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ABSTRACT. We show the linear drift of the Brownian motion on the universal cover of a closed connected Riemannian manifold is C^{k-2} differentiable along any C^k curve in the manifold of C^k metrics with negative sectional curvature. We also show that the stochastic entropy of the Brownian motion is C^1 differentiable along any C^3 curve of C^3 metrics with negative sectional curvature. We formulate the first derivatives of the linear drift and entropy, respectively, and show they are critical at locally symmetric metrics.

CONTENTS

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1. Introduction and statement of results

Let M be an m-dimensional orientable closed connected smooth manifold with fundamental group G. Its universal cover space $\widetilde{M}$ is such that $M = \widetilde{M}/G$.

For $k \in \mathbb{N}$, let $C^k(S^2T^*)$ be the collection of C^k sections of S^2T^*, the bundle of symmetric 2-forms on the tangent space TM. It is a Banach space with the topology of the uniform convergence in k derivatives. The set of all smooth sections of S^2T^*, denoted by $C^\infty(S^2T^*) := \bigcap_{k=0}^\infty C^k(S^2T^*)$, is a Fréchet space whose topology is given by all the C^k-norms. Let $\mathcal{M}^k(M)$ denote the set of C^k Riemannian metrics on M. It is the collection of elements in $C^k(S^2T^*)$ which induces a positive definite inner product on each tangent space T_xM, $x \in M$. The space of all smooth Riemannian metrics $\mathcal{M}^\infty(M) = \bigcap_{k=1}^\infty \mathcal{M}^k(M)$ consists of an open convex positive cone in $C^\infty(S^2T^*)$ and is a Fréchet manifold.

Let $g \in \mathcal{M}^2(M)$ and let $\widetilde{g}$ be its G-invariant extension to $\widetilde{M}$. Consider the Brownian motion on $(\widetilde{M}, \widetilde{g})$ with starting point $x \in \widetilde{M}$. Its density function of the distribution at time $t \in \mathbb{R}_+$, denoted by $p(t, x, y), y \in \widetilde{M}$, is the fundamental solution to the heat equation $\partial u/\partial t = \Delta u$, where $\Delta := \mathrm{Div}\nabla$ is the Laplacian of metric $\widetilde{g}$ on C^2 functions on $\widetilde{M}$. Denote by $\mathrm{Vol}_{\widetilde{g}}$ the Riemannian volume on $(\widetilde{M}, \widetilde{g})$. The *linear drift*

$$\ell := \lim_{t\to+\infty} \frac{1}{t} \int d_{\widetilde{g}}(x, y) p(t, x, y)\ d\mathrm{Vol}_{\widetilde{g}}(y),$$

which was introduced by Guivarc'h ([**Gu**]), tells the average in time of the shift of the Brownian motion from its starting point. The limit ℓ is independent of the choice of x and is well-defined since we have a compact quotient.

Let $\mathfrak{R}^k(M)$ ($k \geqslant 3$ or $k = \infty$) be the submanifold of $\mathcal{M}^k(M)$ made of negatively curved C^k metrics on M. It is open in $\mathcal{M}^k(M)$. For any curve $\lambda \in (-1, 1) \mapsto g^\lambda \in \mathfrak{R}^k(M)$, the linear drift for each (M, g^λ), denoted by ℓ_λ, is positive ([**K1**, Theorem 10]). Our main result in this paper is the following.

Theorem 1.1. *Let M be a closed connected smooth manifold. For any C^k ($k \geqslant 3$) curve $\lambda \in (-1, 1) \mapsto g^\lambda \in \mathfrak{R}^k(M)$, the function $\lambda \mapsto \ell_\lambda$ is C^{k-2} differentiable; for any C^∞ curve $\lambda \in (-1, 1) \mapsto g^\lambda \in \mathfrak{R}^\infty(M)$, the function $\lambda \mapsto \ell_\lambda$ is C^∞ differentiable.*

Our strategy to prove Theorem 1.1 is to use the expression of the linear drift at the infinity boundary of $\widetilde{M}$ and prove the C^{k-2} regularity of the ingredients in that formula.

Let $\widetilde{g}^\lambda$ be the G-invariant extensions of g^λ in $\widetilde{M}$. The *geometric boundary* of $(\widetilde{M}, \widetilde{g}^\lambda)$, denoted $\partial\widetilde{M}^\lambda$, is the collection of equivalent class of unit speed $\widetilde{g}^\lambda$-geodesics that remain a bounded distance apart. Each $\partial\widetilde{M}^\lambda$ can be identified with $\partial\widetilde{M}^0$ (or simply $\partial\widetilde{M}$) since the identity isomorphism from G to itself induces a natural homeomorphism between the two boundaries. For $x \in \widetilde{M}$ and $\xi \in \partial\widetilde{M}$, let $X^\lambda(x, \xi)$ be the initial speed vector of the unit speed $\widetilde{g}^\lambda$-geodesic starting from x belonging to the equivalent class of ξ. Let Div^λ be the

divergence operator of $(\widetilde{M}, \widetilde{g}^\lambda)$. It is true (see Section 2 for a more precise statement) that

$$\ell_\lambda = -\int_{M_0\times\partial\widetilde{M}} \mathrm{Div}^\lambda X^\lambda \, d\tilde{\mathbf{m}}^\lambda, \tag{1.1}$$

where M_0 is a connected fundamental domain and $d\tilde{\mathbf{m}}^\lambda = dx^\lambda \times d\tilde{\mathbf{m}}^\lambda_x$, where dx^λ is proportional $d\mathrm{Vol}_{\widetilde{g}^\lambda}$ and $\tilde{\mathbf{m}}^\lambda_x$ is the hitting probability at $\partial\widetilde{M}$ of the $\widetilde{g}^\lambda$-Brownian motion starting at x.

The term $-\mathrm{Div}^\lambda X^\lambda$ in (1.1) has its geometric feature as being the mean curvature of the strong stable horosphere of the geodesic flow in the metric $\widetilde{g}^\lambda$ (see (2.4)); its regularity in λ can be deduced using the results from [**Co**, **KKPW**, **LMM**] on the Morse correspondence map between the geodesic flows of two negatively curved spaces (Proposition 3.5).

To conclude Theorem 1.1, we show the following on the regularity in λ of the harmonic measure $\mathbf{m}^\lambda := \tilde{\mathbf{m}}^\lambda|_{SM}$, where $SM := M_0 \times \partial\widetilde{M}$ (see Section 3 for precise definitions).

Theorem 1.2. *Let M be a closed connected smooth manifold. For any $g \in \Re^k(M)$, $k \geqslant 3$, there exist a neighborhood $\mathcal{V}_g$ of g in $\Re^k(M)$ and a Banach subspace $\mathcal{H}^0_{\mathfrak{b}}$ of continuous functions on SM such that for any C^k curve $\lambda \in (-1,1) \mapsto g^\lambda \in \mathcal{V}_g$ with $g^0 = g$, the mapping $\lambda \mapsto \mathbf{m}^\lambda$ is C^{k-2} in the weak topology of the dual space $(\mathcal{H}^0_{\mathfrak{b}})^*$.*

For each g^λ, it is defined naturally a one-parameter of actions Q^λ_t ($t \geqslant 0$) on continuous functions f on SM:

$$\mathrm{Q}^\lambda_t(f)(x,\xi) := \int_{M_0\times\partial\widetilde{M}} \widetilde{f}(y,\eta)\mathbf{q}^\lambda(t,(x,\xi),d(y,\eta)), \tag{1.2}$$

where $\mathbf{q}^\lambda$ denotes the transition probability of g^λ-Brownian motion on the stable leaves of SM and $\widetilde{f}$ denotes the G-invariant extension of f to $\widetilde{M}\times\partial\widetilde{M}$. Since (M, g^λ) is negatively curved, it is known ([**L2**]) that each Q^λ_T (for T large) is a contraction on some Banach space $\mathcal{H}^\lambda_{\mathfrak{b}}$ of continuous functions on SM which are Hölder continuous with respect to direction changes and this makes $\mathbf{m}^\lambda$ a fixed point of the dual of $\mathrm{Q}^\lambda_T|_{\mathcal{H}^\lambda_{\mathfrak{b}}}$. The idea to prove Theorem 1.2 is to use the classical perturbation result on a linear contraction in a Banach space ([**Kat**]). Hence, it suffices to find a common Banach space $\mathcal{H}^0_{\mathfrak{b}}$ and a $T > 0$ such that

- all Q^λ_T, $\lambda \in (-1,1)$, are contractions on $\mathcal{H}^0_{\mathfrak{b}}$, uniformly in λ, and
- $\lambda \mapsto \mathrm{Q}^\lambda_T$ is C^{k-2} as maps from $\mathcal{H}^0_{\mathfrak{b}}$ into itself.

To achieve this, we not only need the regularity of the heat kernels $\mathbf{q}^\lambda$ in g^λ, but also need the estimations on its differentials, which we present with full generality as follows.

For each $g = (g_{ij}(x)) \in \mathcal{M}^k(M)$, set $\|g\|_{C^a}$ ($a \leqslant k$) for the C^a-norm of g which involves the bounds of $\{g_{ij}(x)\}$ and of their differentials up to the a-th order. Each C^k curve $\lambda \in (-1,1) \mapsto g^\lambda \in \mathcal{M}^k(M)$ defines a one-parameter of tangent vectors $\mathcal{X}^\lambda = (\mathcal{X}^\lambda_{ij}(x)) \in C^k(S^2T^*)$. Let

$$(\mathcal{X}^\lambda)^{(0)} = \mathcal{X}^\lambda,\ (\mathcal{X}^\lambda)^{(l)} = \big((\mathcal{X}^\lambda)^{(l-1)}\big)^{(1)}_\lambda,\ l = 1, \cdots, k,$$

be its derivatives in λ up to the k-th order. All $(\mathcal{X}^\lambda)^{(l)}$ are elements in $C^k(S^2T^*)$. By $\|(\mathcal{X}^\lambda)^{(l)}\|_{C^a}$ $(a \leqslant k)$, we mean the C^a-norm of $(\mathcal{X}^\lambda)^{(l)}$, which involves the bounds of the $(\mathcal{X}^\lambda)^{(l)}_{ij}(x)$ and of their differentials in x up to the a-th order.

Let $C^{k,\iota}(\widetilde{M})$ denote the collection of C^k functions on $\widetilde{M}$ with Hölder exponent ι. The set of continuous functions on $\widetilde{M}$ is denoted by $C(\widetilde{M})$. For any one-parameter of real functions on $\widetilde{M}$ or real numbers $\lambda \mapsto a^\lambda$, let $(a^\lambda)^{(i)}_\lambda$ denote the i-th differential in λ whenever it exists.

Theorem 1.3. *For any $g \in \mathcal{M}^k(M)$, $k \geqslant 3$, there exist $\iota \in (0,1)$ and a neighborhood $\mathcal{V}_g$ of g in $\mathcal{M}^k(M)$ such that for any C^k curve $\lambda \in (-1,1) \mapsto g^\lambda \in \mathcal{V}_g$ with $g^0 = g$:*

i) *The mappings $\lambda \mapsto p^\lambda(T,x,\cdot)$, $x \in \widetilde{M}, T \in \mathbb{R}_+$, are C^{k-2} in $C^{k,\iota}(\widetilde{M})$.*

ii) *Let $T_0 > 0$ and $q \geqslant 1$. For each i, $1 \leqslant i \leqslant k-2$, l, $0 \leqslant l \leqslant k-2-i$, and $T > T_0$, there exists $c_{\lambda,(l,i)}(q)$ depending on (l,i), q, T, T_0, $\|g^\lambda\|_{C^{l+i+2}}$ and $\{\|(\mathcal{X}^\lambda)^{(j)}\|_{C^{l+i-j+1}}\}_{j\leqslant i-1}$ such that*

$$\left\|\nabla^{(l)}(\ln p^\lambda)^{(i)}_\lambda(T,x,\cdot)\right\|_{L^q} \leqslant c_{\lambda,(l,i)}(q), \tag{1.3}$$

where the L^q-norm is taken with respect to the distribution at T of the $\widetilde{g}^\lambda$-Brownian motion probability.

iii) *Let $T_0 > 0$ and $q \geqslant 1$. For each i, $1 \leqslant i \leqslant k-2$, and $T > T_0$, there exists $c_{\lambda,(i)}(q)$ depending on i, q, T, T_0, $\|g^\lambda\|_{C^{i+2}}$ and $\{\|(\mathcal{X}^\lambda)^{(j)}\|_{C^{i-j+1}}\}_{j\leqslant i-1}$ such that*

$$\left\|\frac{(p^\lambda)^{(i)}_\lambda(T,x,\cdot)}{p^\lambda(T,x,\cdot)}\right\|_{L^q} \leqslant c_{\lambda,(i)}(q). \tag{1.4}$$

iv) *Let $\widetilde{f} \in C(\widetilde{M})$ be uniformly continuous and bounded. Then for any $T > 0$ and i, $1 \leqslant i \leqslant k-2$, the function $\int_{\widetilde{M}}(p^\lambda)^{(i)}_\lambda(T,x,y)\widetilde{f}(y)\ d\mathrm{Vol}_{\widetilde{g}^\lambda}(y)$ belongs to $C(\widetilde{M})$.*

A priori, the derivative in λ of $p^\lambda(t,x,y)$, if it exists, satisfy the equation

$$\left\{\begin{array}{lcl} \frac{\partial}{\partial t}q(t,x,y) & = & \Delta^\lambda_y q(t,x,y) + (\Delta^\lambda_y)^{(1)}_\lambda p^\lambda(t,x,y), \\ q(0,x,y) & \equiv & 0. \end{array}\right. \tag{1.5}$$

Equation (1.5) always has a solution in the distribution sense. Our Theorem 1.3 is that this distribution is given by a function $(p^\lambda(t,x,\cdot))^{(1)} \in C^{k,\iota}(\widetilde{M})$ and that its gradients satisfy (1.3). This does not follow directly from (1.5) since $(\Delta^\lambda_y)^{(1)}_\lambda p^\lambda(t,x,y)$ has singularities as t goes to zero and $y = x$. To get an explicit expression of the solution, we use the stochastic calculus representations of the heat kernel and the Brownian motion. Namely, we find a C^1 vector field $\mathrm{z}^{\lambda,1}_T(y)$ on $\widetilde{M}$ (see (5.15)) such that, for any smooth f on $\widetilde{M}$ with compact support,

$$\left(\int_{\widetilde{M}} f(y)p^\lambda(T,x,y)\ d\mathrm{Vol}^\lambda(y)\right)^{(1)}_\lambda = \int_{\widetilde{M}}\left\langle\nabla^\lambda_y f(y), p^\lambda(T,x,y)\mathrm{z}^{\lambda,1}_T(y)\right\rangle_\lambda\ d\mathrm{Vol}^\lambda(y).$$

So, using the classical integration by parts formula, we obtain

$$\begin{aligned}&\left(\int_{\widetilde{M}} f(y)p^\lambda(T,x,y)\ d\mathrm{Vol}^\lambda(y)\right)'_\lambda\\ &= -\int_{\widetilde{M}} f(y)\left(\left(\mathrm{Div}^\lambda \mathrm{z}_T^{\lambda,1}(y)\right)+\left\langle \mathrm{z}_T^{\lambda,1}(y),\nabla^\lambda \ln p^\lambda(T,x,y)\right\rangle_\lambda\right)p^\lambda(T,x,y)\ d\mathrm{Vol}^\lambda(y).\end{aligned}\tag{1.6}$$

In the same way, we will find C^1 vector fields $\{\mathrm{z}_T^{\lambda,j}(y)\}_{j\leqslant i\leqslant k-2}$ (see (6.6) and (6.12)), which will enter the formulas of $(\ln p^\lambda)^{(i)}_\lambda$ and the gradients $\nabla^{(l)}(\ln p^\lambda)^{(i)}_\lambda$.

We remark that the argument of Theorem 1.3 also applies to the heat kernels on M. But there might be a more direct proof using the compactness of the manifold (see [**M**]).

In showing Theorem 1.1, we also obtain the formula (3.12) for the first order differential of the linear drift under one-parameter deformations of negative curved metrics, which implies the following two theorems.

Theorem 1.4. *(see Corollary 3.10) Let M be a closed connected smooth manifold. Let $g\in\Re^3(M)$ be a negatively curved locally symmetric metric. Then for any C^3 curve $\lambda\in(-1,1)\mapsto g^\lambda\in\Re^3(M)$ with $g^0=g$ and constant volume,*

$$(\ell_\lambda)'_0:=(d\ell_\lambda/d\lambda)|_{\lambda=0}=0.$$

Theorem 1.5. *(see Theorem 3.11) There is a linear functional $\mathcal{L}$ on $C^k(S^2T^*)$ such that for all C^3 curve $\lambda\in(-1,1)\mapsto g^\lambda\in\Re^3(M)$ with $g^0=g$ and constant volume,*

$$(\ell_\lambda)'_0\ =\ \mathcal{L}(\mathcal{X}).$$

A similar approach yields the first order differentiability in λ of the entropy h^λ of the Brownian motion on $(\widetilde{M},\widetilde{g})$, where

$$h^\lambda:=\lim_{t\to\infty}-\frac{1}{t}\int \ln p^\lambda(t,x,y)p^\lambda(t,x,y)\ d\mathrm{Vol}^\lambda(y).$$

Theorem 1.6. *Let M be a closed connected smooth manifold. For any C^3 curve $\lambda\in(-1,1)\mapsto g^\lambda\in\Re^3(M)$, the function $\lambda\mapsto h^\lambda$ is C^1 differentiable and is critical at $\lambda=0$ when g^0 is locally symmetric. Moreover, there is a linear functional $\mathcal{K}$ on $C^k(S^2T^*)$ such that*

$$(h^\lambda)'_0:=(dh^\lambda/d\lambda)|_{\lambda=0}=\mathcal{K}(\mathcal{X}).$$

In our approach, the higher order regularity of $\lambda\mapsto h^\lambda$ and the analysis on the differentials would depend on understanding the regularity of the Martin kernel (see [**Go**] for the random walk case). This will be treated in a subsequent paper ([**LS2**]).

Note that the Hausdorff dimension of the distribution of $\widetilde{\mathbf{m}}^\lambda_x$, denoted by $\dim_{\mathrm{H}}\widetilde{\mathbf{m}}^\lambda_x$, is given by $h^\lambda/(\varkappa\ell^\lambda)$ for a fixed number $\varkappa$ associated with the distance function on the boundary (see (3.1)) ([**L1**]). The following is a corollary of Theorem 1.1 and Theorem 1.6.

Corollary 1.7. *Let M be a closed connected smooth manifold. For any C^3 curve $\lambda \in (-1,1) \mapsto g^\lambda \in \Re^3(M)$ and all $x \in \widetilde{M}$, the function $\lambda \mapsto \dim_{\mathrm{H}} \widetilde{\mathfrak{m}}_x^\lambda$ is C^1 differentiable.*

We arrange the paper as follows. In Section 2, we give some preliminaries. In Section 3, we assume Theorem 1.3 and prove consecutively Theorem 1.2, Theorem 1.1, Theorem 1.4 and Theorem 1.5. Section 4 is for the Eells-Elworthy-Malliavin construction of the stochastic flow corresponding to the Brownian motion and its related dynamical properties. The strategy for proving the first order differentiability in Theorem 1.3 and the $i = 1$ case of (1.3) and (1.4) is explained in Section 5.1. Section 5 is devoted to the details of that proof. The rest of the proof of Theorem 1.3 is by induction on the order of differentiability. See Section 6.1 for the description of the necessary steps and Section 6.2 for their proofs. Finally, in Section 7, we consider the first order regularity of the entropy.

2. Preliminaries

In this section, we introduce the basic notions related to formula (1.1). In the rest of the paper, if it is not specified, we only consider the elements of $\mathcal{M}^k(M)$, $\Re^k(M)$ with $k \geqslant 3$.

2.1. **Jacobi fields and the geodesic flow.** For $g \in \mathcal{M}^k(M)$, let ∇, R be the Levi-Civita connection and the curvature tensor on (M, g) and $(\widetilde{M}, \widetilde{g})$. Recall that a unit speed $\widetilde{g}$-geodesic $t \mapsto \gamma(t) \in \widetilde{M}$ is such that $\nabla_{\dot\gamma}\dot\gamma = 0$, where $\dot\gamma(t) = \nabla_{\frac{\partial}{\partial t}}\gamma(t)$. The Jacobi fields along γ are vector fields $t \mapsto J(t) \in T_{\gamma(t)}\widetilde{M}$ which describe the infinitesimal variations of the geodesics around γ. It is well known that $J(t)$ satisfies the Jacobi equation

$$\nabla_{\dot\gamma(t)}\nabla_{\dot\gamma(t)}J(t) + R(J(t), \dot\gamma(t))\dot\gamma(t) = 0$$

and is uniquely determined by the values of $J(0)$ and $J'(0)$. Let $N(\gamma)$ be the normal bundle of γ, i.e.,

$$N(\gamma) := \bigcup_{t\in\mathbb{R}} N_t(\gamma), \text{ where } N_t(\gamma) = \big\{Y \in T_{\gamma(t)}\widetilde{M} : \ \langle Y, \dot\gamma(t)\rangle = 0\big\}.$$

A $(1,1)$-*tensor* along γ is a family $V = \{V(t),\ t \in \mathbb{R}\}$, where each $V(t)$ is an endomorphism of $N_t(\gamma)$ such that for any family Y_t of parallel vectors along γ, the covariant derivative $\nabla_{\dot\gamma(t)}(V(t)Y_t)$ exists. The curvature tensor R induces a symmetric $(1,1)$-tensor along γ by $R(t)Y = R(Y, \dot\gamma(t))\dot\gamma(t)$. A $(1,1)$-tensor $V(t)$ along γ is called a *Jacobi tensor* if it satisfies

$$\nabla_{\dot\gamma(t)}\nabla_{\dot\gamma(t)}V(t) + R(t)V(t) = 0.$$

If $V(t)$ is a Jacobi tensor along γ, then $V(t)Y_t$ is a Jacobi field for any parallel field Y_t along γ.

The Jacobi fields can also be visualized using the geodesic flow map on the unit tangent bundle. For $x \in \widetilde{M}$ and $\mathrm{v} \in T_x\widetilde{M}$, an element $\mathrm{w} \in T_{\mathrm{v}}T\widetilde{M}$ is *vertical* if its projection on $T_x\widetilde{M}$ vanishes. The vertical subspace V_{v} is identified with $T_x\widetilde{M}$. The connection defines

a *horizontal* complement H_{v}, also identified with $T_x\widetilde{M}$. This gives a horizontal/vertical Whitney sum decomposition

$$TT\widetilde{M} = T\widetilde{M} \oplus T\widetilde{M}.$$

Define the inner product on $TT\widetilde{M}$ by

$$\big\langle (Y_1, Z_1), (Y_2, Z_2) \big\rangle_{\widetilde{g}} := \big\langle Y_1, Y_2 \big\rangle_{\widetilde{g}} + \big\langle Z_1, Z_2 \big\rangle_{\widetilde{g}}.$$

It induces a Riemannian metric on $T\widetilde{M}$, the so-called Sasaki metric. The unit tangent bundle $S\widetilde{M}$ of the universal cover $(\widetilde{M}, \widetilde{g})$ is a subspace of $T\widetilde{M}$ with tangent space

$$T_{(x,\mathrm{v})}S\widetilde{M} = \big\{(Y, Z) :\ Y, Z \in T_x\widetilde{M}, Z \perp \mathrm{v}\big\},\ \text{for } x \in \widetilde{M}, \mathrm{v} \in S_x\widetilde{M}.$$

Assume $\mathbf{v} = (x, \mathrm{v}) \in S\widetilde{M}$ and let $\gamma_{\mathbf{v}}$ be the $\widetilde{g}$-geodesic starting at x with initial velocity v. Horizontal vectors in $T_{\mathbf{v}}S\widetilde{M}$ correspond to pairs $(J(0), 0)$. In particular, the *geodesic spray* $\overline{X}_{\mathbf{v}}$ at $\mathbf{v}$ is the horizontal vector associated with $(\mathrm{v}, 0)$. A vertical vector in $T_{\mathbf{v}}S\widetilde{M}$ is a vector tangent to $S_x\widetilde{M}$, the set of unit tangent vectors at x. It corresponds to a pair $(0, J'(0))$, with $J'(0)$ orthogonal to v. The orthogonal space to $\overline{X}_{\mathbf{v}}$ in $T_{\mathbf{v}}S\widetilde{M}$ corresponds to pairs $(\mathrm{v}_1, \mathrm{v}_2), \mathrm{v}_i \in N_0(\gamma_{\mathbf{v}})$ for $i = 1, 2$.

The vector field $\{\overline{X}_{\mathbf{v}}\}_{\mathbf{v}\in S\widetilde{M}}$ generates the geodesic flow $\{\mathbf{\Phi}_t\}_{t\in\mathbb{R}}$ on the unit tangent bundle, where $\mathbf{\Phi}_t : S\widetilde{M} \to S\widetilde{M},\ \mathbf{v} \mapsto \dot{\gamma}_{\mathbf{v}}(t)$. Any Jacobi field along a geodesic $\gamma_{\mathbf{v}}$ is of the form $D\mathbf{\Phi}_t(\mathbf{w})$, where $\mathbf{w} \in T_{\mathbf{v}}S\widetilde{M}$ is an infinitesimal change of the initial point $\mathbf{v}$. More explicitly, if $(J(0), J'(0))$ is the horizontal/vertical decomposition of $\mathbf{w} \in T_{\mathbf{v}}S\widetilde{M}$, then $(J(t), J'(t))$ is the horizontal/vertical decomposition of $D\mathbf{\Phi}_t(\mathbf{w}) \in T_{\mathbf{\Phi}_t(\mathbf{v})}S\widetilde{M}$.

2.2. **Anosov flow and invariant manifolds.** Assume $g \in \mathfrak{R}^k(M)$. The $\widetilde{g}$-geodesic flow $\mathbf{\Phi}_t$ on $S\widetilde{M}$ has some special properties due the negative curvature nature of the space.

Firstly, $(\widetilde{M}, \widetilde{g})$ has no conjugate points. Hence we can identify $S\widetilde{M}$ with $\widetilde{M} \times \partial\widetilde{M}$ since each pair $(x, \xi) \in \widetilde{M} \times \partial\widetilde{M}$ corresponds to a unique unit speed geodesic $\gamma_{x,\xi}$, which begins at x and is asymptotic to ξ, and the mapping $\partial\widetilde{M} \mapsto S_x\widetilde{M}$ sending ξ to $\dot{\gamma}_{x,\xi}(0)$ is a bijection. In the $(\widetilde{M}, \partial\widetilde{M})$-coordinate, the geodesic flow map $\mathbf{\Phi}_t$ has the expression

$$\mathbf{\Phi}_t(x, \xi) = (\gamma_{x,\xi}(t), \xi),\ \forall (x, \xi) \in S\widetilde{M}.$$

Furthermore, the geodesic flow on $S\widetilde{M}$ is *Anosov*: the tangent bundle $TS\widetilde{M}$ decomposes into the Witney sum of three $D\mathbf{\Phi}_t$-invariant subbundles $\mathbf{E}^{\mathrm{c}} \oplus \mathbf{E}^{\mathrm{ss}} \oplus \mathbf{E}^{\mathrm{su}}$, where $\mathbf{E}^{\mathrm{c}}$ is the 1-dimensional subbundle tangent to the flow and $\mathbf{E}^{\mathrm{ss}}$ and $\mathbf{E}^{\mathrm{su}}$ are the strongly contracting and expanding subbundles, respectively, so that there are constants $C, c > 0$ such that

i) $\|D\mathbf{\Phi}_t\mathbf{w}\| \leqslant Ce^{-ct}\|\mathbf{w}\|$ for $\mathbf{w} \in \mathbf{E}^{\mathrm{ss}}$, $t > 0$.
ii) $\|D\mathbf{\Phi}_t^{-1}\mathbf{w}\| \leqslant Ce^{-ct}\|\mathbf{w}\|$ for $\mathbf{w} \in \mathbf{E}^{\mathrm{su}}$, $t > 0$.

The sub-tangent bundles $\mathbf{E}^{\mathrm{ss}}, \mathbf{E}^{\mathrm{su}}, \mathbf{E}^{\mathrm{c}}$ are the so-called *stable, unstable,* and *central bundles.*

The subbundles $\mathbf{E}^{\mathrm{ss}}$ and $\mathbf{E}^{\mathrm{su}}$ have their characterizations using Jacobi tensors. Assume $\mathbf{v} = (x, \mathrm{v}) \in S\widetilde{M}$. For each $s > 0$, let $S_{\mathbf{v},s}$ be the Jacobi tensor along $\gamma_{\mathbf{v}}$ with the boundary conditions $S_{\mathbf{v},s}(0) = \mathrm{Id}$ and $S_{\mathbf{v},s}(s) = 0$. Since $(\widetilde{M}, \widetilde{g})$ has no conjugate points, the limit $\lim_{s\to+\infty} S_{\mathbf{v},s} =: S_{\mathbf{v}}$ exists ([**Esc**]) and is called the *stable tensor* along the geodesic $\gamma_{\mathbf{v}}$. Similarly, by reversing the time s, we obtain the *unstable tensor* $U_{\mathbf{v}}$ along the geodesic $\gamma_{\mathbf{v}}$. The stable subbundle $\mathbf{E}^{\mathrm{ss}}$ at $\mathbf{v}$ is the graph of the mapping $S'_{\mathbf{v}}(0)$, considered as a map from $\overline{N_0(\gamma_{\mathbf{v}})}$ to V_{v} sending Y to $S'_{\mathbf{v}}(0)Y$, where $\overline{N_0(\gamma_{\mathbf{v}})} := \{\mathbf{w}, \mathbf{w} \in H_{\mathrm{v}}, \mathbf{w} \perp \overline{X}_{\mathbf{v}}\}$. Similarly, the unstable subbundle $\mathbf{E}^{\mathrm{su}}$ at $\mathbf{v}$ is the graph of the mapping $U'_{\mathbf{v}}(0)$ considered as a map from $\overline{N_0(\gamma_{\mathbf{v}})}$ to V_{v}.

Due to the Anosov property of the geodesic flow, the distributions of $\mathbf{E}^{\mathrm{ss}}, \mathbf{E}^{\mathrm{su}}$ (and hence $\mathbf{E}^{\mathrm{c}} \oplus \mathbf{E}^{\mathrm{ss}}, \mathbf{E}^{\mathrm{c}} \oplus \mathbf{E}^{\mathrm{su}}$) are Hölder continuous ([**Ano**], see also [**Ba**, Proposition 4.4]). Hence, the $(1,1)$-tensors $S_{\mathbf{v}}, S'_{\mathbf{v}}, U_{\mathbf{v}}, U'_{\mathbf{v}}$ are also Hölder continuous with respect to $\mathbf{v}$.

Associated with the bundle $\mathbf{E}^{\mathrm{cs}} := \mathbf{E}^{\mathrm{c}} \oplus \mathbf{E}^{\mathrm{ss}}$ are the (weak) *stable manifolds* of $\mathbf{\Phi}_t$:

$$W^{\mathrm{s}}(x,\xi) := \left\{ (y,\eta) \in \widetilde{M} \times \partial\widetilde{M} : \ \limsup_{t\to+\infty} \frac{1}{t} \ln \mathrm{dist}\left(\mathbf{\Phi}_t(y,\eta), \mathbf{\Phi}_t(x,\xi)\right) \leqslant 0 \right\}. \tag{2.1}$$

Each $W^{\mathrm{s}}(x,\xi)$ coincides with the collection of the initial speed vectors of the geodesics asymptotic to ξ and can be identified with $\widetilde{M}$. Associated with $\mathbf{E}^{\mathrm{ss}}$ are the *strong stable manifolds*

$$W^{\mathrm{ss}}(x,\xi) := \left\{ (y,\eta) \in \widetilde{M} \times \partial\widetilde{M} : \ \limsup_{t\to+\infty} \frac{1}{t} \ln \mathrm{dist}\left(\mathbf{\Phi}_t(y,\eta), \mathbf{\Phi}_t(x,\xi)\right) < 0 \right\}. \tag{2.2}$$

Each $W^{\mathrm{ss}}(x,\xi)$, locally, is a C^{k-1} graph from $\mathbf{E}^{\mathrm{ss}}_{(x,\xi)}$ to $\mathbf{E}^{\mathrm{c}}_{(x,\xi)} \oplus \mathbf{E}^{\mathrm{su}}_{(x,\xi)}$ and is tangent to $\mathbf{E}^{\mathrm{ss}}_{(x,\xi)}$ ([**SFL**]). It is true that

$$\mathbf{\Phi}_t\left(W^{\mathrm{ss}}(x,\xi)\right) = W^{\mathrm{ss}}\left(\mathbf{\Phi}_t(x,\xi)\right)$$

and the union of these images is just the stable manifold, i.e.,

$$W^{\mathrm{s}}(x,\xi) = \bigcup_{t\in\mathbb{R}} \mathbf{\Phi}_t\left(W^{\mathrm{ss}}(x,\xi)\right).$$

The weak and strong unstable manifolds, denoted by $W^{\mathrm{u}}(x,\xi)$ and $W^{\mathrm{su}}(x,\xi)$, respectively, can be defined similarly as in (2.1) and (2.2) by reversing the time. They have tangents $\mathbf{E}^{\mathrm{cu}} := \mathbf{E}^{\mathrm{c}} \oplus \mathbf{E}^{\mathrm{su}}$ and $\mathbf{E}^{\mathrm{su}}$, respectively.

The geodesic flow $\mathbf{\Phi}_t$ on $S\widetilde{M}$ naturally descends to the geodesic flow Φ_t on g-unit tangent bundle SM, carrying all the tangent splitting and the corresponding submanifolds downstairs. Indeed, the action of G on the tangent bundle $\mathbf{E}$ (where $\mathbf{E}$ denotes any one of $\mathbf{E}^{\mathrm{ss}}, \mathbf{E}^{\mathrm{su}}$ and $\mathbf{E}^{\mathrm{c}}$) satisfies $\psi(\mathbf{E}(x,\xi)) = \mathbf{E}(D\psi(x,\xi))$ for all $\psi \in G$ so that it defines the $D\Phi_t$-invariant subbundles $E^{\mathrm{ss}}, E^{\mathrm{su}}, E^{\mathrm{c}}$ of TSM, the so-called stable, unstable, and central bundles. We see that E^{c} is tangent to the flow direction and $E^{\mathrm{ss}}, E^{\mathrm{su}}$ are such that

i) $\|D\Phi_t w\| \leqslant Ce^{-ct}\|w\|$ for $w \in E^{\mathrm{ss}}$, $t > 0$.
ii) $\|D\Phi_t^{-1} w\| \leqslant Ce^{-ct}\|w\|$ for $w \in E^{\mathrm{su}}$, $t > 0$.

Similarly, the action of G on the submanifolds W (where W denotes any one of $W^{\mathrm{s}}, W^{\mathrm{ss}}, W^{\mathrm{u}}$ and W^{su}) satisfies $\psi(W(x,\xi)) = W(D\psi(x,\xi))$ for all $\psi \in G$ so that it defines the stable, strong stable, unstable and strong unstable manifolds of the geodesic flow on SM, which have tangents $E^{\mathrm{ss}} \oplus E^{\mathrm{c}}$, E^{ss}, $E^{\mathrm{su}} \oplus E^{\mathrm{c}}$ and E^{su}, respectively. In particular, the collection of $W^{\mathrm{s}}(x,\xi)$ defines a foliation $\mathcal{W} = \{W^{\mathrm{s}}(v)\}_{v \in SM}$ on SM, the so-called *stable foliation* of SM. Each $W^{\mathrm{s}}(x,\xi)$ can be identified with $\widetilde{M} \times \{\xi\}$. Hence the quotients $W^{\mathrm{s}}(v)$ are naturally endowed with the Riemannian metric induced from $\tilde{g}$. They are C^{k-1} immersed submanifold of SM depending continuously on v in the C^{k-1} topology ([**SFL**]).

2.3. **Harmonic measure for the stable foliation.** We continue to assume $g \in \mathfrak{R}^k(M)$. Associated with the stable foliation $\mathcal{W}$ is the harmonic measure which is closely related to the leafwise Brownian motion. Write $\Delta^{\mathcal{W}}$ for the leafwise Laplace operator of $\mathcal{W}$, which acts on functions that are of class C^2 along the leaves of $\mathcal{W}$. A probability measure $\mathbf{m}$ on SM is called *harmonic* if it satisfies, for any C^2 function f on SM,

$$\int_{SM} \Delta^{\mathcal{W}} f \; d\mathbf{m} = 0.$$

Since (M, g) is negatively curved, there is a unique harmonic measure $\mathbf{m}$ associated to the stable foliation ([**Ga**]). Let $\widetilde{\mathbf{m}}$ be the G-invariant extension of $\mathbf{m}$ to $S\widetilde{M}$. It is closely related to the Brownian motion on the stable leaves. For $(x,\xi) \in S\widetilde{M}$, let

$$\mathbf{p}(t, (x,\xi), d(y,\eta)) := p(t,x,y) \; d\mathrm{Vol}_{\tilde{g}}(y)\delta_\xi(\eta),$$

where $\delta_\xi(\eta)$ is the Dirac function at ξ. Then $\mathbf{p}$ is just the transition probability function of the Brownian motion on $W^s(x,\xi) = \widetilde{M} \times \{\xi\}$ starting from (x,ξ). Let $\widetilde{\Omega}_+$ be the space of continuous paths $\omega : [0, +\infty) \to S\widetilde{M}$ equipped with the smallest σ-algebra for which the projections $R_t : \omega \mapsto \omega(t)$ are measurable. Let $\{\mathbb{P}_{(x,\xi)}\}$ be the corresponding Markovian family of $\mathbf{p}$ on Ω_+. Then for every $t > 0$ and every Borel set $A \subset \widetilde{M} \times \partial\widetilde{M}$,

$$\widetilde{\mathbb{P}}_{(x,\xi)} \left(\{\omega \in \Omega_+ : \omega(t) \in A\} \right) = \int_A \mathbf{p}(t, (x,\xi), d(y,\eta)).$$

Proposition 2.1. *([**Ga**]) The following hold true.*

i) *The measure $\widetilde{\mathbf{m}}$ satisfies, for any $f \in C^2(\widetilde{M} \times \partial\widetilde{M})$ with compact support,*

$$\int_{\widetilde{M}\times\partial\widetilde{M}} \left(\int_{\widetilde{M}\times\partial\widetilde{M}} f(y,\eta)\mathbf{p}(t,(x,\xi),d(y,\eta)) \right) \; d\widetilde{\mathbf{m}}(x,\xi) = \int_{\widetilde{M}\times\partial\widetilde{M}} f(x,\xi) \; d\widetilde{\mathbf{m}}(x,\xi).$$

ii) *The measure $\widetilde{\mathbb{P}} = \int \widetilde{\mathbb{P}}_{(x,\xi)} \; d\widetilde{\mathbf{m}}(x,\xi)$ on $\widetilde{\Omega}_+$ is invariant under every t-time shift mapping $\sigma_t : \widetilde{\Omega}_+ \to \widetilde{\Omega}_+$, $\sigma_t(\tilde{\omega}(s)) = \tilde{\omega}(s+t)$, for $s > 0$ and $\tilde{\omega} \in \widetilde{\Omega}_+$.*

iii) *The measure* $\widetilde{\mathbf{m}}$ *can be expressed locally at* $(x,\xi) \in \widetilde{M} \times \partial\widetilde{M}$ *as* $d\widetilde{\mathbf{m}} = dx \times d\widetilde{\mathbf{m}}_x$, *where* dx *is proportional to the volume element and* $\widetilde{\mathbf{m}}_x$ *is the hitting probability at* $\partial\widetilde{M}$ *of the Brownian motion starting at* x.

The group G acts naturally and discretely on the space $\widetilde{\Omega}_+$ with quotient the space Ω_+ of continuous paths in SM, and this action commutes with the shift $\sigma_t, t \geqslant 0$. Therefore, the measure $\widetilde{\mathbb{P}}$ is the extension of a finite, shift invariant measure $\mathbb{P}$ on Ω_+. We identify SM with $M_0 \times \partial\widetilde{M}$, where M_0 is a connected fundamental domain of $(\widetilde{M}, \widetilde{g})$. Hence we can also identify Ω_+ with the lift of its elements in $\widetilde{\Omega}_+$ starting from M_0. We will continue to denote elements in Ω_+ by ω and will clarify the notation whenever there is an ambiguity. In this paper, we normalize the harmonic measure $\mathbf{m}$ to be a probability measure, so that $\mathbb{P}$ is also a probability measure. We denote by $\mathbb{E}_{\mathbb{P}}$ the corresponding expectation symbol.

A nice property for the laminated Brownian motion is that the semi-group $\sigma_t, t \geqslant 0$, of transformations of Ω_+ has strong ergodic properties with respect to the probability $\mathbb{P}$.

Proposition 2.2. *(*[**Ga**]*, cf.* [**LS1**, Proposition 2.3]*) The shift semi-flow* $\sigma_t, t \geqslant 0$, *is mixing on* $(\Omega_+, \mathbb{P})$ *in the sense for any bounded measurable functions* f_1, f_2 *on* Ω_+,

$$\lim_{t\to+\infty} \mathbb{E}_{\mathbb{P}}(f_1\,(f_2 \circ \sigma_t)) \;=\; \mathbb{E}_{\mathbb{P}}(f_1)\mathbb{E}_{\mathbb{P}}(f_2).$$

2.4. **Busemann function and the linear drift.** In this subsection, we derive (1.1).

Let $g \in \Re^k(M)$. For $\mathbf{v} = (x,\xi) \in M_0 \times \partial\widetilde{M}$, the projection on $\widetilde{M}$ of the law of $\mathbb{P}_{\mathbf{v}}$ on $W^s(x,\xi) = \widetilde{M} \times \{\xi\}$ is the same as that of $\mathbb{P}_x$ of the Brownian motion on $\widetilde{M}$ starting from x. For $\omega \in \Omega_+$, we still denote by ω its projection to $\widetilde{M}$. By ergodicity of $\mathbb{P}$ with respect to the shift map σ_t (Proposition 2.2), for $\mathbb{P}$-almost all path $\omega \in \Omega_+$, its leafwise linear drift coincides with ℓ.

Since $\widetilde{g}$ is negatively curved, for $\mathbb{P}$-almost all path ω, $\omega(t)$ tends to a point in the geometric boundary $\partial\widetilde{M}$ ([**K1**]). Write $\omega(\infty) := \lim_{t\to+\infty} \omega(t)$. Roughly speaking, ω follows $\gamma_{\omega(0),\omega(\infty)}$. Hence the drift of $\omega(t)$ from $\omega(0)$ can be measured via its shadow on $\gamma_{\omega(0),\omega(\infty)}$. A candidate function for this measurement is the Busemann function. Let $x_0 \in \widetilde{M}$ be a reference point. For $y, z \in \widetilde{M}$, define

$$b_{x_0,z}(y) := d(y,z) - d(x_0,z).$$

The assignment of $y \mapsto b_{x_0,y}$ is continuous, one-to-one and takes value in a relatively compact set of functions for the topology of uniform convergence on compact subsets of $\widetilde{M}$. The Busemann compactification of $\widetilde{M}$ is the closure of $\widetilde{M}$ for that topology ([**BGS**]) and it coincides with the geometric compactification in the negative curvature case (see [**Ba**]). So for each $\mathbf{v} = (x,\xi) \in \widetilde{M} \times \partial\widetilde{M}$, the function

$$b_{\mathbf{v}}(z) := \lim_{y\to\xi} b_{x,y}(z), \text{ for } z \in \widetilde{M},$$

is well-defined and is called the *Busemann function at* $\mathbf{v}$. It is known ([**EO**]) that, if we consider $b_{\mathbf{v}}$ as a function defined on $W^s(x,\xi)$, then

$$\nabla b_{\mathbf{v}}(z) = -\overline{X}(z,\xi). \tag{2.3}$$

The difference between $b_{\mathbf{v}}(y)$ and $b_{\mathbf{v}}(y')$ is preserved when (y,ξ) and (y',ξ) are driven by the geodesic flow Φ_t. Hence

$$W^{\rm ss}(\mathbf{v}) = \{(y,\xi) :\ b_{\mathbf{v}}(y) = b_{\mathbf{v}}(x)\}.$$

Note that $W^{\rm ss}(\mathbf{v})$ locally is a C^{k-1} graph from $\mathbf{E}^{\rm ss}_{\mathbf{v}}$ to $\mathbf{E}^{\rm c}_{\mathbf{v}} \oplus \mathbf{E}^{\rm su}_{\mathbf{v}}$ and is tangent to $\mathbf{E}^{\rm ss}_{\mathbf{v}}$. So, by the Jacobi tensor characterization of $\mathbf{E}^{\rm ss}_{\mathbf{v}}$ and (2.3), it is true ([**Esc**, **HIH**]) that

$$\nabla_{\mathbf{w}}(\nabla b_{\mathbf{v}})(x) = -S'_{\mathbf{v}}(0)(\mathbf{w}),\ \forall \mathbf{w} \in T_x\widetilde{M}.$$

Thus,

$$\Delta_x b_{\mathbf{v}} = -{\rm Div}\overline{X} = -\text{Trace of } S'_{\mathbf{v}}(0), \tag{2.4}$$

which is the mean curvature of the set of footpoints of $W^{\rm ss}(x,\xi)$. Note that for each $\psi \in G$,

$$b_{(x_0,\psi\xi)}(\psi x) = b_{(x_0,\xi)}(x) + b_{(\psi^{-1}x_0,\xi)}(x_0).$$

Hence $\Delta_x b_{(x_0,\xi)}$ satisfies $\Delta_{\psi x} b_{(x_0,\psi\xi)} = \Delta_x b_{(x_0,\xi)}$ and defines a function $\mathtt{B}$ on the unit tangent bundle SM, which is called the *Laplacian of the Busemann function*. The function $\mathtt{B}$ is a Hölder continuous function on SM by the Hölder continuity of the strong stable tangent bundles ([**Ano**], see Section 2.2).

Now, we can derive the integral formula of the linear drift using the geodesic spray and the harmonic measure ([**K1**]). For $\mathbb{P}$-almost all path $\omega \in \Omega_+$, let $\mathbf{v} := \omega(0)$ and $\eta := \omega(\infty) \in \partial\widetilde{M}$. When t goes to infinity, the process $b_{\mathbf{v}}(\omega(t)) - d(x,\omega(t))$ converges $\mathbb{P}$-a.e. to the a.e. finite number $-2(\xi|\eta)_x$, where the Gromov product $(\cdot|\cdot)_x$ is such that

$$(\xi|\eta)_x := \lim_{y\to\xi, z\to\eta} (y|z)_x \text{ and } (y|z)_x := \frac{1}{2}\left(d(x,y) + d(x,z) - d(y,z)\right). \tag{2.5}$$

So for $\mathbb{P}$-almost all $\omega \in \Omega_+$, we have

$$\lim_{t\to+\infty} \frac{1}{t} b_{\mathbf{v}}(\omega(t)) = \ell.$$

Using the fact that the leafwise Brownian motion has generator Δ and is ergodic with invariant measure $\mathbf{m}$ on SM, we obtain

$$\begin{aligned}
\ell &= \lim_{t\to+\infty} \frac{1}{t}\int_0^t \frac{\partial}{\partial s} b_{\mathbf{v}}(\omega(s))\ ds \\
&= \lim_{t\to+\infty} \frac{1}{t}\int_0^t \Delta b_{\mathbf{v}}(\omega(s))\ ds \ \left(= \int_{M_0\times\partial\widetilde{M}} \Delta b_{\mathbf{v}}\ \ d\mathbf{m}\right) \\
&= -\int_{M_0\times\partial\widetilde{M}} {\rm Div}^{\mathcal{W}}(\overline{X})\ d\mathbf{m}, \qquad (2.6)
\end{aligned}$$

where ${\rm Div}^{\mathcal{W}}$ is the laminated divergence operator for the stable foliation $\mathcal{W}$. Since on each leaf we have ${\rm Div}^{\mathcal{W}}(\overline{X}) = {\rm Div}(X)$, (2.6) reduces to (1.1). But that will not simplify

the discussion of the regularity of the linear drift under metric changes since $\mathrm{Div}(X)$ is essentially a leafwise object. In contrast, (2.6) is more suitable for this purpose because of the natural connection between the geodesic spray $\overline{X}$ and the geodesic flow.

3. Regularity of the linear drift

In this section, we assume Theorem 1.3 holds true. We first prove Theorem 1.1 by showing the regularities of $\mathrm{Div}^{\mathcal{W}}\overline{X}$ and $\mathbf{m}$ under a one-parameter of C^k deformation of metrics in $\mathfrak{R}^k(M)$ and then prove Theorems 1.4 and 1.5.

3.1. **Regularity of the leafwise divergence term $\mathrm{Div}^{\mathcal{W}}\overline{X}$.** Clearly, the geodesic sprays of a metric $g \in \mathfrak{R}^k(M)$ form a C^{k-1} vector field which varies C^{k-1} with respect to C^k metric change. But this does not imply the regularity of $\mathrm{Div}^{\mathcal{W}}\overline{X}$ with respect to the metric changes since we are considering the leafwise divergence.

Laminate $S\widetilde{M} = \widetilde{M} \times \partial\widetilde{M}$ into stable leaves $\{W^{\mathrm{s}}(x,\xi) = \widetilde{M} \times \{\xi\}\}$, where each leaf can be identified with $(\widetilde{M}, \widetilde{g})$, but is only Hölder continuous in the ξ-coordinate (see Section 2.2). Consequently, $\overline{X}(y,\xi) \in TW^{\mathrm{s}}(x,\xi)$ is C^{k-1} in the y-coordinate, but is only Hölder continuous in the ξ-coordinate. Let $g' \in \mathfrak{R}^k(M)$ be another metric. Its geometric boundary $\partial\widetilde{M}_{g'}$ can be identified with $\partial\widetilde{M}$. But the $\widetilde{g}'$-geodesic spray $\overline{X}_{\widetilde{g}'}(x,\xi)$ differs from $\overline{X}(x,\xi)$ and the divergence operator on the $\widetilde{g}'$-stable leaf $W^{\mathrm{s}}_{\widetilde{g}'}(x,\xi)$ differs from that on $W^{\mathrm{s}}(x,\xi)$. Both difference contribute to the change of $(\mathrm{Div}^{\mathcal{W}}\overline{X})(x,\xi)$ in metrics. This, by (2.4), can be understood by a study of the regularity of $\overline{X}$ and $\mathbf{E}^{\mathrm{ss}}$ in $\mathfrak{R}^k(M)$.

Assume $g \in \mathfrak{R}^k(M)$. The set of $\widetilde{g}$-oriented geodesics in $\widetilde{M}$ can be identified with $\partial^2\widetilde{M} := (\partial\widetilde{M} \times \partial\widetilde{M})\backslash\{(\xi,\xi) : \ \xi \in \partial\widetilde{M}\}$. Indeed, for $(x,\xi) \in S\widetilde{M}$, let $\gamma : \mathbb{R} \mapsto \widetilde{M}$ be the unique geodesic with $\dot\gamma(0) = (x,\xi)$ and write $\partial^+\gamma := \lim_{t\to+\infty}\gamma(t)$ and $\partial^-\gamma := \lim_{t\to-\infty}\gamma(t)$. The mapping $\gamma \mapsto (\partial^+\gamma, \partial^-\gamma)$ establishes a homeomorphism between the set of all oriented geodesics in $(\widetilde{M}, \widetilde{g})$ and $\partial^2\widetilde{M}$. Consequently, for any $g' \in \mathfrak{R}^k(M)$, the mapping $D_{g'} : \partial^2(\widetilde{M}) \to \partial^2(\widetilde{M}_{\widetilde{g}'})$ induced from the identity isomorphism from G to itself can be viewed as a homeomorphism between the set of oriented geodesics in $(\widetilde{M}, \widetilde{g})$ and $(\widetilde{M}, \widetilde{g}')$. Further realize points from $S\widetilde{M}_{\widetilde{g}'}$ by pairs (γ, y), where γ is an oriented geodesic and $y \in \gamma$. For g' close to g, we obtain a map $\widetilde{F}_{g'} : S\widetilde{M} \to S\widetilde{M}_{\widetilde{g}'}$ which sends $(\gamma, y) \in S\widetilde{M}$ to

$$\widetilde{F}_{g'}(\gamma, y) = (D_{g'}(\gamma), y'),$$

where y' is the unique intersection point of $D_{g'}(\gamma)$ and the hypersurface $\{\exp_{\widetilde{g}} Y : \ Y \perp \mathbf{v}\}$ with $\mathbf{v}$ being the vector in $S_y\widetilde{M}$ pointing at $\partial^+\gamma$. The map $\widetilde{F}_{g'}$ is a homeomorphism between $S\widetilde{M}$ and $S\widetilde{M}_{\widetilde{g}'}$ which preserves the geodesics, i.e., sending $\widetilde{g}$-geodesics to $\widetilde{g}'$-geodesics, and is referred to as a $(\widetilde{g}, \widetilde{g}')$-*Morse correspondence map.* The restriction of $\widetilde{F}_{g'}$ to geodesics asymptotic to $\xi \in \partial\widetilde{M}$ is a homeomorphism from $W^{\mathrm{s}}_{\widetilde{g}}(x,\xi)$ to $W^{\mathrm{s}}_{\widetilde{g}'}(x,\xi)$. Let $\widetilde{\pi}_{g'} : S\widetilde{M}_{\widetilde{g}'} \to$

$S\widetilde{M}$ be the map sending $\mathbf{v}$ to $\mathbf{v}/\|\mathbf{v}\|_{\tilde{g}}$ which records the direction information points of $S\widetilde{M}_{\tilde{g}'}$ in $S\widetilde{M}$. Then $\widetilde{\pi}_{g'} \circ \widetilde{F}_{g'}$ is a homeomorphism between $S\widetilde{M}$ and itself.

The map $\widetilde{F}_{g'}$ induces a homeomorphism $F_{g'}$ between SM and $SM_{g'}$ which sends g-geodesics to g'-geodesics and is called a (g, g')-*Morse correspondence map*. For any sufficiently small ϵ, if g' is sufficiently close to g, then $F_{g'}$ is such that the footpoint of $F_{g'}(v)$ belongs to the hypersurface of points $\{\exp_g Y : Y \perp v,\ \|Y\|_g < \epsilon\}$, where v is the projection of $\mathbf{v}$ in SM. Let $\pi_{g'} : SM_{g'} \to SM$ be the natural projection map sending v to $v/\|v\|_g$. Then $\pi_{g'} \circ F_{g'}$ is a homeomorphism between SM and itself.

For g' in a small neighborhood of g in $\Re^k(M)$, let $\mathbf{E}_{g'}$ (resp. $E_{g'}$) denote any one of $\mathbf{E}^{\mathrm{ss}}_{g'}, \mathbf{E}^{\mathrm{su}}_{g'}$ and $\mathbf{E}^{\mathrm{c}}_{g'}$ (resp. any one of $E^{\mathrm{ss}}_{g'}, E^{\mathrm{su}}_{g'}$ and $E^{\mathrm{c}}_{g'}$). We also regard $\mathbf{E}_{g'}$ (resp. $E_{g'}$) as a mapping from $S\widetilde{M}_{\tilde{g}'}$ (resp. $SM_{g'}$) to its tangent bundle. Of our special interest, is the regularity of the mappings $g' \mapsto \widetilde{\pi}_{g'} \circ \widetilde{F}_{g'}$, $g' \mapsto D\widetilde{\pi}_{g'} \circ \mathbf{E}_{g'}$. Equivalently, we can consider the regularity of the downstairs mappings $g' \mapsto \pi_{g'} \circ F_{g'}$ and $g' \mapsto D\pi_{g'} \circ E_{g'}$, for which, we can take advantage of the compactness of M to construct certain manifolds of maps so that the implicit function theory applies ([**LMM**, **KKPW**]).

