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# A HAMILTON-JACOBI-BELLMAN APPROACH FOR THE NUMERICAL COMPUTATION OF PROBABILISTIC STATE CONSTRAINED REACHABLE SETS

MOHAMED ASSELLAOU AND ATHENA PICARELLI

ABSTRACT. Aim of this work is to characterise and compute the set of initial conditions for a system of controlled diffusion processes which allow to reach a terminal target satisfying pointwise state constraints with a given probability of success. Defining a suitable auxiliary optimal control problem, the characterization of this set is related to the solution of a particular Hamilton-Jacobi-Bellman equation. A semi-Lagrangian numerical scheme is defined and its convergence to the unique viscosity solution of the equation is proved. The validity of the proposed approach is then tested on some numerical examples.

## INTRODUCTION

We consider the control of stochastic differential equations in  $\mathbb{R}^d$  of the following form

$$\begin{cases} dX(s) = b(s, X(s), u(s))ds + \sigma(s, X(s), u(s))dB(s), & \forall s \in [t, T] \\ X(t) = x \end{cases}. \quad (1)$$

Given a fixed time horizon  $T > 0$ , we aim to characterize the set of initial states from which, with an assigned level of probability, it is possible to reach a target set at time  $T$  satisfying some state constraints along the whole interval  $[t, T]$ .

More precisely, let  $\mathcal{C}$  and  $\mathcal{K}$  be two non-empty subsets of  $\mathbb{R}^d$  representing respectively the target set and the set of state constraints and let  $\rho \in [0, 1)$ . We define the state constrained backward reachable set under probability of success  $\rho$  as the set of initial points  $x \in \mathbb{R}^d$  for which the probability to steer the system (1) towards  $\mathcal{C}$  maintaining the dynamics in the set  $\mathcal{K}$  is higher than  $\rho$ , i.e.

$$\Omega_t^\rho := \left\{ x \in \mathbb{R}^d : \exists u \in \mathcal{U}, \mathbb{P}[X_{t,x}^u(\theta) \in \mathcal{K}, \forall \theta \in [t, T] \text{ and } X_{t,x}^u(T) \in \mathcal{C}] > \rho \right\}, \quad (2)$$

where  $X_{t,x}^u(\cdot)$  represents the strong solution to (1) associated with the control  $u \in \mathcal{U}$ . Assumptions on the coefficients in (1) and on the set of controls  $\mathcal{U}$  will be

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made clear in the next section. Such backward reachable sets play an important role in many applications, as the set  $\Omega_t^\rho$  can be interpreted as a “safety region” for reaching  $\mathcal{C}$  remaining in the set  $\mathcal{K}$ , with confidence  $\rho$ . It turns out that the set  $\Omega_t^\rho$  can be characterized by means of the so-called *level set approach*. At the basis of this approach there is the idea to look at the set of interest, the set  $\Omega_t^\rho$  in our case, as the level set of a certain function solution of a suitable partial differential equation (PDE). Such a characterization of the set is particularly useful in view of its numerical approximation, since it opens the way to the use of a wide choice of numerical methods designed for PDEs. Originally introduced in [23] to model front propagation problems, this approach immediately resulted in a very powerful method for studying backward reachable sets of continuous non-linear dynamical systems under very general conditions. In [15, 22] this idea is used to describe the reachable sets for deterministic problems. The link between stochastic target problems and level set approach is established in [24]. More recently, the level set approach has been extended to the case of state-constrained controlled systems [9, 10] and probabilistic reachability problems [6].

In our case, it is straightforward to see that

$$\Omega_t^\rho = \left\{ x \in \mathbb{R}^d : \vartheta(t, x) > \rho \right\}, \quad (3)$$

where  $\vartheta$  is the value function associated to the following optimal control problem:

$$\vartheta(t, x) := \sup_{u \in \mathcal{U}} \mathbb{E} \left[ \mathbb{1}_{\mathcal{C}}(X_{t,x}^u(T)) \bigwedge \min_{\theta \in [t, T]} \mathbb{1}_{\mathcal{K}}(X_{t,x}^u(\theta)) \right], \quad (4)$$

with the standard notation  $a \wedge b := \min(a, b)$ . In particular, equality (3) describes the evolution over time of the set  $\Omega_t^\rho$  for  $t \in [0, T]$  by means of the function  $\vartheta$ .

We point out that, in the discrete time setting, a similar approach has been considered in [1, 2, 20]. In this case, the value function is obtained recursively by solving the dynamic programming principle. In the present paper, we are interested in the approximation of the probabilistic backward reachable sets for time-continuous stochastic processes by PDE techniques. In the non controlled framework, an alternative numerical algorithm consist in using Monte Carlo simulations to generate a set of trajectories starting from a given initial position. The percentage of trajectories reaching the target without violation of the state constraints gives an approximation of the probability of success when starting from this position. On the other hand, for linear stochastic systems, a bound for the probability of hitting a target can be obtained by using the enclosing hulls of the probability density function for time intervals, see [3, 4]. However, it is worth noticing that these approaches are used to calculate the probabilities of success but do not allow to define the entire set of points that have the same given probability. In addition, Monte-Carlo based methods often require a large number of simulations to obtain a good accuracy. We will use such simulations in Section 4 as a comparison to validate our approach. In the context of financial mathematics,

the problem of characterizing the backward reachable set with a given probability was first introduced by Föllmer and Leukert [17]. This problem was also studied and converted into the class of stochastic target problems by Touzi, Bouchard and Elie in [11]. However in these references the possible presence of state constraints is not taken into account.

In order to apply a dynamic programming approach and characterize the value function  $\vartheta$  as the unique viscosity solution of a Hamilton-Jacobi-Bellman (HJB) equation we face two main difficulties. First, the discontinuous cost functional given by the presence of the indicator functions would require to make use of the notion of discontinuous viscosity solutions. Establish uniqueness results in such a framework is usually a very hard task, so we propose here to work on a regularized version of problem (4). Second, the non commutativity between expectation and maximum operator makes problem (4) not satisfying the natural “Markovian structure” necessary to apply the dynamic programming arguments. We here follow the ideas in [8, 10, 18, 19] and define an auxiliary optimal control problem in an augmented state space and derive the HJB equation for this problem recovering the value function  $\vartheta$  solution of the original problem at a later stage. The obtained HJB equation is setted in a domain and completed with mixed Dirichlet and oblique derivative boundary conditions. We discuss its numerical approximation. We introduce a semi-Lagrangian (SL) approximation scheme which incorporates the aforementioned boundary conditions and we prove its convergence to the viscosity solution following the framework in [7]. We recall that SL scheme for second order HJB equations have been introduced by Menaldi in [21] and then studied by Camilli and Falcone [13]. We refer to [14] and the references therein for an overview. Derivative boundary conditions have been added to the scheme in [10], while the case of mixed Dirichlet-derivative conditions has been recently studied in [19].

