

Evaluation of failure probability under parameter epistemic uncertainty: application to aerospace system reliability assessment

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11 Evaluation of failure probability under parameter enistemic $\frac{11}{12}$ Evaluation of failure probability under parameter epistemic ¹³ uncertainty: Application to aerospace system reliability assessment **1888** 78 14 80

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23 ARTICLE INFO ABSTRACT 89

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25 Article history:
25 Article history: 25 Article history: 25 Article history in a context of 21 and 22 and 23 and 24 and 25 and 2 $\frac{26}{27}$ Received in revised form 13 June 2017 approach" (NRA) is a classical double-loop-approach involving a sampling phase of the parameters and $\frac{27}{27}$ 27 Accepted 13 July 2017
then a reliability analysis for each sampled parameter value. A second approach, called "augmented" 28 Available online xxxx

reliability approach" (ARA), requires to sample both distribution parameters and basic random variables $\frac{29}{\text{Kewounds}}$ conditional to them at the same phase and then integrate simultaneously over both domains. In this 30 Distribution parameter uncertainty **30** article, a numerical comparison is led. Possibilities offered by both approaches are investigated and the ⁹⁶ 31 Rare event simulation **12.1 Contains the ARA** are illustrated through the application on two academic test-cases illustrating 97 32 Predictive failure probability **Several numerical difficulties (low failure** probability, nonlinearity of the limit-state function, correlation 98 33 99 between input basic variables) and two real space system characterization (a launch vehicle stage fallback 34 Space debris collision and a collision probability between a space debris and a satellite estimation) for which 100 only the ARA is tractable. The contraction of the SN and the ARA is tractable. uncertainties affecting probability distribution parameters. The first approach called "nested reliability

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1. Introduction

definitely a well-known issue [4].

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64 130 <http://dx.doi.org/10.1016/j.ast.2017.07.016>65 *intervaluating* and co_lulation of the collection of th

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1. Introduction 106 nesses: high-dimensionality, high-nonlinearities and multi-physics ⁴¹ ¹¹ **107 combination.** All in all, the common similarity between all these ¹⁰⁷ **combination.** All in all, the common similarity between all these ¹⁰⁷ 42
Reliability analysis appears to be one of the dedicated tools codes is their expensive-to-evaluate aspect. On top of that, differ- $_{43}$ Reliability analysis appears to be one of the dedicated tools codes is their expensive-to-evaluate aspect. On top of that, differ- $_{44}$ to quantify the risk of failure for complex aerospace systems re-
 $_{44}$ entity solutions of the way we evaluate the system space is the contract of the contract the way we evaluate the system $_{110}$ $_{45}$ garding the uncertainties affecting their behavior and to help engi-
 $_{45}$ performances (for instance, a number of input variables are uncertained in the uncertainty of the uncer- $_{46}$ neers to make more informed decisions in the design phase. Under $\frac{1}{4}$ tain). These uncertainties can be considered as inherent from na- $_{47}$ safety requirements, one needs to quantify a probability of fail-
 $_{47}$ ture, due to various assumptions or numerical approximations, or $_{113}$ ₄₈ ure p_f. However, failure scenarios possibly impacting the behavior hally steming from measurement errors. From a pragmatic engi- $_{49}$ of a system often lead to rare events, i.e. events associated to a semillering point of view, uncertainties can be separated into two cat- $_{50}$ very low failure probability [1,2]. Estimating such a probability is egories: *aleatory* and *epistemic* [5]. Aleatory uncertainty represents $_{116}$ ₅₁ often burdensome since classical methods such as crude Monte hatural variability which is supposed to be irreducible in a specific 117 $_{52}$ Carlo (CMC) involve a large number of model evaluations which context. Epistemic uncertainty ensues from the lack of knowledge $_{116}$ $_{53}$ make the calculations untractable [3]. Thus, the particular case $\;$ or mathematical simplifications and can be reduced by adding $\;$ $_{119}$ $_{54}$ of coupling between reliability analysis (underlying on multiple influent information or increasing the model ndelity. Both types of $_{120}$ ₅₅ probabilistic analyses) and expensive aerospace computer codes is uncertainty require proper mathematical formalisms, numerical ₁₂₁ $_{56}$ definitely a well-known issue [4]. \hfill \hfill 57 Simulation models used in aerospace engineering became more using a probabilistic framework, several compéting (and comple-58 and more complex over the last decades and reached high fidelity mentary) formalisms are available to model epistemic uncertainty: $_{124}$ 59 representation. However, they suffer from various key computer ill- probabilistic Bayesian analysis [6], interval analysis [7], Dempster– 125 60 126 Shafer's evidence theory [8,9], possibility theory [10], probability-61 **61 127 boxes** [11,12]. This non-exhaustive list mentions multiple frame-62 128 works which are commonly gathered under the global name of 63 129 *imprecise probabilities* [13]. Mixing aleatory and epistemic uncerent sources of uncertainty affect the way we evaluate the system performances (for instance, a number of input variables are uncertain). These uncertainties can be considered as inherent from nature, due to various assumptions or numerical approximations, or finally steming from measurement errors. From a pragmatic engineering point of view, uncertainties can be separated into two categories: *aleatory* and *epistemic* [5]. Aleatory uncertainty represents natural variability which is supposed to be irreducible in a specific context. Epistemic uncertainty ensues from the lack of knowledge or mathematical simplifications and can be reduced by adding more information or increasing the model fidelity. Both types of uncertainty require proper mathematical formalisms, numerical using a probabilistic framework, several competing (and complementary) formalisms are available to model epistemic uncertainty: probabilistic Bayesian analysis [6], interval analysis [7], Dempster–

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В 1950 година в 1950 годин 10 76 21 87 22 88 37 38 $\overline{}$ 104 39 105

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⁸ (NRA), is to consider sampling of uncertain distribution param- tain input variables (denoted as the basic variables in the following, ⁷⁴ ⁹ eters and to perform a nested reliability analysis for each real- see [20] and [21]) of the system. These basic variables are gathered ⁷⁵ ¹⁰ ization of these parameters (see, for example [14] for reliability in a *d*-dimensional random vector **X** of known continuous joint ⁷⁶ 11 assessment under epistemic uncertainty on distribution parame- probability density function (pdf) $f_{\mathbf{X}}:\mathbb{R}^a\to\mathbb{R}_+$. In the space of the the state of the the space of the the space of the space of the space o 11 assessment under epistemic uncertainty on distribution parame-
12 ters given by intervals. [15] in the context of probability-based realizations $\mathbf{x} = (x_1, x_2, ..., x_d)^\top$ of the random vector **X**. failure is 78 ¹³ tolerance analysis of products or [16] for a coupling with Sparse characterized by the use of a function $g:\mathbb{R}^d\to\mathbb{R}$ called the limit-¹⁴ Polynomial Chaos Expansions). Nonetheless, it implies to repeat state function (Isf). A classical formulation for the lsf in the context ⁸⁰ ¹⁵ several times a costly reliability analysis, which can be unafford- of static input-output model can be: ¹⁶ able for most cases in a complex industrial environment. Based **1998** able to the state of the state 17 on the context of complex aerospace systems design and taking $\mathcal{S}(\lambda) = y \ln \frac{1}{\lambda}$ (i) as ¹⁸ into account all the constraints mentioned earlier, assessing relia-
where $v_{th} \in \mathbb{R}$ is a characteristic threshold output value beyond the 84 ¹⁹ bility coupled with consideration on the uncertainty affecting the \qquad one the system falls into a failure state. Thus, one can distinguish \qquad^{85} ²⁰ distribution parameters seems to be quite challenging. However, two domains associated to the behavior of $g(.)$; the *failure domain* 86 ²¹ some researchers proposed methods to incorporate this kind of given by $\mathcal{F}_r = \{x \in \mathcal{D}_x : g(x) < 0\}$, which in fact does include the 21 some researchers proposed methods to incorporate this kind of given by $\mathcal{F}_{\mathbf{x}} = {\mathbf{x} \in \mathcal{D}_{\mathbf{x}} : g(\mathbf{x}) \le 0}$, which in fact does include the $\frac{87}{2}$
22 uncertainty affecting distribution parameters into a mor A classical approach, known as the "*nested reliability approach*" assessment under epistemic uncertainty on distribution parameters given by intervals, [15] in the context of probability-based uncertainty affecting distribution parameters into a more general Bayesian framework [6,17–19].

²⁴ An alternative approach, known as the "*augmented reliability ap-* With no consideration of any distribution parameter uncertainty. ⁹⁰ ²⁵ proach" (ARA), aims at computing a different failure probability the *failure probability p_f* therefore reads: 26 92 (called "*predictive failure probability*") which takes into account the ²⁷ uncertainty in the distribution parameters by considering an "aug-
 $p_c = p \cdot \sigma(\mathbf{X}) < 0$ $\int f_{\mathbf{v}}(\mathbf{x}) d\mathbf{x} = \int \int \tau_{\mathbf{v}}(\mathbf{x}) f_{\mathbf{v}}(\mathbf{x}) d\mathbf{x}$ 28 mented *input* space" of the basic variables with their uncertain dis-²⁹ tribution parameters. Based on these considerations, the aim of the $\mathcal{F}_{\mathbf{x}}$ $\mathcal{F}_{\mathbf{x}}$ $\mathcal{D}_{\mathbf{x}}$ $\mathcal{D}_{\mathbf{x}}$ are $\mathcal{D}_{\mathbf{x}}$ ³⁰ paper is to describe the two approaches (nested vs. augmented) $= \mathbb{E}_{f_v} [\mathbb{1}_{\mathcal{F}_v}(\mathbf{X})]$ (2) ⁹⁶ ⁹⁷ within the same framework and investigate different advantages and the same of the sa 32 offered by ARA. Moreover, explanations are given about the key where $\mathbb{F}_x(\cdot)$ is the multidion function of the familie domain de-33 point of the transformation between the physical space of the basic lined such that $\mathbb{F}_x(x) = 1$ if $x \in \mathcal{F}_x$ and $\mathbb{F}_x(x) = 0$ otherwise. 34 100 input variables and the standard normal space, which is commonly 35 used in reliability analysis, under the consideration of this new using one or the classical methods available in the structural re-36 "augmented space". This article aims at giving a comparison be-
26 11 and the late of 2.1 to do so, two district classes of the life of the life of the life of the life of the those of the those of the those of the thos 37 tween NRA and ARA through numerical application to challenging these been developed. *approximation memous* such as the *First*- 103 38 test-cases representing the main difficulties that aerospace engi-
28 test-cases representing the main difficulties that aerospace engi-39 neering has to face to (nonlinear codes, correlated inputs and a memod (SORN), which both rely on the concept of *MOSt-Probable* 105 40 Tow failure probability to estimate). Another goal is to highlight, The Point (MPP), and simulation members based on Monte Carlo sin-41 through numerical results, advantages and drawbacks of both ap-
11 and the summer conditions of the method of a method of the case of the conditions of the model of the can ind 42 proaches coupled with advanced reliability methods tested on a more advanced sampling-based methods such as *importance sum*-
108 43 benchmark representative of real world aerospace problems for μ ting (15) [3], *Directionul Sumpling* (DS) [20], *Line Sumpling* (ES) or 109

45 111 This paper is organized as follows. Section 2 expounds a bibli-46 112 ography review of reliability assessment under distribution param- 47 eter uncertainty and aims at introducing the formal concepts and $\frac{1}{23,21}$, in the so-called standard normal space (defined as $\frac{113}{2}$ 48 notations. Section 3 defines the two approaches into a common 0 -space) in which an random components of λ become indepen-49 Framework and provides generic algorithms for both methods. Sec-
19 Second General advancation variation in the vector of Annual Case of the Vector of Annual Basic Provider in 50 116 tion 4 will illustrate the benefits of such an augmented approach 51 through a numerical comparison between NRA and ARA on differ-
51 through a numerical comparison between NRA and ARA on differ-
15 iou always required to Serious Definition in the only 112 52 ent test-cases of increasing complexity (from academic toy-cases and *physical space*, denoted as **A**-space). However, most advanced 118 53 119 to real black-box computer codes issued from aerospace research). 54 Section 5 discusses limitations of those approaches and evokes of have some adapted versions of their mittal algorithms, to the 120 55 possible enhancements. A conclusion gathering the most impor-
55 possible enhancements. A conclusion gathering the most impor-56 tant results of this paper is finally given in Section 6.
56 transformation $T : \mathcal{D}_X \to \mathbb{R}^d$ allowing (in terms of probability distant results of this paper is finally given in Section 6.

58 **2. Formulation of failure probability estimation under** $\mathbf{u} = \mathbf{v} \times \mathbf{v} + \mathbf{v} = 1$ and $\mathbf{v} = \mathbf{v} = 1$ and $\mathbf{v} = (2)$ 59 125 **distribution parameter uncertainty**

2.1. Generic time-invariant reliability problem statement

65 by: $y = M(\mathbf{x})$, where $\mathbf{x} \in \mathcal{D}_{\mathbf{X}} \subseteq \mathbb{R}^d$ is a *d*-dimensional vector of the contract of the c A model $\mathcal{M}(\cdot)$ is considered such that it represents a static (i.e.

