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On Veech's proof of Sarnak's theorem on the Möbius flow

el Houcein el Abdalaoui

Abstract

We present Veech's proof of Sarnak's theorem on the Möbius flow which say that there is a unique admissible measure on the Möbius flow. As a consequence, we obtain that Sarnak's conjecture is equivalent to Chowla conjecture with the help of Tao's logarithmic Theorem which assert that the logarithmic Sarnak conjecture is equivalent to logarithmic Chowla conjecture, furthermore, if the even logarithmic Sarnak's conjecture is true then there is a subsequence with logarithmic density one along which Chowla conjecture holds, that is, the Möbius function is quasi-generic.

1. Introduction

In this short note we present Veech's proof of Sarnak's theorem on the Möbius flow [17], [18]. Of-course, this proof is connected to Sarnak and Chowla conjectures. Moreover, let us stress that our exposition is self-contained as much as possible.

Roughly speaking, Chowla conjecture assert that the Liouville function is normal, and Sarnak conjecture assert that the Möbius randomness law holds for any dynamical sequence with zero topological entropy. For more details on the Möbius randomness law we refer to [10].

It is turn out that Veech's proof in combine with the recent result of Tao [15] yields that Sarnak conjecture implies Chowla conjecture. Indeed, Tao's result assert that if the even logarithmic Chowla conjecture holds then there exists a subsequence \mathcal{N} with logarithmic density 1 along which the Chowla conjecture holds, and from Veech's proof we will see that this is enough to conclude that Chowla conjecture holds. We remind that T. Tao obtained as a corollary the recent result of Gomilko-Kwietniak-Lemańczyk [8].

Let us further point out that the proof of Gomilko-Kwietniak-Lemańczyk is based essentially on Tao's theorem on logarithmic Sarnak and Chowla conjectures.

We further notice, as T. Tao pointed out, that the proof of Gomilko-Kwietniak-Lemańczyk use only that the Möbius function is bounded.

Here, as mentioned before, combining Tao's result with Sarnak's theorem as established by W. Veech, we deduce that Sarnak conjecture holds if and only if Chowla conjecture holds.

The more striking result that follows from Veech's proof is the connection between Sarnak conjecture and Hadamard matrix.

We remind that the matrix H of order n is a Hadamard matrix if H is a $n \times n$ matrix with entries ± 1 such that $HH^T = nI_n$, where I_n is the identity matrix. The Hadamard matrix are named after Hadamard since the equality in the famous Hadamard determinant inequality holds if and only if the matrix is a Hadamard matrix.

It is well known that Hadamard matrix exist when $n = 1, 2$ or n is a multiple of 4.

The Hadamard conjecture states that there is a Hadamard matrix for every any multiple of 4. In the opposite direction, the circulant Hadamard matrix conjecture state that the only circulant Hadamard matrix are matrix of order 1 and 4. We recall that a circulant matrix of order m is an $m \times m$ matrix for which each row except the first is a cyclic permutation of the previous row by one position to the right.

The conjectures of Hadamard are two of the most outstanding unsolved problems in mathematics nowadays.

It is well known that the Hadamard matrix is related to the so-called Barker sequences. The Barker sequence is a sequence of ± 1 for which the autocorrelation coefficients are bounded by 1. We remind that the autocorrelation of a sequence $(x_j)_{j=0}^{N-1}$ are given by

$$c_k = \sum_{j=0}^{N-k-1} x_j x_{j+k}, \quad k \geq 1$$

with

$$c_k = \overline{c_{-k}}, \quad \text{if } k < 0.$$

For the special real case we have $c_k = c_{-k}$. To be more precise, it is well known that if a Barker sequence of even length n exists, then so does a circulant Hadamard matrix of order n . But, very recently, the author established that there are only finitely many Barker sequences, that is, Turyn-Golay's conjecture is true [2]. For more details on the Hadamard matrix, we refer to [9].

2. Setup and the main result

The Möbius function μ is related intimately to the Liouville function λ which is defined by $\lambda(n) = 1$ if the number of prime factor of n is even and -1 otherwise. Precisely, the Möbius function μ is given by

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1 \\ \lambda(n), & \text{if } n \text{ is square-free,} \\ 0, & \text{otherwise.} \end{cases}$$

We remind that n is square-free if n has no factor in the subset $\mathcal{P}_2 \stackrel{\text{def}}{=} \{p^2/p \in \mathcal{P}\}$, where as customary, \mathcal{P} denote the subset of prime

numbers.

