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# On causal functions

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**Abstract.** We describe, in the general setting of closed cone fields, the set of causal functions which can be approximated by smooth Lyapounov. We derive several consequences on causality theory.

Dans le contexte général des champs de cônes fermés, on décrit l'ensemble des fonctions causales qui peuvent être approchées par des fonctions de Lyapounov lisses. On en déduit quelques conséquence en théorie de la causalité.

## 1. Definitions and Results

With the recent results [3] on Lyapounov functions and the stable relation of closed cone fields the question has appeared whether the stable relation is the smallest closed and transitive relation  $\mathcal{K}^+$  containing the causal relation of the cone field. The question was answered positively under the assumption of an empty stably recurrent set in [10] for Lorentzian metrics and in [13] for a class of closed cone fields slightly more restricted than the one considered here. A counterexample was given in [8], see Example 6.

The stable relation is equal to  $\mathcal{K}^+$  if every causal function, i.e. continuous and monotone along causal curves, is approximated uniformly on compact sets by Lyapounov functions. It is known that  $\mathcal{K}^+$  is generated by the causal functions [2, 7]. In this article we will introduce a subclass of causal functions (special causal) which are approximated by Lyapounov functions. Further we discuss the consequences for causality theory of closed cone fields.

We work on a Riemannian manifold  $M$ . A *convex cone* of the vector space  $E$  is a convex subset  $C \subset E$  such that  $tx \in C$  for each  $t > 0$  and  $x \in C$ . The convex cone  $C$  is called *regular* if it is not empty and contained in an open half-space, or equivalently if there exists a linear form  $p$  on  $E$  such that  $p \cdot v > 0$  for each  $v \in C$ . The full cone  $C = E$  will be called the *singular* cone.

**Definition 1.** We say the  $C \subset E$  is a *closed cone* if it is a convex cone which is either singular or regular and if  $C \cup \{0\}$  is a closed subset of  $E$ .

Note that the empty set is a closed cone. We call it *degenerate*.

A *cone field*  $\mathcal{C}$  on the manifold  $M$  is a subset of the tangent bundle  $TM$  such that  $\mathcal{C}(x) := T_x M \cap \mathcal{C}$  is a convex cone for each  $x$ .

**Definition 2.** We say that  $\mathcal{C} \subset TM$  is a *closed cone field* if it is a cone field such that  $\mathcal{C} \cup T_0 M$  (the zero section) is a closed subset of  $TM$  and such that  $\mathcal{C}(x)$  is a closed cone for each  $x$ .

The *domain* of  $\mathcal{C}$  is the set of non degenerate points. It is denoted by  $\mathcal{D}(\mathcal{C})$ . The domain of a closed cone field is closed. A cone field is called *non degenerate* if all points are non degenerate, *i.e.* if  $\mathcal{D}(\mathcal{C}) = M$ . The set of singular points of a closed cone field is closed.

For a closed cone field  $\mathcal{C}$ , recall from [3] that the curve  $\gamma : I \rightarrow M$  is  $\mathcal{C}$ -causal (or just causal) if it is locally Lipschitz and if the inclusion  $\dot{\gamma}(t) \in \mathcal{C}(\gamma(t)) \cup \{0\}$  holds for almost all  $t \in I$ . The strictly causal future  $\mathcal{J}_{\mathcal{C}}^{++}(x)$  of  $x$  is the set of points  $y \in M$  such that there exists a non constant causal curve  $\gamma : [0, T] \rightarrow M$  with  $T > 0$  satisfying  $\gamma(0) = x$  and  $\gamma(T) = y$ . The causal future  $\mathcal{J}_{\mathcal{C}}^+(x)$  of  $x$  is the set  $\mathcal{J}_{\mathcal{C}}^+(x) = \mathcal{J}_{\mathcal{C}}^{++}(x) \cup \{x\}$ . The causal past  $\mathcal{J}_{\mathcal{C}}^-(x)$  is the set of points  $x' \in M$  such that  $x \in \mathcal{J}_{\mathcal{C}}^+(x')$ . More generally, for each subset  $A \subset M$ , we denote by  $\mathcal{J}_{\mathcal{C}}^{\pm}(A) := \cup_{x \in A} \mathcal{J}_{\mathcal{C}}^{\pm}(x)$  the causal future and past of  $A$ . We have the inclusion  $\mathcal{J}_{\mathcal{C}}^+(y) \subset \mathcal{J}_{\mathcal{C}}^+(x)$  if  $y \in \mathcal{J}_{\mathcal{C}}^+(x)$ .