¹ tainties can lead to combine some of these frameworks. Choosing In general, this mapping can be either defined using an analyti- 67 ² one of them depends on the available type of data (bounds, distri-cal expression or a numerical model. In our case, this model is a 68 ³ bution, etc.) and has an impact on the propagation of uncertainties computationally expensive simulation code which can lead to con- 69 ⁴ through the model. In this paper, we assume that probability dis- sider it as a black-box function only known pointwise. In the rest ⁷⁰ ⁵ tributions of input data are available but their parameters are not of the paper, $\mathcal{M}(\cdot)$ is supposed to be a deterministic model, i.e. ⁷¹ ⁶ known precisely due to a lack of information or very limited data. In the underlying behavior of the model is not stochastic. The state of the model is not stochastic. In general, this mapping can be either defined using an analytical expression or a numerical model. In our case, this model is a computationally expensive simulation code which can lead to consider it as a *black-box* function only known pointwise. In the rest of the paper, $\mathcal{M}(\cdot)$ is supposed to be a deterministic model, i.e. the underlying behavior of the model is not stochastic.

 7 A classical approach, known as the "nested reliability approach" The scalar random variables X_1, X_2, \ldots, X_d represent the *uncer-* $^{-73}$ *tain* input variables (denoted as the *basic variables* in the following, see [20] and [21]) of the system. These basic variables are gathered in a *d*-dimensional random vector **X** of known continuous joint realizations $\mathbf{x} = (x_1, x_2, ..., x_d)^\top$ of the random vector **X**, failure is characterized by the use of a function $g : \mathbb{R}^d \to \mathbb{R}$ called the *limitstate function* (lsf). A classical formulation for the lsf in the context of static input–output model can be:

$$
g(\mathbf{X}) = y_{\text{th}} - \mathcal{M}(\mathbf{X})
$$
 (1)

23 Bayesian framework [6,17–19].
24 An alternative approach known as the "avermented religibility and the space into two, and the *safe* domain $S_x = \{x \in \mathcal{D}_x : g(x) > 0\}$. where $y_{th} \in \mathbb{R}$ is a characteristic threshold output value beyond the one the system falls into a failure state. Thus, one can distinguish two domains associated to the behavior of *g(*·*)*: the *failure domain limit-state (hyper-)surface* (LSS) $\mathcal{F}_{\mathbf{x}}^0 = {\mathbf{x} \in \mathcal{D}_{\mathbf{x} : g(\mathbf{x}) = 0}$ splitting With no consideration of any distribution parameter uncertainty, the *failure probability* p_f therefore reads:

$$
p_{\mathbf{f}} = \mathbb{P}\left[\mathbf{g}(\mathbf{X}) \leq 0\right] = \int_{\mathcal{F}_{\mathbf{X}}} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} = \int_{\mathcal{D}_{\mathbf{X}}} \mathbb{1}_{\mathcal{F}_{\mathbf{X}}}(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}
$$

$$
= \mathbb{E}_{f_{\mathbf{X}}} \left[\mathbb{1}_{\mathcal{F}_{\mathbf{X}}}(\mathbf{X})\right]
$$
(2)

where $\mathbb{1}_{\mathcal{F}_{\mathbf{x}}}(\cdot)$ is the indicator function of the failure domain defined such that $\mathbb{1}_{\mathcal{F}_{\mathbf{x}}}(\mathbf{x}) = 1$ if $\mathbf{x} \in \mathcal{F}_{\mathbf{x}}$ and $\mathbb{1}_{\mathcal{F}_{\mathbf{x}}}(\mathbf{x}) = 0$ otherwise.

44 which only the ARA is a tractable approach. Finally, estimating such a failure probability can be achieved using one of the classical methods available in the structural reliability literature [22]. To do so, two distinct classes of methods have been developed: *approximation methods* such as the *First-Order Reliability Method* (FORM) and the *Second-Order Reliability Method* (SORM), which both rely on the concept of *Most-Probable Point* (MPP); and *simulation methods* based on Monte Carlo simulations [21]. Among this second class of methods, one can find more advanced sampling-based methods such as *Importance Sampling* (IS) [3], *Directional Sampling* (DS) [20], *Line Sampling* (LS) or *Subset Simulations* (SS) [4].

 57 construction in the construction of Approximation methods (such as FORM/SORM) have been developed, following well-argued mathematical and historical reasons [23,21], in the so-called *standard normal space* (denoted as **U***-space*) in which all random components of **X** become independent standard Gaussian variates gathered in the vector **U**. Among the simulation methods, the use of such a standard normal space is not always required (e.g., CMC method is performed in the original *physical space*, denoted as **X***-space*). However, most advanced sampling-based methods such as those cited above are dedicated, or have some adapted versions of their initial algorithms, to the standard normal space. The general idea is to construct a regular tributions) to get:

$$
\mathbf{U} = T(\mathbf{X}) \Leftrightarrow \mathbf{X} = T^{-1}(\mathbf{U})
$$
\n(3)

60 **126** where $\mathbf{U} = (U_1, U_2, \dots, U_d)^\top$ is a *d*-dimensional standard Gaussian 126 ⁶¹ 2.1. Generic time-invariant reliability problem statement **the vector of independent normal variates** U_i with zero means and ¹²⁷ ⁶² 128 unit standard deviations. Then, one can define a new mapping for ¹²⁸ 63 A model $M(\cdot)$ is considered such that it represents a static (i.e. the 1sf in the standard space considering $G : \mathbb{R}^d \to \mathbb{R}$ defined such ¹²⁹ ⁶⁴ time is not an explicit variable here) input–output system given that: $\frac{130}{2}$ that:

$$
66 \quad \text{input variables and } M: \mathcal{D}_X \to \mathcal{D}_Y \subseteq \mathbb{R} \text{ a given scalar mapping.} \qquad \mathbf{U} \mapsto G(\mathbf{U}) = \left(g \circ T^{-1}\right)(\mathbf{U}) \tag{4}
$$

which allows to rewrite the failure probability:

$$
p_{\mathbf{f}} = \mathbb{P}\left[G(\mathbf{U}) \le 0\right] = \int_{\mathcal{F}_{\mathbf{u}}} \varphi_d(\mathbf{u}) d\mathbf{u} = \int_{\mathbb{R}^d} \mathbb{1}_{\mathcal{F}_{\mathbf{u}}}(\mathbf{u}) \varphi_d(\mathbf{u}) d\mathbf{u}
$$

$$
= \mathbb{E}_{\varphi_d} \left[\mathbb{1}_{\mathcal{F}_{\mathbf{u}}}(\mathbf{U})\right]
$$
(5)

where $\mathcal{F}_{\mathbf{u}} = {\mathbf{u} \in \mathbb{R}^d : G(\mathbf{U}) \leq 0}$ stands for the failure domain in the standard space, $d\mathbf{u} = du_1 du_2 ... du_d$ and $\varphi_d : \mathbb{R}^d \to \mathbb{R}_+$ is the *d*-dimensional standard Gaussian pdf of **U**.

The choice of the transformation $T(\cdot)$ generally depends on the available information. When only the marginal distributions $f_{X_i}(\cdot)$ and the linear correlations are known, following the recommendations in [24,25], one should use the so-called *Nataf transformation* [26]. When the full knowledge of the joint pdf $f_{\mathbf{X}}(\cdot)$ is available, it is advised [27] to better use the so-called *Rosenblatt transformation* [28]. Thus, under the assumption of normal copula [29], without any consideration of parameter uncertainty, both case [30].

2.2. Reliability analysis under distribution parameter uncertainty

24 For complex systems such as aerospace ones, the joint pdf value It becomes a random variable denoted as P_t which depends 90 25 $f_{\mathbf{X}}(\cdot)$ is not accurately known [17,23]. For example, the choice con the realization θ of the random vector of uncertain parameters ⁹¹ 26 of a parametric model for the density $f_{\mathbf{X}}(\cdot)$ can be based on such that: 27 estimation of some distribution parameters (i.e. some moments and the state of the state 28 of the pdf) which can introduce an important bias if the initial $P_f(\theta) = \mathbb{P}[g(\mathbf{X}) \le 0 \mid \Theta = \theta]$ (6a) 94 29 samples only provide some limited information. Moreover, some $\qquad \qquad \qquad \qquad$ $\qquad \qquad$ $\qquad \qquad \qquad$ 95 30 expert-judgment-based assumptions can lead to an a priori choice $\mathcal{I} = \int \mathbb{1}_{\mathcal{F}_x}(\mathbf{X}) f_{\mathbf{X}|\Theta}(\mathbf{x}|\theta \, d\mathbf{X})$ (6D) 96 31 of some values for the parameters instead of others. The perfect $\overrightarrow{p}_{\infty}$ 32 knowledge of the joint pdf $f_{\mathbf{X}}(\cdot)$ would require, from a general $f_{\mathbf{X}}(x)$ $f_{\mathbf{$ 33 point of view, the full knowledge of the marginal pdfs and the $-\frac{\mu}{\lambda}x_0 e^{(\frac{\mu}{\lambda})x} \left(\frac{\nu}{\lambda} - \frac{\nu}{\lambda} \right)$. ³⁴ copula. However, the probabilistic information available about the Hence, by integrating over θ , we get the so-called "predictive failure ¹⁰⁰ ³⁵ input random vector **X** often reduces to the marginal distributions probability" \widetilde{P}_f which is a measure of reliability taking into account 101 ³⁶ and, in the case of dependent inputs, to the imperfect knowledge the effect of the uncertain characterization of distribution parame- ¹⁰² 37 of the linear correlation matrix $\mathbf{R} = [\rho_{ij}]_{i,j \in \{1,\ldots,d\}}$ [29]. Thus, in ad-
ters: 38 104 dition to the first uncertainty level characterizing the basic input ³⁹ variables, uncertainty may also affect both distribution parame- $\bar{P}_f = \mathbb{E}_{f_\Omega} \left[P_f(\Theta) \right]$ (7a) ¹⁰⁵ 40 ters and the dependence structure. Consequently, engineers have \overline{C} 41 to face what we call a *bi-level uncertainty*. In this context, giving $= \int P_f(\theta) f_{\Theta}(\theta) d\theta$ (7b) 107 42 back an hypothetic single measure of reliability taking only one 7 43 109 level into account seems to be inappropriate. $f_{\textbf{X}}(\cdot)$ is not accurately known [17,23]. For example, the choice

44 Such a topic has been early discussed among the structural re-45 liability community, mainly in the first investigations led by Der $= \int \int \int \mathbb{1}_{\mathcal{F}_{\alpha}}(\mathbf{x}) f_{\mathbf{X}|\Theta}(\mathbf{x}|\theta) d\mathbf{x} \int f_{\Theta}(\theta) d\theta$. (7c) 111 46 112 Kiureghian [6,17,31,32] and Ditlevsen [18–20,33]. In their common 47 paper [5], these authors stress the need of a measure of reliabil- \overline{a} \overline{a} ⁴⁸ ity that takes into account parameter uncertainty (the first author Eq. (7c) is the key equation whose solving is under consideration ¹¹⁴ ⁴⁹ proposed in [6] to call it "*predictive reliability measure*", following in this paper. The idea is that it can be numerically solved by two ¹¹⁵ 50 116 Bayesian analysis vocabulary, and provided a formal definition that ⁵¹ will be recalled later). The same states of the set of the set of view, a first way of computing this this thing that the set of ity that takes into account parameter uncertainty (the first author proposed in [6] to call it "*predictive reliability measure*", following will be recalled later).

⁵² Assuming now that **X** is distributed according to the para- integral relies on evaluating pointwise the inner integral for each ¹¹⁸ 53 metric joint pdf $f_{X|\Theta}(\cdot|\cdot)$, each random variable X_i is distributed realization θ of Θ [14–16]: this leads to the nested reliability ap-
⁵³ 54 according to the marginal pdf $f_{X_i|\Theta_i}(\cdot|\cdot)$. In the case of depen-proach (presented in subsection 3.1). The second way consists in 120 ⁵⁵ dent inputs, in the normal copula case, uncertainty affecting the evaluating it by treating both basic variables and uncertain distri- ¹²¹ ⁵⁶ correlation matrix could easily be considered in this framework. bution parameters together and by integrating simultaneously on ¹²² ⁵⁷ However, from a more general point of view, uncertainty affecting both domains (but still respecting the conditioning) as suggested ¹²³ ⁵⁸ the dependence structure (i.e. the copula) is not a widely stud-
 $\frac{17}{1}$: this is the augmented reliability approach (presented in 2²⁴) 59 ied topic in literature. Moreover, from an engineering perspective, subsection 3.2). The next section describes these approaches in 125 60 this problem is really difficult to assess due to the crucial lack details. As a remark, one can notice that this Bayesian frame- 126 61 of information. In this paper, we will only consider distribution work provides here a single reliability measure (the predictive fail- 127 62 parameter uncertainty and let copula structure uncertainty to fu- ure probability). Nevertheless, this quantity can help engineers to 128 ⁶³ ture work. Indeed the vector Θ gathers all distribution parameters anake more informed decisions during the design process and can ¹²⁹ 64 of the corresponding marginals such that $\Theta = (\Theta_1, \Theta_2, ..., \Theta_d)^\top$, be coupled with the classical reliability measure so as to analyze 130 65 where each $\mathbf{\Theta}_i$, $i \in \llbracket 1,d \rrbracket$ is a set of distribution parameters for properly the risk undertaken with a design choice. Decision can be 131 66 the *i*-th marginal (for instance, if $X_i \sim \mathcal{N}(\mu_{X_i}, \sigma_{X_i})$, then $\Theta_i =$ then enlightened by such additional information [37–39]. Assuming now that **X** is distributed according to the parametric joint pdf $f_{\mathbf{X}|\mathbf{\Theta}}(\cdot|\cdot)$, each random variable X_i is distributed according to the marginal pdf $f_{X_i|\Theta_i}(\cdot|\cdot)$. In the case of depenthe dependence structure (i.e. the copula) is not a widely studied topic in literature. Moreover, from an engineering perspective, ture work. Indeed the vector **O** gathers all distribution parameters where each $\mathbf{\Theta}_i, i \in [\![1, d]\!]$ is a set of distribution parameters for