In his seminal paper [13], P. Sarnak makes the following conjecture.

Sarnak conjecture 2.1. *For any dynamical flow (X, T) with topological entropy zero, for any continuous function $f \in C(X)$, for any point $x \in X$,*

$$\frac{1}{N} \sum_{n=1}^N \mu(n) f(T^n x) \xrightarrow{N \rightarrow +\infty} 0. \quad (2.1)$$

The popular Chowla conjecture on the correlation of the Möbius function state that

Chowla conjecture 2.2. *For any $r \geq 0$, $1 \leq a_1 < \dots < a_r$, $i_s \in \{1, 2\}$ not all equal to 2, we have*

$$\sum_{n \leq N} \mu^{i_0}(n) \mu^{i_1}(n + a_1) \cdot \dots \cdot \mu^{i_r}(n + a_r) = o(N). \quad (2.2)$$

This conjecture is related to the weaker conjecture stated in [5]. We refer to [5] for more details.

In his breakthrough paper [16], T. Tao proposed the following logarithmic version of Sarnak and Chowla conjectures.

Logarithmic Sarnak conjecture 2.3. *For any dynamical flow (X, T) with topological entropy zero, for any continuous function $f \in C(X)$, for any point $x \in X$,*

$$\frac{1}{\log(N)} \sum_{n=1}^N \frac{\mu(n) f(T^n x)}{n} \xrightarrow{N \rightarrow +\infty} 0. \quad (2.3)$$

The logarithmic Chowla conjecture can be stated as follows:

Logarithmic Chowla conjecture 2.4. *For any $r \geq 0$, $1 \leq a_1 < \dots < a_r$, $i_s \in \{1, 2\}$ not all equal to 2, we have*

$$\sum_{1 \leq n \leq N} \frac{\mu^{i_0}(n) \mu^{i_1}(n + a_1) \cdot \dots \cdot \mu^{i_r}(n + a_r)}{n} = o(\log(N)). \quad (2.4)$$

We remind that the logarithmic density of a subset $E \subset \mathbb{N}$ is given by the following limit (if it exists)

$$\lim_{N \rightarrow \infty} \frac{1}{\log(N)} \sum_1^N \frac{\mathbb{I}_E(n)}{n}.$$

Let us further notice that one can replace $\log(N)$ by $\ell_N = \sum_{n=1}^N \frac{1}{n}$. Thanks to Euler estimation.

Following L. Mirsky [12] and P. Sarnak [13], the subset $A \subset \mathbb{N}$ is admissible if the cardinality $t(p, A)$ of classes modulo p^2 in A given by

$$t(p, A) \stackrel{\text{def}}{=} |\{z \in \mathbb{Z}/p^2\mathbb{Z} : \exists n \in A, n \equiv z \pmod{p^2}\}|$$

satisfy

$$\forall p \in \mathcal{P}, t(p, A) < p^2. \quad (2.5)$$

In other words, for every prime p the image of A under reduction mod p^2 is proper in $\mathbb{Z}/p^2\mathbb{Z}$.

Let X_3 be the set $\{0, \pm 1\}^{\mathbb{N}}$ and $X_2 \stackrel{\text{def}}{=} \{0, 1\}^{\mathbb{N}}$, and for each $i = 1, 2$, let X_i be equipped with the product topology. Therefore, X_3 and X_2 are a compact set. We denote by $\mathcal{M}_1(X_i)$, $i = 1, 2$, the set of the probability measures on X_i . it is turn out that $\mathcal{M}_1(X_i)$, $i = 1, 2$ is a compact set for the weak-star topology by Banach-Alaoglu-Bourbaki theorem. Let $x \in X_i$, $i = 1, 2$, and for each $N \in \mathbb{N}$, put

$$m_N(x) = \frac{1}{N} \sum_{n=0}^{N-1} \delta_{S^n x},$$

where δ_y is the Dirac measure on y and S is the canonical shift map $(Sx)_n = x_{n+1}$, for each $n \in \mathbb{N}$. Therefore $m_N(x) \in \mathcal{M}_1(X_{A_i})$.

We thus get that the weak-star closure $\mathcal{I}_S(x)$ of the set $\{m_N(x)\}$ is not empty. We further define the square map s on X_3 by $s(x) = (x_n^2)$ for any $x \in X_3$.