The paper is organised as follows. In Section 1 we present the problem and give some preliminary results. The regularized problem is introduced in Section 1.2. Section 2 is devoted to the development of the dynamic programming arguments and the HJB characterization. In Section 3 we discuss the numerical aspects and state the main convergence result. Numerical tests are presented in Section 4

## 1. FORMULATION OF THE PROBLEM AND PRELIMINARY RESULTS

**1.1. Problem formulation.** Let  $\{\Omega, \mathcal{F}_t, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}\}$  be a filtered probability space and  $W(\cdot)$  a given  $p$ -dimensional Brownian motion. Let  $T > 0$ . We denote by  $\mathcal{U}$  the set of all progressively measurable processes valued in  $U \subset \mathbb{R}^m$ ,  $U$  compact set. For any  $u \in \mathcal{U}$ , let us consider the following system of stochastic differential equations (SDEs) in  $\mathbb{R}^d$ :

$$\begin{cases} dX(s) = b(s, X(s), u(s))ds + \sigma(s, X(s), u(s))dB(s), & \forall s \in [t, T] \\ X(t) = x. \end{cases} \quad (5)$$

The following classical assumptions will be considered on the coefficients  $b$  and  $\sigma$ :

**(H1)**  $\sigma : [0, T] \times \mathbb{R}^d \times U \rightarrow \mathbb{R}^{d \times p}$  and  $b : [0, T] \times \mathbb{R}^d \times U \rightarrow \mathbb{R}^d$  are continuous functions and there exist constants  $L, M > 0$  such that

$$|b(t, x, u) - b(t, y, u)| + |\sigma(t, x, u) - \sigma(t, y, u)| \leq L(|x - y|), \quad (6)$$

$$|b(t, x, u)| + |\sigma(t, x, u)| \leq M, \quad (7)$$

for any  $t \in [0, T], x, y \in \mathbb{R}^d$  and  $u \in U$ .

It is well known that, under assumption (H1), for any  $u \in \mathcal{U}$  there is a unique strong solution to (5) [25, p. 42, Thm. 6.3]. We denote by  $X_{t,x}^u(\cdot)$  such a solution.

Let  $\mathcal{C}$  and  $\mathcal{K}$  be nonempty open sets in  $\mathbb{R}^d$ , representing respectively the target set and the set of state constraints. Let  $\rho \in [0, 1)$  an assigned value of success probability. We define the backward reachable set under probability of success  $\rho$ , as the set  $\Omega_t^\rho$  of initial points  $x \in \mathbb{R}^d$  from which it starts a trajectory  $X_{t,x}^u(\cdot)$  such that the probability to reach the target  $\mathcal{C}$  at the final instant  $T$  satisfying the constraint  $\mathcal{K}$  in the interval  $[t, T]$  is at least  $\rho$ , i.e.:

$$\Omega_t^\rho := \left\{ x \in \mathbb{R}^d : \exists u \in \mathcal{U}, \mathbb{P}[X_{t,x}^u(\theta) \in \mathcal{K}, \forall \theta \in [t, T] \text{ and } X_{t,x}^u(T) \in \mathcal{C}] > \rho \right\}. \quad (8)$$

For a given open set  $\mathcal{O} \subseteq \mathbb{R}^d$  we will denote by  $\mathbb{1}_{\mathcal{O}}$  its indicator function, i.e.

$$\mathbb{1}_{\mathcal{O}}(x) := \begin{cases} 1 & \text{if } x \in \mathcal{O} \\ 0 & \text{otherwise} \end{cases}. \quad (9)$$

One can easily verify that

$$\mathbb{1}_{\mathcal{C}}(X_{t,x}^u(T)) \bigwedge \min_{\theta \in [t, T]} \mathbb{1}_{\mathcal{K}}(X_{t,x}^u(\theta)) = \begin{cases} 1 & \text{if } X_{t,x}^u(\theta) \in \mathcal{K} \forall \theta \in [t, T] \text{ and } X_{t,x}^u(T) \in \mathcal{C} \\ 0 & \text{otherwise.} \end{cases} \quad (10)$$

It follows that, for any  $u \in \mathcal{U}$ , the probability in (8) can be expressed by

$$\mathbb{E} \left[ \mathbb{1}_{\mathcal{C}}(X_{t,x}^u(T)) \bigwedge \min_{\theta \in [t, T]} \mathbb{1}_{\mathcal{K}}(X_{t,x}^u(\theta)) \right].$$

As a consequence, it is possible to describe the set  $\Omega_t^\rho$  using optimal control tools just looking at the evolution of the level sets of the following value function:

$$\vartheta(t, x) := \sup_{u \in \mathcal{U}} \mathbb{E} \left[ \mathbb{1}_{\mathcal{C}}(X_{t,x}^u(T)) \bigwedge \min_{\theta \in [t, T]} \mathbb{1}_{\mathcal{K}}(X_{t,x}^u(\theta)) \right]. \quad (11)$$

*Proposition 1.1.* Let assumption (H1) be satisfied. Then, for  $t \in [0, T]$ , we have:

$$\Omega_t^\rho = \{x \in \mathbb{R}^d : \vartheta(t, x) > \rho\}. \quad (12)$$

*Proof.* If  $x \in \Omega_t^\rho$ , thanks to equality (10)

$$\mathbb{E} \left[ \mathbb{1}_{\mathcal{C}}(X_{t,x}^u(T)) \bigwedge \min_{\theta \in [t,T]} \mathbb{1}_{\mathcal{K}}(X_{t,x}^u(\theta)) \right] > \rho$$

for some control  $u \in \mathcal{U}$  and it follows  $\vartheta(t, x) > \rho$ .

Let us now suppose that  $\vartheta(t, x) > \rho$ . By the definition of the supremum and the fact that  $\mathcal{U}$  is a non empty set, one has that, for some control  $\bar{u} \in \mathcal{U}$ ,

$$\mathbb{E} \left[ \mathbb{1}_{\mathcal{C}}(X_{t,x}^{\bar{u}}(T)) \bigwedge \min_{\theta \in [t,T]} \mathbb{1}_{\mathcal{K}}(X_{t,x}^{\bar{u}}(\theta)) \right] > \rho$$

and then, using again (10),  $x \in \Omega_t^\rho$ .  $\square$

Motivated by this result, we are going to focus on the characterization and numerical approximation of the function  $\vartheta$ . Problem (11) is an optimal control problem with a discontinuous cost in a minimum form. This is not a standard formulation in optimal control theory for two main reasons: first, the discontinuity of the cost functional prevents the characterization of (11) as the unique viscosity solution of a HJB equation, second the loss of Markovian structure in the cost, due to the presence of the minimum operator inside the expectation, makes the dynamic programming arguments not directly applicable. We discuss the first issue in the next section.

