Multiple Impacts in Dissipative Granular Chains

Ngoc Son Nguyen, Bernard Brogliato

Multiple Impacts in Granular Chains

N.S. Nguyen and B. Brogliato

This chapter presents general features of the dynamics of chains of aligned balls. The dissipation and the dispersion of the kinetic energy at an impact are studied, independently of any impact law. A dispersion index and a dissipation index are defined, which will be used all through the next chapters. Optimization under constraints is used to analyze the variations of these two indices, where the constraints are imposed by the physics (momentum conservation, energetic consistency, kinematic consistency). A 3-ball chain is analyzed in detail.

2.1 Dynamics of a Chain of Aligned Balls

Let us consider a chain of N balls, each of which is constrained to move on a frictionless straight line in order to ensure colinear collisions between the balls, as illustrated in Figure 2.1. Each ball has a radius R_i and is located at position x_i. The number of contacts s in the chain equals $N-1$. The balls in the chain are indexed as 1, 2, ..., N and the contacts are indexed as 1, 2, ..., $N-1$. Contact i is between balls i and $i+1$. Initially, the first ball moves with a transitional velocity V_s and strikes the other balls that are at rest and barely touch each other. According to Definition 1.1 this is a multiple impact problem where the striked surface has codimension $N-1$. Note that, due to central collisions between balls, there is no rotation of the balls during the impacts. A question that arises here is how we can determine the velocities of the balls after impact. Despite the fact that a chain of balls is apparently simple, the answer to the above question is not simple at all. In the following, we will discuss how the multiple impact problem in a chain of balls can be modeled.

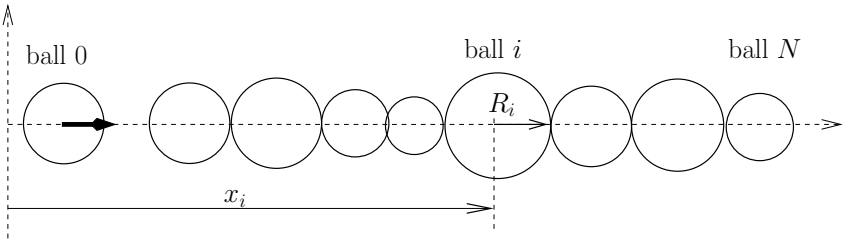


Fig. 2.1 A chain of balls

2.1.1 Lagrangian Dynamics

Let us describe the state of a chain of balls by a generalized coordinate vector: $\boldsymbol{q} = [x_1, x_2, ..., x_N]^T$. This system is subjected to $N-1$ unilateral constraints:

$$g_i(\boldsymbol{q}) = x_{i+1} - x_i - (R_{i+1} + R_i) \geq 0, \ \forall i = 1, 2, ..., N-1. \tag{2.1}$$

These unilateral constraints define the feasible region Φ:

$$\Phi = \{\boldsymbol{q} \in \mathbb{R}^N \, | g_i(\boldsymbol{q}) \geq 0, \ \forall i = 1, 2, ..., N-1\} \tag{2.2}$$

within which the system has to evolve. The right velocity $\boldsymbol{u}^+ = [V_1^+, V_2^+, ..., V_N^+]^T$ is constrained to belong to the convex tangent cone $\mathcal{T}_\Phi(\boldsymbol{q})$ to the feasible region Φ at point $\boldsymbol{q}$:

$$\mathcal{T}_\Phi(\boldsymbol{q}) = \{\boldsymbol{u} \in \mathbb{R}^N \, | \nabla^T g_i \boldsymbol{u} \geq 0, \ i = 1, 2, ..., N-1\}. \tag{2.3}$$

The dynamics of the chain of balls under consideration is described by the Lagrangian equation and the complementarity condition between the gap function $g_i(q)$ and the contact force λ_i:

$$\begin{cases} \boldsymbol{M}\ddot{\boldsymbol{q}}(t) = \boldsymbol{F}_{ext}(t) + \boldsymbol{W}\boldsymbol{\lambda}(t) \\ 0 \leq g_i(\boldsymbol{q}) \perp \lambda_i(t) \geq 0, \ i = 1, 2, ..., N-1, \end{cases} \tag{2.4}$$

where:

- $\boldsymbol{M}$ is the mass matrix defined as:

$$\boldsymbol{M} = \begin{bmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m_N \end{bmatrix}_{N \times N} \tag{2.5}$$

  with m_i being the mass of ball i;
- $\boldsymbol{F}_{ext}$ is the external force applied to the system;

- $\boldsymbol{W}$ is the gradient of the constraints defined in (2.1):

$$\boldsymbol{W} = [\nabla g_1,\ \nabla g_2, ..., \nabla g_{N-1}] = \begin{bmatrix} -1 & 0 & 0 & \cdots & 0 \\ 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{N\times(N-1)} \tag{2.6}$$

- $\boldsymbol{\lambda} = [\lambda_1,\ \lambda_2, ...,\ \lambda_{N-1}]^T$ with λ_i being the normal force at contact i.

Remark 2.1. Chains of aligned balls are therefore a very particular type of unilaterally constrained systems, where gradients of constraints and the mass matrix are constant. Some assumptions usually done in impact mechanics (like a constant position during the impacts, implying constant mass matrix and constraints gradients) are consequently automatically fulfilled.

2.1.2 Impact Equation

The equation describing the impact dynamics can be derived by integrating (2.4) over the impact period:

$$\int_{t^-}^{t^+} \boldsymbol{M}\ddot{\boldsymbol{q}}(t)dt = \int_{t^-}^{t^+} \boldsymbol{F}_{ext}(t)dt + \int_{t^-}^{t^+} \boldsymbol{W}\boldsymbol{\lambda}(t)dt, \tag{2.7}$$

where t^- and t^+ are the instants at the beginning and at the end of the impact process. During the impact process, the contact force between the colliding bodies is impulsive, *it evolves highly over a very brief interval of time*. This is the reason why it is usually adopted in impact mechanics that the contact force is predominant in comparison with other forces. Therefore, the first term in the right-hand side of (2.7) is negligible compared to the second term, thus can be neglected in (2.7). Finally, the impact equation is obtained as:

$$\boldsymbol{M}(\boldsymbol{u}^+ - \boldsymbol{u}^-) = \boldsymbol{W}\boldsymbol{p} \tag{2.8}$$

where $\boldsymbol{p} = [p_1, p_2, ..., p_{N-1}]^T$ with p_i being the impulse at contact i:

$$p_i = \int_{t^-}^{t^+} \lambda_i(t)dt,$$

and $\boldsymbol{u}^-$ and $\boldsymbol{u}^+$ are the pre- and post-impact velocities: $\boldsymbol{u}^- = \dot{\boldsymbol{q}}(t^-)$ and $\boldsymbol{u}^+ = \dot{\boldsymbol{q}}(t^+)$, respectively. It can be noted that there are N equations in (2.8), while there are $2N-1$ unknowns V_1^+, V_2^+,..., V_N^+ and p_1, p_2,..., p_{N-1}. Consequently, one needs to supply $N-1$ independent relations in order to determine uniquely the post-impact velocities V_i^+. This is done by an impact

law. In fact, an impact law relates the post-impact velocity $\boldsymbol{u}^+$ to the pre-impact velocity $\boldsymbol{u}^-$, and to the geometrical and mechanical properties of the chain. Obtaining an impact law is, in general, a hard task and needs to do some hypotheses that are only justified for some particular cases. The post-impact velocity given by any impact law must satisfy the three following consistencies:

