1}{2}, \quad \epsilon + \frac{M}{2} \le M,$$

then $\Phi$ is a contraction on $(X, d)$. This shows the local existence. It remains to show the scattering for small data. As mentioned above, when $\|u_0\|_{\dot{H}^1}$ is small enough, we can take $T^* = \infty$. By Strichartz estimates, we have for $0 < t_1 < t_2$,

$$\begin{aligned}\|e^{it_2P_c}u(t_2) - e^{it_1P_c}u(t_1)\|_{\dot{H}^1} \sim \|e^{it_2P_c}u(t_2) - e^{it_1P_c}u(t_1)\|_{\dot{H}_c^1} &= \Big\| -i\int_{t_1}^{t_2} e^{isP_c}|u(s)|^{\alpha^\star}u(s)ds\Big\|_{\dot{H}_c^1}\\ &\lesssim \||u|^{\alpha^\star}u\|_{L^2([t_1,t_2],\dot{W}_c^{1,\frac{2d}{d+2}})}\\ &\sim \||u|^{\alpha^\star}u\|_{L^2([t_1,t_2],\dot{W}^{1,\frac{2d}{d+2}})}\\ &\lesssim \|u\|_{L^p([t_1,t_2],\dot{W}^{1,q})}^{\alpha^\star+1}.\end{aligned}$$

Similarly,

$$\|e^{it_2P_c}u(t_2) - e^{it_1P_c}u(t_1)\|_{L^2} \lesssim \||u|^{\alpha^\star}u\|_{L^2([t_1,t_2],L^{\frac{2d}{d+2}})} \lesssim \|u\|_{L^p([t_1,t_2],\dot{W}^{1,q})}^{\alpha^\star}\|u\|_{L^p([t_1,t_2],L^q)}.$$

This shows that

$$\|e^{it_2P_c}u(t_2) - e^{it_1P_c}u(t_1)\|_{H^1} \to 0,$$

as $t_1, t_2 \to +\infty$. Thus the limit $u_0^+ : \lim_{t\to+\infty} e^{itP_c}u(t)$ exists in $H^1$. Moreover,

$$u(t) - e^{-itP_c}u_0^+ = -i\int_t^{+\infty} e^{-i(t-s)P_c}|u(s)|^{\alpha^\star}u(s)ds.$$

Estimating as above, we get

$$\lim_{t\to+\infty}\|u(t) - e^{-itP_c}u_0^+\|_{H^1} = 0.$$

The proof is complete. □

## 4. Variational analysis

In this section, we recall the sharp Gagliardo-Nirenberg and the sharp Sobolev embedding inequalities related to the (NLS$_c$).

Let us start with the following sharp Gagliardo-Nirenberg inequality:

$$\|f\|_{L^{\alpha+2}}^{\alpha+2} \le C_{\mathrm{GN}}(c)\|f\|_{L^2}^{\frac{4-(d-2)\alpha}{2}}\|f\|_{\dot{H}_c^1}^{\frac{d\alpha}{2}}. \tag{4.1}$$

The sharp constant $C_{\mathrm{GN}}(c)$ is defined by

$$C_{\mathrm{GN}}(c) := \sup\left\{J_c(f) : f \in H_c^1\backslash\{0\}\right\},$$

where $J_c(f)$ is the Weinstein functional

$$J_c(f) := \|f\|_{L^{\alpha+2}}^{\alpha+2} \div \Big[\|f\|_{L^2}^{\frac{4-(d-2)\alpha}{2}} \|f\|_{\dot H_c^1}^{\frac{d\alpha}{2}}\Big].$$

We also consider the sharp radial Gagliardo-Nirenberg inequality:

$$\|f\|_{L^{\alpha+2}}^{\alpha+2} \le C_{\mathrm{GN}}(c,\mathrm{rad})\|f\|_{L^2}^{\frac{4-(d-2)\alpha}{2}} \|f\|_{\dot H_c^1}^{\frac{d\alpha}{2}}, \quad f \text{ radial}, \tag{4.2}$$

where the sharp constant $C_{\mathrm{GN}}(c,\mathrm{rad})$ is defined by

$$C_{\mathrm{GN}}(c,\mathrm{rad}) := \sup\left\{J_c(f) : f \in H_c^1\backslash\{0\}, f \text{ radial}\right\}.$$

When $c = 0$, Weinstein in [31] proved that the sharp constant $C_{\mathrm{GN}}(0)$ is attained by the function $Q_0$, which is the unique positive radial solution of

$$\Delta Q_0 - Q_0 + Q_0^{\alpha+1} = 0. \tag{4.3}$$

Recently, Killip-Murphy-Visan-Zheng extended Weinstein's result to $c \neq 0$. More precisely, we have the following:

**Theorem 4.1** (Sharp Gagliardo-Nirenberg inequality [17])**.** *Let $d \ge 3, 0 < \alpha < \alpha^\star$ and $c \neq 0$ be such that $c > -\lambda(d)$. Then we have $C_{\mathrm{GN}}(c) \in (0,\infty)$ and*

1. *if $-\lambda(d) < c < 0$, then the equality in (4.1) is attained by a function $Q_c \in H_c^1$, which is a non-zero, non-negative, radial solution to the elliptic equation*

$$-P_cQ_c - Q_c + Q_c^{\alpha+1} = 0. \tag{4.4}$$

2. *if $c > 0$, then $C_{\mathrm{GN}}(c) = C_{\mathrm{GN}}(0)$ and the equality in (4.1) is never attained. However, the constant $C_{\mathrm{GN}}(c,\mathrm{rad})$ is attained by a function $Q_{c,\mathrm{rad}}$ which is a solution to the elliptic equation*

$$-P_cQ_{c,\mathrm{rad}} - Q_{c,\mathrm{rad}} + Q_{c,\mathrm{rad}}^{\alpha+1} = 0. \tag{4.5}$$

*Proof.* In [17, Theorem 3.1], the authors gave the proof for $d = 3$ and $\alpha = 2$. For reader's convenience, we provide some details for the general case. Since $\|f\|_{\dot H_c^1} \sim \|f\|_{\dot H^1}$, we see that $J_c(f) \sim J_0(f)$. Thus the standard Gagliardo-Nirenberg inequality (i.e. (4.1) with $c = 0$) implies $0 < C_{\mathrm{GN}}(c) < \infty$.

Let us consider the case $-\lambda(d) < c < 0$. Let $(f_n)_n \subset H_c^1\backslash\{0\}$ be a maximizing sequence, i.e. $J_c(f_n) \nearrow C_{\mathrm{GN}}(c)$. Let $f_n^*$ be the Schwarz symmetrization of $f_n$ (see e.g. [20]). Using the fact that the Schwarz symmetrization preserves $L^q$ norm and does not increase $\dot H^1$ norm together with the Riesz rearrangement inequality

$$\int c|x|^{-2}|f^*(x)|^2 \le \int c|x|^{-2}|f(x)|^2dx, \tag{4.6}$$

for $c < 0$, we see that $J_c(f_n) \le J_c(f_n^*)$. Thus we may assume that each $f_n$ is radial. Note that (4.6) plays an important role in order to restore the lack of compactness due to translations. We next observe that the functional $J_c$ is invariant under the scaling

$$f_{\lambda,\mu}(x) := \lambda f(\mu x), \quad \lambda,\mu > 0.$$

Indeed, a simple computation shows

$$\|f_{\lambda,\mu}\|_{\dot H_c^1}^2 = \lambda^2\mu^{2-d}\|f\|_{\dot H_c^1}^2, \quad \|f_{\lambda,\mu}\|_{L^2}^2 = \lambda^2\mu^{-d}\|f\|_{L^2}^2, \quad \|f_{\lambda,\mu}\|_{L^{\alpha+2}}^{\alpha+2} = \lambda^{\alpha+2}\mu^{-d}\|f\|_{L^{\alpha+2}}^{\alpha+2}.$$

We thus get $J_c(f_{\lambda,\mu}) = J_c(f)$. We now rescale the sequence $(f_n)_n$ by setting $g_n(x) := \lambda_n f_n(\mu_n x)$, where

$$\lambda_n = \frac{\|f_n\|_{L^2}^{d/2-1}}{\|f_n\|_{\dot{H}_c^1}^{d/2}}, \quad \mu_n = \frac{\|f_n\|_{L^2}}{\|f_n\|_{\dot{H}_c^1}}.$$

It is easy to see that $\|g_n\|_{L^2} = \|g_n\|_{\dot{H}_c^1} = 1$. We thus get a maximizing sequence $(g_n)_n$ of $J_c$, which is bounded in $H_c^1$. We have from the compactness lemma (see e.g. [31]) that $H^1_{\text{rad}}(\mathbb{R}^d) \hookrightarrow L^{\alpha+2}(\mathbb{R}^d)$ compactly for any $0 < \alpha < \frac{4}{d-2}$. Therefore, there exists $g \in H_c^1$ such that, up to a subsequence, $g_n \to g$ strongly in $L^{\alpha+2}$ as well as weakly in $H_c^1$. By the weak convergence, $\|g\|_{L^2} \leq 1$ and $\|g\|_{\dot{H}_c^1} \leq 1$. Hence,

$$C_{\text{GN}}(c) = \lim_{n\to\infty} J_c(g_n) = \|g\|_{L^{\alpha+2}}^{\alpha+2} \leq J_c(g) \leq C_{\text{GN}}(c).$$