**Definition 3.** *The function  $f : M \rightarrow \mathbb{R}$  is said to be causal if it is continuous, and has the property that  $f(y) \geq f(x)$  for each  $y \in \mathcal{J}^+(x)$ . We denote by  $\mathbf{C}$  the set of causal functions.*

Given a causal function  $f$ , the point  $x$  is said to be *neutral* if it is a singular point of  $\mathcal{C}$  or if there exists  $y \in \mathcal{J}^+(x) - \{x\}$  such that  $f(y) = f(x)$ . Otherwise,  $x$  is said to be *strict*.

We say that  $a \in \mathbb{R}$  is a *neutral value* of  $f$  if there exists a neutral point  $x$  such that  $f(x) = a$ . Otherwise,  $a$  is called a *strict value*.

We say that  $f$  is a *special causal function* if it is causal, and if the set of strict values of  $f$  is dense in  $\mathbb{R}$ . We denote by  $\mathbf{S}$  the set of special causal functions. It is classical in the Conley theory of dynamical systems to consider this kind of hypothesis. The terminology *neutral values* is taken from [5].

We say that  $\tau$  is a Lyapounov function if it is smooth and satisfies  $d\tau_x \cdot v > 0$  for each regular point  $x$  of  $\tau$  and each  $v \in \mathcal{C}(x)$ . We denote by  $\mathbf{L}$  the set of Lyapounov functions. We have

$$\mathbf{L} \subset \mathbf{S} \subset \mathbf{C}.$$

The first inclusion following from:

**Lemma 4.** *Each Lyapounov function  $\tau$  is special causal. Moreover, the set of neutral points of  $\tau$  is contained in the set of critical points of  $\tau$ .*

PROOF. If  $\tau$  is a Lyapounov function and  $\gamma$  is a causal curve, then the function  $\tau \circ \gamma$  is Lipschitz and satisfies  $(\tau \circ \gamma)'(t) \geq 0$  almost everywhere, hence it is non-decreasing. This implies that the function  $\tau$  is causal. If there exists  $t > 0$  such that  $\tau \circ \gamma(t) = \tau \circ \gamma(0)$ , then  $\tau \circ \gamma$  is constant on  $[0, t]$ , which implies that  $\gamma(0)$  is a critical point of  $\tau$ . We deduce that the neutral points of  $\tau$  are necessarily critical points, hence that neutral values are critical values. By Sard's Theorem, the set of regular values of  $\tau$  is dense, which implies that the set of strict values is dense.  $\square$

We define the stably recurrent set  $\mathcal{R}$  as the intersection of the sets of critical points of Lyapounov functions. This set is studied in [3]. In the present setting, it is proved there that:

**Proposition 5.** *If  $x \in \mathcal{R}$ , then either  $x$  is a singular point of  $\mathcal{C}$ , or there exists  $y \in \mathcal{J}^+(x) - \{x\}$  such that  $\tau(y) = \tau(x)$  for each  $\tau \in \mathbf{L}$ . As a consequence*

$$\mathcal{R} = \bigcap_{\tau \in \mathbf{L}} \text{neut}(\tau) = \bigcap_{\tau \in \mathbf{L}} \text{crit}(\tau).$$

Moreover, there exists a Lyapounov function  $\tau_0$  such that

$$\mathcal{R} = \text{neut}(\tau_0) = \text{crit}(\tau_0).$$

Our main result is:

**Theorem 1.** For each function  $f \in \mathbf{S}$  and each  $\varepsilon > 0$ , there exists a Lyapounov function  $\tau \in \mathbf{L}$  such that  $|\tau - f|(x) < \varepsilon$  for each  $x$ , and which is regular outside of  $\mathcal{R}$ .

To our knowledge, this statement is new even in the classical Lorentzian setting. See however [4] for results on smoothing causal functions.

The following example, taken from Minguzzi [8, Example 5.1], illustrates that the set  $\mathbf{S}$  could not be replaced by  $\mathbf{C}$  in the above statement.