- **Kinematic consistency** implies that the relative post-impact velocities at all the contacts must be positive or equal to zero:

$$\gamma_i^+ = V_{i+1}^+ - V_i^+ \geq 0, \ \forall i = 1, 2, ..., N-1; \tag{2.9}$$

- **Kinetic consistency** requires that the impulses at all the contacts must be positive (compressive character):

$$p_i \geq 0, \ \forall i = 1, 2, ..., N-1; \tag{2.10}$$

- **Energetic consistency** requires that the post-impact kinetic energy T^+ must be smaller or equal to the pre-impact one T^-:

$$T^+ = \sum_{i=1}^{N} \frac{m_i (V_i^+)^2}{2} \leq T^- = \sum_{i=1}^{N} \frac{m_i (V_i^-)^2}{2}. \tag{2.11}$$

Remark 2.2 (Energy loss during impact). There are several ways to express the kinetic energy loss at an impact. Since we deal with chains of balls the simplest expression is the sum of the individual energies, like in (2.11). In a more general setting, one may start from generalized velocities $\boldsymbol{u}$ and the mass matrix as in (2.8). This allows one to derive equivalent expressions for the kinetic energy loss at an impact instant t:

$$\begin{aligned}
T_L(t) &\stackrel{\Delta}{=} T(t^+) - T(t^-) \\
&= \frac{1}{2}(\boldsymbol{u}^+)^T \boldsymbol{M} \boldsymbol{u}^+ - \frac{1}{2}(\boldsymbol{u}^-)^T \boldsymbol{M} \boldsymbol{u}^- \\
&= \frac{1}{2}(\boldsymbol{u}^+ - \boldsymbol{u}^-)^T \boldsymbol{M} (\boldsymbol{u}^+ + \boldsymbol{u}^-) \\
&= \frac{1}{2}(\boldsymbol{u}^+ + \boldsymbol{u}^-)^T \boldsymbol{W} \boldsymbol{p} \\
&= \frac{1}{2}(\boldsymbol{W}\boldsymbol{p} + \boldsymbol{M}\boldsymbol{u}^-)^T \boldsymbol{M}^{-1} (\boldsymbol{W}\boldsymbol{p} + \boldsymbol{M}\boldsymbol{u}^-) - \frac{1}{2}(\boldsymbol{u}^-)^T \boldsymbol{M} \boldsymbol{u}^- \\
&= \frac{1}{2}\boldsymbol{p}^T \boldsymbol{W}^T \boldsymbol{M}^{-1} \boldsymbol{W} \boldsymbol{p} + \boldsymbol{p}^T \boldsymbol{W}^T \boldsymbol{u}^-.
\end{aligned} \tag{2.12}$$

The symmetric matrix $\boldsymbol{W}^T \boldsymbol{M}^{-1} \boldsymbol{W}$ is called a Delassus' matrix. If the constraints are independent, it is a full-rank matrix. The last expressions in (2.12) combined with the condition $T_L(t) \leq 0$ yields an ellipsoid for the admissible impulse $\boldsymbol{p}$, which is similar to that obtained for the collision between

two bodies [30]. Let $\boldsymbol{\gamma} \triangleq \boldsymbol{W}^T\boldsymbol{u}$ be the vector of normal velocities at the contact/impact points. Using (2.8) to get $\boldsymbol{p} = (\boldsymbol{W}^T\boldsymbol{M}^{-1}\boldsymbol{W})^{-1}(\boldsymbol{\gamma}^+ - \boldsymbol{\gamma}^-)$ and performing some manipulations one obtains:

$$\begin{aligned} T_L(t) &= \frac{1}{2}(\boldsymbol{\gamma}^+ - \boldsymbol{\gamma}^-)^T(\boldsymbol{W}^T\boldsymbol{M}^{-1}\boldsymbol{W})^{-1}(\boldsymbol{\gamma}^+ - \boldsymbol{\gamma}^-) \\ &\quad + (\boldsymbol{\gamma}^+ - \boldsymbol{\gamma}^-)^T(\boldsymbol{W}^T\boldsymbol{M}^{-1}\boldsymbol{W})^{-1}\boldsymbol{\gamma}^- \\ &= \frac{1}{2}(\boldsymbol{\gamma}^+ + \boldsymbol{\gamma}^-)^T\boldsymbol{p}. \end{aligned} \tag{2.13}$$

The last expression is sometimes named the Thomson and Tail formula. Many collision rules relate $\boldsymbol{\gamma}^+$ and $\boldsymbol{\gamma}^-$, so that the expression in (2.13) is useful to study their energetical consistency. For instance, suppose that $\boldsymbol{\gamma}^+ = -\mathcal{E}\boldsymbol{\gamma}^-$ where matrix $\mathcal{E}$ is called *restitution matrix*. Then (2.13) is equivalently rewritten as:

