



HAL
open science

Comparison Between Pierce Equivalent Circuit and Recent Discrete Model for Traveling-Wave Tubes

Damien F G Minenna, Artem G Terentyuk, Frédéric André, Yves Elskens,
Nikita M. Ryskin

► **To cite this version:**

Damien F G Minenna, Artem G Terentyuk, Frédéric André, Yves Elskens, Nikita M. Ryskin. Comparison Between Pierce Equivalent Circuit and Recent Discrete Model for Traveling-Wave Tubes. 2017. hal-01630803v1

HAL Id: hal-01630803

<https://hal.science/hal-01630803v1>

Preprint submitted on 8 Nov 2017 (v1), last revised 3 Apr 2018 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Comparison Between Pierce Equivalent Circuit and Recent Discrete Model for Traveling-Wave Tubes

Damien F. G. Minenna,^{1,2,3,*} Artem G. Terentyuk,^{4,†} Frédéric André,^{3,‡} Yves Elskens,^{2,§} and Nikita M. Ryskin^{4,¶}

¹*Centre National d'Études Spatiales, 31401 Toulouse cedex 9, France*

²*Aix-Marseille University, UMR 7345 CNRS PIIM,*

équipe turbulence plasma, case 322 campus Saint Jérôme,

av. esc. Normandie-Niemen, 13397 Marseille cedex 20, France

³*Thales Electron Devices, rue Latécoère, 2, 78140 Vélizy, France*

⁴*Saratov State University, 410012 Saratov, Russia,*

and with Saratov Branch of the Institute of Radio Engineering and Electronics,

Russian Academy of Sciences, 410019 Saratov, Russia

(Dated: October 31, 2017)

To perform accurate numerical simulations of the traveling-wave tube in time domain, a new approach using field decomposition with large reduction of degrees-of-freedom has been proposed: the discrete model. To assess its validity, we compare it with the well-established Pierce equivalent circuit model in small signal regime. We also discuss associated beam, circuit-beam, and circuit impedances. We demonstrate analytically and with a numerical example that the newly developed discrete model is very close to the Pierce model. Interestingly, small deviations do exist at the edges of the amplification band. We speculate that the deviation from reality is on the Pierce model side, while the discrete model would be more accurate.

PACS numbers: 84.40.Fe (Microwave tubes)
52.40.Mj (Particle beam interaction in plasmas)
52.35.Fp (Electrostatic waves and oscillations)

Keywords: Discrete model, Pierce model, equivalent circuit, dispersion relation, impedances, coupled mode, wave-particle interaction, traveling-wave tube (TWT), time domain, frequency domain, passband, band edge.

I. INTRODUCTION

The recently developed discrete model (a.k.a. Kuznetsov discrete model) [1–3] is a promising tool to analyse devices such as traveling wave tubes (TWTs) beyond what is possible today with the well established Pierce model[4]. It provides an exact reduction of degrees-of-freedom for electromagnetic fields and allows to build both frequency [5] and time domain algorithms [6] that are faster and more accurate alternatives to current PIC algorithms [7]. The discrete model offers several new features compared to Pierce's well-known equivalent circuit model. Most importantly, it is originally in time domain and enables simulating broadband telecom signals for example. Another example are drive-induced oscillations where spurious frequencies are generated very far from the drive frequency in the nonlinear regime. This situation will be accessible to simulation thanks to this new model. Second, the complex structure of stop bands can be accurately described and simulated thus offering a way to progress

on the associated oscillation problems. Fundamentally, the discrete model addresses (and originates from) the general situation of periodic, quasi-periodic and chaotic particles interacting with fields, which is of interest to a broader community of physicists and engineers. Also, three-dimensional simulations are possible with the discrete model. We deepen these aspects in the appendices.

Before addressing these more complex situations, the first question is how the new model compares to the existing one in the simplest case of a single carrier operation (i.e. in frequency domain) in the linear regime, the original background of the Pierce theory. This is the objective of this paper.

In section II, we revisit fundamental definitions of beam, wave and circuit impedances from the Pierce equivalent circuit, starting with a model involving only space charge fields, and then adding circuit fields. In section III, we recall the principles of the discrete model and apply them in the harmonic domain to obtain associated impedances. Finally, we compare both models in section IV. Appendix A revisits the sheath helix approximation using the discrete model. Appendix B compares the TWT discrete model and beam-plasma models.

II. EQUIVALENT CIRCUIT

Developments leading to eqs (1) and (5) below are similar to those used in the coupled wave system of Louisell [8] and to the classical analysis by Gilmour [9]. We re-

* Electronic address: damien.minenna@univ-amu.fr

† Benefited from an Ostrogradski fellowship from the French embassy in Russia.; Acknowledge support from the Russian Science Foundation grant No 17-12-01160.

‡ Electronic address: frederic.andre@thalesgroup.com

§ Electronic address: yves.elskens@univ-amu.fr

¶ Electronic address: ryskinm@info.sgu.ru; Acknowledge support from the Russian Science Foundation grant No 17-12-01160.

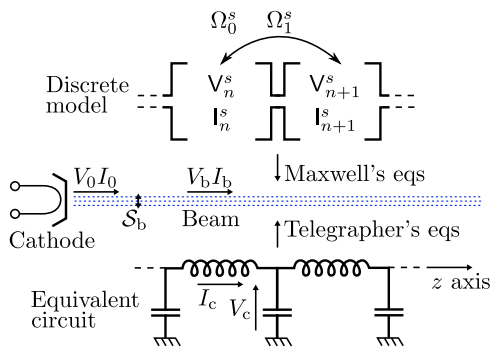


Figure 1. Wave-particles interaction for a periodic slow-wave structure represented using the discrete model (above) and the equivalent circuit (below), along the longitudinal z -axis. The beam is assumed to be a weakly perturbed fluid with section area \mathcal{S}_b . Variables $V_{c,b,0}$ and $I_{c,b,0}$ are the potential and current of the circuit, beam and cathode (dc beam) respectively. V_n^s and I_n^s are the temporal variables of electromagnetic circuit fields (see eqs (21) and (22)) at cell n for the propagation mode s . Ω_m^s is the coupling coefficient between cells at range m .

formulate them to facilitate the comparison with the discrete model analysis and keep our paper self-contained. In particular, after reaching the dispersion relation of the Pierce equivalent circuit, we focus on impedances. A coupled system composed of a beam (b) and a circuit (c) will have two electric potentials V_b and V_c , and two different currents I_b and I_c , leading a priori to four different impedances, respectively the beam impedance Z_b , the circuit-beam impedance Z_{cb} , the beam-circuit impedance Z_{bc} and the circuit impedance Z_c .