Definition 2.5. *An infinite sequence $x = (x_n)_{n \in \mathbb{N}^*} \in X_3$ is said to be admissible if its support $\{n \in \mathbb{N}^* : x_n \neq 0\}$ is admissible. In the same way, a finite block $x_1 \dots x_N \in \{0, \pm 1\}^N$ is admissible if $\{n \in \{1, \dots, N\} : x_n \neq 0\}$ is admissible. In the same manner, we define the admissible sets in X_2 .*

For each $i = 1, 2$, we denote by $X_{\mathcal{A}_i}$ the set of all admissible sequences in X_i . Since a set is admissible if and only if each of its finite subsets is admissible, and a translation of a admissible set is admissible, $X_{\mathcal{A}_i}$ is a closed and shift-invariant subset of X_i , *i.e.* a subshift. We further have that μ^2 is admissible, and $X_{\mathcal{A}_3} = s^{-1}(X_{\mathcal{A}_2})$.

Let us notice that the previous notions has been extended to the so-called \mathcal{B} -free setting by el Abdalaoui-Lemańczyk-de-la-Rue in [1]. Therein, the authors produced a dynamical proof of the Mirsky theorem on the pattern of μ^2 which assert that the indicator function of the square-free integers is generic for the Mirsky measure ν_M , that is, μ^2 is generic for the push-forward measure of the Haar measure μ_h of the group $G = \prod_p \mathbb{Z}/p^2\mathbb{Z}$ under the map $\varphi : G \rightarrow X_2$ defined by

$$\forall g \in G, \quad \varphi(g) \stackrel{\text{def}}{=} \left(f(T^n g) \right)_{n \in \mathbb{N}^*}, \quad (2.6)$$

where T is the translation by $\mathbf{1} \stackrel{\text{def}}{=} (1, 1, \dots)$ and f is defined by

$$f(g) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if there exists } k \geq 1 \text{ such that } g_k = 0, \\ 1 & \text{otherwise.} \end{cases}$$

We thus get that $\mu^2 = (f(T^n \mathbf{O}))$, $\mathbf{O} = (0, 0, \dots)$.

We further have that for each measurable subset $C \subset X_2$, $\nu_M(C) \stackrel{\text{def}}{=} \nu_h(\varphi^{-1}C)$. Then $\nu_M = \varphi(\mu_h)$ is shift-invariant, and it can be shown that

ν_M is concentrated on $X_{\mathcal{A}_2}$. Moreover, the measurable dynamical system $(X_{\mathcal{A}_2}, \nu_M, S)$ is a factor of (G, ν_h, T) . In particular, it is ergodic, and for any $\eta \in \mathcal{I}_S(\boldsymbol{\mu})$, we have $s\eta = \nu_M$, that is, $\eta(s^{-1}A) = \nu_M(A)$, for any Borel set of $X_{\mathcal{A}_2}$. For more details, we refer to [1].

For any finite sets $A, B \subset \mathbb{N}$, we denote by $F_{A,B}$ the function

$$F_{A,B}(x) = \left(\prod_{a \in A} \pi_a(x) \right) \left(\prod_{b \in B} \pi_b(x)^2 \right),$$

where π_n is the n^{th} canonical projection given by $\pi_n(x) = x_n$, $n \in \mathbb{N}$. Obviously, $F_{A,B} \in C(X_{\mathcal{A}_3})$, where $C(X_{\mathcal{A}_3})$ is the space of continuous function on $X_{\mathcal{A}_3}$. We further have $F_{A,B} = F_{A,B \setminus (A \cap B)}$, so we can assume always that A and B are disjoint.

Following W. Veech [17], [18], we introduce also the notion of admissible measure.

Definition 2.6. *A measure $m \in \mathcal{M}_1(X_{\mathcal{A}_3})$ is admissible if*

(i) $Sm = m$, that is, $m(S^{-1}A) = m(A)$, for each Borel set $A \subset X_{\mathcal{A}_3}$.

(ii) $s(m) = \nu_M$, and

$$\int_{X_{\mathcal{A}_3}} F_{A,B}(x) dm(x) = 0,$$

for any $A \neq \emptyset$. and B finite sets of \mathbb{N} .

We are now able to state the main result.

Theorem 2.7 (Sarnak's Theorem on Möbius flow [13]). *There exists a unique admissible measure μ_M on $X_{\mathcal{A}_3}$ which is ergodic with the Pinsker algebra*

$$\mathcal{P}i_{\mu_M} = s^{-1}(\mathcal{B}(\mathcal{A}_2)).$$

Moreover, $\mathbb{E}(\pi_1 | \mathcal{P}i_{\mu_M}) = 0$.