$$\begin{aligned} T_L(t) &= \frac{1}{2}(\boldsymbol{\gamma}^-)^T(\mathcal{E} + I)^T(\boldsymbol{W}^T\boldsymbol{M}^{-1}\boldsymbol{W})^{-1}(\mathcal{E} - I)\boldsymbol{\gamma}^- \\ &= \frac{1}{2}(\boldsymbol{\gamma}^-)^T\mathcal{E}^T(\boldsymbol{W}^T\boldsymbol{M}^{-1}\boldsymbol{W})^{-1}\mathcal{E}\boldsymbol{\gamma}^- - \frac{1}{2}(\boldsymbol{\gamma}^-)^T(\boldsymbol{W}^T\boldsymbol{M}^{-1}\boldsymbol{W})^{-1}\boldsymbol{\gamma}^- \\ &\quad - \frac{1}{2}(\boldsymbol{\gamma}^-)^T\mathcal{E}^T(\boldsymbol{W}^T\boldsymbol{M}^{-1}\boldsymbol{W})^{-1}\boldsymbol{\gamma}^- + \frac{1}{2}(\boldsymbol{\gamma}^-)^T(\boldsymbol{W}^T\boldsymbol{M}^{-1}\boldsymbol{W})^{-1}\mathcal{E}\boldsymbol{\gamma}^- \\ &= \frac{1}{2}(\boldsymbol{\gamma}^-)^T\mathcal{E}^T(\boldsymbol{W}^T\boldsymbol{M}^{-1}\boldsymbol{W})^{-1}\mathcal{E}\boldsymbol{\gamma}^- - \frac{1}{2}(\boldsymbol{\gamma}^-)^T(\boldsymbol{W}^T\boldsymbol{M}^{-1}\boldsymbol{W})^{-1}\boldsymbol{\gamma}^- \\ &= \frac{1}{2}(\boldsymbol{\gamma}^-)^T[\mathcal{E}^T(\boldsymbol{W}^T\boldsymbol{M}^{-1}\boldsymbol{W})^{-1}\mathcal{E} - (\boldsymbol{W}^T\boldsymbol{M}^{-1}\boldsymbol{W})^{-1}]\boldsymbol{\gamma}^-. \end{aligned} \tag{2.14}$$

Note that the last expression in (2.14) is obtained by using the symmetry of the Delassus' matrix, which gives:

$$\begin{aligned} (\boldsymbol{\gamma}^-)^T\mathcal{E}^T(\boldsymbol{W}^T\boldsymbol{M}^{-1}\boldsymbol{W})^{-1}\boldsymbol{\gamma}^- &= ((\boldsymbol{\gamma}^-)^T\mathcal{E}^T(\boldsymbol{W}^T\boldsymbol{M}^{-1}\boldsymbol{W})^{-1}\boldsymbol{\gamma}^-)^T \\ &= (\boldsymbol{\gamma}^-)^T(\boldsymbol{W}^T\boldsymbol{M}^{-1}\boldsymbol{W})^{-1}\mathcal{E}\boldsymbol{\gamma}^-. \end{aligned} \tag{2.15}$$

These expressions can serve as a basis for the energetic consistency analysis. The last expression is the direct generalization of the single impact, frictionless case (see for instance Equation (4.44) in [20]) to the multiple impact case. It is clear that the energetical consistency will imply some kind of smallness of the restitution matrix. However, the Delassus' matrix introduces some distortion compared to the single impact case in which one concludes that the restitution coefficient must be less than or equal to one. Actually, as noted in [6], even if $\mathcal{E}$ is a diagonal matrix with non-negative entries e_i, imposing $0 \leq e_i \leq 1$ is not sufficient to guarantee that $-\mathcal{E}^T(\boldsymbol{W}^T\boldsymbol{M}^{-1}\boldsymbol{W})^{-1}\mathcal{E} + (\boldsymbol{W}^T\boldsymbol{M}^{-1}\boldsymbol{W})^{-1}$ is positive semi definite. It is noteworthy that the copositivity of this matrix is sufficient to guarantee that $T_L \leq 0$ since $\boldsymbol{\gamma}^- \leq \boldsymbol{0}$. Several criteria that guarantee kinetic energy loss have been proposed in [21, 73, 133, 134]. For instance Propositions 7.1, 7.2 and 7.3 in [133] which characterize the positive definiteness of the product of symmetric positive definite matrices, may be used in the first line of (2.14).

2.1.3 Dissipation and Dispersion of Energy

The global energetic behavior of a chain after impact is described by the so-called *kinetic energy ratio*: $KER = T^+/T^-$. The ratio KER is maximum and equal to 1 when the system is conservative and is minimum when the system is purely dissipative. To compute the minimum value of KER for a chain of balls, we need to decompose the kinetic energy of the chain into part $\widehat{T}$ that is invariant during the impact process and part T_{rel} of relative motion as performed in [215]:

$$T = \underbrace{\frac{1}{2}\widehat{m}\widehat{V}^2}_{\widehat{T}} + \underbrace{\frac{1}{2}\sum_{i=1}^{N} m_i (V_i - \widehat{V})^2}_{T_{rel}}, \tag{2.16}$$

where $\widehat{m} = \sum_{i=1}^{N} m_i$ and $\widehat{V}$ is the velocity of the mass center of the chain:

$$\widehat{V} = \frac{1}{\widehat{m}} \sum_{i=1}^{N} m_i V_i = \frac{1}{\widehat{m}} \sum_{i=1}^{N} m_i V_i^-, \tag{2.17}$$

resulting from the conservation of linear momentum of the chain. One can check that T in (2.16) is equal to $\sum_{i=1}^{N} \frac{1}{2} m_i V_i^2$. When the chain is purely dissipative, the kinetic energy of relative motion T_{rel} is entirely dissipated. As a result, the minimum value of KER is:

$$KER_{min} = \frac{\widehat{T}}{T^-} = \frac{1}{\widehat{m}} \frac{\left(\sum_{i=1}^{N} m_i V_i^-\right)^2}{\sum_{i=1}^{N} m_i (V_i^-)^2}. \tag{2.18}$$

For the considered initial condition $V_1^- = V_s,\ V_i^- = 0,\ \forall i = 2,\ 3, ..., N$, we obtain $KER_{min} = m_1/\widehat{m}$.

As mentioned previously, in addition to the principle of conservation of momentum and the global energetic behavior, we need to know the dispersion effect of the chain, *i.e.* the way the kinetic energy is distributed in the chain after impact. To describe the dispersion effect, we propose the following dispersion measure that is similar to the coefficient of variation used in the probability theory and statistics [86]:

$$C_{KE} = \frac{1}{\overline{T}^+} \sqrt{\frac{1}{N} \sum_{i=1}^{N} (T_i^+ - \overline{T}^+)^2}, \tag{2.19}$$

where T_i^+ is the post-impact kinetic energy of ball i ($T_i^+ = m_i (V_i^+)^2/2$), and $\overline{T}^+$ is the mean post-impact kinetic energy:

$$\overline{T}^+ = \frac{1}{N} \sum_{i=1}^{N} T_i^+. \tag{2.20}$$

According to (2.19)*, the higher the value of* C_{KE} *is, the lower the dispersion of energy is.*