¹ which allows to rewrite the failure probability: $(\mu_{X_i}, \sigma_{X_i})^\top$). One can imagine that depending on the distribution 67 2 68 type, all the marginal pdfs will not be defined with the same $3 \qquad p_{\ell} = \mathbb{P}[G(\mathbf{U})] < 0 = \int \varphi_d(\mathbf{u}) d\mathbf{u} = \int \mathbb{1} \mathbb{E}[(\mathbf{u})\varphi_d(\mathbf{u})d\mathbf{u}]$ number of parameters. In this paper, we assume that only a set \mathbb{R}^3 4 **1** *I I I I I I I C* 5 **1 1** \sqrt{u} **1** \mathbb{R}^u and \mathbb{R}^n to consider a general collection of univariate random parameters \mathbb{R}^n 6 $= \mathbb{E}_{\varphi_d} [\mathbb{1}_{\mathcal{F}_{\mathbf{u}}}(\mathbf{U})]$ (5) given by $\mathbf{\Theta} = (\Theta_1, \Theta_2, ..., \Theta_k)^\top \in \mathcal{D}_{\mathbf{\Theta}} \subseteq \mathbb{R}^k$ (which can be either \mathbb{Z}_2 ⁷ The same of the same of the moments or bounds). Consequently, without any loss of generality, $\frac{73}{2}$ 8 where $\mathcal{F}_{\mathbf{u}} = {\mathbf{u} \in \mathbb{R}^d : G(\mathbf{U}) \le 0}$ stands for the failure domain in one can assume the existence of a joint pdf $f_{\mathbf{\Theta}} = \prod_{j=1}^k f_{\Theta_j}$ as a 74 9 the standard space, $du = \frac{du_1 du_2 \ldots du_d}{du_1 du_2 \ldots du_d}$ and $\varphi_d : \mathbb{R}^d \to \mathbb{R}^+$ is the product of the marginal pdfs of the Θ_j [34]. Note here that one 75 10 *a*-dimensional standard Gaussian pur of **U**. $\frac{1}{2}$ could also consider a dependence structure between the distribu-11 11 The choice of the transformation $I(\cdot)$ generally depends on tion parameters. However, the problem would be far more difficult $\frac{77}{2}$ ⁷² the available information. When only the marginal distributions and would imply to have, at minimum, a prior information about ⁷⁸ 13 $J_{X_i}(\cdot)$ and the linear correlations are known, following the recom-such a dependence structure. This topic is beyond the scope of this 14 mendations in $[24,25]$, one should use the so-called *Nataf trans* paper. To get a deeper insight about the practical characterization 80 15 formation [26]. When the full knowledge of the joint part $Jx(\cdot)$ is of $f_{\Theta}(\theta)$ based on available data (which is not the scope of this pa-16 available, it is advised $\lfloor 27 \rfloor$ to better use the so-called *Rosenblutt* per), the reader may refer to [35]. To sum up, in this paper, only a ⁸² ¹⁷ transformation [28]. Inus, under the assumption of normal cop-
prior probability distribution (for instance, following an expert-based ⁸³ 18 Level *(29)*, without any consideration of parameter uncertainty, both independent) will be assumed for Θ without any purpose of Bayesian ⁸⁴ 19 transformations can be used since they are identical in this specific reliability undating [36] reliability updating [36].

20 **Case [30]**. Case [30] **Case 130** and the set of the failure probability can be pro-21 87 posed, following [17]. Indeed, due to this bi-level uncertainty (on 22 2.2. Reliability analysis under distribution parameter uncertainty the vector of basic variables **X** and on the vector of distribution pa- 23 $\qquad \qquad$ rameters Θ), the failure probability p_f is no more a deterministic 88 value. It becomes a random variable, denoted as P_f , which depends such that:

$$
P_{\mathbf{f}}(\boldsymbol{\theta}) = \mathbb{P}\left[\mathbf{g}(\mathbf{X}) \le 0 \mid \boldsymbol{\Theta} = \boldsymbol{\theta}\right]
$$
 (6a)

$$
=\int_{\mathcal{D}_{\mathbf{X}}} \mathbb{1}_{\mathcal{F}_{\mathbf{X}}}(\mathbf{x}) f_{\mathbf{X}|\Theta}(\mathbf{x}|\theta \, \mathrm{d}\mathbf{x})
$$
(6b)

$$
= \mathbb{E}_{f_{\mathbf{X}|\Theta}} \left[\mathbb{1}_{\mathcal{F}_{\mathbf{X}}}(\mathbf{X}) \mid \Theta = \theta \right]. \tag{6c}
$$

Hence, by integrating over *θ*, we get the so-called "*predictive failure probability*" \widetilde{P}_f which is a measure of reliability taking into account the effect of the uncertain characterization of distribution parameters:

$$
\widetilde{P}_{\mathbf{f}} = \mathbb{E}_{f_{\Theta}}\left[P_{\mathbf{f}}(\Theta)\right] \tag{7a}
$$

$$
=\int_{\mathcal{D}_{\Theta}} P_{\mathbf{f}}(\boldsymbol{\theta}) f_{\Theta}(\boldsymbol{\theta}) d\boldsymbol{\theta}
$$
 (7b)

$$
= \int_{\mathcal{D}_{\Theta}} \left(\int_{\mathcal{D}_{\mathbf{X}}} \mathbb{1}_{\mathcal{F}_{\mathbf{X}}}(\mathbf{x}) f_{\mathbf{X}|\Theta}(\mathbf{x}|\theta) d\mathbf{x} \right) f_{\Theta}(\theta) d\theta.
$$
 (7c)

different approaches.

evaluating it by treating both basic variables and uncertain distribution parameters together and by integrating simultaneously on both domains (but still respecting the conditioning) as suggested details. As a remark, one can notice that this Bayesian framework provides here a single reliability measure (the predictive failure probability). Nevertheless, this quantity can help engineers to be coupled with the classical reliability measure so as to analyze then enlightened by such additional information [37–39].

=

¹ Up to now, to our knowledge, several researchers deployed ef- **Algorithm 2** NRA generic box (FORM or SS). $\frac{1}{2}$ 67 ² forts to carry on the way of other approaches to compute this \sqrt{a} $\frac{1}{2}$ $\frac{1}{2$ 3 69 predictive failure probability (see another approach by [40], used 4 in [6] and in [41] only with FORM calculations). Nevertheless, the $\vert\vert$ Denne: classical transformation T to the U-space $\vert\vert$ 70 5 track of exploring the *augmented space* has not been over-exploited $\vert \cdot \vert$ **u** = $T(\mathbf{x})$ (Nataf or Rosenblatt) and its inverse $\vert \cdot \vert$ 71 ⁶ yet. In [42], the author recommended and implemented this strat- $||$ **Start** $||$ **52** ⁷ egy on a fracture mechanics test-case but limited his study to $\vert\vert$ FORM Algorithm / SS Algorithm \vert ⁷³ ⁸ FORM algorithm. All these works mainly focused on providing a $\frac{1}{\sqrt{2}}$ (see [4] for a full description of these algorithms) $\frac{1}{\sqrt{2}}$ ⁷⁴ 9 global reliability index, robust to parameter uncertainty, in the spe- $\frac{1}{2}$ $\frac{1}{2}$ ¹⁰ cific context of FORM. The use of an augmented space has also $\left|\int_{\mathbf{Cot}} \hat{\mathbf{n}}(k)/\mathbf{a}(k)\right|$ 11 been exploited by [43] for design sensitivity purpose while con-¹² sidering uncertain design parameters. More recently, in [44], the the state of the stat ¹³ authors proposed in a broader view on the comprehension and in-14 80 terpretation of the different levels of uncertainty involved in these ¹⁵ calculations and encouraged to use an augmented approach to SS appeared to be a very powerful method to reach estimation ⁸¹ ¹⁶ solve a similar integral problem given in Eq. (7c). However, their of rare event failure probabilities, under the constraint of nonlin- 82 ¹⁷ study did not aim at performing reliability assessment for rare ear lsfs, with a rather moderate computational effort [45]. In brief, 83 ¹⁸ event failure probabilities of some complex simulation codes which the rectangular box can be seen as a non-intrusive plug-in uncer- ⁸⁴ is the scope of the present paper.

3. Description of the two approaches

3.1. The nested reliability approach (NRA)

²⁵ This approach is a nested-loop-based approach since it involves or Rosenblatt ones can be both used, and the distribution parame- ⁹¹ ²⁶ the numerical estimation of two different quantities. The first ter uncertainty does not change anything to their implementation. ⁹² ²⁷ (nested or inside) loop aims at computing a "conditional" failure Nevertheless, one should notice that for each sampled parameter, 93 28 probability whose numerical estimator is denoted as $\hat{P_f}(\theta)$. This the algorithm needs to rebuild and recalculate the transformation 24
29 estimator is a magazine of plichility under the politician 0 of the since it ²⁹ estimator is a measure of reliability under the realization θ of the since it depends on the parameter value. Thus, for complicated ⁹⁵ ³⁰ random vector Θ . The second (outside) loop aims at computing an transformations, with a large number of basic variables, the simu-³¹ estimator of the predictive failure probability, denoted as \widetilde{P}_f , by in-
³² terrative suggestive suggest of the graden vector Q , in graphics it. In this pasted case, we can demonstrate that the estimator $\$ 32 tegrating over the support of the random vector Θ . In practice, it In this nested case, we can demonstrate that the estimator \tilde{P}_f is $\frac{98}{33}$ consists in computing several $\hat{P}_{f}(\theta)$ for a range of realizations θ of unbiased. Moreover, the mean and variance of \hat{P}_{f} are estimated by $\frac{99}{4}$ the water of wareness of the second property of the area in ³⁴ the vector of uncertain parameters Θ . It has been widely used in replication of the algorithm, using the following classical statistics: ³⁵ literature, in various contexts, such as rare event probability es-³⁶ timation with Kriging-based approach in [14], probability-based $10^{N_{rep}}$ \approx (i) ³⁷ tolerance analysis of products in [15] or uncertainty propagation $m_{\tilde{p}} = \frac{1}{N} \sum \widetilde{P}_{f}^{(1)}$ (8) ¹⁰³ ³⁸ using probability-boxes and polynomial chaos expansions in [16]. $\frac{Pf}{P}N_{rep}\frac{N}{i=1}$ 39 105 This approach is a nested-loop-based approach since it involves tegrating over the support of the random vector Θ . In practice, it consists in computing several $\widehat{P}_f(\theta)$ for a range of realizations θ of the vector of uncertain parameters Θ . It has been widely used in

Algorithm 1 Nested reliability approach (NRA) with CMC for probability estimation.

55 A generic implementation of NRA framework coupled with a $K + a$ (K uncertain distribution parameters Θ_j and a random basic Θ_j 56 nested CMC method is given in the Algorithm 1. In the rectan-
 $\frac{V_{\text{ATI}}}{V_{\text{ATI}}}\sim 122$

¹⁹ is the scope of the present paper. **Example 19** tainty propagation code for reliability assessment. An example of a 20 **1.1 20 86** plug-in box (for FORM or SS) is given in the Algorithm 2. In nu-²¹ 3. Description of the two approaches **Exercise 20** 21 merous cases, an additional step is required: the transformation 87 22 to the standard normal space (see subsection 2.1). In the nested 88 ²³ 3.1. The nested reliability approach (NRA) **case, the transformation is already included in the plug-in relia-** 89 24 90 bility rectangular box, i.e. classical transformations such as Nataf SS appeared to be a very powerful method to reach estimation of rare event failure probabilities, under the constraint of nonlinear lsfs, with a rather moderate computational effort $[45]$. In brief, the rectangular box can be seen as a non-intrusive plug-in uncerter uncertainty does not change anything to their implementation. Nevertheless, one should notice that for each sampled parameter, the algorithm needs to rebuild and recalculate the transformation since it depends on the parameter value. Thus, for complicated transformations, with a large number of basic variables, the simulation cost induced can be increased.

$$
m_{\widehat{P}_f} = \frac{1}{N_{rep}} \sum_{i=1}^{N_{rep}} \widehat{P}_f^{(i)}
$$
\n(8)

40 **Algorithm 1** Nested reliability approach (NRA) with CMC for prob-
which is the *sample mean* with N_{rep} the number of replications of 106 $\frac{41}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ the predictive failure probability estimation and S_2^2 the unbiased 107 ⁴² **108 Sumple** variance defined by: $\frac{2}{P_f}$ the *unbiased*

$$
S_{\widehat{\widetilde{P}}_{\rm f}}^2 = \frac{1}{N_{rep} - 1} \sum_{i=1}^{N_{rep}} \left(\widehat{\widetilde{P}}_{\rm f}^{(i)} - m_{\widehat{\widetilde{P}}_{\rm f}}\right)^2.
$$
 (9)

51 10: Predictive failure probability estimator:

¹¹⁷

¹¹⁷ \vec{z} $\vec{z$ $\frac{12}{53}$ $\frac{12}{53}$ $\frac{12}{5}$ $\frac{16}{5}$ for a similar $\frac{11}{21}$ and $\frac{11}{21}$ bution parameters as it appears in Eq. (7c) (see [16] for a similar $\frac{11}{21}$ 54 120 definition). Thus, this augmented input space has a dimension of $k + d$ (*k* uncertain distribution parameters Θ_i and *d* random basic variables *Xi*).