A. Space charge waves

The electron beam is described as a weakly perturbed fluid carrying space charge waves along the longitudinal z coordinate. Time and space dependent variables are expressed, according to the space-time Fourier representation, as $F(z, t) = \Re(\tilde{F}(\beta, \omega) e^{-i\psi})$, with the local phase $\psi = \beta z - \omega t$, where $\beta = \omega/v_{ph}$ is the propagation constant in the longitudinal direction and ω the wave pulsation for the phase velocity v_{ph} . Since we may study non-resonant regimes, one also defines the electronic propagation constant $\beta_e = \omega/v_0$, using the beam velocity. Particle velocities are $v_0 + \Re(\tilde{v} e^{-i\psi})$, where the initial velocity $v_0 = \sqrt{2V_0\eta}$ depends on the cathode (dc beam) potential $V_0 > 0$ and the charge/mass ratio $\eta = |e|/m_e$. Particle charge densities are $\rho_0 + \Re(\tilde{\rho} e^{-i\psi})$, with initial density $\rho_0 = I_0/(v_0\mathcal{S}_b) < 0$, for a cathode (dc beam) current $I_0 < 0$, and section area of the beam \mathcal{S}_b . In the linear regime, the relation between perturbed current density and charge density is $\tilde{J}_z = \rho_0\tilde{v} + v_0\tilde{\rho}$. As a first step, we combine this relation with the continuity equation to

obtain

$$(\omega - \beta v_0) \mathcal{S}_b \tilde{J}_z = -\omega \frac{|I_0|}{2V_0} \tilde{V}_b, \quad (1)$$

with the perturbed beam potential $\tilde{V}_b = v_0\tilde{v}/\eta$. The minus sign comes from the dc current $I_0 < 0$. The continuity equation remains unchanged by the presence of circuit waves, so we will keep eq. (1) in the next section.

On the other hand, if we only consider space charge waves in our system (neglecting metallic boundary conditions for simplicity), the Euler equation for electron motion provides $(i\omega - i\beta v_0)\tilde{v} = -\eta\tilde{E}_{z,sc}$, with the space charge field $\tilde{E}_{z,sc} = i\tilde{J}_z/(\epsilon_0\omega)$ from Poisson and continuity equations. Therefore, this motion equation is rewritten

$$(\omega - \beta v_0) \frac{\tilde{V}_b}{v_0} = \frac{\omega_p^2}{\omega v_0} \frac{2V_0}{|I_0|} \mathcal{S}_b \tilde{J}_z, \quad (2)$$

with the electron plasma pulsation $\omega_p = \sqrt{\eta|\rho_0|/\epsilon_0}$. Equation (2) must be equal to $-\tilde{E}_{z,sc}$. Now, we rewrite the relation between the space charge field and the electron current as

$$\tilde{E}_{z,sc} = -\nabla\tilde{V}_{sc} = i\beta\tilde{V}_{sc} = -i\beta Z_b \tilde{I}_b = -i\beta Z_b \int_{\mathcal{S}_b} \tilde{J}_z dx dy, \quad (3)$$

defining the beam characteristic impedance Z_b , with a minus sign from the negative charge density. Comparing eqs (2) and (3) immediately yields¹

$$Z_b(\beta) = \frac{-\omega_p^2}{\omega\beta v_0} \frac{2V_0}{|I_0|}, \quad (4)$$

as $2V_0/|I_0| = v_0/(|\rho_0|\eta\mathcal{S}_b)$, and if we insert (1) into (3), we have $(\omega - \beta v_0)^2 = \omega_p^2$, viz. the cold² Bohm-Gross dispersion relation [10] for space charge waves only. They are represented in Fig. 2. The ratio V_0/I_0 is the beam impedance in case of unperturbed beam ($\tilde{J}_z = \tilde{V}_b = 0$), so we refer to it as the cathode (dc) impedance.

B. Coupling to slow-wave circuits

Now, we consider the equivalent circuit model (see Fig. 1) provided by [4] in the small signal regime, and we add circuit waves to the previous system. In the motion equation (2), we simply add to the right-hand side the

¹ The minus sign in (1), (3) and (4) come from our notation $I_0 < 0$. This result reads $Z_b = \frac{\omega_p^2}{\omega} \frac{2V_0}{|I_0|}$ in ref. [8], but Louisell was working in the reference frame of the beam instead of the laboratory frame as here.

² In the plasma context, ‘‘cold’’ means neglecting the beam temperature (and pressure) in its ballistic co-moving frame.

term $-\tilde{E}_{z,c}$, corresponding to the electric field from the circuit, and combine eqs (1) and (2) to find

$$\tilde{E}_{z,c} = -i \frac{1}{\omega v_0} \left[(\omega - \beta v_0)^2 - \omega_p^2 \right] \frac{2V_0}{|I_0|} \mathcal{S}_b \tilde{J}_z. \quad (5)$$

This is similar to eq. (3), on replacing the space charge field with the circuit field and the beam impedance with the circuit-beam impedance Z_{cb} corresponding to the response of the circuit potential to the beam current, which is then defined as

$$Z_{cb}(\beta) = \frac{(\omega - \beta v_0)^2 - \omega_p^2}{\omega \beta v_0} \frac{2V_0}{|I_0|}. \quad (6)$$