Minimum and Maximum Values of C_{KE}

We will determine the minimum and the maximum values of C_{KE} that a given chain of balls can reach. We assume that the dissipative property of the chain is unchanged. This means that the parameter KER is the same for any possible impact outcome. For instance, when the chain is purely elastic, any possible impact outcome must produce $KER = 1$. The minimum (resp. maximum) value of C_{KE} is obtained by solving the following minimization (resp. maximization) problem:

$$\begin{aligned} &\text{minimize (maximize) } C_{KE}(V_1^+, V_2^+, ..., V_N^+), \\ &\text{subjected to:} \\ &\begin{cases} h_1 = \sum_{i=1}^{N} m_i V_i - m_1 V_s = 0 \\ h_2 = \sum_{i=1}^{N} m_i V_i^2 - KER(m_1 V_s^2) = 0 \\ f_j = V_{j+1}^+ - V_j^+ \geq 0, \ \forall j = 1, 2, ..., N-1. \end{cases} \end{aligned} \tag{2.21}$$

The problem (2.21) is a nonlinear optimization problem subjected to two equality constraints and $N-1$ inequality constraints. The post-impact velocities of the balls V_1^+, V_2^+,..., V_N^+ are optimization variables. The two equality constraints $h_1 = 0$ and $h_2 = 0$ in (2.21) result from the conservation of momentum and the unchanged global energetic behavior of the chain. The $N-1$ inequality constraints f_1, f_2,...,f_{N-1} correspond to the kinematic consistency for the impact outcome. Note that KER is bounded by 1 as the upper bound and KER_{min} given by (2.18) as the lower bound. When KER is constant, the problem (2.21) can be transformed to the following problem:

$$\begin{aligned} &\text{minimize (maximize) } \mathcal{F}(z_1, z_2, ..., z_N) = \sum_{i=1}^{N} m_{i,1}^2 z_i^4, \\ &\text{subjected to:} \\ &\begin{cases} h_1 = \sum_{i=1}^{N} m_{i,1} z_i - 1 = 0 \\ h_2 = \sum_{i=1}^{N} m_{i,1} z_i^2 - KER = 0 \\ f_j = z_{j+1} - z_j \leq 0, \ \forall j = 1, 2, ..., N-1, \end{cases} \end{aligned} \tag{2.22}$$

where $m_{i,1} = m_i/m_1$, and z_i is the ball velocity normalized by the incident velocity V_s: $z_i = V_i^+/V_s$. One can note that the objective function in (2.22) is convex and the set of feasible solutions defined by the constraints in (2.22) is a non-convex compact set. Moreover, the objective function $\mathcal{F}(z_1, z_2, ..., z_N)$ is bounded. As a result, the problem (2.22) admits at least one solution.

Proposition 2.3. *When the dissipation of the system tends to the maximum value,* i.e. KER *tends to the minimum value* KER_{min} *given by* (2.18)*, the final velocities of the balls tend to the same value* $V_i^+ = V_s/\widehat{m},\ \forall i = 1,\ 2, ..., N$*. This means that when* $KER = KER_{min}$*, any impact law satisfying the energetic behavior of the system produces the same outcome.*

Proof: Let us rewrite the set of constraints in (2.22) as:

$$\begin{cases} \displaystyle\sum_{i=1}^{N} m_{i,1}(z_i - a) = 0 \\ \displaystyle\sum_{i=1}^{N} m_{i,1}(z_i - a)^2 = KER - KER_{min} \\ z_{j+1} - z_j \leq 0,\ \forall j = 1, 2, ..., N-1, \end{cases} \tag{2.23}$$

where a is the velocity of the mass center of the chain normalized by V_s: $a = \widehat{V}/V_s = 1/\widehat{m}$. As can be seen in (2.23), when KER tends to its minimum value KER_{min} the set of feasible solutions z_i tends to the point $z_i = a,\ \forall i = 1,\ 2, ..., N$. Therefore, V_i^+ tends to $V_s/\widehat{m}$ for any $i = 1,\ 2, ..., N$. ■

The optimization problem (2.22) can be properly solved with the Interior Point Method [24] implemented in the solver *fmincon* available in MATLAB or with the global optimization method with polynomials [130] implemented in the solver GLOTIPOLY [87]. We have solved the problem with these two methods and have found that they give very close results. We obtain the minimum and the maximum values $C_{KE,min}$ and $C_{KE,max}$ as functions of the mass ratios $m_{i,1},\ \forall i = 1,\ 2, ..., N$ and the kinetic energy ratio KER.

The optimization problem (2.21) can supply some useful information when designing chains of balls for some engineering purposes, for instance, for transmitting or absorbing efficiently the energy induced by shocks. The zero-dispersion of energy is desirable in the first purpose, whereas the optimal dispersion of energy is desirable in the second one. The first thing we should check in the design is whether or not the mass distribution in the chain allows us to reach the zero or optimal dispersion. To check this, we solve the optimization problem (2.22) with the mass distribution under consideration to obtain the maximum and minimum values $C_{KE,min}$ and $C_{KE,max}$. If $C_{KE,min} = 0$, the designed chain might exhibit the optimal dispersion of energy, and if $C_{KE,max}$ corresponds to the zero-dispersion of energy, the designed chain might exhibit the zero-dispersion of energy. Note that these conditions are only the necessary conditions to obtain the optimal dispersion

and the zero-dispersion of energy. To obtain these propertyies, we need to adjust other parameters of the chain (for example, the contact stiffness).

Let us now illustrate what has been discussed above with three different chains of 7 balls: a tapered chain where the mass of the balls is progressively decreased as $m_i = (1-q)m_{i-1}$ with $0 \leq q < 1$ being the tapering factor; an anti-tapered chain where the mass of balls is progressively increased as $m_i = (1+q)m_{i-1}$ with $q > 0$ being the anti-tapering factor; and a decorated chain where the mass of balls is distributed as $m_1 = m_3 = m_5 = m_7 = m$ and $m_2 = m_4 = m_6 = \alpha m$ with $\alpha > 0$ being the decorating factor. Figure 2.2 shows $C_{KE,min}$ and $C_{KE,max}$ for the three considered chains with different mass distributions. Note that for a chain of 7 elastic balls, the zero-dispersion corresponds to $C_{KE} = \sqrt{6} \approx 2.45$. It can be seen that it is possible to obtain the zero-dispersion with the decorated chain for any value of α. However, this is not possible for the tapered and the anti-tapered chains except when $q = 0$ (a monodisperse chain). As a consequence, the tapered and anti-tapered chains are not suitable for transmitting the energy induced by shocks. On the contrary, the tapered chain might be suitable for absorbing shocks because it is possible to obtain the optimal dispersion ($C_{KE} = 0$) with this kind of chain for some values of the tapering factor q (for example, $q \approx 0.4,\ 2.6$ *etc*). However, this is not the case for the anti-tapered and the decorated chains.