Thus, we have $J_c(g) = \|g\|_{L^{\alpha+2}}^{\alpha+2} = C_{\text{GN}}(c)$ and $\|g\|_{L^2} = \|g\|_{\dot{H}_c^1} = 1$. Therefore, $g$ is a maximizer for the Weinstein functional $J_c$, and so $g$ must satisfy the Euler-Lagrange equation

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} J_c(g + \epsilon h) = 0, \quad \forall h \in C_0^\infty(\mathbb{R}^d \backslash \{0\}).$$

Taking into consideration that $\|g\|_{L^2} = \|g\|_{\dot{H}_c^1} = 1$ and $C_{\text{GN}}(c) = \|g\|_{L^{\alpha+2}}^{\alpha+2}$, we get

$$-\frac{d\alpha}{2} C_{\text{GN}}(c) P_c g - \frac{4-(d-2)\alpha}{2} C_{\text{GN}}(c) g + (\alpha+2) g^{\alpha+1} = 0.$$

If we define $Q_c$ by $g(x) = \lambda Q_c(\mu x)$ with

$$\lambda = \sqrt[\alpha]{\frac{4-(d-2)\alpha}{2(\alpha+2)} C_{\text{GN}}(c)}, \quad \mu = \sqrt{\frac{4-(d-2)\alpha}{d\alpha}},$$

then $Q_c$ solves (4.4). This proves Item 1.

In the case $c > 0$, we consider a sequence $(x_n)_n \subset \mathbb{R}^d$ with $|x_n| \to \infty$. Let $Q_0$ be the unique positive radial solution to (4.3). Using the definition (2.6) and (2.9), we have

$$\|Q_0(\cdot - x_n)\|_{\dot{H}_c^1}^2 = \|\sqrt{P_c}[Q_0(\cdot - x_n)]\|_{L^2}^2 = \|[\sqrt{P_c^n} Q_0](\cdot - x_n)\|_{L^2}^2 \to \|\sqrt{P_c^\infty} Q_0\|_{L^2}^2 = \|Q_0\|_{\dot{H}^1}^2.$$

We thus get

$$J_c(Q_0(\cdot - x_n)) \to J_0(Q_0) = C_{\text{GN}}(0),$$

hence $C_{\text{GN}}(0) \leq C_{\text{GN}}(c)$. Since $c > 0$, it is obvious that $\|f\|_{\dot{H}_x^1} < \|f\|_{\dot{H}_c^1}$ for any $f \in H^1 \backslash \{0\}$. The sharp Gagliardo-Nirenberg inequality for $c = 0$ then implies

$$\|f\|_{L^{\alpha+2}}^{\alpha+2} \leq C_{\text{GN}}(0) \|f\|_{L^2}^{\frac{4-(d-2)\alpha}{2}} \|f\|_{\dot{H}^1}^{\frac{d\alpha}{2}} < C_{\text{GN}}(0) \|f\|_{L^2}^{\frac{4-(d-2)\alpha}{2}} \|f\|_{\dot{H}_c^1}^{\frac{d\alpha}{2}},$$

whence $J_c(f) < C_{\text{GN}}(0)$ for any $f \in H^1 \backslash \{0\}$. Since $H^1$ is equivalent to $H_c^1$, we obtain $C_{\text{GN}}(c) < C_{\text{GN}}(0)$. Therefore, $C_{\text{GN}}(c) = C_{\text{GN}}(0)$. The last estimate also shows that the equality in (4.1) is never attained. Note also that the estimate (4.6) fails to hold true when $c > 0$. If we only consider radial functions, then the estimate (4.6) is obviously holds true. Thus the result for radial functions follows exactly as the case $-\lambda(d) < c < 0$. The proof is complete. □

**Remark 4.2.** 1. When $-\lambda(d) < c < 0$, the proof of Theorem 4.1 shows that there exist solutions to the elliptic equation (4.4), which are non-zero, non-negative and radially symmetric. However, unlike the standard case $c = 0$, we do not know that the uniqueness (up to symmetries) of these solutions. Moreover, any positive maximiser of $J_c$ is radial.

Furthermore, if $Q_c$ is a maximiser of $J_c$, then by multiplying (4.4) with $Q_c$ and $x \cdot \nabla Q_c$ and integrating over $\mathbb{R}^d$, we obtain the following Pohozaev identities:

$$\|Q_c\|^2_{\dot H^1_c} + \|Q_c\|^2_{L^2} - \|Q_c\|^{\alpha+2}_{L^{\alpha+2}} = \frac{d-2}{2}\|Q_c\|^2_{\dot H^1_c} + \frac{d}{2}\|Q_c\|^2_{L^2} - \frac{d}{\alpha+2}\|Q_c\|^{\alpha+2}_{L^{\alpha+2}} = 0.$$

In particular,

$$\|Q_c\|^2_{L^2} = \frac{4-(d-2)\alpha}{d\alpha}\|Q_c\|^2_{\dot H^1_c} = \frac{4-(d-2)\alpha}{2(\alpha+2)}\|Q_c\|^{\alpha+2}_{L^{\alpha+2}}, \tag{4.7}$$

and

$$\begin{aligned} C_{\mathrm{GN}}(c) &= \frac{2(\alpha+2)}{4-(d-2)\alpha}\Big[\frac{4-(d-2)\alpha}{d\alpha}\Big]^{\frac{d\alpha}{4}} \frac{1}{\|Q_c\|^{\alpha}_{L^2}} \\ &= \frac{2(\alpha+2)}{d\alpha}\Big[\frac{d\alpha}{4-(d-2)\alpha}\Big]^{\frac{4-(d-2)\alpha}{4}} \frac{1}{\|Q_c\|^{\alpha}_{\dot H^1_c}} \\ &= \frac{[2(\alpha+2)]^{\frac{\alpha+2}{2}}}{[4-(d-2)\alpha]^{\frac{4-(d-2)\alpha}{4}}[d\alpha]^{\frac{d\alpha}{4}}} \frac{1}{\|Q_c\|^{\frac{\alpha(\alpha+2)}{2}}_{L^{\alpha+2}}}. \end{aligned} \tag{4.8}$$

In particular, all maximizers of $J_c$ have the same $L^2, \dot H^1_c, L^{\alpha+2}$-norms. We also have

$$E_c(Q_c) = \frac{d\alpha-4}{2[4-(d-2)\alpha]}\|Q_c\|^2_{L^2} = \frac{d\alpha-4}{2d\alpha}\|Q_c\|^2_{\dot H^1_c}. \tag{4.9}$$

In particular, in the mass-critical case, i.e. $\alpha = \frac{4}{d}$, we have $E_c(Q_c) = 0$.

2. Since the identities (4.7) – (4.9) hold true for $c = 0$, we have from Theorem 4.1 that for any $c > -\lambda(d)$,

$$\begin{aligned} C_{\mathrm{GN}}(c) &= \frac{2(\alpha+2)}{4-(d-2)\alpha}\Big[\frac{4-(d-2)\alpha}{d\alpha}\Big]^{\frac{d\alpha}{4}} \frac{1}{\|Q_{\overline{c}}\|^{\alpha}_{L^2}} \\ &= \frac{2(\alpha+2)}{d\alpha}\Big[\frac{d\alpha}{4-(d-2)\alpha}\Big]^{\frac{4-(d-2)\alpha}{4}} \frac{1}{\|Q_{\overline{c}}\|^{\alpha}_{\dot H^1_{\overline{c}}}} \\ &= \frac{[2(\alpha+2)]^{\frac{\alpha+2}{2}}}{[4-(d-2)\alpha]^{\frac{4-(d-2)\alpha}{4}}[d\alpha]^{\frac{d\alpha}{4}}} \frac{1}{\|Q_{\overline{c}}\|^{\frac{\alpha(\alpha+2)}{2}}_{L^{\alpha+2}}}, \end{aligned} \tag{4.10}$$

where $\overline{c} = \min\{c, 0\}$.

3. Let $H(c)$ and $K(c)$ be as in (1.11). Using (4.7), (4.8) and (4.9), it is easy to see that

$$H(c) = \frac{d\alpha-4}{2d\alpha}\Big[\frac{d\alpha}{2(\alpha+2)}C_{\mathrm{GN}}(c)\Big]^{-\frac{4}{d\alpha-4}}, \tag{4.11}$$

and

$$K(c) = \Big[\frac{d\alpha}{2(\alpha+2)}C_{\mathrm{GN}}(c)\Big]^{-\frac{2}{d\alpha-4}}. \tag{4.12}$$

In particular,

$$H(c) = \frac{d\alpha-4}{2d\alpha}K(c)^2. \tag{4.13}$$

4. When $c > 0$, we see that the same identities as in (4.7), (4.8), (4.9), (4.10), (4.9), (4.11), (4.12) and (4.13) hold true with $Q_{c,\mathrm{rad}}$, $C_{\mathrm{GN}}(c,\mathrm{rad})$, $H(c,\mathrm{rad})$ and $K(c,\mathrm{rad})$ in place of $Q_c$, $C_{\mathrm{GN}}(c)$, $H(c)$ and $K(c)$ respectively.