At the resonance, where $\beta_e = \beta$ (phase velocity equal to beam velocity), Z_{cb} acts like the beam impedance as if there were only space charge fields. Since we have here circuit waves, we recall the link between eq. (6) and Pierce's circuit impedance [4]

$$Z_c(\beta) = \frac{|\tilde{E}_{z,c}|^2}{2\beta^2 \langle P \rangle} \quad (7)$$

$$= \frac{4V_0}{|I_0|} \mathcal{C}_p^3, \quad (8)$$

with \mathcal{C}_p the Pierce coupling (or gain) parameter, and $\langle P \rangle$ the harmonic power. Eq. (7) comes directly from \tilde{V}_c/\tilde{I}_c , and it is used by Pierce to find eq. (8) where the coupling impedance remains hidden. It would be erroneous to think that for a beamless case ($V_0 = I_0 = 0$), the circuit impedance could be ill-defined: following eq. (7), this is not true. In fact, eq. (8) can only be used for cases with an existing beam: the Pierce parameter compensates the effect of the unperturbed beam impedance. This is why the parameter expressing the coupling of the beam with the circuit is the Pierce coupling parameter \mathcal{C}_p , not the coupling impedance Z_c .

C. Telegrapher's equations

There is another way to find the coupling impedance. The equivalent circuit considered on Fig. 1 is composed of an infinite number of inductances L and capacitances C per unit length, giving the evolution equations of the circuit potential and current from lossless telegrapher's equations (coupled to the beam current)

$$-i\beta\tilde{V}_c = -iL\omega\tilde{I}_c, \quad (9)$$

$$-i\beta\tilde{I}_c = -iC\omega\tilde{V}_c + i\beta\tilde{I}_b. \quad (10)$$

Without beam ($\tilde{I}_b = 0$), the uncoupled circuit propagation constant is $\beta_0 = \omega\sqrt{CL}$, and we find $L\omega = Z_c\beta_0$ when recalling the classical definition of the characteristic impedance $Z_c = \tilde{V}_c/\tilde{I}_c = \sqrt{L/C}$ which Pierces defines as the circuit impedance. Then we merge the two telegrapher's equations and write the circuit-beam impedance

$Z_{cb} = \tilde{V}_c/\tilde{I}_b$, to find

$$Z_{cb}(\beta) = \frac{\beta_0\beta}{\beta_0^2 - \beta^2} Z_c, \quad (11)$$

equal to eq. (6). On combining eqs (6) and (11) with definition (8), we obtain the "hot" linear dispersion relation

$$\mathcal{C}_p^3 = \frac{(\beta_e - \beta)^2 - \beta_p^2}{2\beta_e\beta} \frac{\beta_0^2 - \beta^2}{\beta_0\beta}, \quad (12)$$

as defined (but written differently) in [4], with $\beta_e = \omega/v_0$ and $\beta_p = \omega_p/v_0$. Equation (12) exhibits the product of two fractions: one originating from the beam, and the other one from the circuit. It is of the fourth degree, yielding the four natural modes of propagation. For later use, we rewrite it as

$$\mathcal{C}_p^3 = \frac{(\omega - \beta v_0)^2 - \omega_p^2}{2\omega\beta v_0} \frac{\omega^2 - \beta^2 v_{ph,0}^2}{\omega\beta v_{ph,0}}, \quad (13)$$

with the beamless phase velocity $v_{ph,0} = 1/\sqrt{CL}$.

III. DISCRETE MODEL

A. Time domain discrete model

In this section, we briefly revisit basic equations of the Kuznetsov nonlinear discrete theory [1–3]. In the most general case of any time dependent circuit fields $\mathbf{E}(\mathbf{r}, t)$, $\mathbf{H}(\mathbf{r}, t)$ existing in the delay line (e.g. propagating or evanescent), we are searching an *exact* and *discretized* decomposition of that field. To do so, we proceed in three steps. The first step is that we already know some particular waves propagating in the structure in the form of the propagation modes. The propagation modes are calculated as the eigenvectors of the Helmholtz equation with the Floquet condition at both ends of one period of the structure³. The complex envelopes of the propagation mode are written $\mathbf{E}_\beta^s(\mathbf{r})$ and $\mathbf{H}_\beta^s(\mathbf{r})$ where βd is the phase-shift in the Floquet condition and $s \in \mathbb{N}$ is the label of the mode. Eigenfields \mathbf{E}_β^s and \mathbf{H}_β^s satisfy the normalization⁴

$$N_\beta^s \delta_{s'}^s = \int_{\mathcal{V}_0} \epsilon_0 \mathbf{E}_\beta^s \cdot \mathbf{E}_\beta^{s'*} d^3\mathbf{r} = \int_{\mathcal{V}_0} \mu_0 \mathbf{H}_\beta^s \cdot \mathbf{H}_\beta^{s'*} d^3\mathbf{r}, \quad (14)$$

where \mathcal{V}_0 is the cell volume, and $\delta_{s'}^s$ is the Kronecker symbol.

³ Propagating modes can be computed thanks to general purpose electromagnetic solvers like CST microwave studio or HFSS.

⁴ In [3], this normalisation is chosen equal to the eigenfield pulsation Ω_β^s so that the canonical variables of the Hamiltonian (not discussed here) are the field coefficients V_n^s and I_n^s in (21)-(22); their dimension is then the square root of an action. In [2], this normalisation has the dimension of an energy, and V_n^s and I_n^s become dimensionless.

In a second step, we limit our search for the discretized expansion to the case of fields $\mathbf{E}_\beta(\mathbf{r}, t)$ satisfying the Floquet condition (for a phase-shift βd per period). The propagation modes are eigenvectors of the Helmholtz linear system, with eigenvalues Ω_β^s ,

$$\text{rot } \mathbf{E}_\beta^s(\mathbf{r}) = -\mu_0 \Omega_\beta^s \mathbf{H}_\beta^s(\mathbf{r}), \quad (15)$$

$$\text{rot } \mathbf{H}_\beta^s(\mathbf{r}) = \epsilon_0 \Omega_\beta^s \mathbf{E}_\beta^s(\mathbf{r}). \quad (16)$$

As the Helmholtz operator is hermitian, they constitute a vector basis and we write $\mathbf{V}_\beta^s(t)$ the discretized set of field generalized coordinates:

$$\mathbf{E}_\beta(\mathbf{r}, t) = \sum_s \mathbf{V}_\beta^s(t) \mathbf{E}_\beta^s(\mathbf{r}). \quad (17)$$

This relation is valid in the reference cell \mathcal{V}_0 but all functions satisfy the Floquet condition, so it is valid everywhere.