Following W. Veech [18], the measure μ_M is called Chowla measure.

Remark 2.8. *Furthermore, as pointed out by Veech, the existence of the putative "Chowla measure" does not depend on the Chowla conjecture.*

For the proof of our second main result, we need the following result due to T. Tao [15].

Theorem 2.9 (Tao's theorem on logarithmic and non-logarithmic Chowla conjectures[15]). *Let k be a natural number. Assume that the logarithmically averaged Chowla conjecture is true for $2k$. Then there exists a set \mathcal{N} of natural numbers of logarithmic density 1 such that*

$$\lim_{\substack{N \rightarrow \infty \\ N \in \mathcal{N}}} \frac{1}{N} \sum_1^N \lambda(n+h_1) \dots \lambda(n+h_k) = 0,$$

for any distinct h_1, \dots, h_k .

As a corollary, T. Tao obtain the following

Corollary 2.10 (Gomilko-Kwietniak-Lemańczyk's theorem [8]). *If Sarnak's conjecture holds then there exists a set \mathcal{N} of natural numbers such that for any $r \geq 0$, $1 \leq a_1 < \dots < a_r$, $i_s \in \{1, 2\}$ not all equal to 2, we have*

$$\frac{1}{N} \sum_{n \leq N} \mu^{i_0}(n) \mu^{i_1}(n + a_1) \cdot \dots \cdot \mu^{i_r}(n + a_r) \xrightarrow[N \rightarrow +\infty]{N \in \mathcal{N}} 0. \quad (2.7)$$

Combining Sarnak's Theorem 2.7 with Tao's Theorem 2.9, we get the following

Corollary 2.11. *Sarnak conjecture 2.1 is equivalent to Chowla conjecture 2.2.*

Proof. The proof of the implication follows from Tao's Theorem 2.9 and since the admissible measure is unique. For the converse, there are several proofs by Sarnak [14], Tao [16], Veech [18], and el Abdalaoui-Kualga-Przymus-Lemańczyk-de la Rue[3]. \square

3. Proof of the main result.

We start by proving the following proposition related to Hadamard matrix. For that, let E be a finite nonempty set, and $\mathbb{P}(E)$ be the set of subset of E . For any $A, B \in \mathbb{P}(E)$, put

$$C(A, B) = (-1)^{|A \cap B|},$$

where $|\cdot|$ is the cardinality function. Therefore C is a matrix of order $2^{|E|}$, we further have

Proposition 3.12. *With the notations above,*

$$\det(C) = \begin{cases} 2^{|E|2^{|E|-1}}, & \text{if } |E| > 1 \\ -2, & \text{otherwise.} \end{cases}$$

Moreover, if the vector $(\nu(B))_{B \in \mathbb{P}(E)}$ satisfy

$$\sum_{B \in \mathbb{P}(E)} C(A, B) \nu(B) = \begin{cases} a, & \text{if } A = \emptyset \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\nu(B) = \frac{a}{2^{|E|}}.$$

Proof. The proof of the first part of the proposition can be found in [4, p.42], but for the sake of completeness we include an alternative proof of it.

We start by recalling the Hadamard determinant inequality. Let M be a matrix of order n with real entries and columns m_1, \dots, m_n , then

$$|\det(M)| \leq \prod_{j=1}^n \|m_j\|_2,$$

where $\|\cdot\|_2$ is the usual Euclidean norm. Therefore, if all the entries are in the interval $[-1, 1]$, we get

$$|\det(M)| \leq n^{\frac{n}{2}},$$

with equality if and only if M is a Hadamard matrix. For short and elementary proof of the Hadamard determinant inequality we refer to [4, pp.40-41], [11].