Let $\mathcal{H}^{k-1}(SM)$ be the Banach space of C^{k-1} vector fields on SM endowed with the topology of uniform C^{k-1} convergence on compact subsets. Let $\underline{X}_g$ be the vector field generating the g-geodesic flow. Then $\underline{X}_{g'}$, the projection (via $D\pi_{g'}$) of the generating vector field of the g'-geodesic flow on $SM_{g'}$, belongs to $\mathcal{H}^{k-1}(SM)$ and is C^{k-1} close to $\underline{X}_g$ whenever g' is C^k close to g. For $\alpha \in [0, 1)$, let $C^\alpha(SM, N)$ denote the Banach space of α-Hölder (or continuous for $\alpha = 0$) maps from SM to a Banach space N endowed with the topology given by the α-Hölder norm on SM. Consider

$$C^\alpha_\Phi(SM, SM) := \left\{ F \in C^\alpha(SM, SM) :\ D_\Phi F(v) := \frac{d}{dt} F(\Phi_t(v))\big|_{t=0} \text{ exists and is } \alpha\text{-Hölder} \right\},$$

with the topology of the norm $\|F\| + \|D_\Phi F\|_\alpha$, where $\|\cdot\|_\alpha$ denotes the α-Hölder norm, together with the mapping

$$\Psi :\ \mathcal{H}^{k-1}(SM) \times C^\alpha_\Phi(SM, SM) \times C^\alpha(SM, \mathbb{R}) \to C^\alpha(SM, TSM)$$
$$\Psi(Y, F, f) = Y \circ F - f \cdot D_\Phi F.$$

By hyperbolicity of the g-geodesic flow Φ_t, the implicit function theory applies to Ψ if we further require $F \in C^\alpha_\Phi(SM, SM)$ to be such that the footpoint of $F(v)$ lies in $\{\exp_g(w) :\ w \perp v\}$ for any $v \in SM$. The following structural stability theorem is due to de la Llave-Marco-Moriyón ([**LMM**]) for continuous case and Katok-Knieper-Pollicott-Weiss ([**KKPW**]) for Hölder continuous case.

Proposition 3.1. *(*[**KKPW**, Proposition 2.2]*) For $g \in \Re^k(M)$, there exist $\alpha \in (0, 1)$ and a neighborhood $\mathcal{U} \subset \mathcal{H}^{k-1}(SM)$ of $\underline{X}_g$ and C^{k-2} maps $\mathcal{U} \to C^\alpha_\Phi(SM, SM) :\ Y \mapsto F_Y$ and $\mathcal{U} \to C^\alpha\left(SM, [\frac{1}{2}, +\infty)\right) :\ Y \mapsto f_Y$ such that $Y \circ F_Y = f_Y D_\Phi F$. Moreover, the maps $\mathcal{U} \to C^0_\Phi(SM, SM) :\ Y \mapsto F_Y$ and $\mathcal{U} \to C^0\left(SM, [\frac{1}{2}, +\infty)\right) :\ Y \mapsto f_Y$ are C^{k-1}.*

Define $C^\alpha(S\widetilde{M},N), C^\alpha_\Phi(S\widetilde{M},N)$ analogously as $C^\alpha(SM,N)$, $C^\alpha_\Phi(SM,N)$. A consequence of Proposition 3.1 is

Corollary 3.2. *Assume $g\in\Re^k(M)$. There exist $\alpha\in(0,1)$ and a neighborhood $\mathcal{V}$ of g in $\Re^k(M)$ such that the map $g'\in\mathcal{V}\mapsto \pi_{g'}\circ F_{g'}$ is C^{k-2} into $C^\alpha_\Phi(SM,SM)$ and is C^{k-1} into $C^0_\Phi(SM,SM)$; the map $g'\in\mathcal{V}\mapsto \widetilde{\pi}_{g'}\circ \widetilde{F}_{g'}$ is C^{k-2} into $C^\alpha_\Phi(S\widetilde{M},S\widetilde{M})$ and is C^{k-1} into $C^0_\Phi(S\widetilde{M},S\widetilde{M})$.*

The regularity of $g'\mapsto D\pi_{g'}\circ E_{g'}$ and $g'\mapsto D\widetilde{\pi}_{g'}\circ \mathbf{E}_{g'}$ can be analyzed analogously ([**Co**]). Let $\mathcal{G}$ be the Grassmann bundle of u-planes on TSM, where $u=\dim E^{\rm su}_g$. Let $C^\alpha_\Phi(SM,\mathcal{G})$ be the space of α-Hölder maps $\widehat{F}: SM\to\mathcal{G}, \widehat{F}(v)=(F(v),E(v))$, where $F\in C^\alpha_\Phi(SM,SM)$, with the topology of the α-Hölder norm on $F, D_\Phi F$ and E. Then instead of Ψ, one can consider the maps

$$\widehat{\Psi}_\pm:\ \mathcal{H}^{k-1}(SM)\times C^\alpha_\Phi(SM,\mathcal{G})\times C^\alpha(SM,\mathbb{R})\to C^\alpha(SM,TSM\oplus\mathcal{G})$$
$$\widehat{\Psi}_\pm(Y,\widehat{F},f)=\left(Y\circ F-f\cdot D_\Phi F,\ D\psi_{\tau_Y(v)}\circ F(\Phi_{\pm1}(v))E(\Phi_{\pm1}(v))\right),$$

where ψ_t is the time t map of the flow generated by Y and τ_Y is the time change such that

$$\psi_{\tau_Y(v)}\circ F_Y(\Phi_{\pm1}(v))=F_Y(v),\ \forall v\in SM.$$

Again, by hyperbolicity of the flow generated by Y which is close to $\underline{X}_g$ and the invariance of the corresponding strong stable and unstable bundles, denoted by $E^{\rm ss}_Y, E^{\rm su}_Y$, the implicit function theory applies for $\widehat{\Psi}_+, \widehat{\Psi}_-$ and gives the following.

Proposition 3.3. *([**Co**, Proposition 2.1]) For $g\in\Re^k(M)$, there exist a neighborhood $\mathcal{U}$ of $\underline{X}_g$ in $\mathcal{H}^{k-1}(SM)$ and $\alpha\in(0,1)$ such that the map $\mathcal{U}\to C^\alpha_\Phi(SM,\mathcal{G}):\ Y\mapsto(v\mapsto E_Y\circ F_Y(v))$ is C^{k-3} and the map $\mathcal{U}\to C^0_\Phi(SM,\mathcal{G}):\ Y\mapsto E_Y\circ F_Y$ is C^{k-2}, where $E_Y=E^{\rm ss}_Y$ or $E^{\rm su}_Y$.*

Let $\widetilde{\mathcal{G}}$ be the Grassmann bundle of u-planes on $TS\widetilde{M}$ (where $u=\dim\mathbf{E}^{\rm su}_g$) and define $C^\alpha_\Phi(S\widetilde{M},\widetilde{\mathcal{G}})$ in analogy with $C^\alpha_\Phi(SM,\mathcal{G})$. The following is an application of Proposition 3.3 to the geodesic flows.

Corollary 3.4. *There exist $\alpha\in(0,1)$ and a neighborhood $\mathcal{V}$ of g in $\Re^k(M)$ such that the map $g'\in\mathcal{V}\mapsto D\pi_{g'}\circ E_{g'}\circ F_{g'}$ is C^{k-3} into $C^\alpha_\Phi(SM,\mathcal{G})$ and is C^{k-2} into $C^0_\Phi(SM,\mathcal{G})$, where $E_{g'}$ is any one of $E^{\rm ss}_{g'}, E^{\rm su}_{g'}$ and $E^{\rm c}_{g'}$. Similarly, the map $g'\in\mathcal{V}\mapsto D\widetilde{\pi}_{g'}\circ\mathbf{E}_{g'}\circ\widetilde{F}_{g'}$ is C^{k-3} into $C^\alpha_\Phi(S\widetilde{M},\widetilde{\mathcal{G}})$ and is C^{k-2} into $C^0_\Phi(S\widetilde{M},\widetilde{\mathcal{G}})$, where $\mathbf{E}_{g'}$ is any one of $\mathbf{E}^{\rm ss}_{g'}, \mathbf{E}^{\rm su}_{g'}$ and $\mathbf{E}^{\rm c}_{g'}$.*

For $\lambda\in(-1,1)\mapsto g^\lambda\in\Re^k(M)$, we write $\overline{X}^\lambda$ for the $\widetilde{g}^\lambda$-geodesic spray, $(\mathbf{E}^{\rm ss})^\lambda$ for the $\widetilde{g}^\lambda$-stable bundle and Div^λ for the divergence operator associated with the $\widetilde{g}^\lambda$-stable foliation.

Proposition 3.5. *Let $g\in\Re^k(M)$. There exist $\alpha\in(0,1)$ and a neighborhood $\mathcal{V}_g$ of g in $\Re^k(M)$ such that for any C^k curve $\lambda\in(-1,1)\mapsto g^\lambda\in\mathcal{V}_g$ with $g^0=g$,*

i) $\lambda \mapsto \overline{X}^{\lambda}$ *is* C^{k-3} *into* $C^{\alpha}(\widetilde{M} \times \partial\widetilde{M}, TT\widetilde{M})$ *and is* C^{k-2} *into* $C^{0}(\widetilde{M} \times \partial\widetilde{M}, TT\widetilde{M})$,
ii) $\lambda \mapsto (\mathbf{E}^{\mathrm{ss}})^{\lambda}$ *is* C^{k-3} *into* $C^{\alpha}(\widetilde{M} \times \partial\widetilde{M}, \widetilde{\mathcal{G}})$ *and is* C^{k-2} *into* $C^{0}(\widetilde{M} \times \partial\widetilde{M}, \widetilde{\mathcal{G}})$, *and*
iii) $\lambda \mapsto \mathrm{Div}^{\lambda}\overline{X}^{\lambda}$ *is* C^{k-3} *into* $C^{\alpha}(\widetilde{M} \times \partial\widetilde{M}, \mathbb{R}^{+})$ *and is* C^{k-2} *into* $C^{0}(\widetilde{M} \times \partial\widetilde{M}, \mathbb{R}^{+})$.

Proof. Express the $(\widetilde{g}, \widetilde{g}^{\lambda})$-Morse correspondence map $\widetilde{F}_{g^{\lambda}}$ from $\widetilde{M} \times \partial\widetilde{M}$ to itself as

$$\widetilde{F}^{\lambda}(x,\xi) = \left(f_{\xi}^{\lambda}(x), \xi\right), \ \forall (x,\xi) \in \widetilde{M} \times \partial\widetilde{M},$$

where f_{ξ}^{λ} records the change of the footpoint for the unit vector pointing at ξ in the boundary. For $(x,\xi) \in \widetilde{M} \times \partial\widetilde{M}$, we transform $\overline{X}_{\widetilde{g}}(x,\xi)$ to $\overline{X}_{\widetilde{g}^{\lambda}}(x,\xi)$ in three steps: the first is to follow the footpoint of the inverse of the (g^{λ}, g)-Morse correspondence from $\overline{X}_{\widetilde{g}}(x,\xi)$ to $\overline{X}_{\widetilde{g}}((f_{\xi}^{\lambda})^{-1}(x), \xi)$ with the constraint that the vector remains within $TW^{\mathrm{s}}(x,\xi)$; the second is to use the (g^{λ}, g)-Morse correspondence from $\overline{X}_{\widetilde{g}}((f_{\xi}^{\lambda})^{-1}(x), \xi)$ to $\overline{X}_{\widetilde{g}^{\lambda}}(x,\xi)/\|\overline{X}_{\widetilde{g}^{\lambda}}(x,\xi)\|_{\widetilde{g}}$; the third is to adjust the length of $\overline{X}_{\widetilde{g}^{\lambda}}(x,\xi)/\|\overline{X}_{\widetilde{g}^{\lambda}}(x,\xi)\|_{\widetilde{g}}$ to be one in the metric $\widetilde{g}^{\lambda}$. Hence,

$$\begin{aligned}
&\overline{X}_{\widetilde{g}^{\lambda}}(x,\xi) - \overline{X}_{\widetilde{g}}(x,\xi) \\
&= \left(\overline{X}_{\widetilde{g}^{\lambda}}(x,\xi) - \frac{\overline{X}_{\widetilde{g}^{\lambda}}(x,\xi)}{\|\overline{X}_{\widetilde{g}^{\lambda}}(x,\xi)\|_{\widetilde{g}}}\right) + \left(\frac{\overline{X}_{\widetilde{g}^{\lambda}}(x,\xi)}{\|\overline{X}_{\widetilde{g}^{\lambda}}(x,\xi)\|_{\widetilde{g}}} - \overline{X}_{\widetilde{g}}((f_{\xi}^{\lambda})^{-1}(x),\xi)\right) \\
&\qquad + \left(\overline{X}_{\widetilde{g}}((f_{\xi}^{\lambda})^{-1}(x),\xi) - \overline{X}_{\widetilde{g}}(x,\xi)\right) \\
&=: (a)_{\lambda} + (b)_{\lambda} + (c)_{\lambda}.
\end{aligned}$$

Note that $(a)_0, (b)_0, (c)_0$ are all zero. So the regularity of $\lambda \mapsto \overline{X}^{\lambda}$ will follow from that of $(a)_{\lambda}, (b)_{\lambda}, (c)_{\lambda}$ by Taylor's formula. This is true since $(a)_{\lambda}$ corresponds to length change and is C^{k} in λ, $(b)_{\lambda}$ is C^{k-2} (or C^{k-3}) in λ depending on $\alpha = 0$ (or not) by Corollary 3.2, while $(c)_{\lambda}$ has the same regularity as $(b)_{\lambda}$ since $\overline{X}(x,\xi)$ is C^{k-1} in the x-coordinate.

Similarly, we write $\mathbf{v}^{\lambda} = \overline{X}^{\lambda}(x,\xi)$ and

$$\begin{aligned}
(\mathbf{E}^{\mathrm{ss}})^{\lambda}(\mathbf{v}^{\lambda}) - (\mathbf{E}^{\mathrm{ss}})^{0}(\mathbf{v}^{0}) = &\left((\mathbf{E}^{\mathrm{ss}})^{\lambda}(\mathbf{v}^{\lambda}) - (\mathbf{E}^{\mathrm{ss}})^{0}((f_{\xi}^{\lambda})^{-1}(x),\xi)\right) \\
&+ \left((\mathbf{E}^{\mathrm{ss}})^{0}((f_{\xi}^{\lambda})^{-1}(x),\xi) - (\mathbf{E}^{\mathrm{ss}})^{0}(\mathbf{v}^{0})\right) \\
=: &\ (d)_{\lambda} + (e)_{\lambda}.
\end{aligned}$$

This means we can transport $(\mathbf{E}^{\mathrm{ss}})^{0}(\mathbf{v}^{0})$ to $(\mathbf{E}^{\mathrm{ss}})^{\lambda}(\mathbf{v}^{\lambda})$ in two steps: first is to transport $(\mathbf{E}^{\mathrm{ss}})^{0}(\mathbf{v}^{0})$ to $(\mathbf{E}^{\mathrm{ss}})^{0}((f_{\xi}^{\lambda})^{-1}(x),\xi)$ along the tangent bundle of $W^{\mathrm{s}}(x,\xi)$ and follow the footpoint of the inverse of the $(\widetilde{g}^{\lambda}, \widetilde{g})$-Morse correspondence; the second is to use the Morse correspondence for the stable bundle from $(\mathbf{E}^{\mathrm{ss}})^{0}((f_{\xi}^{\lambda})^{-1}(x),\xi)$ to $(\mathbf{E}^{\mathrm{ss}})^{\lambda}(\mathbf{v}^{\lambda})$. Note that $(d)_0, (e)_0$ are zero. The regularity of $\lambda \mapsto (\mathbf{E}^{\mathrm{ss}})^{\lambda}$ will follow from that of $(d)_{\lambda}, (e)_{\lambda}$ by Taylor's formula, which will follow by Corollary 3.4 if we can show the C^{k-1} dependence of $\mathbf{E}^{\mathrm{ss}}(x,\xi)$ on the x-coordinate. This is true because each $\mathbf{E}^{\mathrm{ss}}(y,\xi)$ is the tangent plane of the strong stable manifold $W^{\mathrm{ss}}(y,\xi)$. Locally, $W^{\mathrm{ss}}(x,\xi)$ is a C^{k-1} graph from $\mathbf{E}^{\mathrm{ss}}_{(x,\xi)}$ to

$\mathbf{E}^{\rm c}_{(x,\xi)} \oplus \mathbf{E}^{\rm su}_{(x,\xi)}$. This means, locally, $y \mapsto \mathbf{E}^{\rm ss}(y,\xi)$ is C^{k-1} along the leaf $W^{\rm ss}(x,\xi)$. On the other hand, by invariance of the strong stable bundle with respect to the geodesic flow, $y \mapsto \mathbf{E}^{\rm ss}(y,\xi)$ is smooth as y varies on the geodesic passing through x asymptotic to ξ. By invariance of the strong stable leaf by geodesic flow, $W^{\rm ss}(x,\xi)$ and the time direction (i.e. the direction of the geodesic spray) consist of a coordinate chart for $W^{\rm s}(x,\xi)$. This shows, locally at x, $y \mapsto \mathbf{E}^{\rm ss}(y,\xi)$ is C^{k-1} along $W^{\rm s}(x,\xi) = \widetilde{M} \times \{\xi\}$.

Finally iii) is just an application of ii) noting that for any $g \in \mathfrak{R}^k(M)$, we have

$$(\mathrm{Div}\overline{X})(x,\xi) = \text{Trace of } S'_{\mathbf{v}}(0),\ \forall \mathbf{v} = (x,\xi) \in \widetilde{M} \times \partial\widetilde{M},$$

and the stable bundle $\mathbf{E}^{\rm ss}$ at $\mathbf{v}$ is the graph of the mapping $S'_{\mathbf{v}}(0)$, considered as a map from $\overline{N_0(\gamma_{\mathbf{v}})}$ to $V_{\rm v}$ sending Y to $S'_{\mathbf{v}}(0)Y$, where $\overline{N_0(\gamma_{\mathbf{v}})} := \{\mathbf{w}, \mathbf{w} \in H_{\rm v}, \mathbf{w} \perp \overline{X}_{\mathbf{v}}\}$. □

3.2. **Regularity of the harmonic measure.** In this subsection, we prove Theorem 1.2 following the sketch that we gave in the Section 1.

For $g^\lambda \in \mathfrak{R}^k(M)$, we introduce a metric on $\partial\widetilde{M}$ as follows. Let $\varkappa > 0$. For $x \in \widetilde{M}$, define

$$d_x^{\varkappa,\lambda}(\zeta,\eta) := e^{-\varkappa(\zeta|\eta)_x^\lambda},\ \forall \zeta,\eta \in \partial\widetilde{M}, \tag{3.1}$$

where $(\cdot|\cdot)_x^\lambda$ is the Gromov product defined in (2.5) for $d_{\widetilde{g}^\lambda}$. If $\varkappa$ is small, each $d_x^{\varkappa,\lambda}(\cdot,\cdot)$ defines a distance on $\partial\widetilde{M}$, the so-called $\varkappa$-Busemann distance ([**K2**]), which is related to the $\widetilde{g}^\lambda$-Busemann functions b^λ since

$$b^\lambda_{\mathbf{v}}(y) = \lim_{\zeta,\eta\to\xi} \left((\zeta|\eta)_y^\lambda - (\zeta|\eta)_x^\lambda \right),\ \text{for any } \mathbf{v} = (x,\xi) \in S\widetilde{M},\ y \in \widetilde{M}. \tag{3.2}$$

Let $\mathsf{b} > 0$. For continuous functions f on $SM = M_0 \times \partial\widetilde{M}$, define

$$\|f\|^\lambda_{\mathsf{b}} := \sup_{x,\xi} |\widetilde{f}(x,\xi)| + \sup_{x,\xi_1,\xi_2} |\widetilde{f}(x,\xi_1) - \widetilde{f}(x,\xi_2)| e^{\mathsf{b}(\xi_1|\xi_2)_x^\lambda}.$$

Let $\mathcal{H}^\lambda_{\mathsf{b}}$ be the Banach space of continuous functions f on SM with $\|f\|^\lambda_{\mathsf{b}} < +\infty$. Elements of $\mathcal{H}^\lambda_{\mathsf{b}}$ are continuous on SM and Hölder continuous with respect to the direction changes.

Recall that the transition probability of the $\widetilde{g}^\lambda$-Brownian motion on the stable leaf $W^{\rm s}_{\widetilde{g}^\lambda}(x,\xi) = \widetilde{M} \times \{\xi\}$ starting from (x,ξ) is given by

$$\mathbf{p}^\lambda(t,(x,\xi),d(y,\eta)) := p^\lambda(t,x,y)\ d\mathrm{Vol}^\lambda(y)\delta_\xi(\eta),$$

where $\{p^\lambda(t,x,\cdot)\}_{x\in\widetilde{M},t\in\mathbb{R}_+}$ is the transition probabilities of the $\widetilde{g}^\lambda$-Brownian motion on $\widetilde{M}$, $\delta_\xi(\eta)$ is the Dirac function at ξ and Vol^λ is the $\widetilde{g}^\lambda$ volume element. Then $\mathbf{p}^\lambda$ descends to be the transition probability of g^λ-Brownian motion the stable leaves of SM: for $(x,\xi),(y,\eta) \in$

$SM = M_0 \times \partial\widetilde{M}$, the transition probability is

$$\begin{aligned}\mathbf{q}^\lambda\left(t,(x,\xi),d(y,\eta)\right) &= \sum_{\beta\in G}\mathbf{p}^\lambda(t,(x,\xi),d(\beta y,\beta\eta))\\ &= \sum_{\beta\in G}p^\lambda(t,x,\beta y)d\mathrm{Vol}^\lambda(y)\delta_\xi(\beta\eta).\end{aligned}$$

Let Q_t^λ ($t\geqslant 0$) be given in (1.2). It defines the action of $[0,+\infty)$ on continuous functions f on SM which describes the $\Delta_{g^\lambda}^{\mathcal{W}}$-diffusion. It was shown in [**L2**] that for sufficiently small $\mathtt{b}>0$, there exists $T>0$ such that Q_T^λ is a contraction on $\mathcal{H}_\mathtt{b}^\lambda$ and hence, as $t\to\infty$, Q_t^λ converges to the mapping $f\mapsto\int f\,d\mathbf{m}^\lambda$ exponentially in t for $f\in\mathcal{H}_\mathtt{b}^\lambda$. Thus, each harmonic measure $\mathbf{m}^\lambda$ is a fixed point of the dual operation $(\mathrm{Q}_T^\lambda)^*$ in the dual space $(\mathcal{H}_\mathtt{b}^\lambda)^*$ with the weak topology, where

$$(\mathrm{Q}_T^\lambda)^*(\mu)(f) := \mu(\mathrm{Q}_T^\lambda(f)),\ \text{for all } \mu\in(\mathcal{H}_\mathtt{b}^\lambda)^*,\ f\in\mathcal{H}_\mathtt{b}^\lambda.$$

The following proposition shows that $\mathcal{H}_\mathtt{b}^\lambda$ can be chosen to be independent of g^λ.

Proposition 3.6. *Let $\mathcal{V}_g$ be as in Theorem 1.3. For every $\mathtt{b}>0$ small enough, there exist $C>0$ and $\Bbbk<1$ such that, for all $\lambda\in(-1,1)$, $t>0$ and $f\in\mathcal{H}_\mathtt{b}^0$,*

$$\left\|\mathrm{Q}_t^\lambda f-\int f\ d\mathbf{m}^\lambda\right\|_\mathtt{b}\leqslant C\Bbbk^t\|f\|_\mathtt{b}.$$

The proof of Proposition 3.6 follows [**L2**, Theorem 3] for an individual metric. The only modification is to find a common Hölder continuous function space independent of the metrics where the contractions (of Hölder norm) happen. Denote d and $(\xi|\eta)_x$ for the $\widetilde{g}^0$ distance and its Gromov product. The key lemma is the following.

Lemma 3.7. *Let $\mathcal{V}_g$ be as in Theorem 1.3. There is a number $\mathtt{b}'>0$ such that for any $\mathtt{b}$, $0<\mathtt{b}<\mathtt{b}'$, there exists $\Bbbk_1<1$ such that for t large enough, $x\in M_0$ and all ξ,η, $\xi\neq\eta$, we have for all $\lambda\in(-1,1)$,*

$$\mathbb{E}_{x,\xi}^\lambda\left(e^{-\mathtt{b}\left((\xi|\eta)_{\lfloor \mathrm{x}_t\rfloor^\lambda}-(\xi|\eta)_x\right)}\right)<\Bbbk_1^t,$$

where $\lfloor \mathrm{x}_t\rfloor^\lambda$ denotes the $\widetilde{g}^\lambda$-Brownian motion on $W^s(x,\xi)$ starting from (x,ξ) and $\mathbb{E}_{x,\xi}^\lambda$ denotes its corresponding expectation.

As a preparation for the proof of Lemma 3.7, define on $M_0\times\partial\widetilde{M}\times\partial\widetilde{M}$ the transition probabilities

$$\mathbf{q}^{2,\lambda}(t,(x,\xi_1,\xi_2),d(y,\eta_1,\eta_2)) := \sum_{\beta\in G}p^\lambda(t,x,\beta y)\ d\mathrm{Vol}^\lambda(y)\delta_{\xi_1}(\beta\eta_1)\delta_{\xi_2}(\beta\eta_2)$$

and the corresponding operator $\mathrm{Q}_t^{2,\lambda}$ on continuous functions on $M_0\times\partial\widetilde{M}\times\partial\widetilde{M}$:

$$\mathrm{Q}_t^{2,\lambda}f(x,\xi_1,\xi_2)=\int f(y,\eta_1,\eta_2)\mathbf{q}^{2,\lambda}\left((x,\xi_1,\xi_2),d(y,\eta_1,\eta_2)\right).$$

By analogy with the case of Q_t^λ, there is a unique $\mathrm{Q}_t^{2,\lambda}$-invariant probability measure on $M_0 \times \partial\widetilde{M} \times \partial\widetilde{M}$ which is related to the harmonic measure $\mathbf{m}^\lambda$ as follows.

Lemma 3.8. *([**L2**, Proposition 1]) For each $g^\lambda \in \Re^k(M)$, with the above notations, there is a unique probability measure $\mathbf{m}^{2,\lambda}$ on $M_0 \times \partial\widetilde{M} \times \partial\widetilde{M}$ satisfying*

$$\int \mathrm{Q}_t^{2,\lambda} f \; d\mathbf{m}^{2,\lambda} = \int f \; d\mathbf{m}^{2,\lambda}$$

for all $f \in C(M_0 \times \partial\widetilde{M} \times \partial\widetilde{M}, \mathbb{R})$ and all positive t. The measure $\mathbf{m}^{2,\lambda}$ is characterized by

$$\int f \; d\mathbf{m}^{2,\lambda} = \int_{M_0 \times \partial\widetilde{M}} f(x,\xi,\xi) \; d\mathbf{m}^\lambda(x,\xi).$$

For $g' \in \Re^k(M)$, let $\widetilde{g}'$ be its G-invariant extension to $\widetilde{M}$, $\mathrm{x}_t^{\widetilde{g}'}(\mathrm{w})$ its Brownian motion on $\widetilde{M}$ and $\mathbf{m}^{\widetilde{g}'}$ its harmonic measure. The following limit exists almost surely:

$$\lim_{t\to+\infty} \frac{1}{t} b_{(x,\xi)}\left(\mathrm{x}_t^{\widetilde{g}'}(\mathrm{w})\right) = \int_{M_0\times\partial\widetilde{M}} \Delta^{\widetilde{g}'} b_{(x,\xi)} \; d\mathbf{m}^{\widetilde{g}'} =: \ell'_{g'}.$$

As $g' \to g$, $\mathbf{m}^{\widetilde{g}'} \to \mathbf{m}$ and hence both $\ell'_{g'}, \ell_{g'}$ converges to ℓ. So we can choose a neighborhood $\mathcal{V}_g$ of g such that $\underline{\ell} := \min_{g'\in\mathcal{V}_g}\{\ell_{g'}, \ell'_{g'}\} > 0$. Consequently, for any curve $\lambda \to g^\lambda \in \mathcal{V}_g$,

$$\min_{\lambda\in(-1,1)} \{\ell_{g^\lambda}, \ell'_{g^\lambda}\} \geqslant \underline{\ell} > 0.$$

Lemma 3.9. *Let $\mathcal{V}_g$ be as in Theorem 1.3. For $T > 0$ large enough, for all $\lambda \in (-1,1)$, $x \in M_0$ and $\xi, \eta \in \partial\widetilde{M}$, $\xi \neq \eta$,*

$$\frac{1}{T}\mathbb{E}_{x,\xi}^\lambda\left((\xi|\eta)_{\mathrm{x}_T^\lambda} - (\xi|\eta)_x\right) \geqslant \frac{1}{4}\underline{\ell}.$$

Proof. We may assume g^λ is defined for $\lambda \in [-1,1]$. Assume the conclusion is not true. Then there exist $\lambda_n \in [-1,1]$, $T_n \in \mathbb{R}_+$, $T_n \to \infty$, and points $x_n, \xi_n, \eta_n, \xi_n \neq \eta_n$, such that

$$\frac{1}{T_n}\mathbb{E}_{x_n,\xi_n}^{\lambda_n}\left((\xi_n|\eta_n)_{\mathrm{x}_{T_n}^{\lambda_n}} - (\xi_n|\eta_n)_{x_n}\right) < \frac{1}{4}\underline{\ell}. \tag{3.3}$$

By definition of the Gromov product $(\cdot|\cdot)$, for all $\xi \neq \eta \in \partial\widetilde{M}$, $y, z \in M_0$ and $\lambda \in [-1,1]$,

$$\left|(\xi|\eta)_y - (\xi|\eta)_z\right| \leqslant 2d(y,z) \leqslant Const. \cdot d^\lambda(y,z),$$

where the constant is independent of λ, ξ, η, y and z. Hence by uniform continuity of $\lambda \mapsto p^\lambda(t,x,\cdot)$ in x, we can find t_0 small enough such that

$$\sup_{\lambda\in[-1,1]} \sup_{0\leqslant t\leqslant t_0} \sup_{x,\xi} \sup_{\eta\neq\xi} \mathbb{E}_{x,\xi}^\lambda\left(\left|(\xi|\eta)_{\lfloor\mathrm{x}_t\rfloor^\lambda} - (\xi|\eta)_x\right|\right) \leqslant \frac{1}{4}\underline{\ell}. \tag{3.4}$$

By using (3.3), (3.4) and suitably relabelling $\lambda_n, x_n, \xi_n, \eta_n$, we can find a sequence $\lambda_j \in [-1,1]$ and a sequence of integers $N_j \to \infty$ and points x_j, ξ_j, η_j such that, for all j,

$$\frac{1}{N_j t_0}\mathbb{E}^{\lambda_j}_{x_j,\xi_j}\left((\xi|\eta)_{\mathrm{x}^{\lambda_j}_{N_j t_0}} - (\xi|\eta)_{x_j}\right) < \frac{1}{2}\underline{\ell}. \tag{3.5}$$

By passing to suitable subsequences, we may also assume that λ_n converges to some $\lambda_0 \in [-1,1]$, as n goes to infinity. For $\lambda \in [-1,1]$, write ϕ^λ for the function on $M_0 \times \partial\widetilde{M} \times \partial\widetilde{M}$ defined for $x \in M_0$ and $\xi, \eta \in \partial\widetilde{M}$, $\xi \neq \eta$, by

$$\phi^\lambda(x,\xi,\eta) = \frac{1}{t_0}\mathbb{E}^\lambda_{x,\xi}\left((\xi|\eta)_{\mathrm{x}^\lambda_{t_0}} - (\xi|\eta)_x\right).$$

Then, by (3.2), ϕ^λ has a continuous extension to the diagonal, still denoted ϕ^λ, given by

$$\phi^\lambda(x,\xi,\xi) = \frac{1}{t_0}\mathbb{E}^\lambda_{x,\xi}\left(b_{(x,\xi)}(\mathrm{x}^\lambda_{t_0})\right).$$

Write $\lfloor \mathrm{x}_t \rfloor^\lambda = \beta^\lambda_t \underline{\mathrm{x}}^\lambda_t$, where $\beta^\lambda_t \in G$ and $\underline{\mathrm{x}}^\lambda_t \in M_0$. Using ϕ^λ, (3.5) shows that there exist sequences $\lambda_j \to \lambda_0$, $N_j \to +\infty$, as $j \to \infty$, and points x_j, ξ_j, η_j, such that for all j,

$$\frac{1}{N_j}\sum_{k=0}^{N_j-1}\mathbb{E}^{\lambda_j}_{x_j,\xi_j}\left(\phi(\underline{\mathrm{x}}^{\lambda_j}_{kt_0}, (\beta^{\lambda_j}_{kt_0})^{-1}\xi_j, (\beta^{\lambda_j}_{kt_0})^{-1}\eta_j)\right) < \frac{1}{2}\underline{\ell}.$$

This means for $\lambda_j, N_j, x_j, \xi_j$ and η_j as above,

$$\frac{1}{N_j}\sum_{k=0}^{N_j-1}\mathrm{Q}^{2,\lambda_j}_{kt_0}\phi^{\lambda_j}(x_j,\xi_j,\eta_j) < \frac{1}{2}\underline{\ell}. \tag{3.6}$$

Define a sequence of probability measures μ_j on $M_0 \times \partial\widetilde{M} \times \partial\widetilde{M}$ by

$$\mu_j := \frac{1}{N_j}\sum_{k=0}^{N_j-1}(\mathrm{Q}^{2,\lambda_j}_{kt_0})^*\left(\delta(x_j,\xi_j,\eta_j)\right) d(\cdot,\cdot,\cdot),$$

where $(\mathrm{Q}^{2,\lambda_j}_{kt_0})^*$ is the dual action of $\mathrm{Q}^{2,\lambda_j}_{kt_0}$ and $\delta(x_j,\xi_j,\eta_j)$ is the Dirac measure at (x_j,ξ_j,η_j). Then,

$$\left\|(\mathrm{Q}^{2,\lambda_j}_{t_0})^*\mu_j - \mu_j\right\| \leqslant \frac{2}{N_j}.$$

Moreover, $(\mathrm{Q}^{2,\lambda_j}_{t_0})^*$ converges to $(\mathrm{Q}^{2,\lambda_0}_{t_0})^*$ in norm as j goes to infinity by Theorem 1.3 since

$$\begin{aligned}
\left\|\mathrm{Q}^{2,\lambda_j}_{t_0} - \mathrm{Q}^{2,\lambda_0}_{t_0}\right\| &\leqslant \sup_{x\in\widetilde{M}}\left|\int_{\widetilde{M}} p^{\lambda_j}(t,x,y)\ d\mathrm{Vol}^{\lambda_j}(y) - p^{\lambda_0}(t,x,y)\ d\mathrm{Vol}^{\lambda_0}(y)\right| \\
&= \sup_{x\in\widetilde{M}}\int_{\lambda_0}^{\lambda_j}\int_{\widetilde{M}}\left|(\ln p^\lambda)^{(1)}_\lambda(t_0,x,y) + (\ln\rho^\lambda)^{(1)}_\lambda(y)\right| p^\lambda(t_0,x,y) d\mathrm{Vol}^\lambda(y)\ d\lambda \\
&\leqslant Const.\,|\lambda_j - \lambda_0|,
\end{aligned}$$

where $\rho^\lambda = d\mathrm{Vol}^\lambda / d\mathrm{Vol}^0$. Consequently, if μ is a weak limit of μ_j, we have

$$(\mathrm{Q}_{t_0}^{2,\lambda_0})^* \mu = \mu.$$

Let $\mu' = (1/t_0) \int_0^{t_0} (\mathrm{Q}_s^{2,\lambda_0})^* \mu \; ds$. The measure μ' is $\mathrm{Q}_t^{2,\lambda_0}$-invariant ($t > 0$) and hence coincides with $\mathbf{m}^{2,\lambda_0}$ by Lemma 3.8. Note that ϕ^{λ_j} converges to ϕ^{λ_0} as j goes to infinity. We conclude from (3.6) that $\int \phi^{\lambda_0} \, d\mu \leqslant \underline{\ell}/2$. Using (3.4) again, we find that

$$\int \phi^{\lambda_0} \, d\mathbf{m}^{2,\lambda_0} \leqslant \frac{3}{4} \underline{\ell}.$$

But, by Lemma 3.8, we also have

$$\int \phi^{\lambda_0} \, d\mathbf{m}^{2,\lambda_0} = \frac{1}{t_0} \int \mathbb{E}_{x,\xi}^{\lambda_0} \left(b_{x,\xi}(\mathrm{x}_{t_0}^{\lambda_0}) \right) \, d\mathbf{m}^{2,\lambda_0} = \lim_{t \to \infty} \frac{1}{t} \int \mathbb{E}_{x,\xi}^{\lambda_0} \left(b_{x,\xi}(\mathrm{x}_t^{\lambda_0}) \right) \, d\mathbf{m}^{\lambda_0} \geqslant \underline{\ell},$$

which is a contradiction. □

Proof of Lemma 3.7. For $\lambda \in (-1, 1)$, $x \in M_0$, $\xi, \eta \in \partial \widetilde{M}$ and $t \in \mathbb{R}_+$, write

$$\psi_{\mathfrak{b}}^\lambda(x, \xi, \eta, t) := \mathbb{E}_{x,\xi}^\lambda \left(e^{-\mathfrak{b}\left((\xi|\eta)_{\lfloor \mathrm{x}_t \rfloor^\lambda} - (\xi|\eta)_x \right)} \right).$$

For each λ and $\mathfrak{b}$, it is true by the Markov property of the $\widetilde{g}^\lambda$-Brownian motion that

$$\sup_{x,\xi,\eta} \psi_{\mathfrak{b}}^\lambda(x, \xi, \eta, t_1 + t_2) \leqslant \sup_{x,\xi,\eta} \psi_{\mathfrak{b}}^\lambda(x, \xi, \eta, t_1) \cdot \sup_{x,\xi,\eta} \psi_{\mathfrak{b}}^\lambda(x, \xi, \eta, t_2).$$

Hence for Lemma 3.7, it suffices to find, for a fixed T and $\mathfrak{b}'$ sufficiently small, positive numbers C' and $\Bbbk'$ such that for all $\lambda \in (-1, 1)$ and $\mathfrak{b} < \mathfrak{b}'$,

$$\sup_{x,\xi,\eta} \sup_{0 \leqslant t < T} \psi_{\mathfrak{b}}^\lambda(x, \xi, \eta, t) \leqslant C', \tag{3.7}$$

$$\sup_{x,\xi,\eta} \psi_{\mathfrak{b}}^\lambda(x, \xi, \eta, T) \leqslant \Bbbk' < 1. \tag{3.8}$$

Let T be as in Lemma 3.9. Note that there is some constant C such that

$$\left| (\xi|\eta)_{\lfloor \mathrm{x}_t \rfloor^\lambda} - (\xi|\eta)_x \right| \leqslant 2d(\lfloor \mathrm{x}_t \rfloor^\lambda, x) \leqslant C d^\lambda(\lfloor \mathrm{x}_t \rfloor^\lambda, x).$$

Using Taylor's expansion of the exponential function, we obtain

$$e^{-\mathfrak{b}((\xi|\eta)_{\lfloor \mathrm{x}_t \rfloor^\lambda} - (\xi|\eta)_x)} \leqslant 1 - \mathfrak{b}((\xi|\eta)_{\lfloor \mathrm{x}_t \rfloor^\lambda} - (\xi|\eta)_x) + (C\mathfrak{b} d^\lambda(\lfloor \mathrm{x}_t \rfloor^\lambda, x))^2 e^{C\mathfrak{b} d^\lambda(\lfloor \mathrm{x}_t \rfloor^\lambda, x)}.$$

Since the metrics $\widetilde{g}^\lambda$ have negative sectional curvatures bounded uniformly away from 0 for all λ, we have the exponential decay of the kernel functions, which implies that there exists some constant C_1 such that for all t, $0 \leqslant t \leqslant T$, and all λ,

$$\mathbb{E}_{x,\xi}^\lambda \left(\left(C d^\lambda(\lfloor \mathrm{x}_t \rfloor^\lambda, x) \right)^2 e^{C d^\lambda(\lfloor \mathrm{x}_t \rfloor^\lambda, x)} \right) < C_1.$$

So, using Lemma 3.9, we obtain for $\mathtt{b} \leqslant 1$,

$$\sup_{0\leqslant t<T} \psi^\lambda_{\mathtt{b}}(x,\xi,\eta,t) \leqslant 1 + \mathtt{b}C_1 + \mathtt{b}^2 C_1,$$
$$\psi^\lambda_{\mathtt{b}}(x,\xi,\eta,T) \leqslant 1 - \frac{1}{4}\mathtt{b}\underline{\ell} + \mathtt{b}^2 C_1.$$

Put $\mathtt{b}' = \min\{1, \underline{\ell}/(8C_1)\}$. We see that (3.7) and (3.8) are satisfied for all $\lambda \in (-1,1)$ and $\mathtt{b} < \mathtt{b}'$ with $C' = 1 + \underline{\ell}/8 + \underline{\ell}^2/(64C_1)$, $\Bbbk' = 1 - \underline{\ell}^2/(64C_1)$. □

Proof of Theorem 1.2. Let $T > 0$ be fixed. Assume $g \in \mathfrak{R}^k(M)$. By Proposition 3.6, there exist some neighborhood $\mathcal{V}_g$ of g in $\mathfrak{R}^k(M)$ such that for any continuous curve $\lambda \mapsto g^\lambda$ in $\mathcal{V}_g$, there is some positive $\mathtt{b}$ and $\Bbbk_0 < 1$ such that for all $f \in \mathcal{H}^0_{\mathtt{b}}, n \in \mathbb{N}$,

$$\left\| (\mathrm{Q}^\lambda_T)^n f - \int f \, d\mathbf{m}^\lambda \right\|_{\mathtt{b}} \leqslant \Bbbk_0^n \|f\|_{\mathtt{b}}. \tag{3.9}$$

(For later consideration, we choose $\mathtt{b}$ to be small such that $2\mathtt{b}$ also fulfills the requirement of Proposition 3.6 and $2\mathtt{b} < \mathtt{b}'$, where $\mathtt{b}'$ is from Lemma 3.7.) The inequality (3.9) means each operator Q^λ_T is a bounded operator on $\mathcal{H}^0_{\mathtt{b}}$, 1 is its isolated eigenvalue and $\mathbf{m}^\lambda$ is the eigenfunction of eigenvalue 1 of the dual operator $(\mathrm{Q}^\lambda_T)^*$. By the classical spectrum theory on operators in Banach space (cf. [**Kat**, Theorem 6.17]), we can decompose $\mathcal{H}^0_{\mathtt{b}}$ into the direct sum of one-dimensional E_ρ associated to the eigenvalue 1, and an infinite-dimensional space $E_{<1}$ on which $(\mathrm{Q}^\lambda_T)^n$ tends exponentially fast to 0. Let $\mathcal{C}$ be any circle around 1 with a small radius. Then the projection of $f \in \mathcal{H}^0_{\mathtt{b}}$ to E_1 is given by

$$\frac{1}{2i\pi}\int_{\mathcal{C}} \left(z\mathrm{Id} - \mathrm{Q}^\lambda_T\right)^{-1} f \, dz.$$

Using this and (3.9), we conclude that the following two functional on $\mathcal{H}^0_{\mathtt{b}}$ coincide:

$$\int \cdot \, d\mathbf{m}^\lambda = \frac{1}{2i\pi}\int_{\mathcal{C}} \left(z\mathrm{Id} - \mathrm{Q}^\lambda_T\right)^{-1} \cdot \, dz$$

For the regularity of $\lambda \mapsto \mathbf{m}^\lambda$, we mean the regularity of $\lambda \mapsto \int \cdot \, \mathbf{m}^\lambda$, which is the composition of two mappings

$$\lambda \mapsto \mathrm{Q}^\lambda_T \quad \text{and} \quad \mathrm{Q}^\lambda_T \mapsto \frac{1}{2i\pi}\int_{\mathcal{C}} \left(z\mathrm{Id} - \mathrm{Q}^\lambda_T\right)^{-1} \cdot \, dz$$

Note that by spectral continuity results for isolated simple eigenvalues (cf. [**Kat**, Theorem 3.11]), for $L \in (\mathcal{H}^0_{\mathtt{b}})^*$ in a small neighborhood of Q^0_T, the mapping

$$L \mapsto \frac{1}{2i\pi}\int_{\mathcal{C}} (z\mathrm{Id} - L)^{-1} \cdot \, dz$$

is analytic. We may assume $\mathcal{V}_g$ is such that all Q^λ_T belong to this neighborhood. Then for the regularity of $\lambda \mapsto \int \cdot \, \mathbf{m}^\lambda$, it remains to show the regularity of the mapping $\lambda \mapsto \mathrm{Q}^\lambda_T$.

For $f \in \mathcal{H}^0_{\mathfrak{b}}$, let $\widetilde{f}$ be its G-invariant extension to $\widetilde{M} \times \partial\widetilde{M}$. Then

$$\mathrm{Q}^\lambda_T f(x,\xi) = \int_{\widetilde{M}} \widetilde{f}(y,\xi) p^\lambda(T,x,y)\ d\mathrm{Vol}^\lambda(y).$$

Put $\rho^\lambda := d\mathrm{Vol}^\lambda / d\mathrm{Vol}^0$. Then $\lambda \mapsto \rho^\lambda$ is C^k in λ in $C^k(\widetilde{M})$. By Theorem 1.3 i) and iii), for every i, $1 \leqslant i \leqslant k-2$, and every $(x,\xi) \in \widetilde{M} \times \partial\widetilde{M}$, the following differential exists:

$$\big(\mathrm{Q}^\lambda_T f(x,\xi)\big)^{(i)}_\lambda = \sum_{j=0}^{i} \binom{i}{j} \int \widetilde{f}(y,\xi) (p^\lambda)^{(j)}_\lambda(T,x,y) (\rho^\lambda)^{(i-j)}_\lambda(y)\ d\mathrm{Vol}^0(y).$$

To conclude this defines the i-th differential of Q^λ_T in λ in $(\mathcal{H}^0_{\mathfrak{b}})^*$, we only need to show it defines a bounded operator from $\mathcal{H}^0_{\mathfrak{b}}$ into itself. For $\mathcal{V}_g$ small, the norms of the differentials $(\ln \rho^\lambda)^{(i)}_\lambda$, $i = 1, \cdots, k-2$, are all bounded. Hence it suffices to consider S^i_λ, where

$$\big(S^i_\lambda f\big)(x,\xi) := \int_{y \in \widetilde{M}} \widetilde{f}(y,\xi) (p^\lambda)^{(i)}_\lambda(T,x,y)\ d\mathrm{Vol}^\lambda(y),$$

and show it is a bounded functional of $\mathcal{H}^0_{\mathfrak{b}}$. For each $\xi \in \widetilde{M}$, $\widetilde{f}(\cdot,\xi)$ is uniformly continuous in x and bounded. Hence Theorem 1.3 iv) applies and shows that $\big(S^i_\lambda f\big)(x,\xi)$ is continuous in x. Using Theorem 1.3 iii), we continue to compute that

$$\big|\big(S^i_\lambda f\big)(x,\xi)\big| \leqslant \|f\|_\infty \cdot \int \frac{(p^\lambda)^{(i)}_\lambda(T,x,y)}{p^\lambda(T,x,y)} p^\lambda(T,x,y)\ d\mathrm{Vol}^\lambda(y) \leqslant c_{\lambda,(i)}(2) \|f\|_{\mathfrak{b}},$$

where $c_{\lambda,(i)}(2)$ is as in (1.4). For the Hölder continuity of $\xi \mapsto \big(S^i_\lambda f\big)(x,\xi)$ and the corresponding Hölder norm estimation, it suffices to show the latter is bounded. By Hölder's inequality, Theorem 1.3 iii) and Lemma 3.7, we obtain

$$\begin{aligned}
&\big|\big(S^i_\lambda f\big)(x,\xi_1) - \big(S^i_\lambda f\big)(x,\xi_2)\big| e^{\mathfrak{b}(\xi_1|\xi_2)_x} \\
&\leqslant \left(\int_{\widetilde{M}} \big|\widetilde{f}(y,\xi_1) - \widetilde{f}(y,\xi_2)\big| \cdot \big|(p^\lambda)^{(i)}_\lambda(T,x,y)\big|\ d\mathrm{Vol}^\lambda(y) \right) e^{\mathfrak{b}(\xi_1|\xi_2)_x} \\
&\leqslant \|f\|_{\mathfrak{b}} \int_{\widetilde{M}} e^{-\mathfrak{b}\left((\xi_1|\xi_2)_y - (\xi_1|\xi_2)_x\right)} \left| \frac{(p^\lambda)^{(i)}_\lambda(T,x,y)}{p^\lambda(T,x,y)} \right| p^\lambda(T,x,y)\ d\mathrm{Vol}^\lambda(y) \\
&= \|f\|_{\mathfrak{b}} \cdot \left(\mathbb{E}^\lambda_{x,\xi_1} \left(e^{-2\mathfrak{b}((\xi_1|\xi_2)_{\lfloor x_T \rfloor^\lambda} - (\xi_1|\xi_2)_x)} \right) \right)^{\frac{1}{2}} \left\| \frac{(p^\lambda)^{(i)}_\lambda(T,x,y)}{p^\lambda(T,x,y)} \right\|_{L^2} \\
&\leqslant c_{\lambda,(i)}(2) (\Bbbk_1^T)^{\frac{1}{2}} \|f\|_{\mathfrak{b}}.
\end{aligned}$$

Altogether, we have that each S^i_λ maps $\mathcal{H}^0_{\mathfrak{b}}$ into itself and is a bounded operator since

$$\begin{aligned}
\|S^i_\lambda f\|_{\mathfrak{b}} &= \sup_{x,\xi} \big|\big(S^i_\lambda f\big)(x,\xi)\big| + \sup_{x,\xi_1,\xi_2} \big|\big(S^i_\lambda f\big)(x,\xi_1) - \big(S^i_\lambda f\big)(x,\xi_2)\big| e^{\mathfrak{b}(\xi_1|\xi_2)_x} \\
&\leqslant c_{\lambda,(i)}(2) \big(1 + (\Bbbk_1^T)^{\frac{1}{2}}\big) \|f\|_{\mathfrak{b}}.
\end{aligned}$$

□

3.3. **Differentials of the linear drift.** We are in a situation to prove Theorem 1.1.

Proof of Theorem 1.1. It suffices to show the first statement.

Let $\mathcal{V}_g$ be such that Proposition 3.5 and Theorem 1.2 hold true. We may also assume the Hölder exponents α of Proposition 3.5 and $\mathfrak{b}$ of Theorem 1.2 coincide. As before, for any C^k curve $\lambda \in (-1,1) \mapsto g^\lambda \in \mathcal{V}_g$ with $g^0 = g$, we write $\overline{X}^\lambda$ for the $\widetilde{g}^\lambda$-geodesic spray, Div^λ for the divergence operator associated with the $\widetilde{g}^\lambda$-stable foliation and $\mathbf{m}^\lambda$ for the g^λ-harmonic measure on SM. Let ℓ_λ be the linear drift of g^λ. By (2.6),

$$\ell_\lambda = -\int_{M_0\times\partial\widetilde{M}} \big(\mathrm{Div}^\lambda \overline{X}^\lambda\big)(x,\xi)\, d\mathbf{m}^\lambda = -L_\lambda\big(\mathrm{Div}^\lambda \overline{X}^\lambda\big). \tag{3.10}$$

By Proposition 3.5 iii), $\lambda \mapsto \mathrm{Div}^\lambda \overline{X}^\lambda$ is C^{k-3} into $C^{\mathfrak{b}}(\widetilde{M}\times\partial\widetilde{M},\mathbb{R}^+)$ and is C^{k-2} into $C^0(\widetilde{M}\times\partial\widetilde{M},\mathbb{R}^+)$. Write $(\mathrm{Div}^\lambda \overline{X}^\lambda)^{(0)}_\lambda = \mathrm{Div}^\lambda \overline{X}^\lambda$ and $(\mathrm{Div}^\lambda \overline{X}^\lambda)^{(i)}_\lambda$, $i = 1,\cdots,k-2$, for its i-th derivative in λ. Then $(\mathrm{Div}^\lambda \overline{X}^\lambda)^{(i)}_\lambda$ belongs to $C^0(\widetilde{M}\times\partial\widetilde{M},\mathbb{R}^+)$ for $i \leqslant k-2$, and belongs to $C^{\mathfrak{b}}(\widetilde{M}\times\partial\widetilde{M},\mathbb{R}^+)$ for $i \leqslant k-3$. Regard each $\mathbf{m}^\lambda$ as a measure on $M_0\times\partial\widetilde{M}$. The operator $L_\lambda := \int_{M_0\times\partial\widetilde{M}} \cdot\, d\mathbf{m}^\lambda$ is an bounded operator on continuous functions on $M_0\times\partial\widetilde{M}$. Moreover, by Theorem 1.2, $\lambda \mapsto L_\lambda$ is C^{k-2} differentiable as elements of $(\mathcal{H}^0_{\mathfrak{b}})^*$. Using these regularities and (3.10), we conclude that the function $\lambda \mapsto \ell_\lambda$ is C^{k-2} differentiable. Denote by $L^{(i)}_\lambda$, $i = 1,\cdots,k-2$, the i-th differential functional of L_λ. Then, for every i, $1 \leqslant i \leqslant k-2$, the i-th differential of ℓ_λ in λ, i.e., $\ell^{(i)}_\lambda$, is given by

$$\ell^{(i)}_\lambda = -\sum_{j=0}^{i} \binom{i}{j} L^{(j)}_\lambda \left((\mathrm{Div}^\lambda \overline{X}^\lambda)^{(i-j)}_\lambda\right). \tag{3.11}$$

□

Specifying (3.11) for $i = 1$, we write:

Corollary 3.10. *Let $g \in \mathfrak{R}^3(M)$. For any C^3 curve $\lambda \in (-1,1) \mapsto g^\lambda \in \mathfrak{R}^3(M)$ with $g^0 = g$ and constant volume, we have*

$$(\ell_\lambda)'_0 = -\int \mathrm{Div}^0 \overline{X}^0 d(\mathbf{m}^\lambda)'_0 - \int \big(\mathrm{Div}^\lambda \overline{X}^\lambda\big)'_0\, d\mathbf{m}^0. \tag{3.12}$$

In particular, if $g = g^0$ is a locally symmetric metric and the volume $\mathrm{Vol}^\lambda(M)$ is constant in λ, then we have $(\ell_\lambda)'_0 = 0$.

Proof. We apply (3.11) for $i = 1$ and $\lambda = 0$. The operator L_0 extends to the harmonic measure $\mathbf{m}^0$ and $L^{(1)}_0$ is a linear functional on the space $\mathcal{H}^0_{\mathfrak{b}}$ that we denote $(\mathbf{m}^\lambda)'_0$. Formula (3.12) follows.