The problem now is that fields in general do not respect the Floquet condition. So our third step is to find an expansion of arbitrary fields over a set of fields satisfying the Floquet condition which would write

$$\mathbf{E}(\mathbf{r}, t) = \int_{\beta d = -\pi}^{\pi} \mathbf{E}_\beta(\mathbf{r}, t) d(\beta d). \quad (18)$$

Since the \mathbf{E}_β would satisfy the Floquet condition, we can rewrite the looked after expansion:

$$\mathbf{E}(\mathbf{r} + n d \mathbf{e}_z, t) = \int_{\beta d = -\pi}^{\pi} \mathbf{E}_\beta(\mathbf{r}, t) e^{-i n \beta d} d(\beta d). \quad (19)$$

Thus $\mathbf{E}(\mathbf{r} + n d \mathbf{e}_z, t)$ is the n^{th} coefficient of the Fourier series expansion of \mathbf{E}_β seen as a function of β , namely

$$\mathbf{E}_\beta(\mathbf{r}, t) = \sum_{n \in \mathbb{N}} \mathbf{E}(\mathbf{r} + n d \mathbf{e}_z, t) e^{i n \beta d}. \quad (20)$$

This yields exactly the looked after \mathbf{E}_β functions which (i) satisfy the Floquet condition and (ii) on which the field is expanded (eq. (18)). The elegant transform (20) into functions satisfying the Floquet condition was introduced by I. Gel'fand [12], and (18) is its inverse transform. It is based on Fourier series and shares many of its properties. In particular, the transform of a product is the convolution of the transforms of its factors. Applying this property to eq. (17) completes our initial search for a discrete model:

$$\mathbf{E}(\mathbf{r}, t) = \sum_{s \in \mathbb{N}} \sum_{n \in \mathbb{Z}} \mathbf{V}_n^s(t) \mathbf{E}_{-n}^s(\mathbf{r}), \quad (21)$$

with \mathbf{V}_n^s the Gel'fand transform of \mathbf{V}_β^s . They are the discrete variables determining the electric field. The magnetic field is also discretized⁵ with its own coordinates

\mathbf{l}_n^s

$$\mathbf{H}(\mathbf{r}, t) = i \sum_{s \in \mathbb{N}} \sum_{n \in \mathbb{Z}} \mathbf{l}_n^s(t) \mathbf{H}_{-n}^s(\mathbf{r}). \quad (22)$$

Note the i factor needed to have real \mathbf{l}_n^s variables instead of purely imaginary one.

The interest of this decomposition appears in eqs (21) and (22). For a single propagating mode, there are $2n_{\text{max}}$ different time variables (a.k.a. degrees of freedom) for the fields in a delay-line of n_{max} periods. In comparison, finite difference techniques used in particle-in-cell codes necessitate several millions degrees of freedom to obtain the same accuracy.

We now introduce the beam. Using Maxwell equations with sources, the field decompositions (21)-(22), and the Helmholtz equations (15)-(16), we find the evolution equations [3]

$$-\sum_{s \in \mathbb{N}} \mathbf{l}_\beta^s \Omega_\beta^s \mathbf{E}_\beta^s = \sum_{s \in \mathbb{N}} \frac{\partial \mathbf{V}_\beta^s}{\partial t} \mathbf{E}_\beta^s + \frac{\mathbf{J}_\beta}{\epsilon_0} - \frac{\partial \nabla \phi_\beta}{\partial t}, \quad (23)$$

$$\sum_{s \in \mathbb{N}} \mathbf{v}_\beta^s \Omega_\beta^s \mathbf{H}_\beta^s = \sum_{s \in \mathbb{N}} \frac{\partial \mathbf{l}_\beta^s}{\partial t} \mathbf{H}_\beta^s, \quad (24)$$

where $\mathbf{J}(\mathbf{r}, t)$ is the 3D charge density and the potential $\phi(\mathbf{r}, t)$ satisfies the Poisson equation $\Delta \phi = -\rho/\epsilon_0$.

B. Harmonic domain discrete model

In small signal regime, the discrete model in harmonic domain couples the charge density $\mathbf{J}(\mathbf{r}, t) = \mathbf{J}_0 + \Re(\tilde{\mathbf{J}}(\mathbf{r}) e^{i\omega t})$, with temporal variables $\mathbf{V}_\beta^s(t) = \tilde{\mathbf{V}}_\beta^s e^{i\omega t}$, and $\mathbf{l}_\beta^s(t) = \tilde{\mathbf{l}}_\beta^s e^{i\omega t}$. From eq. (24) and thanks to the eigenfields orthogonality, we have $\tilde{\mathbf{l}}_\beta^s = -i \Omega_\beta^s \tilde{\mathbf{V}}_\beta^s / \omega$, so the evolution equation (23) becomes

$$\sum_{s \in \mathbb{N}} \frac{(\Omega_\beta^s)^2 - \omega^2}{\omega} \tilde{\mathbf{V}}_\beta^s \mathbf{E}_\beta^s(\mathbf{r}) = \frac{-i}{\epsilon_0} \tilde{\mathbf{J}}_\beta(\mathbf{r}) - \omega \nabla \tilde{\phi}_\beta(\mathbf{r}), \quad (25)$$

where the space charge term $\nabla \tilde{\phi}_\beta$ will disappear under integration over the cell volume thanks to boundary conditions [3]. We dot-multiply eq. (25) by the complex conjugate \mathbf{E}_β^{s*} and integrate over space (viz. we project on the mode (s, β)), to find (for a beam with uniform section and small radius)

$$\frac{(\Omega_\beta^s)^2 - \omega^2}{\omega} \tilde{\mathbf{V}}_\beta^s = -i S_b \int_0^d \tilde{J}_{z, \beta}(z) \mathbf{F}_{z, \beta}^{s*}(z) dz, \quad (26)$$

with $\mathbf{F}_{z, \beta}^{s*}(z) = \mathbf{E}_{z, \beta}^{s*}(z) / N_\beta^s$ related to the vector potential eigenfunction. Eq. (26) from Maxwell equations replaces the telegrapher's equations in the discrete model. We mainly deal with eigenmodes off resonance, so $\Omega_\beta^s \neq \omega$ generally.