Remark 2.4. A chain of balls might exhibit the *zero-dispersion* phenomenon when the last ball takes all the energy of the chain and the other balls stop moving after impact. This phenomenon is also called *dispersion-free* in [89, 193]). In this case, the value of C_{KE} is equal to $\sqrt{N-1}$ where N is the number of balls. In terms of wave propagation, in this situation, there is a solitary wave traveling in the chain without any tail left. We will illustrate such wave propagation in Chapter 6.

2.2 Impact Geometry of a 3-Ball Chain

A 3-ball chain may be considered as the simplest case of a granular chain involving multiple impacts. For this kind of chains, Equation (2.8) can be rewritten as:

$$\begin{cases} m_1(V_1^+ - V_1^-) = -p_1 \\ m_2(V_2^+ - V_2^-) = p_1 - p_2 \\ m_3(V_3^+ - V_3^-) = p_2. \end{cases} \tag{2.24}$$

Let $\gamma_1^+ = V_2^+ - V_1^+$ and $\gamma_2^+ = V_3^+ - V_2^+$ denote post-impact relative velocities at the left and right contacts, respectively. Using the conservation of momentum, one can always obtain the post-impact velocity of each ball from the relative velocities at the two contact points as follows:

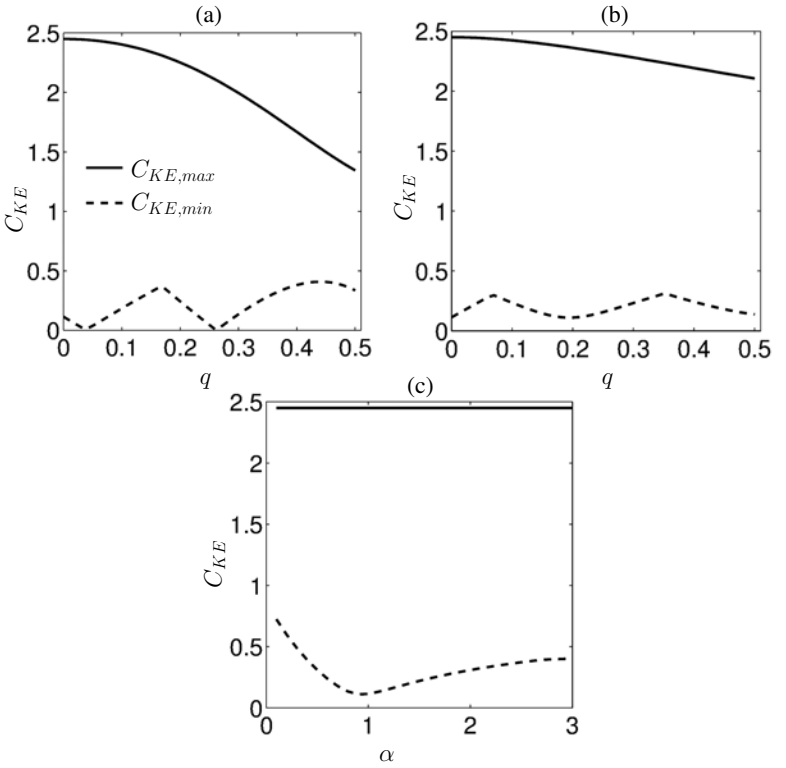


Fig. 2.2 Maximum and minimum values of C_{KE} for chains of 7 balls: (a) for a tapered chain with different values of q, (b) an anti-tapered chain with different values of q and (c) for a decorated chain with different values of α.

$$\begin{cases} V_1^+ = \dfrac{m_1V_1^- + m_2V_2^- + m_3V_3^- - (m_2+m_3)\gamma_1^+ - m_3\gamma_2^+}{m_1+m_2+m_3} \\ V_2^+ = \dfrac{m_1V_1^- + m_2V_2^- + m_3V_3^- + m_1\gamma_1^+ - m_3\gamma_2^+}{m_1+m_2+m_3} \\ V_3^+ = \dfrac{m_1V_1^- + m_2V_2^- + m_3V_3^- + m_1\gamma_1^+ + (m_2+m_3)\gamma_2^+}{m_1+m_2+m_3}. \end{cases} \tag{2.25}$$

Consequently, we can represent the post-impact state of a 3-ball chain in terms of post-impact relative velocities γ_1^+ and γ_2^+.

At various places in this book, we will consider a quantity named *kinetic angle* which is used to describe the coupling between unilateral constraints. It is known to play a significant role in the dynamics of systems with unilateral constraints. In particular, it has a strong influence on the continuity of trajectories with respect to initial data [4,10,64,173]. A detailed definition of the kinetic angle is given in Appendix A.1 (see (A.4)). For a 3-ball chain, the

kinetic angle θ_{12} is related to mass ratios m_2/m_1 and m_2/m_3 by the following relation:

$$\theta_{12} = \arccos\left(\frac{1}{\sqrt{1+\frac{m_2}{m_1}}\sqrt{1+\frac{m_2}{m_3}}}\right). \tag{2.26}$$

As can be seen in (2.26), the kinetic angle $\theta_{12} < \frac{\pi}{2}$ for any m_1, m_2, m_3. We have

$$\theta_{ij} \to \frac{\pi}{2} \text{ when } \frac{m_2}{m_1} \to \infty \text{ or } \frac{m_2}{m_3} \to \infty,$$

and

$$\theta_{12} \to 0 \text{ when } \frac{m_2}{m_1} \to 0 \text{ and } \frac{m_2}{m_3} \to 0.$$

The last property means that if the middle ball is big enough with respect to the two lateral balls, the kinetic angle θ_{12} is near to $\pi/2$. In this case, the two unilateral constraints are orthorgonal (in the sense of the kinetic metric) and the dynamics of the chain is completely decoupled because the Delassus' matrix (defined in Section 2.1.2) is diagonal (see more details in Appendix A.1). Consequently, what happens at one contact does not affect what happens at the other contact. Otherwise, the coupling between two contacts in the chain should be taken into account.

Proposition 2.5. *The impact dynamics in a 3-ball chain is equivalent to that of a particle striking an acute angle if the angle between the two walls is equal to the kinetic angle of the chain.*

The proof can be found in Appendix A.1. In fact, the problem of a particle striking a frictionless corner is a good example to study the behavior of a system with several unilateral constraints near singularites (the corner of the angle constitues a singularity). This problem has been analyzed in [101, 172, 173]. It has been shown that the trajectory of the particle is complex, depending on several parameters: the angle between two walls, the incident angle and the dissipative features at the contacts between particle and walls. In particular, the trajectory may be discontinuous with respect to the initial condition. As a consequence, one can expect that the multiple impact problem in a 3-ball chain possesses all the properties mentioned above.