**Example 6.** We consider the manifold  $M = S^1 \times \mathbb{R}$ , and the constant cone field  $\mathcal{C}(\theta, x) = [0, \infty) \times [0, \infty)$ . The function  $f(\theta, x)$  on  $M$  is causal if and only if it is of the form  $f(\theta, x) = g(x)$  for some non decreasing function  $g : \mathbb{R} \rightarrow \mathbb{R}$ . For such a causal function, all the values in the interval  $g(\mathbb{R})$  are neutral. As a consequence, the function  $f$  is special causal if and only if it is constant.

Obviously,  $\mathbf{S}$ , and hence  $\mathbf{L}$ , is not dense in  $\mathbf{C}$  in this example.

**Corollary 7.**  $\mathcal{R} = \bigcap_{f \in \mathbf{S}} \text{neut}(f)$

PROOF. Since  $\mathbf{L} \subset \mathbf{S}$ , we have the inclusion  $\bigcap_{f \in \mathbf{S}} \text{neut}(f) \subset \mathcal{R}$  by Proposition 5. Conversely, consider  $x \in \mathcal{R}$ . If  $x$  is a singular point of  $\mathcal{C}$ , then it is a neutral point for each causal function. Otherwise, there exists  $y \in \mathcal{J}^+(x) - \{x\}$  such that  $\tau(y) = \tau(x)$  for each  $\tau \in \mathbf{L}$ . The density of  $\mathbf{L}$  in  $\mathbf{S}$  then implies that  $f(y) = f(x)$  for each special causal function  $f$ , hence  $x$  is neutral for  $f$ .  $\square$

In general the set of neutral points of a causal function can be a strict subset of  $\mathcal{R}$ . We recall some definitions before presenting an example. We say that  $f$  is a *time function* if it is a causal function without neutral points. Obviously, time functions are special causal. Note that the existence of a time function with our definitions prevents the presence of singularities of  $\mathcal{C}$ . We say that  $\tau$  is a *temporal function* if it is a Lyapounov function without critical points.

**Example 8.** Consider  $M = S^1 \times \mathbb{R}^2$  with coordinates  $(\theta, x, z)$ . Equip  $M$  with the cone field

$$\mathcal{C}(\theta, x, z) := \{(v_\theta, v_x, v_z) : v_x \geq 0 \text{ and } v_\theta v_x \geq z^2 v_\theta^2 + v_z^2\}.$$

We claim that

$$\bigcap_{f \in \mathbf{C}} \text{neut}(f) = S^1 \times \mathbb{R} \times \{0\} \neq \mathcal{R} = M.$$

PROOF. Under the embedding  $(\theta, x) \mapsto (\theta, x, 0)$  of  $S^1 \times \mathbb{R}$  into  $M$  the cone field  $\mathcal{C}$  pulls back to the cone field considered in Example 6. The coordinate function  $x : M \rightarrow \mathbb{R}$  is a causal, but not a temporal, function of  $(M, \mathcal{C})$ . As seen in Example 6 the set of neutral points of every causal function includes  $S^1 \times \mathbb{R} \times \{0\}$ . Further the neutral points of the function  $x$  are precisely  $S^1 \times \mathbb{R} \times \{0\}$ . Therefore we have

$$\bigcap_{f \in \mathbf{C}} \text{neut}(f) = S^1 \times \mathbb{R} \times \{0\} \subset \mathcal{R}.$$

In particular, all Lyapounov functions are critical, hence constant, on  $S^1 \times \mathbb{R} \times \{0\}$ . The curves  $(\theta(t), x(t), z(t)) = (\theta_0 + t, x_0 + 2t + (t_0 + t)^3 - t_0^3, t_0 + t)$  and  $(\theta(t), x(t), z(t)) = (\theta_0 + t, x_0 + 2t + (t - t_0)^3 + t_0^3, t_0 - t)$  are causal. This implies that for every  $p \in M$  we have

$$\mathcal{J}_\mathcal{C}^\pm(p) \cap (S^1 \times \mathbb{R} \times \{0\}) \neq \emptyset.$$

As a consequence, all Lyapounov functions are constant on  $M$ , hence  $\mathcal{R} = M$ .  $\square$

**Corollary 9.** *There exists a time function if and only if the stably recurrent set is empty. Moreover, each causal function  $f$  can be uniformly approximated by temporal functions  $\tau$ .*

*If  $f$  is a Cauchy time function, then so is the temporal function  $\tau$  (in this case,  $\mathcal{C}$  is globally hyperbolic in the sense of [3]).*

PROOF. The first claim follows directly from the definition and Proposition 5.