57 123 gular box at lines 5–9, one can choose any available reliability 58 method to estimate the conditional failure probability $P_f(\theta)$, from $\frac{f(t) \ln 3}{2}$. Again, in this algorithm, the rectangular box can be re-
50 separatives methods (FORM, SORM) to meet advanced sings and placed by an 59 125 approximation methods (FORM, SORM) to most advanced simu-60 126 lation methods (IS, SS). Nevertheless, it seems more relevant to 61 focus on the ones that are still, up-to-now, the most widely used (see Algorithm 4 as an example). This shows that the ARA does 127 ⁶² either in aerospace industry or in research. Indeed CMC is still bot suffer from any major difference with the classical nested ap- 128 ⁶³ considered as the reference method for validation. FORM offers proach interms of the variety of methods that it can handle. Again, 129 ⁶⁴ wide possibilities for practitioners who want to perform reliabil- as for the NRA, one can demonstrate that the estimator $P_{\rm f}$ is un- 130 65 ity assessment with a low computational cost, even if this method biased. One major difference concerns the transformation to stan- 131 66 only gives the true failure probability for linear lsfs [21]. Finally, dard normal space: since there exists a conditioning between the 132 A generic implementation framework is given in the [Algo](#page-5-0)rithm 3. Again, in this algorithm, the rectangular box can be replaced by any non-intrusive plug-in uncertainty propagation code for reliability assessment as the ones cited previously for the NRA (see Algorithm 4 as an example). This shows that the ARA does not suffer from any major difference with the classical nested approach in terms of the variety of methods that it can handle. Again, as for the NRA, one can demonstrate that the estimator \tilde{P}_f is unbiased. One major difference concerns the transformation to standard normal space: since there exists a conditioning between the

distribution parameters and the basic input variables, Nataf transformation cannot be used anymore and Rosenblatt transformation is the only one that can handle this constraint.

32 Considering uncertainties affecting distribution parameters leads in range of classical problems encountered in reliability assessment 38 33 To adapt the usual kosenblatt transformation. We assume the Joint to difference systems: complex black-box models with numerous the 34 pdf Jx(0(·|·) is known since we know all the marginal pdfs and input variables, high nonlinearities, high computational cost, low the 35 The correlation matrix (or the covariance matrix) giving the linear probability of failure. Moreover, three reliability methods have 101 36 correlation structure between the basic input variables (normal or been tested to calculate the failure probability: CMC, FORM and 102 37 Gaussian copula case [29]). In addition, we know the joint pdr construct SS. The following numerical applications have been implemented the 38 $f_{\Theta}(\cdot)$ as explained previously in subsection 2.2. In this case, un-
in Matlab® and performed using the open source toolbox FERUM 104 39 der the consideration of the *augmented space* of dimension $k + d$ $\frac{1}{2}$ $\frac{1}{4}$ $\frac{1}{4}$ 40 106 (*k* distribution parameters and *d* basic variables), one can apply 41 Rosenblatt transformation [28] first to the *k* components of Θ and θ and θ and comparison metrics 42 108 then to the *d* components of the vector **X**|*-* such that: Considering uncertainties affecting distribution parameters leads to adapt the usual Rosenblatt transformation. We assume the joint pdf $f_{\mathbf{X}|\Theta}(\cdot|\cdot)$ is known since we know all the marginal pdfs and the correlation matrix (or the covariance matrix) giving the linear correlation structure between the basic input variables (normal or Gaussian copula case $[29]$). In addition, we know the joint pdf $f_{\Theta}(\cdot)$ as explained previously in subsection 2.2. In this case, un-

59 (CDF) and $F_{\Theta_i}(\cdot)$, $F_{X_j|\Theta_i}(\cdot|\cdot)$ respectively the marginal CDFs of the clarity and to avoid any confusion, these strategies are presented ₁₂₅ $_{60}$ parameters and the conditional marginal CDFs of the basic vari- and discussed in the dedicated subsections of the test-cases. $_{126}$ 61 ables. In the case of correlated inputs, one can implement this reg- \overline{O} One needs to introduce the comparison metrics used in the fol- $_{62}$ ular transformation following and adapting general formulas given lowing numerical benchmarks. Thus, following [4], we choose in $_{128}$ 63 in [28]. As a remark, one should notice that from a numerical point this paper to characterize the quality of our estimator \widetilde{P}_t^{α} of \widetilde{P}_f 129 $\frac{1}{2}$ of view, the inverse transformation $(T_{aug}^{Ros})^{-1}$ (·) (from the standard botained with the method *M* by the use of three performance met-⁶⁵ normal space to the physical space) can be the most useful (espe- rics computed with respect to the reference calculations, i.e. those ¹³¹ normal space to the physical space) can be the most useful (espe-

3.3. Illustration

 $\frac{3}{16}$ From a numerical point of view, NRA and ARA can be illustrated $\frac{69}{16}$ $\frac{4}{10}$ is out through a two-dimensional example with one uncertain distribu- $\begin{bmatrix} 5 \end{bmatrix}$ $\begin{bmatrix} 2. \text{ L. L. L. R. } 5 \end{bmatrix}$ $\begin{bmatrix} 2. \text{ L. L. R. } 71 \end{bmatrix}$ tion parameter. Let us call X_1 and X_2 the two basic input vari-6 $\begin{vmatrix} 3 & \text{For } i = 1:N_{x,0} \\ 1 & \text{Sounds} & \text{Sounds} \end{vmatrix}$ ables modeled as two Gaussian variates such that *X*₁ ∼ N *(μ*_{*X*₁ = $\frac{72}{73}$} 7
 A: Sample $\mathbf{e}^{(i)}$;
 $\mathbf{e}^{(i)} = \begin{cases} 2\pi i & \text{where } \mathbf{e}^{(i)} \\ \mathbf{e}^{(i)} & \text{where } \mathbf{e}^{(i)} = \mathbf{e}^{(i)} \end{cases}$
 A: Sample $\mathbf{v}^{(i)}$ and $\mathbf{v}^{(i)} = \mathbf{e}^{(i)}$. 8 $\begin{bmatrix} 5 \\ 5 \end{bmatrix}$ Sample $X^{(1)}$ given $\Theta^{(1)} = \Theta^{(1)}$;
 8 $\begin{bmatrix} 5 \\ 6 \end{bmatrix}$ is considered as being uncertain (for example, $\Theta \sim \mathcal{N}(2, 1.5)$). ⁹ $\begin{vmatrix} 6 & \text{Limit-state function evaluation: } g(x^{(1)}) \\ \text{For the sake of clarity, in Fig. 1(a), only three clouds of samples \end{vmatrix}$ 10 $\left|\right|$ 7: Predictive failure probability estimator:
 $\left|\right|$ are plotted for three different values of θ (200 points per cloud). $\widetilde{P}_f = \frac{1}{N_{\rm x,0}} \sum_{i=1}^{N_{\rm x,0}} \mathbb{1}_{\mathcal{F}_{\rm x}}(X^{(i)})$; $\qquad \qquad \qquad$ Indeed such a sequential sampling is the underlying principle of $\frac{77}{20}$ $\frac{12}{12}$ $\frac{1}{12}$ $\frac{1}{28}$ $\frac{1}{28}$ NRA.

 $\frac{13}{14}$ $\frac{9}{14}$ $\frac{1}{14}$ As for ARA, graphical results plotted in Fig. 1(b) bring out the $\frac{79}{96}$ 14 80 underlying principle of this approach: covering in one algorithm $\frac{15}{\text{Algorithm 4 APA} \text{ exercise how (FOPM or SC)}$ 16 **Algorithm 4** ARA generic box (FORM or SS).
points are used, instead of 3×200 points for NRA). One can clearly $\frac{17}{12}$ Being the discussion of the same trend between NRA and ARA, the first one by a 18 **Define**: new transformation T for the augmented space sequential sampling strategy, the second one by a simultaneous $\frac{84}{\text{e}}$ 19 85 sampling over all the dimensions of the augmented input space. In 20 The continuum and and the continuum of the contin 21 $\begin{vmatrix} 1 & u = T(z) \\ u = T(z) \end{vmatrix}$ brief, it appears that ARA offers better space-filling properties than NRA [46].

26 92 To evaluate the efficiency of the ARA, a numerical validation 27 32 benchmark has been performed with a systematic comparison and the systematic comparison \sim 28 94 to the classical NRA. Several test-cases of increasing complex-29 distribution parameters and the basic input variables, Natal trans-only have been chosen (from two analytical cases to two different os 30 Tormation cannot be used anymore and kosemblatt transformation wheal life" aerospace industrial applications) to check the validity the set of the validity 31 31 31 S the only one that can handle this constraint. The methods of the methods. The choice of these test-cases aims at covering a 37 v4.1 [47].

4.1. Methodology and comparison metrics

tive failure probability \tilde{P}_f . These combined approaches (NRA/FORM, NRA/SS, ARA/FORM, ARA/SS) will be respectively compared to a reference estimation performed using CMC (most of the time, a NRA/CMC with a large number of samples on both domains). [Ta](#page-6-0)ble 1 gives a brief overview of the methodology.

58 where $\Phi^{-1}(\cdot)$ is the normal inverse *cumulative density function* specific computational strategies have been set up. For the sake of ₁₂₄ In Table 1, the black squares \blacksquare stand for successful calculations of the test-cases and the crosses \times indicate that FORM is clearly inappropriate since the lsf is known explicitly to be nonlinear. As a remark, one can notice that some specific cases are denoted as specific computational strategies have been set up. For the sake of clarity and to avoid any confusion, these strategies are presented and discussed in the dedicated subsections of the test-cases.

⁶⁶ cially when FORM, SORM or SS methods are used). $\qquad \qquad$ obtained by CMC: $\qquad \qquad$ 132 One needs to introduce the comparison metrics used in the following numerical benchmarks. Thus, following [4], we choose in this paper to characterize the quality of our estimator \tilde{P}_f^M of \tilde{P}_f of \tilde{P}_f obtained with the method *M* by the use of three performance metobtained by CMC:

18 84 $\frac{19}{19}$ (a) that (200 points) cloud, indeed 600 points in today.

Table 1

23 a completed that is a set of the Overall methodology.

2 correlated basic variables, 1 uncertain parameter, $g(\cdot)$ linear, low classical failure probability.

31 97 ^b 8 independent basic variables, 1 uncertain parameter, *g(*·*)* nonlinear, low classical failure probability.

32 98 ^c 6 independent basic variables, 2 uncertain parameters, *g(*·*)* nonlinear, low classical failure probability.

33 99 ^d 6 correlated basic variables, 3 uncertain parameters, *g(*·*)* nonlinear, low classical failure probability.