Let us now consider the sharp Sobolev embedding inequality:

$$\|f\|_{L^{\alpha^\star+2}} \leq C_{\mathrm{SE}}(c)\|f\|_{\dot{H}^1_c}, \tag{4.14}$$

where the sharp constant $C_{\mathrm{SE}}(c)$ is defined by

$$C_{\mathrm{SE}}(c) := \sup\left\{\|f\|_{L^{\alpha^\star+2}} \div \|f\|_{\dot{H}^1_c} \ : \ f \in \dot{H}^1_c\backslash\{0\}\right\}.$$

We also consider the sharp radial Sobolev embedding inequality

$$\|f\|_{L^{\alpha^\star+2}} \leq C_{\mathrm{SE}}(c,\mathrm{rad})\|f\|_{\dot{H}^1_c}, \quad f \text{ radial} \tag{4.15}$$

where the sharp constant $C_{\mathrm{SE}}(c,\mathrm{rad})$ is defined by

$$C_{\mathrm{SE}}(c,\mathrm{rad}) := \sup\left\{\|f\|_{L^{\alpha^\star+2}} \div \|f\|_{\dot{H}^1_c} \ : \ f \in \dot{H}^1_c\backslash\{0\}, f \text{ radial}\right\}.$$

When $c = 0$, it was proved by Aubin [1] and Talenti [30] that the constant $C_{\mathrm{SE}}(0)$ is attained by functions $f(x)$ of a form $\lambda W_0(\mu x + y)$ for some $\lambda \in \mathbb{C}, \mu > 0$ and $y \in \mathbb{R}^d$, where $W_0$ is given in (1.22).

When $c \neq 0$, Killip-Miao-Visan-Zhang-Zheng in [19] proved the following result.

**Theorem 4.3** (Sharp Sobolev embedding inequality [19])**.** *Let $d \geq 3$ and $c \neq 0$ be such that $c > -\lambda(d)$. Then $C_{\mathrm{SE}}(c) \in (0,\infty)$ and*

1. *if $-\lambda(d) < c < 0$, then the equality in (4.14) is attained by functions $f(x)$ of the form $\lambda W_c(\mu x)$ for some $\lambda \in \mathbb{C}$ and some $\mu > 0$, where $W_c$ is given in (1.23).*
2. *if $c > 0$, then $C_{\mathrm{SE}}(c) = C_{\mathrm{SE}}(0)$ and the equality in (4.14) is never attained. However, $C_{\mathrm{SE}}(c,\mathrm{rad})$ is attained by functions $f(x)$ of the form $\lambda W_c(\mu x)$ for some $\lambda \in \mathbb{C}$ and some $\mu > 0$, where $W_c$ is again given in (1.23).*

We refer the reader to [19, Proposition 7.2] for the proof of this result. Note that the non-existence of optimizers to the Sobolev embedding inequality for $c > 0$ is a consequence of the failure of compactness due to translation. If we restrict our consideration to radial functions, the compactness is restored. To end this section, we recall some properties related to $W_c$ (see [19, Section 7] for more details). It is not difficult to verify that $W_c$ solves the elliptic equation

$$P_c W_c = |W_c|^{\alpha^\star} W_c.$$

This implies in particular

$$\|W_c\|^2_{\dot{H}^1_c} = \|W_c\|^{\alpha^\star+2}_{L^{\alpha^\star+2}}. \tag{4.16}$$

Combining with Theorem 4.3, we have for $-\lambda(d) < c < 0$,

$$\|W_c\|^2_{\dot{H}^1_c} = \|W_c\|^{\alpha^\star+2}_{L^{\alpha^\star+2}} = C_{\mathrm{SE}}(c)^{-d}, \tag{4.17}$$

$$E_c(W_c) = \frac{1}{2}\|W_c\|^2_{\dot{H}^1_c} - \frac{1}{\alpha^\star+2}\|W_c\|^{\alpha^\star+2}_{L^{\alpha^\star+2}} = d^{-1}C_{\mathrm{SE}}(c)^{-d}. \tag{4.18}$$

Note that (4.17) and (4.18) hold true for $c = 0$. In particular, we have for any $c \neq 0$ satisfying $c > -\lambda(d)$,

$$C_{\mathrm{SE}}(c) = \|W_{\overline{c}}\|^{-\frac{2}{d}}_{\dot{H}^1_{\overline{c}}} = \|W_{\overline{c}}\|^{-\frac{\alpha^\star+2}{d}}_{L^{\alpha^\star+2}} = [dE_{\overline{c}}(W_{\overline{c}})]^{-\frac{1}{d}}. \tag{4.19}$$

Similarly, we have for $c > 0$ that

$$\|W_c\|_{\dot{H}^1_c}^2 = \|W_c\|_{L^{\alpha^\star+2}}^{\alpha^\star+2} = C_{\mathrm{SE}}(c, \mathrm{rad})^{-d}, \tag{4.20}$$

$$E_c(W_c) = \frac{1}{2}\|W_c\|_{\dot{H}^1_c}^2 - \frac{1}{\alpha^\star+2}\|W_c\|_{L^{\alpha^\star+2}}^{\alpha^\star+2} = d^{-1}C_{\mathrm{SE}}(c, \mathrm{rad})^{-d}. \tag{4.21}$$

$$C_{\mathrm{SE}}(c, \mathrm{rad}) = \|W_c\|_{\dot{H}^1_c}^{-\frac{2}{d}} = \|W_c\|_{L^{\alpha^\star+2}}^{-\frac{\alpha^\star+2}{d}} = [dE_c(W_c)]^{-\frac{1}{d}}. \tag{4.22}$$

## 5. Virial identities

In this section, we derive virial identities and localized virial estimates associated to the $(\mathrm{NLS}_c)$. Given a smooth real valued function $\chi$, we define the virial potential by

$$V_\chi(t) := \int \chi(x)|u(t,x)|^2 dx. \tag{5.1}$$

By a direct computation, we have the following result.

**Lemma 5.1.** *Let $d \geq 3$ and $c > -\lambda(d)$. If $u : I \times \mathbb{R}^d \to \mathbb{C}$ is a smooth-in-time and Schwartz-in-space solution to*

$$i\partial_t u - P_c u = N(u),$$

*with $N(u)$ satisfying* $\mathrm{Im}\ (N(u)\overline{u}) = 0$*, then we have for any $t \in I$,*

$$\frac{d}{dt}V_\chi(t) = 2\int_{\mathbb{R}^d} \nabla\chi(x) \cdot \mathrm{Im}\ (\overline{u}(t,x)\nabla u(t,x))dx, \tag{5.2}$$

*and*

$$\begin{aligned}\frac{d^2}{dt^2}V_\chi(t) = &-\int \Delta^2\chi(x)|u(t,x)|^2dx + 4\sum_{j,k=1}^d \int \partial^2_{jk}\chi(x)\mathrm{Re}\ (\partial_k u(t,x)\partial_j\overline{u}(t,x))dx \\ &+ 4c\int \nabla\chi(x)\cdot\frac{x}{|x|^4}|u(t,x)|^2dx + 2\int \nabla\chi(x)\cdot\{N(u),u\}_p(t,x)dx,\end{aligned} \tag{5.3}$$

*where $\{f,g\}_p := \mathrm{Re}\ (f\nabla\overline{g} - g\nabla\overline{f})$ is the momentum bracket.*

We note that if $N(u) = -|u|^\alpha u$, then

$$\{N(u),u\}_p = \frac{\alpha}{\alpha+2}\nabla(|u|^{\alpha+2}).$$

Using this fact, we immediately have the following result.

**Corollary 5.2.** *Let $d \geq 3$ and $c > -\lambda(d)$. If $u : I \times \mathbb{R}^d \to \mathbb{C}$ is a smooth-in-time and Schwartz-in-space solution to the $(\mathrm{NLS}_c)$, then we have for any $t \in I$,*

$$\begin{aligned}\frac{d^2}{dt^2}V_\chi(t) = &-\int \Delta^2\chi(x)|u(t,x)|^2dx + 4\sum_{j,k=1}^d \int \partial^2_{jk}\chi(x)\mathrm{Re}\ (\partial_k u(t,x)\partial_j\overline{u}(t,x))dx \\ &+ 4c\int \nabla\chi(x)\cdot\frac{x}{|x|^4}|u(t,x)|^2dx - \frac{2\alpha}{\alpha+2}\int \Delta\chi(x)|u(t,x)|^{\alpha+2}dx.\end{aligned} \tag{5.4}$$

We now have the following standard virial identity for the $(\mathrm{NLS}_c)$.