Let v^λ be the volume entropy of $(\widetilde{M}, \widetilde{g}^\lambda)$,

$$v^\lambda := \lim_{R\to\infty} \frac{1}{R} \ln \mathrm{Vol}^\lambda B(x,R),$$

where $B(x,R)$ is the ball of radius R about x in $\widetilde{M}$. We know by [**K1**] that for all λ, $\ell^\lambda \leqslant v^\lambda$ and by [**BCG**] that the volume entropy of a negatively curved locally symmetric space achieves its minimum over all metrics of the same volume on that space. Since $\lambda \mapsto \ell_\lambda$ and $\lambda \mapsto v^\lambda$ are differentiable at 0 (by Theorem 1.1 and by [**KKPW**]), the derivative has to be 0. □

We develop formula (3.12). The vector $(\overline{X}^\lambda)'_0$ is a vertical vector given by [**LS1**, Proposition 4.5]. For $\mathbf{v} = (x,\xi)$, it is the sum of $\left(\|\overline{X}^\lambda\|_{\widetilde{g}^0}\right)'_0(\mathbf{v})\overline{X}^0(\mathbf{v})$ and of a vector $Y(\mathbf{v})$ orthogonal to $\mathbf{v}$, where Y is a C^1 vector field along the stable manifolds. Let u_0 be the function such that

$$\Delta u_0 = -\mathrm{Div}\overline{X} - \ell, \text{ (see [**LS1**, (5.12)])}.$$

Theorem 3.11. *Let M be a closed connected smooth manifold and let $g \in \Re^3(M)$. For any C^3 curve $\lambda \in (-1,1) \mapsto g^\lambda \in \Re^3(M)$ with $g^0 = g$ and constant volume,*

$$\begin{aligned}(\ell_\lambda)'_0 \;=\; & \int \left(-\frac{1}{2}\langle \nabla \mathrm{trace}\mathcal{X}, \overline{X}\rangle + \frac{1}{2}\mathcal{X}(\overline{X},\overline{X})\mathrm{Div}(\overline{X}) + \frac{1}{2}\langle \nabla(\mathcal{X}(\overline{X},\overline{X})), \overline{X}\rangle - \mathrm{Div}Y\right)\, d\mathbf{m} \\ & + \int \left(-\frac{1}{2}\langle \nabla \mathrm{trace}\mathcal{X}, \nabla u_0\rangle + \mathrm{Div}\big(\mathcal{X}(\nabla u_0)\big)\right)\, d\mathbf{m}, \qquad (3.13)\end{aligned}$$

where we omit the index 0 for $\nabla^0, \overline{X}^0, \langle\cdot,\cdot\rangle_0, \mathrm{Div}^0$ and $\mathbf{m}^0$ at g^0, and where $\mathcal{X}(\cdot)$ is considered as the $(1,1)$-form in $\widetilde{M}$ such that $\langle \mathcal{X}(Z), Z'\rangle = \mathcal{X}(Z,Z')$. In particular, there is a linear functional $\mathcal{L}$ on $C^k(S^2T^*)$ such that $(\ell_\lambda)'_0 = \mathcal{L}(\mathcal{X})$.

Proof. To obtain (3.13), we use the decomposition of $(\ell_\lambda)'_0$ given by (3.12) as above:

$$(\ell_\lambda)'_0 \;=\; -\int (\mathrm{Div}^\lambda \overline{X})'_0\, d\mathbf{m} - \int (\mathrm{Div}\overline{X}^\lambda)'_0\, d\mathbf{m} - \int \mathrm{Div}\overline{X}\, d(\mathbf{m}^\lambda)'_0$$

and study the three terms successively.

Firstly, we have $(\mathrm{Div}^\lambda \overline{X})'_0 = \frac{1}{2}\langle \nabla(\mathrm{Trace}\mathcal{X}), \overline{X}\rangle$.

Then, for $\mathbf{v} = (x,\xi)$, $(\overline{X}^\lambda)'_0$ is the sum of $\left(\|\overline{X}^\lambda\|\right)'_0(\mathbf{v})\overline{X}(\mathbf{v})$ and $Y(\mathbf{v})$. Hence,

$$(\mathrm{Div}\overline{X}^\lambda)'_0 = \mathrm{Div}Y + \mathrm{Div}\left(\left(\|\overline{X}^\lambda\|\right)'_0(\mathbf{v})\overline{X}(\mathbf{v})\right).$$

Since $\|\overline{X}^\lambda\|^2_{g^\lambda} = 1$, we have

$$\left(\|\overline{X}^\lambda\|\right)'_0(\mathbf{v}) \;=\; -\frac{1}{2}\mathcal{X}(\overline{X}(\mathbf{v}), \overline{X}(\mathbf{v})).$$

Thus,

$$\mathrm{Div}\left(\left(\|\overline{X}^{\lambda}\|\right)_0'(\mathbf{v})\overline{X}(\mathbf{v})\right) = -\frac{1}{2}\mathcal{X}(\overline{X}(\mathbf{v}),\overline{X}(\mathbf{v}))\mathrm{Div}(\overline{X}(\mathbf{v})) - \frac{1}{2}\langle\nabla(\mathcal{X}(\overline{X}(\mathbf{v}),\overline{X}(\mathbf{v}))),\overline{X}(\mathbf{v})\rangle.$$

Lastly, we discuss the term $\int \mathrm{Div}\overline{X}\ d(\mathbf{m}^{\lambda})_0'$. Recall that, by Theorem 1.2, $\lambda \mapsto \mathbf{m}^{\lambda}$ is differentiable at 0, with derivative $(\mathbf{m}^{\lambda})_0' \in (\mathcal{H}^0_{\mathfrak{b}})^*$ (denoted as an integral). It follows that, for f smooth on SM,

$$\int (\Delta^{\lambda})_0' f\ d\mathbf{m} \ +\ \int \Delta f\ d(\mathbf{m}^{\lambda})_0' \ =\ 0. \tag{3.14}$$

The equation (3.14) extends to functions f that are of class C^2 along the stable leaves with globally continuous second order derivatives. In particular, (3.14) applies to the function u_0 and therefore,

$$\int \mathrm{Div}\overline{X}\ d(\mathbf{m}^{\lambda})_0' = \int (\Delta^{\lambda})_0' u_0\ d\mathbf{m} = \int \left(\frac{1}{2}\langle\nabla u_0, \nabla \mathrm{trace}\mathcal{X}\rangle - \mathrm{Div}\big(\mathcal{X}(\nabla u_0)\big)\right)\ d\mathbf{m}.$$

To show $(\ell_{\lambda})_0'$ is linear in $\mathcal{X}$, it remains to consider $\int \mathrm{Div}Y\, d\mathbf{m}$. If we denote $k(x,y,\xi)$ the continuous version of the density $\big(dm_y/dm_x\big)(\xi)$ (see e.g. [**LS1**, Proposition 2.2]), the integration by parts formula yields

$$\int \mathrm{Div}Y\ d\mathbf{m} \ =\ -\int \langle Y, \nabla_y \ln k(x,y,\xi)|_{y=x}\rangle\ d\mathbf{m}. \tag{3.15}$$

We recall from [**LS1**, Proposition 4.5] the construction of the vector field Y. Let $\mathbf{v} \in TM$. We define the vector $\Upsilon(\mathbf{v}) \in TTM$ as the vertical vector with vertical component given by

$$\Upsilon(\mathbf{v}) \ :=\ \big(\nabla^{\lambda}_{\mathbf{v}}(\mathbf{v})\big)_0' - \big\langle\big(\nabla^{\lambda}_{\mathbf{v}}(\mathbf{v})\big)_0', \mathbf{v}\big\rangle.$$

Clearly, for all $\mathbf{v} \in SM$, $\Upsilon(\mathbf{v})$ depends linearly of $\mathcal{X}$, and $\sup_{\mathbf{v}} \|\Upsilon(\mathbf{v})\|$ is bounded by $C\|\mathcal{X}\|_{C^1}$. In order to obtain $Y(\mathbf{v})$, we consider the orbit $\mathbf{\Phi}_s(\mathbf{v}), s \geqslant 0$, under the geodesic flow. For each $s \geqslant 0$, we decompose $\Upsilon(\mathbf{\Phi}_s(\mathbf{v}))$ into a sum of its unstable part $\Upsilon(\mathbf{\Phi}_s(\mathbf{v}))^{\mathrm{u}}$ and its stable part. The vector $Y(\mathbf{v})$ is the vertical part of

$$\int_0^{\infty} (D\mathbf{\Phi}_s)^{-1}\Upsilon(\mathbf{\Phi}_s(\mathbf{v}))^{\mathrm{u}}\, ds.$$

Since the geodesic flow is Anosov, there are $C, \tau > 0$ such that $(D\mathbf{\Phi}_s)^{-1}$ restricted to the unstable manifold has norm smaller than $Ce^{-\tau s}$. It follows that the expression $\int \mathrm{Div}Y\, d\mathbf{m}$ is linear in $\mathcal{X}$ and bounded by $C\|\mathcal{X}\|_{C^1}$. □

Remark 3.12. We can also verify that the formula (3.13) gives indeed 0 in the case when $g = g^0$ is locally symmetric.

Assume that g is a locally symmetric metric, then $\mathrm{Div}\overline{X}$ is the constant $-\ell$ and the measure $\mathbf{m}$ is the normalized Liouville measure. Since the measures $\mathbf{m}^{\lambda}$ are normalized

(and the constant functions belong to the space $\mathcal{H}^0_{\mathfrak{b}}$), $\int \mathrm{Div}\overline{X} d(\mathbf{m}^\lambda)'_0 = 0$ and formula (3.13) reduces to

$$\begin{aligned}(\ell_\lambda)'_0 &= \int \left(-\frac{1}{2}\langle \nabla \mathrm{trace}\mathcal{X}, \overline{X}\rangle + \frac{1}{2}\mathcal{X}(\overline{X},\overline{X})\mathrm{Div}(\overline{X}) + \frac{1}{2}\langle \nabla(\mathcal{X}(\overline{X},\overline{X})), \overline{X}\rangle - \mathrm{Div}Y\right) d\mathbf{m} \\ &=: \int (\mathrm{(I)} + \mathrm{(II)} + \mathrm{(III)} + \mathrm{(IV)})\ d\mathbf{m}.\end{aligned}$$

Since $\mathrm{trace}\mathcal{X}, \mathcal{X}(\overline{X},\overline{X})$ are functions on SM and we integrate with respect to the invariant Liouville measure, the integrals of (I), (III) vanish. Since $\widetilde{g}$ is a symmetric space, the $k(x,y,\xi)$ in formula (3.15) is given by $-\ell_0 b_{(x,\xi)}(y)$, where $b_{(x,\xi)}$ is the Busemann function (see Section 2.4). It follows that $\nabla_y \ln k(x,y,\xi)|_{y=x} = \ell\overline{X}(\mathbf{v})$. Since $Y(\mathbf{v})$ is orthogonal to $\overline{X}(\mathbf{v})$, the integral $\int \mathrm{(IV)}\ d\mathbf{m}$ vanishes as well. Remains to consider

$$\int \mathrm{(II)}\ d\mathbf{m} = -\frac{1}{2}\ell \int \mathcal{X}(\overline{X},\overline{X})\ d\mathbf{m} = -\frac{1}{2}\frac{\ell}{m}\int_{M_0} \mathrm{trace}\mathcal{X}\ \frac{d\mathrm{Vol}_{\widetilde{g}}}{\mathrm{Vol}_{\widetilde{g}}(M_0)} = -\frac{\ell}{m}\frac{\left(\mathrm{Vol}_{\widetilde{g}^\lambda}(M_0)\right)'_0}{\mathrm{Vol}_{\widetilde{g}}(M_0)},$$

where $\mathrm{Vol}_{\widetilde{g}}$ is the Riemannian volume. So, $\int \mathrm{(II)}\ d\mathbf{m}$ vanishes since the volume is constant.

4. Brownian motion and stochastic flows

In this section, we recall the Eells-Elworthy-Malliavin construction of the Brownian motion on a manifold through a stochastic differential equation (SDE) on the orthogonal frame bundle and of the associated stochastic flow (see Proposition 4.28). We give estimations on the growth in time of the derivatives of this stochastic flow. We will need in Sections 5 and 6 both uniform estimations and estimations in average with respect to the Brownian motion and Brownian bridge distributions in the non-compact case.

4.1. **Parallelism and the Brownian motion.** Let N be a C^∞ n-dimensional Riemannian manifold. A differential form ϑ on N with values in $\mathbb{R}^n$ is called a *parallelism differential form* ([**Ma**]), if it realizes for every $u \in \mathrm{N}$ an isomorphism of $T_u\mathrm{N}$ on $\mathbb{R}^n$. A parallelism differential form ϑ is called C^k if it is a C^k section of the frame bundle space $\mathcal{F}(\mathrm{N})$ of N.

Let $f : [0,+\infty) \to \mathbb{R}^n$ be a C^2 curve. It defines a one-parameter of continuous vectors $\{(df/dt)|_{t=\tau}\}_{\tau\in[0,+\infty)}$. Let ϑ be a C^1 parallelism differential form. It, together with f, defines a C^1 vector field on $\mathrm{N}\times\mathbb{R}^+$:

$$Z^f_{t,u} := \vartheta_u^{-1}(\frac{df}{dt}),\ \forall u \in \mathrm{N}, t \in \mathbb{R}^+.$$

By the classical theory of ordinary differential equation, there exists a flow $F_{f,t}$ generated by $Z^f_{t,u}$, which solves Cauchy's problem

$$\frac{d}{dt}\left(F_{f,t}(u_0)\right) = Z^f_{t,u(t)},\ \text{where}\ \ u(t) = F_{f,t}(u_0) \text{ and } F_{f,0}(u_0) = u_0 \in \mathrm{N}.$$

The orbit of each $u_0 \in \mathrm{N}$ under $F_{f,t}$ is an analogue of the curve f since the velocity at time τ is just the preimage of $(df/dt)|_{t=\tau}$ by ϑ. Moreover, the time t map $F_{f,t}$ depends C^1

on the initial point u_0. The variation of $F_{f,t}(u_0)$ with respect to u_0 reflects the geometric difference between N and $\mathbb{R}^n$ and the pull back of the tangent map of $F_{f,t}$ in $\mathbb{R}^n$ via ϑ can be formulated using the equation of $d\vartheta$ ([**Ma**, Proposition 3.2]). In general, if f is a C^{k+1} curve in $\mathbb{R}^n$ and ϑ is C^k, then the flow generated by $Z^f_{t,u}$ depends C^k on the initial point.

In case N is the frame bundle space of $\widetilde{M}$, there are plenty of parallelism differential forms using the dual form and the connection forms. Recall that a *frame* u for $T_x\widetilde{M}$, $x \in \widetilde{M}$, is an ordered basis $u = (u_1, \cdots, u_m)$ for $T_x\widetilde{M}$, which defines a linear isomorphism form $\mathbb{R}^m$ to $T_x\widetilde{M}$ by letting $u(y) := \sum_{i=1}^m y^i u_i$, for $y = (y^i) \in \mathbb{R}^m$. The set of all frames u for all tangent spaces $T_x\widetilde{M}$, denoted by $\mathcal{F}(\widetilde{M})$, is a C^∞ manifold. The *dual form* (or the canonical form) on $\mathcal{F}(\widetilde{M})$ is an $\mathbb{R}^m$-valued 1-form defined by

$$\theta_u(Y) := u^{-1}(\pi_* Y),\ \forall Y \in T_u\mathcal{F}(\widetilde{M}).$$

Its kernel is the *vertical vector bundle* of $T\mathcal{F}(\widetilde{M})$:

$$VT\mathcal{F}(\widetilde{M}) := \big\{Y \in T\mathcal{F}(\widetilde{M}) :\ \pi_* Y = 0\big\}.$$

For $A \in \mathfrak{gl}(m, \mathbb{R})$, let A^* be the vector field on $\mathcal{F}(\widetilde{M})$ with $A^*(u) = \dot{\gamma}_u(0)$, where $\gamma_u(t) = R_{\exp(tA)}u$ and R_a denotes the right action by a. A C^k *(Ehresmann) affine connection* ϖ for $(\mathcal{F}(\widetilde{M}), \pi, \cdot)$ is a C^k $\mathfrak{gl}(m, \mathbb{R})$-valued 1-form on $\mathcal{F}(\widetilde{M})$ satisfying

$$\begin{aligned} \varpi(A^*(u)) &= A,\ \forall A \in \mathfrak{gl}(m, \mathbb{R}), \\ \varpi((R_a)_* Y) &= Ad(a^{-1})\varpi(Y),\ \forall a \in GL(m, \mathbb{R}),\ Y \in T\mathcal{F}(\widetilde{M}). \end{aligned}$$

Each C^k affine connection form ϖ of $\mathcal{F}(\widetilde{M})$ assigns a unique C^k-distributed complementary horizontal vector bundle $HT\mathcal{F}(\widetilde{M})$, the kernel of ϖ, which is invariant under the right action of $GL(m, \mathbb{R})$. Each ϖ induces the notion of covariant derivative ∇, D on vector fields and forms on $\mathcal{F}(\widetilde{M})$, respectively. Let T, R be the corresponding torsion tensor and curvature tensor and $\mathbf{\Theta} := D\theta$, $\mathbf{\Omega} := D\varpi$ be the torsion form and curvature form. Then

$$\begin{aligned} T(X, Y) &= u(\mathbf{\Theta}(\breve{X}, \breve{Y})), \\ R(X, Y)Z &= u(\mathbf{\Omega}(\breve{X}, \breve{Y}) \cdot (u^{-1}Z)), \end{aligned}$$

where $\breve{X}, \breve{Y}, \breve{Z} \in T_u\mathcal{F}(\widetilde{M})$ are any vectors which project to $X, Y, Z \in T_x\widetilde{M}$, respectively, and $u \in \mathcal{F}_x(M)$ can be chosen arbitrarily. Any pair (θ, ϖ) is a parallelism differential form for $\mathcal{F}(\widetilde{M})$. It satisfies the following structure equations (cf. [**Sp**, p. 327]):

$$d\theta(Y_1, Y_2) = -\left\{\varpi(Y_1) \cdot \theta(Y_2) - \varpi(Y_2) \cdot \theta(Y_1)\right\} + \mathbf{\Theta}(Y_1, Y_2), \tag{4.1}$$

$$d\varpi(Y_1, Y_2) = -\left[\varpi(Y_1), \varpi(Y_2)\right] + \mathbf{\Omega}(Y_1, Y_2), \tag{4.2}$$

where $Y_1, Y_2 \in T_u\mathcal{F}(\widetilde{M})$ and $\varpi(Y_1) \cdot \theta(Y_2)$ is the action of the matrix $\varpi(Y_1)$ on $\theta(Y_2) \in \mathbb{R}^m$.

For $g \in \mathcal{M}^k(M)$, let $\mathcal{O}^{\tilde{g}}(\widetilde{M}) \subset \mathcal{F}(\widetilde{M})$ be the collection of $\tilde{g}$-orthogonal frames, the so-called *orthogonal frame bundle space* of $(\widetilde{M}, \tilde{g})$. Each $u \in \mathcal{O}^{\tilde{g}}_x(\widetilde{M})$ defines an isometry

from $\mathbb{R}^m$ with the classical Euclidean metric to $(T_x\widetilde{M}, \tilde{g})$. Let ϖ be the unique torsion free connection form on $\mathcal{F}(\widetilde{M})$ which induces the $\tilde{g}$-connection ∇ and curvature tensor R. Then $\varpi = (\varpi^i_j), \mathbf{\Omega} = (\Omega^i_j)$ satisfy

$$\varpi^i_j = \sum_k \mathbf{\Gamma}^i_{kj}\theta^k,\ \mathbf{\Omega}^i_j = \frac{1}{2}\sum_{k,l}\mathbf{R}^i_{jkl}\theta^k \wedge \theta^l,$$

where $\mathbf{\Gamma}$ and $\mathbf{R}$ are ∇ and R read in the frame u. The structural equations (4.1) and (4.2) of (θ, ϖ) are reduced to

$$d\theta^i(Y_1, Y_2) = -\left(\varpi^i_j(Y_1)\theta^j(Y_2) - \varpi^i_j(Y_2)\theta^j(Y_1)\right), \tag{4.3}$$

$$d\varpi^i_j(Y_1, Y_2) = -\left(\varpi^i_q(Y_1)\varpi^q_j(Y_2) - \varpi^i_q(Y_2)\varpi^q_j(Y_1)\right) + \mathbf{R}^i_{jkl}\theta^k(Y_1)\theta^l(Y_2), \tag{4.4}$$

where $Y_1, Y_2 \in T_u\mathcal{F}(\widetilde{M})$ and $u \in \mathcal{F}(\widetilde{M})$. The restriction of (θ, ϖ) to $\mathcal{O}^{\tilde{g}}(\widetilde{M})$ also defines a parallelism differential form. For instance, we can use this parallelism to recover the geodesic flow on $S\widetilde{M}$. Let $f : [0, +\infty) \to \mathcal{O}(\mathbb{R}^m)$ be a half line with $df/dt \equiv (\vec{e}, 0)$ for some unit vector $e \in \mathbb{R}^m$. It defines a C^{k-1} vector field on $\mathcal{O}^{\tilde{g}}(\widetilde{M}) \times \mathbb{R}^+$ by letting

$$Z^f_{t,u} := (\theta, \varpi)^{-1}_u(\frac{df}{dt}),\ \forall u \in \mathcal{O}^{\tilde{g}}(\widetilde{M}),$$

where each $Z^f_{t,u}$ is just the lift of ue to $HT\mathcal{F}(\widetilde{M})$. Let $F_{e,t}$ denote the flow generated by $Z^f_{t,u}$ with $df/dt \equiv (e, 0)$. It projects to the $\tilde{g}$-geodesic flow on $S\widetilde{M}$ and the orbit of $u \in \mathcal{O}^{\tilde{g}}(\widetilde{M})$ under it is the parallel transportation of u along the unit speed geodesic γ_{ue}.

The key point of the Eells-Elworthy-Malliavin construction of the Brownian motion on a Riemannian manifold is to realize it as a transportation of the $\mathbb{R}^m$-Brownian motion using the parallelism differential form of the orthogonal frame bundle.

Let Θ_+ be the space of continuous paths $w : [0, +\infty) \to \mathbb{R}^m$, equipped with the smallest σ-algebra $\mathcal{F}$ for which the projections $R_t : w \mapsto w(t)$ are measurable. The sub σ-algebras $\{\mathcal{F}_t\}_{t\in\mathbb{R}^+}$ of $\mathcal{F}$ is an increasing sequence such that $\{R_s\}_{s\leqslant t}$ are measurable in $\mathcal{F}_t$. An $\mathbb{R}^m$-Brownian motion is a continuous time random process $\{B_t :\ B_t(w) = w(t)\}_{t\in\mathbb{R}^+}$ on Θ_+ with distribution Q so that the induced actions $\mathrm{Q}_t : (\mathrm{Q}_t\varphi)(x) = \mathbb{E}_x(\varphi(B_t(w)))$ on smooth functions φ form a semigroup with Euclidean Laplacian Δ_{Eu} as being the infinitesimal generator ($\lim_{t\to 0}(\mathrm{Q}_t\varphi - \varphi)/t = \Delta_{\mathrm{Eu}}\varphi$ whenever $\varphi \in C^2_c(\mathbb{R}^m)$, the collection of C^2 functions on $\mathbb{R}^m$ with compact support). In other words,

$$B_t = (B^1_t, \cdots, B^m_t), \tag{4.5}$$

where all B^i_t are independent 1-dimensional Brownian motions on $\mathbb{R}$ with time t transition probability $(4\pi t)^{-\frac{1}{2}}e^{-\frac{(x_i-y_i)^2}{4t}}$ between points x_i and y_i in $\mathbb{R}$. In the language of Stratonovich stochastic differential equation (SDE), (4.5) is

$$dB_t = \sum_{i=1}^m e_i(B_t) \circ dB^i_t,$$

where $\{e_i = \partial/\partial x_i\}$ is an orthogonal chart of $\mathbb{R}^m$, which means for all $\varphi \in C_c^\infty(\mathbb{R}^m)$, the collection of C^∞ functions on $\mathbb{R}^m$ with compact support, and for all $t \in \mathbb{R}^+$,

$$\varphi(B_t) = \varphi(B_0) + \int_0^t \sum_{i=1}^m e_i\varphi(B_s) \circ dB_s^i.$$

Fix a C^∞ function c, with support contained in the unit interval $[0,1]$ with integral 1. For each $\epsilon > 0$, let $\mathrm{c}_\epsilon(\tau) := \epsilon^{-1}\mathrm{c}(\epsilon^{-1}\tau)$ be an approximate unit function. For any sample path $t \mapsto w(t) = (w^1(t), \cdots, w^m(t))$ of B, we can smooth it using c_ϵ by letting

$$w_\epsilon^i(t) := \int_0^\epsilon w^i(t+s)\mathrm{c}_\epsilon(s)\, ds, \quad i = 1, \cdots, m.$$

Let $w_\epsilon(t) = (w_\epsilon^1(t), \cdots, w_\epsilon^m(t))$. We see that $t \mapsto w_\epsilon(t)$ is smooth and satisfies

$$\lim_{\epsilon \to 0} \sup_{t \in \mathbb{R}^+} \big\| w_\epsilon(t) - w(t) \big\| = 0.$$

As w varies, $B_t^\epsilon :\ w \mapsto w_\epsilon(t)$ defines an $\mathcal{F}_{t+\epsilon}$-measurable process on Θ_+. Each B_t^ϵ solves

$$\frac{d}{dt}(\widetilde{B}_t) = \sum_{i=1}^m e_i(\widetilde{B}_t) \cdot \frac{d}{dt}(w_\epsilon^i(t))$$

and, almost surely, the limit of B_t^ϵ (as $\epsilon \to 0$) gives the Brownian motion B_t ([**Ma**]).

Given a sample path w of B_t starting from the origin, the smoothed curve w_ϵ has its lift in $\mathcal{O}(\mathbb{R}^m)$ with tangent vectors $(dw_\epsilon/dt, 0)$. Let $g \in \mathcal{M}^k(M)$ and let θ, ϖ, H be the associated dual form, $\widetilde{g}$-connection form and horizontal lift map. Consider the C^{k-1} vector field on $\mathcal{O}^{\widetilde{g}}(\widetilde{M}) \times \mathbb{R}^+$:

$$Z_{t,u}^{f,\epsilon} := (\theta, \varpi)_u^{-1}(\frac{dw_\epsilon}{dt}, 0),\ \forall u \in \mathcal{O}^{\widetilde{g}}(\widetilde{M}).$$

We see that

$$Z_{t,u}^{f,\epsilon} = \sum_{i=1}^m H(u, e_i) \cdot \frac{dw_\epsilon^i}{dt},$$

where $H(u, e_i)$ is horizontal lift of ue_i to $HT\mathcal{F}(\widetilde{M})$. Let $\Phi_{f,t}^\epsilon$ be the flow generated by $Z_{t,u}^{f,\epsilon}$. For $u \in \mathcal{O}^{\widetilde{g}}(\widetilde{M})$, its orbit $u^\epsilon(t)$ under $\Phi_{f,t}^\epsilon$ solves the differential equation

$$\frac{du^\epsilon(t)}{dt} = \sum_{i=1}^m H(u^\epsilon(t), e_i) \cdot \frac{dw_\epsilon^i}{dt}. \tag{4.6}$$

The projection of the orbit $t \mapsto u^\epsilon(t)$ to $\widetilde{M}$ has tangent $u^\epsilon(t)(dw_\epsilon/dt)$ at time t and is an analog of the curve w_ϵ. As w varies, the distribution of the projection of $u^\epsilon(t)$ on $\widetilde{M}$ simulates the distribution of the $\mathbb{R}^m$ Brownian motion. As ϵ tends to 0, almost surely, the differential system (4.6) tends to

$$d\mathrm{u}_t = \sum_{i=1}^m H(\mathrm{u}_t, e_i) \circ dB_t^i(w), \tag{4.7}$$

which means for all smooth function φ on $\mathcal{O}^{\tilde{g}}(\widetilde{M})$,

$$\varphi(\mathrm{u}_t) = \varphi(\mathrm{u}_0) + \int_0^t \sum_{i=1}^m (H(\mathrm{u}_s, e_i)\varphi)(\mathrm{u}_s) \circ dB_s^i,\ 0 \leqslant t < \infty.$$

Since the vector fields $H(\cdot, e_i)$ are C^{k-1}, for any initial u_0, there exists a unique solution $\mathrm{u} = (\mathrm{u}_t)_{t\in\mathbb{R}_+}$ to (4.7), which is continuous in (t, u_0) for all $t \in \mathbb{R}_+$ (see Proposition 4.1).

Recall that the generator A of u_t is such that

$$\varphi(\mathrm{u}_t) - \varphi(\mathrm{u}_0) - \int_0^t \mathrm{A}\varphi(\mathrm{u}_s)\ ds$$

is a local martingale for all smooth φ. By Itô's formula, we see that

$$\mathrm{A} = \sum_{i=1}^m H(\cdot, e_i)^2,$$

which is the Bochner horizontal Laplacian $\Delta_{\mathcal{O}^{\tilde{g}}(\widetilde{M})}$. It is a lift of the Laplacian Δ in the sense that for any smooth function $\underline{\varphi}$ on $\widetilde{M}$ and its lift φ to $\mathcal{O}^{\tilde{g}}(\widetilde{M})$,

$$\Delta_{\mathcal{O}^{\tilde{g}}(\widetilde{M})}\varphi(u) = \Delta\underline{\varphi}(\pi u). \tag{4.8}$$

Let $\mathrm{x} = (\mathrm{x}_t)_{t\in\mathbb{R}_+}$ be the projection on $\widetilde{M}$ of the solution $\mathrm{u} = (\mathrm{u}_t)_{t\in\mathbb{R}_+}$ of (4.7) with initial value $\mathrm{u}_0 \in \mathcal{O}^{\tilde{g}}_{\mathrm{x}_0}(\widetilde{M})$. It defines a measurable map from orbits in Θ_+ starting from the origin to $C_{\mathrm{x}_0}(\mathbb{R}^+, \widetilde{M})$, the space of continuous paths on $\widetilde{M}$ starting from x_0. As x_0 varies, $\mathrm{Q}(\mathrm{x}^{-1})$ gives a distribution in the space of continuous paths on $\widetilde{M}$. For $\tau \in \mathbb{R}_+$, let $C_{\mathrm{x}_0}([0,\tau], \widetilde{M})$ be the collection of continuous paths $\rho : [0,\tau] \to \widetilde{M}$ with $\rho(0) = \mathrm{x}_0$. Then x also induces a measurable map $\mathrm{x}_{[0,\tau]} :\ \Theta_+ \to C_{\mathrm{x}_0}([0,\tau], \widetilde{M})$ sending w to $(\mathrm{x}_t(w))_{t\in[0,\tau]}$. So,

$$\mathbb{P}_\tau := \mathrm{Q}(\mathrm{x}^{-1}_{[0,\tau]})$$

gives the distribution probability of paths $\mathrm{x}(w)$ on $\widetilde{M}$ up to time τ and this distribution is independent of the choice of the initial orthogonal frame u_0 that projects to x_0. Since x has generator Δ by (4.8), it visualizes the Brownian motion on $\widetilde{M}$. This is the Eells-Elworthy-Malliavin's approach to obtain the Brownian motion on a manifold (cf. [**Elw**]).

4.2. **A stochastic analogue of the geodesic flow.** The regularity of the Brownian companion process u_t with respect to its initials u_0 can be understood by general theory on stochastic flows associated to SDEs.

Let $X_1, \cdots, X_d$ be bounded vector fields on a smooth finite dimensional Riemannian manifold $(\mathrm{N}, \langle\cdot,\cdot\rangle)$. Let $(z_t)_{t\in\mathbb{R}_+} = (z_t^1, \cdots, z_t^d)$ be a continuous stochastic process on $\mathbb{R}^d$. An N-valued semimartingale $(x_t)_{t\in\mathbb{R}_+}$ defined up to a stopping time τ is said to be a solution

of the following Stratonovich SDE

$$dx_t = \sum_{i=1}^{d} X_i(x_t) \circ dz_t^i, \tag{4.9}$$

if for all $\psi \in C^\infty(\mathrm{N})$,

$$\psi(x_t) = \psi(x_0) + \int_0^t \sum_{i=1}^{d} X_i\psi(x_s) \circ dz_s^i,\ 0 \leqslant t < \tau.$$

The solution to (4.9) always exists when all X_i are C^1 bounded ([**Elw**]). Note that $\mathbf{X} = (X_1, \cdots, X_d)$ is a linear isomorphism from $\mathbb{R}^d$ to $T\mathrm{N}$. So, x_t is a parallel transportation of z_t to the manifold N via $\mathbf{X}$. The pair $(\mathbf{X}, (z_t)_{t\in\mathbb{R}_+})$ is called a *stochastic dynamical system* (SDS) on N ([**Elw**]) and it is said to be C^j if all X_i are C^j bounded. Using $\mathbf{X}$, we also write (4.9) as

$$dx_t = \mathbf{X}(x_t) \circ dz_t.$$

The mapping

$$F_t(\cdot, w) :\ x_0(w) \mapsto x_t(w)$$

has the following regularity with respect to the starting point $x_0(w)$.

Proposition 4.1. *(*[**Elw**, Theorem 3, Chapter VIII]*) Let $(\mathbf{X}, (z_t)_{t\in\mathbb{R}_+})$ be a C^j SDS on* N*. There is a version of the explosion time map $x \mapsto \tau^x$, defined for $x \in \mathrm{N}$, and a version of $F_t(x, w)$, defined when $t \in [0, \tau^x(w))$, such that if $\mathrm{N}(t, w) = \{x \in \mathrm{N} :\ t < \tau^x\}$, then the following are true for each $(t, w) \in \mathbb{R}_+ \times \Theta_+$.*

i) *The set $\mathrm{N}(t, w)$ is open in* N*.*
ii) *The map $F_t(x, w) :\ \mathrm{N}(t, w) \to \mathrm{N}$ is C^{j-1} and is a diffeomorphism onto an open subset of* N*. Moreover, the map $\tau \mapsto F_\tau(\cdot, w)$ of $[0, t]$ into C^{j-1} mappings of $\mathrm{N}(t, w)$ is continuous.*

Corollary 4.2. *Let $g \in \mathcal{M}^k(M)$ $(k \geqslant 3)$. There is a version of the solution flow*

$$F_t(\cdot, w) :\ \mathrm{u}_0(w) \to \mathrm{u}_t(w),\ t \in \mathbb{R}_+,$$

to (4.7) in $\mathcal{F}(\widetilde{M})$, which is a C^{k-2} diffeomorphism into $\mathcal{F}(\widetilde{M})$ and is continuous in t.

Proof. Each $x \in \widetilde{M}$ has infinite distance to the boundary. Hence each solution process u to (4.7) with $\mathrm{u}_0 \in \mathcal{F}_x(\widetilde{M})$ projects to be a diffusion process on $\widetilde{M}$ starting from x and has infinity explosion time. Since $g \in \mathcal{M}^k(M)$, the vector fields $H(\cdot, e_i)$, $i = 1, 2, \cdots, m$, on $\mathcal{O}^{\tilde{g}}(\widetilde{M})$ are all C^{k-1} bounded with respect to the $\tilde{g}$ metric. So $F_t(\cdot, w) : \mathrm{u}_0(w) \mapsto \mathrm{u}_t(w)$ is C^{k-2} with respect to the initial points u_0 and is continuous in t by Proposition 4.1. □

For $l \leqslant j-1$, the l-th tangent map of F_t in Proposition 4.1, denoted by $D^{(l)}F_t(\cdot, w)$, can be formulated and its norm can be estimated if N is equipped with a reference connection.

Proposition 4.3. *([**Elw**]) Let $(\mathbf{X}, (z_t)_{t\in\mathbb{R}_+})$ be a C^j SDS on* N*. Assume there is a Levi-Civita connection ∇ induced by some metric such that the covariant derivatives $\nabla^\iota X_i$, $\iota = 0, 1, \cdots, j$, $i = 1, \cdots, d$, are bounded and the curvature tensor R of ∇ and its first $j-1$ derivatives are bounded. The following hold true.*

i) *There is a version of $\{F_t(\cdot, w)\}$ such that almost surely, for $l \leqslant j-1$, $t \in \mathbb{R}_+$ and $\mathsf{v}_0(w) \in T^{(l)}\mathrm{N}$, $\mathsf{v}_t(w) := [D^{(l)}F_t(\cdot, w)]\mathsf{v}_0(w)$ satisfies the Stratonovich SDE*

$$d\mathsf{v}_t = \sum_{i=1}^{d} [D^{(l)}X_i(x_t)]\mathsf{v}_t \circ dz_t^i,$$

where, if we denote by F_t^i the deterministic flow map generated by the vector field X_i and $D^{(l)}F_t^i$ its l-th differential map, then for $\mathsf{v} \in T^{(l)}N$ with footpoint $x \in \mathrm{N}$,

$$[D^{(l)}X_i(x)]\mathsf{v} := \frac{D}{dt}([D^{(l)}F_t^i]\mathsf{v}).$$

ii) *For any $q \in [1, \infty)$, there is a bounded function $c_l(t, q)$, which depends on t, q, and the bounds of $\nabla^\iota X_i$ and $\nabla^{\iota-1}R$, $\iota \leqslant l+1$, such that $\|[D^{(l)}F_t(\cdot, w)]\|_{L^q} < c_l(t, q)$.*

Proposition 4.3 applies to the flow map corresponding to (4.7). So we can formulate the SDEs of $\{[D^{(l)}F_t(\cdot, w)]\}$. We will use them to specify $c_l(t, q)$ and give a more detailed study of their norm growths in time for later use.

Let $F_t(\cdot, w)$ be as in Corollary 4.2. The first order tangent map $D^{(1)}F_t(\mathrm{u}_0, w)$ records the first order infinitesimal response of $F_t(\mathrm{u}_0, w)$ to the change of initial point u_0. Let $\mathcal{C} : (-1, 1) \mapsto \mathcal{F}(\widetilde{M})$ be a differential curve with $\mathcal{C}(0) = \mathrm{u}_0, \mathcal{C}'(0) = \mathsf{v}$. Then

$$\mathsf{v}_t := \left[D^{(1)}F_t(\mathrm{u}_0, w)\right]\mathsf{v} = \frac{D}{\partial s}F_t(\mathcal{C}(s), w)\Big|_{s=0}.$$

The SDEs of v_t can be formulated using the parallelism form (θ, ϖ) as follows.

Lemma 4.4. *([**Ma**, Theorem 5.1]) Let $F_t(\cdot, w)$ be as in Corollary 4.2.*

i) *For any $\mathsf{v} \in T_{\mathrm{u}_0}\mathcal{F}(\widetilde{M})$, v_t satisfies the Statonovich SDE*

$$d\mathsf{v}_t(w) = \nabla(\mathsf{v}_t(w))H(\mathrm{u}_t, \circ dB_t).$$

ii) *Consider the map*

$$\widetilde{[D^{(1)}F_t(\mathrm{u}_0, w)]} := (\theta, \varpi)_{\mathrm{u}_t} \circ [D^{(1)}F_t(\mathrm{u}_0, w)] \circ (\theta, \varpi)_{\mathrm{u}_0}^{-1}.$$

For $(z(0), \mathbf{z}(0)) := (z^i(0), \mathbf{z}_j^l(0)) \in T\mathcal{F}(\mathbb{R}^m)$, $(z(t), \mathbf{z}(t)) := \widetilde{[D^{(1)}F_t(\mathrm{u}_0, w)]}(z(0), \mathbf{z}(0))$ satisfies the Stratonovich SDE

$$\begin{cases} dz(t) = \mathbf{z}(t) \circ dB_t(w), \\ d\mathbf{z}(t) = \mathrm{u}_t^{-1}R\left(\mathrm{u}_t \circ dB_t(w), \mathrm{u}_t z(t)\right)\mathrm{u}_t. \end{cases} \tag{4.10}$$

iii) *The Itô form of (4.10) is*

$$\begin{cases} dz(t) = \mathbf{z}(t)\ dB_t(w) + \mathrm{Ric}(\mathrm{u}_t z(t))\ dt, \\ d\mathbf{z}(t) = \mathrm{u}_t^{-1} R\left(\mathrm{u}_t dB_t(w), \mathrm{u}_t z(t)\right)\mathrm{u}_t + \mathrm{u}_t^{-1} R\left(\mathrm{u}_t e_i, \mathrm{u}_t \mathbf{z}(t) e_i\right)\mathrm{u}_t\ dt \\ \qquad\qquad +\mathrm{u}_t^{-1}\left(\nabla(\mathrm{u}_t e_i)R\right)\left(\mathrm{u}_t e_i, \mathrm{u}_t z(t)\right)\mathrm{u}_t\ dt, \end{cases} \tag{4.11}$$

where the summation $\Sigma_{i=1}^m$ is omitted in (4.11) for simplicity and

$$\mathrm{Ric}(\mathrm{u}z) := \sum_{i=1}^m \mathrm{u}^{-1} R(\mathrm{u}e_i, \mathrm{u}z)\mathrm{u}e_i, \forall \mathrm{u} \in \mathcal{F}(\widetilde{M}), z \in \mathbb{R}^m. \tag{4.12}$$

For $\underline{t}, \bar{t}$, $0 \leqslant \underline{t} < \bar{t} \leqslant T$, let $F_{\underline{t},\bar{t}}(\cdot, w)$ be the flow map of (4.7) sending $\mathrm{u}_{\underline{t}}$ to $\mathrm{u}_{\bar{t}}$. Then

$$F_{\bar{t}}(\mathrm{u}_0, w) \equiv F_{0,\bar{t}}(\mathrm{u}_0, w) = F_{\underline{t},\bar{t}}(\mathrm{u}_{\underline{t}}, w) \circ F_{0,\underline{t}}(\mathrm{u}_0, w).$$

Let $[D^{(l)} F_{\underline{t},\bar{t}}(\cdot, w)]$ ($l \leqslant k-2$) be the l-th tangent map of $F_{\underline{t},\bar{t}}$. When $l = 1$, let

$$\widetilde{[D^{(1)} F_{\underline{t},\bar{t}}}(\mathrm{u}_0, w)] := (\theta, \varpi)_{\mathrm{u}_{\bar{t}}} \circ [D^{(1)} F_{\underline{t},\bar{t}}(\mathrm{u}_0, w)] \circ (\theta, \varpi)^{-1}_{\mathrm{u}_{\underline{t}}}.$$

Then $[D^{(1)} F_{\underline{t},\bar{t}}]$ (resp. $\widetilde{[D^{(1)} F_{\underline{t},\bar{t}}]}$) satisfies the same SDE as $[D^{(1)} F_{0,t}]$ (resp. $\widetilde{[D^{(1)} F_{0,t}]}$).

To describe $\big[D^{(2)} F_t(\mathrm{u}_0, w)\big]$, we can follow [**Elw**] to use the horizontal/vertical Whitney sum decomposition of $T_{(u,\mathsf{v})} T\mathcal{F}(\widetilde{M}) = T_u \mathcal{F}(\widetilde{M}) \times T_u \mathcal{F}(\widetilde{M})$ with respect to the Levi-Civita connection. The second order tangent vector

$$(u, \mathsf{v}; \mathbb{V}_0, \mathbb{V}_1) \in T_{(u,\mathsf{v})} T\mathrm{N}$$

is in one-to-one correspondence with the Jacobi field $Y(s)$ along the geodesic $s \mapsto \mathcal{C}(s) := \exp(s\mathsf{v})$ with $Y(0) = \mathbb{V}_0, \nabla Y(0) = \mathbb{V}_1$, where $Y(0)$ tells the infinitesimal change of $\mathcal{C}(0)$ (i.e., the horizontal part change of $(\mathcal{C}(0), \mathcal{C}'(0))$) and $\nabla Y(0)$ tells the the infinitesimal change of $\mathcal{C}'(0)$ along the geodesic from u_0 with initial velocity $\mathbb{V}_0$ (i.e., the vertical part change of $(\mathcal{C}(0), \mathcal{C}'(0))$). For the geodesic $\tau \mapsto \mathcal{C}_1(\tau) := \exp(\tau \mathbb{V}_1)$, let $\mathsf{v}_{/\!/}(\tau)$ be the parallel transportation of v along $\mathcal{C}_1$ to the point $\mathcal{C}_1(\tau)$ and define

$$\nabla_{\mathbb{V}_0}\big[D^{(1)} F_t(\mathrm{u}_0, w)\big](\mathsf{v}) = \frac{D}{\partial \tau}\big[D^{(1)} F_t(\mathcal{C}_1(\tau), w)\big](\mathsf{v}_{/\!/}(\tau))\Big|_{\tau=0}. \tag{4.13}$$

Then for almost all w,

$$\big[D^{(2)} F_t(\mathrm{u}_0, w)\big](\mathrm{u}_0, \mathsf{v}; \mathbb{V}_0, \mathbb{V}_1) = \Big(\big[D^{(1)} F_t(\mathrm{u}_0, w)\big](\mathrm{u}_0; \mathsf{v}); \big[D^{(2)} F_t(\mathrm{u}_0, w)\big](\mathbb{V}_0, \mathbb{V}_1)\Big),$$

where

$$\begin{aligned} &\big[D^{(2)} F_t(\mathrm{u}_0, w)\big](\mathbb{V}_0, \mathbb{V}_1) \\ &\qquad = \Big(\big[D^{(1)} F_t(\mathrm{u}_0, w)\big](\mathbb{V}_0), \nabla_{\mathbb{V}_0}\big[D^{(1)} F_t(\mathrm{u}_0, w)\big](\mathsf{v}) + \big[D^{(1)} F_t(\mathrm{u}_0, w)\big](\mathbb{V}_1)\Big). \end{aligned}$$

By Lemma 4.4, to describe $\big[D^{(2)} F_t(\cdot, w)\big](\mathbb{V}_0, \mathbb{V}_1)$, it remains to identify

$$\mathcal{V}_t(\mathsf{v}, \mathbb{V}_0, w) := \nabla_{\mathbb{V}_0}\big[D^{(1)} F_t(\mathrm{u}_0, w)\big](\mathsf{v}).$$

Lemma 4.5. *([**Elw**, Lemma 5B, Chapter VIII]) Let $g \in \mathcal{M}^k(M)$ $(k \geqslant 4)$. For $\mathsf{v} \in T_{\mathrm{u}_0}\mathcal{F}(\widetilde{M})$, $(\mathbb{V}_0, 0) \in T_{(\mathrm{u}_0,\mathsf{v})}T\mathcal{F}(\widetilde{M})$, let $\mathsf{v}_t := [D^{(1)}F_t(\mathrm{u}_0, w)]\mathsf{v}$, $\mathbb{V}_t := [D^{(1)}F_t(\mathrm{u}_0, w)]\mathbb{V}_0$.*

i) *On $T\mathcal{F}(\widetilde{M})$, the process $\mathcal{V}_t := \mathcal{V}_t(\mathsf{v}, \mathbb{V}_0, w)$ satisfies the Stratonovich SDE*

$$d\mathcal{V}_t = \nabla(\mathcal{V}_t)H(\mathrm{u}_t, \circ dB_t) + \nabla^{(2)}(\mathsf{v}_t, \mathbb{V}_t)H(\mathrm{u}_t, \circ dB_t) + R(H(\mathrm{u}_t, \circ dB_t), \mathbb{V}_t)\mathsf{v}_t.$$

ii) *On $T\mathcal{F}(\mathbb{R}^m)$, the process $(\theta, \varpi)_{\mathrm{u}_t}(\mathcal{V}_t)$ satisfies the Stratonovich SDE*

$$\begin{aligned} d\left((\theta, \varpi)_{\mathrm{u}_t}(\mathcal{V}_t)\right) &= \left(\varpi(\mathcal{V}_t) \circ dB_t(w), \mathrm{u}_t^{-1}R(\mathrm{u}_t \circ dB_t, \theta(\mathcal{V}_t))\mathrm{u}_t\right) \\ &\quad + (\theta, \varpi)_{\mathrm{u}_t}\left(\nabla^{(2)}(\mathsf{v}_t, \mathbb{V}_t)H(\mathrm{u}_t, \circ dB_t(w)) + R(H(\mathrm{u}_t, \circ dB_t(w)), \mathbb{V}_t)\mathsf{v}_t\right). \end{aligned} \tag{4.14}$$

iii) *The Itô form of (4.14) is*

$$d\theta(\mathcal{V}_t) = \varpi(\mathcal{V}_t)dB_t(w) + \mathrm{Ric}(\mathrm{u}_t\theta(\mathcal{V}_t))dt + \Phi_\theta(\mathsf{v}_t, \mathbb{V}_t, dB_t, dt), \tag{4.15}$$

$$\begin{aligned} d\varpi(\mathcal{V}_t) = {} & \mathrm{u}_t^{-1}R\left(\mathrm{u}_t dB_t(w), \mathrm{u}_t\theta(\mathcal{V}_t)\right)\mathrm{u}_t + \mathrm{u}_t^{-1}R\left(\mathrm{u}_t e_i, \mathrm{u}_t\varpi(\mathcal{V}_t)e_i\right)\mathrm{u}_t\ dt \\ & + \mathrm{u}_t^{-1}\left(\nabla(\mathrm{u}_t e_i)R\right)\left(\mathrm{u}_t e_i, \mathrm{u}_t\theta(\mathcal{V}_t)\right)\mathrm{u}_t\ dt + \Phi_\varpi(\mathsf{v}_t, \mathbb{V}_t, dB_t, dt), \end{aligned} \tag{4.16}$$

where the summation $\Sigma_{i=1}^m$ is omitted and

$$\begin{aligned} \Phi_\theta(\mathsf{v}_t, \mathbb{V}_t, dB_t, dt) := {} & \theta\left(\nabla^{(2)}(\mathsf{v}_t, \mathbb{V}_t)H(\mathrm{u}_t, dB_t(w)) + R(H(\mathrm{u}_t, dB_t(w)), \mathbb{V}_t)\mathsf{v}_t\right) \\ & + 2\varpi\left(\nabla^{(2)}(\mathsf{v}_t, \mathbb{V}_t)H(\mathrm{u}_t, e_i) + R(H(\mathrm{u}_t, e_i), \mathbb{V}_t)\mathsf{v}_t\right)e_i\ dt \\ & + \theta\left(\Big[H(\mathrm{u}_t, e_i), \nabla^{(2)}(\mathsf{v}_t, \mathbb{V}_t)H(\mathrm{u}_t, e_i) + R(H(\mathrm{u}_t, e_i), \mathbb{V}_t)\mathsf{v}_t\Big]\right)\ dt, \\ \Phi_\varpi(\mathsf{v}_t, \mathbb{V}_t, dB_t, dt) := {} & \varpi\left(\nabla^{(2)}(\mathsf{v}_t, \mathbb{V}_t)H(\mathrm{u}_t, dB_t(w)) + R(H(\mathrm{u}_t, dB_t(w)), \mathbb{V}_t)\mathsf{v}_t\right) \\ & + 2\mathrm{u}_t^{-1}R\left(\mathrm{u}_t e_i, \mathrm{u}_t\theta\big(\nabla^{(2)}(\mathsf{v}_t, \mathbb{V}_t)H(\mathrm{u}_t, e_i) + R(H(\mathrm{u}_t, e_i), \mathbb{V}_t)\mathsf{v}_t\big)\right)\mathrm{u}_t dt \\ & + \varpi\left(\Big[H(\mathrm{u}_t, e_i), \nabla^{(2)}(\mathsf{v}_t, \mathbb{V}_t)H(\mathrm{u}_t, e_i) + R(H(\mathrm{u}_t, e_i), \mathbb{V}_t)\mathsf{v}_t\Big]\right)\ dt. \end{aligned}$$

A corollary of Lemma 4.5 is that we can describe $\mathcal{V}_t$ (resp. $(\theta, \varpi)_{\mathrm{u}_t}(\mathcal{V}_t)$) using the tangent maps $\left[D^{(1)}F_{\underline{t},\bar{t}}(\mathrm{u}_0, w)\right]$ (resp. $\left[\widetilde{D^{(1)}F_{\underline{t},\bar{t}}}(\mathrm{u}_0, w)\right]$) by a stochastic version of the variation of constant method, i.e., a stochastic Duhamel principle.