• the *Relative standard Error* (RE):

$$
RE\left[\widehat{P}_{f}^{M}\right] = \frac{\sqrt{V\left[\widehat{P}_{f}^{M}\right]} }{E\left[\widehat{P}_{f}^{M}\right]}
$$
\n
$$
V^{M} = \frac{\left(1 - \widehat{P}_{f}^{M}\right)}{V^{M} - \left(1 - \widehat{P}_{f}^{M}\right)}
$$
\n
$$
(14)
$$
\n
$$
(14)
$$
\n
$$
V^{M} = \frac{\left(1 - \widehat{P}_{f}^{M}\right)}{V^{M} - \left(1 - \widehat{P}_{f}^{M}\right)}
$$
\n
$$
(15)
$$

 \mathcal{H} \mathcal{H} \mathcal{H} \mathcal{H} \mathcal{H} \mathcal{H} \mathcal{H} and \mathcal{H} \mathcal{H} are estimated on the sample ob where $\mathbb{E}\left[\begin{matrix} \widetilde{P}^{M} \\ \widetilde{P}_{f} \end{matrix}\right]$ and $\mathbb{V}\left[\begin{matrix} \widehat{P}^{M} \\ \widehat{P}_{f} \end{matrix}\right]$ are estimated on the sample ob-
45 A value of $\nu^{M} > 1$ indicates that the method M is more ef-
45 A value of $\nu^{M} > 1$ indicat $\begin{bmatrix} M \\ f \end{bmatrix}$ and $\mathbb{V} \left[\widehat{P}_{f}^{M} \right]$ $\binom{M}{f}$ are estimated on the sample ob-

49 115 50 116 *4.2. Application on two academic test-cases* 51 117 52 118 *4.2.1. A first R* − *S example with correlated basic variables and low* RB *P M* f = E *P M* f [−] *^P CMC* f *P CMC* f *.* (12)

 54 120 μ 200 μ 200 μ 200 μ 200 μ 200 μ 30 Ω 30 Ω 30 μ 30 It gives a description of how close the estimate \widehat{P}_{f}^{M} is close to the reference value \widehat{P}_{f}^{CMC} . In the following (see Tables 3 to 6),

NRA/CMC and ARA/CMC) defined such that:

$$
v^M = \frac{N_{sim}^{CMC}}{N_{sim}^{M}} \tag{13}
$$
 The left thus reads:
\n $v^M = \frac{N_{sim}^{CMC}}{N_{sim}^{M}} \tag{15}$

 35 • the *Relative standard Error* (RE): where $N_{\text{sim}}^{\text{CMC}}$ is the required number of CMC simulations to get 101 36 102 37 RE $\begin{bmatrix} \widetilde{P}_{f}^{CMC} \\ P_{f} \end{bmatrix} = RE \begin{bmatrix} \widetilde{P}_{f}^{M} \\ P_{f} \end{bmatrix}$. Thus, the v^{M} ratio can be rewritten 103 38 104 38 104 104 38 38 38 39 394 $RE \left[\widehat{P} \right]$ *CMC* f ٦ f . Thus, the *ν^M* ratio can be rewritten as:

$$
RE\left[\begin{array}{c}\widetilde{P}_{f}^{M}\end{array}\right]=\frac{V}{E\left[\begin{array}{c}\widetilde{P}_{f}^{M}\end{array}\right]}
$$
(11)
\n
$$
v^{M}=\frac{\left(1-\widehat{\widetilde{P}}_{f}^{M}\right)}{N_{\text{sim}}^{M}\times\widehat{P}_{f}^{M}\times RE\left[\widehat{P}_{f}^{M}\right]^{2}}.
$$
(14)

\n
$$
45
$$

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\n $$

53 119 *failure probability*

55 121 56 122 57 RB for NRA is computed with reference to the quantity $P_{f,ref}$ and the stimate a low failure probability with respect to a given sim-
Continued by a gafanger CMC or another method when CMC to estimate a low failure 58 (estimated by a reference CMC or another method when CMC $\frac{1}{2}$ reference the insult data The reference follows in 124 $\frac{59}{25}$ is untractable) while RB for ARA is computed with reference $\frac{125}{25}$ and $\frac{125}{25}$ is untractable) while RB for ARA is computed with reference $\frac{125}{25}$ and $\frac{125}{25}$ and $\frac{125}{25}$ and $\frac{125}{$ 60 126 to *^P ARA/CMC* ^f to make the comparison representative; ⁶¹ • the *efficiency* v^M relatively to CMC estimate (respectively tained using FORM). The correlation coefficient $\rho = 0.9$ expresses $\frac{127}{100}$ **the efficiency** v^M relatively to CMC estimate (respectively tained using FORM). The correlation coefficient $\rho = 0.9$ expresses 128 63 129 is considered when the sollicitation *S* overcomes the resource *R*. 64 130 *Description.* The aim of this first academic test-case is to check the validity of the two approaches regarding two difficulties: assuming a strong correlation in the input probabilistic model and trying to estimate a low failure probability with respect to a given simulation budget. Table 2 gives the input data. The reference failure probability without parameter uncertainty is $p_{f,ref} = 8.84 \times 10^{-8}$ (because of the linear lsf, the true failure probability can be obthe linear correlation between the two basic variables. The failure The lsf thus reads:

$$
y = \frac{1}{N_{\text{c}}^M} \qquad (15) \qquad g(\mathbf{X}) = R - S = X_1 - X_2. \tag{15} \tag{15} \tag{15} \tag{16} \tag{16} \qquad \text{16} \qquad \text{17} \qquad \text{18} \qquad \text{18} \qquad \text{18} \qquad \text{19} \qquad \text{19} \qquad \text{18} \qquad \text{18} \qquad \text{19} \qquad \text{19} \qquad \text{10} \qquad \text{10} \qquad \text{10} \qquad \text{11} \qquad \text{11} \qquad \text{12} \qquad \text{13} \qquad \text{13} \qquad \text{14} \qquad \text{15} \qquad \text{16} \qquad \text{16} \qquad \text{17} \qquad \text{18} \qquad \text{19} \qquad \text{19} \qquad \text{10} \qquad \text{10} \qquad \text{11} \qquad \text{12} \qquad \text{13} \qquad \text{14} \qquad \text{15} \qquad \text{16} \qquad \text{17} \qquad \text{18} \qquad \text{18} \qquad \text{19} \qquad \text{19} \qquad \text{10} \qquad \text{10} \qquad \text{11} \qquad \text{11} \qquad \text{12} \qquad \text{13} \qquad \text{14} \qquad \text{15} \qquad \text{16} \qquad \text{16} \qquad \text{17} \qquad \text{18} \qquad \text{18} \qquad \text{19} \qquad \text{19} \qquad \text{19} \qquad \text{10} \qquad \text{10} \qquad \text{11} \qquad \text{11} \qquad \text{12} \qquad \text{13} \qquad \text{14} \qquad \text{15} \qquad \text{16} \qquad \text{17} \qquad \text{18} \qquad \text{18} \qquad \text{19} \qquad \text{19} \qquad \text{10} \qquad \text{10} \qquad \text{11} \qquad \text{11} \qquad \text{12} \qquad \text{1
$$

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Table 2

2 Second-order statistics and distributions of input random variables for the *R* − *S* Second in Second-order statistics and distributions of input random variables for the *R* − *S* Second-order statistics and distribu test-case.

Variable X_i^a	Distribution	Mean μ_{X_i}	Std σ_{X_i}
$=$ K	Normal	. .	- $\overline{}$ 5/4
$\lambda_2 = \lambda$	Normal	uncertain ^D μ_{X_2}	\sim $\sqrt{2}$ ، ت ∵v J
$\Theta = \mu_{X_2}$	Normal		

11 77 The conditional failure probability, i.e. *P*f*(* = *θ)*, can be written ¹² in its integral form since the joint conditional pdf $f_{\mathbf{X}|\Theta}(\cdot|\cdot)$ can be $\left|\begin{array}{c} | & | & | & | \end{array}\right|$ 13 analytically derived. One gets: the control of the con

15
$$
P_f(\theta = \mu_{X_2})
$$

\n16 $\frac{1}{17}$ $= \int_{\frac{1}{x_2} \pi \sigma_{X_1} \sigma_{X_2} \sqrt{1 - \rho^2}}$
\n17 $\approx \frac{1}{x_2}$ $\frac{1}{2\pi \sigma_{X_1} \sigma_{X_2} \sqrt{1 - \rho^2}}$
\n18 $\int_{\frac{1}{x_2} \sigma_{X_1} \sigma_{X_2} \sqrt{1 - \rho^2}}{1 - \frac{1}{2(1 - \rho^2)}$
\n20 $\times \exp \left[-\frac{1}{2(1 - \rho^2)}\right]$
\n21 $\times \left(\frac{(x_1 - \mu_{X_1})^2}{\sigma_{X_1}^2} - \frac{2\rho(x_1 - \mu_{X_1})(x_2 - \theta)}{\sigma_{X_1} \sigma_{X_2}} + \frac{(x_2 - \theta)^2}{\sigma_{X_2}^2}\right)\Big] dx$
\n22 $\pi \int_{\frac{1}{x_2} \pi \sigma_{X_1} \sigma_{X_2} \sigma_{X_1} \sigma_{X_2} \sigma_{X_2} \sigma_{X_1} \sigma_{X_2} \sigma_{X_2} \sigma_{X_1} \sigma_{X_2} \sigma_{X_1} \sigma_{X_2} \sigma_{X_1} \sigma_{X_2} \sigma_{X_2} \sigma_{X_1} \sigma_{X_1} \sigma_{X_2} \sigma_{X_1} \sigma_{X_2} \sigma_{X_1} \sigma_{X_1} \sigma_{X_1} \sigma_{X_1} \sigma_{X_1} \sigma_{X_1} \sigma_{X_1} \sigma_{X_1} \sigma_{X_1} \sigma_{X_$

27 In the specific case of two correlated normal variables and a linear $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 &$

$$
P_{f}(\theta = \mu_{X_2}) = \Phi(-\beta_C) = \Phi\left(-\frac{\mu_{X_1} - \theta}{\sqrt{\sigma_{X_1}^2 + \sigma_{X_2}^2 - 2\rho\sigma_{X_1}\sigma_{X_2}}}\right),\tag{17}
$$

35 101 where *β^C* is the *Cornell reliability index* [48]. This simple closed-36 form solution can be used to check and validate numerical results $||\cdot||_1$ $||\cdot||$ $||\cdot||$ $||\cdot||$ $||\cdot||$ $||\cdot||$ $||\cdot||$ $||\cdot||$ 37 obtained for this elementary test-case. The control of the control of

39 Results. Table 3 illustrates that NRA and ARA give similar results for $\begin{array}{c} | \\ | \infty | \infty | \infty \end{array}$ 40 $\frac{1}{2}$ $\frac{1}{2}$ 41 all the methods (except NRA/CMC which suffers here from a lack $\begin{array}{c|c} | & | & | \times \end{array}$ 42 of points while computing the integral over \mathcal{D}_{Θ}) it demonstrates $\begin{bmatrix} \frac{11}{3} & \frac{11}{3} & \frac{11}{3} \\ \frac{11}{3} & \frac{11}{3} & \frac{11}{3} \\ \frac{11}{3} & \frac{11}{3} & \frac{11}{3} \end{bmatrix}$ 43 that ARA can handle both rare event probabilities and strong corre-
 $\left|\frac{\mathfrak{S}}{2}\right| = \left|\frac{\mathfrak{S}}{2} \times \mathfrak{S}\right| = \frac{\mathfrak{S}}{2}$ $\frac{44}{16}$ include the seriously challenges other methods since it has a very small num-
 $\frac{8}{16} \begin{bmatrix} \frac{8}{16} & \frac{8}{16} \\ \frac{8}{16} & \frac{8}{16} \end{bmatrix}$ As beriodally changes offer memods since it has a very sinal right.
 $\begin{bmatrix} 1 & 1 \\ 2 & 5 \end{bmatrix}$ if $\begin{bmatrix} 2 & 5 \\ 2 & 5 \end{bmatrix}$ if $\begin{bmatrix} 1 & 1 \\ 2 & 5 \end{bmatrix}$ if $\begin{bmatrix} 2 & 11 \\ 2 & 5 \end{bmatrix}$ if $\begin{bmatrix} 1 & 11 \\ 2 & 5 \end{bmatrix}$ if $\begin{b$ $\frac{46}{47}$ gives exact results since the lsf is linear. On the other hand, ARA/SS $\frac{112}{47}$ $\frac{112}{47}$ $\frac{112}{47}$ $\frac{112}{47}$ $\frac{112}{47}$ 47 113 definitely gives promising results compared to ARA/CMC since the 48 114 *v* value ($\nu = 54.44$) is high. In a classical context of rare event $\begin{array}{c|c|c|c|c|c|c|c|c} & & & & & & \hline &$ 50 (often encountered in aerospace engineering), one can see the su-
50 $\left[\begin{array}{c|c} \end{array}\right] \left[\begin{array}{c|c} \end{array}\right] \left[\begin{array}{ccc} \end{array}\right]$ 51 117 periority of ARA (coupled with FORM or SS) compared to other 52 118 NRA-coupled methods. This test-case serves as a preliminary ver- $\frac{1}{53}$ ification for ARA before testing it on a real aerospace simulation $\frac{1}{52}$ $\frac{1}{52}$ 54 120
ceals how high can be the variations between the classical failure
ceals how high can be the variations between the classical failure 55 121 probability estimate and the predictive one considering parameters 56 Presence, Commercial Commercial Constructions per annual construction of the construction of the process of estimating the predictive failure probability. Moreover, for almost lation between basic input variables. On the one hand, ARA/FORM code such as those presented in subsections 4.3 and 4.4. It also re-

4.2.2. A nonlinear oscillator

60 126 *Description.* This nonlinear oscillator is a well-known structural re-61 liability test-case firstly proposed in [49] and then used for bench- with $|\overline{G}|\leq |\overline{G}|\leq |\overline{G}|\leq |\overline{G}|\leq |G|$ 62 Individually purposes in [2,50,51]. The aim here, is to assess reliability $\tilde{\phi}$ and $\tilde{\phi}$ are $\tilde{\phi}$ and $\tilde{\phi}$ are $\tilde{\phi}$ and 63 129 of a two-degree-of-freedom primary-secondary system, as shown 64 in Fig. 2, under a white noise base acceleration. The basic variables $\begin{bmatrix} 8 \\ 2 \\ 3 \end{bmatrix}$ $\begin{bmatrix} 8 \\ 2 \\ 3 \end{bmatrix}$ $\begin{bmatrix} 36 \\ 2 \\ 3 \end{bmatrix}$ $\begin{bmatrix} 36 \\ 2 \\ 3 \end{bmatrix}$ $\begin{bmatrix} 136 \\ 2 \\ 3 \end{bmatrix}$ $\begin{bmatrix} 136 \\ 2 \\ 3 \end{bmatrix}$ $\begin{$ 65 characterizing the physical behavior are the masses m_p and m_s , $\frac{1}{2}$ $\frac{1}{2}$ 66 spring stiffnesses *k_p* and *k_s*, natural frequencies $\omega_p = (k_p/m_p)^{1/2}$ $\mathbb{E} \left[\frac{1}{2} \right]$

11 **Fig. 2.** Two-degree-of-freedom damped oscillator with primary and secondary sys-**begine that in the secondary of the secondary sys-**12 tems. The set of the tems.