**Lemma 5.3.** *Let $d \geq 3$ and $c > -\lambda(d)$. Let $u_0 \in H^1$ be such that $|x|u_0 \in L^2$ and $u : I \times \mathbb{R}^d \to \mathbb{C}$ the corresponding solution to the $(\mathrm{NLS}_c)$. Then, $|x|u \in C(I, L^2)$. Moreover, for any $t \in I$,*

$$\frac{d^2}{dt^2}\|xu(t)\|_{L^2}^2 = 8\|u(t)\|_{\dot{H}^1_c}^2 - \frac{4d\alpha}{\alpha+2}\|u(t)\|_{L^{\alpha+2}}^{\alpha+2}. \tag{5.5}$$

*Proof.* The first claim follows from the standard approximation argument, we omit the proof and refer the reader to [5, Proposition 6.5.1] for more details. It remains to show (5.5). Applying Corollary 5.2 with $\chi(x) = |x|^2$, we have

$$\begin{aligned}\frac{d^2}{dt^2}V_{|x|^2}(t) = \frac{d^2}{dt^2}\|xu(t)\|_{L^2}^2 &= 8\int |\nabla u(t,x)|^2 + c|x|^{-2}|u(t,x)|^2dx - \frac{4d\alpha}{\alpha+2}\int |u(t,x)|^{\alpha+2}dx \\ &= 8\|u(t)\|_{\dot H_c^1}^2 - \frac{4d\alpha}{\alpha+2}\|u(t)\|_{L^{\alpha+2}}^{\alpha+2}.\end{aligned}$$

This gives (5.5). □

In order to prove the blowup for the (NLS$_c$) with radial data, we need localized virial estimates. To do so, we introduce the smooth, non-negative function $\theta : [0,\infty) \to [0,\infty)$ satisfying

$$\theta(r) = \begin{cases} r^2 & \text{if } 0 \le r \le 1, \\ \text{const.} & \text{if } r \ge 2, \end{cases} \qquad \text{and} \quad \theta''(r) \le 2 \quad \text{for } r \ge 0. \tag{5.6}$$

Note that the precise constant here is not important. For $R > 1$, we define the radial function

$$\varphi_R(x) = \varphi_R(r) := R^2\theta(r/R), \quad r = |x|. \tag{5.7}$$

It is easy to see that

$$2 - \varphi_R''(r) \ge 0, \quad 2 - \frac{\varphi_R'(r)}{r} \ge 0, \quad 2d - \Delta\varphi_R(x) \ge 0. \tag{5.8}$$

Here the last inequality follows from the fact $\Delta = \partial_r^2 + \frac{d-1}{r}\partial_r$.

**Lemma 5.4.** *Let $d \ge 3, c > -\lambda(d), R > 1$ and $\varphi_R$ be as in (5.7). Let $u : I \times \mathbb{R}^d \to \mathbb{C}$ be a radial solution to the* (NLS$_c$)*. Then for any $t \in I$,*

$$\frac{d^2}{dt^2}V_{\varphi_R}(t) \le 8\|u(t)\|_{\dot H_c^1}^2 - \frac{4d\alpha}{\alpha+2}\|u(t)\|_{L^{\alpha+2}}^{\alpha+2} + O\Big(R^{-2} + R^{-\frac{(d-1)\alpha}{2}}\|u(t)\|_{\dot H_c^1}^{\frac{\alpha}{2}}\Big). \tag{5.9}$$

*Proof.* We apply (5.4) for $\chi(x) = \varphi_R(x)$ to get

$$\begin{aligned}\frac{d^2}{dt^2}V_{\varphi_R}(t) = &-\int \Delta^2\varphi_R(x)|u(t,x)|^2dx + 4\sum_{j,k=1}^d \int \partial_{jk}^2\varphi_R(x)\mathrm{Re}\,(\partial_k u(t,x)\partial_j\overline{u}(t,x))dx \\ &+ 4c\int \nabla\varphi_R(x)\cdot\frac{x}{|x|^4}|u(t,x)|^2dx - \frac{2\alpha}{\alpha+2}\int \Delta\varphi_R(x)|u(t,x)|^{\alpha+2}dx.\end{aligned}$$

Since $\varphi_R(x) = |x|^2$ for $|x| \le R$, we use (5.5) to have

$$\begin{aligned}\frac{d^2}{dt^2}V_{\varphi_R}(t) = &\, 8\|u(t)\|_{\dot H_c^1}^2 - \frac{4d\alpha}{\alpha+2}\|u(t)\|_{L^{\alpha+2}}^{\alpha+2} - 8\|u(t)\|_{\dot H_c^1(|x|>R)}^2 + \frac{4d\alpha}{\alpha+2}\|u(t)\|_{L^{\alpha+2}(|x|>R)}^{\alpha+2} \\ &- \int_{|x|>R}\Delta^2\varphi_R|u(t)|^2dx + 4\sum_{j,k=1}^d\int_{|x|>R}\partial_{jk}^2\varphi_R\mathrm{Re}\,(\partial_k u(t)\partial_j\overline{u}(t))dx \\ &+4c\int_{|x|>R}\nabla\varphi_R\cdot\frac{x}{|x|^4}|u(t)|^2dx - \frac{2\alpha}{\alpha+2}\int_{|x|>R}\Delta\varphi_R|u(t)|^{\alpha+2}dx.\end{aligned} \tag{5.10}$$

Since $|\Delta\varphi_R| \lesssim 1$ and $|\Delta^2\varphi_R| \lesssim R^{-2}$, we have

$$\begin{aligned}\frac{d^2}{dt^2}V_{\varphi_R}(t) = 8\|u(t)\|_{\dot H^1_c}^2 - \frac{4d\alpha}{\alpha+2}\|u(t)\|_{L^{\alpha+2}}^{\alpha+2} + 4\sum_{j,k=1}^d \int_{|x|>R} \partial^2_{jk}\varphi_R \mathrm{Re}\,(\partial_k u(t)\partial_j \overline{u}(t))dx \\ +4c\int_{|x|>R} \nabla\varphi_R\cdot\frac{x}{|x|^4}|u(t)|^2dx - 8\|u(t)\|^2_{\dot H^1_c(|x|>R)} \\ +O\Big(\int_{|x|>R} R^{-2}|u(t)|^2 + |u(t)|^{\alpha+2}dx\Big).\end{aligned}$$

Using (5.8) and the fact that

$$\partial_j = \frac{x_j}{r}\partial_r, \quad \partial^2_{jk} = \Big(\frac{\delta_{jk}}{r} - \frac{x_jx_k}{r^3}\Big)\partial_r + \frac{x_jx_k}{r^2}\partial_r^2,$$

we see that

$$\sum_{j,k=1}^d \partial^2_{jk}\varphi_R\partial_k u\partial_j\overline{u} = \varphi_R''(r)|\partial_r u|^2 \le 2|\partial_r u|^2 = 2|\nabla u|^2,$$

and

$$\nabla\varphi_R\cdot x = \varphi_R'\frac{x}{r}\cdot x = \varphi_R' r \le 2r^2 = 2|x|^2.$$

Therefore

$$4\sum_{j,k=1}^d \int_{|x|>R} \partial^2_{jk}\varphi_R \mathrm{Re}\,(\partial_k u\partial_j\overline{u})dx + 4c\int_{|x|>R}\nabla\varphi_R\cdot x|x|^{-4}|u|^2dx - 8\|u(t)\|^2_{\dot H^1_c(|x|>R)} \le 0.$$

The conservation of mass then implies

$$\begin{aligned}\frac{d^2}{dt^2}V_{\varphi_R}(t) &\le 8\|u(t)\|^2_{\dot H^1_c} - \frac{4d\alpha}{\alpha+2}\|u(t)\|^{\alpha+2}_{L^{\alpha+2}} + O\Big(\int_{|x|>R} R^{-2}|u(t)|^2 + |u(t)|^{\alpha+2}dx\Big) \\ &\le 8\|u(t)\|^2_{\dot H^1_c} - \frac{4d\alpha}{\alpha+2}\|u(t)\|^{\alpha+2}_{L^{\alpha+2}} + O\Big(R^{-2} + \|u(t)\|^{\alpha+2}_{L^{\alpha+2}(|x|>R)}\Big).\end{aligned}$$

It remains to bound $\|u(t)\|^{\alpha+2}_{L^{\alpha+2}(|x|>R)}$. To do this, we recall the following radial Sobolev embedding ([27, 6]).

**Lemma 5.5** (Radial Sobolev embedding [27, 6])**.** *Let $d \ge 2$ and $\frac{1}{2} \le s < 1$. Then for any radial function $f$,*

$$\sup_{x\neq 0}|x|^{\frac{d-2s}{2}}|f(x)| \le C(d,s)\|f\|_{L^2}^{1-s}\|f\|_{\dot H^1}^s. \tag{5.11}$$

*Moreover, the above inequality also holds for $d \ge 3$ and $s = 1$.*

Since $\dot H^1 \sim \dot H^1_c$, we have in particular

$$\sup_{x\neq 0}|x|^{\frac{d-1}{2}}|f(x)| \lesssim \|f\|_{L^2}^{\frac12}\|f\|_{\dot H^1_c}^{\frac12}. \tag{5.12}$$

Using (5.12) and the conservation of mass, we estimate

$$\begin{aligned}\|u(t)\|^{\alpha+2}_{L^{\alpha+2}(|x|>R)} &\le \Big(\sup_{|x|>R}|u(t)|\Big)^\alpha\|u(t)\|^2_{L^2} \\ &\lesssim R^{-\frac{(d-1)\alpha}{2}}\Big(\sup_{|x|>R}|x|^{\frac{d-1}{2}}|u(t)|\Big)^\alpha\|u(t)\|^2_{L^2} \\ &\lesssim R^{-\frac{(d-1)\alpha}{2}}\|u(t)\|^{\frac{\alpha}{2}}_{\dot H^1_c}\|u(t)\|^{\frac{\alpha}{2}+2}_{L^2} \lesssim R^{-\frac{(d-1)\alpha}{2}}\|u(t)\|^{\frac{\alpha}{2}}_{\dot H^1_c}.\end{aligned}$$

The proof is complete. □

The localized virial estimate given in Lemma 5.4 is not enough to show blowup solutions in the mass-critical case, i.e. $\alpha = \alpha_\star$. In this case, we need a refined version of Lemma 5.4. We follow the argument of [23] (see also [3]).