Corollary 4.6. *Let $g \in \mathcal{M}^k(M)$ $(k \geqslant 4)$ and let v_t, $\mathbb{V}_t$ and $\mathcal{V}_t$ be as in Lemma 4.5.*

i) $$\mathcal{V}_t = \int_0^t \left[D^{(1)}F_{\tau,t}(\mathrm{u}_\tau, w)\right]\left(\nabla^{(2)}(\mathsf{v}_\tau, \mathbb{V}_\tau)H(\mathrm{u}_\tau, e_i) + R(H(\mathrm{u}_\tau, e_i), \mathbb{V}_\tau)\mathsf{v}_\tau\right) \circ dB_\tau^i.$$

ii) $(\theta, \varpi)_{\mathrm{u}_t}(\mathcal{V}_t) =$

$$\int_0^t \left[\widetilde{D^{(1)}F_{\tau,t}}(\mathrm{u}_\tau, w)\right](\theta, \varpi)_{\mathrm{u}_\tau}\left(\nabla^{(2)}(\mathsf{v}_\tau, \mathbb{V}_\tau)H(\mathrm{u}_\tau, e_i) + R(H(\mathrm{u}_\tau, e_i), \mathbb{V}_\tau)\mathsf{v}_\tau\right) \circ dB_\tau^i. \tag{4.17}$$

iii) *The Itô form of (4.17) is*

$$(\theta,\varpi)_{\mathrm{u}_t}(\mathcal{V}_t)=\int_0^t\left[\widetilde{D^{(1)}F_{\tau,t}}(\mathrm{u}_\tau,w)\right]\left(\widetilde{\Phi}_\theta(\mathsf{v}_\tau,\mathbb{V}_\tau,dB_\tau,d\tau),\widetilde{\Phi}_\varpi(\mathsf{v}_\tau,\mathbb{V}_\tau,dB_\tau,d\tau)\right), \tag{4.18}$$

where

$$\begin{aligned}\widetilde{\Phi}_\theta(\mathsf{v}_\tau,\mathbb{V}_\tau,dB_\tau,d\tau)&=\Phi_\theta(\mathsf{v}_\tau,\mathbb{V}_\tau,dB_\tau,d\tau)\\&\quad-\varpi\left(\nabla^{(2)}(\mathsf{v}_\tau,\mathbb{V}_\tau)H(\mathrm{u}_\tau,e_i)+R(H(\mathrm{u}_\tau,e_i),\mathbb{V}_\tau)\mathsf{v}_\tau\right)e_i d\tau,\\\widetilde{\Phi}_\varpi(\mathsf{v}_\tau,\mathbb{V}_\tau,dB_\tau,d\tau)&=\Phi_\varpi(\mathsf{v}_\tau,\mathbb{V}_\tau,dB_\tau,d\tau)\\&\quad-\mathrm{u}_\tau^{-1}R\left(\mathrm{u}_\tau e_i,\mathrm{u}_\tau\theta\big(\nabla^{(2)}(\mathsf{v}_\tau,\mathbb{V}_\tau)H(\mathrm{u}_\tau,e_i)+R(H(\mathrm{u}_\tau,e_i),\mathbb{V}_\tau)\mathsf{v}_\tau\big)\right)\mathrm{u}_\tau d\tau.\end{aligned}$$

Proof. For i) and ii), it suffices to show i) since it implies ii) by applying the (θ,ϖ) map. Regard the tangent map $\left[D^{(1)}F_t(\mathrm{u}_0,w)\right]$ as a random matrix solution $\mathbf{y}_t(w)$ to

$$d\mathbf{y}_t(w)=\nabla(\mathbf{y}_t(w))H(\mathrm{u}_t,\circ dB_t),\quad \mathbf{y}_0=\mathrm{Id}.$$

Put

$$\upsilon_t:=\mathcal{V}_0+\int_0^t\left[D^{(1)}F_\tau(\mathrm{u}_0,w)\right]^{-1}\left(\nabla^{(2)}(\mathsf{v}_\tau,\mathbb{V}_\tau)H(\mathrm{u}_\tau,\circ dB_\tau)+R(H(\mathrm{u}_\tau,\circ dB_\tau),\mathbb{V}_\tau)\mathsf{v}_\tau\right).$$

Then the differentiation rule of Stratonovich integral shows that

$$\begin{aligned}d(\mathbf{y}_t\upsilon_t)&=(\circ d\mathbf{y}_t)\upsilon_t+\mathbf{y}_t\circ d\upsilon_t\\&=\nabla(\mathbf{y}_t(w)\upsilon_t)H(\mathrm{u}_t,\circ dB_t)+\nabla^{(2)}(\mathsf{v}_t,\mathbb{V}_t)H(\mathrm{u}_t,\circ dB_t)+R(H(\mathrm{u}_t,\circ dB_t),\mathbb{V}_t)\mathsf{v}_t,\end{aligned}$$

where d should be understood as the covariant derivative. Since $\mathbf{y}_0\upsilon_0=\mathcal{V}_0=0$, we obtain

$$\mathcal{V}_t=\mathbf{y}_t\upsilon_t=\int_0^t\left[D^{(1)}F_{\tau,t}(\mathrm{u}_\tau,w)\right]\left(\nabla^{(2)}(\mathsf{v}_\tau,\mathbb{V}_\tau)H(\mathrm{u}_\tau,e_i)+R(H(\mathrm{u}_\tau,e_i),\mathbb{V}_\tau)\mathsf{v}_\tau\right)\circ dB_\tau^i.$$

Regard $\left[\widetilde{D^{(1)}F_t}(\cdot,w)\right]$ as a matrix solution $\mathbf{y}_t(w)$ to (4.11) with $\mathbf{y}_0=\mathrm{Id}$. Put

$$\widetilde{\upsilon}_t:=\mathcal{V}_0+\int_0^t\left[\widetilde{D^{(1)}F_\tau}(\mathrm{u}_0,w)\right]^{-1}\left(\widetilde{\Phi}_\theta(\mathsf{v}_\tau,\mathbb{V}_\tau,dB_\tau,d\tau),\widetilde{\Phi}_\varpi(\mathsf{v}_\tau,\mathbb{V}_\tau,dB_\tau,d\tau)\right).$$

Write $\mathbf{y}_t\upsilon_t:=((\mathbf{y}_t\upsilon_t)_\theta,(\mathbf{y}_t\upsilon_t)_\varpi)$, where $(\mathbf{y}_t\upsilon_t)_\theta\in\mathbb{R}^m$ and $(\mathbf{y}_t\upsilon_t)_\varpi\in\mathcal{F}(\mathbb{R}^m)$. Then the Itô form infinitesimal differentiation rule shows that

$$\begin{aligned}d(\mathbf{y}_t\widetilde{\upsilon}_t)&=(d\mathbf{y}_t)\widetilde{\upsilon}_t+\mathbf{y}_t d\widetilde{\upsilon}_t+d\mathbf{y}_t\cdot d\widetilde{\upsilon}_t\\&=\big((\mathbf{y}_t\widetilde{\upsilon}_t)_\varpi dB_t(w)+\mathrm{Ric}(\mathrm{u}_t(\mathbf{y}_t\widetilde{\upsilon}_t)_\theta)dt+\Phi_\theta(\mathsf{v}_t,\mathbb{V}_t,dB_t,dt),\\&\quad\mathrm{u}_t^{-1}R\left(\mathrm{u}_t dB_t(w),\mathrm{u}_t(\mathbf{y}_t\widetilde{\upsilon}_t)_\theta\right)\mathrm{u}_t+\mathrm{u}_t^{-1}R\left(\mathrm{u}_t e_i,\mathrm{u}_t(\mathbf{y}_t\widetilde{\upsilon}_t)_\varpi e_i\right)\mathrm{u}_t\ dt\\&\quad+\mathrm{u}_t^{-1}\nabla R(\mathrm{u}_t e_i)\left(\mathrm{u}_t e_i,\mathrm{u}_t(\mathbf{y}_t\widetilde{\upsilon}_t)_\theta\right)\mathrm{u}_t\ dt+\Phi_\varpi(\mathsf{v}_t,\mathbb{V}_t,dB_t,dt)\big).\end{aligned}$$

This means $\mathbf{y}_t\widetilde{\upsilon}_t$ with $\widetilde{\upsilon}_0=(0,0)$ solves (4.15) and (4.16). Thus (4.18) holds true. □

For $\big[D^{(2)}F_t(\mathrm{u}_0,w)\big]$ on $T_{(\mathrm{u},\mathrm{v})}T_{\mathrm{u}}\mathcal{F}(\widetilde{M}) = T_{\mathrm{u}}\mathcal{F}(\widetilde{M})\times T_{\mathrm{u}}\mathcal{F}(\widetilde{M})$, we can define its Euclidean companion map $\widetilde{\big[D^{(2)}F_t(\mathrm{u}_0,w)\big]}$ on $T\mathcal{F}(\mathbb{R}^m)\times T\mathcal{F}(\mathbb{R}^m)$ as follows. For $\big(\underline{\mathbb{V}}_0,\underline{\mathbb{V}}_1\big)\in T\mathcal{F}(\mathbb{R}^m)\times T\mathcal{F}(\mathbb{R}^m)$, let $(\mathbb{V}_0,\mathbb{V}_1) := (((\theta,\varpi)_{\mathrm{u}_0})^{-1}(\underline{\mathbb{V}}_0),((\theta,\varpi)_{\mathrm{u}_0})^{-1}(\underline{\mathbb{V}}_1))$. Let $\mathbb{V}_{i,t} := [D^{(1)}F_t(\mathrm{u}_0,w)]\mathbb{V}_i$ for $i=0,1$ and let $\mathrm{v}_t,\mathcal{V}_t$ be defined as in Lemma 4.5. Then

$$\widetilde{\big[D^{(2)}F_t(\mathrm{u}_0,w)\big]}\big(\underline{\mathbb{V}}_0,\underline{\mathbb{V}}_1\big) := \big((\theta,\varpi)(\mathbb{V}_{0,t}),(\theta,\varpi)(\mathbb{V}_{1,t}+\mathcal{V}_t)\big).$$

We can continue the above discussion to formulate $\big[D^{(l)}F_t(\cdot,w)\big]$, $3\leqslant l\leqslant k-2$. Put

$$\begin{cases} \big(\mathrm{u}^{(2)};\mathsf{v}^{(2)}\big) =: (\mathrm{u},\mathsf{v};\mathbb{V}_0,\mathbb{V}_1), \\ \big(\mathrm{u}^{(l)};\mathsf{v}^{(l)}\big) =: \big(\mathrm{u}^{(l-1)},\mathsf{v}^{(l-1)};\mathbb{V}_0^{(l-1)},\mathbb{V}_1^{(l-1)}\big),\ \forall(\mathbb{V}_0^{(l-1)},\mathbb{V}_1^{(l-1)})\in T_{u^{(l-1)}}T^{l-1}\mathcal{F}(\widetilde{M}). \end{cases}$$

Then,

$$\begin{aligned}\Big[D^{(l)}F_t(\mathrm{u}_0,w)\Big]\big(\mathrm{u}^{(l)};\mathsf{v}^{(l)}\big) =& \Big(\Big[D^{(l-1)}F_t(\mathrm{u}_0,w)\Big]\big(u^{(l-1)},\mathsf{v}^{(l-1)}\big);\ \big[D^{(l-1)}F_t(\mathrm{u}_0,w)\big](\mathbb{V}_0^{(l-1)}),\\ & \nabla_{\mathbb{V}_0^{(l-1)}}\big[D^{(l-1)}F_t(\mathrm{u}_0,w)\big](\mathsf{v}^{(l-1)}) + \big[D^{(l-1)}F_t(\mathrm{u}_0,w)\big](\mathbb{V}_1^{(l-1)})\Big)\end{aligned} \tag{4.19}$$

and the covariant derivative term $\nabla_{\mathbb{V}_0^{(l-1)}}\big[D^{(l-1)}F_t(\mathrm{u}_0,w)\big](\mathsf{v}^{(l-1)})$ involves a combination of the l'-th ($l'\leqslant l-1$) covariant derivatives

$$\nabla_{\mathbb{V}_{0,l'}}\nabla_{\mathbb{V}_{0,l'-1}}\cdots\nabla_{\mathbb{V}_{0,0}}\Big[D^{(1)}F_t(\mathrm{u},w)\Big](\mathsf{v}),\ \forall\mathsf{v},\mathbb{V}_{0,0},\cdots,\mathbb{V}_{0,l'}\in T_{\mathrm{u}}\mathcal{F}(\widetilde{M}), \tag{4.20}$$

where for $l'=1$, (4.20) was given in (4.13), and for $l'>1$, let $\tau\mapsto\mathcal{C}_{l'}(\tau) := \exp(\tau\mathbb{V}_{0,l'})$ be the geodesic passing through u and let $\mathsf{v}_{/\!/}(\tau),\mathbb{V}_{0,0\,/\!/}(\tau),\cdots,\mathbb{V}_{0,l'-1\,/\!/}(\tau)$ be the parallel transportations of $\mathsf{v},\mathbb{V}_{0,0},\cdots,\mathbb{V}_{0,l'-1}$ along $\mathcal{C}_{l'}$ to the point $\mathcal{C}_{l'}(\tau)$, then

$$\begin{aligned}&\nabla_{\mathbb{V}_{0,l'}}\nabla_{\mathbb{V}_{0,l'-1}}\cdots\nabla_{\mathbb{V}_{0,0}}\Big[D^{(1)}F_t(\mathrm{u},w)\Big](\mathsf{v})\\ &\quad= \frac{D}{\partial\tau}\Big(\nabla_{\mathbb{V}_{0,l'-1\,/\!/}(\tau)}\cdots\nabla_{\mathbb{V}_{0,0\,/\!/}(\tau)}\Big[D^{(1)}F_t(\mathcal{C}_{l'}(\tau),w)\Big](\mathsf{v}_{/\!/}(\tau))\Big)\Big|_{\tau=0}.\end{aligned}$$

The Stratonovich SDE of (4.20) involves $\{\nabla^\iota H\}_{\iota\leqslant l'}$ and $\{\nabla^\iota R\}_{\iota\leqslant l'-1}$. But the Itô SDE of (4.20) involves $\{\nabla^\iota H\}_{\iota\leqslant l'+1}$ and $\{\nabla^\iota R\}_{\iota\leqslant l'}$.

By Corollary 4.2, all the tangent maps $[D^{(l)}F_t(\cdot,w)]$ are invertible. The inverse maps $[D^{(l)}F_t(\cdot,w)]^{-1}$ can be formulated by the same equation as $[D^{(l)}F_t(\cdot,w)]$, but using the backward infinitesimals $d\overleftarrow{B}_\tau,-d\tau$ instead of $dB_\tau,d\tau$. We skip the details.

4.3. **Growth of the stochastic tangent maps in time.** We use the above SDEs to estimate the L^q-norm ($q\geqslant 1$) of $\|[D^{(l)}F_{\underline{t},\bar{t}}(\mathrm{u},w)]\|$ and $\sup_{0\leqslant\underline{t}<\bar{t}\leqslant T}\|[D^{(l)}F_{\underline{t},\bar{t}}(\mathrm{u},w)]\|$.

A useful tool to the L^q-norm estimations of stochastic integrals is Burkholder's inequality which can be obtained using Itô's formula for $|\cdot|^q$ and Doob's inequality of martingales.

Lemma 4.7. *(cf.* [**Kun2**, Theorem 2.3.12]*) For an $\mathcal{F}_\tau$-adapted $\mathbb{R}^m$ or $\mathcal{O}(\mathbb{R}^m)$ process f_τ,*

$$\mathbb{E}\left(\left|\int_t^{t'}\langle f_\tau, dB_\tau\rangle\right|^q\right) \leqslant \mathtt{C}_1(q)\cdot\mathbb{E}\left|\int_t^{t'}|f_\tau|^2\,d\tau\right|^{\frac{q}{2}},\ \forall q\geqslant 2, \tag{4.21}$$

where $\mathtt{C}_1(q)=(\frac{1}{2}q(q-1)(q/(q-1))^{q-2})^{\frac{q}{2}}$.

(When $q=2$, the inequality in (4.21) becomes an equality and is referred to as the isometry property of Brownian motion.)

We would also like to list a simple fact that will be used from time to time for computations in the remaining paper: for any $q\geqslant 1$ and $a_1,\cdots,a_{i_0}\in\mathbb{R}_+\cup\{0\}$, $i_0\in\mathbb{N}$,

$$(\sum_{i=1}^{i_0}a_i)^q\leqslant (i_0)^{q-1}\sum_{i=1}^{i_0}a_i^q. \tag{4.22}$$

We first consider the L^q-norm ($q\geqslant 1$) estimations of $\|[D^{(l)}F_{\underline{t},\bar{t}}(\mathrm{u},w)]\|$.

Proposition 4.8. *Let $g\in\mathcal{M}^k(M)$ ($k\geqslant 3$). For $x\in\widetilde{M}$, $T\in\mathbb{R}_+$, let $\{\mathrm{u_t}\}_{t\in[0,T]}$ be the solution to (4.7) in $\mathcal{O}^{\tilde{g}}(\widetilde{M})$ with $\mathrm{u}_0\in\mathcal{O}^{\tilde{g}}_x(\widetilde{M})$. Then for every l, $1\leqslant l\leqslant k-2$, and $q\geqslant 1$, there exist $\underline{c}(l,q)>0$, which depends on l,q and the norm bounds of $\{\nabla^{l'}H\}_{l'\leqslant l+1},\{\nabla^{l'}R\}_{l'\leqslant l}$, and $c(l,q)>0$, which depends on l,q and the norm bounds of $\{\nabla^{l'}H\}_{l'\leqslant 2},\nabla R$, such that*

$$\mathbb{E}\left\|\left[D^{(l)}F_{\underline{t},\bar{t}}(\mathrm{u}_{\underline{t}},w)\right]^{\pm 1}\right\|^q<\underline{c}(l,q)e^{c(l,q)T},\ \forall 0\leqslant\underline{t}<\bar{t}\leqslant T, \tag{4.23}$$

where we simply use the upper-script ± 1 to denote the map itself and its inverse.

Proof. Using the cocycle property of the map F_t, we obtain

$$\left[D^{(l)}F_{\underline{t},\bar{t}}(\mathrm{u}_{\underline{t}},w)\right]=\left[D^{(l)}F_{\bar{t}}(\mathrm{u}_0,w)\right]\circ\left[D^{(l)}F_{\underline{t}}(\mathrm{u}_0,w)\right]^{-1}.$$

So, by Hölder's inequality, it suffices to verify (4.23) for $q\geqslant 2$ and $0=\underline{t}<\bar{t}=t\leqslant T$. We only consider the estimation for the forward tangent maps since the estimations on the inverse tangent maps can be obtained similarly using the backward SDEs.

When $l=1$, (4.23) was inspected in [**Elw**, Proposition 5A, Chapter VIII]. Using this and the SDEs in Section 4.2, we can obtain all the other estimations in (4.23). Recall that

$$\left[D^{(2)}F_t(\mathrm{u}_0,w)\right](\mathrm{u}_0,\mathsf{v};\mathbb{V}_0,\mathbb{V}_1)=$$
$$\left(\left[D^{(1)}F_t(\mathrm{u}_0,w)\right](\mathrm{u}_0;\mathsf{v}),\left[D^{(1)}F_t(\mathrm{u}_0,w)\right](\mathbb{V}_0),\nabla_{\mathbb{V}_0}\left[D^{(1)}F_t(\mathrm{u}_0,w)\right](\mathsf{v})+\left[D^{(1)}F_t(\mathrm{u}_0,w)\right](\mathbb{V}_1)\right).$$

To verify (4.23) for $l=2$, it remains to analyze the L^q-norm ($q\geqslant 2$) of the mapping

$$(\mathsf{v},\mathbb{V}_0)\mapsto\mathcal{V}_t(\mathsf{v},\mathbb{V}_0,w)=\nabla_{\mathbb{V}_0}\left[D^{(1)}F_t(\mathrm{u}_0,w)\right](\mathsf{v}).$$

Equivalently, we can consider the mapping $(\mathsf{v}, \mathbb{V}_0) \mapsto (\theta, \varpi)\big(\mathcal{V}_t\big)$ for $\mathsf{v}, \mathbb{V}_0$ with norm 1. Put $\mathsf{v}_\tau := [D^{(1)}F_\tau(\mathrm{u}_0, w)]\mathsf{v}$, $\mathbb{V}_\tau := [D^{(1)}F_\tau(\mathrm{u}_0, w)]\mathbb{V}_0$. Let

$$
\begin{aligned}
(4.24)\qquad \mathrm{A}_\tau(w) &:= (\theta, \varpi)_{\mathrm{u}_\tau}\left(\nabla^{(2)}(\mathsf{v}_\tau, \mathbb{V}_\tau)H(\mathrm{u}_\tau, \cdot) + R(H(\mathrm{u}_\tau, \cdot), \mathbb{V}_\tau)\mathsf{v}_\tau\right),\\
\mathrm{B}_\tau(w) &:= (\theta, \varpi)_{\mathrm{u}_\tau}\left(\Big[H(\mathrm{u}_\tau, e_i), \nabla^{(2)}(\mathsf{v}_\tau, \mathbb{V}_\tau)H(\mathrm{u}_\tau, e_i) + R(H(\mathrm{u}_\tau, e_i), \mathbb{V}_\tau)\mathsf{v}_\tau\Big]\right),\\
\mathrm{C}_\tau(w) &:= \Big(\varpi\left(\nabla^{(2)}(\mathsf{v}_\tau, \mathbb{V}_\tau)H(\mathrm{u}_\tau, e_i) + R(H(\mathrm{u}_\tau, e_i), \mathbb{V}_\tau)\mathsf{v}_\tau\right)e_i\ ,\\
&\qquad \mathrm{u}_\tau^{-1}R\left(\mathrm{u}_\tau e_i, \mathrm{u}_\tau\theta\big(\nabla^{(2)}(\mathsf{v}_\tau, \mathbb{V}_\tau)H(\mathrm{u}_\tau, e_i) + R(H(\mathrm{u}_\tau, e_i), \mathbb{V}_\tau)\mathsf{v}_\tau\big)\right)\mathrm{u}_\tau\Big)
\end{aligned}
$$

and let

$$
\begin{aligned}
\widetilde{\mathrm{A}}_t(w) &:= \int_0^t \big[\widetilde{D^{(1)}F_{\tau,t}}(\mathrm{u}_\tau, w)\big]\mathrm{A}_\tau(w)\ dB_\tau(w),\\
\widetilde{\mathrm{B}}_t(w) &:= \int_0^t \big[\widetilde{D^{(1)}F_{\tau,t}}(\mathrm{u}_\tau, w)\big]\mathrm{B}_\tau(w)\ d\tau,\ \widetilde{\mathrm{C}}_t(w) := \int_0^t \big[\widetilde{D^{(1)}F_{\tau,t}}(\mathrm{u}_\tau, w)\big]\mathrm{C}_\tau(w)\ d\tau.
\end{aligned}
$$

By Corollary 4.6,

$$
(\theta, \varpi)_{\mathrm{u}_t}(\mathcal{V}_t) = \widetilde{\mathrm{A}}_t(w) + \widetilde{\mathrm{B}}_t(w) + \widetilde{\mathrm{C}}_t(w).
$$

Using (4.22), we obtain

$$
3^{1-q}\mathbb{E}\big\|(\theta, \varpi)_{\mathrm{u}_t}(\mathcal{V}_t)\big\|^q \leqslant\ \mathbb{E}\big\|\widetilde{\mathrm{A}}_t(w)\big\|^q + \mathbb{E}\big\|\widetilde{\mathrm{B}}_t(w)\big\|^q + \mathbb{E}\big\|\widetilde{\mathrm{C}}_t(w)\big\|^q =: (\mathrm{A}) + (\mathrm{B}) + (\mathrm{C}).
$$

For (A), it is true by Burkholder's inequality and Hölder's inequality that

$$
\begin{aligned}
(\mathrm{A}) \leqslant& \mathtt{C}_1(q)\mathbb{E}\left|\int_0^t \Big\|\big[\widetilde{D^{(1)}F_{\tau,t}}(\mathrm{u}_\tau, w)\big]\mathrm{A}_\tau(w)\Big\|^2\ d\tau\right|^{\frac{q}{2}}\\
\leqslant& T^{\frac{1}{2}q-1}\mathtt{C}_1(q)\int_0^t \mathbb{E}\Big\|\big[\widetilde{D^{(1)}F_{\tau,t}}(\mathrm{u}_\tau, w)\big]\mathrm{A}_\tau(w)\Big\|^q\ d\tau\\
\leqslant& T^{\frac{1}{2}q-1}\mathtt{C}_1(q)\int_0^t \left(\mathbb{E}\Big\|\big[\widetilde{D^{(1)}F_{\tau,t}}(\mathrm{u}_\tau, w)\big]\Big\|^{2q}\right)^{\frac{1}{2}}\left(\mathbb{E}\left\|\mathrm{A}_\tau(w)\right\|^{2q}\right)^{\frac{1}{2}}\ d\tau.
\end{aligned}
$$

Using (4.24), we continue to compute that

$$
\mathbb{E}\left\|\mathrm{A}_\tau(w)\right\|^{2q} \leqslant C_{\mathrm{A}}^{2q}\left(\mathbb{E}\left\|\mathsf{v}_\tau(w)\right\|^{4q}\right)^{\frac{1}{2}}\cdot\left(\mathbb{E}\left\|\mathbb{V}_\tau(w)\right\|^{4q}\right)^{\frac{1}{2}} \leqslant (C'_{\mathrm{A}})^{2q}\mathbb{E}\Big\|\big[\widetilde{D^{(1)}F_\tau}(\mathrm{u}_0, w)\big]\Big\|^{4q},
$$

where $C_{\mathrm{A}}, C'_{\mathrm{A}}$ depend on the norm bounds of $\{\nabla^l H\}_{l\leqslant 2}$ and $\{\nabla^l R\}_{l\leqslant 1}$. Hence, using (4.23) with $l = 1$, we obtain

$$
(\mathrm{A}) \leqslant \mathtt{C}_1(q)(C'_{\mathrm{A}}\sqrt{T})^q\sqrt{c_1(2q)c_1(4q)}e^{\frac{1}{2}(c_1(2q)+c_1(4q))T}.
$$

Using Hölder's inequality and (4.23) with $l = 1$, we obtain

$$\begin{aligned}(\mathrm{B}) \leqslant & T^{q-1}\int_0^t \left(\mathbb{E}\left\|\widetilde{\left[D^{(1)}F_{\tau,t}(\mathrm{u}_\tau, w)\right]}\right\|^{2q}\right)^{\frac{1}{2}} \left(\mathbb{E}\left\|\mathrm{B}_\tau(w)\right\|^{2q}\right)^{\frac{1}{2}}\, d\tau \\ \leqslant & (CT)^q\sqrt{c_1(2q)c_1(4q)}e^{\frac{1}{2}(c_1(2q)+c_1(4q))T},\end{aligned}$$

where C depends on the norm bounds of $\{\nabla^l H\}_{l\leqslant 3}$ and $\{\nabla^l R\}_{l\leqslant 2}$, and the same inequality holds true for (C). This shows (4.23) for $\left[D^{(2)}F_t(\mathrm{u}_0, w)\right]$.

Let $l \geqslant 3$. Assume (4.23) holds true up to the $(l-1)$-th differentials. For the estimation on l-th differential, by the inductive definition of $\left[D^{(l)}F_t(\mathrm{u}_0, w)\right]$ (see (4.19)), it remains to estimate, for $\mathsf{v}^{(l-1)}, \mathbb{V}_0^{(l-1)}$ with norm 1, $\mathbb{E}\sup_{0\leqslant t\leqslant T}\left\|\mathcal{V}_t^{(l-1)}\right\|^q$, where

$$\mathcal{V}_t^{(l-1)} := \nabla_{\mathbb{V}_0^{(l-1)}}\left[D^{(l-1)}F_t(\mathrm{u}_0, w)\right](\mathsf{v}^{(l-1)}).$$

As in Corollary 4.6, we can formulate $\mathcal{V}_t^{(l-1)}$ using $\left[D^{(l-1)}F_{\tau,t}(\mathrm{u}_\tau, w)\right]$ by Duhamel's principle. Using the inductive assumption on (4.23) and following the argument in step 2, we can obtain $\underline{c}(l,q), c(l,q)$ of the prescribed type such that

$$\mathbb{E}\left\|\nabla_{(\cdot)}\left[D^{(l-1)}F_t(\mathrm{u}_0, w)\right](\cdot)\right\|^q < \underline{c}(l,q)e^{c(l,q)T}.$$

□

We continue to give the L^q-norm ($q \geqslant 1$) estimations of $\sup_{0\leqslant\underline{t}<\bar{t}\leqslant T}\|[D^{(l)}F_{\underline{t},\bar{t}}(\mathrm{u}, w)]\|$.

Recall the Dambis-Dubins-Schwarz Theorem which relates local martingales with Brownian motion using Lévy's characterization (see Section 4.4).

Lemma 4.9. *(cf.* [**RY**, Theorems 1.6 &1.7, p. 181]*) If* M *is a* $(\Omega, \mathtt{F}, \mathtt{P})$*-continuous local martingale vanishing at* 0*. Let* $T_t = \inf\{s : \langle \mathsf{M}, \mathsf{M}\rangle_s > t\}$*.*

i) *If* $\langle \mathsf{M}, \mathsf{M}\rangle_\infty = \infty$*, then* $\mathsf{B}_t = \mathsf{M}_{T_t}$ *is a* $(\mathtt{F}_{T_t})$*-Brownian motion and* $\mathsf{M}_t = B_{\langle \mathsf{M},\mathsf{M}\rangle_t}$*.*

ii) *If* $\langle \mathsf{M}, \mathsf{M}\rangle_\infty < \infty$*, then there exist an enlargement* $(\widetilde{\Omega}, \widetilde{\mathtt{F}}, \widetilde{\mathtt{P}})$ *of* $(\Omega, \mathtt{F}, \mathtt{P})$ *and a Brownian motion* $\widetilde{\mathsf{B}}$ *on* $\widetilde{\Omega}$ *independent of* M *such that the process*

$$\mathsf{B}_t = \begin{cases} \mathsf{M}_{T_t}, & \text{if } t < \langle \mathsf{M}, \mathsf{M}\rangle_\infty, \\ \mathsf{M}_\infty + \widetilde{\mathsf{B}}_{t-\langle \mathsf{M},\mathsf{M}\rangle_\infty}, & \text{if } t \geqslant \langle \mathsf{M}, \mathsf{M}\rangle_\infty \end{cases}$$

is a standard linear Brownian motion. The process W *given by*

$$\mathsf{W}_t = \begin{cases} \mathsf{M}_{T_t}, & \text{if } t < \langle \mathsf{M}, \mathsf{M}\rangle_\infty, \\ \mathsf{M}_\infty, & \text{if } t \geqslant \langle \mathsf{M}, \mathsf{M}\rangle_\infty \end{cases}$$

is a $(\widetilde{\mathtt{F}}_t)$*-Brownian motion stopped at* $\langle \mathsf{M}, \mathsf{M}\rangle_\infty$*.*

Given an $(\Omega, \mathtt{F}, \mathtt{P})$-Brownian motion B, we know that for almost all w, $t \mapsto \mathsf{B}_t(w)$ is not differentiable, but is $\mathtt{a}$-Hölder continuous for every $\mathtt{a} \in (0, 1/2)$. Let $\mathsf{T} > 0$ be fixed. Define

$$\|\mathsf{B}_{[0,\mathsf{T}]}(w)\|_{\mathtt{a}} := \sup_{0 \leqslant t \leqslant \mathsf{T}} |\mathsf{B}_t(w)| + \sup_{0 \leqslant t < t' \leqslant \mathsf{T}} \frac{|\mathsf{B}_{t'}(w) - \mathsf{B}_t(w)|}{|t' - t|^{\mathtt{a}}}. \tag{4.25}$$

The following result of Skorokhod ([**Sk**]) is a weaker form of the Fernique Theorem ([**Fe**]).

Lemma 4.10. *([**Sk**]) Let B be an $(\Omega, \mathtt{F}, \mathtt{P})$-Brownian motion. For any $\mathtt{a} \in (0, 1/2)$, there exists $\epsilon > 0$ such that $\mathbb{E}\left(e^{\epsilon \|\mathsf{B}_{[0,\mathsf{T}]}\|_{\mathtt{a}}}\right) < \infty$.*

Remark 4.11. The original proof of Lemma 4.10 is for $\mathsf{T} = 1$. In general, for any $t > 0$ and $a > 0$, B_t has the same distribution as $\sqrt{a}\mathsf{B}_{t/a}$. In particular, this holds for $a = \mathsf{T}$. A simple calculation shows that $\|\mathsf{B}_{[0,\mathsf{T}]}\|_{\mathtt{a}} \leqslant (\sqrt{\mathsf{T}} + \sqrt{\mathsf{T}}/\mathsf{T}^{\mathtt{a}})\|\mathsf{B}_{[0,1]}\|_{\mathtt{a}}$. Hence the Lemma is true for every T with $\epsilon(\mathsf{T}) = \min\{\epsilon(1)/2, \epsilon(1)/(\sqrt{\mathsf{T}} + \sqrt{\mathsf{T}}/\mathsf{T}^{\mathtt{a}})\}$.

Proposition 4.12. *Let $g \in \mathcal{M}^k(M)$ with $k \geqslant 3$. For $x \in \widetilde{M}$ and $T \in \mathbb{R}_+$, let $\{\mathrm{u}_\mathrm{t}\}_{t \in [0,T]}$ be the solution to (4.7) in $\mathcal{O}^{\tilde{g}}(\widetilde{M})$ with $\mathrm{u}_0 \in \mathcal{O}^{\tilde{g}}_x(\widetilde{M})$. Then for every l, $1 \leqslant l \leqslant k-2$, and $q \geqslant 1$, there exist $\underline{c}_l(q) > 0$, which depends on l, q and the norm bounds of $\{\nabla^{l'} H\}_{l' \leqslant l+1}, \{\nabla^{l'} R\}_{l' \leqslant l}$, and $c_l(q) > 0$, which depends on l, q and the norm bounds of $\{\nabla^{l'} H\}_{l' \leqslant 2}, \nabla R$, such that*

$$\mathbb{E} \sup_{0 \leqslant \underline{t} < \bar{t} \leqslant T} \left\| \left[D^{(l)} F_{\underline{t},\bar{t}}(\mathrm{u}_{\underline{t}}, w) \right]^{\pm 1} \right\|^q < \underline{c}_l(q) e^{c_l(q) T^3}. \tag{4.26}$$

Proof. We show (4.26) by induction. At each step, we only check the bound for the tangent map since the estimation on the inverse map can be obtained analogously using its SDE.

We begin with the $l = 1$ case. It suffices to do the computation for $\big[\widetilde{D^{(1)} F_{\underline{t},\bar{t}}}(\cdot, w)\big]$. Following [**Ma**, Theorem 5.1] (see Lemma 4.4), the solutions to (4.11) can be understood using multiplicative stochastic integral in Ito's form. For each $j \in \{1, 2, \cdots, m\}$ and $\mathrm{u} \in \mathcal{F}(\widetilde{M})$, define a $m(m+1) \times m(m+1)$ matrix $\mathbf{M}_j(\mathrm{u})$, which is an endomorphism from $T_o\mathcal{F}(\mathbb{R}^m)$ to itself, such that for $(z, \mathbf{z}) \in T_o\mathcal{F}(\mathbb{R}^m)$,

$$\mathbf{M}_j(\mathrm{u})\big((z, \mathbf{z})\big) = \left((\mathbf{z}^\iota_j)_{\iota=1}^m, (\mathbf{R}^\iota_{q,j,l}(\mathrm{u}) z^l)_{\iota,q=1}^m \right). \tag{4.27}$$

Define another $m(m+1) \times m(m+1)$ matrix $\mathbf{N}(\mathrm{u})$ (or an endomorphism from $T_o\mathcal{F}(\mathbb{R}^m)$ to itself) such that for $(z, \mathbf{z}) \in T_o\mathcal{F}(\mathbb{R}^m)$,

$$\mathbf{N}(\mathrm{u})\big((z, \mathbf{z})\big) = \left(0, \mathbf{N}^\iota_{q,l}(\mathrm{u}) z^l\right), \tag{4.28}$$

where

$$\mathbf{N}^\iota_{q,l}(\mathrm{u}) := \sum_{j=1}^m \big\langle (\nabla(\mathrm{u}e_j) R\,(\mathrm{u}e_j, \mathrm{u}e_l)\,\mathrm{u}e_q, \mathrm{u}e_\iota \big\rangle.$$

Using $\mathbf{M}, \mathbf{N}$, we conclude from (4.11) that the Itô form of the SDE of $\bar{\mathbf{z}}_t := (z_t, \mathbf{z}_t)$ is

$$d\bar{\mathbf{z}}_t(w) = \sum_{j=1}^{m} \Big(\mathbf{M}_j(\mathrm{u}_t)\bar{\mathbf{z}}_t(w)\ dB_t^j(w) + [\mathbf{M}_j(\mathrm{u}_t)]^2\bar{\mathbf{z}}_t(w)\ dt \Big) + \mathbf{N}(\mathrm{u}_t)\bar{\mathbf{z}}_t(w)\ dt,$$

which means, in terms of the multiplicative stochastic integral,

$$\bar{\mathbf{z}}_t = e^{\left\{\int_0^t \mathbf{M}_j(\mathrm{u}_\tau)\ dB_\tau^j(w) + \mathbf{N}(\mathrm{u}_\tau)\ d\tau\right\}}\bar{\mathbf{z}}_0. \tag{4.29}$$

(The coefficient of $\mathbf{N}$ in (4.29) is different from that in [**Ma**, Theorem 5.1] since we are considering Brownian motion with generator Δ instead of $\Delta/2$.) Hence,

$$\Big\| \big[\widetilde{D^{(1)}F_{\underline{t},\bar{t}}}(\mathrm{u}_{\underline{t}}, w) \big] \Big\| \leqslant e^{\left\| \int_{\underline{t}}^{\bar{t}} \mathbf{M}_j(\mathrm{u}_\tau)\ dB_\tau^j(w) \right\|} \cdot e^{\int_{\underline{t}}^{\bar{t}} \|\mathbf{N}(\mathrm{u}_\tau)\|\ d\tau}. \tag{4.30}$$

Clearly,

$$\mathbb{E} \sup_{0 \leqslant \underline{t} < \bar{t} \leqslant T} e^{q\int_{\underline{t}}^{\bar{t}} \|\mathbf{N}(\mathrm{u}_\tau)\|\ d\tau} \leqslant \mathbb{E}\big(e^{q\int_0^T \|\mathbf{N}(\mathrm{u}_\tau)\|\ d\tau} \big) < e^{CTq},$$

where C is some constant depends on the norm bound of ∇R. Consider the matrix valued process

$$\mathsf{M}_t(w) := \int_0^t \mathbf{M}_j(\mathrm{u}_\tau)\ dB_\tau^j.$$

It is a continuous martingale with $\mathsf{M}_0 = 0$ and with the quadratic variation $\langle \mathsf{M}, \mathsf{M} \rangle_t \leqslant C_1 t$ for some constant C_1 which depends on the norm bound of R. By Lemma 4.9, there exist a continuous martingale $\widetilde{\mathsf{M}}$ and a Brownian motion B on an enlargement $(\widetilde{\Theta}_+, \widetilde{\mathcal{F}}, \widetilde{\mathrm{Q}})$ of $(\Theta_+, \mathcal{F}, \mathrm{Q})$ so that $\widetilde{\mathsf{M}}$ has the same law as M and

$$\widetilde{\mathsf{M}}_t = \mathsf{B}_{\langle \widetilde{\mathsf{M}}, \widetilde{\mathsf{M}} \rangle_t}.$$

Fix $\mathtt{a} \in (0, 1/2)$ and consider $\|\mathsf{B}_{[0, \frac{1}{2}C_1 T]}\|_{\mathtt{a}}$. By Lemma 4.10 and Remark 4.11, there is $\epsilon = \min\{\epsilon(1)/2, \epsilon(1)/(\sqrt{T} + \sqrt{T}/T^{\mathtt{a}})\}$ for some constant $\epsilon(1)$ such that

$$\mathbb{E}_{\widetilde{\mathrm{Q}}}\big(e^{\epsilon \left\| \mathsf{B}_{[0, \frac{1}{2}C_1 T]} \right\|_{\mathtt{a}}} \big) < \widetilde{C_1} < \infty,$$

where $\widetilde{C_1}$ only depends on C_1 and $\mathtt{a}$, and is independent of T. Let $t_1 = (\frac{1}{4}\epsilon q^{-1})^{\frac{1}{\mathtt{a}}}$ and let

$$i_1(\tau) := \max\{i \in \mathbb{N} :\ it_1 < \tau\},\ \forall \tau \in [0, T].$$

We can chop the integral expression of $\mathsf{M}_t - \mathsf{M}_{\underline{t}}$ into sub-integrals on successive intervals of length not exceeding t_1 as

$$\begin{aligned} \mathsf{M}_t - \mathsf{M}_{\underline{t}} &= \sum_{i_1(\underline{t}) < i \leqslant i_1(t)} \int_{(i-1)t_1}^{it_1} \mathbf{M}_j(\mathrm{u}_\tau) dB_\tau^j + \left(\int_{i_1(\underline{t})t_1}^{\underline{t}} + \int_{i_1(t)t_1}^{t} \right) \mathbf{M}_j(\mathrm{u}_\tau) dB_\tau^j \\ &= \sum_{i_1(\underline{t}) < i \leqslant i_1(t)} \big(\mathsf{M}_{it_1} - \mathsf{M}_{(i-1)t_1} \big) + \big(\mathsf{M}_{\underline{t}} - \mathsf{M}_{i_1(\underline{t})t_1} \big) + \big(\mathsf{M}_t - \mathsf{M}_{i_1(t)t_1} \big). \end{aligned} \tag{4.31}$$

By the definition of the Hölder norm $\|\cdot\|_{\mathsf{a}}$ (see (4.25)), for any $0 \leqslant \underline{\tau} < \tau \leqslant T$,

$$\|\mathsf{M}_{\underline{\tau}} - \mathsf{M}_{\tau}\| \leqslant (\tau - \underline{\tau})^{\mathsf{a}} \big\|\mathsf{B}_{[0,\frac{1}{2}C_1T]}\big\|_{\mathsf{a}}.$$

In particular, we have

$$\|\mathsf{M}_{\underline{t}} - \mathsf{M}_{i_1(\underline{t})t_1}\|,\ \|\mathsf{M}_t - \mathsf{M}_{i_1(t)t_1}\| \leqslant t_1^{\mathsf{a}} \big\|\mathsf{B}_{[0,\frac{1}{2}C_1T]}\big\|_{\mathsf{a}}. \tag{4.32}$$

Hence we can conclude from (4.31) and (4.32) that

$$\sup_{0\leqslant \underline{t}<\bar{t}\leqslant T} e^{4\left\|q\int_{\underline{t}}^{\bar{t}} \mathbf{M}_j(\mathrm{u}_\tau)\ dB_\tau^j(w)\right\|} \leqslant \Big(\prod_{i=1}^{i_1(T)} e^{q\left\|\mathsf{M}_{it_1}-\mathsf{M}_{(i-1)t_1}\right\|}\Big) \cdot e^{2qt_1^{\mathsf{a}}\left\|\mathsf{B}_{[0,\frac{1}{2}C_1T]}\right\|_{\mathsf{a}}}.$$

Using Hölder's inequality and then the Markov property of M_t, we obtain

$$\begin{aligned}
\mathbb{E} \sup_{0\leqslant \underline{t}<\bar{t}\leqslant T} e^{4\left\|q\int_{\underline{t}}^{\bar{t}} \mathbf{M}_j(\mathrm{u}_\tau)\ dB_\tau^j(w)\right\|} &\leqslant \mathbb{E}_{\widetilde{\mathbb{Q}}}\left(\prod_{i=1}^{i_1(T)} e^{2q\left\|\mathsf{M}_{it_1}-\mathsf{M}_{(i-1)t_1}\right\|}\right) \cdot \mathbb{E}_{\widetilde{\mathbb{Q}}}\left(e^{4qt_1^{\mathsf{a}}\left\|\mathsf{B}_{[0,\frac{1}{2}C_1T]}\right\|_{\mathsf{a}}}\right) \\
&\leqslant \prod_{i=1}^{i_1(T)} \mathbb{E}_{\widetilde{\mathbb{Q}}}\left(e^{2q\left\|\mathsf{M}_{it_1}-\mathsf{M}_{(i-1)t_1}\right\|}\right) \cdot \mathbb{E}_{\widetilde{\mathbb{Q}}}\left(e^{\epsilon\left\|\mathsf{B}_{[0,\frac{1}{2}C_1T]}\right\|_{\mathsf{a}}}\right) \\
&\leqslant \left(\mathbb{E}_{\widetilde{\mathbb{Q}}}\left(e^{\epsilon\left\|\mathsf{B}_{[0,\frac{1}{2}C_1T]}\right\|_{\mathsf{a}}}\right)\right)^{i_1(T)+1}.
\end{aligned}$$

By our choice of ϵ,

$$i_1(T) \leqslant \begin{cases} (8q\epsilon(1)^{-1})^{\frac{1}{\mathsf{a}}}T, & \text{if } T \leqslant 1, \\ (8q\epsilon(1)^{-1})^{\frac{1}{\mathsf{a}}}T^{1+\frac{1}{2\mathsf{a}}}, & \text{if } T > 1. \end{cases}$$

Hence, if we take $\mathsf{a} \in (1/4, 1/2)$, then $i_1(T) \leqslant (8q\epsilon(1)^{-1})^{\frac{1}{\mathsf{a}}}(1+T^3)$. This shows there is some constant $c(q)$ which depends on q and the norm bound of R such that

$$\mathbb{E} \sup_{0\leqslant \underline{t}<\bar{t}\leqslant T} e^{4\left\|q\int_{\underline{t}}^{\bar{t}} \mathbf{M}_j(\mathrm{u}_\tau)\ dB_\tau^j(w)\right\|} < e^{c(q)(1+T^3)}.$$

When $l = 2$, it is true by the cocycle property of the map F_t that

$$\sup_{0\leqslant \underline{t}<\bar{t}\leqslant T} \Big\|\big[D^{(2)}F_{\underline{t},\bar{t}}(\mathrm{u}_{\underline{t}}, w)\big]\Big\|^q \leqslant \sup_{0\leqslant \bar{t}\leqslant T} \Big\|\big[D^{(2)}F_{\bar{t}}(\mathrm{u}_0, w)\big]\Big\|^q \cdot \sup_{0\leqslant \underline{t}\leqslant T} \Big\|\big[D^{(2)}F_{\underline{t}}(\mathrm{u}_0, w)\big]^{-1}\Big\|^q.$$

Using Hölder's inequality, we see that to conclude (4.26) for the $i = 2$ case, it suffices to show there are $\underline{c}_2(q), c_2(q)$ of the types that were prescribed in the proposition such that

$$\mathbb{E} \sup_{0\leqslant t\leqslant T} \Big\|\big[D^{(2)}F_t(\mathrm{u}_0, w)\big]^{\pm 1}\Big\|^{2q} < \underline{c}_2(q)e^{c_2(q)T^3}. \tag{4.33}$$

Since (4.26) is valid for the $l = 1$ case, to obtain (4.33), it remains to analyze

$$\mathbb{E} \sup_{0\leqslant t\leqslant T} \|(\theta, \varpi)_{\mathrm{u}_t}(\mathcal{V}_t)\|^{2q},$$

where $\mathcal{V}_t := \nabla_{\mathbb{V}_0}\big[D^{(1)}F_t(\mathrm{u}_0, w)\big](\mathtt{v})$ and $\mathtt{v}, \mathbb{V}_0$ have norm 1. Using the notations in the proof of Proposition 4.8, we have

$$\begin{aligned}3^{1-2q}\mathbb{E}\sup_{0\leqslant t\leqslant T}\|(\theta,\varpi)_{\mathrm{u}_t}(\mathcal{V}_t)\|^{2q} &\leqslant \mathbb{E}\sup_{0\leqslant t\leqslant T}\Big\|\widetilde{\mathrm{A}}_t(w)\Big\|^{2q} + \mathbb{E}\sup_{0\leqslant t\leqslant T}\Big\|\widetilde{\mathrm{B}}_t(w)\Big\|^{2q} + \mathbb{E}\sup_{0\leqslant t\leqslant T}\Big\|\widetilde{\mathrm{C}}_t(w)\Big\|^{2q}\\ &=: (\widetilde{\mathrm{A}}) + (\widetilde{\mathrm{B}}) + (\widetilde{\mathrm{C}}).\end{aligned}$$

For $(\widetilde{\mathrm{A}})$, it is true by Doob's inequality of sub-martingales and Burkholder's inequality that

$$\begin{aligned}(\widetilde{\mathrm{A}}) \leqslant& \mathtt{C}(2q)\mathbb{E}\left\|\int_0^T \big[\widetilde{D^{(1)}F_{\tau,T}}(\mathrm{u}_\tau, w)\big]\mathrm{A}_\tau(w)\ dB_\tau(w)\right\|^{2q}\\ \leqslant& \mathtt{C}(2q)\mathtt{C}_1(2q)\mathbb{E}\left|\int_0^T \Big\|\big[\widetilde{D^{(1)}F_{\tau,T}}(\mathrm{u}_\tau, w)\big]\mathrm{A}_\tau(w)\Big\|^2\ d\tau\right|^q\\ \leqslant& \mathtt{C}(2q)\mathtt{C}_1(2q)T^q\left(\mathbb{E}\sup_{0\leqslant \underline{t}<\bar{t}\leqslant T}\Big\|\big[\widetilde{D^{(1)}F_{\underline{t},\bar{t}}}(\mathrm{u}_{\underline{t}}, w)\big]\Big\|^{4q}\right)^{\frac{1}{2}}\cdot\left(\mathbb{E}\sup_{0\leqslant\tau\leqslant T}\|\mathrm{A}_\tau(w)\|^{4q}\right)^{\frac{1}{2}},\end{aligned}$$

where $\mathtt{C}(q) := (q/q-1)^q$ and $\mathtt{C}_1(q)$ is given in Lemma 4.7. Using (4.24), we compute that

$$\begin{aligned}\mathbb{E}\sup_{0\leqslant\tau\leqslant T}\|\mathrm{A}_\tau(w)\|^{4q} \leqslant& C_{\mathrm{A}}^{4q}\left(\mathbb{E}\sup_{0\leqslant\tau\leqslant T}\|\mathtt{v}_\tau(w)\|^{8q}\right)^{\frac{1}{2}}\cdot\left(\mathbb{E}\sup_{0\leqslant\tau\leqslant T}\|\mathbb{V}_\tau(w)\|^{8q}\right)^{\frac{1}{2}}\\ \leqslant& (C'_{\mathrm{A}})^{4q}\mathbb{E}\sup_{0\leqslant\underline{t}<\bar{t}\leqslant T}\Big\|\big[\widetilde{D^{(1)}F_{\underline{t},\bar{t}}}(\mathrm{u}_{\underline{t}}, w)\big]\Big\|^{8q},\end{aligned}$$

where $C_{\mathrm{A}}, C'_{\mathrm{A}}$ depend on the norm bounds of $\{\nabla^l H\}_{l\leqslant 2}$ and $\{\nabla^l R\}_{l\leqslant 1}$. With (4.26) for $l = 1$, we conclude that

$$(\widetilde{\mathrm{A}}) \leqslant C'(q)(C'_{\mathrm{A}}\sqrt{T})^{2q}\sqrt{c_1(4q)c_1(8q)}e^{\frac{1}{2}(c_1(4q)+c_1(8q))T^3}.$$

Using (4.26) with $l = 1$ and Hölder's inequality, we have

$$\begin{aligned}(\widetilde{\mathrm{B}}) \leqslant& T^{2q}\left(\mathbb{E}\sup_{0\leqslant\underline{t}<\bar{t}\leqslant T}\Big\|\big[\widetilde{D^{(1)}F_{\underline{t},\bar{t}}}(\mathrm{u}_{\underline{t}}, w)\big]\Big\|^{4q}\right)^{\frac{1}{2}}\cdot\left(\mathbb{E}\sup_{0\leqslant\tau\leqslant T}\|\mathrm{B}_\tau(w)\|^{4q}\right)^{\frac{1}{2}}\\ \leqslant& (CT)^{2q}\left(\mathbb{E}\sup_{0\leqslant\underline{t}<\bar{t}\leqslant T}\Big\|\big[\widetilde{D^{(1)}F_{\underline{t},\bar{t}}}(\mathrm{u}_{\underline{t}}, w)\big]\Big\|^{4q}\right)^{\frac{1}{2}}\cdot\left(\mathbb{E}\sup_{0\leqslant\underline{t}<\bar{t}\leqslant T}\Big\|\big[\widetilde{D^{(1)}F_{\underline{t},\bar{t}}}(\mathrm{u}_{\underline{t}}, w)\big]\Big\|^{8q}\right)^{\frac{1}{2}}\\ \leqslant& (CT)^{2q}\sqrt{c_1(4q)c_1(8q)}e^{\frac{1}{2}(c_1(4q)+c_1(8q))T^3}\end{aligned}$$

and the same inequality holds true for (C), where C depends on the norm bounds of $\{\nabla^l H\}_{l\leqslant 3}$, $\{\nabla^l R\}_{l\leqslant 2}$. This shows (4.33) for $\big[D^{(2)}F_t(\mathrm{u}_0, w)\big]$. Similarly, we can obtain the estimation on $\big[D^{(2)}F_t(\mathrm{u}_0, w)\big]^{-1}$. This finishes the proof of (4.26) for the $l = 2$ case.

Let $l \geqslant 3$. Assume (4.26) holds true for tangents up to the $(l-1)$-th order. For the estimation on l-th tangent map, using the cocycle property of F_t, it suffices to show

$$\mathbb{E} \sup_{0\leqslant t\leqslant T} \left\| \left[D^{(l)}F_t(\mathrm{u}_0, w)\right]^{\pm 1} \right\|^q < \underline{c}_l(q) e^{c_l(q)T^3},$$

where $\underline{c}_l(q), c_l(q)$ are the same types of the constants that were prescribed in the propsition. By the inductive definition of $\left[D^{(l)}F_t(\mathrm{u}_0, w)\right]$ (see (4.19)), it remains to show

$$\mathbb{E} \sup_{0\leqslant t\leqslant T} \left\| \nabla_{(\cdot)} \left[D^{(l-1)}F_t(\mathrm{u}_0, w)\right](\cdot) \right\|^q < \underline{c}_l(q) e^{c_l(q)T^3}. \tag{4.34}$$

This can be done as in the $l = 2$ case by formulating $\mathcal{V}_t^{(l-1)}$ in terms of $\left[D^{(l-1)}F_{\tau,t}(\mathrm{u}_\tau, w)\right]$ by Duhamel's principle and using the inductive assumption on (4.26). □

4.4. **Brownian bridge and conditional estimations.** We want to further estimate the growth of (4.26) with respect to Brownian bridge distributions using their SDEs, which can be derived from the classical Cameron-Martin-Girsanov formula.

We begin with some classical estimations on heat kernels in the non-compact case.