**1.2. Regularized Problem.** The discontinuity introduced by the presence of the indicator functions and the consequent necessity of dealing with the notion of discontinuous viscosity solutions (see for instance [16, Section VII.4] for their definition) pose nontrivial issues when attempting to establish uniqueness results for the associated HJB equation. To overcome this difficulty, from now on we will work with a regularized version of the cost functional in (11). In particular, observing that the indicator functions  $\mathbb{1}_{\mathcal{C}}$  and  $\mathbb{1}_{\mathcal{K}}$  can be written as

$$\mathbb{1}_{\mathcal{C}}(z) = \begin{cases} 1 & \text{if } d_{\mathcal{C}}(z) \leq 0 \\ 0 & \text{if } d_{\mathcal{C}}(z) > 0 \end{cases}, \quad \mathbb{1}_{\mathcal{K}}(z) = \begin{cases} 1 & \text{if } d_{\mathcal{K}}(z) \leq 0 \\ 0 & \text{if } d_{\mathcal{K}}(z) > 0 \end{cases} \quad (13)$$

where  $d_{\mathcal{C}}$  and  $d_{\mathcal{K}}$  are respectively the signed distance function to  $\partial\mathcal{C}$  and  $\partial\mathcal{K}$ , we consider the following regularized functions  $\phi_{\mathcal{C}}^\epsilon$  and  $\phi_{\mathcal{K}}^\epsilon$  (see Figure. 1):

$$\phi_{\mathcal{C}}^\epsilon(x) := \min(1, \max(0, -\frac{1}{\epsilon}d_{\mathcal{C}}(x))), \quad \phi_{\mathcal{K}}^\epsilon(x) := \min(1, \max(0, -\frac{1}{\epsilon}d_{\mathcal{K}}(x))). \quad (14)$$

The following optimal control problem is considered:

$$\vartheta^\epsilon(t, x) := \sup_{u \in \mathcal{U}} \mathbb{E} \left[ \phi_{\mathcal{C}}^\epsilon(X_{t,x}^u(T)) \bigwedge \min_{\theta \in [t,T]} \phi_{\mathcal{K}}^\epsilon(X_{t,x}^u(\theta)) \right] \quad (15)$$

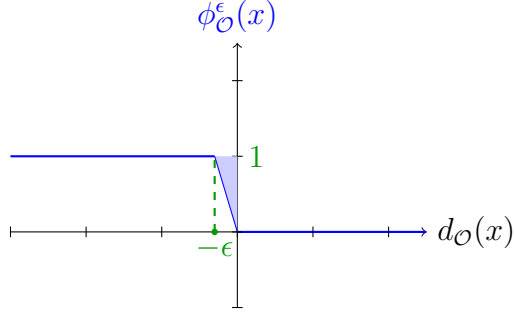


FIGURE 1. Regularization of the indicator function in the case  $\mathcal{O} = (-\infty, 0)$

*Remark 1.2.* Note that the choice of  $\phi^\epsilon$  is such that  $\phi^\epsilon \leq \mathbb{1}$ , which implies

$$\vartheta^\epsilon(t, x) > \rho \Rightarrow \vartheta(t, x) > \rho.$$

Hence if we are able to find a numerical approximation of  $\tilde{\vartheta}^\epsilon$  of  $\vartheta^\epsilon$  up to some controlled error  $\eta > 0$ , we will have  $\tilde{\vartheta}^\epsilon(t, x) > \rho + \eta \Rightarrow \vartheta^\epsilon(t, x) > \rho \Rightarrow \vartheta(t, x) > \rho$ .

This regularization allow us to deal with a continuous cost and also to obtain the following regularity result for the associated value function:

*Proposition 1.3.* Let assumption (H1) be satisfied and let  $\epsilon > 0$ . The value function  $\vartheta^\epsilon$  is Lipschitz continuous with respect to  $x$  and  $\frac{1}{2}$ -Hölder continuous with respect to  $t$ , i.e. there exists a constant  $L_\epsilon > 0$  such that

$$|\vartheta^\epsilon(t, x) - \vartheta^\epsilon(s, y)| \leq L_\epsilon (|x - y| + |t - s|^{\frac{1}{2}}(1 + |x| + |y|))$$

for any  $t, s \in [0, T]$ ,  $x, y \in \mathbb{R}^d$ .

*Proof.* Let be  $0 \leq t \leq s \leq T$ ,  $x, y \in \mathbb{R}^d$ . Thanks to the property of minimum operator  $|(a \wedge b) - (c \wedge d)| \leq |a - c| \vee |b - d|$ , one has:

$$\begin{aligned} & |\vartheta^\epsilon(t, x) - \vartheta^\epsilon(t, y)| \tag{16} \\ & \leq \sup_{u \in \mathcal{U}} \mathbb{E} \left[ \left| (\phi_{\mathcal{C}}^\epsilon(X_{t,x}^u(T)) \wedge \min_{\theta \in [t, T]} \phi_{\mathcal{K}}^\epsilon(X_{t,x}^u(\theta)) \wedge y) \right. \right. \\ & \quad \left. \left. - (\phi_{\mathcal{C}}^\epsilon(X_{t,y}^u(T)) \wedge \min_{\theta \in [t, T]} \phi_{\mathcal{K}}^\epsilon(X_{t,y}^u(\theta)) \wedge y) \right| \right] \\ & \leq \sup_{u \in \mathcal{U}} \mathbb{E} \left[ \left| \phi_{\mathcal{C}}^\epsilon(X_{t,x}^u(T)) - \phi_{\mathcal{C}}^\epsilon(X_{t,y}^u(T)) \right| \vee \max_{\theta \in [t, T]} \left| \phi_{\mathcal{K}}^\epsilon(X_{t,x}^u(\theta)) - \phi_{\mathcal{K}}^\epsilon(X_{t,y}^u(\theta)) \right| \right] \end{aligned}$$

and

$$\begin{aligned} & |\vartheta^\epsilon(t, x) - \vartheta^\epsilon(s, x)| \tag{17} \\ & \leq \sup_{u \in \mathcal{U}} \mathbb{E} \left[ \left| \phi_{\mathcal{C}}^\epsilon(X_{s, X_{t,x}^u(s)}^u(T)) - \phi_{\mathcal{C}}^\epsilon(X_{s,x}^u(T)) \right| \vee \max_{\theta \in [t, s]} \left| \phi_{\mathcal{K}}^\epsilon(X_{t,x}^u(\theta)) - \phi_{\mathcal{K}}^\epsilon(x) \right| \right. \\ & \quad \left. \vee \max_{\theta \in [s, T]} \left| \phi_{\mathcal{K}}^\epsilon(X_{s, X_{t,x}^u(s)}^u(\theta)) - \phi_{\mathcal{K}}^\epsilon(X_{s,x}^u(\theta)) \right| \right]. \end{aligned}$$

It can be easily verified that  $\phi_{\mathcal{C}}^\epsilon$  and  $\phi_{\mathcal{K}}^\epsilon$  are Lipschitz continuous functions with Lipschitz constant  $1/\epsilon$ . Then by (16) and (17) we get

$$|\vartheta^\epsilon(t, x) - \vartheta^\epsilon(t, y)| \leq \frac{1}{\epsilon} \sup_{u \in \mathcal{U}} \mathbb{E} \left[ \max_{\theta \in [t, T]} |X_{t,x}^u(\theta) - X_{t,y}^u(\theta)| \right]$$

and

$$|\vartheta^\epsilon(t, x) - \vartheta^\epsilon(s, x)| \leq \frac{1}{\epsilon} \sup_{u \in \mathcal{U}} \mathbb{E} \left[ \max_{\theta \in [t, s]} |X_{t,x}^u(\theta) - x| \bigvee \max_{\theta \in [s, T]} |X_{s, X_{t,x}^u(s)}^u(\theta) - x| \right].$$