**Lemma 5.6.** *Let $d \geq 3, c > -\lambda(d), R > 1$ and $\varphi_R$ be as in (5.7). Let $u : I \times \mathbb{R}^d \to \mathbb{C}$ be a radial solution to the mass-critical* (NLS$_c$)*, i.e. $\alpha = \alpha_\star$. Then for any $\epsilon > 0$ and any $t \in I$,*

$$\frac{d^2}{dt^2} V_{\varphi_R}(t) \leq 16E_c(u_0) - 4\int_{|x|>R} \Big(\chi_{1,R} - \frac{\epsilon}{d+2}\chi_{2,R}^{\frac{d}{2}}\Big)|\nabla u(t)|^2 dx + O\Big(R^{-2} + \epsilon R^{-2} + \epsilon^{-\frac{2}{d-2}} R^{-2}\Big), \tag{5.13}$$

*where*

$$\chi_{1,R} = 2 - \varphi_R'', \quad \chi_{2,R} = 2d - \Delta\varphi_R. \tag{5.14}$$

*Proof.* Using (5.10) with $\alpha = \alpha_\star = \frac{4}{d}$ and $\sum_{j,k} \partial_{jk}^2 \varphi_R \partial_k u \partial_j \overline{u} = \varphi_R'' |\partial_r u|^2$ and rewriting $\varphi_R'' = 2 - (2 - \varphi_R'')$ and $\Delta\varphi_R = 2d - (2d - \Delta\varphi_R)$, we have

$$\begin{aligned}
\frac{d^2}{dt^2} V_{\varphi_R}(t) = 16E_c(u(t)) - \int_{|x|>R} \Delta^2\varphi_R |u(t)|^2 dx - 4\int_{|x|>R} (2 - \varphi_R'')|\partial_r u(t)|^2 dx \\
+ \frac{4}{d+2}\int_{|x|>R} (2d - \Delta\varphi_R)|u(t)|^{\frac{4}{d}+2} dx \\
+8\int_{|x|>R} |\partial_r u(t)|^2 dx + 4c\int_{|x|>R} \nabla\varphi_R \cdot x|x|^{-4}|u(t)|^2 dx - 8\|u(t)\|_{\dot{H}_c^1(|x|>R)}^2 \\
\leq 16E_c(u_0) + O(R^{-2}) - 4\int_{|x|>R} \chi_{1,R}|\nabla u(t)|^2 dx + \frac{4}{d+2}\int_{|x|>R} \chi_{2,R}|u(t)|^{\frac{4}{d}+2} dx.
\end{aligned}$$

We now bound the last term. Using the radial Sobolev embedding (5.11) with $s = 1$ and the conservation of mass, we estimate

$$\begin{aligned}
\int_{|x|>R} \chi_{2,R}|u(t)|^{\frac{4}{d}+2} dx &= \int_{|x|>R} |\chi_{2,R}^{\frac{d}{4}} u(t)|^{\frac{4}{d}} |u(t)|^2 dx \\
&\leq \Big(\sup_{|x|>R} |\chi_{2,R}^{\frac{d}{4}}(x) u(t,x)|\Big)^{\frac{4}{d}} \|u(t)\|_{L^2}^2 \\
&\lesssim R^{-\frac{2(d-2)}{d}} \Big\|\nabla\Big(\chi_{2,R}^{\frac{d}{4}} u(t)\Big)\Big\|_{L^2}^{\frac{4}{d}} \|u(t)\|_{L^2}^2 \\
&\lesssim R^{-\frac{2(d-2)}{d}} \Big\|\nabla\Big(\chi_{2,R}^{\frac{d}{4}} u(t)\Big)\Big\|_{L^2}^{\frac{4}{d}}.
\end{aligned}$$

We next use the Young inequality $ab \lesssim \epsilon a^p + \epsilon^{-\frac{q}{p}} b^q$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $\epsilon > 0$ an arbitrary real number to have

$$R^{-\frac{2(d-2)}{d}} \Big\|\nabla\Big(\chi_{2,R}^{\frac{d}{4}} u(t)\Big)\Big\|_{L^2}^{\frac{4}{d}} \lesssim \epsilon \Big\|\nabla\Big(\chi_{2,R}^{\frac{d}{4}} u(t)\Big)\Big\|_{L^2}^2 + O\Big(\epsilon^{-\frac{2}{d-2}} R^{-2}\Big).$$

Here we apply the Young inequality with $p = \frac{d}{2}$ and $q = \frac{d}{d-2}$. It is not hard to check $|\nabla(\chi_{2,R}^{d/4})| \lesssim R^{-1}$ for $|x| > R$. Thus the conservation of mass implies

$$\Big\|\nabla\Big(\chi_{2,R}^{\frac{d}{4}} u(t)\Big)\Big\|_{L^2}^2 \lesssim R^{-2} + \Big\|\chi_{2,R}^{\frac{d}{4}} \nabla u(t)\Big\|_{L^2}^2.$$

Combining the above estimates, we prove (5.13). □

## 6. Global existence

In this section, we give the proofs of global existence given Theorem 1.3 and Theorem 1.6.

6.1. **Mass-critical case.** Thanks to the local well-posedness given in Theorem 3.1, it suffices to bound $\|u(t)\|_{H^1_c}$ for all $t$ in the existence time. Applying (4.10) with $\alpha = \alpha_\star$, we see that

$$C_{\mathrm{GN}}(c) = \frac{\alpha_\star + 2}{2\|Q_{\bar{c}}\|_{L^2}^{\alpha_\star}}.$$

By the definition of energy, we have

$$\|u(t)\|_{\dot{H}^1_c}^2 = 2E_c(u(t)) + \frac{2}{\alpha_\star + 2}\|u(t)\|_{L^{\alpha_\star+2}}^{\alpha_\star+2}.$$

The sharp Gagliardo-Nirenberg inequality and the conservations of mass and energy imply

$$\begin{aligned}\|u(t)\|_{\dot{H}^1_c}^2 &\leq 2E_c(u(t)) + \frac{2}{\alpha_\star + 2}C_{\mathrm{GN}}(c)\|u(t)\|_{L^2}^{\alpha_\star}\|u(t)\|_{\dot{H}^1_c}^2 \\ &= 2E_c(u_0) + \frac{2}{\alpha_\star + 2}C_{\mathrm{GN}}(c)\|u_0|_{L^2}^{\alpha_\star}\|u(t)\|_{\dot{H}^1_c}^2 \\ &= 2E_c(u_0) + \Big(\frac{\|u_0\|_{L^2}}{\|Q_{\bar{c}}\|_{L^2}}\Big)^{\alpha_\star}\|u(t)\|_{\dot{H}^1_c}^2.\end{aligned}$$

Thus,

$$\Big[1 - \Big(\frac{\|u_0\|_{L^2}}{\|Q_{\bar{c}}\|_{L^2}}\Big)^{\alpha_\star}\Big]\|u(t)\|_{\dot{H}^1_c}^2 \leq 2E_c(u_0).$$

Since $\|u_0\|_{L^2} < \|Q_{\bar{c}}\|_{L^2}$, the above estimate shows the boundedness of $\|u(t)\|_{\dot{H}^1_c}$. Hence $\|u(t)\|_{H^1_c}$ is bounded by the conservation of mass. This proves the global existence of Theorem 1.3.

**Remark 6.1.** Let us show Item 3 of Remark 1.4. Let $-\lambda(d) < c < 0$ and $M_c > \|Q_c\|_{L^2}$. Let $\lambda = M_c/\|Q_c\|_{L^2} > 1$. Set $u_0(x) = \lambda Q_c(x)$. We have $\|u_0\|_{L^2} = M_c$ and

$$\begin{aligned}E_c(u_0) = E_c(\lambda Q_c) &= \frac{\lambda^2}{2}\|Q_c\|_{\dot{H}^1_c}^2 - \frac{\lambda^{\alpha_\star+2}}{\alpha_\star + 2}\|Q_c\|_{L^{\alpha_\star+2}}^{\alpha_\star+2} \\ &= \lambda^{\alpha_\star+2}E_c(Q_c) - \frac{\lambda^{\alpha_\star+2} - \lambda^2}{2}\|Q_c\|_{\dot{H}^1_c}^2.\end{aligned}$$

Since $E_c(Q_c) = 0$ and $\lambda > 1$, we see that $E_c(u_0) < 0$. On the other hand, it is obvious that $u_0$ is radial. Thus by Item 2 of Theorem 1.3, we see that the corresponding solution with initial data $u_0$ blows up in finite time.