Let us next consider a causal function  $f$ . Since  $\mathcal{R}$  is empty, there exists a time function  $g'$ . The function  $g := \varepsilon \arctan \circ g'$  is a time function taking values in  $] -\varepsilon, \varepsilon[$ . The function  $f + g$  is then a time function, hence it belongs to  $\mathbf{S}$ . By Theorem 1, there exists a temporal function  $\tau$  such that  $|\tau - (f + g)| < \varepsilon$ , hence  $|\tau - f| < 2\varepsilon$ .

If  $f$  and  $g$  are two time functions such that  $f - g$  is bounded, and if  $f$  is Cauchy, then so is  $g$ . Indeed, for each inextendible causal curve  $\gamma$ , we have  $\sup(f \circ \gamma) = +\infty$ , hence  $\sup(g \circ \gamma) = +\infty$  and  $\inf(f \circ \gamma) = -\infty$ , hence  $\inf(g \circ \gamma) = \infty$ .  $\square$

Let us discuss the implications of our result in terms of causality relations. The stable (also called Seifert) relation  $\mathcal{F}^+(x)$  is defined by

$$\mathcal{F}^+(x) := \{y : \tau(y) \geq \tau(x) \forall \tau \in \mathbf{L}\}$$

This relation can also be defined in terms of cone enlargements, see [3] for the present generality. It is an obvious consequence of Theorem 1 that

$$\mathcal{F}^+(x) = \{y : \tau(y) \geq \tau(x) \forall \tau \in \mathbf{S}\}.$$

On the other hand, it is a general fact that

$$\mathcal{K}^+(x) = \{y : \tau(y) \geq \tau(x) \forall \tau \in \mathbf{C}\} \subset \mathcal{F}^+(x)$$

is the smallest transitive closed relation containing  $\mathcal{J}^+$ . This follows from the Auslander-Levin Theorem, [2, 7], [1, Theorem 2], as was noticed in [11, Theorem 3].

In Example 6 we have  $\mathcal{K}^+(\theta, x) = \mathcal{J}^+(\theta, x) = \mathbb{S}^1 \times [x, \infty)$  and  $\mathcal{F}^+(\theta, x) = M$  for all  $(\theta, x) \in M$ . This shows that equality does not hold in general between  $\mathcal{K}^+$  and  $\mathcal{F}^+$ . In the case where  $\mathcal{R}$  is empty, however, we have by Corollary 9:

**Proposition 10.** *If  $\mathcal{R}$  is empty,  $\mathcal{K} = \mathcal{F}$ .*

The equality  $\mathcal{K} = \mathcal{F}$  is proved in [10] in the classical Lorentzian setting, and very recently [13, Theorem 85] under assumptions slightly stronger than the present ones.

Let us finally recover and slightly generalize some results of Minguzzi on the equivalence between  $\mathcal{K}$ -causality and stable causality. We say that the cone field is  $\mathcal{K}$ -causal if  $\mathcal{K}^+(x) \cap \mathcal{K}^-(x) = \{x\}$  for each  $x$ .  $\mathcal{K}$ -causality implies the that for all  $x$  and all  $y \in \mathcal{J}^+(x) \setminus \{x\}$  there exists  $f \in \mathbf{C}$  with  $f(y) > f(x)$ . Recall that the cone field is *stably causal* if  $\mathcal{R}$  is empty. This is equivalent to  $\mathcal{F}$ -causality, that is to the property that  $\mathcal{F}^+(x) \cap \mathcal{F}^-(x) = \{x\}$  for each  $x$ , as follows from Proposition 5, see also [15, 6, 9, 3].

**Proposition 11.** *The following properties are equivalent:*

- *The cone field  $\mathcal{C}$  is stably causal.*
- *There exists a temporal function.*
- *There exists a time function.*
- *The cone field is  $\mathcal{K}$ -causal.*

PROOF. We have already proved the equivalence of the three first points.