Lemma 4.13. *(*[**Sa**, Theorem 6.1]*) Let $g \in \mathcal{M}^k(M)$ and let $p(t, x, y)$ be the heat kernel functions of the $\widetilde{g}$-Brownian motion on $\widetilde{M}$. There exist constants b_1, c_1, c_2, κ_1 (depending on n and curvature bound) such that for any $t > 0$ and $x, y \in \widetilde{M}$, we have*

$$p(t, x, y) \leqslant \frac{1}{\sqrt{\mathrm{Vol(x}, \sqrt{\mathrm{t}})\mathrm{Vol(y}, \sqrt{\mathrm{t}})}} e^{c_1(1+b_1 t+\sqrt{\kappa_1 t}) - \frac{d^2(x,y)}{c_2 t}}. \tag{4.35}$$

For later use, we would like to state a simplified rough version of (4.35): there are constants $\underline{c}_0$ (which depends on $\|g^\lambda\|_{C^0}$) and c_0 (which depends on $\|g^\lambda\|_{C^2}$) such that

$$p(T, x, y) \leqslant \underline{c}_0 T^{-m} e^{c_0(1+T)}. \tag{4.36}$$

Lemma 4.14. *(*[**Li**, Theorem 1.5]*) Let $g \in \mathcal{M}^k(M)$ and let $p(t, x, y)$ be the heat kernel functions of the $\widetilde{g}$-Brownian motion on $\widetilde{M}$. Let $T > 0$. There are constants $c(i, T)$, $i \leqslant k-2$, which depend on i, T and the curvature and its derivatives up to i-th order, such that, for all $(t, x, y) \in (0, T] \times \widetilde{M} \times \widetilde{M}$, the i-th covariant derivative of $\ln p$ satisfies*

$$\|\nabla^{(i)} \ln p(t, x, y)\| \leqslant c(i, T) \left(\frac{1}{t} d_{\widetilde{g}}(x, y) + \frac{1}{\sqrt{t}} \right)^i. \tag{4.37}$$

Let $T > 0$. For $x, y \in \widetilde{M}$, the distribution of the *Brownian bridge* from x to y in time T, i.e., the Brownian motion starting from x conditioning on paths that are at y at time T, is

$$\mathbb{P}_{x,y,T}(\cdot) := \mathbb{E}_{\mathbb{P}_x}\left(\cdot \middle| \mathrm{x}_T = y\right).$$

It is a probability on the bridge space

$$C_{x,y}([0, T], \widetilde{M}) := \{w \in C_x([0, T], M) : \ w_0 = x, w_T = y\}.$$

Proposition 4.15. *Write* $\mathbb{P}^*_{x,y,T} := p(T,x,y)\mathbb{P}_{x,y,T}$. *Fix* $T_0 > 0$. *For any* $q \in \mathbb{R}_+$ *and* $T > T_0$, *there exists* c *depending on* q, T, T_0 *and* $\|g\|_{C^2}$ *such that for all* $x, y \in \widetilde{M}$,

$$\mathbb{E}_{\mathbb{P}^*_{x,y,T}} e^{q\int_0^T \|\nabla \ln p(T-t,\mathrm{x}_t,y)\|\ dt} \leqslant e^{c(1+d(x,y))}. \tag{4.38}$$

Proof. By (4.37), there is some $\bar{c}$ which depends on $\|g\|_{C^2}$ and T such that

$$\int_0^T \|\nabla \ln p(T-\tau,\mathrm{x}_\tau,y)\|\ d\tau \leqslant \bar{c}\sqrt{T} + \bar{c}\int_0^T \frac{1}{T-\tau} d(\mathrm{x}_\tau,y)\ d\tau.$$

Hence it is true by Hölder's inequality that for $t_0 \in (0, \min\{1, T_0/2\})$ small,

$$\begin{aligned}\left(\mathbb{E}_{\mathbb{P}^*_{x,y,T}} e^{q\int_0^T \|\nabla \ln p(T-t,\mathrm{x}_t,y)\|\ dt}\right)^2 &\leqslant e^{2\bar{c}q\sqrt{T}} \mathbb{E}_{\mathbb{P}^*_{x,y,T}} e^{2\bar{c}q\int_0^{t_0} \frac{1}{\tau} d(\mathrm{y}_\tau,y)\ d\tau} \cdot \mathbb{E}_{\mathbb{P}^*_{x,y,T}} e^{2\bar{c}q\int_0^{T-t_0} \frac{1}{t_0} d(\mathrm{x}_\tau,y)\ d\tau} \\ &=: e^{2\bar{c}q\sqrt{T}} (\mathtt{E})(t_0)(\mathtt{F})(t_0),\end{aligned}$$

where $(\mathrm{y}_t)_{t\in[0,T]}$ denotes the Brownian motion starting from $y \in \widetilde{M}$. Let $t_0 < T_0/2$. Then for $(\mathtt{E})(t_0)$, by (4.36), we have

$$\begin{aligned}(\mathtt{E})(t_0) &= \mathbb{E}_{\mathbb{P}^*_{y,x,T}} e^{2\bar{c}q\int_0^{t_0} \frac{1}{\tau} d(\mathrm{y}_\tau,y)\ d\tau} \\ &= \mathbb{E}_{\mathbb{P}_y}\left(e^{2\bar{c}q\int_0^{t_0} \frac{1}{\tau} d(\mathrm{y}_\tau,y)\ d\tau} \cdot p(T-t_0,\mathrm{y}_{t_0},x)\right) \\ &\leqslant \underline{c}_0 2^m T_0^{-m} e^{c_0(1+T)} \mathbb{E}_{\mathbb{P}_y} e^{2\bar{c}q\int_0^{t_0} \frac{1}{\tau} d(\mathrm{y}_\tau,y)\ d\tau}.\end{aligned}$$

To show there is some small $t_0 > 0$ (depending on q, T, T_0 and $\|g\|_{C^2}$) such that $(\mathtt{E})(t_0)$ is bounded, we can use a trick from Driver ([**D**, Lemma 3.8]) to compare it with Euclidean Brownian motions. Find finite many smooth functions $\{f_i\}_{i=1}^l$ on $\widetilde{M}$ with $f_i(y) = 0$ and $d(z,y) \leqslant \sum_{i=1}^l |f_i(z)|$ for all $z \in \widetilde{M}$, where all f_i have bounded first and second order differentials on $\widetilde{M}$. So for an upper bound estimation of $(\mathtt{E})(t_0)$, it suffices to consider

$$\mathbb{E}_{\mathbb{P}_y} e^{2\bar{c}q\int_0^{t_0} \frac{1}{\tau}|f(\mathrm{y}_\tau)|\ d\tau} =: (\mathtt{E}')(t_0)$$

for any C^2 function f on $\widetilde{M}$ with bounded differentials up to second order. Let $(\mathrm{y}, \mho, B)$ be the triple which defines the Brownian motion on $\widetilde{M}$ starting from y. By Itô's formula,

$$|f(\mathrm{y}_t)| \leqslant \left|\int_0^t \mho_\tau^{-1}\nabla f(\mathrm{y}_\tau)\ dB_\tau + \int_0^t \Delta f(\mathrm{y}_\tau)\ d\tau\right| \leqslant \left|\int_0^t \mho_\tau^{-1}\nabla f(\mathrm{y}_\tau)\ dB_\tau\right| + C(f)t,$$

where $C(f)$ is some constant which bounds $|\Delta f|$. Write

$$\mathsf{M}'_t := \int_0^t \mho_\tau^{-1}\nabla f(\mathrm{y}_\tau)\ dB_\tau.$$

The process M'_t is a continuous martingale with $\mathsf{M}'_0 = 0$ and has quadratic variation $\langle \mathsf{M}', \mathsf{M}'\rangle_t \leqslant C't$ for some constant C' which depends on the bound of $|\nabla f|$. By Lemma

4.9, there exist a continuous martingale $\widetilde{\mathsf{M}}'$ and a Brownian motion B' on an enlargement $(\widetilde{\Omega}', \widetilde{\mathcal{F}}', \widetilde{\mathbb{P}}')$ of $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\widetilde{\mathsf{M}}'$ has the same law as M' and

$$\widetilde{\mathsf{M}}'_t = \mathsf{B}'_{\langle \widetilde{\mathsf{M}}', \widetilde{\mathsf{M}}' \rangle_t}.$$

Let $\mathtt{a} \in (0, 1/2)$. By Lemma 4.10, there is some $\epsilon' > 0$ which depends on $\|g\|_{C^2}$ such that $\mathbb{E}_{\widetilde{\mathbb{P}}'}\big(e^{\epsilon' \left\| \mathsf{B}'_{[0, \frac{1}{2}C']} \right\|_{\mathtt{a}}}\big)$ is finite. By the definition of the Hölder norm $\|\cdot\|_{\mathtt{a}}$,

$$\int_0^{t_0} \frac{1}{\tau} \|\mathsf{M}'_\tau\| \, d\tau \leqslant \left\|\mathsf{B}'_{[0, \frac{1}{2}C']}\right\|_{\mathtt{a}} \cdot \int_0^{t_0} \tau^{\mathtt{a}-1} \, d\tau \leqslant \frac{1}{\mathtt{a}} t_0^{\mathtt{a}} \left\|\mathsf{B}'_{[0, \frac{1}{2}C']}\right\|_{\mathtt{a}}.$$

Hence, for $t_0 = \min\{1, T_0/2, (\mathtt{a}(2\bar{c}q)^{-1}\epsilon')^{\frac{1}{\mathtt{a}}}\}$, we have

$$(\mathtt{E}')(t_0) \leqslant e^{2\bar{c}qt_0 C(f)} \cdot \mathbb{E}_{\mathbb{P}_x}\big(e^{2\bar{c}q \left\|\mathsf{M}'_{t_0}\right\|}\big) \leqslant \mathbb{E}_{\widetilde{\mathbb{P}}'}\big(e^{\epsilon' \left\|\mathsf{B}'_{[0, \frac{1}{2}C']}\right\|_{\mathtt{a}}}\big) < \infty.$$

For $(\mathtt{F})(t_0)$, by symmetry of the bridge distribution,

$$\begin{aligned}
(\mathtt{F})(t_0) &= \mathbb{E}_{\mathbb{P}^*_{x,y,T}} e^{2\bar{c}q \int_0^{\frac{1}{2}T} \frac{1}{t_0} d(\mathrm{x}_\tau, y) \, d\tau + 2\bar{c}q \int_{\frac{1}{2}T}^{T-t_0} \frac{1}{t_0} d(\mathrm{y}_{T-\tau}, y) \, d\tau} \\
&\leqslant e^{\bar{c}qt_0^{-1} T d(x,y)} \left(\mathbb{E}_{\mathbb{P}^*_{x,y,T}} e^{4\bar{c}q \int_0^{\frac{1}{2}T} \frac{1}{t_0} d(\mathrm{x}_\tau, x) \, d\tau} \right)^{\frac{1}{2}} \left(\mathbb{E}_{\mathbb{P}^*_{y,x,T}} e^{4\bar{c}q \int_0^{\frac{1}{2}T} \frac{1}{t_0} d(\mathrm{y}_\tau, y) \, d\tau} \right)^{\frac{1}{2}}.
\end{aligned}$$

By (4.35) and Markov property of p (see (4.40)),

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}^*_{y,x,T}} e^{4\bar{c}q \int_0^{\frac{1}{2}T} \frac{1}{t_0} d(\mathrm{y}_\tau, y) \, d\tau} &= \mathbb{E}_{\mathbb{P}_y} \left(e^{4\bar{c}qt_0^{-1} \int_0^{\frac{1}{2}T} d(\mathrm{y}_\tau, y) \, d\tau} \cdot p(\frac{1}{2}T, \mathrm{y}_{\frac{1}{2}T}, x) \right) \\
&\leqslant \underline{c}_0 2^m T_0^{-m} e^{c_0(1+T)} \mathbb{E}_{\mathbb{P}_y} e^{4\bar{c}qt_0^{-1} \int_0^{\frac{1}{2}T} d(\mathrm{y}_\tau, y) \, d\tau}.
\end{aligned}$$

Let $t'_0 < \min\{1, T_0/2\}$ be small. Partition $[0, T/2]$ into $0 = \tau^0 < \tau^1 < \cdots < \tau^N < T/2$, where $\tau^i := it'_0$ and $N := \max\{i, \tau^i < T/2\}$, and chop the integral $\int_0^{\frac{1}{2}T} d(\mathrm{y}_\tau, y) \, d\tau$ into pieces accordingly. Using the triangle inequality, we obtain

$$\begin{aligned}
\int_0^{\frac{1}{2}T} d(\mathrm{y}_\tau, y) \, d\tau &\leqslant \sum_{i=1}^N \left(\int_{\tau^{i-1}}^{\tau^i} d(\mathrm{y}_t, \mathrm{y}_{\tau^{i-1}}) \, dt + t'_0 d(\mathrm{y}_{\tau^{i-1}}, y) \right) + \int_{\tau^N}^{\frac{1}{2}T} d(\mathrm{y}_t, \mathrm{y}_{\tau^N}) \, dt + t'_0 d(\mathrm{y}_{\tau^N}, y) \\
&\leqslant \sum_{i=1}^N \left(\int_{\tau^{i-1}}^{\tau^i} d(\mathrm{y}_t, \mathrm{y}_{\tau^{i-1}}) \, dt + (N+1-i) t'_0 d(\mathrm{y}_{\tau^i}, \mathrm{y}_{\tau^{i-1}}) \right) + \int_{\tau^N}^{\frac{1}{2}T} d(\mathrm{y}_t, \mathrm{y}_{\tau^N}) \, dt.
\end{aligned}$$

Using the Markov property of the Brownian motion and Hölder's inequality, we see that

$$\begin{aligned}
\left(\mathbb{E}_{\mathbb{P}_y} e^{4\bar{c}qt_0^{-1} \int_0^{\frac{1}{2}T} d(\mathrm{y}_\tau, y) \, d\tau} \right)^2 &\leqslant \left(\sup_{y' \in \widetilde{M}} \mathbb{E}_{\mathbb{P}_{y'}} e^{\bar{q} \int_0^{t'_0} d(\mathrm{y}'_t, y') \, dt} \right)^{N+1} \cdot \prod_{i=1}^N \sup_{y' \in \widetilde{M}} \mathbb{E}_{\mathbb{P}_{y'}} e^{\bar{q}t'_0 (N+1-i) d(\mathrm{y}'_{t'_0}, y')} \\
&=: \big((\mathtt{F})_0\big)^{N+1} \prod_{i=1}^N (\mathtt{F})_i,
\end{aligned}$$

where $\bar{q} := 8\bar{c}qt_0^{-1}$, y'_t is the Brownian motion on $\widetilde{M}$ which starts from y' and $\mathbb{P}_{y'}$ is its distribution probability. For $(\mathtt{F})_0$, we estimate as in the first part. Note that $(\widetilde{M}, \widetilde{g})$ is the universal cover of the compact space (M, g), although the choice of the f_i may differ from point to point, we can ensure their differentials up to second order are uniformly bounded. So we can choose t'_0 (for instance, $t'_0 = \min\{1, T_0, \epsilon'/\bar{q}\}$) such that $(\mathtt{F})_0$ is bounded. Fix such a t'_0 and estimate $(\mathtt{F})_i$ using Lemma 4.13. We obtain some constant $c(t'_0)$ such that

$$(\mathtt{F})_i \leqslant c(t'_0) e^{c_2 \bar{q}^2 (t'_0)^3 (N+1-i)^2},$$

where c_2 is as in (4.35). Hence, there is some constant $c(\bar{q})$ depending on q, T, T_0 and $\|g\|_{C^2}$ such that

$$\mathbb{E}_{\mathbb{P}_y}\big(e^{4\bar{c}q' t_0^{-1} \int_0^{\frac{1}{2}T} d(\mathrm{y}_\tau, y)\ d\tau}\big) \leqslant e^{c(\bar{q})}.$$

So,

$$(\mathtt{F})(t_0) \leqslant \underline{c}_0 2^m T_0^{-m} e^{c(\bar{q}) + c_0(1+T) + \bar{c}qt_0^{-1} T d(x,y)}.$$

Putting the estimations on $(\mathtt{E})(t_0), (\mathtt{F})(t_0)$ together, we obtain (4.38). □

Consider the Wiener space $C_0([0,T], \mathbb{R}^m)$ with the standard filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and let $(B_t)_{t\in[0,T]}$ be an $(\mathcal{F}_t)$-Brownian motion starting from 0 with respect to a probability measure Q on $\mathcal{F}_T$. Let $f : [0,T] \mapsto \mathbb{R}^m$ be square integrable with respect to Lebesgue measure. Define a random process $(\mathrm{M}_t)_{t\in[0,T]}$ on $[0,T]$ satisfying $\mathrm{M}_0 = 1$ and Itô's SDE

$$d\mathrm{M}_t = \frac{1}{2}\mathrm{M}_t \langle f_t, dB_t\rangle.$$

Then

$$\mathrm{M}_t = e^{\{\frac{1}{2}\int_0^t \langle f_\tau, dB_\tau(w)\rangle - \frac{1}{4}\int_0^t |f_\tau|^2\ d\tau\}}.$$

Since $\mathbb{E}_{\mathrm{Q}}(e^{\frac{1}{4}\int_0^t |f_\tau|^2\ d\tau})$, $t \leqslant T$, are all finite, we have by Novikov ([**N**]), that $(\mathrm{M}_t)_{t\in[0,T]}$ is a continuous $(\mathcal{F}_t)_{t\in[0,T]}$-martingale, i.e.,

$$\mathbb{E}_{\mathrm{Q}}(\mathrm{M}_t) = 1, \quad \forall t \in [0,T].$$

For $t \in [0,T]$, let $\widetilde{\mathrm{Q}}_t$ be the probability on $C_0([0,T], \mathbb{R}^m)$, which is absolutely continuous with respect to Q with

$$\frac{d\widetilde{\mathrm{Q}}_t}{d\mathrm{Q}}(w) = \mathrm{M}_t(w).$$

Since M_t is a martingale, the projection of $\widetilde{\mathrm{Q}}_t$ on $\mathcal{F}_\tau$, $\tau < t$, is given by the same formula. The classical Cameron-Martin-Girsanov Theorem ([**CM1**, **CM2**, **Gi**]) says that the process $(B_t - \int_0^t f_\tau\ d\tau)_{t\in[0,T]}$ is a Brownian motion with respect to $\widetilde{\mathrm{Q}}_T$. In other words, we have that the probability Q on Wiener space is quasi-invariant under the transformation $\mathbf{T} : C_0([0,T], \mathbb{R}^m) \to C_0([0,T], \mathbb{R}^m) :\ \mathrm{w} \ \mapsto \mathrm{w} + \int_0^\cdot f_\tau\ d\tau$ with

$$\frac{d\mathrm{Q}\circ\mathbf{T}^{-1}}{d\mathrm{Q}}(w) = e^{\{\frac{1}{2}\int_0^T \langle f_\tau(w),\ dB_\tau(w)\rangle - \frac{1}{4}\int_0^T |f_\tau(w)|^2\ d\tau\}}. \tag{4.39}$$

As in the compact case (see [**Hs3**, Theorem 5.4.4]), we can deduce the SDE of the Brownian bridge on $\widetilde{M}$ from the Cameron-Martin-Girsanov Theorem. Let $(\mathrm{x}_t, \mathrm{u}_t)_{t\in[0,T]}$ be the stochastic pair which defines the Brownian motion starting from x up to time T. By the Markov property of p,

$$\left.\frac{d\mathbb{P}_{x,y,T}}{d\mathbb{P}_x}\right|_{\mathcal{F}_t} = \frac{p(T-t,\mathrm{x}_t,y)}{p(T,x,y)} = \frac{\widetilde{p}(T-t,\mathrm{u}_t,y)}{p(T,x,y)} =: \Xi_t,\ \forall t\in[0,T), \tag{4.40}$$

where $\widetilde{p}(t,\mathrm{u},y) := p(t,\pi(\mathrm{u}),y)$. Using (4.7) and the heat equation, one can calculate using Itô's formula to obtain

$$d\ln\Xi_t = \langle \mathrm{u}_t^{-1}\nabla^H \ln\widetilde{p}(T-t,\mathrm{u}_t,y), dB_t\rangle - \|\nabla^H\ln\widetilde{p}(T-t,\mathrm{u}_t,y)\|^2\ dt.$$

Hence

$$\left.\frac{d\mathbb{P}_{x,y,T}}{d\mathbb{P}_x}\right|_{\mathcal{F}_t} = e^{\left\{\int_0^t\langle \mathrm{u}_t^{-1}\nabla^H\ln\widetilde{p}(T-\tau,\mathrm{u}_\tau,y),dB_\tau\rangle - \int_0^t\|\nabla^H\ln\widetilde{p}(T-\tau,\mathrm{u}_\tau,y)\|^2 d\tau\right\}}.$$

Comparing this with (4.39), it implies

$$b_t := B_t - 2\int_0^t \mathrm{u}_\tau^{-1}\nabla^H\ln\widetilde{p}(T-\tau,\mathrm{u}_\tau,y)\ d\tau$$

is a Brownian motion with respect to $\mathbb{P}_{x,y,T}$ and hence Proposition 4.16 holds for $t\in[0,T)$. One can conclude that $\{\mathtt{U}_t\}_{t\in[0,T]}$ is a semi-martingale on $[0,T]$ since

$$\mathbb{E}_{\mathbb{P}_{y,x,T}}\left(\int_0^T \|\nabla\ln p(T-\tau,\mathrm{x}_\tau,y)\|d\tau\right) < \infty$$

is also true on the non-compact universal cover space $(\widetilde{M},\widetilde{g})$ by (4.38). In summary,

Lemma 4.16. *There is a Brownian motion $(b_t)_{t\in[0,T)}$ such that the horizontal lift $\mathtt{U}$ of the Brownian bridge $\mathtt{x}$ is a semi-martingale on $[0,T]$ which satisfies the SDE*

$$d\mathtt{U}_t = H(\mathtt{U}_t, e_i)\circ\left(db_t^i + 2H(\mathtt{U}_t,e_i)\ln\widetilde{p}_M(T-t,\mathtt{U}_t,x)dt\right). \tag{4.41}$$

In other words, the anti-development of the Brownian bridge $\mathtt{x}$ (i.e., the pre-image via parallelism, see Section 5.2 for more precise definition) is

$$W_t = b_t + 2\int_0^t \mathtt{U}_\tau^{-1}\nabla\ln p(T-\tau,\mathtt{x}_\tau,y)\ d\tau.$$

Now, we can use Proposition 4.15 and Lemma 4.16 to derive a bridge version of (4.26).

Proposition 4.17. *Let $g\in\mathcal{M}^k(M)$ ($k\geqslant 3$). For $x\in\widetilde{M}$, let $(\mathrm{u}_t)_{t\in[0,T]}$ be the solution to (4.7) in $\mathcal{O}^{\widetilde{g}}(\widetilde{M})$ with $\mathrm{u}_0\in\mathcal{O}_x^{\widetilde{g}}(\widetilde{M})$. For every $T_0>0$, l, $1\leqslant l\leqslant k-2$, $q\geqslant 1$ and $T>T_0$, there exist $\underline{c}_l'(q)>0$, which depends on l,q and the norm bounds of $\{\nabla^{l'}H\}_{l'\leqslant l}, \{\nabla^{l'}R\}_{l'\leqslant l}$, and $c_l'(q)>0$, which depends on l,q,T,T_0 and the norm bounds of $\{\nabla^{l'}R\}_{l'\leqslant 1}$, such that*

$$\mathbb{E}_{\mathbb{P}^*_{x,y,T}}\sup_{0\leqslant\underline{t}<\bar{t}\leqslant T}\left\|\left[D^{(l)}F_{\underline{t},\bar{t}}(\mathrm{u}_{\underline{t}},w)\right]^{\pm 1}\right\|^q < \underline{c}_l'(q)e^{c_l'(q)(1+d(x,y))},\ \forall x,y\in\widetilde{M}. \tag{4.42}$$

Proof. We show (4.42) by induction and at each step we only verify it for $\big[D^{(l)}F_{\underline{t},\bar{t}}(\mathrm{u}_{\underline{t}},w)\big]$.

When $l=1$, it suffices to do the estimation for $\big[\widetilde{D^{(1)}F_{\underline{t},\bar{t}}}(\cdot,w)\big]$. Using (4.30), we obtain

$$\mathbb{E}_{\mathbb{P}^*_{x,y,T}}\sup_{0\leqslant\underline{t}<\bar{t}\leqslant T}\Big\|\big[\widetilde{D^{(1)}F_{\underline{t},\bar{t}}}(\mathrm{u}_{\underline{t}},w)\big]\Big\|^q\leqslant\ e^{CTq}\mathbb{E}_{\mathbb{P}^*_{x,y,T}}\sup_{0\leqslant\underline{t}<\bar{t}\leqslant T}e^{q\left\|\int_{\underline{t}}^{\bar{t}}\mathbf{M}_j(\mathrm{u}_\tau)\ dB^j_\tau(w)\right\|}=:\ e^{CTq}(\mathbf{M}),$$

where C is some constant depending on the norm bound of ∇R. By Lemma 4.16,

$$dB_\tau=db_\tau+2\mathrm{u}_\tau^{-1}\nabla\ln p(T-\tau,\mathrm{x}_\tau,y)\ d\tau, \tag{4.43}$$

where $(b_\tau)_{\tau\in[0,T)}$ is a Brownian motion for $\mathbb{P}_{x,y,T}$. So,

$$\begin{aligned}(\mathbf{M})^2=&\left(\mathbb{E}_{\mathbb{P}^*_{x,y,T}}\sup_{0\leqslant\underline{t}<\bar{t}\leqslant T}e^{q\left\|\int_{\underline{t}}^{\bar{t}}\mathbf{M}_j(\mathrm{u}_\tau)\ (db_\tau(w)+2\mathrm{u}_\tau^{-1}\nabla\ln p(T-\tau,\mathrm{x}_\tau,y)\ d\tau)^j\right\|}\right)^2\\ \leqslant&\mathbb{E}_{\mathbb{P}^*_{x,y,T}}\sup_{0\leqslant\underline{t}<\bar{t}\leqslant T}e^{2q\left\|\int_{\underline{t}}^{\bar{t}}\mathbf{M}_j(\mathrm{u}_\tau)\ db^j_\tau(w)\right\|}\cdot\mathbb{E}_{\mathbb{P}^*_{x,y,T}}\sup_{0\leqslant\underline{t}<\bar{t}\leqslant T}e^{4q\left\|\int_{\underline{t}}^{\bar{t}}\mathbf{M}_j(\mathrm{u}_\tau)\ (\mathrm{u}_\tau^{-1}\nabla\ln p(T-\tau,\mathrm{x}_\tau,y))^j\ d\tau\right\|}\\ =:&(\mathbf{M})_1\cdot(\mathbf{M})_2.\end{aligned}$$

For $(\mathbf{M})_1$, the first step argument in Proposition 4.12 applies and shows that there is some constant $c(q)$ which depends on the norm bound of R and q such that

$$(\mathbf{M})_1<e^{c(q)(1+T^3)}.$$

For $(\mathbf{M})_2$, using the uniform boundedness of $\|\mathbf{M}_j(\mathrm{u})\|$ for $\mathrm{u}\in\mathcal{O}^{\widetilde{g}}(\widetilde{M})$, we obtain some $\widetilde{c}$ depending on the norm bound of R such that

$$(\mathbf{M})_2\leqslant\mathbb{E}_{\mathbb{P}^*_{y,x,T}}e^{\widetilde{c}q\int_0^T\|\nabla\ln p(T-t,\mathrm{x}_t,y)\|\ dt}.$$

So, by Proposition 4.15, there is some $\underline{c}$ depending on $\|g\|_{C^0}$ and some other $c(q)$ depending on $\widetilde{c},q,T,T_0$ and $\|g\|_{C^2}$ such that

$$(\mathbf{M})_2\leqslant e^{c(q)(1+d(x,y))}.$$

Putting all the estimations together, we obtain (4.42) for $i=1$.

When $l=2$, it suffices to give a conditional version of (4.33), which by the decomposition of $[D^{(2)}F_t]$ and the conclusion of (4.42) with $l=1$, can be reduced to the estimation of

$$(\mathbf{V}):=\mathbb{E}_{\mathbb{P}^*_{x,y,T}}\sup_{0\leqslant t\leqslant T}\|(\theta,\varpi)_{\mathrm{u}_t}(\mathcal{V}_t)\|^{2q}.$$

Let $\mathrm{A},\mathrm{B},\mathrm{C},\widetilde{\mathrm{A}},\widetilde{\mathrm{B}},\widetilde{\mathrm{C}}$ be given in the proof of Proposition 4.12. Then

$$\begin{aligned}3^{1-2q}(\mathbf{V})\leqslant&\mathbb{E}_{\mathbb{P}^*_{x,y,T}}\sup_{0\leqslant t\leqslant T}\left\|\widetilde{\mathrm{A}}_t(w)\right\|^{2q}+\mathbb{E}_{\mathbb{P}^*_{x,y,T}}\sup_{0\leqslant t\leqslant T}\left\|\widetilde{\mathrm{B}}_t(w)\right\|^{2q}+\mathbb{E}_{\mathbb{P}^*_{x,y,T}}\sup_{0\leqslant t\leqslant T}\left\|\widetilde{\mathrm{C}}_t(w)\right\|^{2q}\\ =:&\ (\mathbf{A})+(\mathbf{B})+(\mathbf{C}).\end{aligned}$$

For $(\mathbf{B})$, following its non-conditional estimation in the proof of Proposition 4.12, we obtain

$$\begin{aligned}(\mathbf{B}) \leqslant & (CT)^{2q}\left(\mathbb{E}_{x,y,T}^{\mathbb{P}^*}\sup_{0\leqslant \underline{t}<\bar{t}\leqslant T}\left\|\big[\widetilde{D^{(1)}F}_{\underline{t},\bar{t}}(\mathrm{u}_{\underline{t}},w)\big]\right\|^{4q}\mathbb{E}_{x,y,T}^{\mathbb{P}^*}\sup_{0\leqslant \underline{t}<\bar{t}\leqslant T}\left\|\big[\widetilde{D^{(1)}F}_{\underline{t},\bar{t}}(\mathrm{u}_{\underline{t}},w)\big]\right\|^{8q}\right)^{\frac{1}{2}}\\ \leqslant & (CT)^{2q}\sqrt{\underline{c}_1'(4q)\underline{c}_1'(8q)}e^{\frac{1}{2}(c_1'(4q)+c_1'(8q))(1+d(x,y))},\end{aligned}$$

where C depends on the norm bounds of $\{\nabla^l H\}_{l\leqslant 3}, \{\nabla^l R\}_{l\leqslant 2}$ and $\underline{c}_1', c_1'$ are from (4.42) for $l=1$, and the constants can be chosen such that the same bound is valid for $(\mathbf{C})$. For $(\mathbf{A})$, we use (4.43). Let

$$\begin{aligned}\overline{\mathrm{A}}_t^1(w) &:= \int_0^t \big[\widetilde{D^{(1)}F}_{\tau,t}(\mathrm{u}_\tau,w)\big]\mathrm{A}_\tau(w)\ db_\tau(w),\\ \overline{\mathrm{A}}_t^2(w) &:= \int_0^t \big[\widetilde{D^{(1)}F}_{\tau,t}(\mathrm{u}_\tau,w)\big]\mathrm{A}_\tau(w)\ 2(\lfloor \mathrm{u}_\tau\rceil^\lambda)^{-1}\nabla^\lambda \ln p(T-\tau,\lfloor \mathrm{x}_\tau\rceil^\lambda,y)\ d\tau.\end{aligned}$$

Then,

$$2^{1-2q}(\mathbf{A}) \leqslant \mathbb{E}_{x,y,T}^{\mathbb{P}^*}\sup_{0\leqslant t\leqslant T}\left\|\overline{\mathrm{A}}_t^1(w)\right\|^{2q} + \mathbb{E}_{x,y,T}^{\mathbb{P}^*}\sup_{0\leqslant t\leqslant T}\left\|\overline{\mathrm{A}}_t^2(w)\right\|^{2q} =: (\mathbf{A})_1 + (\mathbf{A})_2.$$

Using the Brownian character of b_τ with respect to $\mathbb{P}_{x,y,T}$, we can estimate $(\mathbf{A})_1$ as in the non-conditional case using Doob's inequality of sub-martingales and Burkholder's inequality. This gives

$$(\mathbf{A})_1 \leqslant \mathtt{C}(2q)\mathtt{C}_1(2q)T^q\left(\mathbb{E}_{x,y,T}^{\mathbb{P}^*}\sup_{0\leqslant \tau<t\leqslant T}\left\|\big[\widetilde{D^{(1)}F}_{\tau,t}(\mathrm{u}_\tau,w)\big]\right\|^{4q}\right)^{\frac{1}{2}}\cdot\left(\mathbb{E}_{x,y,T}^{\mathbb{P}^*}\sup_{0\leqslant \tau\leqslant T}\|\mathrm{A}_\tau\|^{4q}\right)^{\frac{1}{2}},$$

where $\mathtt{C}, \mathtt{C}_1$ are as in the proof of Proposition 4.12. Using (4.24), we compute that

$$\mathbb{E}_{x,y,T}^{\mathbb{P}^*}\sup_{0\leqslant \tau\leqslant T}\|\mathrm{A}_\tau\|^{jq} \leqslant (C_{\mathrm{A}}')^{jq}\mathbb{E}_{x,y,T}^{\mathbb{P}^*}\sup_{0\leqslant \tau\leqslant T}\left\|\big[\widetilde{D^{(1)}F_{0,\tau}}(\mathrm{u}_0,w)\big]\right\|^{2jq},\ \forall j\in\mathbb{N}, \tag{4.44}$$

where C_{A}' depends on the norm bounds of $\{\nabla^l H\}_{l\leqslant 2}, \{\nabla^l R\}_{l\leqslant 1}$. Hence, by (4.42) for $l=1$,

$$(\mathbf{A})_1 \leqslant C'(q)(C_{\mathrm{A}}'\sqrt{T})^{2q}\sqrt{c_1'(4q)c_1'(8q)}e^{\frac{1}{2}(c_1'(4q)+c_1'(8q))(1+d_{\tilde{g}}(x,y))}.$$

For $(\mathbf{A})_2$, we have

$$\begin{aligned}\big((\mathbf{A})_2\big)^3 \leqslant & \mathbb{E}_{x,y,T}^{\mathbb{P}^*}\sup_{0\leqslant \tau<t\leqslant T}\left\|\big[\widetilde{D^{(1)}F}_{\tau,t}(\mathrm{u}_\tau,w)\big]\right\|^{6q}\cdot\mathbb{E}_{x,y,T}^{\mathbb{P}^*}\sup_{0\leqslant \tau\leqslant T}\|\mathrm{A}_\tau\|^{6q}\\ & \cdot\mathbb{E}_{x,y,T}^{\mathbb{P}^*}\left|\int_0^T\|\nabla\ln p(T-\tau,\mathrm{x}_\tau,y)\|\ d\tau\right|^{6q}.\end{aligned}$$

Note that

$$\mathbb{E}_{x,y,T}^{\mathbb{P}^*}\left|\int_0^T\|\nabla\ln p(T-\tau,\mathrm{x}_\tau,y)\|\ d\tau\right|^{6q} \leqslant \mathbb{E}_{x,y,T}^{\mathbb{P}^*}e^{6q\int_0^T\|\nabla\ln p(T-t,\mathrm{x}_t,y)\|\ dt}.$$

So by Proposition 4.15, (4.44) and (4.42) for $l = 1$, we compute that

$$(\mathbf{A})_2 \leqslant (C'_{\mathrm{A}})^{2q} \sqrt[3]{\underline{c}'_1(6q)\underline{c}'_1(12q)\underline{c}(6q)}\, e^{\frac{1}{3}(c'_1(6q)+c'_1(12q)+c(6q))(1+d(x,y))}.$$

Hence $(\mathbf{V})$ has the same type of bound as in (4.42) for $l = 2$ as claimed.

Assume we have shown (4.42) for $l \leqslant l_0 - 1 \leqslant k - 3$. Using the induction assumption and (4.19), we can reduce the estimation of (4.26) at $l = l_0$ to the conditional estimation of (4.34), which can be done exactly as in the $l = 2$ step. □

4.5. **Regularity of the stochastic analogue of the geodesic flow.** Finally, we employ the SDE theory in the previous subsections of this section to discuss the regularity of the Brownian companion process u with respect to metric changes.

Let $\lambda \in (-1, 1) \mapsto g^\lambda \in \mathcal{M}^k(M)$ be a C^k curve. Each lifted metric $\widetilde{g}^\lambda$ in $\widetilde{M}$ determines a horizontal space $H^\lambda T\mathcal{F}(\widetilde{M})$ of the frame bundle space. For any $u \in \mathcal{F}(\widetilde{M})$, let $H^\lambda(u, e_i)$, $i = 1, \cdots, m$, be the vector in $H^\lambda_u T\mathcal{F}(\widetilde{M})$ which projects to ue_i. Since $g^\lambda \in \mathcal{M}^k(M)$, the map $u \mapsto H^\lambda(u, e_i), u \in \mathcal{F}(\widetilde{M})$, is C^{k-1} bounded. Hence the SDE

$$d\lfloor \mathrm{u}_t \rceil^\lambda = \sum_{i=1}^m H^\lambda(\lfloor \mathrm{u}_t \rceil^\lambda, e_i) \circ dB^i_t(w) \tag{4.45}$$

is solvable in $\mathcal{F}(\widetilde{M})$ for any initial $\lfloor \mathrm{u}_0 \rceil^\lambda \in \mathcal{F}_x(\widetilde{M})$, $x \in \widetilde{M}$. In particular, if $\lfloor \mathrm{u}_0 \rceil^\lambda \in \mathcal{O}^{\widetilde{g}^\lambda}_x(\widetilde{M})$, $\lfloor \mathrm{u}_t \rceil^\lambda$ remains in $\mathcal{O}^{\widetilde{g}^\lambda}(\widetilde{M})$ and its projection to $\widetilde{M}$ gives the stochastic process of the $\widetilde{g}^\lambda$-Brownian motion starting from x. Let $\lfloor F_t \rceil^\lambda : \lfloor \mathrm{u}_0 \rceil^\lambda \mapsto \lfloor \mathrm{u}_t \rceil^\lambda$ denote the flow map associated to (4.45). Let $[D^{(l)} \lfloor F_t \rceil^\lambda(\cdot, w)]$, $1 \leqslant l \leqslant k - 2$, be the l-th tangent map of $\lfloor F_t \rceil^\lambda$ and denote by $\widetilde{[D^{(l)} \lfloor F_t \rceil^\lambda(\cdot, w)]}$ its pull back map in $T^l\mathcal{F}(\mathbb{R}^m)$ via the map (θ, ϖ). They have the following regularity in λ by applying Proposition 4.1.

Lemma 4.18. *Let $\lambda \in (-1, 1) \mapsto g^\lambda \in \mathcal{M}^k(M)$ $(k \geqslant 3)$ be a C^k curve. Assume $H^\lambda(\cdot, e_i)$ has bounded norms (independent of λ) for the covariant derivatives up to the $(k-1)$-th order with respect to the reference metric $\widetilde{g}^0$.*

i) *Let $\lambda \mapsto \lfloor \mathrm{u}_0 \rceil^\lambda$ be a C^{k-2} curve in $\mathcal{F}(\widetilde{M})$ and let $\{\lfloor \mathrm{u}_t \rceil^\lambda\}_{t \in \mathbb{R}_+}$ be the solution to (4.45) with initial value $\lfloor \mathrm{u}_0 \rceil^\lambda$. Then there is a version of the solution to (4.45) such that almost surely, $\lfloor \mathrm{u}_t \rceil^\lambda(w)$ is C^{k-2} in λ for any $t \in \mathbb{R}_+$.*

ii) *For each l, $1 \leqslant l \leqslant k - 2$, the tangent map $[D^{(l)} \lfloor F_t \rceil^\lambda(\cdot, w)]$ is C^{k-2-l} in λ. In particular, for any $\mathrm{v} \in T^l\mathcal{F}(\widetilde{M})$, the map $\lambda \mapsto \big[D^{(l)} \lfloor F_t \rceil^\lambda(\cdot, w)\big] \mathrm{v}$ is C^{k-2-l}.*

Proof. Consider the stochastic process $\{\widetilde{\mathrm{u}}_t\}_{t \in \mathbb{R}_+}$ on $\mathcal{F}(\widetilde{M}) \times (-1, 1)$ with

$$d\widetilde{\mathrm{u}}_t = \sum_{i=1}^m \widetilde{H}_i(\widetilde{\mathrm{u}}_t) \circ dB^i_t(w), \text{ where } \widetilde{H}_i = (H^\lambda(\cdot, e_i), 0). \tag{4.46}$$

It has the solution $\widetilde{\mathrm{u}}_t = (\lfloor \mathrm{u}_t \rceil^\lambda, \lambda)$ for $\widetilde{\mathrm{u}}_0 = (\lfloor \mathrm{u}_0 \rceil^\lambda, \lambda)$, where $\lfloor \mathrm{u}_t \rceil^\lambda$ is the solution of (4.45) with initial value $\lfloor \mathrm{u}_0 \rceil^\lambda$. Since (4.46) is a C^{k-1} SDS on $\mathcal{F}(\widetilde{M}) \times (-1,1)$, we have by Proposition 4.1 ii) that for almost all w, the mapping $\widetilde{\mathrm{u}}_0(w) \to \widetilde{\mathrm{u}}_t(w)$ is C^{k-2}. Consequently, for any C^{k-2} curve $\lambda \mapsto \lfloor \mathrm{u}_0 \rceil^\lambda$, $\lfloor \mathrm{u}_t \rceil^\lambda(w)$ is C^{k-2} in λ for almost all w.

For each l, $1 \leqslant l \leqslant k-2$, the SDE of $[D^{(l)} \lfloor F_t \rceil^\lambda(\cdot, w)]$ was given in Section 4.1 and it forms a C^{k-1-l} SDS on $T\mathcal{F}(\widetilde{M})$. As in Lemma 4.18, we can treat the one-parameter SDEs of $[D^{(l)} \lfloor F_t \rceil^\lambda(\cdot, w)]$ as a C^{k-1-l} SDS on $T\mathcal{F}(\widetilde{M}) \times (-1,1)$ when $\lambda \mapsto g^\lambda$ is C^k in $\mathcal{M}^k(M)$. So Proposition 4.1 applies and shows ii). □

For $0 \leqslant \underline{t} < \bar{t} < \infty$, let $\lfloor F_{\underline{t},\bar{t}} \rceil^\lambda : \lfloor \mathrm{u}_{\underline{t}} \rceil^\lambda \mapsto \lfloor \mathrm{u}_{\bar{t}} \rceil^\lambda$ denote the flow map associated to (4.45) and let $[D^{(l)} \lfloor F_{\underline{t},\bar{t}} \rceil^\lambda(\cdot, w)]$, $l \leqslant k-2$, be its l-th tangent map. As a corollary of the cocycle property of $\lfloor F_{\underline{t},\bar{t}} \rceil^\lambda$ and Lemma 4.18, $\big[D^{(l)} \lfloor F_{\underline{t},\bar{t}} \rceil^\lambda(\cdot, w)\big]$ is C^{k-2-l} differentiable in λ and we denote its j-th differential by $\big([D^{(l)} \lfloor F_{\underline{t},\bar{t}} \rceil^\lambda(\cdot, w)]\big)^{(j)}_\lambda$ for $j \leqslant k-2-l$. Let $\lfloor \mathrm{u}_t \rceil^\lambda$ be as in Lemma 4.18 and let $(\lfloor \mathrm{u}_t \rceil^\lambda)^{(j)}_\lambda$, $j \leqslant k-2$, be its j-th differential in λ. We identify

$$(\lfloor \mathrm{u}_{\bar{t}} \rceil^\lambda)^{(j)}_\lambda = \big([D^{(0)} \lfloor F_{\underline{t},\bar{t}} \rceil^\lambda(\lfloor \mathrm{u}_{\underline{t}} \rceil^\lambda, w)]\big)^{(j)}_\lambda.$$

In the following, we show the L^q-norm bounds in Propositions 4.8, 4.12 and 4.17 are also valid for $\big([D^{(l)} \lfloor F_{\underline{t},\bar{t}} \rceil^\lambda(\cdot, w)]\big)^{(j)}_\lambda$ by a detailed analysis of their SDEs.

Endow $\mathcal{F}(\widetilde{M}) \times (-1,1)$ with the product metric $d_{\tilde{g}^0} \times d_{(-1,1)}$, where $d_{\tilde{g}^0}$ is the induced metric of $d_{\tilde{g}^0}$ in $\mathcal{F}(\widetilde{M})$ and $d_{(-1,1)}$ is canonical. Let ∇ be the $\tilde{g}^0$ Levi-Civita connection and θ, ϖ be the associated canonical form and curvature form. Let $(H^\lambda)^{(j)}_\lambda(u, \cdot)$, $j \leqslant k-2$, be the j-th differential in λ of the maps $H^\lambda(u, \cdot)$. The SDEs of $\big([D^{(l)} \lfloor F_{\underline{t},\bar{t}} \rceil^\lambda(\cdot, w)]\big)^{(j)}_\lambda$ can be formulated by using Proposition 4.3. We state them as follows.

Lemma 4.19. *Let $\lfloor \mathrm{u}_t \rceil^\lambda$ be as in Lemma 4.18.*

i) *The Stratonovich SDE of $(\lfloor \mathrm{u}_t \rceil^\lambda)^{(1)}_\lambda$ in $T\mathcal{F}(\widetilde{M})$ is*

$$d(\lfloor \mathrm{u}_t \rceil^\lambda)^{(1)}_\lambda = \nabla((\lfloor \mathrm{u}_t \rceil^\lambda)^{(1)}_\lambda) H^\lambda(\lfloor \mathrm{u}_t \rceil^\lambda, \circ dB_t) + (H^\lambda)^{(1)}_\lambda(\lfloor \mathrm{u}_t \rceil^\lambda, \circ dB_t).$$

ii) *The Stratonovich SDE of $(\theta, \varpi)_{\lfloor \mathrm{u}_t \rceil^\lambda}\big((\lfloor \mathrm{u}_t \rceil^\lambda)^{(1)}_\lambda\big)$ in $T\mathcal{F}(\mathbb{R}^m)$ is*

$$\begin{aligned}
d\big(\theta(\lfloor \mathrm{u}_t \rceil^\lambda)^{(1)}_\lambda\big) &= d\theta\left(H^\lambda(\lfloor \mathrm{u}_t \rceil^\lambda, \circ dB_t), (\lfloor \mathrm{u}_t \rceil^\lambda)^{(1)}_\lambda\right), \\
d\big(\varpi(\lfloor \mathrm{u}_t \rceil^\lambda)^{(1)}_\lambda\big) &= d\varpi\left(H^\lambda(\lfloor \mathrm{u}_t \rceil^\lambda, \circ dB_t), (\lfloor \mathrm{u}_t \rceil^\lambda)^{(1)}_\lambda\right) + \nabla((\lfloor \mathrm{u}_t \rceil^\lambda)^{(1)}_\lambda)\left(\varpi\big(H^\lambda(\lfloor \mathrm{u}_t \rceil^\lambda, \circ dB_t)\big)\right) \\
&\quad + \varpi\left((H^\lambda)^{(1)}_\lambda(\lfloor \mathrm{u}_t \rceil^\lambda, \circ dB_t)\right).
\end{aligned}$$

iii) *The Itô SDE of* $(\theta,\varpi)_{\lfloor \mathrm{u}_t\rfloor^\lambda}\big((\lfloor \mathrm{u}_t\rfloor^\lambda)^{(1)}_\lambda\big)$ *in* $T\mathcal{F}(\mathbb{R}^m)$ *is*

$$\begin{aligned}
d\big(\theta(\lfloor \mathrm{u}_t\rfloor^\lambda)^{(1)}_\lambda\big) = \;& d\theta\left(H^\lambda(\lfloor \mathrm{u}_t\rfloor^\lambda, dB_t), (\lfloor \mathrm{u}_t\rfloor^\lambda)^{(1)}_\lambda\right)\\
&+\big(\nabla(H^\lambda(\lfloor \mathrm{u}_t\rfloor^\lambda, e_i))d\theta\big)\left(H^\lambda(\lfloor \mathrm{u}_t\rfloor^\lambda, e_i), (\lfloor \mathrm{u}_t\rfloor^\lambda)^{(1)}_\lambda\right)\, dt\\
&+d\theta\left(\nabla(H^\lambda(\lfloor \mathrm{u}_t\rfloor^\lambda, e_i))H^\lambda(\lfloor \mathrm{u}_t\rfloor^\lambda, e_i), (\lfloor \mathrm{u}_t\rfloor^\lambda)^{(1)}_\lambda\right)\, dt\\
&+d\theta\left(H^\lambda(\lfloor \mathrm{u}_t\rfloor^\lambda, e_i), \nabla((\lfloor \mathrm{u}_t\rfloor^\lambda)^{(1)}_\lambda)H^\lambda(\lfloor \mathrm{u}_t\rfloor^\lambda, e_i) + (H^\lambda)^{(1)}_\lambda(\lfloor \mathrm{u}_t\rfloor^\lambda, e_i)\right)\, dt,\\
d\big(\varpi(\lfloor \mathrm{u}_t\rfloor^\lambda)^{(1)}_\lambda\big) = \;& d\varpi\left(H^\lambda(\lfloor \mathrm{u}_t\rfloor^\lambda, dB_t), (\lfloor \mathrm{u}_t\rfloor^\lambda)^{(1)}_\lambda\right)\\
&+\big(\nabla(H^\lambda(\lfloor \mathrm{u}_t\rfloor^\lambda, e_i))d\varpi\big)\left(H^\lambda(\lfloor \mathrm{u}_t\rfloor^\lambda, e_i), (\lfloor \mathrm{u}_t\rfloor^\lambda)^{(1)}_\lambda\right)\, dt\\
&+d\varpi\left(\nabla(H^\lambda(\lfloor \mathrm{u}_t\rfloor^\lambda, e_i))H^\lambda(\lfloor \mathrm{u}_t\rfloor^\lambda, e_i), (\lfloor \mathrm{u}_t\rfloor^\lambda)^{(1)}_\lambda\right)\, dt\\
&+d\varpi\left(H^\lambda(\lfloor \mathrm{u}_t\rfloor^\lambda, e_i), \nabla((\lfloor \mathrm{u}_t\rfloor^\lambda)^{(1)}_\lambda)H^\lambda(\lfloor \mathrm{u}_t\rfloor^\lambda, e_i) + (H^\lambda)^{(1)}_\lambda(\lfloor \mathrm{u}_t\rfloor^\lambda, e_i)\right)\, dt\\
&+\nabla((\lfloor \mathrm{u}_t\rfloor^\lambda)^{(1)}_\lambda)\left(\varpi\big(H^\lambda(\lfloor \mathrm{u}_t\rfloor^\lambda, dB_t)\big)\right) + \varpi\left((H^\lambda)^{(1)}_\lambda(\lfloor \mathrm{u}_t\rfloor^\lambda, dB_t)\right)\\
&+\nabla(H^\lambda(\lfloor \mathrm{u}_t\rfloor^\lambda, e_i))\left(\nabla((\lfloor \mathrm{u}_t\rfloor^\lambda)^{(1)}_\lambda)\left(\varpi\big(H^\lambda(\lfloor \mathrm{u}_t\rfloor^\lambda, e_i)\big)\right)\right)\, dt\\
&+\nabla(H^\lambda(\lfloor \mathrm{u}_t\rfloor^\lambda, e_i))\left(\varpi\left((H^\lambda)^{(1)}_\lambda(\lfloor \mathrm{u}_t\rfloor^\lambda, e_i)\right)\right)\, dt.
\end{aligned}$$

Note that $\mathrm{Ker}(\theta) = VT\mathcal{F}(\widetilde{M}), \mathrm{Ker}(\varpi) = HT\mathcal{F}(\widetilde{M})$ and for any $\mathrm{v}^1, \mathrm{v}^2 \in HT\mathcal{F}(\widetilde{M}), \mathrm{v}^3 \in VT\mathcal{F}(\widetilde{M})$, the bracket $[\cdot,\cdot]$ satisfies the property (cf. [**Hs3**, Lemma 5.5.1])

$$[\mathrm{v}^1, \mathrm{v}^2] \in VT\mathcal{F}(\widetilde{M}),\ [\mathrm{v}^1, \mathrm{v}^3] \in HT\mathcal{F}(\widetilde{M}).$$

Using these facts, we can simplify the SDEs of $(\theta,\varpi)\big((\lfloor \mathrm{u}_t\rfloor^\lambda)^{(1)}_\lambda\big)$ at $\lambda = 0$.

Corollary 4.20. *Let* $\lfloor \mathrm{u}_t\rfloor^\lambda$ *be as in Lemma 4.18.*

i) *The Stratonovich SDE of* $(\theta,\varpi)_{\lfloor \mathrm{u}_t\rfloor^0}\big((\lfloor \mathrm{u}_t\rfloor^\lambda)^{(1)}_0\big)$ *on* $T\mathcal{F}(\mathbb{R}^m)$ *is*

$$\begin{aligned}
d\big(\theta(\lfloor \mathrm{u}_t\rfloor^\lambda)^{(1)}_0\big) &= \varpi(\lfloor \mathrm{u}_t\rfloor^\lambda)^{(1)}_0) \circ dB_t,\\
d\big(\varpi(\lfloor \mathrm{u}_t\rfloor^\lambda)^{(1)}_0\big) &= (\lfloor \mathrm{u}_t\rfloor^0)^{-1}R\big(\lfloor \mathrm{u}_t\rfloor^0 \circ dB_t, \theta(\lfloor \mathrm{u}_t\rfloor^\lambda)^{(1)}_0)\big)\lfloor \mathrm{u}_t\rfloor^0 + \varpi\big((H^\lambda)^{(1)}_0(\lfloor \mathrm{u}_t\rfloor^0, \circ dB_t)\big).
\end{aligned}$$

ii) *The Itô SDE of* $(\theta,\varpi)_{\lfloor \mathrm{u}_t\rfloor^0}\big((\lfloor \mathrm{u}_t\rfloor^\lambda)^{(1)}_0\big)$ *on* $T\mathcal{F}(\mathbb{R}^m)$ *is*

$$\begin{aligned}
d\big(\theta(\lfloor \mathrm{u}_t\rfloor^\lambda)^{(1)}_0\big) &= \varpi(\lfloor \mathrm{u}_t\rfloor^\lambda)^{(1)}_0)\ dB_t + \mathrm{Ric}(\mathrm{u}_t\theta(\lfloor \mathrm{u}_t\rfloor^\lambda)^{(1)}_0)\ dt + \varpi\big((H^\lambda)^{(1)}_0(\lfloor \mathrm{u}_t\rfloor^0, e_i)\big)e_i\ dt,\\
d\big(\varpi(\lfloor \mathrm{u}_t\rfloor^\lambda)^{(1)}_0\big) &= (\lfloor \mathrm{u}_t\rfloor^0)^{-1}R\big(\lfloor \mathrm{u}_t\rfloor^0 dB_t, \lfloor \mathrm{u}_t\rfloor^0\theta(\lfloor \mathrm{u}_t\rfloor^\lambda)^{(1)}_0)\big)\lfloor \mathrm{u}_t\rfloor^0 + \varpi\big((H^\lambda)^{(1)}_0(\lfloor \mathrm{u}_t\rfloor^0, dB_t)\big)\\
&\quad + (\lfloor \mathrm{u}_t\rfloor^0)^{-1}R\left(\lfloor \mathrm{u}_t\rfloor^0 e_i, \lfloor \mathrm{u}_t\rfloor^0\varpi(\lfloor \mathrm{u}_t\rfloor^\lambda)^{(1)}_0)e_i\right)\lfloor \mathrm{u}_t\rfloor^0\ dt\\
&\quad + (\lfloor \mathrm{u}_t\rfloor^0)^{-1}\big(\nabla(\lfloor \mathrm{u}_t\rfloor^0 e_i)R\big)\left(\lfloor \mathrm{u}_t\rfloor^0 e_i, \lfloor \mathrm{u}_t\rfloor^0\theta(\lfloor \mathrm{u}_t\rfloor^\lambda)^{(1)}_0)\right)\lfloor \mathrm{u}_t\rfloor^0\ dt.
\end{aligned}$$

Using Corollary 4.20 and Itô's formula, we can express $(\lfloor \mathrm{u}_t\rfloor^\lambda)^{(1)}_0$ using $D^{(1)}\lfloor F_{\tau,t}\rfloor^0(\cdot, w)$ by Duhamel's principle. This can be verified as in Corollary 4.6. We omit the proof.