Under assumption (H1), there exist some constant  $C > 0$  such that for any  $0 \leq t \leq s \leq T$ ,  $x, y \in \mathbb{R}^d$  the unique strong solution to equation (5) satisfies

$$\mathbb{E} \left[ \sup_{\theta \in [t, s]} |X_{t,x}^u(\theta) - X_{t,y}^u(\theta)|^2 \right] \leq C|x - y|^2, \quad (18)$$

$$\mathbb{E} \left[ \sup_{\theta \in [t, s]} |X_{t,x}^u(\theta) - X_{s,x}^u(\theta)|^2 \right] \leq C(1 + |x|^2) |t - s|. \quad (19)$$

Hence, the result follows just taking  $L_\epsilon := C/\epsilon$ .  $\square$

*Remark 1.4.* It has been proved in [6, Theorem 3.1] that, if  $\mathcal{K} = \mathbb{R}^d$  and  $\mathcal{C}$  is a non empty, convex set with a  $C^1$  regular boundary, under the uniform ellipticity condition, for some  $\alpha > 0$ ,  $\forall (t, x, u) \in (0, T) \times \mathbb{R}^d \times U$ ,

$$\sigma(t, x, u)\sigma(t, x, u)^T \geq \alpha 1_d. \quad (20)$$

where  $1_d$  is the identity matrix, the following holds:

$$|\vartheta(t, x) - \vartheta^\epsilon(t, x)| \leq C \frac{1 + |x|^2 + |\log \epsilon|}{(T - t)^d} \epsilon \quad (21)$$

for some constant  $C$  depending only on  $\alpha$  and the constants in assumption (H1). We conjecture that analogous estimates can be obtain in the general case  $\mathcal{K} \neq \mathbb{R}^d$ , but a rigorous proof of this fact is still material of an ongoing research.

## 2. DYNAMIC PROGRAMMING AND HAMILTON-JACOBI-BELLMAN EQUATION

Aim of this section is to characterize the function  $\vartheta^\epsilon$  as a (viscosity) solution to a suitable HJB equation. For doing this, we closely follow the dynamic programming arguments recently developed in [10, 18] for optimal control problems with a cost depending on a running maximum. Therefore, in order to directly use those results in our framework, we will rewrite the optimal control problem (15) by means of the cost functional

$$J(t, x, u) := \mathbb{E} \left[ -\phi_{\mathcal{C}}^\epsilon(X_{t,x}^u(T)) \bigvee \max_{\theta \in [t, T]} -\phi_{\mathcal{K}}^\epsilon(X_{t,x}^u(\theta)) \right] \quad (22)$$



such that the following holds

$$\vartheta^\epsilon(t, x) = - \inf_{u \in \mathcal{U}} J(t, x, u).$$

The presence of the maximum operator inside the expectation, makes the cost in (22) non-Markovian preventing the direct use of the Dynamic Programming Principle (DPP), which is the first fundamental result towards the HJB characterisation. A classical strategy to overcome this difficulty consists in adding an auxiliary variable  $y$  that, roughly speaking, gets rid of the non-Markovian component of the cost. This has been originally used in [8] where an approximation technique of the  $L^\infty$ -norm is used, whereas in [10, 18] the HJB equation is derived without making use of any approximation.

Let us introduce the following value function:

$$w^\epsilon(t, x, y) := \inf_{u \in \mathcal{U}} \mathbb{E} \left[ - \phi_{\mathcal{C}}^\epsilon(X_{t,x}^u(T)) \bigvee \max_{\theta \in [t, T]} - \phi_{\mathcal{K}}^\epsilon(X_{t,x}^u(\theta)) \bigvee y \right]. \quad (23)$$

Defined the process

$$Y_{t,x,y}^u(\cdot) := \max_{s \in [t, \cdot]} - \phi_{\mathcal{K}}^\epsilon(X_{t,x}^u(s)) \bigvee y,$$

the value function (23) can also be written as

$$w^\epsilon(t, x, y) = \inf_{u \in \mathcal{U}} \mathbb{E} \left[ - \phi_{\mathcal{C}}^\epsilon(X_{t,x}^u(T)) \bigvee Y_{t,x,y}^u(T) \right].$$

Observe that the following property holds:

$$\vartheta^\epsilon(t, x) = -w^\epsilon(t, x, -1), \quad (24)$$

so from now on only the optimal control problem (23) will be taken into account, since the corresponding value of the function  $\vartheta^\epsilon$  can be derived by the previous equality. The following property is satisfied:

*Proposition 2.1.* Let assumption (H1) be satisfied. Then, there exists a constant  $C > 0$  such that for any  $\epsilon > 0$ ,  $t, s \in [0, T]$ ,  $(x, y), (x', y') \in \mathbb{R}^{d+1}$  one has

$$|w^\epsilon(t, x, y) - w^\epsilon(s, x', y')| \leq \frac{C}{\epsilon} \left( |x - x'| + |y - y'| + |t - s|^{\frac{1}{2}} (1 + |x| \vee |x'|) \right).$$

Moreover, for any family of stopping times  $\{\tau^u, u \in \mathcal{U}\}$  with values in  $[t, T]$  one has

$$w^\epsilon(t, x, y) = \inf_{u \in \mathcal{U}} \mathbb{E} \left[ w^\epsilon(\tau^u, X_{t,x}^u(\tau^u), Y_{t,x,y}^u(\tau^u)) \right] \quad (25)$$

for any  $(t, x, y) \in [0, T] \times \mathbb{R}^{d+1}$ .

*Proof.* The regularity of  $w^\epsilon$  with respect to  $t$  and  $x$  can be proved as in Proposition 1.3, while the Lipschitzianity with respect to  $y$  is trivial.

Thanks to the regularity of  $w^\epsilon$ , the DPP (25) follows by arguments similar to [12]

observing that for the couple of variables  $(X_{t,x}^u(\cdot), Y_{t,x,y}^u(\cdot))$  the following property holds:

$$\begin{pmatrix} X_{t,x}^u(s) \\ Y_{t,x,y}^u(s) \end{pmatrix} = \begin{pmatrix} X_{\theta, X_{t,x}^u(\theta)}^u(s) \\ Y_{\theta, X_{t,x}^u(\theta), Y_{t,x,y}^u(\theta)}^u(s) \end{pmatrix} \quad \text{a.s.}$$

for any  $t \leq \theta \leq s \leq T$  (with  $\theta$  possibly a stopping time). We remand to [10] for a sketch of the proof showing how the arguments in [12] adapt to our case.  $\square$

**2.1. HJB equation.** Proposition 2.1 is the main tool for proving the next result that characterizes  $w^\epsilon$  as the unique solution, in the viscosity sense, of a suitable HJB equation. In the sequel we will restrict our domain to

$$D := \{(x, y) \in \mathbb{R}^{d+1} : -\phi_{\mathcal{K}}^\epsilon(x) < y < 0\}.$$

Indeed, the knowledge of  $w^\epsilon$  in  $\overline{D}$  is sufficient to characterize it everywhere thanks to the following relation:

$$\begin{aligned} w^\epsilon(t, x, y) &= y && \text{for any } y \geq 0 \\ w^\epsilon(t, x, y) &= w^\epsilon(x, -\phi_{\mathcal{K}}^\epsilon(x)) && \text{for any } y \leq -\phi_{\mathcal{K}}^\epsilon(x). \end{aligned} \quad (26)$$