Since  $\mathcal{K}^+(x) \cap \mathcal{K}^-(x) \subset \mathcal{F}^+(x) \cap \mathcal{F}^-(x)$ , any of the first three conditions implies the last point.

Conversely, let us consider the set  $\mathbf{C}([0, 1])$  of causal functions taking values in  $[0, 1]$ , endowed with the topology of uniform convergence on compact sets. It is a separable metric space, as a subset of the separable metric space of all continuous functions with the topology of uniform convergence on compact sets. We consider a dense sequence  $f_i \in \mathbf{C}([0, 1])$  and  $f := \sum_{i \geq 1} 2^{-i} f_i$ , which is an element of  $\mathbf{C}$ . If the cone field is  $\mathcal{K}$ -causal, then  $f$  is a time function. Indeed, if  $y \in \mathcal{F}^+(x) - \{x\}$ , then  $f_i(y) \geq f_i(x)$  for each  $i$ , and there exists  $j$  such that  $f_j(y) > f_j(x)$ . As a consequence, we get that  $f(y) > f(x)$ .  $\square$

## 2. Proof of Theorem 1

We need more definitions from [3].

We say that  $\mathcal{E} \subset TM$  is an open cone field if it is a cone field which is open as a subset of  $TM$ . Then  $\mathcal{E}(x)$  is an open cone for each  $x$ . We say that  $\mathcal{E}$  is an open enlargement of  $\mathcal{C}$  if it is an open cone field containing  $\mathcal{C}$ .

Given an open cone field  $\mathcal{E}$ , we say that the curve  $\gamma : I \rightarrow M$  is  $\mathcal{E}$ -timelike (or just timelike) if it is piecewise smooth and if  $\dot{\gamma}(t) \in \mathcal{E}(\gamma(t))$  for all  $t$  in  $I$  (at non smooth points, the inclusion is required to hold for left and right differentials). The *chronological future*  $\mathcal{I}_{\mathcal{E}}^+(x)$  of  $x$  is the set of points  $y \in M$  such that there exists a non constant timelike curve  $\gamma : [0, T] \rightarrow M$  satisfying  $\gamma(0) = x$  and  $\gamma(T) = y$ . The *chronological past*  $\mathcal{I}_{\mathcal{E}}^-(x)$  of  $x$  is the set of points  $x' \in M$  such that  $x \in \mathcal{I}_{\mathcal{E}}^+(x')$ . More generally, for each subset  $A \subset M$ , we denote by  $\mathcal{I}_{\mathcal{E}}^{\pm}(A) := \cup_{x \in A} \mathcal{I}_{\mathcal{E}}^{\pm}(x)$  the *chronological future and past of A*. The chronological future can also be defined using only smooth timelike curves, see [3, Lemma 22].

**Proposition 12.** *Let  $f$  be a special causal function, and let  $a \in \mathbb{R}$  a strict value of  $f$ . Then for each open interval  $]a_-, a_+[ \ni a$ , there exists a Lyapounov function  $\tau : M \rightarrow ]a_-, a_+[$  such that  $\tau(x) = a_+$  if  $f(x) \geq a_+$  and  $\tau(x) = a_-$  if  $f(x) \leq a_-$  and all values in  $]a_-, a_+[$  are regular for  $\tau$ .*

PROOF OF THEOREM 1 FROM PROPOSITION 12: We fix a special causal function  $f$  and  $\varepsilon > 0$ . For each  $k$ , there exists a Lyapounov function  $\tau_k : M \rightarrow [k\varepsilon, (k+1)\varepsilon]$  such that  $\tau_k = (k+1)\varepsilon$  on  $\{f \geq (k+1)\varepsilon\}$  and  $\tau_k = k\varepsilon$  on  $\{f \leq k\varepsilon\}$ . The function  $\tau$  which is equal to  $\tau_k$  on the set  $f^{-1}([k\varepsilon, (k+1)\varepsilon])$  for each  $k$  is a Lyapounov function, whose set of critical values is  $\varepsilon\mathbb{Z}$ , and which satisfies  $|f - \tau| < \varepsilon$ .