We thus need to check that C is a Hadamard matrix. For that, we proceed by induction. For $n = 1$, the matrix is given by

$$C = \begin{pmatrix} C(\emptyset, \emptyset) & C(\emptyset, \{1\}) \\ C(\{1\}, \emptyset) & C(\{1\}, \{1\}) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Assume that the property is true for $n \geq 1$, and let $E_{n+1} = \{1, 2, \dots, n+1\} = E_n \cup \{n+1\}$. We assume that the subsets of E_{n+1} are ordered as those of E_n . Notice that this does not affect our proof since the determinant does not depend upon any ordering of the elements of 2^E . It follows that the resulting $2^{n+1} \times 2^{n+1}$ matrix has block form

$$C_{n+1} = \begin{pmatrix} C_n & C_n \\ C_n & -C_n \end{pmatrix}.$$

We thus get, by Sylvester observation, that C_{n+1} is a Hadamard matrix. For the second part, let

$$\begin{pmatrix} p \\ q \end{pmatrix} \in \mathbb{R}^{2^n} \times \mathbb{R}^{2^n}$$

such that

$$C_{n+1} \begin{pmatrix} p \\ q \end{pmatrix} = a \cdot \delta_\emptyset(A).$$

Then, for $n = 1$, we have

$$\begin{cases} p + q = a, & \text{if } A = \emptyset \\ p - q = 0, & \text{if not.} \end{cases}$$

Obviously, we get $p = q = \frac{a}{2}$. Assume that the property is true for n . Then

$$p + q = \left(\frac{a}{2^n}, \dots, \frac{a}{2^n} \right).$$

Moreover, since $\det(C_n) \neq 0$, we get $p = q$, that is,

$$p = q = \left(\frac{a}{2^{n+1}}, \dots, \frac{a}{2^{n+1}} \right).$$

The proof of the lemma is complete. \square

For the proof of the Sarnak's theorem 2.7, we need also to characterize the Chowla measure. For that, let us put

$$\mathcal{Q}r_n = \left\{ x \in X_2 \mid \text{supp}(x) \text{ is admissible} \right\}, \quad n > 0$$

and

$$C(x) = \bigcap_{j=1}^n \left\{ y \in X_{\mathcal{A}_2} \mid \pi_j(x) = x_j \right\}.$$

Define the partition $\mathcal{P}r_n$ by

$$\mathcal{P}r_n = \left\{ C(x) \mid x \in \mathcal{Q}r_n \right\}.$$

It follows that any $C(x) \in \mathcal{P}r_n$ admits a partition into $2^{|\text{supp}(x)|}$ "cylinder" since

$$s^{-1}(C(x)) \subset s^{-1}(X_{A_2}) = X_{A_3}.$$

More precisely, if $A \subset \text{supp}(x)$, then A can be seen as a element $y(A) \in X_{A_2}$. We thus denote by $C(x, y(A))$ the subset of X_{A_3} such that $z \in C(x, y(A))$ if and only if the first n coordinates of z are -1 on A , 1 on $\text{supp}(x) \setminus A$ and 0 on $[1, n] \setminus \text{supp}(x)$. This allows us to see that

$$s^{-1}(C(x)) = \bigcup_{A \subset \text{supp}(x)} C(x, y(A)).$$

Now, for any $A \subset \text{supp}(x)$, put

$$G_{A, \text{supp}(x) \setminus A} = F_{A, \text{supp}(x) \setminus A} \prod_{c \in [1, n] \setminus \text{supp}(x)} (1 - \pi_c(y)^2).$$

It is straightforward that $G_{A, \text{supp}(x) \setminus A} \in C(X_{A_3})$. Moreover,

$$G_{A, \text{supp}(x) \setminus A} |_{s^{-1}(C(x))} = F_{A, \text{supp}(x) \setminus A} |_{s^{-1}(C(x))} \quad (3.8)$$

and $G_{A, \text{supp}(x) \setminus A}$ is identically null on $X_{A_3} \setminus s^{-1}(C(x))$.

Expand the product in the definition of $G_{A, \text{supp}(x) \setminus A}$, we get

$$G_{A, \text{supp}(x) \setminus A} = \sum_{B \subset [1, n] \setminus \text{supp}(x)} (-1)^{|B|} F_{A, \text{supp}(x) \setminus A \cup B}.$$

Now, let m be an admissible measure and assume that $A \neq \emptyset$. Then

$$\int_{X_{A_3}} G_{A, \text{supp}(x) \setminus A} m(dz) = \int_{s^{-1}C(x)} F_{A, \text{supp}(x) \setminus A}(z) m(dz) \quad (3.9)$$

$$= 0. \quad (3.10)$$

This gives, for $A \subset \text{supp}(x)$ and $A \neq \emptyset$,

$$\sum_{B \subset \text{supp}(x)} (-1)^{|A \cap B|} m(C(x, y(B))) = 0,$$

since $F_{A, \text{supp}(x) \setminus A}$ is constant on each "cylinder" set $C(x, y(B))$ with the constant value equal to $(-1)^{|A \cap B|}$.