Corollary 4.21. *Let $\lfloor \mathrm{u}_t \rceil^\lambda$ be as in Lemma 4.18.*

i) *On $T\mathcal{F}(\widetilde{M})$,*

$$(\lfloor \mathrm{u}_t \rceil^\lambda)_0^{(1)} = \Big[D^{(1)} \lfloor F_t \rceil^0 (\lfloor \mathrm{u}_0 \rceil^0, w) \Big] (\lfloor \mathrm{u}_0 \rceil^\lambda)_0^{(1)} + \mathrm{V}_c((\lfloor \mathrm{u}_t \rceil^\lambda)_0^{(1)}),$$

where

$$\mathrm{V}_c((\lfloor \mathrm{u}_t \rceil^\lambda)_0^{(1)}) := \int_0^t \Big[D^{(1)} \lfloor F_{\tau,t} \rceil^0 (\lfloor \mathrm{u}_\tau \rceil^0, w) \Big] (H^\lambda)_0^{(1)} (\lfloor \mathrm{u}_\tau \rceil^0, \circ dB_\tau(w)).$$

ii) *On $T\mathcal{F}(\mathbb{R}^m)$, the Itô form of $\widetilde{(\lfloor \mathrm{u}_t \rceil^\lambda)_0^{(1)}} := (\theta, \varpi)_{\lfloor \mathrm{u}_t \rceil^0} \Big((\lfloor \mathrm{u}_t \rceil^\lambda)_0^{(1)} \Big)$ is given by*

$$\widetilde{(\lfloor \mathrm{u}_t \rceil^\lambda)_0^{(1)}} = \Big[\widetilde{D^{(1)} \lfloor F_t \rceil^0} (\lfloor \mathrm{u}_0 \rceil^\lambda, w) \Big] \widetilde{(\lfloor \mathrm{u}_0 \rceil^\lambda)_0^{(1)}} + \widetilde{\mathrm{V}}_c((\lfloor \mathrm{u}_t \rceil^\lambda)_0^{(1)}),$$

where

$$\widetilde{\mathrm{V}}_c((\lfloor \mathrm{u}_t \rceil^\lambda)_0^{(1)}) := \int_0^t \Big[\widetilde{D^{(1)} \lfloor F_{\tau,t} \rceil^0} (\lfloor \mathrm{u}_\tau \rceil^0, w) \Big] \Big(\varpi\big((H^\lambda)_0^{(1)} (\lfloor \mathrm{u}_\tau \rceil^0, e_i) \big) e_i \; d\tau, \varpi\big((H^\lambda)_0^{(1)} (\lfloor \mathrm{u}_\tau \rceil^0, dB_\tau) \big) \Big).$$

To describe the second order differential of $\lfloor \mathrm{u}_t \rceil^\lambda$ in λ, we use the horizontal/vertical sum decomposition of $TT\mathcal{F}(\widetilde{M})$ of $\widetilde{g}^0$. By Lemma 4.19, it remains to find the SDEs of

$$(\lfloor \mathrm{u}_t \rceil^\lambda)_\lambda^{(2)} := \frac{D}{d\lambda} \Big((\lfloor \mathrm{u}_t \rceil^\lambda)_\lambda^{(1)} \Big) = \nabla((\lfloor \mathrm{u}_t \rceil^\lambda)_0^{(1)}) (\lfloor \mathrm{u}_t \rceil^\lambda)_0^{(1)}.$$

Lemma 4.22. *Let $\lambda \in (-1,1) \mapsto g^\lambda \in \mathcal{M}^k(M)$ be a C^k curve with $k \geqslant 4$. Let $\lambda \mapsto \lfloor \mathrm{u}_0 \rceil^\lambda$ be C^{k-2} and let $\lfloor \mathrm{u}_t \rceil^\lambda$ be as in Lemma 4.18 with $(\lfloor \mathrm{u}_t \rceil^\lambda)_\lambda^{(1)}$, $(\lfloor \mathrm{u}_t \rceil^\lambda)_\lambda^{(2)}$ defined as above.*

i) *The Stratonovich SDE of $(\lfloor \mathrm{u}_t \rceil^\lambda)_\lambda^{(2)}$ on $T\mathcal{F}(\widetilde{M})$ is*

$$\begin{aligned} d\big((\lfloor \mathrm{u}_t \rceil^\lambda)_\lambda^{(2)} \big) =& \nabla\big((\lfloor \mathrm{u}_t \rceil^\lambda)_\lambda^{(2)} \big) H^\lambda (\lfloor \mathrm{u}_t \rceil^\lambda, \circ dB_t) + \nabla^{(2)}\big((\lfloor \mathrm{u}_t \rceil^\lambda)_\lambda^{(1)}, (\lfloor \mathrm{u}_t \rceil^\lambda)_\lambda^{(1)} \big) H^\lambda (\lfloor \mathrm{u}_t \rceil^\lambda, \circ dB_t) \\ &+ R\big(H^\lambda (\lfloor \mathrm{u}_t \rceil^\lambda, \circ dB_t), (\lfloor \mathrm{u}_t \rceil^\lambda)_\lambda^{(1)} \big) (\lfloor \mathrm{u}_t \rceil^\lambda)_\lambda^{(1)} \\ &+ 2\nabla\big((\lfloor \mathrm{u}_t \rceil^\lambda)_\lambda^{(1)} \big) \big(H^\lambda \big)_\lambda^{(1)} (\lfloor \mathrm{u}_t \rceil^\lambda, \circ dB_t) + \big(H^\lambda \big)_\lambda^{(2)} (\lfloor \mathrm{u}_t \rceil^\lambda, \circ dB_t). \end{aligned}$$

ii) *The Stratonovich SDE of $(\theta, \varpi)_{\lfloor \mathrm{u}_t \rceil^\lambda} \big((\lfloor \mathrm{u}_t \rceil^\lambda)_\lambda^{(2)} \big)$ on $T\mathcal{F}(\mathbb{R}^m)$ is*

$$\begin{aligned} & d\Big(\big(\theta, \varpi \big) \big((\lfloor \mathrm{u}_t \rceil^\lambda)_\lambda^{(2)} \big) \Big) \\ &= d\big(\theta, \varpi \big) \Big(H^\lambda (\lfloor \mathrm{u}_t \rceil^\lambda, \circ dB_t), (\lfloor \mathrm{u}_t \rceil^\lambda)_\lambda^{(2)} \Big) + \nabla((\lfloor \mathrm{u}_t \rceil^\lambda)_\lambda^{(2)}) \Big(\big(\theta, \varpi \big) \big(H^\lambda (\lfloor \mathrm{u}_t \rceil^\lambda, \circ dB_t) \big) \Big) \\ &\quad + \big(\theta, \varpi \big) \Big(\nabla^{(2)}\big((\lfloor \mathrm{u}_t \rceil^\lambda)_\lambda^{(1)}, (\lfloor \mathrm{u}_t \rceil^\lambda)_\lambda^{(1)} \big) H^\lambda (\lfloor \mathrm{u}_t \rceil^\lambda, \circ dB_t) \Big) \\ &\quad + \big(\theta, \varpi \big) \Big(R\big(H^\lambda (\lfloor \mathrm{u}_t \rceil^\lambda, \circ dB_t), (\lfloor \mathrm{u}_t \rceil^\lambda)_\lambda^{(1)} \big) (\lfloor \mathrm{u}_t \rceil^\lambda)_\lambda^{(1)} \Big) \\ &\quad + \big(\theta, \varpi \big) \Big(2\nabla\big((\lfloor \mathrm{u}_t \rceil^\lambda)_\lambda^{(1)} \big) \big(H^\lambda \big)_\lambda^{(1)} (\lfloor \mathrm{u}_t \rceil^\lambda, \circ dB_t) + \big(H^\lambda \big)_\lambda^{(2)} (\lfloor \mathrm{u}_t \rceil^\lambda, \circ dB_t) \Big). \end{aligned}$$

iii) *The Itô SDE of* $(\theta,\varpi)_{\lfloor \mathrm{u}_t\rfloor^\lambda}\big((\lfloor \mathrm{u}_t\rfloor^\lambda)^{(2)}_\lambda\big)$ *on* $T\mathcal{F}(\mathbb{R}^m)$ *is*

$$
\begin{aligned}
&d\Big((\theta,\varpi)\big((\lfloor \mathrm{u}_t\rfloor^\lambda)^{(2)}_\lambda\big)\Big)\\
&= d\big(\theta,\varpi\big)\Big(H^\lambda(\lfloor \mathrm{u}_t\rfloor^\lambda,dB_t),(\lfloor \mathrm{u}_t\rfloor^\lambda)^{(2)}_\lambda\Big)+\nabla((\lfloor \mathrm{u}_t\rfloor^\lambda)^{(2)}_\lambda)\big((\theta,\varpi)\big(H^\lambda(\lfloor \mathrm{u}_t\rfloor^\lambda,dB_t)\big)\big)\\
&\quad+\big(\theta,\varpi\big)\Big(\nabla^{(2)}\big((\lfloor \mathrm{u}_t\rfloor^\lambda)^{(1)}_\lambda,(\lfloor \mathrm{u}_t\rfloor^\lambda)^{(1)}_\lambda\big)H^\lambda(\lfloor \mathrm{u}_t\rfloor^\lambda,dB_t)\Big)\\
&\quad+\big(\theta,\varpi\big)\Big(R\big(H^\lambda(\lfloor \mathrm{u}_t\rfloor^\lambda,dB_t),(\lfloor \mathrm{u}_t\rfloor^\lambda)^{(1)}_\lambda\big)(\lfloor \mathrm{u}_t\rfloor^\lambda)^{(1)}_\lambda\Big)\\
&\quad+\big(\theta,\varpi\big)\Big(2\nabla\big((\lfloor \mathrm{u}_t\rfloor^\lambda)^{(1)}_\lambda\big)\big(H^\lambda\big)^{(1)}_\lambda(\lfloor \mathrm{u}_t\rfloor^\lambda,dB_t)+\big(H^\lambda\big)^{(2)}_\lambda(\lfloor \mathrm{u}_t\rfloor^\lambda,dB_t)\Big)\\
&\quad+\nabla(H^\lambda(\lfloor \mathrm{u}_t\rfloor^\lambda,e_i))\Big\{d\big(\theta,\varpi\big)\Big(H^\lambda(\lfloor \mathrm{u}_t\rfloor^\lambda,e_i),(\lfloor \mathrm{u}_t\rfloor^\lambda)^{(2)}_\lambda\Big)\\
&\qquad\qquad+\nabla((\lfloor \mathrm{u}_t\rfloor^\lambda)^{(2)}_\lambda)\big((\theta,\varpi\big)\big(H^\lambda(\lfloor \mathrm{u}_t\rfloor^\lambda,e_i)\big)\big)\\
&\qquad\qquad+\big(\theta,\varpi\big)\Big(\nabla^{(2)}\big((\lfloor \mathrm{u}_t\rfloor^\lambda)^{(1)}_\lambda,(\lfloor \mathrm{u}_t\rfloor^\lambda)^{(1)}_\lambda\big)H^\lambda(\lfloor \mathrm{u}_t\rfloor^\lambda,e_i)\Big)\\
&\qquad\qquad+\big(\theta,\varpi\big)\Big(R\big(H^\lambda(\lfloor \mathrm{u}_t\rfloor^\lambda,e_i),(\lfloor \mathrm{u}_t\rfloor^\lambda)^{(1)}_\lambda\big)(\lfloor \mathrm{u}_t\rfloor^\lambda)^{(1)}_\lambda\Big)\\
&\qquad\qquad+\big(\theta,\varpi\big)\Big(2\nabla\big((\lfloor \mathrm{u}_t\rfloor^\lambda)^{(1)}_\lambda\big)\big(H^\lambda\big)^{(1)}_\lambda(\lfloor \mathrm{u}_t\rfloor^\lambda,e_i)+\big(H^\lambda\big)^{(2)}_\lambda(\lfloor \mathrm{u}_t\rfloor^\lambda,e_i)\Big)\Big\}\,dt.
\end{aligned}
$$

Again, we can simplify the SDEs in Lemma 4.22 at $\lambda=0$.

Corollary 4.23. *We retain all the notations in Lemma 4.22.*

i) *The Stratonovich SDE of* $(\theta,\varpi)_{\lfloor \mathrm{u}_t\rfloor^0}\big((\lfloor \mathrm{u}_t\rfloor^\lambda)^{(2)}_0\big)$ *on* $T\mathcal{F}(\mathbb{R}^m)$ *is*

$$
\begin{aligned}
d\Big((\theta,\varpi)\big((\lfloor \mathrm{u}_t\rfloor^\lambda)^{(2)}_0\big)\Big) &= \Big(\varpi(\lfloor \mathrm{u}_t\rfloor^\lambda)^{(2)}_0)\circ dB_t,(\lfloor \mathrm{u}_t\rfloor^0)^{-1}R\big(\lfloor \mathrm{u}_t\rfloor^0\circ dB_t,\theta(\lfloor \mathrm{u}_t\rfloor^\lambda)^{(2)}_0)\big)\lfloor \mathrm{u}_t\rfloor^0\Big)\\
&\quad+\big(\theta,\varpi\big)\Big(\nabla^{(2)}\big((\lfloor \mathrm{u}_t\rfloor^\lambda)^{(1)}_0,(\lfloor \mathrm{u}_t\rfloor^\lambda)^{(1)}_0\big)H^0(\lfloor \mathrm{u}_t\rfloor^0,\circ dB_t)\Big)\\
&\quad+\big(\theta,\varpi\big)\Big(R\big(H^0(\lfloor \mathrm{u}_t\rfloor^0,\circ dB_t),(\lfloor \mathrm{u}_t\rfloor^\lambda)^{(1)}_0\big)(\lfloor \mathrm{u}_t\rfloor^\lambda)^{(1)}_0\Big)\\
&\quad+\big(\theta,\varpi\big)\Big(2\nabla\big((\lfloor \mathrm{u}_t\rfloor^\lambda)^{(1)}_0\big)\big(H^\lambda\big)^{(1)}_0(\lfloor \mathrm{u}_t\rfloor^0,\circ dB_t)+\big(H^\lambda\big)^{(2)}_0(\lfloor \mathrm{u}_t\rfloor^0,\circ dB_t)\Big).
\end{aligned}
$$

ii) *The Itô SDE of* $\theta_{\lfloor \mathrm{u}_t\rfloor^0}\big((\lfloor \mathrm{u}_t\rfloor^\lambda)^{(2)}_0\big)$ *on* $T\mathcal{F}(\mathbb{R}^m)$ *is*

$$
\begin{aligned}
d\big(\theta((\lfloor \mathrm{u}_t\rfloor^\lambda)^{(2)}_0)\big) =&\varpi((\lfloor \mathrm{u}_t\rfloor^\lambda)^{(2)}_0)dB_t+\mathrm{Ric}(\lfloor \mathrm{u}_t\rfloor^0\theta((\lfloor \mathrm{u}_t\rfloor^\lambda)^{(2)}_0))dt\\
&+\Phi_\theta((\lfloor \mathrm{u}_t\rfloor^\lambda)^{(1)}_0,(\lfloor \mathrm{u}_t\rfloor^\lambda)^{(1)}_0,dB_t,dt)+\Phi^{0,2}_\theta((\lfloor \mathrm{u}_t\rfloor^\lambda)^{(1)}_0,dB_t,dt),
\end{aligned}
$$

where $\Phi_\theta(\cdot,\cdot,dB_t,dt)$ *is given in (4.15) for* $\lfloor \mathrm{u}_t\rfloor^0$ *and* [1]

$$
\begin{aligned}
&\Phi^{0,2}_\theta((\lfloor \mathrm{u}_t\rfloor^\lambda)^{(1)}_0,dB_t,dt)\\
&\quad:=\varpi\Big(2\nabla\big((\lfloor \mathrm{u}_t\rfloor^\lambda)^{(1)}_0\big)\big(H^\lambda\big)^{(1)}_0(\lfloor \mathrm{u}_t\rfloor^0,e_i)+\big(H^\lambda\big)^{(2)}_0(\lfloor \mathrm{u}_t\rfloor^0,e_i)\Big)e_i\;dt\\
&\qquad+\theta\Big(\Big[H(\lfloor \mathrm{u}_t\rfloor^0,e_i),2\nabla\big((\lfloor \mathrm{u}_t\rfloor^\lambda)^{(1)}_0\big)\big(H^\lambda\big)^{(1)}_0(\lfloor \mathrm{u}_t\rfloor^0,e_i)+\big(H^\lambda\big)^{(2)}_0(\lfloor \mathrm{u}_t\rfloor^0,e_i)\Big]\Big)\;dt,
\end{aligned}
$$

[1]The upper script ^{0,2} is to indicate that $\Phi^{0,2}_\theta,\Phi^{0,2}_\varpi$ are associated with $\big([D^{(0)}\lfloor F_t(\cdot,w)]\big)^{(2)}_\lambda$.

and the Itô SDE of $\varpi_{\lfloor \mathrm{u}_t\rfloor^0}\big((\lfloor \mathrm{u}_t\rfloor^\lambda)_0^{(2)}\big)$ *on* $T\mathcal{F}(\mathbb{R}^m)$ *is*

$$\begin{aligned} d\big(\varpi((\lfloor \mathrm{u}_t\rfloor^\lambda)_0^{(2)})\big) = {} & (\lfloor \mathrm{u}_t\rfloor^0)^{-1} R\left(\lfloor \mathrm{u}_t\rfloor^0 dB_t, \lfloor \mathrm{u}_t\rfloor^0 \theta((\lfloor \mathrm{u}_t\rfloor^\lambda)_0^{(2)})\right)\lfloor \mathrm{u}_t\rfloor^0 \\ & + (\lfloor \mathrm{u}_t\rfloor^0)^{-1} R\left(\lfloor \mathrm{u}_t\rfloor^0 e_i, \lfloor \mathrm{u}_t\rfloor^0 \varpi((\lfloor \mathrm{u}_t\rfloor^\lambda)_0^{(2)}) e_i\right)\lfloor \mathrm{u}_t\rfloor^0\, dt \\ & + (\lfloor \mathrm{u}_t\rfloor^0)^{-1}\big(\nabla(\lfloor \mathrm{u}_t\rfloor^0 e_i)R\big)\left(\lfloor \mathrm{u}_t\rfloor^0 e_i, \lfloor \mathrm{u}_t\rfloor^0 \theta((\lfloor \mathrm{u}_t\rfloor^\lambda)_0^{(2)})\right)\lfloor \mathrm{u}_t\rfloor^0\, dt \\ & + \Phi_\varpi((\lfloor \mathrm{u}_t\rfloor^\lambda)_0^{(1)}, (\lfloor \mathrm{u}_t\rfloor^\lambda)_0^{(1)}, dB_t, dt) + \Phi^{0,2}_\varpi((\lfloor \mathrm{u}_t\rfloor^\lambda)_0^{(1)}, dB_t, dt), \end{aligned}$$

where $\Phi_\theta(\cdot,\cdot,dB_t,dt)$ *is given in (4.15) for* $\lfloor \mathrm{u}_t\rfloor^0$ *and*

$$\Phi^{0,2}_\varpi((\lfloor \mathrm{u}_t\rfloor^\lambda)_0^{(1)}, dB_t, dt) := \varpi\left(2\nabla\big((\lfloor \mathrm{u}_t\rfloor^\lambda)_0^{(1)}\big)\big(H^\lambda\big)_0^{(1)}(\lfloor \mathrm{u}_t\rfloor^0, dB_t) + \big(H^\lambda\big)_0^{(2)}(\lfloor \mathrm{u}_t\rfloor^0, dB_t)\right).$$

By Corollary 4.23 with Lemma 4.5, we can formulate $(\lfloor \mathrm{u}_t\rfloor^\lambda)_0^{(2)}$ and $(\theta,\varpi)(\lfloor \mathrm{u}_t\rfloor^\lambda)_0^{(2)}$ using $D^{(1)}\lfloor F_{\tau,t}\rfloor^0(\cdot,w)$ by stochastic Duhamel principle. We only state the conclusion.

Corollary 4.24. *We retain all the notations in Lemma 4.22.*

i) *On* $T\mathcal{F}(\widetilde{M})$,

$$\begin{aligned} (\lfloor \mathrm{u}_t\rfloor^\lambda)_0^{(2)} = {} & \Big[D^{(1)}\lfloor F_t\rfloor^0(\lfloor \mathrm{u}_0\rfloor^0, w)\Big](\lfloor \mathrm{u}_0\rfloor^\lambda)_0^{(2)} + \nabla_{(\lfloor \mathrm{u}_0\rfloor^\lambda)_0^{(1)}}\Big[D^{(1)}\lfloor F_t\rfloor^0(\lfloor \mathrm{u}_0\rfloor^0, w)\Big]\big((\lfloor \mathrm{u}_0\rfloor^\lambda)_0^{(1)}\big) \\ & + \mathrm{V}_c\big((\lfloor \mathrm{u}_t\rfloor^\lambda)_0^{(2)}\big), \end{aligned}$$

where

$$\begin{aligned} \mathrm{V}_c\big((\lfloor \mathrm{u}_t\rfloor^\lambda)_0^{(2)}\big) = \int_0^t & \Big[D^{(1)}\lfloor F_{\tau,t}\rfloor^0(\lfloor \mathrm{u}_\tau\rfloor^0, w)\Big]\Big\{\nabla^{(2)}\big((\lfloor \mathrm{u}_\tau\rfloor^\lambda)_0^{(1)}, (\lfloor \mathrm{u}_\tau\rfloor^\lambda)_0^{(1)}\big)H^0(\lfloor \mathrm{u}_\tau\rfloor^0, \circ dB_\tau) \\ & - \nabla^{(2)}\big([D^{(1)}\lfloor F_\tau\rfloor^0](\lfloor \mathrm{u}_0\rfloor^\lambda)_0^{(1)}, [D^{(1)}\lfloor F_\tau\rfloor^0](\lfloor \mathrm{u}_0\rfloor^\lambda)_0^{(1)}\big)H^0(\lfloor \mathrm{u}_\tau\rfloor^0, \circ dB_\tau) \\ & + R\big(H^0(\lfloor \mathrm{u}_\tau\rfloor^0, \circ dB_\tau), (\lfloor \mathrm{u}_\tau\rfloor^\lambda)_0^{(1)}\big)(\lfloor \mathrm{u}_\tau\rfloor^\lambda)_0^{(1)} \\ & - R\big(H^0(\lfloor \mathrm{u}_\tau\rfloor^0, \circ dB_\tau), [D^{(1)}\lfloor F_\tau\rfloor^0](\lfloor \mathrm{u}_0\rfloor^\lambda)_0^{(1)}\big)[D^{(1)}\lfloor F_\tau\rfloor^0](\lfloor \mathrm{u}_0\rfloor^\lambda)_0^{(1)} \\ & + 2\nabla\big((\lfloor \mathrm{u}_\tau\rfloor^\lambda)_0^{(1)}\big)\big(H^\lambda\big)_0^{(1)}(\lfloor \mathrm{u}_\tau\rfloor^0, \circ dB_t) + \big(H^\lambda\big)_0^{(2)}(\lfloor \mathrm{u}_\tau\rfloor^0, \circ dB_\tau)\Big\}. \end{aligned}$$

ii) *On* $T\mathcal{F}(\mathbb{R}^m)$, *the Itô form of* $(\widetilde{\lfloor \mathrm{u}_t\rfloor^\lambda})_0^{(2)} := (\theta,\varpi)_{\lfloor \mathrm{u}_t\rfloor^0}\big((\lfloor \mathrm{u}_t\rfloor^\lambda)_0^{(2)}\big)$ *is*

$$\begin{aligned} (\widetilde{\lfloor \mathrm{u}_t\rfloor^\lambda})_0^{(2)} = {} & \Big[D^{(1)}\widetilde{\lfloor F_t\rfloor}^0(\lfloor \mathrm{u}_0\rfloor^\lambda, w)\Big](\widetilde{\lfloor \mathrm{u}_0\rfloor^\lambda})_0^{(2)} \\ & + (\theta,\varpi)\Big(\nabla_{(\lfloor \mathrm{u}_0\rfloor^\lambda)_0^{(1)}}\Big[D^{(1)}\lfloor F_t\rfloor^0(\lfloor \mathrm{u}_0\rfloor^\lambda, w)\Big]\big((\lfloor \mathrm{u}_0\rfloor^\lambda)_0^{(1)}\big)\Big) + \widetilde{\mathrm{V}_c}((\lfloor \mathrm{u}_t\rfloor^\lambda)_0^{(2)}), \end{aligned}$$

where

$$\begin{aligned}\widetilde{\mathrm{V}_c}((\lfloor \mathrm{u}_t\rceil^\lambda)_0^{(2)}) :=& \int_0^t \Big[\widetilde{D^{(1)}\lfloor F_{\tau,t}\rceil^0}(\lfloor \mathrm{u}_\tau\rceil^0, w)\Big]\Big\{\big(\Phi_\theta, \Phi_\varpi\big)\big((\lfloor \mathrm{u}_\tau\rceil^\lambda)_0^{(1)}, (\lfloor \mathrm{u}_\tau\rceil^\lambda)_0^{(1)}, dB_\tau, d\tau\big)\\ &- \big(\Phi_\theta, \Phi_\varpi\big)\big([D^1\lfloor F_\tau\rceil^0](\lfloor \mathrm{u}_0\rceil^\lambda)_0^{(1)}, [D^1\lfloor F_\tau\rceil^0](\lfloor \mathrm{u}_0\rceil^\lambda)_0^{(1)}, dB_\tau, d\tau\big)\\ &+ \big(\Phi_\theta^{0,2}, \Phi_\varpi^{0,2}\big)((\lfloor \mathrm{u}_\tau\rceil^\lambda)_0^{(1)}, dB_\tau, d\tau) - \Big(\varpi\big(\Phi_{e_i}^{0,2}\big((\lfloor \mathrm{u}_\tau\rceil^\lambda)_0^{(1)}, (\lfloor \mathrm{u}_\tau\rceil^\lambda)_0^{(1)}\big)\big)e_i,\\ &+ (\lfloor \mathrm{u}_\tau\rceil^0)^{-1}R\big(\lfloor \mathrm{u}_\tau\rceil^0 e_i, \lfloor \mathrm{u}_\tau\rceil^0\theta\big(\Phi_{e_i}^{0,2}\big((\lfloor \mathrm{u}_\tau\rceil^\lambda)_0^{(1)}, (\lfloor \mathrm{u}_\tau\rceil^\lambda)_0^{(1)}\big)\big)\lfloor \mathrm{u}_\tau\rceil^0\Big)\ d\tau\Big\},\end{aligned}$$

$$\begin{aligned}\Phi_{e_i}^{0,2}\big((\lfloor \mathrm{u}_\tau\rceil^\lambda)_0^{(1)}, (\lfloor \mathrm{u}_\tau\rceil^\lambda)_0^{(1)}\big) :=& \nabla^{(2)}\big((\lfloor \mathrm{u}_\tau\rceil^\lambda)_0^{(1)}, (\lfloor \mathrm{u}_\tau\rceil^\lambda)_0^{(1)}\big)H^0(\lfloor \mathrm{u}_\tau\rceil^0, e_i)\\ &- \nabla^{(2)}\big([D^{(1)}\lfloor F_\tau\rceil^0](\lfloor \mathrm{u}_0\rceil^\lambda)_0^{(1)}, [D^{(1)}\lfloor F_\tau\rceil^0](\lfloor \mathrm{u}_0\rceil^\lambda)_0^{(1)}\big)H^0(\lfloor \mathrm{u}_\tau\rceil^0, e_i)\\ &+ R\big(H^0(\lfloor \mathrm{u}_\tau\rceil^0, e_i), (\lfloor \mathrm{u}_\tau\rceil^\lambda)_0^{(1)}\big)(\lfloor \mathrm{u}_\tau\rceil^\lambda)_0^{(1)}\\ &- R\big(H^0(\lfloor \mathrm{u}_\tau\rceil^0, e_i), [D^{(1)}\lfloor F_\tau\rceil^0](\lfloor \mathrm{u}_0\rceil^\lambda)_0^{(1)}\big)[D^{(1)}\lfloor F_\tau\rceil^0](\lfloor \mathrm{u}_0\rceil^\lambda)_0^{(1)}\\ &+ 2\nabla\big((\lfloor \mathrm{u}_\tau\rceil^\lambda)_0^{(1)}\big)\big(H^\lambda\big)_0^{(1)}(\lfloor \mathrm{u}_\tau\rceil^0, e_i) + \big(H^\lambda\big)_0^{(2)}(\lfloor \mathrm{u}_\tau\rceil^0, e_i).\end{aligned}$$

Let $[D^{(2)}\lfloor F_t\rceil^0(\cdot, w)]$ be the restriction of the second order tangent map of $\lfloor F_t\rceil^0$ on the space $T_{(\lfloor \mathrm{u}\rceil^0, (\lfloor \mathrm{u}\rceil^\lambda)_\lambda^{(1)})}T_{\lfloor \mathrm{u}\rceil^0}\mathcal{F}(\widetilde{M})$. We can deduce from Corollary 4.21 and Corollary 4.24 that

$$\begin{aligned}\Big((\lfloor \mathrm{u}_t\rceil^\lambda)_0^{(1)}, (\lfloor \mathrm{u}_t\rceil^\lambda)_0^{(2)}\Big) =& \ [D^{(2)}\lfloor F_t\rceil^0(\cdot, w)]\Big((\lfloor \mathrm{u}_0\rceil^\lambda)_0^{(1)}, (\lfloor \mathrm{u}_0\rceil^\lambda)_0^{(2)}\Big)\\ &+ \Big(\mathrm{V}_c((\lfloor \mathrm{u}_t\rceil^\lambda)_0^{(1)}), \mathrm{V}_c((\lfloor \mathrm{u}_t\rceil^\lambda)_0^{(2)})\Big),\\ \big((\widetilde{\lfloor \mathrm{u}_t\rceil^\lambda})_0^{(1)}, (\widetilde{\lfloor \mathrm{u}_t\rceil^\lambda})_0^{(2)}\big) =& \ [\widetilde{D^{(2)}\lfloor F_t\rceil^0}(\cdot, w)]\big((\widetilde{\lfloor \mathrm{u}_0\rceil^\lambda})_0^{(1)}, (\widetilde{\lfloor \mathrm{u}_0\rceil^\lambda})_0^{(2)}\big)\\ &+ \Big(\widetilde{\mathrm{V}_c}((\lfloor \mathrm{u}_t\rceil^\lambda)_0^{(1)}), \widetilde{\mathrm{V}_c}((\lfloor \mathrm{u}_t\rceil^\lambda)_0^{(2)})\Big).\end{aligned} \tag{4.47}$$

Continuing the discussions in Lemma 4.19 and Lemma 4.22, we can derive the SDEs for the differentials $(\lfloor \mathrm{u}_t\rceil^\lambda)_\lambda^{(j)}$, $3 \leqslant j \leqslant k-2$, and their pull back $(\widetilde{\lfloor \mathrm{u}_t\rceil^\lambda})_\lambda^{(j)}$ via (θ, ϖ)-map, whose Itô form involves $\{\nabla^{(l')}(H^\lambda)_\lambda^{(j')}\}_{j' \leqslant j, l'+j' \leqslant j}, \{\nabla^{(l')}R^\lambda\}_{l' \leqslant j}$. We omit the details.

The SDEs of $\big([D^{(l)}\lfloor F_t\rceil^\lambda(\cdot, w)]\big)_\lambda^{(j)}$ can be formulated as in Section 4.2 by analogy with the deterministic case. We only state the SDEs for the $(l, j) = (1, 1)$ case using the reference connection of $\widetilde{g}^0$, whose calculations can be done as in Lemma 4.5.

Lemma 4.25. *Let $\lambda \in (-1, 1) \mapsto g^\lambda \in \mathcal{M}^k(M)$ be a C^k curve with $k \geqslant 4$. Let $\lambda \mapsto (\lfloor \mathrm{u}_0\rceil^\lambda, \mathrm{v}_0^\lambda) \in T\mathcal{F}(\widetilde{M})$ be C^1 and write*

$$\big(\lfloor \mathrm{u}_t\rceil^\lambda, \mathrm{v}_t^\lambda\big) := \Big(\lfloor F_t\rceil^\lambda(\lfloor \mathrm{u}_0\rceil^\lambda), \big[D^{(1)}\lfloor F_t\rceil^\lambda(\lfloor \mathrm{u}_0\rceil^\lambda, w)\big]\mathrm{v}_0^\lambda\Big), \quad (\mathrm{v}_t^\lambda)_\lambda^{(1)} := \nabla\big((\lfloor \mathrm{u}_t\rceil^\lambda)_\lambda^{(1)}\big)\mathrm{v}_t^\lambda.$$

i) *The process* $(\mathrm{v}_t^\lambda)_\lambda^{(1)}$ *satisfies the Stratonovich SDE*

$$\begin{aligned} d(\mathrm{v}_t^\lambda)_\lambda^{(1)} =& \nabla\big((\mathrm{v}_t^\lambda)_\lambda^{(1)}\big)H^\lambda(\lfloor \mathrm{u}_t\rfloor^\lambda, \circ dB_t) + \nabla^{(2)}\big(\mathrm{v}_t^\lambda, (\lfloor \mathrm{u}_t\rfloor^\lambda)_\lambda^{(1)}\big)H^\lambda(\lfloor \mathrm{u}_t\rfloor^\lambda, \circ dB_t) \\ &+ R\big(H^\lambda(\lfloor \mathrm{u}_t\rfloor^\lambda, e), (\lfloor \mathrm{u}_t\rfloor^\lambda)_\lambda^{(1)}\big)\mathrm{v}_t^\lambda + \nabla(\mathrm{v}_t^\lambda)(H^\lambda)_\lambda^{(1)}(\lfloor \mathrm{u}_t\rfloor^\lambda, \circ dB_t). \end{aligned}$$

ii) *The process* $(\theta, \varpi)_{\lfloor \mathrm{u}_t\rfloor^\lambda}\big((\mathrm{v}_t^\lambda)_\lambda^{(1)}\big)$ *satisfies the Stratonovich SDE*

$$\begin{aligned} &d\big((\theta,\varpi)(\mathrm{v}_t^\lambda)_\lambda^{(1)}\big) \\ &= d(\theta,\varpi)\Big(H^\lambda(\lfloor \mathrm{u}_t\rfloor^\lambda, \circ dB_t), (\mathrm{v}_t^\lambda)_\lambda^{(1)}\Big) + \nabla((\mathrm{v}_t^\lambda)_\lambda^{(1)})\Big((\theta,\varpi)\big(H^\lambda(\lfloor \mathrm{u}_t\rfloor^\lambda, \circ dB_t)\big)\Big) \\ &\quad + (\theta,\varpi)\Big(\nabla^{(2)}\big(\mathrm{v}_t^\lambda, (\lfloor \mathrm{u}_t\rfloor^\lambda)_\lambda^{(1)}\big)H^\lambda(\lfloor \mathrm{u}_t\rfloor^\lambda, \circ dB_t) + R\big(H^\lambda(\lfloor \mathrm{u}_t\rfloor^\lambda, \circ dB_t), (\lfloor \mathrm{u}_t\rfloor^\lambda)_\lambda^{(1)}\big)\mathrm{v}_t^\lambda\Big) \\ &\quad + (\theta,\varpi)\Big(\nabla(\mathrm{v}_t^\lambda)(H^\lambda)_\lambda^{(1)}(\lfloor \mathrm{u}_t\rfloor^\lambda, \circ dB_t)\Big). \end{aligned}$$

iii) *The Itô SDE of the process* $(\theta, \varpi)_{\lfloor \mathrm{u}_t\rfloor^\lambda}\big((\mathrm{v}_t^\lambda)_\lambda^{(1)}\big)$ *is*

$$\begin{aligned} &d\big((\theta,\varpi)(\mathrm{v}_t^\lambda)_\lambda^{(1)}\big) \\ &= d(\theta,\varpi)\Big(H^\lambda(\lfloor \mathrm{u}_t\rfloor^\lambda, dB_t), (\mathrm{v}_t^\lambda)_\lambda^{(1)}\Big) + \nabla((\mathrm{v}_t^\lambda)_\lambda^{(1)})\big((\theta,\varpi)\big(H^\lambda(\lfloor \mathrm{u}_t\rfloor^\lambda, dB_t)\big)\big) \\ &\quad + (\theta,\varpi)\Big(\nabla^{(2)}\big(\mathrm{v}_t^\lambda, (\lfloor \mathrm{u}_t\rfloor^\lambda)_\lambda^{(1)}\big)H^\lambda(\lfloor \mathrm{u}_t\rfloor^\lambda, dB_t) + R\big(H^\lambda(\lfloor \mathrm{u}_t\rfloor^\lambda, e), (\lfloor \mathrm{u}_t\rfloor^\lambda)_\lambda^{(1)}\big)\mathrm{v}_t^\lambda\Big) \\ &\quad + (\theta,\varpi)\Big(\nabla(\mathrm{v}_t^\lambda)(H^\lambda)_\lambda^{(1)}(\lfloor \mathrm{u}_t\rfloor^\lambda, dB_t)\Big) + \nabla(H^\lambda(\lfloor \mathrm{u}_t\rfloor^\lambda, e_i))\Big\{d(\theta,\varpi)\big(H^\lambda(\lfloor \mathrm{u}_t\rfloor^\lambda, e_i), (\mathrm{v}_t^\lambda)_\lambda^{(1)}\big) \\ &\quad + \nabla\big((\mathrm{v}_t^\lambda)_\lambda^{(1)}\big)\big((\theta,\varpi)\big(H^\lambda(\lfloor \mathrm{u}_t\rfloor^\lambda, e_i)\big)\big) + (\theta,\varpi)\big(\nabla^{(2)}\big(\mathrm{v}_t^\lambda, (\lfloor \mathrm{u}_t\rfloor^\lambda)_\lambda^{(1)}\big)H^\lambda(\lfloor \mathrm{u}_t\rfloor^\lambda, e_i)\big) \\ &\quad + R\big(H^\lambda(\lfloor \mathrm{u}_t\rfloor^\lambda, e_i), (\lfloor \mathrm{u}_t\rfloor^\lambda)_\lambda^{(1)}\big)\mathrm{v}_t^\lambda + \big(\theta,\varpi\big)\big(\nabla(\mathrm{v}_t^\lambda)\big(H^\lambda\big)_\lambda^{(1)}(\lfloor \mathrm{u}_t\rfloor^\lambda, e_i)\big)\Big\}\ dt. \end{aligned}$$

As before, the formulas in Lemma 4.25 can be simplified at $\lambda = 0$.

Corollary 4.26. *Let* $\lambda \in (-1,1) \mapsto g^\lambda \in \mathcal{M}^k(M)$ *be a* C^k *curve with* $k \geqslant 4$ *and let* $\lambda \mapsto (\lfloor \mathrm{u}_0\rfloor^\lambda, \mathrm{v}_0^\lambda) \in T\mathcal{F}(\widetilde{M})$ *be* C^{k-2-i}*. We retain all the notations in Lemma 4.25.*

i) *The process* $(\theta, \varpi)_{\lfloor \mathrm{u}_t\rfloor^\lambda}\big((\mathrm{v}_t^\lambda)_0^{(1)}\big)$ *satisfies the Stratonovich SDE*

$$\begin{aligned} d\big((\theta,\varpi)((\mathrm{v}_t^\lambda)_0^{(1)})\big) =& \Big(\varpi((\mathrm{v}_t^\lambda)_0^{(1)}) \circ dB_t, (\lfloor \mathrm{u}_t\rfloor^0)^{-1}R\big(\lfloor \mathrm{u}_t\rfloor^0 \circ dB_t, \theta((\mathrm{v}_t^\lambda)_0^{(1)})\big)\lfloor \mathrm{u}_t\rfloor^0\Big) \\ &+ (\theta,\varpi)\Big(\nabla^{(2)}\big(\mathrm{v}_t^\lambda, (\lfloor \mathrm{u}_t\rfloor^\lambda)_\lambda^{(1)}\big)H^\lambda(\lfloor \mathrm{u}_t\rfloor^\lambda, \circ dB_t)\Big) \\ &+ (\theta,\varpi)\Big(R\big(H^\lambda(\lfloor \mathrm{u}_t\rfloor^\lambda, \circ dB_t), (\lfloor \mathrm{u}_t\rfloor^\lambda)_\lambda^{(1)}\big)\mathrm{v}_t^\lambda + \nabla(\mathrm{v}_t^\lambda)(H^\lambda)_\lambda^{(1)}(\lfloor \mathrm{u}_t\rfloor^\lambda, \circ dB_t)\Big). \end{aligned}$$

ii) *The Itô SDE of the process* $\theta_{\lfloor \mathrm{u}_t\rfloor^\lambda}\big((\mathrm{v}_t^\lambda)_0^{(1)}\big)$ *in* $T\mathcal{F}(\mathbb{R}^n)$ *is*

$$\begin{aligned} d\big(\theta((\mathrm{v}_t^\lambda)_0^{(1)})\big) =& \varpi((\mathrm{v}_t^\lambda)_0^{(1)})dB_t + \mathrm{Ric}(\lfloor \mathrm{u}_t\rfloor^0\theta((\mathrm{v}_t^\lambda)_0^{(1)}))dt \\ &+ \Phi_\theta(\mathrm{v}_t^\lambda, (\lfloor \mathrm{u}_t\rfloor^\lambda)_0^{(1)}, dB_t, dt) + \Phi_\theta^{1,1}(\mathrm{v}_t^\lambda, (\lfloor \mathrm{u}_t\rfloor^\lambda)_0^{(1)}, dB_t, dt), \end{aligned}$$

where $\Phi_\theta(\cdot,\cdot,dB_t,dt)$ *is given in (4.15) associated to* $\lfloor \mathrm{u}_t\rfloor^0$ *and* [2]

$$\begin{aligned}\Phi_\theta^{1,1}(\mathrm{v}_t^0,(\lfloor \mathrm{u}_t\rfloor^\lambda)_0^{(1)},dB_t,dt) :=& 2\varpi\big(\nabla(\mathrm{v}_t^0)(H^\lambda)_0^{(1)}(\lfloor \mathrm{u}_t\rfloor^0,e_i)\big)e_i dt\\ &+\theta\left(\Big[H^0(\lfloor \mathrm{u}_t\rfloor^0,e_i),\nabla(\mathrm{v}_t^0)(H^\lambda)_0^{(1)}(\lfloor \mathrm{u}_t\rfloor^0,e_i)\Big]\right)dt.\end{aligned}$$

The Itô SDE of the process $\varpi_{\lfloor \mathrm{u}_t\rfloor^\lambda}\big((\mathrm{v}_t^\lambda)_0^{(1)}\big)$ *in* $T\mathcal{F}(\mathbb{R}^n)$ *is*

$$\begin{aligned}d\big(\varpi((\mathrm{v}_t^\lambda)_0^{(1)})\big) =& (\lfloor \mathrm{u}_t\rfloor^0)^{-1}R\left(\lfloor \mathrm{u}_t\rfloor^0 dB_t,\lfloor \mathrm{u}_t\rfloor^0\theta((\mathrm{v}_t^\lambda)_0^{(1)})\right)\lfloor \mathrm{u}_t\rfloor^0\\ &+(\lfloor \mathrm{u}_t\rfloor^0)^{-1}R\left(\lfloor \mathrm{u}_t\rfloor^0 e_i,\lfloor \mathrm{u}_t\rfloor^0\varpi((\mathrm{v}_t^\lambda)_0^{(1)})e_i\right)\lfloor \mathrm{u}_t\rfloor^0\ dt\\ &+(\lfloor \mathrm{u}_t\rfloor^0)^{-1}\big(\nabla(\lfloor \mathrm{u}_t\rfloor^0 e_i)R\big)\left(\lfloor \mathrm{u}_t\rfloor^0 e_i,\lfloor \mathrm{u}_t\rfloor^0\theta((\mathrm{v}_t^\lambda)_0^{(1)})\right)\lfloor \mathrm{u}_t\rfloor^0\ dt\\ &+\Phi_\varpi(\mathrm{v}_t^\lambda,(\lfloor \mathrm{u}_t\rfloor^\lambda)_0^{(1)},dB_t,dt)+\Phi_\varpi^{1,1}(\mathrm{v}_t^\lambda,(\lfloor \mathrm{u}_t\rfloor^\lambda)_0^{(1)},dB_t,dt),\end{aligned}$$

where $\Phi_\varpi(\cdot,\cdot,dB_t,dt)$ *is given in (4.16) associated to* $\lfloor \mathrm{u}_t\rfloor^0$ *and*

$$\Phi_\varpi^{1,1}(\mathrm{v}_t^0,(\lfloor \mathrm{u}_t\rfloor^\lambda)_0^{(1)},dB_t,dt) :=\varpi\left(\nabla(\mathrm{v}_t^0)(H^\lambda)_0^{(1)}(\lfloor \mathrm{u}_t\rfloor^0,dB_t)\right).$$

We can formulate $(\mathrm{v}_t^\lambda)_0^{(1)}$ and $(\theta,\varpi)(\mathrm{v}_t^\lambda)_0^{(1)}$ by stochastic Duhamel principle.

Corollary 4.27. *We retain all the notations in Corollary 4.26.*

i) *The process* $(\mathrm{v}_t^\lambda)_0^{(1)}$ *has the expression*

$$(\mathrm{v}_t^\lambda)_0^{(1)}=\big[D^{(1)}\lfloor F_t\rfloor^0(\lfloor \mathrm{u}_0\rfloor^0,w)\big]((\mathrm{v}_0^\lambda)_0^{(1)})+\nabla((\lfloor \mathrm{u}_0\rfloor^\lambda)_0^{(1)})\big[D^{(1)}\lfloor F_t\rfloor^0(\lfloor \mathrm{u}_0\rfloor^0,w)\big](\mathsf{v}_0^0)+\mathrm{V}_c((\mathrm{v}_t^\lambda)_0^{(1)}),$$

where

$$\mathrm{V}_c((\mathrm{v}_t^\lambda)_0^{(1)}) := \int_0^t\big[D^{(1)}\lfloor F_{\tau,t}\rfloor^0(\lfloor \mathrm{u}_\tau\rfloor^0,w)\big]\left(\nabla(\mathrm{v}_\tau^0)(H^\lambda)_0^{(1)}(\lfloor \mathrm{u}_\tau\rfloor^0,\circ dB_\tau)\right).$$

ii) *On* $T\mathcal{F}(\mathbb{R}^n)$*, the process* $\widetilde{(\mathrm{v}_t^\lambda)_0^{(1)}} := (\theta,\varpi)_{\lfloor \mathrm{u}_t\rfloor^\lambda}\big((\mathrm{v}_t^\lambda)_0^{(1)}\big)$ *has the expression*

$$\begin{aligned}\widetilde{(\mathrm{v}_t^\lambda)_0^{(1)}} =&[\widetilde{D^{(1)}\lfloor F_t\rfloor}^0(\lfloor \mathrm{u}_0\rfloor^0,w)]\widetilde{(\mathrm{v}_0^\lambda)_0^{(1)}}+(\theta,\varpi)\left(\nabla((\lfloor \mathrm{u}_0\rfloor^\lambda)_0^{(1)})\big[D^{(1)}\lfloor F_t\rfloor^0(\lfloor \mathrm{u}_0\rfloor^0,w)\big](\mathsf{v}_0^0)\right)\\ &+\widetilde{\mathrm{V}}_c((\mathrm{v}_t^\lambda)_0^{(1)}),\end{aligned}$$

where

$$\begin{aligned}\widetilde{\mathrm{V}}_c((\mathrm{v}_t^\lambda)_0^{(1)}) =& \int_0^t\Big[\widetilde{D^{(1)}\lfloor F_{\tau,t}\rfloor}^0(\lfloor \mathrm{u}_\tau\rfloor^0,w)\Big]\Big\{\big(\Phi_\theta^{1,1},\Phi_\varpi^{1,1}\big)(\mathrm{v}_\tau^0,(\lfloor \mathrm{u}_\tau\rfloor^\lambda)_0^{(1)},dB_\tau,d\tau)\\ &-\Big(\varpi\big(\nabla(\mathrm{v}_\tau^0)(H^\lambda)_0^{(1)}(\lfloor \mathrm{u}_\tau\rfloor^0,e_i)\big)e_i,\ 0\Big)\ d\tau\Big\}.\end{aligned}$$

[2]We use the upper script ^{1,1} to indicate the functions $\Phi_\theta^{1,1},\Phi_\varpi^{1,1}$ are associated with $\big([D^{(1)}\lfloor F_t(\cdot,w)\rfloor]\big)_\lambda^{(1)}$.

Proof. By a comparison of the SDEs in Lemma 4.25 and Corollary 4.26 with those in Lemma 4.5, we can compute as in Corollary 4.6 to derive i) and ii). We note that for ii), $\widetilde{\mathrm{V}}_c((\mathrm{v}_t^\lambda)_0^{(1)})$ has an extra term

$$-\Big(0,\ (\lfloor \mathrm{u}_\tau\rfloor^0)^{-1}R\Big(\lfloor \mathrm{u}_\tau\rfloor^0 e_i, \lfloor \mathrm{u}_\tau\rfloor^0\theta\big(\nabla(\mathrm{v}_\tau^0)(H^\lambda)_0^{(1)}(\lfloor \mathrm{u}_\tau\rfloor^0, e_i)\big)\Big)\lfloor \mathrm{u}_\tau\rfloor^0\Big)\ d\tau,$$

which turns out to be zero since $\theta\big(\nabla(\mathrm{v}_\tau^0)(H^\lambda)_0^{(1)}(\lfloor \mathrm{u}_\tau\rfloor^0, e_i)\big)$ is zero. □

We are in a situation to state the norm estimations on the differential processes.

Proposition 4.28. *Let $\lambda \mapsto g^\lambda \in \mathcal{M}^k(M)$ be a C^k curve with $k \geqslant 3$. Let $x \in \widetilde{M}$ and $\lambda \mapsto \lfloor \mathrm{u}_0\rfloor^\lambda \in \mathcal{O}_x^{\tilde{g}^\lambda}(\widetilde{M})$ be a C^{k-2} curve in $\mathcal{F}(\widetilde{M})$ and let $\{\lfloor \mathrm{u}_t\rfloor^\lambda\}_{t\in[0,T]}$ be the solution to (4.45) with initial $\lfloor \mathrm{u}_0\rfloor^\lambda$.*

i) *For every $q \geqslant 1$ and (l,j) with $1 \leqslant l+j \leqslant k-2$, there exist $\underline{c}_{(l,j)}(q)$ depending on q and the norm bounds of $\{\nabla^{(l')}(H^\lambda)_\lambda^{(j')}\}_{j'\leqslant j, l'+j'\leqslant l+j}, \{\nabla^{(l')}R^\lambda\}_{l'\leqslant l}$, and $c_{(l,j)}(q)$ depending on $(l,j), q$ and the norm bounds of $\{\nabla^{(l')}R^\lambda\}_{l'\leqslant 1}$, such that*

$$\mathbb{E}\left\|\Big(\big[D^{(l)}\lfloor F_{\underline{t},\bar{t}}\rfloor^\lambda(\mathrm{u}_{\underline{t}}, w)\big]\Big)_\lambda^{(j)}\right\|^q \leqslant \underline{c}_{(l,j)}(q)e^{c_{(l,j)}(q)T},\ \forall T\in\mathbb{R}_+, \tag{4.48}$$

$$\mathbb{E}\sup_{0\leqslant\underline{t}<\bar{t}\leqslant T}\left\|\Big(\big[D^{(l)}\lfloor F_{\underline{t},\bar{t}}\rfloor^\lambda(\mathrm{u}_{\underline{t}}, w)\big]\Big)_\lambda^{(j)}\right\|^q \leqslant \underline{c}_{(l,j)}(q)e^{c_{(l,j)}(q)T^3},\ \forall T\in\mathbb{R}_+. \tag{4.49}$$

ii) *Let $T_0 > 0$. For each $q \geqslant 1$, (l,j) with $1 \leqslant l+j \leqslant k-2$ and $T > T_0$, there exist $\underline{c}'_{(l,j)}(q)$, which depends on q and the norm bounds of $\{\nabla^{(l')}(H^\lambda)_\lambda^{(j')}\}_{j'\leqslant j, l'+j'\leqslant l+j}$, $\{\nabla^{(l')}R^\lambda\}_{l'\leqslant l}$, and $c'_{(l,j)}(q)$, which depends on $(l,j), q, T, T_0$ and the norm bounds of $\{\nabla^{(l')}R^\lambda\}_{l'\leqslant 1}$, such that, for any $x, y \in \widetilde{M}$,*

$$\mathbb{E}_{\mathbb{P}^{\lambda,*}_{x,y,T}}\sup_{0\leqslant\underline{t}<\bar{t}\leqslant T}\left\|\Big(\big[D^{(l)}\lfloor F_{\underline{t},\bar{t}}\rfloor^\lambda(\mathrm{u}_{\underline{t}}, w)\big]\Big)_\lambda^{(j)}\right\|^q \leqslant \underline{c}'_{(l,j)}(q)e^{c'_{(l,j)}(q)(1+d_{\tilde{g}^\lambda}(x,y))}. \tag{4.50}$$

Proof. We can show (4.48) and (4.49) in a similar way by using Proposition 4.8 and Proposition 4.12, respectively. Hence, we only present the proof for (4.48).

For $(l,j)=(l,0)$, (4.48) is (4.23). For $(l,j)=(0,j)$, without loss of generality, we verify it at $\lambda=0$ and we can do the estimations for $(\widetilde{\lfloor \mathrm{u}_{\bar{t}}\rceil^{\lambda}})_0^{(j)}$ by induction. By Corollary 4.21,

$$\begin{aligned}(\widetilde{\lfloor \mathrm{u}_{\bar{t}}\rceil^{\lambda}})_0^{(1)} &= \Big[D^{(1)}\widetilde{\lfloor F_{0,\bar{t}}\rceil}^0(\lfloor \mathrm{u}_0\rceil^0,w)\Big]\,(\widetilde{\lfloor \mathrm{u}_0\rceil^{\lambda}})_0^{(1)}\\ &\quad+\int_0^{\bar{t}}\Big[D^{(1)}\widetilde{\lfloor F_{\tau,\bar{t}}\rceil}^0(\lfloor \mathrm{u}_\tau\rceil^0,w)\Big]\Big(\varpi\big((H^{\lambda})_0^{(1)}(\lfloor \mathrm{u}_\tau\rceil^0,e_i)\big)e_i\ d\tau,\ 0\Big)\\ &\quad+\int_0^{\bar{t}}\Big[D^{(1)}\widetilde{\lfloor F_{\tau,\bar{t}}\rceil}^0(\lfloor \mathrm{u}_\tau\rceil^0,w)\Big]\Big(0,\varpi\big((H^{\lambda})_0^{(1)}(\lfloor \mathrm{u}_\tau\rceil^0,dB_\tau)\big)\Big)\\ &=:(\mathrm{I})_{\bar{t}}+(\mathrm{II})_{\bar{t}}+(\mathrm{III})_{\bar{t}}.\end{aligned}\tag{4.51}$$

Hence, by (4.22),

$$3^{1-q}\mathbb{E}\left\|(\widetilde{\lfloor \mathrm{u}_{\bar{t}}\rceil^{\lambda}})_0^{(1)}\right\|^q\leqslant\mathbb{E}\|(\mathrm{I})_{\bar{t}}\|^q+\mathbb{E}\|(\mathrm{II})_{\bar{t}}\|^q+\mathbb{E}\|(\mathrm{III})_{\bar{t}}\|^q.$$

Clearly,

$$\mathbb{E}\|(\mathrm{I})_{\bar{t}}\|^q\leqslant\mathbb{E}\left\|\Big[D^{(1)}\widetilde{\lfloor F_{0,\bar{t}}\rceil}^0(\lfloor \mathrm{u}_0\rceil^0,w)\Big]\right\|^q\cdot\left\|(\lfloor \mathrm{u}_0\rceil^{\lambda})_0^{(1)}\right\|^q\leqslant\underline{c}(1,q)e^{c(1,q)T}\left\|(\lfloor \mathrm{u}_0\rceil^{\lambda})_0^{(1)}\right\|^q,$$

where $\underline{c}(1,q),c(1,q)$ are the constants given in Proposition 4.8. For $(\mathrm{II})_{\bar{t}}$, we have

$$\begin{aligned}\mathbb{E}\|(\mathrm{II})_{\bar{t}}\|^q&\leqslant T^{q-1}\int_0^{\bar{t}}\left(\mathbb{E}\left\|\Big[D^{(1)}\lfloor F_{\tau,\bar{t}}\rceil^{\lambda}(\lfloor \mathrm{u}_\tau\rceil^{\lambda},w)\Big]\right\|^{2q}\cdot\mathbb{E}\left\|\varpi\big((H^{\lambda})_0^{(1)}(\lfloor \mathrm{u}_\tau\rceil^0,e_i)\big)e_i\right\|^{2q}\right)^{\frac{1}{2}}\ d\tau\\ &\leqslant(C_1T)^q\sqrt{\underline{c}(1,2q)}e^{\frac{1}{2}c(1,2q)T},\end{aligned}$$

where C_1 is a bound for the norms of $\{\nabla^{(\iota)}(H^{\lambda})_{\lambda}^{(i)}\}_{\iota+i\leqslant1}$ and $\{\nabla^{(\iota)}R^{\lambda}\}_{\iota\leqslant1}$. Using Burkholder's inequality, we obtain

$$\big(\mathbb{E}\|(\mathrm{III})_{\bar{t}}\|^q\big)^2\leqslant\mathtt{C}_1(2q)\mathbb{E}\left|\int_0^{\bar{t}}\left\|\Big[D^{(1)}\widetilde{\lfloor F_{\tau,\bar{t}}\rceil}^0(\lfloor \mathrm{u}_\tau\rceil^0,w)\Big]\Big(0,\varpi\big((H^{\lambda})_0^{(1)}(\lfloor \mathrm{u}_\tau\rceil^0,\cdot)\big)\Big)\right\|^2\ d\tau\right|^q.$$

Then, using (4.23) and Hölder's inequality, we continue to estimate that

$$\mathbb{E}\|(\mathrm{III})_{\bar{t}}\|^q\leqslant\sqrt{\mathtt{C}_1(2q)}(C_1\sqrt{T})^q\sqrt[4]{\underline{c}(1,2q)}e^{\frac{1}{4}c(1,2q)T}.$$

Putting all these estimations together shows (4.48) for the $(l,j)=(0,1)$ case.