In order to control the critical set, we consider a Lyapounov function  $\tau_0 : M \rightarrow [0, \varepsilon]$  which is regular outside of  $\mathcal{R}$ . Then  $\tau + \tau_0$  is regular outside of  $\mathcal{R}$  and  $|f - (\tau + \tau_0)| < 2\varepsilon$ .  $\square$

PROOF OF PROPOSITION 12: Let  $a \in \mathbb{R}$  be a strict value of the special causal function  $f : M \rightarrow \mathbb{R}$ . Then  $U := \{f > a\}$  is open with  $\mathcal{J}^{++}(\bar{U}) \subset U$  and  $\partial U$  consists only of regular points of  $\mathcal{C}$ . Especially there exists a neighborhood of  $\partial U$  consisting only of regular points of  $\mathcal{C}$  since the singular set of  $\mathcal{C}$  is closed. Recall from [3] the definition of a trapping domain:

**Definition 13.** *An open set  $A \subset M$  is a trapping domain for the open cone field  $\mathcal{E}$  if  $\mathcal{I}_{\mathcal{E}}^+(A) \subset A$ .  $A$  is a trapping domain for the closed cone field  $\mathcal{C}$  if it is a trapping domain for some open enlargement  $\mathcal{E}$  of  $\mathcal{C}$ . We say that the domain is locally trapping at a point  $x \in \partial A$  if there exists an open neighborhood  $U$  of  $x$  such that  $A \cap U$  is a trapping domain for  $\mathcal{C}|_U$ .*

We proved in [3, Lemma 35] that a domain is trapping if and only if it is locally trapping at each point of its boundary.

**Proposition 14.** *Let  $U$  be an open set such that  $\mathcal{J}^{++}(\bar{U}) \subset U$  and assume that  $\partial U$  consists only of regular points of  $\mathcal{C}$ . Further let  $F_i$  be a closed set contained in  $U$ , and let  $F_e$  be a closed set disjoint from  $\bar{U}$ . Then there exists a trapping domain  $A$  such that  $F_i \subset A$  and  $F_e$  is disjoint from  $\bar{A}$ .*

PROOF OF PROPOSITION 14: Set  $W := M \setminus \bar{U}$ . For an open cone field  $\mathcal{E}$  denote with  $\mathcal{E}_W$  the open cone field which is equal to  $\mathcal{E}$  on  $W$  and empty on  $\bar{U}$ . We denote by  $\partial^- \mathcal{I}_{\mathcal{E}_W}^+(y)$  the part of the boundary of  $\mathcal{I}_{\mathcal{E}_W}^+(y)$  which is contained in  $W$ . Note that the open set  $\mathcal{I}_{\mathcal{E}_W}^+(y)$  is locally trapping at each point of  $\partial^- \mathcal{I}_{\mathcal{E}_W}^+(y)$  and  $\mathcal{I}_{\mathcal{E}_W}^+(y) \subset W$ .

**Lemma 15.** *For each  $x \in \partial U$ , there exists  $y_x \in W \cap \mathcal{J}^-(x)$  and an enlargement  $\mathcal{E}^x$  such that  $V_x := \mathcal{I}_{\mathcal{E}_W}^+(y_x)$  is bounded in  $M$  and disjoint from  $F_e$ .*

PROOF OF LEMMA 15: Let us fix  $x \in \partial U$ . We consider a bounded open neighborhood  $V$  of  $x$  in  $M$ , which is disjoint from  $F_e$  and does not contain any singular point of  $\mathcal{C}$ . We furthermore assume that there exists a temporal function  $\tau$  on  $V$ , which satisfies  $d\tau_z \cdot v > 2|v|_z$  for each  $z \in V$  and  $v \in \mathcal{C}(z)$ . We will prove that there exists  $y^x \in W \cap \mathcal{J}^-(x)$  and an open enlargement  $\mathcal{E}^x$  of  $\mathcal{C}$  such that  $\mathcal{I}_{\mathcal{E}_W}^+(y^x) \subset V$ . Since  $V$  is bounded so is  $\mathcal{I}_{\mathcal{E}_W}^+(y^x)$ . Further  $\mathcal{I}_{\mathcal{E}_W}^+(y^x)$  is disjoint from  $F_e$  because  $V$  is.