14 80 and *ω^s* = *(ks/ms)*¹*/*² and damping ratios *ζ^p* and *ζ^s* , where the sub-15 scripts *p* and *s* respectively refer to the primary and secondary the set of the second $\vert \cdot \vert$ in the second secondary the second secondary the second s 16 oscillators. If F_s denotes the force capacity of the secondary spring, $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ 82 17 then the reliability of the system can be evaluated using the fol-
17 $\left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right]$ $\left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right]$ $\left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right]$ 33 18 lowing lsf [49,52]: 84

20
$$
g(\mathbf{X}) = F_s - 3k_s
$$

\n21 $\times \sqrt{\frac{\pi S_0}{4\zeta_s \omega_s^2} \left[\frac{\zeta_a \zeta_s}{\zeta_p \zeta_s (4\zeta_a^2 + r^2) + \gamma \zeta_a^2} \frac{(\zeta_p \omega_p^3 + \zeta_s \omega_s^3) \omega_p}{4\zeta \omega_a^4} \right]}$
\n22 $\times \sqrt{\frac{\pi S_0}{4\zeta_s \omega_s^2} \left[\frac{\zeta_p \zeta_s}{\zeta_p \zeta_s (4\zeta_a^2 + r^2) + \gamma \zeta_a^2} \frac{(\zeta_p \omega_p^3 + \zeta_s \omega_s^3) \omega_p}{4\zeta \omega_a^4} \right]}$
\n23 $\frac{1}{\zeta_s} \left[\frac{\zeta_p}{\zeta_p} \frac{\zeta_a \zeta_s}{\zeta_p} \frac{\omega_p}{\zeta_p} \right]$
\n24 $\frac{1}{\zeta_s} \left[\frac{\zeta_p}{\zeta_p} \frac{\omega_p}{\zeta_p} \frac{\omega_p}{\zeta_p} \right]$
\n25 $\frac{1}{\zeta_s} \left[\frac{\zeta_p}{\zeta_p} \frac{\omega_p}{\zeta_p} \frac{\omega_p}{\zeta_p} \right]$
\n26 $\frac{1}{\zeta_s} \left[\frac{\zeta_p}{\zeta_p} \frac{\omega_p}{\zeta_p} \frac{\omega_p}{\zeta_p} \frac{\omega_p}{\zeta_p} \right]$

²⁶ where *S*₀ is the intensity of the white noise, $\gamma = m_s/m_p$ the mass $\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}$ 27 ratio, $\omega_a = (\omega_p + \omega_s)/2$ the average frequency ratio, $\zeta_a = (\zeta_p +$ ²⁸ ζ ^{*s*}/2 the average damping ratio and *r* = $(\omega_p - \omega_s)/\omega_a$ a tuning ζ and ζ = ζ 29 95 parameter. The probabilistic model for **X** is detailed in Table 7.

30 The two interesting characteristics of this application test-case $\begin{bmatrix} \mathbb{R} & \kappa & 0 \\ \kappa & \kappa & 0 \end{bmatrix}$ 31 97 are its set of non-normal basic random variables and the fact that ³² it suffers from a highly nonlinear limit-state surface [50] (which $\begin{array}{ccc} | & | & | \end{array}$ 33 prevents from using any FORM-based approach). Moreover, follow-
24 **20 Prevents from using any FORM-based approach**). Moreover, follow- $\frac{34}{\sqrt{2}}$ ing [51], it seems relevant to consider the mean of the force capac- $\frac{35}{36}$ ity μ_{X_7} as the most influent distribution parameter on the failure $\left| \begin{array}{ccc} | & | & | \ \frac{1}{36} & | & | \ \frac{1}{36} & | & | \end{array} \right|$ $\frac{36}{37}$ probability. The nominal value for μ_{X_7} is chosen to be 21.5 N so $\left[\begin{array}{c|c} \frac{1}{2} & \frac{1}{2} \end{array}\right]$ $\frac{37}{2}$ as to reach a reference probability without parameter uncertainty $\begin{array}{ccc} \begin{array}{ccc} \end{array} & \end{array}$ $p_{\text{f,ref}}$ equal to 4.75 × 10⁻⁵ [51]. $p_{\text{f,ref}}$ **104** $p_{\text{f,ref}}$ **104** $p_{\text{f,ref}}$ **104** $p_{\text{f,ref}}$ **104 105 106 106 106 106 106 106 106 106 106 106 106 106 106 106 106 106 106** 39 105

40 106 *Results.* Numerical results summarized in Table 7 show that, for the 41 107 same simulation budget, ARA/CMC is more accurate than NRA/CMC 42 to estimate the predictive failure probability (the reference result $\vert \vert$ $\vert \Omega$ \geq \vert Ω \geq \vert Ω \geq \vert \vert \sim Ω $\widetilde{P}_{f,\text{ref}}$ is provided below the table). As for ARA/SS, a significant $\left|\sum_{\alpha=1}^{\infty}\left|\sum_{\alpha=1}^{\infty}\frac{x}{\alpha}\right|$ $\sum_{\alpha=1}^{\infty}\left|\sum_{\alpha=1}^{\infty}\frac{y}{\alpha}\right|$ $\sum_{\alpha=1}^{\infty}\left|\sum_{\alpha=1}^{\infty}\left|\sum_{\alpha=1}^{\infty}\frac{y}{\alpha}\right|\right|$ $\left|\sum_{\alpha=1}^$ 44 gain is noticeable referring to the high *v* values compared to unity $\mathbb{E} \left[\mathbb{E} \right] \mathbb{E} \left[\mathbb{E} \right] \mathbb{E} \left[\mathbb{E} \right]$ $\mathbb{E} \left[\mathbb{E} \right]$ and $\mathbb{E} \left[\mathbb{E} \right]$ 45 111 (*ν >* 10). In brief, that means the ARA/SS is very efficient to treat 46 this problem compared to a classical Monte Carlo approach. A fi-47 in al remark concerns the comparison between the two reference $\left| \begin{array}{cc} | & | & | \\ | & | & | \end{array} \right| \left| \begin{array}{cc} | & | & | \\ | & | & | \end{array} \right| \left| \begin{array}{cc} | & | & | \\ | & | & | \end{array} \right|$ 48 probabilities $p_{f,ref} = 4.75 \times 10^{-5}$ and $\widehat{P}_{f,ref} = 1.55 \times 10^{-4}$; one can
49 associated in this case, considering uncertainty on a distribution of As that, in this case, considering uncertainty on a distribution pa-50 116 51 important indicator for design or re-design purposes. $\begin{bmatrix} 51 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 2 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 2 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 3 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$

53 119 *4.3. Application on a launch vehicle stage fallback zone estimation*

56 Description. Space launcher complexity arises from the coupling be-
 $\begin{array}{c} \begin{array}{c} \uparrow \\ \downarrow \end{array} \end{array}$ $\begin{array}{c} \downarrow \\ \downarrow \end{array}$ $\begin{array}{c} \downarrow \\ \downarrow \end{array}$ $\begin{array}{c} \downarrow \\ \downarrow \end{array}$ 57 tween several subsystems such as stages or boosters and other $\left[\begin{array}{c|c} x & x \end{array}\right]$ $\left[\begin{array}{ccc|c} x & x \end{array}\right]$ $\left[\begin{array}{ccc|c} x & x \end{array}\right]$ 58 embedded systems. Optimal trajectory assessment is a key disci-
 $\frac{9}{2}$ | | | $\frac{9}{2}$ | | $\frac{1}{2}$ | 59 pline since it is one of the cornerstones of the mission success $\overline{5}$ | | $\overline{5}$ | $\overline{2}$ $\overline{3}$ $\overline{2}$ $\overline{5}$ | $\overline{2}$ $\overline{3}$ | $\overline{2}$ | $\overline{3}$ | $\overline{2}$ $\overline{3}$ | $\overline{2}$ | $\overline{3}$ | $\overline{2$ 60 126 (for ascent as well as for re-entry trajectories). However, during 61 the real flight, aleatory uncertainties can affect the different flight $\frac{1}{2}$ $\frac{1}{$ 62 128 phases at different levels (e.g., on the dynamics perturbations or 63 stage combustion) and be combined to lead to a failure state of $E|_F = |E|_F = |E|_F = E|_F = E|_F = 1$ ⁶⁴ the space vehicle trajectory. After their propelled phase, the dif- **اَنْتَ مَارَّةِ وَالْقَادِمَةِ** وَالْقَا 65 ferent stages reach successively their separation altitudes and may 131 로 주 131

NRAc 1*.*18 × 10−4 8*.*82 × 10−11 7*.*97 × 10−2 9*.*16 × 10−3 1*.*11 × 10−4 1*.*15 75*.*32 − 1*.*13 × 10−4 4*.*98 × 10−9 0*.*63 −6*.*09 × 10−2 5*.*65 × 10−2 ARA 1*.*27 × 10−4 1*.*05 × 10−10 8*.*07 × 10−2 8*.*28 × 10−5 − −−0*.*35 − 1*.*25 × 10−4 1*.*52 × 10−9 0*.*31 −1*.*57 × 10−2 20*.*6 Approach CMC^a FORM Results for the launch vehicle stage fallback test-case. **Table 5** 66 fall back into the ocean (see Fig. 3). Such a dynamic phase is of the 그래 아니다. 그래 아이들은 그래 그래도 그래도 아이들은 그래도 아이들 abPlease cite this article in press as: V. Chabridon et al., Evaluation of failure probability under parameter epistemic uncertainty: Application to aerospace system reliability assessment, Aerosp. Sci. Technol. (2017), http://dx.doi.org/10.1016/j.ast.2017.07.016

 $m_{P_f}^{\alpha}$

NRA:

NRA:

 $N_{\theta} = 10^3$ samples,

 $N_{\theta} = 10^3$ samples,

 $N_{\bf x} = 10^3$ samples | ARA:

 $N_{\bf x} = 10^3$ samples/step | ARA:

 $N_{\mathbf{x},\theta} = 10^6$ samples.

 $N_{\mathbf{x},\theta} = 10^3$ samples/step.

*S*²

*P*f RE

 m_P° \hat{r}

 S^2

P f RE RB *ν mP*f

*S*²

RE RB *ν*
*P_f P*_{*P*}

21 87 **Fig. 3.** Illustration scheme of the first stage fallback phase into the Atlantic Ocean. 22 88

Table 8

23 and 22 and 23 and 24 and 25 and 26 and 27 and 28 and 29 and 29 and 20 and **Table 7**

24 Second-order statistics and distributions of input random variables for the nonlinear Second-order statistics and distributions of input random variables for the launch 90 25 oscillator test-case. 291 and 2012 of the stage in the control of the Second-order statistics and distributions of input random variables for the nonlinear oscillator test-case.

26	Variable X_i^a	Distribution	Mean μ_{X_i}	$C_v \, \delta_{X_i}$	Variable X_i^a	Distribution	Mean μ_{X_i}	Std σ_{X_i}	92
27	$X_1 = m_n$ (kg)	Lognormal	1.5	10%	$X_1 = \Delta h$ (m)	Normal		1650	93
28	$X_2 = m_s$ (kg)	Lognormal	0.01	10%	$X_2 = \Delta v$ (m s ⁻¹)	Normal	μ_{X_2} uncertain ^b	3.7	94
29	$X_3 = k_n$ (Nm ⁻¹)	Lognormal		20%	$X_3 = \Delta \gamma$ (rad)	Normal	μ_{X_3} uncertain	0.001	95
30	$X_4 = k_s$ (N m ⁻¹)	Lognormal	0.01	20%	$X_4 = \Delta \psi$ (rad)	Normal		0.0018	96
31	$X_5 = \zeta_p(1)$	Lognormal	0.05	40%	$X_5 = \Delta m$ (kg)	Normal		70	97
	$X_6 = \zeta_5(1)$	Lognormal	0.02	50%	$X_6 = \Delta C_d$ (1)	Normal		0.1	
32	$X_7 = F_s(N)$	Lognormal	μ_{X_7} uncertain ^c	10%	$\Theta_2 = \mu_{X_2}$ (ms ⁻¹)	Normal		3.7	98
33	$X_8 = S_0$ (m s ⁻²)	Lognormal	100	10%	$\Theta_3 = \mu_{X_3}$ (rad)	Normal		0.001	99
34	$\Theta = \mu_{X_7}$ (N)	Normal	21.5	10%	^a The basic variables are independent.				100

^a The basic variables are independent.

36 ^b Note that the coefficient of variation (*C_v*) of *X_i* is defined by $\delta_{X_i} = \sigma_{X_i}/|\mu_{X_i}|$ 102 for $\mu_{X_i} \neq 0$.

^c For a fixed value $\mu_{X_7} = 21.5$, $p_{f,\text{ref}} = 4.75 \times 10^{-5}$.