When $(l,j)=(0,2)$, we deduce from (4.26) and (4.47) that

$$2^{1-q}\mathbb{E}\left\|(\widetilde{\lfloor \mathrm{u}_{\bar{t}}\rceil^{\lambda}})_{\lambda}^{(2)}\right\|^q\leqslant\underline{c}(2,q)e^{c(2,q)T}\left\|(\lfloor \mathrm{u}_0\rceil^{\lambda})_0^{(1)}\right\|^q\left\|(\lfloor \mathrm{u}_0\rceil^{\lambda})_0^{(2)}\right\|^q+\mathbb{E}\left\|\widetilde{\mathrm{V}}_c((\lfloor \mathrm{u}_{\bar{t}}\rceil^{\lambda})_0^{(2)})\right\|^q.$$

Using the the Itô expression of $\widetilde{\mathrm{V}}_c((\lfloor \mathrm{u}_{\bar{t}}\rceil^{\lambda})_0^{(2)})$ in Corollary 4.24, (4.26) and (4.48) for the $(l,j)=(0,1)$ case, we can follow the calculations for $(\widetilde{\lfloor \mathrm{u}_{\bar{t}}\rceil^{\lambda}})_0^{(1)}$ to show

$$\mathbb{E}\left\|\widetilde{\mathrm{V}}_c((\lfloor \mathrm{u}_{\bar{t}}\rceil^{\lambda})_0^{(2)})\right\|^q<\underline{c}^{\mathrm{v}}_{(0,2)}(q)e^{c^{\mathrm{v}}_{(0,2)}(q)T},$$

where $\underline{c}^{\mathrm{v}}_{(0,2)}(q), c^{\mathrm{v}}_{(0,2)}(q)$ are of the same types as $\underline{c}_{(0,2)}(q), c_{(0,2)}(q)$ prescribed in (4.48). This finishes the proof of (4.48) for the $(l,j)=(0,2)$ case.

Assume we have shown (4.48) for all $(0,j)$, $j<j_0\leqslant k-2$. For the j_0 step, following Corollary 4.21 and Corollary 4.24, we obtain

$$\begin{aligned}\big(\widetilde{(\lfloor \mathrm{u}_{\bar{t}}\rceil^{\lambda})_0^{(1)}},\cdots,\widetilde{(\lfloor \mathrm{u}_{\bar{t}}\rceil^{\lambda})_0^{(j_0)}}\big)=&\big[\widetilde{D^{(j_0)}\lfloor F_{\bar{t}}\rceil^0}(\cdot,w)\big]\Big((\widetilde{\lfloor \mathrm{u}_0\rceil^{\lambda})_0^{(1)}},\cdots,\widetilde{(\lfloor \mathrm{u}_{\bar{t}}\rceil^{\lambda})_0^{(j_0)}}\Big)\\&+\Big(\widetilde{\mathrm{V}}_c((\lfloor \mathrm{u}_{\bar{t}}\rceil^{\lambda})_0^{(1)}),\cdots,\widetilde{\mathrm{V}}_c((\lfloor \mathrm{u}_{\bar{t}}\rceil^{\lambda})_0^{(j_0)})\Big),\end{aligned}$$

where $\widetilde{\mathrm{V}}_c((\lfloor \mathrm{u}_{\bar{t}}\rceil^{\lambda})_0^{(j)})$ is the 'variation of constant part' for $\widetilde{(\lfloor \mathrm{u}_{\bar{t}}\rceil^{\lambda})_0^{(j)}}$ whose Itô form can be expressed using $\big[\widetilde{D^{(1)}\lfloor F_{\tau,\bar{t}}\rceil^0}(\lfloor \mathrm{u}_\tau\rceil^0,w)\big]$, $\{\nabla^{(\iota)}(H^\lambda)^{(i)}_\lambda\}_{i\leqslant j,\iota+i\leqslant j}$, $\{\nabla^{(\iota)}R^\lambda\}_{\iota\leqslant j}$ and $\{\widetilde{(\lfloor \mathrm{u}_{\bar{t}}\rceil^{\lambda})_0^{(\iota)}}\}_{\iota<j}$. So, using Proposition 4.8 and the inductive estimation assumptions on $\{\widetilde{(\lfloor \mathrm{u}_{\bar{t}}\rceil^{\lambda})_\lambda^{(j)}}\}_{j<j_0}$, the same computations as above will derive (4.48) for the $j=j_0$ case.

For (l,j) with $l\neq 0$, it is equivalent to consider (4.48) for $\big(\widetilde{[D^{(l)}\lfloor F_{\underline{t},\bar{t}}\rceil^\lambda}(\cdot,w)]\big)^{(j)}_\lambda$. Without loss of generality, we check it at $\lambda=0$. Consider the case $(l,j)=(1,1)$ first. Let $\lambda\mapsto \mathrm{v}^\lambda_{\underline{t}}\in T_{(\lfloor \mathrm{u}_{\underline{t}}\rceil^\lambda)_0^{(1)}}\mathcal{F}(\widetilde{M})$ be C^1 with $(\mathrm{v}^\lambda_{\underline{t}})^{(1)}_0$ of norm 1. Corollary 4.27 shows that

$$\begin{aligned}\widetilde{(\mathrm{v}^\lambda_{\bar{t}})_0^{(1)}}=&(\theta,\varpi)\Big([D^{(1)}\lfloor F_{\underline{t},\bar{t}}\rceil^0(\lfloor \mathrm{u}_{\underline{t}}\rceil^0,w)]((\mathrm{v}^\lambda_{\underline{t}})_0^{(1)})+\nabla((\lfloor \mathrm{u}_{\underline{t}}\rceil^\lambda)_0^{(1)})\big[D^{(1)}\lfloor F_{\underline{t},\bar{t}}\rceil^0(\lfloor \mathrm{u}_{\underline{t}}\rceil^0,w)\big](\mathrm{v}^0_{\underline{t}})\Big)\\&+\widetilde{\mathrm{V}}_c((\mathrm{v}^\lambda_{\bar{t}})_0^{(1)}),\end{aligned}$$

where

$$\begin{aligned}\widetilde{\mathrm{V}}_c((\mathrm{v}^\lambda_{\bar{t}})_0^{(1)})=\int_{\underline{t}}^{\bar{t}}\Big[\widetilde{D^{(1)}\lfloor F_{\tau,\bar{t}}\rceil^0}(\lfloor \mathrm{u}_\tau\rceil^0,w)\Big]\Big\{\Big(\Phi^{1,1}_\theta,\Phi^{1,1}_\varpi\Big)(\mathrm{v}^0_\tau,(\lfloor \mathrm{u}_\tau\rceil^\lambda)_0^{(1)},dB_\tau,d\tau)\\-\Big(\varpi\big(\nabla(\mathrm{v}^0_\tau)(H^\lambda)_0^{(1)}(\lfloor \mathrm{u}_\tau\rceil^0,e_i)\big)e_i,\ 0\Big)\ d\tau\Big\}.\end{aligned}$$

This, together with Lemma 4.5, implies that

$$\begin{aligned}\big(\widetilde{(\lfloor \mathrm{u}_{\bar{t}}\rceil^\lambda)_0^{(1)}},\widetilde{(\lfloor \mathrm{v}_{\bar{t}}\rceil^\lambda)_0^{(1)}}\big)\ =\ &\Big[\widetilde{D^{(2)}\lfloor F_{\underline{t},\bar{t}}\rceil^0}(\lfloor \mathrm{u}_{\underline{t}}\rceil^0,w)\Big]\Big(\widetilde{(\lfloor \mathrm{u}_{\underline{t}}\rceil^\lambda)_0^{(1)}},\widetilde{(\lfloor \mathrm{v}_{\underline{t}}\rceil^\lambda)_0^{(1)}}\Big)\\&+\Big(\widetilde{\mathrm{V}}_c((\lfloor \mathrm{u}_{\bar{t}}\rceil^\lambda)_0^{(1)}),\widetilde{\mathrm{V}}_c((\lfloor \mathrm{v}_{\bar{t}}\rceil^\lambda)_0^{(1)})\Big).\end{aligned}$$

So,

$$\Big\|\widetilde{(\lfloor \mathrm{v}_{\bar{t}}\rceil^\lambda)_0^{(1)}}\Big\|\leqslant\Big\|\Big[\widetilde{D^{(2)}\lfloor F_{\underline{t},\bar{t}}\rceil^0}(\lfloor \mathrm{u}_{\underline{t}}\rceil^0,w)\Big]\Big\|\cdot\Big\|\widetilde{(\lfloor \mathrm{u}_{\underline{t}}\rceil^\lambda)_0^{(1)}}\Big\|+\Big\|\widetilde{\mathrm{V}}_c((\lfloor \mathrm{v}_{\bar{t}}\rceil^\lambda)_0^{(1)})\Big\|,$$

where $\|\widetilde{\mathrm{V}}_c((\lfloor \mathrm{v}_{\bar{t}}\rceil^\lambda)_0^{(1)})\|$ is interpreted as the norm of the mapping $(\mathrm{v}^\lambda_{\underline{t}},(\mathrm{v}^\lambda_{\underline{t}})^{(1)}_0)\mapsto(\mathrm{v}^\lambda_{\bar{t}},(\mathrm{v}^\lambda_{\bar{t}})^{(1)}_0)$. Using (4.26) and (4.48) for $\widetilde{D^{(1)}\lfloor F_{\underline{t},\bar{t}}\rceil^0}$ and $\widetilde{(\lfloor \mathrm{u}_t\rceil^\lambda)_0^{(1)}}$, and the above expression of $\widetilde{\mathrm{V}}_c((\mathrm{v}^\lambda_{\bar{t}})^{(1)}_0)$,

we can compute as in the proof of Proposition 4.8 to deduce that

$$\mathbb{E}\left\|\widetilde{\mathrm{V}_c}((\lfloor \mathrm{v}_{\bar{t}}\rceil^\lambda)_0^{(1)})\right\| \leqslant \underline{c}^{\mathrm{v}}_{(1,1)}(q)e^{c^{\mathrm{v}}_{(1,1)}(q)T},$$

where $\underline{c}^{\mathrm{v}}_{(1,1)}(q), c^{\mathrm{v}}_{(1,1)}(q)$ are of the same types as $\underline{c}_{(1,1)}(q), c_{(1,1)}(q)$ prescribed in (4.48). Then using (4.26) and (4.48) for $\widetilde{D^{(2)}\lfloor F_{\underline{t},\bar{t}}\rceil^0}$ and $\widetilde{(\lfloor \mathrm{u}_t\rceil^\lambda)_0^{(1)}}$, we obtain

$$\begin{aligned}
&2^{1-q}\mathbb{E}\left\|\left([\widetilde{D^{(1)}\lfloor F_{\underline{t},\bar{t}}\rceil}^\lambda(\cdot,w)]\right)_0^{(1)}\right\|^q \\
&\qquad\leqslant \sqrt{\underline{c}_{(0,1)}(2q)\underline{c}_{(2,0)}(2q)}e^{\frac{1}{2}(c_{(0,1)}(2q)+c_{(2,0)}(2q))(1+T^3)} + \underline{c}^{\mathrm{v}}_{(1,1)}(q)e^{c^{\mathrm{v}}_{(1,1)}(q)(1+T^3)}.
\end{aligned}$$

For (l,j) with $l,j\geqslant 1$ and $3\leqslant l+j\leqslant k-2$, we can continue the discussion in Lemma 4.25 to formulate the Itô SDEs for $\left([\widetilde{D^{(l)}\lfloor F_t\rceil}^\lambda(\cdot,w)]\right)_\lambda^{(j)}$ as a sum of $[\widetilde{D^{(l+j)}\lfloor F_t\rceil}^\lambda(\cdot,w)]$ and a variation of constant part using $[D^{(1)}\lfloor F_{\tau,t}\rceil^\lambda(\cdot,w)]$, $\{\nabla^{(l')}(H^\lambda)_\lambda^{(j')}\}_{j'\leqslant j, l'+j'\leqslant l+j}$ and $\{\nabla^{(l')}R^\lambda\}_{l'\leqslant l}$. Then using (4.26) and the inductive assumption on (4.48), we can argue exactly as in the case $(l,j)=(1,1)$ to show (4.48) is valid.

For ii), it can be reduced to the estimations in (4.42) by using Lemma 4.16 and Proposition 4.15. We show ii) by an induction on $l+j$. For $(l,j)=(0,1)$, we compute

$$(\mathbf{U}) := \mathbb{E}_{\mathbb{P}^{0,*}_{x,y,T}}\sup_{0\leqslant\bar{t}\leqslant T}\left\|(\widetilde{\lfloor \mathrm{u}_{\bar{t}}\rceil^\lambda})_0^{(1)}\right\|^q.$$

Let $(\mathrm{I})_{\bar{t}}$, $(\mathrm{II})_{\bar{t}}$ and $(\mathrm{III})_{\bar{t}}$ be as in (4.51). It is true by (4.51) and (4.22) that

$$\begin{aligned}
3^{1-q}(\mathbf{U}) &\leqslant \mathbb{E}_{\mathbb{P}^{0,*}_{x,y,T}}\sup_{0\leqslant\bar{t}\leqslant T}\|(\mathrm{I})_{\bar{t}}\|^q + \mathbb{E}_{\mathbb{P}^{0,*}_{x,y,T}}\sup_{0\leqslant\bar{t}\leqslant T}\|(\mathrm{II})_{\bar{t}}\|^q + \mathbb{E}_{\mathbb{P}^{0,*}_{x,y,T}}\sup_{0\leqslant\bar{t}\leqslant T}\|(\mathrm{III})_{\bar{t}}\|^q \\
&=: (\mathbf{I})+(\mathbf{II})+(\mathbf{III}).
\end{aligned}$$

Hence $(\mathbf{U})$ will satisfy the bound estimation in (4.50) if $(\mathbf{I}),(\mathbf{II})$ and $(\mathbf{III})$ do. Let $\underline{c}'_1(q), c'_1(q)$ be given in Proposition 4.17 such that (4.42) holds with $l=1$. Then

$$\begin{aligned}
(\mathbf{I}) &\leqslant \mathbb{E}_{\mathbb{P}^{0,*}_{x,y,T}}\sup_{0\leqslant\bar{t}\leqslant T}\left\|\left[\widetilde{D^{(1)}\lfloor F_{0,\bar{t}}\rceil}^0(\lfloor \mathrm{u}_0\rceil^0,w)\right]\right\|^q\cdot\left\|(\lfloor \mathrm{u}_0\rceil^\lambda)_0^{(1)}\right\|^q \\
&\leqslant \underline{c}'_1(q)e^{c'_1(q)(1+d_{\tilde{g}^0}(x,y))}\left\|(\lfloor \mathrm{u}_0\rceil^\lambda)_0^{(1)}\right\|^q.
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
(\mathbf{II})^2 &\leqslant T^{2q}\mathbb{E}_{\mathbb{P}^{0,*}_{x,y,T}}\sup_{0\leqslant\tau<\bar{t}\leqslant T}\left\|\left[D^{(1)}\lfloor F_{\tau,\bar{t}}\rceil^\lambda(\lfloor \mathrm{u}_\tau\rceil^\lambda,w)\right]\right\|^{2q}\mathbb{E}_{\mathbb{P}^{0,*}_{x,y,T}}\sup_{0\leqslant\tau\leqslant T}\left\|\varpi\big((H^\lambda)_0^{(1)}(\lfloor \mathrm{u}_\tau\rceil^0,e_i)\big)e_i\right\|^{2q} \\
&\leqslant p^0(T,x,y)(C_1T)^{2q}\underline{c}'_1(2q)e^{c'_1(2q)(1+d_{\tilde{g}^0}(x,y))}.
\end{aligned}$$

Let $\underline{c}_0, c_0$ be given in (4.36). Then

$$(\mathbf{II}) \leqslant \sqrt{\underline{c}_0\underline{c}'_1(2q)}T_0^{-\frac{1}{2}m}(C_1T)^q e^{\frac{1}{2}c_0(1+T)+\frac{1}{2}c'_1(2q)(1+d_{\tilde{g}^0}(x,y))},$$

which is of the same bound type as in (4.50). It remains to consider (**III**). Again, we can use Lemma 4.16. Recall that

$$dB_\tau = db_\tau + 2(\lfloor \mathrm{u}_\tau \rceil^0)^{-1} \nabla \ln p^0(T-\tau, \lfloor \mathrm{x}_\tau \rceil^0, y)\ d\tau,$$

where $(b_\tau)_{\tau\in[0,T)}$ is a Brownian motion for $\mathbb{P}^0_{x,y,T}$. Let

$$(\mathrm{IV})_{\bar t} := \int_0^{\bar t} \left[\widetilde{D^{(1)}\lfloor F_{\tau,\bar t}\rceil^0}(\lfloor \mathrm{u}_\tau \rceil^0, w)\right]\left(0, \varpi\big((H^\lambda)_0^{(1)}(\lfloor \mathrm{u}_\tau \rceil^0, db_\tau)\big)\right),$$

$$(\mathrm{V})_{\bar t} := \int_0^{\bar t} \left[\widetilde{D^{(1)}\lfloor F_{\tau,\bar t}\rceil^0}(\lfloor \mathrm{u}_\tau \rceil^0, w)\right]\left(0, \varpi\big((H^\lambda)_0^{(1)}(\lfloor \mathrm{u}_\tau \rceil^0, 2(\lfloor \mathrm{u}_\tau \rceil^0)^{-1} \nabla \ln p^0(T-\tau, \lfloor \mathrm{x}_\tau \rceil^0, y)\ d\tau)\big)\right).$$

Then

$$2^{1-q}(\mathbf{III}) \leqslant \mathbb{E}_{\mathbb{P}^{0,*}_{x,y,T}} \sup_{0\leqslant \bar t\leqslant T} \|(\mathrm{IV})_{\bar t}\|^q + \mathbb{E}_{\mathbb{P}^{0,*}_{x,y,T}} \sup_{0\leqslant \bar t\leqslant T} \|(\mathrm{V})_{\bar t}\|^q =: (\mathbf{IV}) + (\mathbf{V}).$$

For (**IV**), using the Brownian character of b_τ with respect to $\mathbb{P}^0_{x,y,T}$, Doob's inequality of sub-martingales and Burkholder's inequality, we obtain

$$\begin{aligned}(\mathbf{IV})^2 \leqslant& \mathtt{C}(2q)\mathtt{C}_1(2q)\mathbb{E}_{\mathbb{P}^{0,*}_{x,y,T}} \left| \int_0^T \left\| \left[\widetilde{D^{(1)}\lfloor F_{\tau,T}\rceil^0}(\lfloor \mathrm{u}_\tau \rceil^0, w)\right]\left(0, \varpi\big((H^\lambda)_0^{(1)}(\lfloor \mathrm{u}_\tau \rceil^0, \cdot)\big)\right)\right\|^2\ d\tau \right|^q \\ \leqslant& \mathtt{C}(2q)\mathtt{C}_1(2q)(C_1^2T)^q \left(\mathbb{E}_{\mathbb{P}^{0,*}_{x,y,T}} \sup_{0\leqslant\tau<\bar t\leqslant T} \left\| \left[D^{(1)}\lfloor F_{\tau,\bar t}\rceil^\lambda(\lfloor \mathrm{u}_\tau \rceil^\lambda, w)\right]\right\|^{4q} \right)^{\frac12}.\end{aligned}$$

Using (4.42),

$$(\mathbf{IV}) \leqslant \sqrt{\mathtt{C}(2q)\mathtt{C}_1(2q)}(C_1\sqrt{T})^q \sqrt[4]{\underline{c}_1'(2q)} e^{\frac14 c_1'(2q)(1+d_{\tilde g^0}(x,y))}.$$

For (**V**), we have

$$(\mathbf{V}) \leqslant (2C_1)^q \mathbb{E}_{\mathbb{P}^{0,*}_{x,y,T}} \sup_{0\leqslant\tau<\bar t\leqslant T} \left\| \left[D^{(1)}\lfloor F_{\tau,\bar t}\rceil^0(\lfloor \mathrm{u}_\tau \rceil^0, w)\right]\right\|^q \cdot \left| \int_0^T \|\nabla \ln p^0(T-\tau, \lfloor \mathrm{x}_\tau \rceil^0, y)\|\ d\tau \right|^q.$$

Using Hölder's inequality, (4.42) and (4.38), we continue to estimate that

$$\begin{aligned}(\mathbf{V}) \leqslant& (2C_1)^q \sqrt{\underline{c}_1'(2q)} e^{\frac12 c_1'(2q)(T^3+Td_{\tilde g^0}(x,y))} \left(\mathbb{E}_{\mathbb{P}^{0,*}_{x,y,T}} e^{2q\int_0^T \|\nabla \ln p^0(T-\tau, \lfloor \mathrm{x}_\tau \rceil^0, y)\|\ d\tau} \right)^{\frac12} \\ \leqslant& \underline{c} T_0^{-m} (2C_1)^q \sqrt{\underline{c}_1'(2q)} e^{c(1+T)+\frac12 c_1'(2q))(1+d_{\tilde g^0}(x,y))},\end{aligned}$$

where $\underline{c}, c$ are as in (4.38). Altogether, this shows (**U**) has a bound type as in (4.50) and finishes the proof of (4.50) for $(l,j) = (0,1)$. The other cases of (4.50) can be handled by a similar conditional treatment parallel to (4.49).

Assume ii) holds true for (l,j) with $l+j = i-1$. For (l,j) with $l+j=i$, the cases $(l,j) = (i,0)$ were verified in Proposition 4.17. Then we continue to check the cases $(l,j) = (0,i)$, which will follow from (4.50) with $(l,j) = (1,0)$ and a variation of constant part estimation which can treated using (4.50) for $(l,j) = (0,j)$ with $j \leqslant i-1$ and (4.49)

with $(l,j)=(1,0)$. The remaining estimations in (4.42) for (l,j) with $l+j=i$ and $l\leqslant i-1$ will follow from (4.50) with $(l,j)=(i,0),(0,j), j\leqslant i$, as in the non-conditional case. □

5. The first differential of the heat kernels in metrics

Our main result in this section is a first step of the proof of Theorem 1.3.

Theorem 5.1. *For any $g^0=g\in\mathcal{M}^k(M)$ ($k\geqslant 3$), there exist $\iota\in(0,1)$ and a neighborhood $\mathcal{V}_g$ of g in $\mathcal{M}^k(M)$ such that the following hold true for any C^k curve $\lambda\in(-1,1)\mapsto g^\lambda\in\mathcal{V}_g$.*

i) *For any $x\in\widetilde{M}$ and $T\in\mathbb{R}_+$, $\lambda\mapsto p^\lambda(T,x,\cdot)$ is C^1 in $C^{k,\iota}(\widetilde{M})$ with*

$$(\ln p^\lambda)^{(1)}_\lambda(T,x,y)+(\ln\rho^\lambda)^{(1)}_\lambda(y)=\phi^1_\lambda(T,x,y), \tag{5.1}$$

where $\rho^\lambda(y)=(d\mathrm{Vol}^\lambda/d\mathrm{Vol}^0)(y)$ and ϕ^1_λ is as in (5.16).

ii) *Let $T_0>0$. For $q\geqslant 1$ and j, $0\leqslant l\leqslant k-3$, there are constants $c_{\lambda,(j,1)}(q)$ which depend on q, T, T_0, $\|g^\lambda\|_{C^{l+3}}$ and $\|\mathcal{X}^\lambda\|_{C^{l+2}}$ such that for all $x\in\widetilde{M}$ and $T>T_0$,*

$$\left\|\nabla^{(l)}(\ln p^\lambda)^{(1)}_\lambda(T,x,\cdot)\right\|_{L^q}\leqslant c_{\lambda,(l,1)}(q). \tag{5.2}$$

iii) *The function $x\mapsto\int_{\widetilde{M}}(p^\lambda)^{(1)}_\lambda(T,x,y)\widetilde{f}(y)\,d\mathrm{Vol}_{\widetilde{g}^\lambda}(y)$ is continuous for any uniformly continuous and bounded $\widetilde{f}\in C(\widetilde{M})$.*

5.1. **Strategy.** We show Theorem 5.1 by describing the C^1 vector field $\mathrm{z}^{\lambda,1}_T$ such that (1.6) holds true. Before that, let us recall some classical results for parabolic equations.

Let $\mathcal{D}\subset\mathcal{D}_1\times\mathcal{D}_2$ with $\mathcal{D}_1$ being a bounded interval of $\mathbb{R}_+$ and D_2 being a bounded connected open domain of $\widetilde{M}$. For $g\in\mathcal{M}^k(M)$, consider the parabolic equation

$$\mathrm{L}q:=(\frac{\partial}{\partial t}-\Delta)q=r, \tag{5.3}$$

where Δ is the $\widetilde{g}$-Laplacian on C^2 functions on $\widetilde{M}$ and r is a continuous function on $\mathcal{D}$. By a solution q to (5.3), we mean a function q on $\mathcal{D}$ which satisfies (5.3) and all the derivatives of which appear in $\mathrm{L}q$ are continuous functions on $\mathcal{D}$. Such a q can be smoother, depending on the regularities of both L and r. For instance, q is C^∞ if both both L and r are C^∞. In our case, L varies C^{k-2} Hölder with respect to base points and q is mostly C^k Hölder in general even in case r is smooth.

Lemma 5.2. *(*[**Fr**, Theorem 11, p.74]*) Let* L *be given in (5.3) which is C^{k-2} and Hölder continuous with exponent ι. Assume r in (5.3) is such that*

$$D^n_xD^l_tr,\ 0\leqslant n+2l\leqslant k-2, l\leqslant l',$$

are Hölder continuous with exponent ι, where D^b_a means the b-th differential form with respect to the a-coordinate. If q is a solution to (5.2), then

$$D^n_xD^l_tq,\ 0\leqslant n+2l\leqslant k, l\leqslant l'+1,$$

exist and are Hölder continuous with exponent ι.

In particular, if r is ι-Hölder and q solves (5.3), Lemma 5.2 shows that all the differentials of q up to the second order (where $\partial/\partial t$ is considered as second order differential) exist and are ι-Hölder. The next lemma from [**Fr**] further shows these differentials have bounds completely determined by the bounds of q and r. For $P = (\tau, x) \in \mathcal{D}$, define

$$d_P = \sup_{Q \in \mathcal{D}(\tau)} d(P, Q),$$

where $\mathcal{D}(\tau)$ is the intersection of the boundary of $\mathcal{D}$ with the half-space $t \leqslant \tau$. For a function f on $\mathcal{D}$ and any non-negative integers n, j and for $\iota \in (0, 1)$, define

$$|f|_{n,j} = \sum_{l=0}^{j} N_{n,l}[f], \quad |f|_{n,j+\iota} = |f|_{n,j} + \sum_{0}^{j} N_{n,l+\iota}[f],$$

where

$$\begin{aligned} N_{n,l}[f] &= \sum \sup_{P \in \mathcal{D}} \left\{ d_P^{n+l} |D_x^l f(P)| \right\}, \\ N_{n,l+\iota}[f] &= \sum \sup_{P,Q \in \mathcal{D}} \left\{ \min\{d_P^{n+l+\iota}, d_Q^{n+l+\iota}\} \cdot \frac{|D_x^l f(P) - D_x^l f(Q)|}{d(P, Q)^\iota} \right\}, \end{aligned}$$

and the summation is over all the differentials of order l.

Lemma 5.3. *(*[**Fr**, Theorem 1, p.92]*) Let* L *be as in Lemma 5.2. There exists some geometric constant κ (which depends on ι, $\|g\|_{C^1}$) such that if $|r|_{2,\iota} < +\infty$ and q is a bounded solution to (5.3) and all its derivatives appearing in* Lq *are ι Hölder, then*

$$|q|_{0,2+\iota} < \kappa(|q|_{0,0} + |r|_{2,\iota}).$$

(Both Lemma 5.2 and Lemma 5.3 were stated in [**Fr**] for domains in the Euclidean case. They apply to the manifold case since (5.3) can be treated locally in coordinate charts.)

A companion notion of a solution to a parabolic equation is a solution as distribution. Recall that the distributions on the domain $\mathcal{D}$ are the linear continuous functionals on the test function space $C_c^\infty(\mathcal{D})$ of compactly supported smooth functions on $\mathcal{D}$. Given a distribution q on $\mathcal{D}$, one can define its weak derivative of any order α, denoted by $D^{\alpha,\mathrm{w}}q$, as a distribution on $\mathcal{D}$ by letting

$$(D^{\alpha,\mathrm{w}}q)(f) := (-1)^{|\alpha|} q(D^\alpha f), \ \forall f \in C_c^\infty(\mathcal{D}).$$

Any locally integrable function $q \in L^1_{\mathrm{loc}}(\mathcal{D})$ can be identified with a distribution by letting

$$q(f) := \int_{\mathcal{D}} q f dt \times d\mathrm{Vol}, \ \forall f \in C_c^\infty(\mathcal{D}),$$

and hence its weak derivatives of any order always exist. Let L be as in (5.3). The L distributional derivative of a distribution q on $\mathcal{D}$ will be denoted by $\mathrm{L}^{\mathrm{w}}q$. Using mollifier and Lemma 5.3, we have the following classical result.

Lemma 5.4. *Let $g \in C^k(M)$ and let* L *be as in (5.3). Assume $r \in C^{0,\iota}(\mathcal{D})$ for some $\iota > 0$ with $|r|_{2,\iota} < \infty$. Then for any $q \in C(\mathcal{D})$,*

$$\mathrm{L}^{\mathrm{w}} q = r \implies \mathrm{L} q = r.$$

As a corollary of the above lemmas, we have the following.

Lemma 5.5. *Assume there are locally L^1 integrable functions $\{\phi^1_\lambda(T,x,y)\}_{x\in\widetilde{M},T\in\mathbb{R}_+}$ on $\widetilde{M}$ which are continuous in λ-parameter and are continuous in (T,y)-parameter, locally uniformly in λ, such that, for any $f \in C^\infty_c(\widetilde{M})$,*

$$\left(\int_{\widetilde{M}} f(y) p^\lambda(T,x,y)\ d\mathrm{Vol}^\lambda(y)\right)^{(1)}_\lambda = \int_{\widetilde{M}} f(y)\phi^1_\lambda(T,x,y) p^\lambda(T,x,y)\ d\mathrm{Vol}^\lambda(y). \tag{5.4}$$

Then, Theorem 5.1 i) *holds true.*

Proof. Let $T \in \mathbb{R}_+$ and $x \in \widetilde{M}$. If (5.4) is true, then for any $f \in C^\infty_c(\widetilde{M})$,

$$\begin{aligned}
&\int_{\widetilde{M}} f(y)\left(p^\lambda(T,x,y)\rho^\lambda(y) - p^0(T,x,y)\rho^0(y)\right) d\mathrm{Vol}^0(y)\\
&= \int_0^\lambda \int_{\widetilde{M}} f(y)\phi^1_{\tilde\lambda}(T,x,y) p^{\tilde\lambda}(T,x,y)\rho^{\tilde\lambda}(y)\ d\mathrm{Vol}^0(y) d\tilde\lambda\\
&= \int_{\widetilde{M}} f(y)\left(\int_0^\lambda \phi^1_{\tilde\lambda}(T,x,y) p^{\tilde\lambda}(T,x,y)\rho^{\tilde\lambda}(y) d\tilde\lambda\right)\ d\mathrm{Vol}^0(y), \qquad (5.5)
\end{aligned}$$

where $\rho^{\tilde\lambda} = d\mathrm{Vol}^{\tilde\lambda}/d\mathrm{Vol}^0$ and the second equality holds by Fubini theorem. Note that if a continuous function ϕ is such that $\int_{\widetilde{M}} \phi(y) f(y)\ d\mathrm{Vol}(y) = 0$ for all $f \in C^\infty_c(\widetilde{M})$ and a volume element Vol of a C^2 Riemannian metric, then ϕ is zero. Hence we can conclude from (5.5) that

$$p^\lambda(T,x,y)\rho^\lambda(y) - p^0(T,x,y)\rho^0(y) = \int_0^\lambda \phi^1_{\tilde\lambda}(T,x,y) p^{\tilde\lambda}(T,x,y)\rho^{\tilde\lambda}(y) d\tilde\lambda \tag{5.6}$$

since the functions appearing on both sides are all continuous in y-variable and λ-variable. Since $\lambda \mapsto \rho^\lambda$ is C^k, (5.6) implies the existence of $(p^\lambda)^{(1)}_\lambda(T,x,y)$ for every y and

$$(p^\lambda)^{(1)}_\lambda(T,x,y)\cdot\rho^\lambda(y) + p^\lambda(T,x,y)\cdot(\rho^\lambda)^{(1)}_\lambda(y) = \phi^1_\lambda(T,x,y)p^\lambda(T,x,y)\rho^\lambda(y). \tag{5.7}$$

Then (5.7) implies that $(p^\lambda)^{(1)}_\lambda(\cdot,x,\cdot)$ is a continuous function on $\mathbb{R}_+ \times \widetilde{M}$ since we have the continuity in the (T,y)-coordinate of both $p^\lambda(T,x,y)$ and $\phi^1_\lambda(T,x,y)$ by assumption.

Shrinking the neighborhood $\mathcal{V}_g$ of g if necessary, we may assume there is $\iota > 0$ such that $p^\lambda(T,x,\cdot) \in C^{k,\iota}(\widetilde{M})$ for all λ. Since it is a local problem, for $(T,y) \in \mathbb{R}_+ \times \widetilde{M}$, we can also restrict ourselves to a bounded domain $\mathcal{D}$ containing (T,y). Note that $\mathrm{L}^\lambda p^\lambda = 0$. Lemma

5.3 implies $|p^\lambda(T,x,\cdot)|_{0,2+\iota}<\infty$ on $\mathcal{D}$. For each $x\in\widetilde{M}$, since $(p^\lambda)^{(1)}_\lambda(T,x,y)$ is continuous in (T,y), its weak derivatives in (T,y) of any order are well-defined. So

$$\mathrm{L}^{\lambda,\mathrm{w}}(p^\lambda)^{(1)}_\lambda(T,x,\cdot)=(\mathrm{L}^\lambda)^{(1),\mathrm{w}}p^\lambda(T,x,\cdot)=(\mathrm{L}^\lambda)^{(1)}_\lambda p^\lambda(T,x,\cdot). \tag{5.8}$$

We can handle the equation locally. Shrinking the domain $\mathcal{D}$ to $\mathcal{D}_1$ if necessary, we deduce from $|p^\lambda(T,x,\cdot)|_{0,2+\iota}<\infty$ on $\mathcal{D}$ that $|(\mathrm{L}^\lambda)^{(1)}_\lambda p^\lambda(T,x,\cdot)|_{2,\iota}<\infty$ on $\mathcal{D}_1$. Since $(p^\lambda)^{(1)}_\lambda(T,x,\cdot)$ is continuous, Lemma 5.4 implies that (5.8) holds true in the usual sense, i.e.,

$$\mathrm{L}^\lambda(p^\lambda)^{(1)}_\lambda(T,x,\cdot)=-(\mathrm{L}^\lambda)^{(1)}_\lambda p^\lambda(T,x,\cdot). \tag{5.9}$$

Then we can apply Lemma 5.3 to conclude that $|(p^\lambda)^{(1)}_\lambda(T,x,\cdot)|_{0,2+\iota}<\infty$ on $\mathcal{D}_1$ and apply Lemma 5.2 to conclude that $(p^\lambda)^{(1)}_\lambda(T,x,\cdot)\in C^{k,\iota}(\mathcal{D}_1)$. The norm of $(p^\lambda)^{(1)}_\lambda(T,x,\cdot)$ in $C^{k,\iota}(\mathcal{D}_1)$ are locally uniformly bounded in λ by using (5.9), Lemma 5.2 and Lemma 5.3. So the continuity of $\lambda\mapsto(p^\lambda)^{(1)}_\lambda(T,x,\cdot)$ in $C(\widetilde{M})$ is improved to the continuity in $C^{k,\iota}(\widetilde{M})$. □

For Theorem 5.1, it remains to find a candidate $\phi^1_\lambda(T,x,y)$ for Lemma 5.5. Let $x\in\widetilde{M}$ and let $\lfloor \mathrm{u}_0\rceil^\lambda\in\mathcal{O}^{\tilde{g}^\lambda}_x(\widetilde{M})$. Recall that the solution to the SDE

$$d\lfloor \mathrm{u}_t\rceil^\lambda=\sum_{i=1}^m H^\lambda(\lfloor \mathrm{u}_t\rceil^\lambda,e_i)\circ dB^i_t(w)$$

with initial value $\lfloor \mathrm{u}_0\rceil^\lambda$ projects to be the Brownian motion $\lfloor \mathrm{x}_t\rceil^\lambda$ on $\widetilde{M}$ starting from x and the heat kernel function $p^\lambda(T,x,\cdot)$ is just the density of the distribution of $w\mapsto\lfloor \mathrm{x}_T\rceil^\lambda(w)$ under Q. Hence for any $f\in C^\infty_c(\widetilde{M})$, we have

$$\int_{\widetilde{M}} f(y)p^\lambda(T,x,y)\ d\mathrm{Vol}^\lambda(y)=\mathbb{E}\big(f(\lfloor \mathrm{x}_T\rceil^\lambda(w))\big)$$

and the equality continues to hold if we differentiate both sides in λ. Choose $\lambda\mapsto\lfloor \mathrm{u}_0\rceil^\lambda$ to be a C^{k-2} curve. By Lemma 4.18, for almost all w and all $t\in\mathbb{R}_+$, $\lambda\mapsto\lfloor \mathrm{u}_t\rceil^\lambda(w)$ is C^{k-2}. By Proposition 4.28, the differentials $(\lfloor \mathrm{u}_t\rceil^\lambda)^{(j)}_\lambda(w)$, $j\leqslant k-2$, are L^1 integrable, uniformly in λ. Hence,

$$\begin{aligned}&\Big(\mathbb{E}\big(f(\lfloor \mathrm{x}_T\rceil^{\tilde{\lambda}}(w))\big)\Big)^{(1)}_\lambda\\ &=\ \mathbb{E}\left(\Big\langle\nabla^\lambda_{\lfloor \mathrm{x}_T\rceil^\lambda(w)}(f\circ\pi)(\lfloor \mathrm{u}_T\rceil^\lambda(w)),(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda(w)\Big\rangle_\lambda\right) &(5.10)\\ &=\ \int_{\widetilde{M}}\mathbb{E}\left(\big\langle\nabla^\lambda_y f(y),D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda(w)\big\rangle_\lambda\Big|\ \lfloor \mathrm{x}_T\rceil^\lambda(w)=y\right)\cdot p^\lambda(T,x,y)\ d\mathrm{Vol}^\lambda(y). &(5.11)\end{aligned}$$

Note that (5.11) holds for every choice of $\lfloor \mathrm{u}_0\rceil^\lambda\in\mathcal{O}^{\tilde{g}^\lambda}(\widetilde{M})$ at λ. For some technical considerations which we will mention later, we choose $\lfloor \mathrm{u}_0\rceil^\lambda$ in $\mathcal{O}^{\tilde{g}^\lambda}(\widetilde{M})$ at random with a uniform distribution normalized to be probability 1 and then choose

$$\lfloor \mathrm{u}_0\rceil^{\tilde{\lambda}}=\lfloor \overline{\mathrm{u}}_0\rceil^{\tilde{\lambda}}(\lfloor \overline{\mathrm{u}}_0\rceil^\lambda)^{-1}\lfloor \mathrm{u}_0\rceil^\lambda,\ \tilde{\lambda}\in(-1,1),$$

where $\lfloor \overline{\mathrm{u}}_0 \rceil^{\tilde{\lambda}}$ is some fixed C^k curve in $\mathcal{F}(\widetilde{M})$ with $\lfloor \overline{\mathrm{u}}_0 \rceil^{\tilde{\lambda}} \in \mathcal{O}^{\tilde{g}^{\tilde{\lambda}}}(\widetilde{M})$. Write $\overline{\mathbb{E}}$ for the new expectation when the random choices of $\lfloor \mathrm{u}_0 \rceil^{\lambda}$ are taken into account. Then

(5.12)

$$\left(\mathbb{E} f(\lfloor \mathrm{x}_T \rceil^{\tilde{\lambda}})\right)^{(1)}_{\lambda} = \int_{\widetilde{M}} \overline{\mathbb{E}} \left(\left\langle \nabla^{\lambda}_y f(y), D\pi(\lfloor \mathrm{u}_T \rceil^{\lambda})^{(1)}_{\lambda}(w) \right\rangle_{\lambda} \Big| \lfloor \mathrm{x}_T \rceil^{\lambda}(w) = y \right) \cdot p^{\lambda}(T,x,y)\ d\mathrm{Vol}^{\lambda}(y).$$

For any C^k bounded vector field Y on $\widetilde{M}$, let

$$\overline{\Phi}^1_{\lambda}(Y)(y) := \overline{\mathbb{E}} \left(\Phi^1_{\lambda}(Y,w) \big| \lfloor \mathrm{x}_T \rceil^{\lambda}(w) = y \right), \tag{5.13}$$

where

$$\Phi^1_{\lambda}(Y,w) := \left\langle Y(\lfloor \mathrm{x}_T \rceil^{\lambda}(w)), D\pi(\lfloor \mathrm{u}_T \rceil^{\lambda})^{(1)}_{\lambda}(w) \right\rangle_{\lambda}. \tag{5.14}$$

We will show the linear functional $\overline{\Phi}^1_{\lambda}$ is such that $\overline{\Phi}^1_{\lambda}(Y)$ is C^1 in y variable, from which we can deduce that

$$\mathrm{z}^{\lambda,1}_T(y) := \overline{\mathbb{E}} \left(D\pi(\lfloor \mathrm{u}_T \rceil^{\lambda})^{(1)}_{\lambda}(w) \Big| \lfloor \mathrm{x}_T \rceil^{\lambda}(w) = y \right) \tag{5.15}$$

is a C^1 vector field on $\widetilde{M}$. Hence, we can apply the the classical integration by parts formula to (5.12) and compute that

$$\begin{aligned} &\left(\mathbb{E}(f(\lfloor \mathrm{x}_T \rceil^{\tilde{\lambda}}(w)))\right)^{(1)}_{\lambda} \\ &\quad = -\int_{\widetilde{M}} f(y) \left((\mathrm{Div}^{\lambda} \mathrm{z}^{\lambda,1}_T(y)) + \left\langle \mathrm{z}^{\lambda,1}_T(y), \nabla^{\lambda} \ln p^{\lambda}(T,x,y) \right\rangle_{\lambda} \right) p^{\lambda}(T,x,y)\ d\mathrm{Vol}^{\lambda}(y). \end{aligned}$$

This gives a candidate of ϕ^1_{λ} for Lemma 5.5 as

$$\phi^1_{\lambda}(T,x,y) := -\left((\mathrm{Div}^{\lambda} \mathrm{z}^{\lambda,1}_T(y)) + \left\langle \mathrm{z}^{\lambda,1}_T(y), \nabla^{\lambda} \ln p^{\lambda}(T,x,y) \right\rangle_{\lambda} \right). \tag{5.16}$$

If we can further verify that $\mathrm{z}^{\lambda,1}_T(y)$, $\mathrm{Div}^{\lambda} \mathrm{z}^{\lambda,1}_T(y)$ are both uniformly continuous in T and y, we can apply Lemma 5.5 to obtain a continuous version of $(p^{\lambda})^{(1)}_{\lambda}(T,x,\cdot)$ in distribution.

To justify that (5.16) is well-defined, we need to show the C^1 dependence of $\overline{\Phi}^1_{\lambda}(Y)(y)$ in y-variable. Let V be a smooth bounded vector field on $\widetilde{M}$ and let F^s be its flow map. To compare $\overline{\Phi}^1_{\lambda}(Y)(F^s(y))$ with $\overline{\Phi}^1_{\lambda}(Y)(y)$, our strategy is to extend F^s on $\lfloor \mathrm{x}_T \rceil^{\lambda}(w)$, the endpoint of Brownian motion paths at time T, to be a 'flow' $\mathbf{F}^s$ on Brownian paths up to time T. Let $\overline{\mathbb{P}}^{\lambda}_x$ denote the product of the probability $\mathbb{P}^{\lambda}_x$ with the uniform probability on $\mathcal{O}^{\tilde{g}^{\lambda}}(\widetilde{M})$ for the choice of $\lfloor \mathrm{u}_0 \rceil^{\lambda}$. We will ensure the maps $\mathbf{F}^s$ are such that $\overline{\mathbb{P}}^{\lambda}_x \circ \mathbf{F}^s$ are absolutely continuous with respect to $\overline{\mathbb{P}}^{\lambda}_x$. Clearly, for any bounded measurable function f on $\widetilde{M}$,

$$\overline{\mathbb{E}} \left(\Phi^1_{\lambda}(Y,w) f(\lfloor \mathrm{x}_T \rceil^{\lambda}(w)) \right) = \overline{\mathbb{E}} \left(\overline{\Phi}^1_{\lambda}(Y)(y) f(y) \right) \tag{5.17}$$

$$= \int \overline{\Phi}^1_{\lambda}(Y)(F^s(y)) f(F^s(y)) p^{\lambda}(T,x,F^s y)\ d\mathrm{Vol}^{\lambda}(F^s(y)), \tag{5.18}$$

where the first equality holds by the definition of conditional measures and the second equality holds by changing the variable to $F^s(y)$. The left hand side of (5.17), after a change of variable under $\mathbf{F}^s$, is equal to

$$\begin{aligned}&\overline{\mathbb{E}}\left(\Phi^1_\lambda\circ\mathbf{F}^s\cdot f\circ\mathbf{F}^s\cdot\frac{d\overline{\mathbb{P}}^\lambda_x\circ\mathbf{F}^s}{d\overline{\mathbb{P}}^\lambda_x}\right)\\ &=\int\overline{\mathbb{E}}\left(\Phi^1_\lambda\circ\mathbf{F}^s\cdot\frac{d\overline{\mathbb{P}}^\lambda_x\circ\mathbf{F}^s}{d\overline{\mathbb{P}}^\lambda_x}\middle|\lfloor \mathrm{x}_T\rceil^\lambda(w)=y\right)\cdot f(F^s(y))p^\lambda(T,x,y)\ d\mathrm{Vol}^\lambda(y).\end{aligned}\tag{5.19}$$

Since f is arbitrary, a comparison of (5.18) with (5.19) implies that

$$\begin{aligned}\overline{\Phi}^1_\lambda(Y)(F^s(y))=&\overline{\mathbb{E}}\left(\Phi^1_\lambda\middle|\lfloor \mathrm{x}_T\rceil^\lambda(w)=F^s(y)\right)\\ =&\overline{\mathbb{E}}\left(\Phi^1_\lambda\circ\mathbf{F}^s\cdot\frac{d\overline{\mathbb{P}}^\lambda_x\circ\mathbf{F}^s}{d\overline{\mathbb{P}}^\lambda_x}\middle|\lfloor \mathrm{x}_T\rceil^\lambda(w)=y\right)\frac{p^\lambda(T,x,y)}{p^\lambda(T,x,F^sy)}\frac{d\mathrm{Vol}^\lambda}{d\mathrm{Vol}^\lambda\circ F^s}(y).\end{aligned}\tag{5.20}$$

Note that $p^\lambda(T,x,y)$ and the volume element Vol^λ are C^k in y variable. So the differentiability in the s parameter of $\overline{\Phi}^1_\lambda(Y)(F^s(y))$ will follow from the differentiability in the s parameter of

$$\overline{\mathbb{E}}\left(\Phi^1_\lambda\circ\mathbf{F}^s\cdot\frac{d\overline{\mathbb{P}}^\lambda_x\circ\mathbf{F}^s}{d\overline{\mathbb{P}}^\lambda_x}\middle|\lfloor \mathrm{x}_T\rceil^\lambda(w)=y\right).\tag{5.21}$$

In order to show this differentiability in s, we will show that our one-parameter family of maps $\mathbf{F}^s$ satisfy the following properties (see Proposition 5.23, Proposition 5.29 and Proposition 5.30), where all the integrals are taken with respect to $\overline{\mathbb{P}}^\lambda_x$ conditioned on $\lfloor \mathrm{x}_T\rceil^\lambda(w)=y$.

i) $\overline{\mathbb{P}}^\lambda_x\circ\mathbf{F}^s$ is absolutely continuous with respect to $\overline{\mathbb{P}}^\lambda_x$ and the Radon-Nikodyn derivative $d\overline{\mathbb{P}}^\lambda_x\circ\mathbf{F}^s/d\overline{\mathbb{P}}^\lambda_x$ is L^q integrable for $q\geqslant 1$, locally uniformly in the s parameter,
ii) the differential of $d\overline{\mathbb{P}}^\lambda_x\circ\mathbf{F}^s/d\overline{\mathbb{P}}^\lambda_x$ in s is $\overline{\mathcal{E}^s_T}\cdot(d\overline{\mathbb{P}}^\lambda_x\circ\mathbf{F}^s/d\overline{\mathbb{P}}^\lambda_x)$, where $\overline{\mathcal{E}^s_T}$ is L^q integrable for $q\geqslant 1$, locally uniformly in the s parameter, and
iii) $(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda\circ\mathbf{F}^s$ is differentiable in s with the differential stochastic process L^q integrable for $q\geqslant 1$, locally uniformly in the s parameter.

With these three properties, we will obtain

$$\left(\Phi^1_\lambda\circ\mathbf{F}^s\cdot\frac{d\overline{\mathbb{P}}^\lambda_x\circ\mathbf{F}^s}{d\overline{\mathbb{P}}^\lambda_x}\right)'_s=\Phi^1_\lambda\circ\mathbf{F}^s\cdot\overline{\mathcal{E}^s_T}\cdot\frac{d\overline{\mathbb{P}}^\lambda_x\circ\mathbf{F}^s}{d\overline{\mathbb{P}}^\lambda_x}+\left(\Phi^1_\lambda\circ\mathbf{F}^s\right)'_s\frac{d\overline{\mathbb{P}}^\lambda_x\circ\mathbf{F}^s}{d\overline{\mathbb{P}}^\lambda_x}$$

and this differential is absolutely integrable, locally uniformly in the s parameter. Hence (5.21) is differentiable in the s parameter and we are allowed to take the differential inside the expectation sign. The uniform continuity of $\mathrm{z}^{\lambda,1}_T(y)$ and $\mathrm{Div}^\lambda\mathrm{z}^{\lambda,1}_T(y)$ in T and y will follow from (see the proof of Theorem 5.1 with $k=3$)

iv) the uniform continuity in T and y of

$$\overline{\mathbb{E}}\big(\,\Phi^1_\lambda\big|\,\lfloor \mathrm{x}_T\rceil^\lambda(w)=y\big)\ \text{ and }\ \overline{\mathbb{E}}\left(\big(\Phi^1_\lambda\circ\mathbf{F}^s\cdot\frac{d\overline{\mathbb{P}}^\lambda_x\circ\mathbf{F}^s}{d\overline{\mathbb{P}}^\lambda_x}\big)'_0\,\middle|\,\lfloor \mathrm{x}_T\rceil^\lambda(w)=y\right).$$

The major part of the remaining subsections is devoted to the construction of $\mathbf{F}^s$ and the verification of its properties i)-iv) mentioned above, which will conclude i) of Theorem 5.1. We will discuss Theorem 5.1 ii) and iii) in the last subsection.

Fix $T>0$. For each $y\in\widetilde{M}$, we will construct a one-parameter of maps $\mathbf{F}^s_y$ on Brownian motion paths starting from y up to time T with $\mathbf{F}^s_{y,x}$ being its conditional map on paths that will arrive at x in time T. We will achieve this in two steps: one for the SDE description of $\mathbf{F}^s_y$ and the other for its existence by Picard's iteration argument. The desired map $\mathbf{F}^s$ will be the collection of all $\mathbf{F}^s_{\mathrm{x}_T,x}$. But, we need to justify the meaning of $\Phi^1_\lambda\circ\mathbf{F}^s$ and $d\overline{\mathbb{P}}^\lambda_x\circ\mathbf{F}^s/d\overline{\mathbb{P}}^\lambda_x$ since Φ^1_λ and $\overline{\mathbb{P}}^\lambda_x$ are associated with the diffusion paths from x. This and the verification of i)-iv) will be done in Sections 5.4 and 5.5. Finally, in Section 5.6, we will show the assumption of Lemma 5.5 is satisfied and will give the estimations in (5.2) by an analysis of $\mathrm{z}^{\lambda,1}_T(y)$ and $\mathrm{Div}^\lambda\mathrm{z}^{\lambda,1}_T(y)$ using the SDE theory.

5.2. **A description of $\mathbf{F}^s_y$.** In this part, we fix $T\in\mathbb{R}^+$. Let $y\in\widetilde{M}$ and $\overline{\beta}_0\in\mathcal{O}^{\tilde{g}}_y(\widetilde{M})$. For a smooth segment $t\mapsto\alpha_t=(\alpha_{t,1},\cdots,\alpha_{t,m})\in\mathbb{R}^m, t\in[0,T]$, with $\alpha_0=o$, let $\overline{\beta}=(\overline{\beta}_t)_{t\in[0,T]}$ in $\mathcal{O}^{\tilde{g}}(\widetilde{M})$ be the unique smooth segment with initial $\overline{\beta}_0$ satisfying the differential equation

$$\nabla_{\frac{D}{\partial t}}\overline{\beta}_t=\sum_{i=1}^m H(\overline{\beta}_t,e_i)\cdot\frac{d\alpha_{t,i}}{dt}.$$

In the language of Section 4, this means that $\overline{\beta}$ is the transportation (or development) of α in $\mathcal{O}^{\tilde{g}}(\widetilde{M})$ with starting point $\overline{\beta}_0$ using the parallelism differential form (θ,ϖ). The Itô map $\mathcal{I}_{\overline{\beta}_0}:\ C^\infty_o([0,T],\mathbb{R}^m)\to C^\infty_y([0,T],\widetilde{M})$ is given by

$$\mathcal{I}_{\overline{\beta}_0}(\alpha):=\pi(\overline{\beta})=\beta. \tag{5.22}$$

It is invertible since α for (5.22) can be uniquely determined by the equation

$$\frac{d\alpha_t}{dt}=(\overline{\beta}_t)^{-1}\nabla_{\frac{D}{\partial t}}\beta_t,$$

where $\overline{\beta}\subset\mathcal{O}^{\tilde{g}}(\widetilde{M})$ is the horizontal lift of β with initial value $\overline{\beta}_0$, i.e.,

$$\nabla_{\frac{D}{\partial t}}\overline{\beta}_t=H(\overline{\beta}_t,\overline{\beta}_t^{-1}\nabla_{\frac{D}{\partial t}}\beta_t).$$

For $\beta\in C^\infty_y([0,T],\widetilde{M})$, its $\mathcal{I}$-preimage $\mathcal{I}^{-1}_{\overline{\beta}_0}(\beta)$ is called the *anti-development* of β in $\mathbb{R}^m$.