We consider a nested sequence of open enlargements  $\mathcal{E}^n$  with  $\mathcal{C} \subset \mathcal{E}^n \subset \mathcal{E}$  and  $\cap_n \mathcal{E}^n = \mathcal{C}$ . The existence of such a sequence is proved in [3, Lemma 20]. We set  $L = d(x, \partial V)/2$  where  $d$  denotes the distance with respect to the Riemannian metric.

We first prove the existence of a  $\mathcal{C}$ -causal curve  $\eta : [-\delta, 0]$ , parametrized by arclength, such that  $\eta(0) = x$  and  $\eta(t) \in W$  for each  $t \in [-\delta, 0[$ . For this, we consider a sequence  $\eta_n : [-L, 0] \rightarrow V$  of  $\mathcal{E}^n$ -timelike curves parametrized by arclength. At the limit (possibly along a subsequence), we obtain a curve  $\tilde{\eta} : [-L, 0] \rightarrow V$ , which is  $\mathcal{C}$  causal according to the limit curve Lemma [3, Lemma 23].

Since  $\cap \mathcal{E}^n = \mathcal{C}$ , the curves  $\eta_n$  satisfy  $d\tau_{\eta_n(t)} \cdot \dot{\eta}_n(t) \geq |\dot{\eta}_n(t)| = 1$  for each  $t \in [-L, 0]$  when  $n$  is large enough, hence  $\tau \circ \eta_n(-t) \leq \tau \circ \eta_n(0) - t$  for each  $t \in [0, L]$ . At the limit, we deduce that the curve  $\tilde{\eta}$  is not constant. We parametrize it by arclength and get  $\eta$ . Since  $\mathcal{J}^{++}(\bar{U}) \subset U$  and  $\eta(0) \notin U$ , we obtain that  $\eta(t) \in W$  for  $t < 0$ .

We now consider a sequence  $y_n \in \mathcal{J}^-(x) \cap W$  converging to  $x$ . These points can be taken on the curve  $\eta$ . We claim that  $\mathcal{I}_{\mathcal{E}_W}^+(y_n)$  is contained in  $V$  when  $n$  is large enough. If we assume the contrary, there exists for each  $n$  large enough a smooth  $\mathcal{E}^n$ -timelike curve  $\gamma_n : [0, L] \rightarrow V \cap W$  parametrized by arclength and starting at  $y_n$ . As above, these curves converge uniformly to a non constant limit  $\gamma : [0, L] \rightarrow \bar{W}$  which satisfies  $\gamma(0) = x$ . The fact that  $\gamma(L)$  belongs to  $\bar{W} = M \setminus U$  is in contradiction with the hypothesis  $\mathcal{J}^{++}(x) \subset U$ . Last  $\bar{W} = M \setminus U$  follows from the assumption that  $a$  is a strict value of  $f$ .  $\square$

In order to finish the proof of Proposition 12 we invoke two propositions from [3]:

**Proposition 16.** *Let  $A$  be a trapping domain, let  $F_i$  be a closed set contained in  $A$ , let  $F_e$  be a closed set disjoint from  $\bar{A}$ , and let  $\theta$  be a point in the boundary of  $A$ .*

*Then there exists a smooth (near  $\mathcal{D}(\mathcal{C})$ ) trapping domain  $A'$  which contains  $F_i$ , whose boundary contains  $\theta$ , and whose closure is disjoint from  $F_e$ .*

**Proposition 17.** *Let  $A$  be a smooth trapping domain, then there exists a (smooth) Lyapounov function  $\tau' : M \rightarrow [-1, 1]$  such that  $A = \{\tau' > 0\}$  and all values in  $] -1, 1[$  are regular values of  $A$  (hence  $\partial A = \{\tau' = 0\}$ ).*

*If  $F_i$  and  $F_e$  are closed sets contained in  $A$  and disjoint from  $\bar{A}$ , respectively, we can moreover impose that  $\tau' = 1$  on  $F_i$  and  $\tau' = -1$  on  $F_e$ .*

Here we consider  $F_i := \{f \geq a_+\}$  and  $F_e := \{f \leq a_-\}$ . The claimed Lyapounov function  $\tau$  is defined as

$$\tau := \frac{1}{2}[(a_+ - a_-)\tau' + (a_+ + a_-)].$$

$\square$

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