40 106 utmost importance in terms of launcher safety since the conse-41 quence of a mistake in the prediction of the fallback zone can be $g(\mathbf{A}) = q_{\text{safe}} - \mathcal{M}(\mathbf{A}) = q_{\text{safe}} - D_{\text{code}}.$ ⁴² dramatic in terms of human security and environmental impact. The numerical experiment the threshold safety distance decay of 108 43 For that reason, the handling of uncertainties plays a crucial role $\frac{1}{10}$ is chosen to be equal to 20 km so as to reach a reference prob-⁴⁴ in the comprehension and prediction of the global system behav-
ability without parameter uncertainty n_0 , equal to 2.31 \times 10⁻⁷ ⁴⁵ ior. That is the reason why it is of utmost importance to take it (estimated by CMC with 10⁸ samples and confirmed by SS with ¹¹¹ 46 into account during the reliability analysis. 112 into a symple (5.60) into account during the reliability analysis.

47 113 The black-box model M*(*·*)* considered here is ^a trajectory sim-⁴⁸ ulation code of the dynamic fallback phase of a generic launcher Results. Numerical results gathered in Table 5 show that both 114 49 first stage. For the interested reader, two different but close test-
NRA/CMC and ARA/CMC give similar results and manage to cor- 115 50 cases (with different numerical values) are used in [4] and in [53]. Fectly estimate the predictive failure probability (whose reference 116 51 As an input vector of the simulation code, the following basic vari- value is given in Table 5). NRA/SS and ARA/SS even if a signifi- 117 52 ables representing the initial conditions and the launch vehicle cant value of the efficiency ($\nu > 20$) for ARA/SS which indicates 118 53 characteristics will be passed through the code: the state of ARA/SS with such an industrial test-
53 characteristics will be passed through the code: the code of the code of the with promising is the use of ARA/SS with ulation code of the dynamic fallback phase of a generic launcher first stage. For the interested reader, two different but close testcases (with different numerical values) are used in $[4]$ and in $[53]$. As an input vector of the simulation code, the following basic variables representing the initial conditions and the launch vehicle characteristics will be passed through the code:

*X*₂: velocity perturbation at separation (∆v (ms^{−1}));

*X*₃: flight path angle perturbation at separation (
$$
\Delta \gamma
$$
 (rad));

*X*₄: azimuth angle perturbation at separation ($\Delta \psi$ (rad));

 X_5 : propellant mass perturbation at separation (Δm (kg));

 X_6 : drag force error perturbation (ΔC_d dimensionless).

 62 Moreover, in this case, the mean values of the basic variables X_2 ond integration loop over D_{θ} to get the reference predictive failure 128 ⁶³ and X_3 are considered as uncertain (see Table 8) as they are phys-
 $\frac{125}{120}$ is a contributed in the contract of results as a comparable assessment the simulation budget allocation (between the set of $\frac{125}{$ 64 ical quantities difficult to measure and to control in reality. As an problem concerning the simulation budget allocation (between the 130 65 output, the code will give back the distance $D_{code} = \mathcal{M}(\mathbf{X})$, which two domains, $D_{\mathbf{X}}$ and D_{Θ}) can be an obstacle to an accurate estiand X_3 are considered as uncertain (see Table 8) as they are physical quantities difficult to measure and to control in reality. As an is also a random variable, between the theoretical fallback posi-

vehicle stage fallback test-case.

Variable X_i^a	Distribution	Mean μ_{X_i}	Std σ_{X_i}
$X_1 = \Delta h$ (m)	Normal	0	1650
$X_2 = \Delta v$ (ms ⁻¹)	Normal	μ_{X_2} uncertain ^b	3.7
$X_3 = \Delta \gamma$ (rad)	Normal	μ_{X_2} uncertain	0.001
$X_4 = \Delta \psi$ (rad)	Normal	0	0.0018
$X_5 = \Delta m$ (kg)	Normal		70
$X_6 = \Delta C_d$ (1)	Normal	0	0.1
$\Theta_2 = \mu_{X_2}$ (ms ⁻¹)	Normal		3.7
$\Theta_3 = \mu_{X_3}$ (rad)	Normal		0.001

35 a The basic variables are independent. $\frac{b}{2}$ For fixed values $\mu_{X_2} = 0$ and $\mu_{X_3} = 0$, $p_{f,ref} = 2.31 \times 10^{-7}$.

 37 103 $\mu_{X_i} = 0$. 37 103 $\mu_{X_i} = 0$ tion into the ocean and the estimated one due to the uncertainty $\frac{38}{104}$ 104 a the value $\mu_{X_7} = 21.5$, $\mu_{\text{tree}} = 4.73 \times 10^{-4}$. $\frac{39}{20}$ exceeds a threshold safety distance d_{safe} :

$$
g(\mathbf{X}) = d_{\text{safe}} - \mathcal{M}(\mathbf{X}) = d_{\text{safe}} - D_{\text{code}}.\tag{19}
$$

In the numerical experiment, the threshold safety distance d_{safe} is chosen to be equal to 20 km so as to reach a reference probability without parameter uncertainty $p_{f,ref}$ equal to 2.31 × 10⁻⁷ (estimated by CMC with 10^8 samples and confirmed by SS with 10^3 samples/step).

 120 case. As for NRA/FORM and ARA/FORM, they both give poor results. X_1 : stage altitude perturbation at separation (Δh (m)): ARA/FORM gives at least an order of magnitude of the predictive 121 X_1 : stage altitude perturbation at separation (Δh (m)); ARA/FORM gives at least an order of magnitude of the predictive the term order of magnitude of the predictive the state of the predictive the state of the r failure probability quite close to the reference one.

 $57 \chi_3$: flight path angle perturbation at separation ($\Delta \nu$ (rad)): Through this test-case, one can illustrate the budget allocation 123 X_4 ; azimuth angle perturbation at separation ($\Delta\psi$ (rad)); being problem which appears in NRA. Classical reference failure prob- X_5 : propellant mass perturbation at separation (Δm (kg)); ability without parameter uncertainty $p_{f,\text{ref}}$ is very low (order of antistanting) X_6 ; drag force error perturbation (ΔG_d dimensionless). The magnitude of 10^{-7} , see Table 8) and can be time-consuming to the 127 get, especially with an expensive computer model. Adding a sec- is also a random variable, between the theoretical fallback posi- mation of the predictive failure probability when using CMC-based 132 ond integration loop over \mathcal{D}_{θ} to get the reference predictive failure two domains, \mathcal{D}_X and \mathcal{D}_Θ) can be an obstacle to an accurate esti-

¹⁴ quadrature scheme-based DOE over \mathcal{D}_{Θ} . The idea is to approximate and the interval of the interval of the idea is to approximate and the interval of the idea is to approximate and the idea is to approximate and a *k*-variate integral over $\mathcal{D}_{\boldsymbol{\Theta}} \subseteq \mathbb{R}^k$ of the form:

$$
\mathcal{I}[P_f(\theta)] = \int_{\mathcal{D}_{\Theta}} P_f(\theta) f_{\Theta}(\theta) d\theta
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\mathcal{I}[P_f(\theta)] = \int_{\mathcal{D}_{\Theta}} P_f(\theta) f_{\Theta}(\theta) d\theta
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\mathcal{I}[P_f(\theta)] = \int_{\mathcal{D}_{\Theta}} P_f(\theta) f_{\Theta}(\theta) d\theta
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\mathcal{I}[P_f(\theta)] = \int_{\mathcal{D}_{\Theta}} P_f(\theta) f_{\Theta}(\theta) d\theta
$$
\n
$$
\mathcal{I}[P_f(\theta)] = \int_{\mathcal{D}_{\Theta}} P
$$

20 where $f_{\Theta}(\theta) ≡ w(\theta)$ is a density (or weight) function which $\frac{\text{test-case.}}{\text{test-case.}}$ that [54]:

25
$$
\mathcal{I}[P_f(\theta)]
$$

\n26 $M_1 M_2 M_2 \rightarrow M_k$
\n27 $\approx \sum_{j_1=1}^{M_1} \sum_{j_2=1}^{M_2} \cdots \sum_{j_k=1}^{M_k} (w_{j_1} \otimes w_{j_2} \otimes \cdots \otimes w_{j_k})$
\n28 $\approx \sum_{j_1=1}^{M_1} \sum_{j_2=1}^{M_2} \cdots \sum_{j_k=1}^{M_k} (w_{j_1} \otimes w_{j_2} \otimes \cdots \otimes w_{j_k})$
\n29 $\times P_f(\theta_1^{(j_1)}, \theta_2^{(j_2)}, \dots, \theta_k^{(j_k)})$
\n20 $\times \sum_{j_1=1}^{M_2} \sum_{j_2=1}^{M_2} \cdots \sum_{j_k=1}^{M_k} (w_{j_1} \otimes w_{j_2} \otimes \cdots \otimes w_{j_k})$
\n21 $\frac{(\sum_{j=1}^{M_1} \sum_{j_k=1}^{M_k} (w_{j_1} \otimes w_{j_2} \otimes \cdots \otimes w_{j_k})}{\sum_{j_1=1}^{M_1} \sum_{j_2=1}^{M_k} (w_{j_1} \otimes \cdots \otimes w_{j_k})}$
\n22 $\frac{(\sum_{j=1}^{M_k} \sum_{j_k=1}^{M_k} (w_{j_1} \otimes w_{j_2} \otimes \cdots \otimes w_{j_k})}{\sum_{j_1=1}^{M_k} \sum_{j_k=1}^{M_k} (w_{j_1} \otimes \cdots \otimes w_{j_k})}$
\n23 $\frac{(\sum_{j=1}^{M_k} \sum_{j_k=1}^{M_k} (w_{j_1} \otimes \cdots \otimes w_{j_k})}{\sum_{j_k=1}^{M_k} \sum_{j_k=1}^{M_k} (w_{j_1} \otimes \cdots \otimes w_{j_k})}$
\n24 $\frac{(\sum_{j=1}^{M_k} \sum_{j_k=1}^{M_k} (w_{j_1} \otimes \cdots \otimes w_{j_k})}{\sum_{j_1=1}^{M_k} \sum_{j_k=1}^{M_k} (w_{j_1} \otimes \cd$

31 with *μ*_{*i*} the weights and \otimes the tensor product operator. The in b For fixed values $\mu_{X_1} = \mu_{X_2} = \mu_{X_3} = 0$, $p_{f,\text{ref}} = 3.4 \times 10^{-4}$. $\frac{32}{22}$ dices M_1, \ldots, M_k represents the number of points in each dimen- $\frac{33}{21}$ sion. In the specific case of launch vehicle stage fallback, we have here is a deterministic model $\mathcal{M}(\cdot)$ computing the minimum dis-46 \ldots \ldots \approx \ldots \approx \ldots \ldots \approx \ldots \ldots quality *Predictive failure probability* $P_{f,ref}$ *.* A last remark concerns the fact $P_{f,ref}$ and $P_{f,ref}$ 48 and drilling only two parameters out of six basic variables as being 10.97 0.12 0.04 -0.03 -0.90 $_{49}$ uncertain implies to increase the failure probability of three loga-
 $_{49}$ 1 0.10 0.73 0 0 \vert $_{50}$ rithmic decades in terms of magnitude compared to the classical $_{50}$ and $_{-0.14}$ $_{-0.14}$ $_{-0.47}$ $_{-0.11}$ $_{-0.23}$ $_{116}$ $\begin{array}{ccc} 51 & \text{reference estimate.} \end{array}$ Again, that emphasizes how crucial taking dis-
 $\begin{array}{ccc} \mathbf{R} = \begin{array}{ccc} 1 & -0.79 & -0.81 \end{array} \end{array}$ $_{52}$ tribution parameters uncertainty is during the reliability analysis $_{\rm (sym.)}$ (sym.) 1 0.83 with w_i the weights and \otimes the tensor product operator. The insion. In the specific case of launch vehicle stage fallback, we have two uncertain distribution parameters, which means that the integration domain is \mathbb{R}^2 . The quadrature type is chosen to be a *Gauss–Hermite quadrature scheme*, which means that we use Gaussian weights [55,56]. Depending on the problem dimensionality, one can choose an accuracy level *Macc* which allows to integrate complete polynomials of total order 2*Macc* − 1 exactly. Here, we chose an accuracy level $M_{acc} = 14$ so as to provide enough samples (here, it corresponds exactly to 1009 samples [55]) to cover the domain \mathcal{D}_{Θ} . Such a choice is constrained by the expensive aspect of the computer code. However, for different applications, one could choose another accuracy level. Finally, coupling this DOE with a SS method with 10^4 samples/step allows to estimate the reference that taking only two parameters out of six basic variables as being phase.

54 120 55 121 *4.4. Application on estimation of collision probability between orbiting objects*

58 124 *Description.* Because of the drastic growth of the number of or- $\frac{1}{25}$ biting objects (cataloged and uncataloged space debris) over the $\frac{1}{25}$ coulse a replace the space debris in the space of the sp 60 past few decades, the number of potential collision between satel-
 $\frac{1}{2}$ and $\frac{1}{2}$ (cotinizated by CMG with 106 semples and separated by CMG with 106 semples and separated 126 $\frac{61}{100}$ lites and other orbiting objects increased (see Fig. 4). Space debris $\frac{1}{1000}$ is the sc with 10⁴ complements and community that the sc with 10⁴ complements and community that the sc with 10⁴ comple 62 surveillance and management is one of the key issue and is di-
62 surveillance and management is one of the key issue and is di-⁶³ rectly linked to the rare event probability estimation topic. Results. This industrial test-case can be considered as the worst ¹²⁹ rectly linked to the rare event probability estimation topic.