2.2.1 $C_{KE,max}$ and $C_{KE,min}$

Let us now determine the maximum and minimum values $C_{KE,max}$ and $C_{KE,min}$ of the dispersion measure for a 3-ball chain by solving the optimization problem (2.21). From Proposition 2.3, one can obtain that when KER tends to its minimum value, C_{KE} must tend to the following value:

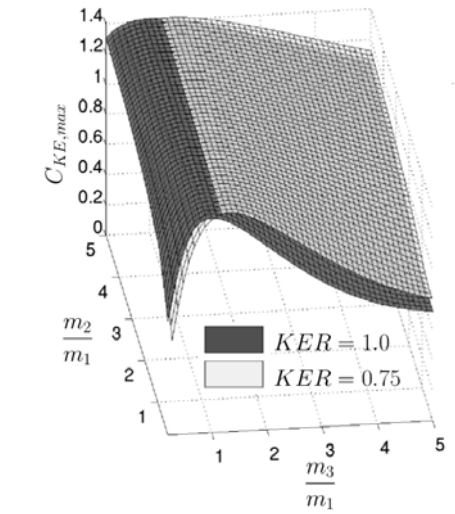
$$\lim_{KER \to KER_{min}} C_{KE} = \frac{\sqrt{2(m_1^2+m_2^2+m_3^2-m_1m_2-m_1m_3-m_2m_3)}}{\widehat{m}}. \tag{2.27}$$

Figures 2.3 and 2.4 show respectively $C_{KE,max}$ and $C_{KE,min}$ versus $m_{2,1}$ and $m_{3,1}$ for $KER = 1$ and $KER = 0.75$. The right-panel of each figure corresponds to a cut-off of the left-panel at the two planes $m_{2,1} = 1$ and $m_{2,1} = 3$. It can be seen that $C_{KE,max}$ and $C_{KE,min}$ depend significantly on the mass distribution and on the energetic behavior of the chain. The highest value of C_{KE} is $\sqrt{2}$ (1.4142, approximately), which corresponds to the zero-dispersion of energy. For $KER = 1$ (for a conservative chain), the zero-dispersion may be reached when the mass ratio $m_{3,1} = 1$, whatever the mass ratio $m_{2,1}$. More interestingly, a dissipative chain may also exhibit the zero-dispersion. For instance a chain with $KER = 0.75$ may exhibit the zero-dispersion when the mass ratio $m_{3,1} \approx 1.3$, whatever the mass ratio $m_{2,1}$. As shown in Figure 2.4, the lowest value of C_{KE} is 0, corresponding to the outcome $T_1^+ = T_2^+ = T_3^+ = T^+/3$ (the three balls have the same post-impact kinetic energy). In this case, the chain under consideration exhibits the uniform dispersion of energy after the shock. This property is very desirable when we want to design a chain of balls that is able to absorb efficiently the energy induced by shocks. Note that one might obtain the uniform dispersion of energy ($C_{KE} = 0$) even with purely elastic chains ($KER = 1$) if the mass of the balls is suitably distributed, for instance, if $m_{2,1} = 3$ and $m_{3,1} = 1$ (see the right panel of Figure 2.4).

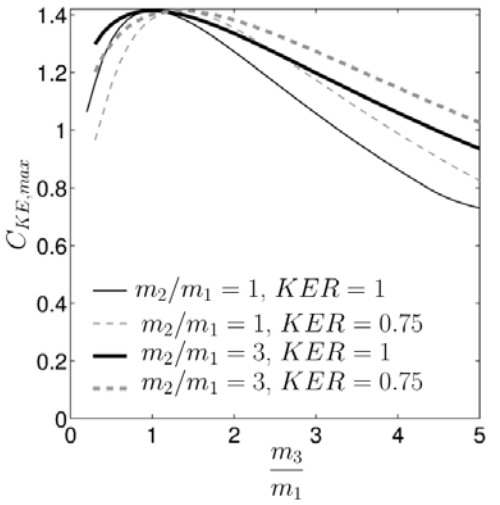
Figure 2.5 shows the relations of $C_{KE,min}$ and $C_{KE,max}$ to KER for a *decorated* 3-ball chain ($m_1 = m_3$) with different values of m_2/m_1. It can be seen that $C_{KE,max}$ increases, in general, with KER except for the case $m_2/m_1 = 5$ where a decrease in $C_{KE,max}$ for small values of KER is observed. $C_{KE,min}$ may decrease when KER increases. The domain of admissible values of C_{KE} tends to be reduced when KER tends to KER_{min}. It should be noted that obtaining $C_{KE,min} = 0$) when changing the dissipative property (KER) is possible for some but not all mass distributions. As can be seen in Figure 2.5, $C_{KE,min} = 0$ when $m_2/m_1 = 1$ and $KER = 1/3$ (minimum value of KER for a monodisperse 3-ball chain) or when $m_2/m_1 = 5$ and $KER \approx 0.6$. It is worth noting that the dissipation of energy (characterized by KER) is independent of the dispersion of energy (characterized by C_{KE}), however the latter is somehow affected by the former.

As mentioned previously, information on the dispersion of energy of the system is needed in addition to its global energetic behavior in order to determine the impact outcome of a 3-ball chain. The following conjecture is made on the dispersion of energy of the system in order to get an admissible impact outcome without introducing any supplementary parameters. It may be seen as an implicit formulation of impact laws through an optimization problem.

Conjecture 2.6. Given a 3-ball chain with a fixed global energetic behavior, the multiple impact occurs in this system in such a way that it maximizes (or minimizes) the dispersion of energy.