To complete our model, we take the same weakly perturbed electron beam as in section II, so we end with the

⁵ Ref. [2] uses $\mathbf{V}_\beta^s = -\mathbf{l}_\beta^s$ but this is misleading [11]. We also use $-\pi \leq \beta d \leq \pi$ instead of $0 \leq \beta d \leq 2\pi$.

same linear equation (5) but this time with the circuit field (21) and the charge density spatially modulated in z by $\hat{J}_z(z) = \hat{J}_z e^{-i\beta z}$. Then using the inverse Fourier transform, we obtain

$$\begin{aligned} \hat{J}_{z,\beta'}(z) &= \sum_{n \in \mathbb{Z}} \tilde{J}_z(z + nd) e^{in\beta' d} = \sum_{n \in \mathbb{Z}} \hat{J}_z e^{-i\beta z + i(\beta' - \beta)nd} \\ &= \hat{J}_z e^{-i\beta z} \sum_{p \in (2\pi/d)\mathbb{Z}} \delta\left(\frac{\beta' - \beta - p}{2\pi} d\right), \end{aligned} \quad (27)$$

for any wave number β' , where δ is Dirac's distribution. The same is performed for electric field coefficient, $\mathbf{V}_{\beta'}^s = \sum_p \hat{V}_p^s \delta(\beta' - \beta - p) \frac{2\pi}{d}$, and (26) becomes

$$\frac{(\Omega_{\beta+p}^s)^2 - \omega^2}{\omega} \hat{V}_p^s = -iS_b \hat{J}_z \int_0^d e^{-i(\beta+p)z} \mathbf{F}_{z,\beta+p}^{s*}(z) dz. \quad (28)$$

In the circuit field (21), the integration on βd ranges only over $[-\pi, \pi]$, so the sum reduces to the single term $p = 0$ (viz. only one band, $s = 0$, matters for the waves). As the Gel'fand eigenfield must respect $\mathbf{E}_{z,\beta}^s(z) = \hat{\mathbf{E}}_z^s e^{-i\beta z}$ (and using $\mathbf{F}_{z,\beta}^{s*}(0,0,z) = \hat{\mathbf{E}}_z^{s*} e^{i\beta z} / N_\beta^s$), we have the perturbed circuit field

$$\tilde{\mathbf{E}}_{z,c} = \sum_{s \in \mathbb{N}} \hat{V}_s^s \mathbf{E}_{z,\beta}^s(z) e^{i\beta z} = \sum_{s \in \mathbb{N}} \hat{V}_s^s \hat{\mathbf{E}}_z^s. \quad (29)$$

So we finally reach a new expression for (28)

$$\frac{(\Omega_\beta^s)^2 - \omega^2}{\omega} \hat{V}_s^s = -iS_b \hat{J}_z d \frac{\hat{\mathbf{E}}_z^{s*}}{N_\beta^s}, \quad (30)$$

and we rewrite eq. (5) with eqs (27) and (29).

On the other hand, we can rewrite the circuit impedance from the discrete model as [15]

$$Z_c(\beta) = \frac{|\mathbf{E}_{z,\beta}^s(r=0)|^2 d}{\beta^2 v_g N_\beta^s}, \quad (31)$$

where $v_g(s, \beta)$ is the group velocity, and we can compare eq. (31) to the equivalent circuit impedance (7). We remark that this wave impedance tends to infinity at the passband edges where the group velocity vanishes. An advantage of (31) is that it involves only experimentally known cold values, providing values for $\mathbf{E}_{z,\beta}^s(r=0)$ (and its values in the n -representation) from Z_c . Following the definition (3), but for the circuit field and the circuit-beam impedance, we rewrite the latter for the discrete model

$$Z_{cb}(\beta) = -\frac{i}{\beta} \frac{\hat{V}_s^s \hat{\mathbf{E}}_z^s}{S_b \hat{J}_z}, \quad (32)$$

for a beam with uniform section. We insert this relation in eq. (30) and use eq. (31) to find a new expression

$$Z_{cb}(\beta) = \frac{\omega \beta v_g}{\omega^2 - (\Omega_\beta^s)^2} Z_c, \quad (33)$$

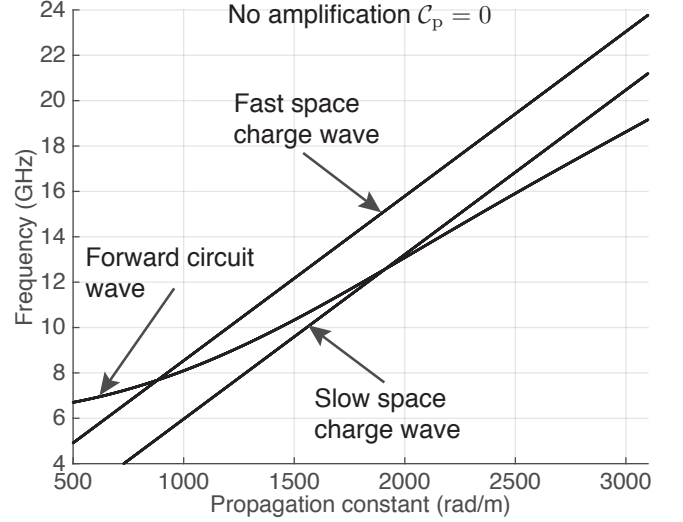


Figure 2. Dispersion diagrams without coupling $\mathcal{C}_p = 0$ (as if the dc current tends to zero) for the equivalent circuit model (eq. (13)), or equivalently for the harmonic domain discrete model (eq. (34)). Solutions are purely real values. The forward circuit wave is simply the “cold” dispersion relation.