Based on this observation, we are going to prove that  $w^\epsilon$  is the unique solution of the HJB equation in the domain  $\overline{D}$  under additional mixed derivative-Dirichlet boundary conditions. Let

$$\Gamma_1 := \{(x, y) \in \overline{D} : y = 0\}; \quad \Gamma_2 := \{(x, y) \in \overline{D} : y = -\phi_{\mathcal{K}}^\epsilon(x)\}, \quad (27)$$

one has the following result:

*Theorem 2.2.* Let assumption (H1) be satisfied. Then,  $w^\epsilon$  is the unique continuous viscosity solution of the following HJB equation

$$\begin{cases} -\partial_t w + H(t, x, D_x w, D_x^2 w) = 0 & [0, T) \times D \\ w = 0 & [0, T) \times \Gamma_1 \\ -\partial_y w = 0 & [0, T) \times \Gamma_2 \\ w(T, x, y) = w_0(x, y) & \overline{D} \end{cases} \quad (28)$$

with

$$H(t, x, p, Q) := \sup_{u \in U} \left\{ -b(t, x, u)p - \frac{1}{2} \text{Tr}[\sigma \sigma^T](t, x, u)Q \right\} \quad (29)$$

and

$$w_0(x, y) := -\phi_{\mathcal{C}}^\epsilon(x) \bigvee -\phi_{\mathcal{K}}^\epsilon(x) \bigvee y.$$

We need here to specify in which sense boundary conditions are satisfied. In particular, we ask the Dirichlet boundary conditions on  $\Gamma_1$  to be satisfied in the strong sense, whereas the derivative conditions on  $\Gamma_2$  are considered in the (weak) viscosity sense. We point out that in our case the strong constant Dirichlet boundary condition on  $\Gamma_1$  is compatible with the homogeneous derivative condition on  $\Gamma_2$ . This prevents possible problems related with mixed boundary conditions at the junctions where different components of the boundary cross.

*Definition 2.3.* A USC function  $\underline{w}$  (resp. LSC function  $\overline{w}$ ) on  $[0, T] \times \overline{D}$  is a viscosity sub-solution (resp. super-solution) of (28), if for every function  $\varphi \in C^{1,2}([0, T] \times \overline{D})$ , at each maximum (resp. minimum) point  $(t, x, y)$  of  $\underline{w} - \varphi$  (resp.  $\overline{w} - \varphi$ ) the following inequality holds

$$\begin{cases} -\partial_t \varphi + H(t, x, D_x \varphi, D_x^2 \varphi) \leq 0 & [0, T] \times D \\ \varphi \leq 0 & [0, T] \times \Gamma_1 \\ \min(-\partial_y \varphi, -\partial_t \varphi + H(t, x, D_x \varphi, D_x^2 \varphi)) \leq 0 & [0, T] \times \Gamma_2 \\ \varphi(T, x, y) \leq w_0(x, y) & \overline{D} \end{cases}$$

(resp.

$$\begin{cases} -\partial_t \varphi + H(t, x, D_x \varphi, D_x^2 \varphi) \geq 0 & [0, T] \times D \\ \varphi \geq 0 & [0, T] \times \Gamma_1 \\ \max(-\partial_y \varphi, -\partial_t \varphi + H(t, x, D_x \varphi, D_x^2 \varphi)) \geq 0 & [0, T] \times \Gamma_2 \\ \varphi(T, x, y) \geq w_0(x, y) & \overline{D}. \end{cases}$$

A continuous function  $w$  on  $[0, T] \times \overline{D}$  is a viscosity solution of (28) if it is both a sub- and super-solution.

*Proof of Theorem 2.2.* The Dirichlet and terminal conditions are ensured by the very definition of  $w^\epsilon$  and its continuity. In particular, the continuity allows the conditions to be considered in the strong sense. Then, we only need to prove the result in  $[0, T] \times (D \cup \Gamma_2)$ .

The proof of sub- and supersolution properties follows quite straightforward by the the arguments in [10, Theorem 3.2] and [19, Theorem 4.1], so we don't reproduce it here.

Uniqueness of the solution relies on comparison results for sub and super solution. The proof can be found in [18, Appendix A]. We point out that the fact of considering Dirichlet conditions in a strong sense is an important requirement for the proof of the comparison principle.  $\square$

### 3. NUMERICAL APPROXIMATION

In this section we discuss an approximation scheme for the unique continuous viscosity solution  $w^\epsilon$  to the equation

$$\partial_t w + H(t, x, D_x w, D_x^2 w) = 0 \quad -\phi_{\mathcal{K}}^\epsilon(x) < y < 0, \quad t \in (0, T] \quad (30a)$$

$$w = 0 \quad y = 0, \quad t \in (0, T] \quad (30b)$$

$$-\partial_y w = 0 \quad y = -\phi_{\mathcal{K}}^\epsilon(x), \quad t \in (0, T] \quad (30c)$$

with initial data

$$w(0, x, y) = w_0^\epsilon(x, y) \quad -\phi_{\mathcal{K}}^\epsilon(x) \leq y \leq 0 \quad (30d)$$

(the convenient change of variable  $t \rightarrow T - t$  has been here applied).

In [10, Section 4.1] a general convergence result for numerical schemes approximating HJB equations under oblique derivative boundary conditions as (30c) is

provided. Those arguments can be easily modified in order to prove convergence also in presence of the additional Dirichlet boundary condition (30b) (see also [19]). Following the same lines, we present here a semi-Lagrangian (SL) scheme for the approximation of (30). The same scheme will be used in the numerical experiments in Section 4.

Let  $N \geq 1$  be an integer (number of time steps), and let

$$h := \frac{T}{N} \quad \text{and} \quad t_n := n\Delta t$$

for  $n = 0, \dots, N$ . Let  $\Delta x = (\Delta x_1, \dots, \Delta x_d) \in (\mathbb{R}_+^*)^d$  and  $\Delta y > 0$ , and let  $\mathcal{G}_\rho$  (where  $\rho \equiv (\Delta x, \Delta y)$ ) be the space grid

$$\mathcal{G}_\rho := \{(x_i, y_j) = (i\Delta x, j\Delta y), \text{ for } (i, j) \in \mathbb{Z}^d \times \mathbb{Z}\}.$$

The grid is considered uniform for simplicity of presentation. We also assume that the discretization in the  $y$  coordinate is aligned with the boundary of the domain, this allows us to get the Dirichlet condition exactly.

We look for a fully discrete scheme for the viscosity solution of (30) on the time-space grid  $\{t_0, \dots, t_N\} \times (\mathcal{G}_\rho \cap \bar{D})$ . Following the ideas in [10, 19] the numerical scheme is defined starting from a standard scheme for (30a), which is then mixed with a step of “projection” on  $\Gamma_2$  and the use of the Dirichlet condition on  $\Gamma_1$ . The approximation of equation (30a) we consider is the SL scheme proposed by Camilli and Falcone [13] and also used in [14]. We recall that first schemes of this type have been introduced by Menaldi in [21].