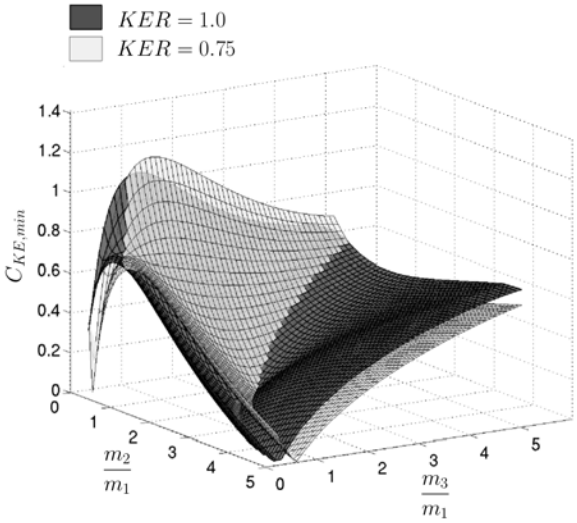


(a)

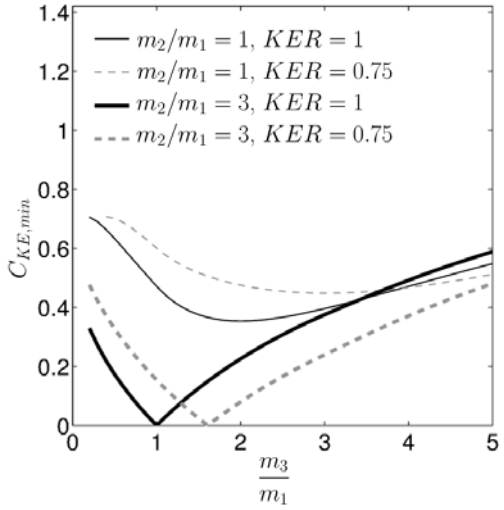


(b)

Fig. 2.3 Maximum value of C_{KE} versus mass ratios m_2/m_1 and m_3/m_1 for two values of KER: $KER = 1$ and $KER = 0.75$



(a)



(b)

Fig. 2.4 Minimum value of C_{KE} versus mass ratios m_2/m_1 and m_3/m_1 for two values of KER: $KER = 1$ and $KER = 0.75$

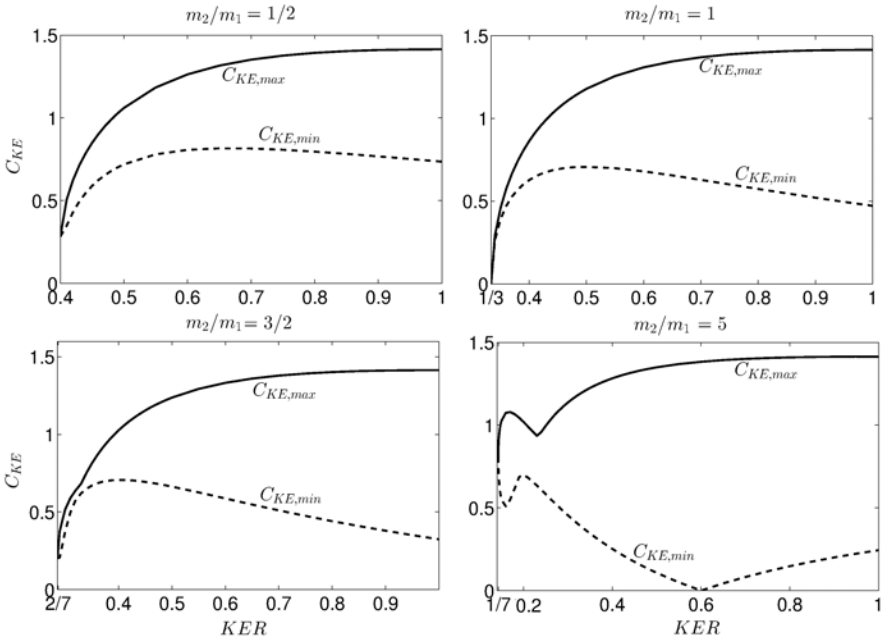


Fig. 2.5 Dependence of $C_{KER,min}$ and $C_{KE,max}$ on KER for different values of m_2/m_1 while $m_1 = m_3$

Applying this conjecture to a monodisperse elastic 3-ball chain ($KER = 1$), one can determine the impact outcome by minimizing (or maximizing) the dispersion measure of C_{KE} defined in (2.19). One gets either the outcome $V_1^+ = -1/3V_s$, $V_2^+ = V_3^+ = 2/3V_s$ that is the minimizer of C_{KE} ($C_{KE,min} = \sqrt{2}/3$) or the outcome $V_1^+ = V_2^+ = 0$, $V_3^+ = V_s$ that is the maximizer of C_{KE} ($C_{KE,max} = \sqrt{2}$). We will see later in Sections 3.1, 3.2 and 3.4 that the first outcome is also the solution given by impact models of type "simultaneous collisions", while the second is also given by models of type "sequential collisions". Clearly Conjecture 2.6 is, in general, not satisfied and even contradicted by experiments.

In fact, the wave effect highly affects the dispersion behavior as can be observed in two experiments on 3-ball monodisperse chains presented in [178], which can be considered as benchmarks to analyze the multiple impact problem. They are described as follows:

- Experiment No 1: the balls are all made of a highly elastic steel. The first ball strikes with a velocity $V_s = 25$ mm/s the two other stationary balls that are in contact. After impact, the velocities of the balls are $V_1^+ \approx -0.0605V_s$, $V_2^+ \approx 0.1049V_s$, $V_3^+ \approx 0.9978V_s$.
- Experiment No 2: this experiment is similar to the experiment No 1 except that a thick coat of grease is applied on both contacts in order to dissipate the energy of relative motion at the contact points by means of viscosity

of the grease. In this case, the balls are stuck together and move with the same velocity after impact, *i.e.* $V_1^+ = V_2^+ = V_3^+ = V_s/3$.

We can see that the dispersion is very small in the first experiment: the kinetic energy is almost entirely transmitted to the last ball. In this case, we obtain $C_{KE} = 1.38$ that is very close to the maximum value of $\sqrt{2}$. In fact, many authors have shown that for a chain of identical elastic balls, a very little but non zero dispersion of energy is always present after impact [90,100]. For this kind of granular chain, the wave effect is so high that a soliton is formed and propagates [164,166], leading to a low value of the dispersion of energy. In the second experiment, the wave is highly damped by the viscosity of the grease so the kinetic energy is distributed uniformly through all the balls, and we obtain the maximum dispersion effect ($C_{KE} = 0$).

2.2.2 Characterization of the Impact Geometry

In the following, we will characterize the impact geometry for a 3-ball chain using the kinetic energy ratio KER et the dispersion measure C_{KE}. It should be noted that, for a 3-ball chain, if an outcome satisfies the kinematic consistency, it satisfies the kinetic consistency. Consequently, the kinematic and energetic consistencies (2.9) and (2.11) define the admissible domain for the final velocities of the balls. A point in the admissible domain is accessed from the values of KER and C_{KE}. Figure 2.6 shows the impact geometry for a monodisperse 3-ball chain with the initial condition $V_1^- = V_s$, $V_2^- = V_3^- = 0$. The global energetic behavior is represented by isolines of KER, and the dispersion effect is represented by isolines of C_{KE}. As a result, the isolines of KER describe the *impact intensity*, and the isolines of C_{KE} describe the *impact topology*. The isolines of KER span from left to right with increasing values from 1/3 to 1, while the isolines of KER span from bottom to top with increasing values from 0 to $\sqrt{2}$.