enabling us to compare the equivalent circuit–beam coupling impedance (6) with the circuit impedance (8). Substituting eqs (6) and (33) in (8), the “hot” linear dispersion relation of the discrete model becomes

$$\mathcal{C}_p^3 = \frac{(\omega - \beta v_0)^2 - \omega_p^2}{2\omega \beta v_0} \frac{\omega^2 - (\Omega_\beta^s)^2}{\omega \beta v_g}. \quad (34)$$

IV. COMPARISON

To compare accurately both models from sections II and III, we take a phase velocity $v_{ph,0}$ depending on our “cold” dispersion relation, instead of taking it constant, as in Pierce's theory.

A. Without amplification

We consider the case when the Pierce parameter tends to zero ($\mathcal{C}_p \rightarrow 0$), as if the dc current also tends to zero ($I_0 \rightarrow 0$). The four solutions of the dispersion relations (13) and (34) of the equivalent circuit and discrete models are identical. Solutions for forward and return circuit waves are $\omega = \pm \beta v_{ph,0}$. For the discrete model, we have $\omega = \pm \Omega_\beta^s$ but because we take the same “cold” dispersion relation and because $\mathcal{C}_p \rightarrow 0$, we can take $\Omega_\beta^s = \beta v_{ph,0}$, leading to identical results for both models. Solutions for the slow and fast beam waves are $\omega = \beta v_0 \pm \omega_p$. Those solutions are presented in Fig. 2.

B. With amplification

First, we notice that relations (13) and (34) coincide when using the first order linear approximation for numerators of the second fractions. Indeed, near the wave resonance (when $\omega \simeq \Omega_\beta^s$, viz. $\beta \simeq \beta_0$), Taylor expansion yields

$$\omega^2 - \beta^2 v_{\text{ph},0}^2 = 2\omega v_{\text{ph},0} (\beta - \beta_0) + \dots \quad (35)$$

$$\omega^2 - (\Omega_\beta^s)^2 = 2\omega v_g (\beta - \beta_0) + \dots \quad (36)$$

This approximation leads to the conclusion that the harmonic domain discrete model provides the same results as the equivalent circuit model when the dispersion diagram is a slight perturbation of the un-coupled waves, which is the case for practical devices.

But, outside this approximation, we expect small variations between the two models. The maximum distance between un-coupled and coupled waves occurs at the amplification band edges where mode coalescence takes place. To assess them on an example, we take the “cold” dispersion relation of a TWT and we solve the previous equation.

As the independent variable in (13) and (34) is the propagation constant β , amplification is considered in time, with complex frequencies $\omega(\beta)$ whose imaginary parts are growth rates. The tube passband is defined when non-zero growth rates occur. A symbolic solver provides solutions for the four waves as represented with Fig. 3. We immediately see the close similarity between both models as their solutions are almost superposed. The upper curve stands for the fast space charge wave, lower curve depicts the slow space charge wave, and between them we see the forward circuit wave. The backward circuit wave, with negative frequencies or negative propagation constants, is not shown. From 10 to 18 GHz, real solutions for the slow space charge wave and the forward circuit wave are superposed, and for both waves, we have non-zero imaginary parts: this defines the passband of the tube.

A zoom at band edges of Fig. 3 is presented in Fig. 4. As expected, small differences occur when we study the band edge vicinity. Similar differences, but with other dispersion relations, were found in [13]. The main difference is the size of the passband: larger for the equivalent circuit.

V. CONCLUSION AND PERSPECTIVES

We first presented another way to find the “hot” dispersion relation of the Pierce equivalent circuit, using (less usual) beam and circuit-beam impedances. After recalling the basis of the discrete model, we computed its “hot” dispersion relation in linear harmonic domain. Finally, an analytical comparison shows that both models lead to similar results, which validates the discrete model in small signal regime.

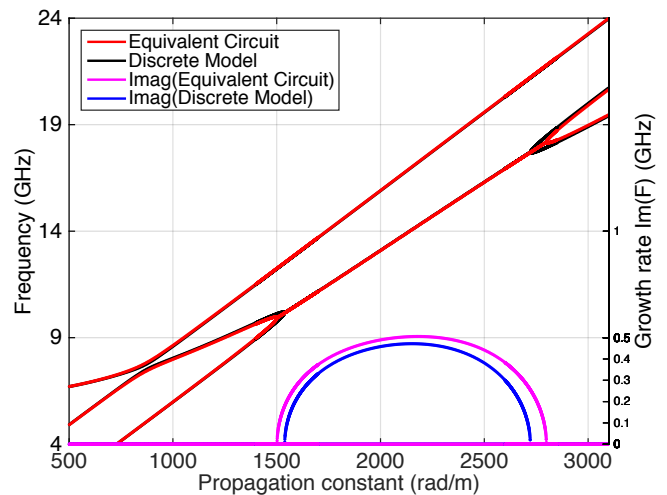


Figure 3. “Hot” linear dispersion diagrams $C_p > 0$; left ordinate axis : real part of the frequency; right axis : imaginary part. Equivalent circuit model (13) is in red for the real part and in magenta for the imaginary part. Harmonic domain discrete model (34) is in black for the real part and in blue for the imaginary part. Both models yield almost identical results, with fast space charge wave, slow space charge wave, and forward circuit wave, except at band edges. Tube passband from 10 to 18 GHz.