We next show for $c > 0$ that if $u_0$ is radial and satisfies $\|u_0\|_{L^2} < \|Q_{c,\mathrm{rad}}\|_{L^2}$, then the corresponding solution exists globally. It follows similarly as the beginning of Subsection 6.1 by using the sharp radial Gagliardo-Nirenberg inequality

$$\|f\|_{L^{\alpha_\star+2}}^{\alpha_\star+2} \leq C_{\mathrm{GN}}(c, \mathrm{rad})\|f\|_{L^2}^{\alpha_\star}\|f\|_{\dot{H}^1_c}^2, \quad f \text{ radial}.$$

Note also that by Item 4 of Remark 4.2, we have

$$C_{\mathrm{GN}}(c, \mathrm{rad}) = \frac{\alpha_\star + 2}{2\|Q_{c,\mathrm{rad}}\|_{L^2}^{\alpha_\star}}.$$

To complete the proof of Item 3, we show that for any $M_c > \|Q_{c,\mathrm{rad}}\|_{L^2}$, there exists $u_0 \in H^1$ radial satisfying $\|u_0\|_{L^2} = M_c$ and the corresponding solution blows up in finite time. We proceed

as above. Let $\lambda = M_c/\|Q_{c,\mathrm{rad}}\|_{L^2} > 1$ and set $u_0(x) = \lambda Q_{c,\mathrm{rad}}(x)$. We see that $\|u_0\|_{L^2} = M_c$ and

$$\begin{aligned} E_c(u_0) = E_c(\lambda Q_{c,\mathrm{rad}}) &== \frac{\lambda^2}{2}\|Q_{c,\mathrm{rad}}\|_{\dot H^1_c}^2 - \frac{\lambda^{\alpha_\star+2}}{\alpha_\star+2}\|Q_{c,\mathrm{rad}}\|_{L^{\alpha_\star+2}}^{\alpha_\star+2} \\ &= \lambda^{\alpha_\star+2}E_c(Q_{c,\mathrm{rad}}) - \frac{\lambda^{\alpha_\star+2}-\lambda^2}{2}\|Q_{c,\mathrm{rad}}\|_{\dot H^1_c}^2. \end{aligned}$$

Since $Q_{c,\mathrm{rad}}$ is a solution to the (4.5), we see that $E_c(Q_{c,\mathrm{rad}}) = 0$. This shows that $E_c(u_0) < 0$. Thus the corresponding solution blows up in finite time.

**Remark 6.2.** Let us show Item 5 of Remark 1.4, that is to show when $c > 0$ there exists a radial blowup solution to the mass-critical ($\mathrm{NLS}_c$) with $\|u_0\|_{L^2} = \|Q_{c,\mathrm{rad}}\|_{L^2}$. Since $Q_{c,\mathrm{rad}}$ is a solution to the elliptic equation

$$-P_cQ_{c,\mathrm{rad}} - Q_{c,\mathrm{rad}} + Q_{c,\mathrm{rad}}^{\alpha_\star+1} = 0,$$

it is easy to see that $u(t) = e^{it}Q_{c,\mathrm{rad}}$ is a solution to the mass-critical ($\mathrm{NLS}_c$). Then a direct computation shows that for any $0 < T < +\infty$, the function

$$u_T(t,x) = \frac{1}{|t-T|^{d/2}}e^{-i\frac{|x|^2}{4(t-T)}+\frac{i}{t-T}}Q_{c,\mathrm{rad}}\Big(\frac{x}{t-T}\Big)$$

is also a solution to the mass-critical ($\mathrm{NLS}_c$) which blows up at $T$ and $\|u_T(0)\|_{L^2} = \|Q_{c,\mathrm{rad}}\|_{L^2}$.

6.2. **Intercritical case.** Again thanks to the local well-posedness of the ($\mathrm{NLS}_c$) given in Theorem 3.1. It suffices to show that $\|u(t)\|_{H^1_c}$ is bounded as long as $t$ belongs to the existence time. Let $u_0 \in H^1$ be such that (1.12) and (1.13) hold. By the definition of energy and multiplying both sides of $E_c(u(t))$ by $M(u(t))^\sigma$, the sharp Gagliardo-Nirenberg inequality (4.1) implies

$$\begin{aligned} E_c(u(t))M(u(t))^\sigma &= \frac{1}{2}\Big(\|u(t)\|_{\dot H^1_c}\|u(t)\|_{L^2}^\sigma\Big)^2 - \frac{1}{\alpha+2}\|u(t)\|_{L^{\alpha+2}}^{\alpha+2}\|u(t)\|_{L^2}^{2\sigma} \\ &\geq \frac{1}{2}\Big(\|u(t)\|_{\dot H^1_c}\|u(t)\|_{L^2}^\sigma\Big)^2 - \frac{C_{\mathrm{GN}}(c)}{\alpha+2}\|u(t)\|_{L^2}^{\frac{4-(d-2)\alpha}{2}+2\sigma}\|u(t)\|_{\dot H^1_c}^{\frac{d\alpha}{2}} \\ &= f(\|u(t)\|_{\dot H^1_c}\|u(t)\|_{L^2}^\sigma), \end{aligned} \tag{6.1}$$

where

$$f(x) = \frac{1}{2}x^2 - \frac{C_{\mathrm{GN}}(c)}{\alpha+2}x^{\frac{d\alpha}{2}}. \tag{6.2}$$

Using (4.12) and (4.13), we see that

$$f(K(c)) = \frac{d\alpha-4}{2d\alpha}K(c)^2 = H(c). \tag{6.3}$$

We have from (6.1), the conservations of mass and energy and the assumption (1.12) that

$$f(\|u(t)\|_{\dot H^1_c}\|u(t)\|_{L^2}^\sigma) \leq E_c(u_0)M(u_0)^\sigma < H(c). \tag{6.4}$$

Using this together with (1.13), (6.3) and (6.4), the continuity argument shows

$$\|u(t)\|_{\dot H^1_c}\|u(t)\|_{L^2}^\sigma < K(c),$$

for any $t$ as long as the solution exists. The conservation of mass then implies the boundedness of $\|u(t)\|_{H^1_c}$.

The global existence of Theorem 1.8 is proved similarly as above using Item 4 of Remark 4.2.

## 7. Blowup

This section is devoted to the proofs of blowup solutions given in Theorem 1.3, Theorem 1.6 and Theorem 1.12.

7.1. **Mass-critical case.** Let us consider the case $E_c(u_0) < 0$ and $xu_0 \in L^2$. By the standard virial identity (5.5),

$$\frac{d^2}{dt^2}\|xu(t)\|_{L^2}^2 = 8\|u(t)\|_{\dot H_c^1}^2 - \frac{4d\alpha_\star}{\alpha_\star+2}\|u(t)\|_{L^{\alpha_\star+2}}^{\alpha_\star+2} = 16E_c(u_0) < 0.$$

By the classical argument of Glassey [11], it follows that the solution $u$ blows up in finite time.

We next consider the case $E_c(u_0) < 0$ and $u_0$ is radial. Applying the localized virial estimate (5.13), we have

$$\frac{d^2}{dt^2}V_{\varphi_R}(t) \le 16E_c(u_0) - 4\int_{|x|>R}\Big(\chi_{1,R} - \frac{\epsilon}{d+2}\chi_{2,R}^{\frac{d}{2}}\Big)|\nabla u(t)|^2dx + O\Big(R^{-2} + \epsilon^{-\frac{2}{d-2}}R^{-2}\Big),$$

where $\chi_{1,R} = 2 - \varphi_R''$ and $\chi_{2,R} = 2d - \Delta\varphi_R$. We seek for a radial function $\varphi_R$ defined by (5.7) so that

$$\chi_{1,R} - \frac{\epsilon}{d+2}\chi_{2,R}^{\frac{d}{2}} \ge 0, \quad \forall r > R, \tag{7.1}$$

for a sufficiently small $\epsilon > 0$. If (7.1) is satisfied, then by choosing $R > 1$ sufficiently large depending on $\epsilon$, we see that

$$\frac{d^2}{dt^2}V_{\varphi_R}(t) \le 8E_c(u_0) < 0,$$

for any $t$ in the existence time. This shows that the solution $u$ must blow up in finite time. It remains to show (7.1). To do so, we follow the argument of [23]. Let us define the smooth function

$$\vartheta(r) := \begin{cases} 2r & \text{if } 0 \le r \le 1, \\ 2[r-(r-1)^3] & \text{if } 1 < r \le 1+1/\sqrt{3}, \\ \vartheta' < 0 & \text{if } 1+1/\sqrt{3} < r < 2, \\ 0 & \text{if } r \ge 2, \end{cases}$$

and

$$\theta(r) := \int_0^r \vartheta(s)ds.$$

It is easy to see that $\theta$ satisfies (5.6). Define $\varphi_R$ as in (5.7). We will show that (7.1) holds true for this choice of $\varphi_R$. Indeed, by definition,

$$\varphi_R'(r) = R\theta'(r/R) = R\vartheta(r/R), \quad \varphi_R''(r) = \theta''(r/R) = \vartheta'(r/R), \quad \Delta\varphi_R(x) = \varphi_R''(r) + \frac{d-1}{r}\varphi_R'(r).$$

When $r > (1+1/\sqrt{3})R$, we see that $\vartheta'(r/R) \le 0$, so $\chi_{1,R}(r) = 2 - \varphi_R''(r) \ge 2$. We also have $\chi_{2,R}(r) \le C$ for some constant $C > 0$. Thus by choosing $\epsilon > 0$ small enough, we have (7.1).