Let  $\sigma^u = \sigma(\cdot, \cdot, u)$  and  $b^u = b(\cdot, \cdot, u)$ , and let  $(\sigma_k^u)_{k=1, \dots, p}$  denote the column vectors of  $\sigma^u$ . We consider the following operator  $\mathcal{T}$ :

$$\mathcal{T}(\varphi)(t, x, y) = \min_{u \in U} \frac{1}{2p} \left( \sum_{k=1, \dots, 2p} [\varphi(t, \cdot, y)](x + hb^u(t, x) + \sqrt{h}\bar{\sigma}_k^u(t, x)) \right) \quad (31)$$

with the following vector definition in  $\mathbb{R}^d$ :

$$\bar{\sigma}_{2k-j}^u := \sqrt{p} (-1)^j \sigma_k^u \quad (32)$$

for  $k = 1, \dots, p$  and  $j \in \{0, 1\}$ . Now  $[\cdot] \equiv [\cdot]_x$  stands for a monotone,  $P_1$  interpolation operator on the  $x$ -grid  $(x_i)$ , satisfying in particular:

$$\begin{cases} (i) [\varphi](x_i) = \varphi(x_i), \forall i, \\ (ii) |[\varphi](x) - \varphi(x)| \leq C|\Delta x|^2 \|D_x^2 \varphi\|_\infty \text{ for any } \varphi \in C^2(\mathbb{R}^d, \mathbb{R}), \\ (iii) \text{ for any functions } \varphi, \psi : \mathbb{R}^d \rightarrow \mathbb{R}, \varphi \leq \psi \Rightarrow [\varphi] \leq [\psi]. \end{cases} \quad (33)$$

The numerical scheme is defined as follows:

**Algorithm.** Initialization step, for  $n = 0$ , for all  $i, j$ :

$$W_{i,j}^0 = w_0^\epsilon(x_i, y_j).$$

Then, for  $n = 0, \dots, N - 1$ :

**Step 1:** Compute  $W_{i,j}^{n+1} = \mathcal{T}(W)(t_n, x_i, y_j)$ , for all  $(x_i, y_j) \in \mathcal{G}_\rho \cap \overline{D}$ ;

**Step 2:** Assign  $W_{i,j}^{n+1} = W_{i,j_{x_i}}^n$ , for all  $(x_i, y_j) : y_j \leq -\phi_{\mathcal{K}}^\epsilon(x_i)$ ;

$W_{i,j}^{n+1} = y_j$ , for all  $(x_i, y_j) : y_j \geq 0$ ;

where for every  $x \in \mathbb{R}^d$ ,  $j_x \in \mathbb{Z}$  is defined by

$$j_x := \min \{j \in \mathbb{Z} : j\Delta y \geq -\phi_{\mathcal{K}}^\epsilon(x)\}$$

and we also used the following short notation

$$W_{i,j}^n = W(t_n, x_i, y_j).$$

Hereafter we will denote by  $W$  the solution of the numerical scheme defined by the algorithm above on  $\{t_0, \dots, t_N\} \times \mathcal{G}_\rho$  and by  $W^\rho$  its continuous extension to  $[0, T] \times \mathbb{R}^d \times \mathbb{R}$  obtained by linear interpolation. We point out that the necessity of defining  $W$  outside  $\overline{D}$  comes from the fact that the SL scheme involves values outside the domain. However, this is not a issue in virtue of (26).

*Remark 3.1.* The core of the scheme in Step 1 can be written as

$$S(t, x, y, W_{i,j}^{n+1}, W) := W_{i,j}^{n+1} - \mathcal{T}(W)(t_n, x_i, y_j) = 0.$$

It is immediate to verify that  $S$  is monotone in the sense of Barles and Souganidis [7], i.e. for every  $h, \rho > 0$ ,  $r \in \mathbb{R}$ , for all function  $\phi, \psi$  such that  $\phi \geq \psi$ , there holds

$$S(t, x, y, r, \phi) \leq S(t, x, y, r, \psi).$$

Moreover, the choice of  $\bar{\sigma}_k^u$  in (32) leads to the following consistency estimate, for any  $\varphi \in C^{2,4}((0, T) \times \mathbb{R}^d \times \mathbb{R})$ :

$$\begin{aligned} & \left| \frac{1}{h} S(t, x, y, \varphi(t, x, y), \phi) - \left( \partial_t \varphi + H(t, x, D_x \varphi, D_x^2 \varphi) \right) \right| \\ & \leq C_1 \left( |b^u(t, x)|^2 \|D_x^2 \varphi\|_\infty + |b^u(t, x)| |\sigma^u(t, x)|^2 \|D_x^3 \varphi\|_\infty + |\sigma^u(t, x)|^4 \|D_x^4 \varphi\|_\infty \right. \\ & \quad \left. \|\partial_{tt}^2 \varphi\|_\infty \right) h + C_2 \|D_x^2 \varphi\|_\infty \frac{|\Delta x|^2}{h}. \end{aligned}$$

These are classical properties of SL schemes, see [14] for instance. In particular, the error term in  $\frac{|\Delta x|^2}{h}$  comes the interpolation error estimate (ii), and the term in  $h$  from classical Taylor expansions. Moreover it is easy to verify that the scheme admits a bounded solution in  $\{t_0, \dots, t_N\} \times (\mathcal{G}_\rho \cap \overline{D})$ , so that the scheme is also stable. We recall that monotonicity, consistency and stability are the fundamental property necessary for proving convergence of numerical schemes in the framework of viscosity solutions, see [7].

**Theorem 3.1.** Let assumption (H1) be satisfied. Let  $W^\rho$  be solution of the scheme defined by Algorithm 1, where  $\mathcal{T}$  is the SL scheme (31)-(32). Then, if

$$\frac{|\Delta x|^2}{\Delta t} \rightarrow 0 \tag{34}$$

$W^\rho$  converges to  $w^\epsilon$  in  $[0, T] \times \overline{D}$ .

*Proof.* The proof follows the strategy in [7]. Let us define for  $(t, x, y) \in [0, T] \times \overline{D}$

$$\begin{aligned}\overline{W}(t, x, y) &:= \limsup_{\substack{[0, T] \times \overline{D} \ni (s, \xi, \gamma) \rightarrow (t, x, y) \\ \rho \rightarrow 0}} W^\rho(s, \xi, \gamma), \\ \underline{W}(t, x, y) &:= \liminf_{\substack{[0, T] \times \overline{D} \ni (s, \xi, \gamma) \rightarrow (t, x, y) \\ \rho \rightarrow 0}} W^\rho(s, \xi, \gamma).\end{aligned}$$

If  $(t, x, y) \in [0, T] \times (D \cup \Gamma_2)$  the local uniform convergence follows by the proof of Theorem 4.1 in [10]. It remains to prove that  $\overline{W}$  and  $\underline{W}$  satisfy the Dirichlet condition pointwise on  $\Gamma_1$ . For this purpose it is worth to observe that  $W$  is Lipschitz continuous in  $y$ , i.e. there exists some constant  $L > 0$  such that

$$|W^\rho(t, x, y) - W^\rho(t, x, y')| \leq L|y - y'|$$

for any  $t \in [0, T], x \in \mathbb{R}^d, y, y' \in \mathbb{R}$ . Therefore, on  $\Gamma_1$  (i.e for  $y = 0$ ), one has

$$|W^\rho(s, \xi, \eta) - (-1)| = |W^\rho(s, \xi, \eta) - W^\rho(s, \xi, 0)| \leq L|\eta|$$

so that

$$\lim_{\substack{[0, T] \times \overline{D} \ni (s, \xi, \gamma) \rightarrow (s, x, y) \\ \rho \rightarrow 0}} W^\rho(s, \xi, \gamma) = -1$$

and the convergence in the whole domain  $[0, T] \times \overline{D}$  is proved.  $\square$

#### 4. NUMERICAL TESTS

In this section we present some numerical tests for probabilistic reachability problems in presence of state constraints. To solve the HJB equation (30), we use the fully discrete SL scheme introduced in Section 3 implemented on the ROC-HJ solver available at the following link <https://uma.ensta-paristech.fr/soft/ROC-HJ/>. The minimum in (31) is performed on a subset of control values  $\{u_1, \dots, u_{N_u}\}$  that represents a discretization of  $U$  with a mesh size  $\Delta u$ . Relying on the error analysis in [6], in all the simulations the regularization parameter will be chosen to be  $\epsilon = \frac{1}{\Delta x}$ .