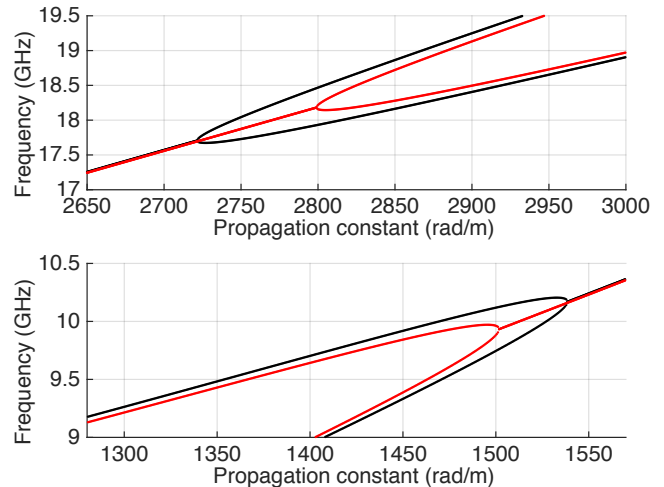


Figure 4. Zoom near band edges on “hot” linear dispersion diagrams of Fig. 3. Equivalent circuit passband starts lower in frequency, near 10 GHz, when discrete model passband starts at 10.2 GHz. Equivalent circuit passband stops higher at 18.1 GHz, when the discrete model stops at 17.7 GHz.

However, small measurable differences do exist between the models so that one of them (or both) must deviate from the experiment. We suggested elsewhere [11] that the Pierce model slightly violates Maxwell equations when coupling is strong. On the other hand, we see no approximation in the discrete model except the truncation on the number of modes which is a sensible approxima-

tion. Based on these arguments, we speculate that the Pierce model is more likely to contain approximations than the discrete model, especially because discrepancies are stronger near the band edges, where the Pierce coupling impedance tends to infinity. According to us, a major advantage of the discrete model is its validity near band edges as well as in the center of passband.

In the continuity of this work, frequency [5] and time domain [6] simulations are currently investigated in large signal regime.

Appendix A: Sheath helix approximation

Now, we present an application of the discrete model on the sheath helix model [4], a three-dimensional traveling-wave tube model. Starting from the real periodic structure instead of the equivalent circuit, we obtain the circuit impedance from the tube geometry. First of all, derivation of the normalisation (14) provides [6] the group velocity

$$v_g(s, \beta) = \frac{d}{N_\beta^s} \int_S \Re(\mathbf{E}_\beta^{s*}(\mathbf{r}) \times \mathbf{H}_\beta^s(\mathbf{r})) \cdot \mathbf{e}_z dS, \quad (\text{A1})$$

where the surface integral is equal to $1/d$ times the cell volume integral. In the sheath helix model, we use only one propagation mode (we omit superscript $s = 0$), without space harmonics. The flux of the Poynting vector in the harmonic discrete model along the z -axis reads [14, 15]

$$\langle \mathcal{P} \rangle = \frac{1}{2} \Re \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{V}_\beta^* \tilde{i}_\beta \frac{1}{d} K_\beta c^2 d(\beta d), \quad (\text{A2})$$

with the geometric propagation factor K_β resulting from (A1)

$$K_\beta c^2 = v_g N_\beta, \quad (\text{A3})$$

with c the speed of light. Knowing the 3D boundary conditions, one can write solutions of the Helmholtz equations (15) and (16) for $\mathbf{E}_\beta(\mathbf{r})$ and $\mathbf{H}_\beta(\mathbf{r})$, and provide a definition for each eigenfield leading to

$$K_\beta c^2 = \frac{\beta N_\beta \epsilon_0}{\gamma^2} \pi a^2 |\mathbf{E}_{z,\beta}^s(r=0)|^2 \mathcal{F}(\gamma a), \quad (\text{A4})$$

with πa^2 the disc section area of the helix, with the transverse propagation constant $\gamma = \sqrt{\beta^2 - (\Omega_\beta/c)^2}$, and with the dimensionless impedance reduction factor [16]

$$\begin{aligned} \mathcal{F}(\gamma a) = & \left(1 + \frac{I_0 K_1}{I_1 K_0}\right) (I_1^2 - I_0 I_2) \\ & + \left(\frac{I_0}{K_0}\right)^2 \left(1 + \frac{I_1 K_0}{I_0 K_1}\right) (K_0 K_2 - K_1^2), \end{aligned} \quad (\text{A5})$$

where $I_m = I_m(\gamma a)$ and $K_m = K_m(\gamma a)$ are modified Bessel functions of the m^{th} order of the first and second kinds respectively. Using (7), we finally recover the

circuit impedance in the thick beam model as

$$Z_c(\beta) = \frac{1}{\pi a^2 \epsilon_0} \frac{\gamma^2}{\Omega_\beta \beta^3} [\mathcal{F}(\gamma a)]^{-1}, \quad (\text{A6})$$

depending on the helix geometry [4, 17]. A similar development can be done for any tube geometries from the discrete model, so it is well adapted to investigating 3D structures.

Appendix B: Beam-plasma systems

Because they generate only little noise, traveling-wave tubes have also proved to be good tools for plasma physics (beyond the fact that the beam is already a plasma). In the classic beam-plasma system [18–21], waves are propagated using the classic plasma itself. To study this system, we substitute the propagating medium with a slow-wave structure like in a TWT. Following [19], considering the power definition, in harmonic domain, for a one-dimensional plasma $\langle \mathcal{P} \rangle = \int v_g \mathcal{E} dS$, where v_g is the group velocity, and \mathcal{E} is the wave energy density of the plasma given by

$$\mathcal{E} = \frac{\epsilon_0 \omega}{2} \frac{\partial}{\partial \omega} (\epsilon(\beta, \omega)) \Big|_{\omega, \beta_0} \langle E_{z,c}^2 \rangle \quad (\text{B1})$$

from the average squared electric field, with $\epsilon(\beta, \omega)$ the plasma dielectric function. Thus

$$\langle \mathcal{P} \rangle = \frac{-\pi a^2 \epsilon_0 \omega}{2} \frac{\partial}{\partial \beta} (\epsilon(\beta, \omega)) \Big|_{\beta_0} |E_{z,c}|^2, \quad (\text{B2})$$

because $v_g = \frac{\partial \omega}{\partial \beta}$. Using (7), we make the link between the beam-plasma system and the beam-slow-wave structure system by setting the plasma impedance

$$Z_p = \frac{-1}{\pi a^2 \epsilon_0} \frac{2}{\omega \beta^2} \left[\frac{\partial}{\partial \beta} (\epsilon(\beta, \omega)) \Big|_{\beta_0} \right]^{-1} \quad (\text{B3})$$

equal to Z_c . Eq. (B3) can also be obtained from computing circuit potentials of both systems, as done in [22]. Then the Pierce parameter becomes

$$C_p^3 = \frac{-\omega_p^2}{\omega \beta^2 v_0} \left[\frac{\partial}{\partial \beta} (\epsilon(\beta, \omega)) \Big|_{\beta_0} \right]^{-1}. \quad (\text{B4})$$