When $R < r \le (1+1/\sqrt{3})R$, we have

$$\chi_{1,R}(r) = 6\Big(\frac{r}{R}-1\Big)^2, \quad \chi_{2,R}(r) = 6\Big(\frac{r}{R}-1\Big)^2\Big[1 + \frac{(d-1)(r/R-1)}{3r/R}\Big] < 6\Big(\frac{r}{R}-1\Big)^2\Big(1+\frac{d-1}{3\sqrt{3}}\Big).$$

Since $0 < r/R - 1 < 1/\sqrt{3}$, we can choose $\epsilon > 0$ small enough, for instance,

$$\epsilon < (d+2)\Big(1+\frac{d-1}{3\sqrt{3}}\Big)^{-d/2}$$

to get (7.1). The proof is complete. □

**Remark 7.1.** We now show Item 4 of Remark 1.4 that is to show the condition $E_c(u_0) < 0$ is a sufficient condition but it is not necessary. Let $E_c > 0$. We find data $u_0 \in H^1$ so that $E_c(u_0) = E_c$ and the corresponding solution $u$ blows up in finite time. We follow the standard argument (see e.g. [5, Remark 6.5.8]). Using the standard virial identity with $\alpha = \alpha_\star$, we have

$$\frac{d^2}{dt^2}\|xu(t)\|_{L^2}^2 = 16E_c(u_0),$$

hence

$$\|xu(t)\|_{L^2}^2 = 8t^2E_c(u_0) + 4t\Big(\mathrm{Im}\int \overline{u}_0 x\cdot\nabla u_0 dx\Big) + \|xu_0\|_{L^2}^2 =: f(t).$$

Note that if $f(t)$ takes negative values, then the solution $u$ must blow up in finite time. In order to make $f(t)$ takes negative values, we need

$$\Big(\mathrm{Im}\int \overline{u}_0 x\cdot\nabla u_0 dx\Big)^2 > 2E_c(u_0)\|xu_0\|_{L^2}^2. \tag{7.2}$$

Now fix $\theta \in C_0^\infty(\mathbb{R}^d)$ a real-valued function and set $\psi(x) = e^{-i|x|^2}\theta(x)$. We see that $\psi \in C_0^\infty(\mathbb{R}^d)$ and

$$\mathrm{Im}\int \overline{\psi} x\cdot\nabla\psi dx = -2\int |x|^2\theta^2(x)dx < 0.$$

We now set

$$A = \frac{1}{2}\|\psi\|_{\dot H_c^1}^2, \quad B = \frac{1}{\alpha_\star+2}\|\psi\|_{L^{\alpha_\star+2}}^{\alpha_\star+2},$$
$$C = \|x\psi\|_{L^2}^2, \quad D = -\mathrm{Im}\int \overline{\psi} x\cdot\nabla\psi dx.$$

Let $\lambda, \mu > 0$ be chosen later and set $u_0(x) = \lambda\psi(\mu x)$. We will choose $\lambda, \mu > 0$ so that $E_c(u_0) = E_c$ and (7.2) holds true. A direct computation shows

$$E_c(u_0) = \lambda^2\mu^2\mu^{-d}\frac{1}{2}\|\psi\|_{\dot H_c^1}^2 - \lambda^{\alpha_\star+2}\mu^{-d}\frac{1}{\alpha_\star+2}\|\psi\|_{L^{\alpha_\star+2}}^{\alpha_\star+2} = \lambda^2\mu^{2-d}\Big(A - \frac{\lambda^{\alpha_\star}}{\mu^2}B\Big),$$

and

$$\mathrm{Im}\int \overline{u}_0 x\cdot\nabla u_0 dx = \lambda^2\mu^{-d}\mathrm{Im}\int \overline{\psi} x\cdot\nabla\psi dx = -\lambda^2\mu^{-d}D,$$

and

$$\|xu_0\|_{L^2}^2 = \lambda^2\mu^{-d-2}\|x\psi\|_{L^2}^2 = \lambda^2\mu^{-d-2}C.$$

Thus, the conditions $E_c(u_0) = E_c$ and (7.2) yield

$$\lambda^2\mu^{2-d}\Big(A - \frac{\lambda^{\alpha_\star}}{\mu^2}B\Big) = E_c, \tag{7.3}$$

$$\frac{D^2}{C} > 2\Big(A - \frac{\lambda^{\alpha_\star}}{\mu^2}B\Big). \tag{7.4}$$

Fix $0 < \epsilon < \min\Big\{A, \frac{D^2}{2C}\Big\}$ and choose

$$\frac{\lambda^{\alpha_\star}}{\mu^2}B = A - \epsilon.$$

It is obvious that (7.4) is satisfied. Condition (7.3) implies

$$\epsilon\lambda^2\mu^{2-d} = E_c \quad \text{or} \quad \epsilon\Big(\frac{B}{A-\epsilon}\Big)^{\frac{2-d}{2}}\lambda^{2+\frac{(2-d)\alpha_\star}{2}} = E_c.$$

This holds true by choosing a suitable value of $\lambda$.

7.2. **Intercritical case.** We firstly show (1.16). We have from (6.1) that

$$f(\|u(t)\|_{\dot H^1_c}\|u(t)\|_{L^2}^\sigma) \le E_c(u(t))M(u(t))^\sigma,$$

where $f$ is defined as in (6.2). Note that $f(K(c)) = H(c)$. By our assumption (1.12), we have

$$f(\|u(t)\|_{\dot H^1_c}\|u(t)\|_{L^2}^\sigma) < H(c).$$

Using (1.15) and the continuity argument, we get

$$\|u(t)\|_{\dot H^1_c}\|u(t)\|_{L^2}^\sigma > K(c),$$

for any $t$ in the existence time. This proves (1.16).

We next pick $\delta > 0$ small enough so that

$$E_c(u_0)M(u_0)^\sigma \le (1-\delta)H(c). \tag{7.5}$$

This implies

$$f(\|u(t)\|_{\dot H^1_c}\|u(t)\|_{L^2}^\sigma) \le (1-\delta)H(c). \tag{7.6}$$

Using (6.2), (4.12) and (4.13), we have from (7.6) that

$$\frac{d\alpha}{d\alpha-4}\Big(\frac{\|u(t)\|_{\dot H^1_c}\|u(t)\|_{L^2}^\sigma}{K(c)}\Big)^2 - \frac{4}{d\alpha-4}\Big(\frac{\|u(t)\|_{\dot H^1_c}\|u(t)\|_{L^2}^\sigma}{K(c)}\Big)^{\frac{d\alpha}{2}} \le 1-\delta.$$

The continuity argument shows that there exists $\delta' > 0$ depending on $\delta$ so that

$$\frac{\|u(t)\|_{\dot H^1_c}\|u(t)\|_{L^2}^\sigma}{K(c)} \ge 1+\delta' \quad \text{or} \quad \|u(t)\|_{\dot H^1_c}\|u(t)\|_{L^2}^\sigma \ge (1+\delta')K(c). \tag{7.7}$$

We also have for $\epsilon > 0$ small enough,

$$8\|u(t)\|_{\dot H^1_c}^2 - \frac{4d\alpha}{\alpha+2}\|u(t)\|_{L^{\alpha+2}}^{\alpha+2} + \epsilon\|u(t)\|_{\dot H^1_c}^2 \le -c < 0, \tag{7.8}$$

for any $t$ in the existence time. Indeed, multiplying the left hand side of (7.8) with a conserved quantity $M(u(t))^\sigma$, we get

$$\text{LHS}(7.8) \times M(u(t))^\sigma = 4d\alpha E_c(u(t))M(u(t))^\sigma + (8+\epsilon-2d\alpha)\|u(t)\|_{\dot H^1_c}^2 M(u(t))^\sigma.$$

The conservations of mass and energy, (7.5), (7.7) and (4.13) then yield

$$\begin{aligned}\text{LHS}(7.8) \times M(u_0)^\sigma &\le 4d\alpha(1-\delta)H(c) + (8+\epsilon-2d\alpha)(1+\delta')^2K(c)^2\\ &= 2(d\alpha-4)(1-\delta)K(c)^2 + (8+\epsilon-2d\alpha)(1+\delta')^2K(c)^2\\ &= K(c)^2\Big[2(d\alpha-4)(1-\delta-(1+\delta')^2) + \epsilon(1+\delta')^2\Big].\end{aligned}$$

By taking $\epsilon > 0$ small enough, we prove (7.8).