**4.1. Example 1.** We consider the following controlled stochastic system:

$$dX(s) = \begin{pmatrix} -1 & -4 \\ 4 & -1 \end{pmatrix} (X(s) + u(s)) ds + \begin{pmatrix} 0.7 & 0 \\ 0 & 0.7 \end{pmatrix} \begin{pmatrix} dB_1(s) \\ dB_2(s) \end{pmatrix} \quad (35)$$

where  $u(s) = \begin{pmatrix} u_1(s) \\ u_2(s) \end{pmatrix}$ ,  $u_i(s) \in [-0.1, 0.1]$ , for  $i = 1, 2$  and  $B_1, B_2$  are two independent Brownian motions.

The linear system (35) has been used in [6] to validate the HJB approach in the characterization of an approximated probabilistic reachable set without state constraints and in [5] to illustrate an approximation of the probability of reaching a

target by using enclosing hulls of probability density functions.

We set  $T = 1.75$  and define the target  $\mathcal{C} := (0.5, 1.5)^2$  (green square in Figure 2). The constraint is given by the presence of an obstacle, represented in Figure 2 (right) by the red rectangle, i.e.  $\mathcal{K} := \mathbb{R}^2 \setminus ([-6, -2] \times [2, 4])$ . We compute the set  $\Omega_t^\rho$  (blue region) for  $t = 0.75$  and  $\rho = 0.4$  in presence (Figure 2, right) or not (Figure 2, left) of the obstacle. To approximate the auxiliary function  $w^\epsilon$  solution to (30), the numerical simulation is performed on a computational domain  $[-8, 8]^2 \times [-1, 0]$ . The corresponding values of  $\vartheta^\epsilon$  are then obtained using relation (24).

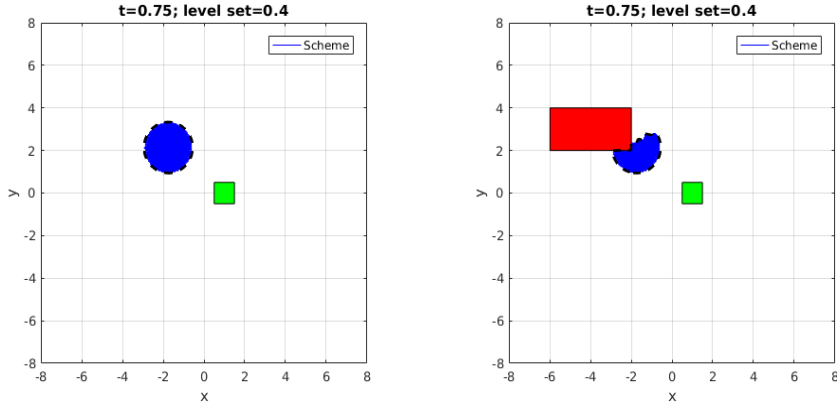


FIGURE 2. (Example 1) Backward reachable sets at  $t = 0.75$  for a time horizon  $T = 1.75$  and  $\rho = 0.4$  without (left) and with (right) obstacle. The target set, the obstacle and the backward reachable set  $\Omega_{0.75}^{0.4}$  are represented respectively by the green square, the red rectangle and the blue region.

Figure 3 (top) shows the set  $\Omega_t^\rho$  for  $\rho = 0.4$  at different time  $t \in \{0.25; 0\}$  in presence of the obstacle. Then, in Figure 3 (bottom) we simulate different optimal paths starting from a given point of the backward reachable set using the algorithm described below.

**Algorithm (trajectory reconstruction)** Initialization: Set  $X_0 = \bar{x}$ .

For  $k = 0$  to  $N - 1$ :

**Step 1:** Compute optimal control at  $t = t_k$ :

$$u^k = \arg \min_{u \in \{u_1, \dots, u_{N_u}\}} \mathbb{E}[W(t_{k+1}, X_{k+1}^u, -1)] \quad (36)$$

where for  $u_i \in \{u_1, \dots, u_{N_u}\}$

$$X_{k+1} := X_k + b(t_k, X_k, u_i)h + \sigma(t_k, X_k, u_i)\sqrt{h}\xi, \quad (37)$$

here  $\xi := (\xi_1, \xi_2)$  with  $\xi_i$  ( $i = 1, 2$ ) random variables following a  $N(0, 1)$  distribution.

**Step 2:** Compute the point of the optimal trajectory:

$$X_{k+1} := X_k + b(t_k, X_k, u^k)h + \sigma(t_k, X_k, u^k)\sqrt{h}\xi \quad (38)$$

where again  $\xi_i \sim N(0, 1)$  for  $i = 1, 2$ .

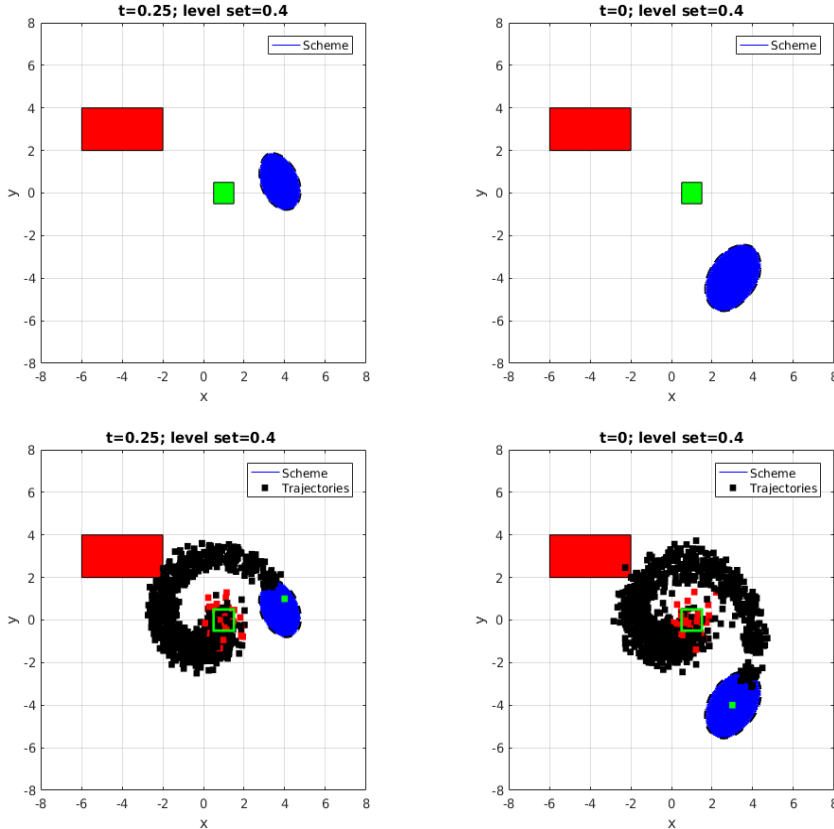


FIGURE 3. (Example 1). Top: backward reachable set (blue region) at times  $t \in \{0.25, 0\}$  for a final time horizon  $T = 1.75$  in presence of the obstacle (red rectangle). Bottom: reconstruction of some optimal paths starting from point  $\bar{x} = (4, 1)$  (bottom, left) and  $\bar{x} = (3, -4)$  (bottom, right).