 65 tween a space debris and a satellite, both orbiting around an with three uncertain distribution parameters. Numerical results are 131 ⁶⁶ Earth-centered inertial reference frame. The dynamical model used given in Table 6. Again, since we know the conditional marginal ¹³² In this test-case, the failure scenario concerns a collision be-

15 a k-variate integral over $D_{\Theta} \subseteq \mathbb{R}^k$ of the form: **Fig. 4.** Picture extracted from a video (*The story of space debris*) by ESA[©] [\(http://](http://www.esa.int) ⁸¹ 16 www.esa.int). 82 www.esa.int).

Table 9

19 19 85 86 86 10 10 Second-order statistics and distributions of input random variables for the collision as test-case.

³⁴ two uncertain distribution parameters, which means that the in-
 $\frac{100}{2}$ tance D_{\min} between the debris and the satellite during a given the satellite during a given the 35 tegration domain is \mathbb{R}^2 . The guadrature type is chosen to be a time span τ . For the interested reader, two other applications ¹⁰¹ 36 Cause-Hermite quadrature scheme, which means that we use Cause. (with different numerical values) of a similar test-case are treated 102 37 sian weights [55.56]. Depending on the problem dimensionality 10 and 10 and in [34] and another approach to estimate the probabil- 38 009 can choose an accuracy level M, which allows to integrate ity of collision can be found in [57]. Assuming that the position 104 ³⁹ complete polynomials of total order $2M = 1$ exactly. Here we and the speed of the satellite are perfectly known, the input vec- 40 chose an accuracy level $M = 14$ so as to provide enough samples tor of basic variables gathers the three components of the space 106 ⁴¹ (here it corresponds exactly to 1009 samples [55]) to cover the do-

debris position vector and the three components of its speed vec-
 $\frac{107}{2}$ $\frac{42}{108}$ main \mathcal{D}_{Ω} Such a choice is constrained by the expensive aspect of the space the space debris position vector $\frac{108}{108}$ ⁴³ the computer code However for different applications, one could are uncertain (see Table 9) because of the difficulty to measure 109 ⁴⁴ choose inother accuracy level Finally coupling this DOE with a courately these quantities. As for the correlation structure of the ¹¹⁰ ⁴⁵ SS method with 10⁴ samples/step allows to estimate the reference basic variables, it is given by the following linear correlation ma $trix \mathbf{R} = [\rho_{ij}]_{i,j \in \{1,...,d\}}$:

53 119 **R** = ⎡ ⎢ ⎢ ⎢ ⎢ ⎢ ⎣ 1 0*.*97 0*.*12 0*.*84 −0*.*85 −0*.*98 1 0*.*10 0*.*73 0 0 1 −0*.*14 −0*.*47 −0*.*11 1 −0*.*79 −0*.*81 *(*sym*.)* 1 0*.*83 1 ⎤ ⎥ ⎥ ⎥ ⎥ ⎥ ⎦ *.* (22)

Thus the lsf can be expressed as follows:

 $g(\mathbf{X}) = d_{\text{collision}} - \mathcal{M}(\mathbf{X}) = d_{\text{collision}} - D_{\text{min}}.$ (23) ¹²²

57 123 In the numerical experiment, the threshold collision distance $d_{\text{collision}}$ is chosen to be equal to 20 m so as to reach a reference probability without parameter uncertainty *p*f*,*ref equal to 3*.*4 × 10−⁴ (estimated by CMC with 106 samples and confirmed with a SS with 10^4 samples/step).

⁶⁴ In this test-case, the failure scenario concerns a collision be- case here since it involves six correlated basic random variables ¹³⁰ with three uncertain distribution parameters. Numerical results are given in Table 6. Again, since we know the conditional marginal

¹ pdfs and the covariance matrix, we can apply the same approach have asymmetric consequences from a risk management perspec- ⁶⁷ 2 using the adapted Rosenblatt transformation. Here, none of the tive. Thus, a more conservative estimate (e.g., a quantile) could be 68 ³ nested approaches are accessible computationally speaking since combined with P_f to more accurately characterize the probabil- ⁶⁹
4. The model is very synepsive to systems. This is vyane if we want a ity of interes ⁴ the model is very expensive to evaluate. This is worse if we want ity of interest. However, estimating such an indicator is possible 70 5 to get a reference result using a huge CMC. Moreover, the strategy with NRA but, as mentioned previously, with a computational cost 71 6 using quadrature-based scheme and subset simulations as used in incompatible with industrial considerations. Further investigations 72 ⁷ subsection 4.3 is not conceivable here since it would require to are required to make the calculation of quantiles possible with 73 8 truncate the scheme at a very high order so as to get enough ARA. In order to reduce the computational cost of these estima- 74 ⁹ points over \mathcal{D}_{Θ} to get relevant results. For all these reasons, ARA tions, one possible enhancement track could be to use a surrogate 75 ¹⁰ seems to be the only alternative to assess reliability under distribu- model [4]. However, the use of surrogate models also raises several ⁷⁶ ¹¹ tion parameter uncertainty regarding a realistic simulation budget difficulties. Firstly, the construction of such a model over the whole 77 ¹² constraint. ARA/CMC gives a result confirmed by ARA/SS with $10³$ definition domain (required for quantile estimation mentioned pre-¹³ samples/step. However, the v factor is just above one and does viously) of the input random variables can be a challenging task. ⁷⁹ ¹⁴ not indicate a huge efficiency compared to ARA/CMC. However, Secondly, the performance of a surrogate mainly depends on the 80 ¹⁵ ARA/SS offers still more tuning possibilities in the algorithm and tuning of hyper-parameters. Thirdly, the surrogate model induces 81 ¹⁶ can handle lower probabilities. ARA/FORM definitely fails to assess its own uncertainty that has to be handled. ¹⁷ a correct probability. As a consequence, ARA/SS remains the only **17** a correct probability. As a consequence, ARA/SS remains the only ¹⁸ solution to possibly assess reliability considering uncertain distri- **6. Conclusions** ¹⁹ bution parameters. Even if the efficiency is not as good as the one **1958** ²⁰ obtained for previous test-cases, it still highlights the fact that this $\frac{1}{2}$ In this article, we compared two different approaches to handle $\frac{86}{1}$ ²¹ method allows benefits compared to ARA/CMC. The reliability assessment under distribution parameter uncertainty. If 87 using the adapted Rosenblatt transformation. Here, none of the nested approaches are accessible computationally speaking since the model is very expensive to evaluate. This is worse if we want to get a reference result using a huge CMC. Moreover, the strategy using quadrature-based scheme and subset simulations as used in subsection 4.3 is not conceivable here since it would require to truncate the scheme at a very high order so as to get enough points over \mathcal{D}_{Θ} to get relevant results. For all these reasons, ARA seems to be the only alternative to assess reliability under distribution parameter uncertainty regarding a realistic simulation budget constraint. ARA/CMC gives a result confirmed by ARA/SS with $10³$ samples/step. However, the *ν* factor is just above one and does not indicate a huge efficiency compared to ARA/CMC. However, ARA/SS offers still more tuning possibilities in the algorithm and

4.5. Synthesis about aerospace test-cases

The aim of this subsection is to give a synthesis for the inter-NRA and ARA. According to the numerical results, one can sum up the following characteristics:

- ARA leads to more accurate results than NRA with respect to a given simulation budget;
- only ARA is able to handle very expensive simulation codes;
- NRA suffers from the "budget allocation" problem;
- ARA requires to adapt the Rosenblatt transformation so as to use the classical reliability methods in the standard normal space.

 $_{45}$ (regardless the rareness of the failure probability). Combining sev-
 $_{45}$ areas. $_{46}$ eral characteristics lead to deduce which optimal method should $_{112}$ ₄₇ be used regarding all these simulation constraints. **Conflict of interest statement** that the state of the statement

48 114 49 115 **5. Limitations and possible enhancements**

51 As described in previous sections, reliability assessment un-**Acknowledgements** As described in previous that 52 118 der parameter uncertainty involves mainly two components: the ⁵³ choice of an estimator for the failure probability (here, the pre-
⁵³ choice of an estimator for the failure probability (here, the pre-
The authors would primarily like to thank the editor and the 54 dictive failure probability $P_{\rm f}$) and the choice of a numerical strat-anonymous reviewer for the constructive remarks and valuable 120 ⁵⁵ egy to get an estimate of it (i.e. NRA vs. ARA). Concerning the suggestions. The first author is enrolled in a PhD program co- 121 56 first point, *P_f* corresponds to the mean failure probability over funded by ONERA – The French Aerospace Lab and SIGMA Clermont. 122
57 all the conditional failure amphabilities B (0), As discussed in 129 and Flatin 57 all the conditional failure probabilities $P_f(\theta)$. As discussed in [38, Their financial supports are gratefully acknowledged. $\qquad \qquad \qquad$ 123 58 124 [39\],](#page-12-0) such a predictive estimator appears to be, from the *statisti-*59 125 *cal decision theory* [59] point of view, associated to a *quadratic cost* 60 126 *function* whose aim is to quantify the impact of a mis-estimation ⁶¹ through P_f with respect to the true failure probability p_f (probabil- [1] S.-K. Au, J.L. Beck, Estimation of small failure probabilities in high dimensions ¹²⁷
62 its aktained with a fall lynevyladge of the gra 62 128 ity obtained with a full knowledge of the probability distribution 63 of **X** and with a perfect computer model). However, as pointed [2] J.-M. Bournet, F. Deneeger, M. Lemaire, Assessing small failure probabilities by 129 64 out in $[39]$, since the associated quadratic cost function is sym-
 $\frac{(2011) 242.552}{(2011) 242.552}$ 65 metric, both under- and over-estimations have the same costs, $\frac{131 \text{ R.V.}$ Rubinstein, D.P. Kroese, Simulation and the Monte Carlo Method, second 131 ⁶⁶ theoretically speaking. Nevertheless, this symmetry of costs may edition, Wiley, 2008. The state of dictive failure probability \tilde{P}_f) and the choice of a numerical strategy to get an estimate of it (i.e. NRA vs. ARA). Concerning the

have asymmetric consequences from a risk management perspecits own uncertainty that has to be handled.

6. Conclusions

 $\frac{22}{100}$ the first one, the nested reliability approach (NRA) is widely used $\frac{88}{100}$ ²³ 4.5. Synthesis about aerospace test-cases and simple to set up, it definitely crashes with both the curse of $\frac{89}{60}$ 24 90 dimensionality and simulation budget considerations. The second ²⁵ The aim of this subsection is to give a synthesis for the inter-
{on} the augmented reliability approach (ARA) relies on the defini-26 $\frac{26}{27}$ ested reader to get the main advantages and drawbacks of both tion of an augmented input vector of uncertain distribution param- $^{27}{22}$ RRA and ARA. According to the numerical results, one can sum up eters and the basic input variables. For ARA, numerical sampling $^{93}_{27}$ $\frac{28}{100}$ the following characteristics: 29 dom variables conditioned on uncertain distribution parameters. $\frac{30}{24}$ • ARA leads to more accurate results than NRA with respect to The main principles of both methods have been presented into $\frac{96}{24}$ $\frac{31}{20}$ a given simulation budget;
a unified common framework. Specific attention has been given $\frac{32}{100}$ \bullet only ARA is able to handle very expensive simulation codes; to the algorithmic links and differences existing between these $\frac{98}{100}$ $\frac{33}{24}$ • $\frac{33}{24}$ $\frac{34}{\pi}$ • The strict strict from the budget uncellion problem,
 $\frac{34}{\pi}$ • ARA requires to adapt the Rosenblatt transformation so as to a mation with ARA have been evoked. Then, a comparison between $\frac{35}{25}$ are the classical reliability methods in the standard normal NRA and ARA has been carried out through application on both $\frac{101}{100}$ $\frac{36}{22}$ space. Space the classical remaining incritions in the standard homin $\frac{36}{2}$ academic and industrial test-cases (launch vehicle stage fallback $\frac{102}{100}$ 37 103 zone estimation and collision probability estimation between or-³⁸
As a final remark concerning the coupling between ARA and reli-
thing object and space debris), the final ones being representative $_{39}$ As a final remark concerning the coupling between ARA and reli-
 $\frac{105}{9}$ between the space debt is he final ones being respectively and space debtis), the final ones being representative $_{40}$ ability methods, one can notice that ARA/SS seems to be the most of the complex simulation codes used in aerospace engineering. $_{41}$ generic method since it can handle most of the difficulties encoun-
 $_{41}$ and study showed the benefits of using ARA with dedicated rare $_{107}$ $_{42}$ tered in complex simulation codes. However, other methods can be event probability estimation methods such as Subset Simulations $_{108}$ $_{43}$ used if some specific characteristics are preponderant. For instance, tor complex models with nonlinear lsis and correlated inputs. Sev-₄₄ a linear limit-state function should lead to use ARA/FORM instead eral enhancements raised in Section 5 are current open research In this article, we compared two different approaches to handle reliability assessment under distribution parameter uncertainty. If tion of an augmented input vector of uncertain distribution paramof the complex simulation codes used in aerospace engineering. This study showed the benefits of using ARA with dedicated rare event probability estimation methods such as Subset Simulations for complex models with nonlinear lsfs and correlated inputs. Several enhancements raised in Section 5 are current open research tracks.

Conflict of interest statement

None declared.

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