Let us consider the case $xu_0 \in L^2$ satisfying (1.12) and (1.15). By the standard virial identity (5.5) and (7.8),

$$\frac{d^2}{dt^2}\|xu(t)\|_{L^2}^2 = 8\|u(t)\|_{\dot H^1_c}^2 - \frac{4d\alpha}{\alpha+2}\|u(t)\|_{L^{\alpha+2}}^{\alpha+2} \le -c < 0.$$

This shows that the solution blows up in finite time.

We now consider the case $u_0$ is radial, and satisfies (1.12) and (1.15). Using the localized virial estimate (5.9), we have

$$\frac{d^2}{dt^2}V_{\varphi_R}(t) \le 8\|u(t)\|_{\dot H^1_c}^2 - \frac{4d\alpha}{\alpha+2}\|u(t)\|_{L^{\alpha+2}}^{\alpha+2} + O\Big(R^{-2} + R^{-\frac{(d-1)\alpha}{2}}\|u(t)\|_{\dot H^1_c}^{\frac{\alpha}{2}}\Big).$$

We next use the Young inequality to bound

$$R^{-\frac{(d-1)\alpha}{2}}\|u(t)\|_{\dot H^1_c}^{\frac{\alpha}{2}} \lesssim \epsilon\|u(t)\|_{\dot H^1_c}^2 + \epsilon^{-\frac{\alpha}{4-\alpha}}R^{-\frac{2(d-1)\alpha}{4-\alpha}},$$

for $\epsilon > 0$ an arbitrary real number. We thus get

$$\frac{d^2}{dt^2}V_{\varphi_R}(t) \le 8\|u(t)\|_{\dot H^1_c}^2 - \frac{4d\alpha}{\alpha+2}\|u(t)\|_{L^{\alpha+2}}^{\alpha+2} + \epsilon\|u(t)\|_{\dot H^1_c}^2 + O\Big(R^{-2} + \epsilon^{-\frac{\alpha}{4-\alpha}}R^{-\frac{2(d-1)\alpha}{4-\alpha}}\Big).$$

By taking $\epsilon > 0$ small enough and $R > 1$ large enough depending on $\epsilon$, we obtain from (7.8) that

$$\frac{d^2}{dt^2}V_{\varphi_R}(t) \le -c/2 < 0.$$

This shows that the solution must blow up in finite time.

The blowup of Theorem 1.8 follows by the same argument as above and Item 4 of Remark 4.2.

7.3. **Energy-critical case.** By definition of the energy and the sharp Sobolev embedding inequality (4.14),

$$\begin{aligned} E_c(u(t)) &= \frac{1}{2}\|u(t)\|_{\dot H^1_c}^2 - \frac{1}{\alpha^\star+2}\|u(t)\|_{L^{\alpha^\star+2}}^{\alpha^\star+2} \\ &\le \frac{1}{2}\|u(t)\|_{\dot H^1_c}^2 - \frac{[C_{\mathrm{SE}}(c)]^{\alpha^\star+2}}{\alpha^\star+2}\|u(t)\|_{\dot H^1_c}^{\alpha^\star+2} =: g(\|u(t)\|_{\dot H^1_c}), \end{aligned}$$

where

$$g(y) = \frac{1}{2}y^2 - \frac{[C_{\mathrm{SE}}(c)]^{\alpha^\star+2}}{\alpha^\star+2}y^{\alpha^\star+2}. \tag{7.9}$$

We have from (4.19) that

$$g(\|W_{\overline{c}}\|_{\dot H^1_{\overline{c}}}) = E_{\overline{c}}(W_{\overline{c}}).$$

By the conservation of energy and the assumption $E_c(u_0) < E_{\overline{c}}(W_{\overline{c}})$,

$$g(\|u(t)\|_{\dot H^1_c}) \le E_c(u(t)) = E_c(u_0) < E_{\overline{c}}(W_{\overline{c}}).$$

We thus have from the assumption $\|u_0\|_{\dot H^1_c} > \|W_{\overline{c}}\|_{\dot H^1_{\overline{c}}}$ and the continuity argument that

$$\|u(t)\|_{\dot H^1_c} > \|W_{\overline{c}}\|_{\dot H^1_{\overline{c}}}, \tag{7.10}$$

for any $t$ as long as the solution exists. We next improve (7.10) as follows. Pick $\delta > 0$ small enough so that

$$E_c(u_0) \le (1-\delta)E_{\overline{c}}(W_{\overline{c}}). \tag{7.11}$$

This implies

$$g(\|u(t)\|_{\dot H^1_c}) \le (1-\delta)E_{\overline{c}}(W_{\overline{c}}). \tag{7.12}$$

Using (7.9) and (4.19), we have from (7.12) that

$$\frac{d}{2}\Big(\frac{\|u(t)\|_{\dot H^1_c}}{\|W_{\overline{c}}\|_{\dot H^1_{\overline{c}}}}\Big)^2 - \frac{d-2}{2}\Big(\frac{\|u(t)\|_{\dot H^1_c}}{\|W_{\overline{c}}\|_{\dot H^1_{\overline{c}}}}\Big)^{\alpha^\star+2} \le 1-\delta.$$

The continuity argument shows that there exists $\delta' > 0$ depending on $\delta$ so that

$$\frac{\|u(t)\|_{\dot H^1_c}}{\|W_{\overline{c}}\|_{\dot H^1_{\overline{c}}}} \ge 1+\delta' \quad \text{or} \quad \|u(t)\|_{\dot H^1_c} \ge (1+\delta')\|W_{\overline{c}}\|_{\dot H^1_{\overline{c}}}. \tag{7.13}$$

We also have for $\epsilon > 0$ small enough,

$$8\|u(t)\|_{\dot H^1_c}^2 - \frac{4d\alpha^\star}{\alpha^\star+2}\|u(t)\|_{L^{\alpha^\star+2}}^{\alpha^\star+2} + \epsilon\|u(t)\|_{\dot H^1_c}^2 \le -c < 0, \tag{7.14}$$

for any $t$ in the existence time. Indeed,

$$\mathrm{LHS}(7.14) = 4d\alpha^\star E_c(u(t)) + (8+\epsilon-2d\alpha^\star)\|u(t)\|_{\dot H^1_c}^2.$$

The conservations of mass and energy, (7.11), (7.13), (4.17) and (4.18) then yield

$$\begin{aligned}\mathrm{LHS}(7.14) &\le 4d\alpha^\star(1-\delta)E_{\overline{c}}(W_{\overline{c}}) + (8+\epsilon-2d\alpha^\star)(1+\delta')^2\|W_{\overline{c}}\|_{\dot H^1_{\overline{c}}}^2\\ &= \frac{16}{d-2}(1-\delta)\|W_{\overline{c}}\|_{\dot H^1_{\overline{c}}}^2 + \Big(-\frac{16}{d-2}+\epsilon\Big)(1+\delta')^2\|W_{\overline{c}}\|_{\dot H^1_{\overline{c}}}^2\\ &= \|W_{\overline{c}}\|_{\dot H^1_{\overline{c}}}^2\Big[\frac{16}{d-2}(1-\delta-(1+\delta')^2)+\epsilon(1+\delta')^2\Big].\end{aligned}$$

By taking $\epsilon > 0$ small enough, we prove (7.14).

Let us consider the case $xu_0 \in L^2$ satisfying $E_c(u_0) < E_{\overline{c}}(W_{\overline{c}})$ and $\|u_0\|_{\dot H^1_c} > \|W_{\overline{c}}\|_{\dot H^1_{\overline{c}}}$. By the standard virial identity (5.5) and (7.14),

$$\frac{d^2}{dt^2}\|xu(t)\|_{L^2}^2 = 8\|u(t)\|_{\dot H^1_c}^2 - \frac{4d\alpha^\star}{\alpha^\star+2}\|u(t)\|_{L^{\alpha^\star+2}}^{\alpha^\star+2} \le -c < 0.$$

This shows that the solution blows up in finite time.

We now consider the case $u_0$ is radial, and satisfies $E_c(u_0) < E_{\overline{c}}(W_{\overline{c}})$ and $\|u_0\|_{\dot H^1_c} > \|W_{\overline{c}}\|_{\dot H^1_{\overline{c}}}$. Using the localized virial estimate (5.9), we have

$$\frac{d^2}{dt^2}V_{\varphi_R}(t) \le 8\|u(t)\|_{\dot H^1_c}^2 - \frac{4d\alpha^\star}{\alpha^\star+2}\|u(t)\|_{L^{\alpha^\star+2}}^{\alpha^\star+2} + O\Big(R^{-2} + R^{-\frac{(d-1)\alpha^\star}{2}}\|u(t)\|_{\dot H^1_c}^{\frac{\alpha^\star}{2}}\Big).$$

Using the fact $\frac{\alpha^\star}{2} = \frac{2}{d-2} \le 2$, the uniform bound (7.10) and (7.14), we see that for $R > 1$ large enough,

$$\frac{d^2}{dt^2}V_{\varphi_R}(t) \le -c/2 < 0.$$

Therefore, the solution must blow up in finite time.

The blowup of Theorem 1.14 follows by the same argument as above and (4.20) – (4.22).

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