In order to validate our approach, we compare the value of the scheme in a given point with the percentage of trajectories reaching the target without hitting the obstacle. We consider the case  $t = 0.25$  and two different starting points  $\bar{x}_1 := (4.0, 1.0)^T$  and  $\bar{x}_2 := (3.0, 0.0)^T$ . The approximation of the level-set function obtained by numerically solving the corresponding HJB equation on the grid  $\Delta x_1 = \Delta x_2 = 0.0125, \Delta y = 0.1, h = 0.025$  at points  $\bar{x}_1$  and  $\bar{x}_2$  is respectively  $-W(t, \bar{x}_1, -1) \simeq 0.459 \pm 0.004$  and  $-W(t, \bar{x}_2, -1) \simeq 0.404 \pm 0.009$ .



The results of Monte Carlo simulations are reported in Table 1. One can conclude that the approximated value of the level set function belongs in each case to the confidence interval.

	$M$	$p$	C.I.	MC-error
$\bar{x}_1$	6000	0.4630	(0.4504, 0.4756)	0.0126
	12000	0.4624	(0.4535, 0.4713)	0.0089
	25000	0.4603	(0.4541, 0.4664)	0.0062
	50000	0.4618	(0.4574, 0.4661)	0.0045
	100000	0.4628	(0.4597, 0.4659)	0.0031
$\bar{x}_2$	6000	0.3915	(0.3593, 0.4037)	0.0122
	12000	0.3991	(0.3705, 0.4078)	0.0087
	25000	0.4026	(0.3966, 0.4087)	0.0061
	50000	0.4016	(0.3996, 0.4081)	0.0043
	100000	0.4015	(0.3985, 0.4045)	0.0030

TABLE 1. (Example 1) Percentage  $p$  of  $M$  simulated trajectories that reach the target set without hitting the obstacle, with a corresponding confidence interval (C.I.), and a Monte Carlo error estimate (MC-error)

4.2. **Example 2.** We now test our method on the same example used in [10, Section 6]. Let us consider the following dynamics:

$$dX(s) = u(s) \begin{pmatrix} 1 \\ 0 \end{pmatrix} ds + u(s)\sigma(X(s))dB(s), \quad s \geq t, \quad (39)$$

where  $B$  is a one-dimensional Brownian motion,  $U = [0, 1] \subset \mathbb{R}$  and the volatility  $\sigma(x)$  is given by

$$\sigma(x) := 5 d_{\Theta}(x) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

where  $d_{\Theta}$  denotes the distance function to the set

$$\Theta := \{(x_1, x_2), |x_2| \geq 0.3\}.$$

The target set is  $\mathcal{C} = (0, 0.4) \times (-0.5, 0.5)$  (green rectangle in Figure 4) and the state constraint is  $\mathcal{K} = \mathbb{R}^2 \setminus ([-0.4, 0.2] \times [-0.1, 0.1])$  (i.e. the entire space except the red square obstacle in Figure 4). We fix  $T = 0.5$  and consider the computational domain  $[-1, 1]^2 \times [-1, 0]$ .

The strong degeneracy of the diffusion term in this example allowed in [10] to obtain the “almost sure” backward reachable set, corresponding here to the case

$\rho = 1$ . Figure 4 shows the approximation of  $\Omega_t^\rho$  for  $t = 0$  and different levels  $\rho \in [0, 1, 0.9]$ . The black region corresponds to the exact backward reachable set for  $\rho = 1$ . Indeed, due to the simple dynamics considered it is possible for this example to infer the exact set  $\Omega_0^1$ , i.e. the set of points from where the target is reached and the constraint satisfied with probability one, see [10] for a further discussion. One can observe that as  $\rho$  approaches the value 1, we recover the results obtained in [10]. A loss of precision appears at corners. This was already noticed in [10] and it is due to the smoothing effects of the diffusion term (see [10, Figure 2, Section 6]) which can be reduced with the refinement of the mesh.

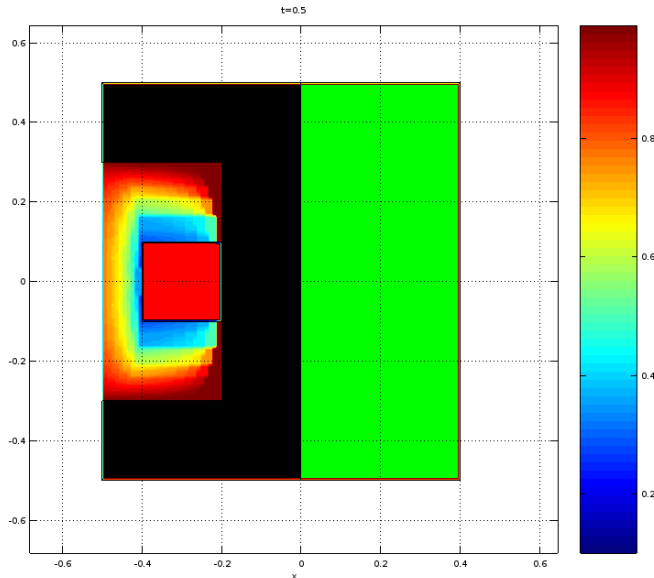


FIGURE 4. (Example 2) approximation of  $\Omega_t^\rho$  at  $t = 0$  for different levels  $\rho \in [0, 1, 0.9]$  computed with  $\Delta x_1 = \Delta x_2 = 0.005, \Delta y = 0.1, h = 0.01$  (same mesh parameters used in [10]).

## 5. CONCLUSIONS

In this paper we have used the HJB theory for characterising the probabilistic backward reachable set for a system of controlled diffusions in presence of state constraints. We have shown that such a set is a level set of the value function associated to a suitable optimal control problem. To deal with the discontinuity of the cost functional associated to this problem, arising from the use of indicator functions for representing probabilities, we have defined a regularised problem. Precise estimates of the error introduced by this regularisation are still object of ongoing research.

Following the approach in [10, 18, 19], for the regularised problem we have obtained a characterization by a HJB equation with mixed Dirichlet-derivative

boundary conditions. We have defined a fully discrete SL approximation scheme and we have proved its convergence to the unique viscosity solution of the equation. Then, we have used such a scheme in order to validate our approach on some numerical tests. We focused on the examples studied in [6] and [10], adding state constraints to the first one and variable levels of probability to the second one. More complex tests on concrete models are a promising future direction of work.

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