We immediately see the analogy between the helix slow-wave structure circuit impedance (A6), depending on the tube geometry, and the plasma impedance (B3), depending on the plasma dielectric function. It is because waves in a TWT are expressed thank to the dispersion relation and the way they are coupled with the beam. Ref. [18] takes $\beta = \beta_e = \omega/v_0$, so the linear Landau growth rate is $\gamma_{\text{max}} = (n_p/n_b)^{1/3} \sqrt{3} C_p \omega/2^{1/3}$, with n_b and n_p the beam and plasma densities. Analogy of the beam-plasma and the TWT slow-wave structure is allowed by replacing the dielectric function by the geometric factor contained in the circuit impedance.

ACKNOWLEDGEMENT

The authors would like to thank F. Doveil for his suggestions.

-
- [1] S. P. Kuznetsov, “On one form of excitation equations of a periodic waveguide”, *Sov. J. Commun. Technol. Electron.*, vol 25, 1980, 419-421.
- [2] N. M. Ryskin, V. N. Titov and A. V. Yakovlev, “Nonstationary nonlinear discrete model of a coupled-cavity traveling-wave-tube amplifier”, *IEEE Trans. Electron Devices*, vol 56, 2009, 928-934, DOI: 10.1109/TED.2009.2016690.
- [3] F. André, P. Bernardi, N. M. Ryskin, F. Doveil and Y. Elskens, “Hamiltonian description of self-consistent wave-particle dynamics in a periodic structure”, *Europhys. Lett.*, vol 103, 2013, 28004 (5 pp), DOI: 10.1209/0295-5075/103/28004.
- [4] J. R. Pierce, *Traveling wave tubes* (Van Nostrand, New York) 1950.
- [5] A. Terentyuk, *Discrete model of the folded waveguide slow-wave structure*, Master thesis (Saratov Univ., Saratov) 2016, in Russian.
- [6] D. F. G. Minenna, *Description hamiltonienne de l’interaction ondes–électrons dans un guide d’onde périodique*, Master thesis (Aix-Marseille Univ., Marseille) 2016, in French.
- [7] F. André, S. Théveny, F. Doveil and Y. Elskens, “First comparison of new TWT discrete model with existing models”, *IVEC 2015* (Beijing).
- [8] W. H. Louisell, *Coupled mode and parametric electronics* (Wiley, New York) 1960.
- [9] A. S. Gilmour, Jr., *Principles of Traveling Wave Tubes* (Artech House, Boston) 1994.
- [10] D. Bohm and E. P. Gross, “Theory of Plasma Oscillations. A. Origin of Medium-Like Behavior”, *Phys. Rev.*, vol 75(12), 1949, 1851-1864, DOI: 10.1103/physrev.75.1851.
- [11] S. Théveny, F. André and Y. Elskens, “On frequency and time domain models of traveling wave tubes”, 2016, preprint, url: <https://hal.archives-ouvertes.fr/hal-01340471>.
- [12] I. M. Gel’fand, “Eigenfunction expansions for an equation with periodic coefficients”, *Dokl. Akad. Nauk. SSSR*, vol 73, 1950, 1117-1120.
- [13] S. Théveny, *Approches fréquentielle et temporelle de la dynamique des tubes à ondes progressives*, Ph.D. thesis (Aix-Marseille Univ., Marseille) 2016, in French.
- [14] D. F. G. Minenna, Y. Elskens and F. André, “Electron-wave momentum exchange and time domain simulations applied to traveling wave tubes”, *IVEC 2017* (London).
- [15] D. F. G. Minenna, Y. Elskens and F. André, “Electromagnetic power and wave-particle momentum exchange from a hamiltonian approach”, to be published.
- [16] J. R. Pierce, “Theory of the beam-type traveling-wave tube”, *Proc. IRE*, vol 35, 1947, 111-123, DOI: 10.1109/JRPROC.1947.226217.
- [17] P. K. Tien, “Traveling-wave tube helix impedance”, *Proc. IRE*, vol 41, 1953, 1617-1623, DOI: 10.1109/jrproc.1953.274189.
- [18] T. M. O’Neil, J. H. Winfrey and J. H. Malmberg, “Nonlinear interaction of a small cold beam and a plasma”, *Phys. Fluids*, vol 14, 1971, 1204-1212, DOI: 10.1063/1.1693587.
- [19] S. I. Tsunoda, *Wave enhancement due to a static electric field*, Ph.D. thesis (Univ. California at San Diego, La Jolla, California) 1982.
- [20] S. I. Tsunoda and J. H. Malmberg, “Effect of a static electric field on the trapping of beam electrons in a slow wave structure”, *Phys. Rev. Lett.*, vol. 49, 1982, 546-549, DOI: 10.1103/PhysRevLett.49.546.
- [21] F. Doveil, A. Macor and A. Aïssi, “Observation of hamiltonian chaos and its control in wave particle interaction”, *Plasma Phys. Control. Fusion*, vol 49, 2007, 125-135, DOI: 10.1088/0741-3335/49/12B/S12.
- [22] D. Guyomarc’h, *Un tube à onde progressive pour l’étude de la turbulence plasma*, Ph.D. thesis (Univ. Provence, Aix-Marseilles I) 1996, in French.