We proceed now to evaluate the expression when $A = \emptyset$. Since $sm = \nu_M$, we obtain

$$\nu_M(C(x)) = \sum_{B \subset \text{supp}(x)} m(C(x, y(B))) = 0.$$

This combined with Proposition 3.12 yields that for any $C(x) \in \mathcal{P}r_n$, for any $B \subset \text{supp}(x)$, we have

$$m(C(x, y(B))) = \frac{\nu_M(C(x))}{2^{|\text{supp}(x)|}}.$$

Summarizing, we conclude that m is completely determined on the partition $\mathcal{P}r_n$, i.e., if an admissible measure exists, then it is unique.

We proceed now to the proof of Sarnak's theorem 2.7.

Consider the canonical dynamical system $(X_{\mathcal{A}_2} \times \{\pm 1\}^{\mathbb{N}}, S \times S, \nu_M \otimes m_B(\frac{1}{2}, \frac{1}{2}))$, where $m_B(\frac{1}{2}, \frac{1}{2})$ is the Bernoulli measure. Therefore, by Furstenberg theorem (Proposition I.3 in [6]), the dynamical system $(X_{\mathcal{A}_2} \times \{\pm 1\}^{\mathbb{N}}, S \times S, \nu_M \otimes m_B(\frac{1}{2}, \frac{1}{2}))$, is ergodic. We further have, by Theorem 18.13 from [7, p.325], that the Pinsker algebra satisfies

$$\mathcal{P}i_{\nu_M \times m_B(\frac{1}{2}, \frac{1}{2})} = \mathcal{P}i_{\nu_M} \otimes \mathcal{P}i_{m_B(\frac{1}{2}, \frac{1}{2})} = \mathcal{B}(X_{\mathcal{A}_2}) \times \{\emptyset, \{\pm 1\}^{\mathbb{N}}\}$$

up to the null set with respect to $\nu_M \otimes m_B(\frac{1}{2}, \frac{1}{2})$.

Now we define a coordinate-wise multiplicative map $\Pi : X_{\mathcal{A}_2} \times \{\pm 1\}^{\mathbb{N}} \rightarrow X_{\mathcal{A}_3}$ by

$$\Pi(x, \omega) = x.\omega,$$

that is,

$$\pi_n(\Pi(x, \omega)) = x_n \omega_n, \quad n > 0.$$

Therefore the dynamical system $(X_{\mathcal{A}_3}, S, \mu_M = \Pi(\nu_M \otimes m_B(\frac{1}{2}, \frac{1}{2})))$, where μ_M is the push-forward measure under Π , satisfies

- $s\mu_M = \nu_M$, and
- For any $A \neq \emptyset$, we have

$$\int_{X_{\mathcal{A}_3}} F_{A,B}(z) \mu_M(dz) = 0.$$

Whence, μ_M is admissible, ergodic and $\mathcal{P}i_{\mu_M} = s^{-1}\mathcal{B}(X_{\mathcal{A}_2})$ up to μ_M null set. This last fact follows from the following

$$\Pi^{-1}\left(\bigcap_n S^{-n}\mathcal{B}(X_{\mathcal{A}_3})\right) \subseteq \bigcap_n (S \times S)^{-n}\mathcal{B}(X_{\mathcal{A}_2} \times \{\pm 1\}^{\mathbb{N}}).$$

To finish the proof, we need only to notice that $\pi_1 = F_{\{1\}, \emptyset}$, and for any finite set $B \subset \mathbb{N}$,

$$\int_{X_{\mathcal{A}_3}} \pi_1(y) F_{\emptyset, B}(y) \mu_M(dy) = \int_{X_{\mathcal{A}_3}} F_{\{1\}, B}(y) \mu_M(dy).$$

We thus conclude that

$$\mathbb{E}(\pi_1 | \mathcal{P}i_{\mu_M}) = 0,$$

up to μ_M null sets, since the family $\{F_{A,B}\}$ are dense in $\mathbb{C}(X_{\mathcal{A}_3})$, by the classical Stone-Weierstrass theorem.

Question. Let $O(\boldsymbol{\mu}) \subset X_{\mathcal{A}_3}$ be the orbit closure of $\boldsymbol{\mu}$ under the left shift S . Do we have that $O(\boldsymbol{\mu}) = X_{\mathcal{A}_3}$?

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¹W. Veech in his letter indicated to me that there is only four persons in the world who has a copy of his notes including me.

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