There are three particular points on the outcome domain. The points A and B correspond to the points A and B shown in Figure 1.2, at which the energy is conserved during the impact ($KER = 1$). The point A corresponds to the outcome $V_1^+ = V_2^+ = 0, V_3^+ = V_s$ so the dispersion of energy is zero at this point (C_{KE} takes its maximum value of $\sqrt{2}$ when $KER = 1$). Consequently, the point A is called *the dispersion-free point*. On the other hand, the point B corresponds to the outcome $V_1^+ = -V_s/3$, $V_2^+ = V_3^+ = 2V_s/3$, hence the dispersion of energy at this point is maximum for the elastic behavior (C_{KE} takes its minimum value of $\sqrt{2}/3$ when $KER = 1$) . It can be noted that, for an elastic monodisperse 3-ball chain, the impact outcome spans the isoline of $KER = 1$ between the extreme points A and B. The point O at the origin corresponds to the outcome $V_1^+ = V_2^+ = V_3^+ = V_s/3$ (the three balls are stuck together after impact) so the dissipation and the dispersion of energy at this point are maximum ($C_{KE} = 0$ and $KER = 1/3$). This is the impact outcome for a purely dissipative monodisperse 3-ball chain.

For a 3-ball chain, one might observe three motion patterns of the balls after impact. The motion pattern I corresponds to the case when the first ball rebounds, while the two last balls move forward after impact. The motion pattern II corresponds to the case when all the balls moves forward after the impact. For the motion pattern III, the two first balls rebound backward, while the last ball moves forward. It is easy to prove that, for a monodisperse 3-ball chain, there exist only the motion patterns I and II. As can be seen in Figure 2.6, the whole admissible impact outcome domain for a monodisperse 3-ball chain can be subdivided into two regions corresponding to the motion patterns I and II. On the whole, the motion pattern I is dominant when the chain is highly elastic, *i.e.* $KER > 0.5$, while the motion pattern II is dominant when the chain is highly dissipative. One can remark that a highly elastic behavior can exhibit the pattern II when the dispersion effect is low. The zero dispersion and dissipation point constitutes a limit between the patterns I and II, while the maximum dissipation point belongs to the pattern II.

The method of characterizing the impact geometry presented above is similar to that introduced in [72, §5.5], in which the impact topology is characterized by the impulse ratio p_2/p_1 (see also Figure 16.2 in [74]). It is worth noting that this method is no longer valid for a chain of $N \geq 4$ balls because the two parameters KER and C_{KE} are no longer sufficient to determine uniquely the post-impact velocity of the balls.

Remark 2.7. KER and C_{KE} isolines shown in Figure 2.6 will be used in the sequel for several other similar figures. For the sake of simplicity of the presentation, their associated values are displayed only in Figures 2.6 and 5.1.

After having described the geometry of impact, we are now going to address another question: how can the impact outcome be determined for a given monodisperse 3-ball chain? In other words, which point in the admissible domain shown in Figure 2.6 corresponds to the chain under consideration? An impact law is aimed at answering this question. In order to determine the right impact outcome, an impact law should model well the energetic behaviour (parameter KER) and the wave effect (parameter C_{KE}). Generally, an impact law must possess the following properties [138]:

(1) The outcome given by the impact law must respect the kinematic, kinetic, and energetic consistencies. Moreover, it must be able to span the whole admissible domain of the outcome.
(2) The parameters considered in the impact law must possess clear physical meanings. They should be related to geometrical and material characteristics of the system. Moreover, they should be properly identified from independent experiments.
(3) The outcome given by the impact must be close to the experimental observation.

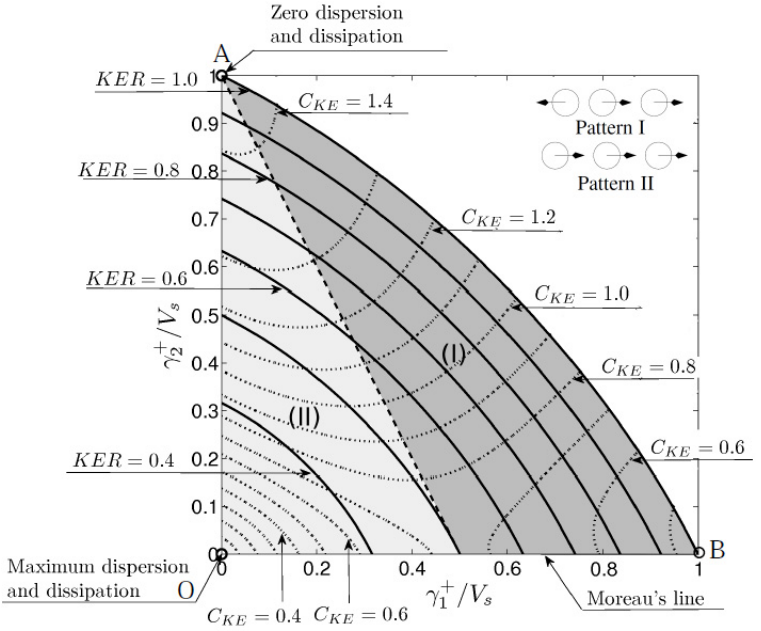


Fig. 2.6 Impact geometry for a monodisperse chain of three balls with the initial condition $V_1^- = V_s$, $V_2^- = V_3^- = 0$. Moreau's line shown in this figure will be explained in Section 3.2.

(4) The impact law must be numerically tractable. In other words, it must be solved by efficient numerical schemes.

Some of these statements may need to be refined. For instance, the notion of closeness in item (3) is vague. Depending on the application, some results may be considered as being close enough to experiments, or as being quite inaccurate. Consider, for example, a monodisperse conservative chain of aligned balls that is collided at one end by another identical ball. Classically, the last ball of the chain will take about 98% of the kinetic energy. Any simplified collision rule which assigns 100% of the energy to this last ball and neglects the post-impact motion of the other balls, may be considered as good (if the motion of the other balls after the impact does is not of interest), or bad (if the dispersion of the energy within the chain is a crucial matter for the application under consideration). See Section 4.2.9 for further arguments. As mentioned previously, several impact laws have been proposed to solve the multiple impact problem. In the next chapter, we will analyze in details some of them for a 3-ball chain.