For a smooth segment (or curve) $\beta = (\beta_t)_{t\in[0,T]}$ on $\widetilde{M}$, the classical *parallel transportation map* $/\!\!/^{\beta}_{t_1,t_2}$ of tangent vectors along the segments $(\beta_t)_{t\in[t_1,t_2]}$ $(0 \leqslant t_1 \leqslant t_2 \leqslant T)$ is given by

$$/\!\!/^{\beta}_{t_1,t_2}(\mathbf{v}) = \overline{\beta}_{t_2} \circ \overline{\beta}_{t_1}^{-1}(\mathbf{v}),\ \forall \mathbf{v} \in T_{\beta_{t_1}}\widetilde{M},$$

where $\overline{\beta}$ is a horizontal lift of β. This definition is independent of the horizontal lift chosen since if $\overline{\beta}'$ is another horizontal lift of β, then $\overline{\beta}'_t = \overline{\beta}_t \overline{\beta}_0{}^{-1} \overline{\beta}'_0$ for $t \in [0,T]$ and hence

$$\overline{\beta}'_{t_2} \circ (\overline{\beta}'_{t_1})^{-1} = \overline{\beta}_{t_2} \overline{\beta}_0{}^{-1} \overline{\beta}'_0 \circ \big((\overline{\beta}'_0)^{-1} \overline{\beta}_0 \overline{\beta}_{t_1}^{-1}\big) = \overline{\beta}_{t_2} \circ \overline{\beta}_{t_1}^{-1}.$$

The Itô map and the parallel transportation map can also be defined in the stochastic case. Call an $\mathcal{O}^{\tilde{g}}(\widetilde{M})$-valued continuous stochastic process $\overline{\beta} = (\overline{\beta}_t)_{t\in[0,T]}$ *horizontal* if there exists a $\mathbb{R}^m$-valued continuous stochastic process $\alpha = (\alpha_{t,1}, \cdots, \alpha_{t,m})_{t\in[0,T]}$ with $\alpha_0 = o$ such that $\overline{\beta}$ solves the Stratonovich SDE

$$d\overline{\beta}_t = \sum_{i=1}^{m} H(\overline{\beta}_t, e_i) \circ d\alpha_{t,i}. \tag{5.23}$$

For a continuous stochastic process $(\beta_t)_{t\in[0,T]}$ on $\widetilde{M}$, its *horizontal lifts* are those horizontal processes $\overline{\beta}$ in $\mathcal{O}^{\tilde{g}}(\widetilde{M})$ projecting to it and its *anti-developments* in $\mathbb{R}^m$ are those α satisfying (5.23) (cf. [**Hs3**]). For a fixed $y \in \widetilde{M}$ and $\overline{\beta}_0 \in \mathcal{O}^{\tilde{g}}_y(\widetilde{M})$, (5.23) is uniquely solvable for every semi-martingale α and the *Itô map*

$$\mathcal{I}_{\overline{\beta}_0}(\alpha) := \pi(\overline{\beta}) = \beta$$

is well-defined. In the sprit of Section 4, $\mathcal{I}_{\overline{\beta}_0}(\alpha)$ is the projection process of a transportation (or development) of α in $\mathcal{O}^{\tilde{g}}(\widetilde{M})$ using the parallelism differential form (θ, ϖ). The one-to-one correspondence between α, β, and $\overline{\beta}$ for semi-martingales is discussed in [**Hs3**].

For a semi-martingale $\beta = (\beta_t)_{t\in[0,T]}$ on $\widetilde{M}$, its horizontal lifts are uniquely determined by the distribution of $\overline{\beta}_0$ (cf. [**Hs3**, Theorem 2.3.5]). Hence, for almost all $\mathrm{w} \in \Theta_+$, we can define a stochastic 'parallel transportation map' $/\!\!/^{\beta}_{t_1,t_2}$ of tangent vectors along the path segments $(\beta_t(\mathrm{w}))_{t\in[t_1,t_2]}$ $(0 \leqslant t_1 \leqslant t_2 \leqslant T)$ by letting

$$/\!\!/^{\beta}_{t_1,t_2}(\mathbf{v}) := \overline{\beta}_{t_2} \circ \overline{\beta}_{t_1}^{-1}(\mathbf{v}),\ \forall \mathbf{v} \in T_{\beta_{t_1}(\mathrm{w})}\widetilde{M}.$$

As in the deterministic case, this definition is independent of the horizontal lift $\overline{\beta}$ chosen. Each $/\!\!/^{\beta}_{t_1,t_2}$ is an isometry between $T_{\beta_{t_1}(\mathrm{w})}\widetilde{M}$ and $T_{\beta_{t_2}(\mathrm{w})}\widetilde{M}$ with the inverse map

$$(/\!\!/^{\beta}_{t_1,t_2})^{-1}(\mathbf{v}') := \overline{\beta}_{t_1} \circ \overline{\beta}_{t_2}^{-1}(\mathbf{v}'),\ \forall \mathbf{v}' \in T_{\beta_{t_2}(\mathrm{w})}\widetilde{M}.$$

Moreover, the parallel transportation maps $/\!\!/^{\beta}_{t_1,t_2}$ also satisfy the cocycle property

$$/\!\!/^{\beta}_{t_1,t_3} = /\!\!/^{\beta}_{t_3,t_2} \circ /\!\!/^{\beta}_{t_1,t_2},\ \forall 0 \leqslant t_1 \leqslant t_2 \leqslant t_3 \leqslant T.$$

Let V be a smooth bounded vector field on $\widetilde{M}$. For each $y \in \widetilde{M}$, we obtain a smooth curve $s \mapsto F^s(y), s \in \mathbb{R}$, with

$$\frac{dF^s(y)}{ds} = \mathrm{V}(F^s(y)).$$

Let $T > 0$ be fixed and let $\{(\mathrm{y}_t, \mathfrak{O}_t)\}_{t\in[0,T]}$ be a stochastic pair which defines the $\widetilde{g}$-Brownian motion starting from y. The mapping $\mathrm{w} \mapsto (\mathrm{y}_t(\mathrm{w}))_{t\in[0,T]}$ gives the distribution of Brownian paths up to time T in $C_y([0,T], \widetilde{M})$, where we use w to differ it from w for $\lfloor \mathrm{x}_t \rceil^\lambda$. We want to construct a one-parameter of mappings $\mathbf{F}^s_y$ on Brownian distributions $(\mathrm{y}_t)_{t\in[0,T]}$ so that

$$\mathrm{y}^s_t(\mathrm{w}) := \left(\mathbf{F}^s_y(\mathrm{y}_{[0,T]}(\mathrm{w}))\right)(t), \ \forall t \in [0,T],$$

is differentiable in s 'direction' for almost all w with initial restriction $d\mathrm{y}^s_0/ds = \mathrm{V}(F^s(y))$.

Choose a C^1 function $\mathtt{s} : [0,T] \to \mathbb{R}_+$ with

$$\mathtt{s}(0) = 1, \mathtt{s}(T) = 0, \ \overline{\lim_{t\to T}} \frac{1}{T-t}\mathtt{s}(t) < \infty. \tag{5.24}$$

For almost all w, we obtain a vector field along the paths $\mathrm{y}_{[0,T]}(\mathrm{w})$ with

$$\Upsilon_{\mathrm{V},\mathrm{y}}(t) := \mathtt{s}(t) \cdot /\!\!/_{0,t}(\mathrm{V}(y)), \ t \in [0,T],$$

where

$$/\!\!/_{t_1,t_2}(\mathbf{v}) := \mathfrak{O}_{t_2} \circ \mathfrak{O}^{-1}_{t_1}(\mathbf{v}), \ \forall \mathbf{v} \in T_{\mathrm{y}_{t_1}(\mathrm{w})}\widetilde{M}, \ 0 \leqslant t_1 \leqslant t_2 \leqslant T.$$

Our desired maps $\mathbf{F}^s_y$ on $(\mathrm{y}_t(\mathrm{w}))_{t\in[0,T]}$ are such that $(\mathrm{y}^s_t(\mathrm{w}))_{t\in[0,T]}$ satisfy the equation

$$\frac{d\mathrm{y}^s_t(\mathrm{w})}{ds} = \Upsilon_{\mathrm{V},\mathrm{y}^s}(t), \tag{5.25}$$

where

$$\Upsilon_{\mathrm{V},\mathrm{y}^s}(t) := \mathtt{s}(t) \cdot /\!\!/^s_{0,t}(\mathrm{V}(\mathrm{y}^s_0)), \ t \in [0,T],$$

and $/\!\!/^s$ denotes the parallel transportation map for the process y^s. The length of tangent vectors remain unchanged under parallel transportations. Hence

$$\|\Upsilon_{\mathrm{V},\mathrm{y}^s}(t)\| = \mathtt{s}(t) \cdot \|\mathrm{V}(\mathrm{y}^s_0)\| \leqslant \mathtt{s}(t) \sup\{\|V(y)\|\},$$

which tends to zero of order $(T-t)$ as $t \to T$ by our choice of $\mathtt{s}$. So, if the processes y^s exist, the ending points y^s_T remain in y_T.

Remark 5.6. In [**Hs3**], Hsu introduced a class of maps for the Brownian motion starting from some point y on a compact manifold: in our notation,

$$\Upsilon_{\mathrm{V},\mathrm{y}^s}(t) = \mathfrak{O}_t(\dot{h}(t)),$$

where h is a fixed $\mathbb{R}^m$ valued curve from the Euclidean Cameron-Martin space, i.e., the completion of the space of smooth paths $h : [0,T] \mapsto \mathbb{R}^m$ starting from origin with the Hilbert norm $|h| = (\int_0^1 |\dot{h}(t)|^2)^{\frac{1}{2}}$. In his construction, the initial point y^s_0 remain unchanged since h starts from 0 and hence the equations of all y^s can be transferred back to $\mathbb{R}^m$ using a single Itô map at y. In contrast, in our construction, our manifold is non-compact and we use a vector field V on the manifold instead of a Euclidean Cameron-Martin space element h to generate the random vector field $\Upsilon_{\mathrm{V},\mathrm{y}^s}$. Our ends y^s_T remain unchanged for almost

all paths since $\mathtt{s}(t)$ tends to zero as t goes to T; while the initials y_0^s changes as s varies so the Itô transfer map of y^s to $\mathbb{R}^m$ also changes with s. The C^1 requirement of $\mathtt{s}(t)$ is stronger than the L^2 integrability of the differentials of $h(t)$. This is to guarantee that we can obtain a continuous version of the resulting process y_t^s (and all other related processes) in the parameter (t,s) (see Theorem 5.17), which is not true for general h.

We will solve the SDE (5.25) by identifying the anti-developments $\alpha_t^s = \mathcal{I}_{\mathfrak{V}_0^s}^{-1}(\mathrm{y}_t^s)$ using Picard's iteration method, where $\mathfrak{V}_0^s$ is the parallel transportation of $\mathfrak{V}_0$ along the curve $(F^s(y))_{s\in\mathbb{R}_+}$. In many places, the transferred equations using V only differ in notations from that for the case of h in [**Hs3**]. But, technically, we have to write every steps in details since the construction is different, the footpoints of the Itô maps are shifting, and we need more regularity of y_t^s and also more information of the associated random structures.

We first consider (5.25) for smooth paths. Let $y \in \widetilde{M}$ and $\overline{\beta}_0 \in \mathcal{O}_y^{\tilde{g}}(\widetilde{M})$ be fixed. For $\beta = (\beta_t) \in C_y^\infty([0,T], \widetilde{M})$, the equation

$$\frac{\partial \beta_t^s}{\partial s} = \Upsilon_{\mathrm{V},\beta^s}(t) := \mathtt{s}(t) \cdot /\!\!/_{0,t}^s(\mathrm{V}(F^s y)),\ \beta^0 = \beta, \tag{5.26}$$

where $/\!\!/_{0,t}^s := /\!\!/_{0,t}^{\beta^s}$, is always solvable. Consider $\alpha_t^s = \mathcal{I}_{\overline{\beta}_0^s}^{-1}(\beta_t^s)$, where $\overline{\beta}_0^s$ is the parallel transportation of $\overline{\beta}_0$ along the curve $s \mapsto F^s(y)$. Then $(\partial \alpha_t^s/\partial s)|_{s=0}$ differs from $\overline{\beta}_t^{-1}(\Upsilon_{\mathrm{V},\beta^0}(t))$ by an integral of curvature term, which can be determined by a standard calculation exactly as in [**Hs1**, Theorem 2.1]. We give the proof for completeness.

Lemma 5.7. *Let* V *be a smooth bounded vector field on* $\widetilde{M}$*. For* $\beta \in C_y^\infty([0,T], \widetilde{M})$*, let* β^s *be the solution to (5.26) and let* $\overline{\beta}^s$ *be its horizontal lift in* $\mathcal{O}^{\tilde{g}}(\widetilde{M})$ *with initial point* $\overline{\beta}_0^s$*.*

i) *The differential* $(\alpha_t^s)'_s := \partial \alpha_t^s/\partial s$ *is given by*

$$(\alpha_t^s)'_s = \Upsilon_{\mathrm{V},\alpha^s}(t) := \int_0^t \mathtt{s}'(\tau)(\overline{\beta}_0^s)^{-1}\big(\mathrm{V}(\beta_0^s)\big)\ d\tau - \int_0^t K_{\mathrm{V},\alpha^s}(\tau)\ d\alpha_\tau^s,$$

where

$$K_{\mathrm{V},\alpha^s}(\tau) = \int_0^\tau (\overline{\beta}_\varsigma^s)^{-1} R\big(\overline{\beta}_\varsigma^s(\frac{\partial \alpha_\varsigma^s}{\partial \varsigma}), \Upsilon_{\mathrm{V},\beta^s}(\varsigma)\big)\overline{\beta}_\varsigma^s\ d\varsigma.$$

ii) *The differential* $(\overline{\beta}_t^s)'_s := \nabla_{\frac{D}{\partial s}} \overline{\beta}_t^s$ *satisfies the equation*

$$\begin{cases} \nabla_{\frac{D}{\partial t}}\big(\theta\big((\overline{\beta}_t^s)'_s\big)\big) = \mathtt{s}'(t)(\overline{\beta}_0^s)^{-1} V(F^s y), \\ \nabla_{\frac{D}{\partial t}}\big(\varpi\big((\overline{\beta}_t^s)'_s\big)\big) = (\overline{\beta}_t^s)^{-1} R\left(\overline{\beta}_t^s(\frac{\partial \alpha_t^s}{\partial t}), \overline{\beta}_t^s(\overline{\beta}_0^s)^{-1}\mathtt{s}(t)\mathrm{V}(F^s y)\right)\overline{\beta}_t^s. \end{cases} \tag{5.27}$$

Proof. For i), we have

$$\frac{\partial}{\partial t}\left(\frac{\partial \alpha_t^s}{\partial s}\right) = \frac{\partial}{\partial s}\left(\frac{\partial \alpha_t^s}{\partial t}\right) = \nabla_{\frac{D}{\partial s}}\left(\theta\big(\nabla_{\frac{D}{\partial t}}\overline{\beta}_t^s\big)\right).$$

Using the exterior differentiation formula in covariant derivative (cf. [**GHL**]) and the structure equation (4.3) for θ, we obtain

$$\begin{aligned}\nabla_{\frac{D}{\partial s}}\Big(\theta\big(\nabla_{\frac{D}{\partial t}}\overline{\beta}^s_t\big)\Big) &= \nabla_{\frac{D}{\partial t}}\Big(\theta\big(\nabla_{\frac{D}{\partial s}}\overline{\beta}^s_t\big)\Big) + d\theta\Big(\nabla_{\frac{D}{\partial s}}\overline{\beta}^s_t, \nabla_{\frac{D}{\partial t}}\overline{\beta}^s_t\Big)\\ &= \mathtt{s}'(t)(\overline{\beta}^s_0)^{-1}(\mathrm{V}(\beta^s_0)) - \varpi\big(\nabla_{\frac{D}{\partial s}}\overline{\beta}^s_t\big)(\frac{\partial\alpha^s_t}{\partial t}).\end{aligned}$$

We continue to compute that

$$\varpi\big(\nabla_{\frac{D}{\partial s}}\overline{\beta}^s_t\big) = \int_0^t \nabla_{\frac{D}{\partial\tau}}\Big(\varpi\big(\nabla_{\frac{D}{\partial s}}\overline{\beta}^s_\tau\big)\Big)\ d\tau,$$

where, by using the exterior differentiation formula, $\mathrm{Ker}(\varpi) = HT\mathcal{F}(\widetilde{M})$ and (4.4),

$$\begin{aligned}\nabla_{\frac{D}{\partial\tau}}\Big(\varpi\big(\nabla_{\frac{D}{\partial s}}\overline{\beta}^s_\tau\big)\Big) &= \nabla_{\frac{D}{\partial s}}\Big(\varpi\big(\nabla_{\frac{D}{\partial\tau}}\overline{\beta}^s_\tau\big)\Big) + d\varpi\Big(\nabla_{\frac{D}{\partial s}}\overline{\beta}^s_\tau, \nabla_{\frac{D}{\partial\tau}}\overline{\beta}^s_\tau\Big)\\ &= \Omega\left(H(\overline{\beta}^s_\tau, (\frac{\partial\alpha^s_\tau}{\partial\tau})), H\left(\overline{\beta}^s_\tau, (\overline{\beta}^s_0)^{-1}[\mathtt{s}(\tau)\mathrm{V}(\beta^s_0)]\right)\right)\\ &= (\overline{\beta}^s_\tau)^{-1}R\left(\overline{\beta}^s_\tau(\frac{\partial\alpha^s_\tau}{\partial\tau}), \Upsilon_{\mathrm{V},\beta^s}(\tau)\right)\overline{\beta}^s_\tau. \qquad (5.28)\end{aligned}$$

For (5.27), the first equation is true by the construction. The second equation holds by (5.28) since $\Upsilon_{\mathrm{V},\beta^s}(\tau) = \overline{\beta}^s_\tau(\overline{\beta}^s_0)^{-1}\mathtt{s}(\tau)\mathrm{V}(F^s y)$. □

For every smooth segment $\alpha = (\alpha_t)_{t\in[0,T]}$ in $\mathbb{R}^m$, consider the associated flow maps $\{F^\alpha_{t_1,t_2}\}_{0\leqslant t_1<t_2\leqslant T}$ for the transportation of α to $\widetilde{M}$ using the parallelism differential form (θ,ϖ), where $F^\alpha_{t_1,t_2}:\mathcal{F}(\widetilde{M})\to\mathcal{F}(\widetilde{M})$; $\overline{\beta}^\alpha_{t_1}\mapsto\overline{\beta}^\alpha_{t_2}$ with $(\overline{\beta}^\alpha_t)_{t\in[t_1,t_2]}$ solving the equation

$$\nabla_{\frac{D}{\partial t}}\overline{\beta}^\alpha_t = H(\overline{\beta}^\alpha_t, \frac{d\alpha_t}{dt}).$$

Each $F^\alpha_{t_1,t_2}$ is a C^{k-1} diffeomorphism since H is C^{k-1} and α is smooth. Let $DF^\alpha_{t_1,t_2}$ be the tangent map of $F^\alpha_{t_1,t_2}$. It can be read in the (θ,ϖ)-coordinate as follows.

Lemma 5.8. *(*[**Ma**, Proposition 3.2]*) Let $\alpha = (\alpha_t)_{t\in[0,T]}\subset\mathbb{R}^m$ be a smooth segment. For any $t_1\in[0,T]$ and $\mathtt{v}^\alpha_{t_1}\in T_{\overline{\beta}^\alpha_{t_1}}\mathcal{F}(\widetilde{M})$, let $\mathtt{v}^\alpha_t := \Big[DF^\alpha_{t_1,t}\big(\overline{\beta}^\alpha_\tau, \mathrm{w}\big)\Big]\mathtt{v}^\alpha_{t_1}$ for $t\in[t_1,T]$. Then $\mathtt{v}^\alpha_t$ satisfies the equation*

$$\nabla_{\frac{D}{\partial t}}\big(\mathtt{v}^\alpha_t\big) = \big(\nabla(\mathtt{v}^\alpha_t)H(\overline{\beta}^\alpha_t,\cdot)\big)\frac{d\alpha_t}{dt}. \qquad (5.29)$$

In the (θ,ϖ)-coordinate, we have

$$\begin{cases}\frac{d}{dt}\big(\theta\big(\mathtt{v}^\alpha_t\big)\big) = \varpi\big(\mathtt{v}^\alpha_t\big)\frac{d\alpha_t}{dt},\\ \frac{d}{dt}\big(\varpi\big(\mathtt{v}^\alpha_t\big)\big) = (\overline{\beta}^\alpha_t)^{-1}R\Big(\overline{\beta}^\alpha_t\frac{d\alpha_t}{dt}, \overline{\beta}^\alpha_t\big(\theta\big(\mathtt{v}^\alpha_t\big)\big)\Big)\overline{\beta}^\alpha_t.\end{cases}$$

Let $\alpha^s = (\alpha_t^s)_{t\in[0,T]}$ be a one-parameter of smooth segments of curves in $\mathbb{R}^m$. For any t_1, t_2 with $0 \leqslant t_1 < t_2 \leqslant T$, $s \mapsto DF^s_{t_1,t_2} := DF^{\alpha^s}_{t_1,t_2}$ is said to be C^1 in s if the image curve $s \mapsto [DF^s_{t_1,t_2}]\mathsf{v}^s_{t_1}$ is C^1 for any C^1 curve $s \mapsto \mathsf{v}^s_{t_1} \in T_{\overline{\beta}^s_{t_1}}\mathcal{F}(\widetilde{M})$.

Lemma 5.9. *Let $\alpha^s_t = \mathcal{I}^{-1}_{\overline{\beta}^s_0}(\beta^s_t)$, where β^s are given in Lemma 5.7. The tangent maps $(DF^s_{t_1,t_2})_{0\leqslant t_1<t_2\leqslant T}$ are C^1 in s. Let $s \mapsto \mathsf{v}^s_{t_1} \in T_{\overline{\beta}^s_{t_1}}\mathcal{F}(\widetilde{M})$ be C^1. Then the differential $(\mathsf{v}^s_t)'_s := \nabla_{D/\partial s}\mathsf{v}^s_t$, where $\mathsf{v}^s_t := \big[DF^s_{t_1,t}\big(\overline{\beta}^s_\tau, \mathrm{w}\big)\big]\mathsf{v}^s_{t_1}$ for $t \in [t_1, t_2]$, solves the equation*

$$\nabla_{\frac{D}{\partial t}}(\mathsf{v}^s_t)'_s = \big(\nabla((\mathsf{v}^s_t)'_s)H(\overline{\beta}^s_t,\cdot)\big)\frac{\partial\alpha^s_t}{\partial t} + \circledast\big(\mathsf{v}^s_t, (\overline{\beta}^s_t)'_s\big), \tag{5.30}$$

where

$$\circledast\big(\mathsf{v}^s_t, (\overline{\beta}^s_t)'_s\big) = \nabla^{(2)}\big(\mathsf{v}^s_t, (\overline{\beta}^s_t)'_s\big)H(\overline{\beta}^s_t, \frac{\partial\alpha^s_t}{\partial t}) + \nabla(\mathsf{v}^s_t)H\big(\overline{\beta}^s_t, \Upsilon_{\mathrm{V},\alpha^s}(t)\big) + R\left(H(\overline{\beta}^s_t, \frac{\partial\alpha^s_t}{\partial t}), (\overline{\beta}^s_t)'_s\right)\mathsf{v}^s_t.$$

In the (θ, ϖ)-coordinate, we have

$$\begin{cases} \nabla_{\frac{D}{\partial t}}\big(\theta\big((\mathsf{v}^s_t)'_s\big)\big) = \varpi\big((\mathsf{v}^s_t)'_s\big)\frac{\partial\alpha^s_t}{\partial t} + \theta\left(\circledast\big(\mathsf{v}^s_t, (\overline{\beta}^s_t)'_s\big)\right), \\ \nabla_{\frac{D}{\partial t}}\big(\varpi\big((\mathsf{v}^s_t)'_s\big)\big) = (\overline{\beta}^s_t)^{-1}R\left(\overline{\beta}^s_t\frac{\partial\alpha^s_t}{\partial t}, \overline{\beta}^s_t\theta\big((\mathsf{v}^s_t)'_s\big)\right)\overline{\beta}^s_t + \varpi\left(\circledast\big(\mathsf{v}^s_t, (\overline{\beta}^s_t)'_s\big)\right). \end{cases}$$

Proof. Using (5.29), we obtain

$$\begin{aligned} \nabla_{\frac{D}{\partial t}}\nabla_{\frac{D}{\partial s}}\mathsf{v}^s_t &= \nabla_{\frac{D}{\partial s}}\nabla_{\frac{D}{\partial t}}\mathsf{v}^s_t + R\left(H(\overline{\beta}^s_t, \frac{\partial\alpha^s_t}{\partial t}), (\overline{\beta}^s_t)'_s\right)\mathsf{v}^s_t \\ &= \big(\nabla((\mathsf{v}^s_t)'_s)H(\overline{\beta}^s_t,\cdot)\big)\frac{\partial\alpha^s_t}{\partial t} + \circledast\big(\mathsf{v}^s_t, (\overline{\beta}^s_t)'_s\big). \end{aligned}$$

Using (5.30) and the structure equation (4.3) for θ, we continue to compute that

$$\begin{aligned} \nabla_{\frac{D}{\partial t}}\big(\theta\big((\mathsf{v}^s_t)'_s\big)\big) &= \big(\nabla_{\frac{D}{\partial t}}\theta\big)\big((\mathsf{v}^s_t)'_s\big) + \theta\left(\big(\nabla((\mathsf{v}^s_t)'_s)H(\overline{\beta}^s_t,\cdot)\big)\frac{\partial\alpha^s_t}{\partial t}\right) + \theta\left(\circledast\big(\mathsf{v}^s_t, (\overline{\beta}^s_t)'_s\big)\right) \\ &= d\theta\left(H(\overline{\beta}^s_t, \frac{\partial\alpha^s_t}{\partial t}), (\mathsf{v}^s_t)'_s\right) + \theta\left(\circledast\big(\mathsf{v}^s_t, (\overline{\beta}^s_t)'_s\big)\right) \\ &= \varpi\big((\mathsf{v}^s_t)'_s\big)\frac{\partial\alpha^s_t}{\partial t} + \theta\left(\circledast\big(\mathsf{v}^s_t, (\overline{\beta}^s_t)'_s\big)\right). \end{aligned}$$

Similarly, using (5.30) and the structure equation (4.4) for ϖ, we obtain

$$\begin{aligned} \nabla_{\frac{D}{\partial t}}\big(\varpi\big((\mathsf{v}^s_t)'_s\big)\big) &= d\varpi\left(H(\overline{\beta}^s_t, \frac{\partial\alpha^s_t}{\partial t}), (\mathsf{v}^s_t)'_s\right) + \varpi\left(\circledast\big(\mathsf{v}^s_t, (\overline{\beta}^s_t)'_s\big)\right) \\ &= (\overline{\beta}^s_t)^{-1}R\left(\overline{\beta}^s_t\frac{\partial\alpha^s_t}{\partial t}, \overline{\beta}^s_t\theta\big((\mathsf{v}^s_t)'_s\big)\right)\overline{\beta}^s_t + \varpi\left(\circledast\big(\mathsf{v}^s_t, (\overline{\beta}^s_t)'_s\big)\right). \end{aligned}$$

□

We will solve (5.25) by identifying the anti-development of α^s of y^s in the set

$$\mathcal{A} := \left\{ \alpha_t = \int_0^t O_\tau dB_\tau + \int_0^t \mathtt{g}_\tau \, d\tau, \quad t \in [0,T] \right\},$$

where O_τ is an $\mathcal{O}(\mathbb{R}^m)$ valued $\mathcal{F}_\tau$-adapted process, $\mathtt{g}_\tau$ is a $\mathbb{R}^m$ valued $\mathcal{F}_\tau$-adapted process with $|\mathtt{g}| \leqslant \text{Const.} \sup |\mathrm{V}|$ and $\{\mathcal{F}_t\}_{t\in\mathbb{R}^+}$ is the filtration of the Brownian motion in $\mathbb{R}^m$. We see that $\mathcal{A}$ is a complete infinite dimensional Banach space under the norm

$$\|\alpha\|_{\infty,T} := \sqrt{\|\mathtt{g}\|^2_{\infty,T} + \|O\|^2_{\infty,T}},$$

where

$$\|\mathtt{g}\|^2_{\infty,T} = \mathbb{E} \sup_{t\in[0,T]} |\mathtt{g}_t|^2, \ \|O\|^2_{\infty,T} = \mathbb{E} \sup_{t\in[0,T]} |O_t|^2.$$

Let V and $\mathtt{s}$ be as above. For $\alpha \in \mathcal{A}$, let $\overline{\beta}$ be a horizontal process in $\mathcal{O}^{\tilde{g}}(\widetilde{M})$ with projection $\beta = \mathcal{I}_{\overline{\beta}_0}(\alpha)$ on $\widetilde{M}$. For $t \in [0,T]$, put

$$\Upsilon_{\mathrm{V},\beta}(t) := \mathtt{s}(t) \cdot /\!\!/^{\beta}_{0,t}(\mathrm{V}(\beta_0)) = \overline{\beta}_t \overline{\beta}_0^{-1}[\mathtt{s}(t)\mathrm{V}(\beta_0)].$$

We define

$$\underline{\Upsilon}_{\mathrm{V},\alpha}(t) := \int_0^t \mathtt{s}'(\tau)\overline{\beta}_0^{-1}(\mathrm{V}(\beta_0)) \, d\tau - \int_0^t K_{\mathrm{V},\alpha}(\tau) \circ d\alpha_\tau,$$

where $\circ$ denotes the Stratonovich stochastic integral, and

$$K_{\mathrm{V},\alpha}(\tau) := \int_0^\tau \overline{\beta}_\varsigma^{-1} R\left(\overline{\beta}_\varsigma(\circ d\alpha_\varsigma), \Upsilon_{\mathrm{V},\beta}(\varsigma)\right) \overline{\beta}_\varsigma. \tag{5.31}$$

Lemma 5.10. *For $\alpha \in \mathcal{A}$, the Itô forms of $\underline{\Upsilon}_{\mathrm{V},\alpha}$, $K_{\mathrm{V},\alpha}$ are*

$$\begin{aligned} \underline{\Upsilon}_{\mathrm{V},\alpha}(t) &= \int_0^t \left\{ \overline{\beta}_0^{-1}[\mathtt{s}'(\tau)\mathrm{V}(\beta_0)] - \mathrm{Ric}\left(\Upsilon_{\mathrm{V},\beta}(\tau)\right) \right\} \, d\tau - \int_0^t \langle K_{\mathrm{V},\alpha}(\tau), d\alpha_\tau \rangle \\ &=: \mathrm{R}_{\mathrm{V},\alpha}(t) - \int_0^t \langle K_{\mathrm{V},\alpha}(\tau), d\alpha_\tau \rangle, \end{aligned} \tag{5.32}$$

where Ric *was defined in (4.12) and*

$$K_{\mathrm{V},\alpha}(t) = \int_0^t \overline{\beta}_\tau{}^{-1} R\left(\overline{\beta}_\tau d\alpha_\tau, \Upsilon_{\mathrm{V},\beta}(\tau)\right) \overline{\beta}_\tau + \int_0^t \overline{\beta}_\tau{}^{-1} \left(\nabla(\overline{\beta}_\tau e_i)R\right)\left(\overline{\beta}_\tau e_i, \Upsilon_{\mathrm{V},\beta}(\tau)\right) \overline{\beta}_\tau \, d\tau. \tag{5.33}$$

Proof. Using Itô's formula, we can identify the Itô integral expression of $\underline{\Upsilon}_{\mathrm{V},\alpha}$ as

$$\underline{\Upsilon}_{\mathrm{V},\alpha}(t) = \int_0^t \mathtt{s}'(\tau)\overline{\beta}_0^{-1}(\mathrm{V}(\beta_0)) \, d\tau - \int_0^t \langle K_{\mathrm{V},\alpha}(\tau), d\alpha_\tau \rangle - \frac{1}{2}\int_0^t \langle dK_{\mathrm{V},\alpha}(\tau), \circ d\alpha_\tau \rangle.$$

Let $\alpha_t = \alpha_{t,1}e_1 + \cdots + \alpha_{t,m}e_m$, where $\{e_1, \cdots, e_m\}$ is the standard orthogonal base of $\mathbb{R}^m$. Since $\alpha \in \mathcal{A}$, we see that $\langle \circ d\alpha_{t,i}, \circ d\alpha_{t,i}\rangle = 2dt$. So, using (5.31), we obtain

$$\begin{aligned}\frac{1}{2}\int_0^t \langle dK_{\mathrm{V},\alpha}(\tau), \circ d\alpha_\tau\rangle &= \frac{1}{2}\int_0^t \langle \overline{\beta}_\tau^{-1} R\left(\overline{\beta}_\tau(\circ d\alpha_\tau), \Upsilon_{\mathrm{V},\beta}(\tau)\right)\overline{\beta}_\tau, \circ d\alpha_\tau\rangle \\ &= \int_0^t \overline{\beta}_\tau^{-1} R\left(\overline{\beta}_\tau e_i, \Upsilon_{\mathrm{V},\beta}(\tau)\right)\overline{\beta}_\tau e_i \; d\tau \\ &= \int_0^t \mathrm{Ric}\left(\Upsilon_{\mathrm{V},\beta}(\tau)\right) \; d\tau.\end{aligned}$$

The Itô integral expression for $K_{\mathrm{V},\alpha}(t)$ can be obtained similarly using Itô's formula. □

We want to solve (5.25) with y^0 being the Brownian motion on $\widetilde{M}$ starting from y.

Lemma 5.11. *Let* V *be a smooth bounded vector field on* $\widetilde{M}$ *with flow maps* $(F^s)_{s\in\mathbb{R}_+}$. *For* $y \in \widetilde{M}$, *let* $(\mathfrak{V}_0^s \in \mathcal{O}^{\tilde{g}}_{F^s y}(\widetilde{M}))_{s\in\mathbb{R}_+}$ *be a solution to* $d\mathfrak{V}_0^s/ds = H(\mathfrak{V}_0^s, (\mathfrak{V}_0^s)^{-1}\mathrm{V}(F^s y))$.

i) *Let* $\alpha^s = \int_0 O_\tau^s dB_\tau + \int_0 \mathfrak{g}_\tau^s \; d\tau \in \mathcal{A}$ *be a one-parameter of stochastic processes with* $\alpha_t^0 = B_t$. *Then* α^s *solves* $d\alpha_t^s(\mathrm{w})/ds = \underline{\Upsilon}_{\mathrm{V},\alpha^s}(t)$ *iff*

$$O_\tau^s = \mathrm{Id} - \int_0^s K_{\mathrm{V},\alpha^\jmath}(\tau) O_\tau^\jmath \; d\jmath, \tag{5.34}$$

$$\mathfrak{g}_\tau^s = O_\tau^s \int_0^s [O_\tau^\jmath]^{-1}\left\{(\mathfrak{V}_0^\jmath)^{-1}\left[\mathfrak{s}'(\tau)\mathrm{V}(F^\jmath y)\right] - \mathrm{Ric}\left(\mathfrak{V}_\tau^\jmath(\mathfrak{V}_0^\jmath)^{-1}[\mathfrak{s}(\tau)\mathrm{V}(F^\jmath y)]\right)\right\} \; d\jmath. \tag{5.35}$$

ii) *Let* α^s *be as in* i) *and let* $\mathfrak{V}^s$ *be its horizontal lift in* $\mathcal{O}^{\tilde{g}}(\widetilde{M})$ *with initial* $\mathfrak{V}_0^s$. *Then* $\mathfrak{V}^s$ *is differentiable in* s *iff the following SDE is uniquely solvable with initial* $(\mathfrak{V}_0^s)'_s$*:*

$$\begin{cases} d\theta(Y_t^s) = \varpi\big(Y_t^s\big) \circ d\alpha_t^s + \circ d\underline{\Upsilon}_{\mathrm{V},\alpha^s}, \\ d\varpi(Y_t^s) = (\mathfrak{V}_t^s)^{-1} R\big(\mathfrak{V}_t^s \circ d\alpha_t^s, \mathfrak{V}_t^s\theta(Y_t^s)\big)\mathfrak{V}_t^s. \end{cases} \tag{5.36}$$

iii) *Let* $\alpha^s, \mathfrak{V}^s$ *be as in* i), ii). *Then* $s \mapsto \mathrm{y}^s = \mathcal{I}_{\mathfrak{V}_0^s}(\alpha^s)$ *has the differential process* $\Upsilon_{\mathrm{V},\mathrm{y}^s}$.

Proof. By analogy with the deterministic case (Lemma 5.7), we have y^s solves (5.25) iff α^s solves $d\alpha_t^s(\mathrm{w})/ds = \underline{\Upsilon}_{\mathrm{V},\alpha^s}(t)$, which means

$$\begin{aligned}\alpha_t^s - \alpha_t^0 &= \int_0^s\int_0^t (\mathfrak{V}_0^\jmath)^{-1}[\mathfrak{s}'(\tau)\mathrm{V}(F^\jmath y)] \; d\tau \; d\jmath - \int_0^s\int_0^t \langle K_{\mathrm{V},\alpha^\jmath}(\tau), d\alpha_\tau^\jmath\rangle \; d\jmath \\ &\quad - \int_0^s\int_0^t \mathrm{Ric}\left(\mathfrak{V}_\tau^\jmath(\mathfrak{V}_0^\jmath)^{-1}[\mathfrak{s}(\tau)\mathrm{V}(F^\jmath y)]\right) \; d\tau \; d\jmath \\ &= \int_0^s\int_0^t (\mathrm{u}_T^\jmath)^{-1}[\mathfrak{s}'(\tau)\mathrm{V}(F^\jmath y)] \; d\tau \; d\jmath - \int_0^s\int_0^t \mathrm{Ric}\left(\mathfrak{V}_\tau^\jmath(\mathfrak{V}_0^\jmath)^{-1}[\mathfrak{s}(\tau)\mathrm{V}(F^\jmath y)]\right) \; d\tau \; d\jmath \\ &\quad - \int_0^s\int_0^t K_{\mathrm{V},\alpha^\jmath}(\tau)\mathfrak{g}_\tau^\jmath \; d\tau \; d\jmath - \int_0^s\int_0^t K_{\mathrm{V},\alpha^\jmath}(\tau) O_\tau^\jmath \; dB_\tau \; d\jmath.\end{aligned}$$

Note that $\alpha_t^0 = B_t$ and hence $O^0 = \mathrm{Id}, \mathfrak{g}^0 = 0$. So a comparison of the above expression with the the assumption that $\alpha_t^s = \int_0^t O_\tau^s dB_\tau + \int_0^t \mathfrak{g}_\tau^s \, d\tau$ gives (5.34) and

$$\mathfrak{g}_\tau^s = \int_0^s (\mathcal{U}_0^\jmath)^{-1}[\mathfrak{s}'(\tau)\mathrm{V}(F^\jmath y)] \, d\jmath - \int_0^s \mathrm{Ric}\left(\mathcal{U}_\tau^\jmath (\mathcal{U}_0^\jmath)^{-1}[\mathfrak{s}(\tau)\mathrm{V}(F^\jmath y)]\right) \, d\jmath - \int_0^s K_{\mathrm{V},\alpha^\jmath}(\tau)\mathfrak{g}_\tau^\jmath \, d\jmath.$$

Hence by the variation of constants method (i.e., Duhamel's principle), we obtain (5.35).

Let α^s be as in i) which solves $d\alpha_t^s(\mathrm{w})/ds = \underline{\Upsilon}_{\mathrm{V},\alpha^s}(t)$. Then $\mathcal{U}^s$ is differentiable in s iff the following SDE is solvable with initial $(\mathcal{U}_0^s)_s'$:

$$dY_t^s = \big(\nabla(Y_t^s)H\big)(\mathcal{U}_t^s, \circ d\alpha_t^s) + H(\mathcal{U}_t^s, (\circ d\alpha_t^s)_s'). \tag{5.37}$$

Writing (5.37) in the (θ, ϖ)-coordinate, we have

$$\begin{aligned} d(\theta(Y_t^s)) &= d\theta(\circ d\mathcal{U}_t^s, Y_t^s) + \theta\big(H(\mathcal{U}_t^s, (\circ d\alpha_t^s)_s')\big) = \omega(Y_t^s) \circ d\alpha_t^s + \circ d\underline{\Upsilon}_{\mathrm{V},\alpha^\jmath} \\ d(\varpi(Y_t^s)) &= \Omega\left(H(\mathcal{U}_t^s, \circ d\alpha_t^s), H(\mathcal{U}_t^s, \theta(Y_t^s))\right) = (\mathcal{U}_t^s)^{-1} R\big(\mathcal{U}_t^s \circ d\alpha_t^s, \mathcal{U}_t^s \theta(Y_t^s)\big)\mathcal{U}_t^s. \end{aligned}$$

Let $\alpha^s, \mathcal{U}^s$ be such that i), ii) hold true. For iii), it suffices to check the equality $\nabla_{D/\partial s}(\pi(\mathcal{U}^s)) = \Upsilon_{\mathrm{V},\mathrm{y}^s}$. Let $Z_t^\jmath := \theta\left((\mathcal{U}_t^s)_\jmath'\right)$. By (5.36),

$$Z_t^\jmath = \underline{\Upsilon}_{\mathrm{V},\alpha^\jmath}(t) - \underline{\Upsilon}_{\mathrm{V},\alpha^\jmath}(0) + \int_0^t \left(\int_0^\tau (\mathcal{U}_{\tau'}^\jmath)^{-1} R\big(\mathcal{U}_{\tau'}^\jmath \circ d\alpha_{\tau'}^\jmath, \mathcal{U}_{\tau'}^\jmath Z_{\tau'}^\jmath\big)\mathcal{U}_{\tau'}^\jmath \right) \circ d\alpha_\tau^s.$$

Write $\mathcal{Z}_t^\jmath := Z_t^\jmath - \int_0^t (\mathcal{U}_0^\jmath)^{-1}[\mathfrak{s}'(\tau)\mathrm{V}(F^\jmath y)] \, d\tau$. Then we have

$$\mathcal{Z}_t^\jmath = \int_0^t \left(\int_0^\tau (\mathcal{U}_{\tau'}^\jmath)^{-1} R\big(\mathcal{U}_{\tau'}^\jmath \circ d\alpha_{\tau'}^\jmath, \mathcal{U}_{\tau'}^\jmath \mathcal{Z}_{\tau'}^\jmath\big)\mathcal{U}_{\tau'}^\jmath \right) \circ d\alpha_\tau^s.$$

Using Itô's formula for $|\cdot|^2 = \langle \cdot, \cdot \rangle$ or the isometry property of Brownian motion, we can find some constant $C(s_0, T)$ depending on R, s_0, T such that

$$\mathbb{E}((\mathcal{Z}_t^\jmath)^2) \leqslant C(s_0, T) \int_0^t \mathbb{E}((\mathcal{Z}_\tau^\jmath)^2) \, d\tau.$$

This gives $\mathcal{Z}_t^\jmath = 0$ by Gronwall's Lemma (see Lemma 5.16). Thus $(\mathrm{y}^s)_s' = \Upsilon_{\mathrm{V},\mathrm{y}^s}$. □

Corollary 5.12. *Let $\mathcal{U}^s$ be as in* ii) *of Lemma 5.11. Then $Y^s = (\mathcal{U}^s)_s'$ is given by*

$$\begin{cases} d\theta(Y_t^s) = \mathfrak{s}'(t)(\mathcal{U}_0^s)^{-1} V(F^s y) \, dt, \\ d\varpi(Y_t^s) = (\mathcal{U}_t^s)^{-1} R\big(\mathcal{U}_t^s \circ d\alpha_t^s, \mathfrak{s}(t)\mathcal{U}_t^s (\mathcal{U}_0^s)^{-1} V(F^s y)\big)\mathcal{U}_t^s, \end{cases} \tag{5.38}$$

whose Itô form is

$$\begin{cases} d\theta(Y_t^s) = \mathfrak{s}'(t)(\mathcal{U}_0^s)^{-1} V(F^s y) \, dt, \\ d\varpi(Y_t^s) = (\mathcal{U}_t^s)^{-1} R\big(\mathcal{U}_t^s d\alpha_t^s, \mathfrak{s}(t)\mathcal{U}_t^s (\mathcal{U}_0^s)^{-1} V(F^s y)\big)\mathcal{U}_t^s \\ \qquad\qquad + (\mathcal{U}_t^s)^{-1} (\nabla(\mathcal{U}_t^s e_i) R) \left(\mathcal{U}_t^s e_i, \mathfrak{s}(t)\mathcal{U}_t^s (\mathcal{U}_0^s)^{-1} V(F^s y)\right) \mathcal{U}_t^s \, dt. \end{cases} \tag{5.39}$$

Proof. Note that $\mathcal{U}^s$ is a horizontal lift of y^s. Reporting this and $(\mathrm{y}^s)_s' = \Upsilon_{\mathrm{V},\mathrm{y}^s}$ in (5.36) shows (5.38). Then (5.39) follows by applying the Itô formula. □

For $\alpha = (\alpha_{t,1}, \cdots, \alpha_{t,m}) \in \mathcal{A}$, consider the associated flow maps $\{F^\alpha_{t_1,t_2}\}_{0\leqslant t_1<t_2\leqslant T}$, where $F^\alpha_{t_1,t_2} : \mathcal{F}(\widetilde{M}) \to \mathcal{F}(\widetilde{M})$; $\overline{\beta}^\alpha_{t_1} \mapsto \overline{\beta}^\alpha_{t_2}$, with $(\overline{\beta}^\alpha_t)_{t\in[t_1,t_2]}$ solving the Stratonovich SDE

$$d\overline{\beta}^\alpha_t = H(\overline{\beta}^\alpha_t, \circ d\alpha_t). \tag{5.40}$$

By Proposition 4.3, $F^\alpha_{t_1,t_2}$ are C^{k-2} diffeomorphisms for almost all w and the first order tangent map $DF^\alpha_{t_1,t_2}$ satisfies the following (see also Lemma 4.4).

Lemma 5.13. *Let $\alpha \in \mathcal{A}$. For almost all* w*, any $t_1 \in [0,T]$ and $\mathsf{v}^\alpha_{t_1} \in T_{\overline{\beta}^\alpha_{t_1}}\mathcal{F}(\widetilde{M})$, $\mathsf{v}^\alpha_t := \big[DF^\alpha_{t_1,t}(\overline{\beta}^\alpha_{t_1}, \mathrm{w})\big]\mathsf{v}^\alpha_{t_1}, t \in [t_1,T]$ satisfies the Stratonovich SDE*

$$d\mathsf{v}^\alpha_t = \big(\nabla(\mathsf{v}^\alpha_t)H\big)(\overline{\beta}^\alpha_t, \circ d\alpha_t).$$

In the (θ,ϖ)-coordinate, we have

$$\begin{cases} d\left(\theta\big(\mathsf{v}^\alpha_t\big)\right) = \varpi\big(\mathsf{v}^\alpha_t\big) \circ d\alpha_t, \\ d\left(\varpi\big(\mathsf{v}^\alpha_t\big)\right) = (\overline{\beta}^\alpha_t)^{-1} R\left(\overline{\beta}^\alpha_t \circ d\alpha_t, \overline{\beta}^\alpha_t\left(\theta\big(\mathsf{v}^\alpha_t\big)\right)\right)\overline{\beta}^\alpha_t \end{cases}$$

and its Itô form is

$$\begin{cases} d\theta(\mathsf{v}^\alpha_t) = \varpi\big(\mathsf{v}^\alpha_t\big)d\alpha_t + \mathrm{Ric}\big(\overline{\beta}^\alpha_t\theta(\mathsf{v}^\alpha_t)\big)\ dt, \\ d\varpi(\mathsf{v}^\alpha_t) = (\overline{\beta}^\alpha_t)^{-1}R\left(\overline{\beta}^\alpha_t d\alpha_t, \overline{\beta}^\alpha_t\theta(\mathsf{v}^\alpha_t)\right)\overline{\beta}^\alpha_t + (\overline{\beta}^\alpha_t)^{-1}R\left(\overline{\beta}^\alpha_t e_i, \varpi(\mathsf{v}^\alpha_t)e_i\right)\overline{\beta}^\alpha_t\ dt \\ \qquad\qquad +(\overline{\beta}^\alpha_t)^{-1}(\nabla(\overline{\beta}^\alpha_t e_i)R)\left(\overline{\beta}^\alpha_t e_i, \overline{\beta}^\alpha_t\theta(\mathsf{v}^\alpha_t)\right)\overline{\beta}^\alpha_t\ dt. \end{cases}$$

Let $\alpha^s \in \mathcal{A}$ be a one-parameter of random processes. We abbreviate

$$\overline{\beta}^s_t := \overline{\beta}^{\alpha^s}_t,\ F^s_{t_1,t_2} := F^{\alpha^s}_{t_1,t_2},\ DF^s_{t_1,t_2} := DF^{\alpha^s}_{t_1,t_2}.$$

The maps $(DF^s_{t_1,t_2})_{0\leqslant t_1<t_2\leqslant T}$ are said to be C^1 in s if, for almost all w and any $(\upsilon^s_{t_1}, Q^s_{t_1}) \in T_{\overline{\beta}^s}\mathcal{F}(\widetilde{M})$ which is C^1 in s, $[DF^s_{t_1,t_2}](\upsilon^s_{t_1}, Q^s_{t_1})$ is also C^1 in s. The following can be formulated using Lemma 5.13 and Itô's formula by analogy with Lemma 5.9.

Lemma 5.14. *Let α^s, y^s and $\mathcal{U}^s$ be as in Lemma 5.11. Then $(DF^s_{t_1,t_2})_{0\leqslant t_1<t_2\leqslant T}$ are C^1 in s iff for any $\mathsf{v}^s_{t_1} \in T_{\mathcal{U}^s_{t_1}}\mathcal{F}(\widetilde{M})$ C^1 in s, there is a unique $(\upsilon^s_t)_{t\in[t_1,t_2]}$ continuous in (t,s), with $\upsilon^s_{t_1} = \nabla_{D/\partial s}\mathsf{v}^s_{t_1}$, that solves the SDE*

$$d\upsilon^s_t = \big(\nabla(\upsilon^s_t)H\big)(\mathcal{U}^s_t, \circ d\alpha^s_t) + \circledast\big(\mathsf{v}^s_t, (\mathcal{U}^s_t)'_s\big), \tag{5.41}$$

where

$$\begin{aligned} \circledast\big(\mathsf{v}^s_t, (\mathcal{U}^s_t)'_s\big) =& \nabla^{(2)}\big(\mathsf{v}^s_t, (\mathcal{U}^s_t)'_s\big)H(\mathcal{U}^s_t, \circ d\alpha^s_t) + \nabla(\mathsf{v}^s_t)H\big(\mathcal{U}^s_t, \circ d\Upsilon_{\mathrm{V},\alpha^s}(t)\big) \\ &+ R\left(H(\mathcal{U}^s_t, \circ d\alpha^s_t), (\mathcal{U}^s_t)'_s\right)\mathsf{v}^s_t. \end{aligned}$$

In the (θ,ϖ)-coordinate, (5.41) is

$$\begin{cases} d\big(\theta\big(\upsilon^s_t\big)\big) = \varpi\big(\upsilon^s_t\big) \circ d\alpha^s_t + \theta\left(\circledast\big(\mathsf{v}^s_t, (\mathcal{U}^s_t)'_s\big)\right), \\ d\big(\varpi\big(\upsilon^s_t\big)\big) = (\mathcal{U}^s_t)^{-1}R\left(\mathcal{U}^s_t \circ d\alpha^s_t, \mathcal{U}^s_t\theta\big(\upsilon^s_t\big)\right)\mathcal{U}^s_t + \varpi\left(\circledast\big(\mathsf{v}^s_t, (\mathcal{U}^s_t)'_s\big)\right). \end{cases} \tag{5.42}$$

The Itô form of (5.42) is

(5.43)

$$\begin{cases} d\big(\theta(\upsilon_t^s)\big) &= \varpi\big(\upsilon_t^s\big)d\alpha_t^s + \mathrm{Ric}\big(\mathcal{U}_t^s\theta(\upsilon_t^s)\big)\ dt + \theta\left(\circledast_{\mathrm{I}}\big(\mathsf{v}_t^s,(\mathcal{U}_t^s)'_s\big)\right) + \circledast_{\mathrm{A}}^{\theta}\big(\mathsf{v}_t^s,(\mathcal{U}_t^s)'_s\big),\\ d\big(\varpi\big(\upsilon_t^s\big)\big) &= (\mathcal{U}_t^s)^{-1}R\left(\mathcal{U}_t^s d\alpha_t^s,\mathcal{U}_t^s\theta\big(\upsilon_t^s\big)\right)\mathcal{U}_t^s + (\mathcal{U}_t^s)^{-1}R\left(\mathcal{U}_t^s e_i,\mathcal{U}_t^s\varpi(\upsilon_t^s)e_i\right)\mathcal{U}_t^s\ dt\\ &\quad +(\mathcal{U}_t^s)^{-1}(\nabla(\mathcal{U}_t^s e_i)R)\left(\mathcal{U}_t^s e_i,\mathcal{U}_t^s\theta(\upsilon_t^s)\right)\mathcal{U}_t^s\ dt\\ &\quad +\varpi\left(\circledast_{\mathrm{I}}\big(\mathsf{v}_t^s,(\mathcal{U}_t^s)'_s\big)\right)+\circledast_{\mathrm{A}}^{\varpi}\big(\mathsf{v}_t^s,(\mathcal{U}_t^s)'_s\big), \end{cases}$$

where $\circledast_{\mathrm{I}}\big(\mathsf{v}_t^s,(\mathcal{U}_t^s)'_s\big)$ *is* $\circledast\big(\mathsf{v}_t^s,(\mathcal{U}_t^s)'_s\big)$ *with* $\circ d\alpha_t^s$ *replaced by the Itô infinitesimal* $d\alpha_t^s$,

$$\begin{aligned} \circledast_{\mathrm{A}}^{\theta}\big(\mathsf{v}_t^s,(\mathcal{U}_t^s)'_s\big) &= 2\varpi\left(\circledast\big(\mathsf{v}_t^s,(\mathcal{U}_t^s)'_s,e_i\big)\right)e_i\ dt+\theta\left(\big[H(\mathcal{U}_t^s,e_i),\circledast\big(\mathsf{v}_t^s,(\mathcal{U}_t^s)'_s,e_i\big)\big]\right)\ dt,\\ \circledast_{\mathrm{A}}^{\varpi}\big(\mathsf{v}_t^s,(\mathcal{U}_t^s)'_s\big) &= 2(\mathcal{U}_t^s)^{-1}R\big(\mathcal{U}_t^s e_i,\mathcal{U}_t^s\theta\big(\circledast\left(\mathsf{v}_t^s,(\mathcal{U}_t^s)'_s,e_i\right)\big)\big)\mathcal{U}_t^s\ dt\\ &\quad +\varpi\left(\big[H(\mathcal{U}_t^s,e_i),\circledast\big(\mathsf{v}_t^s,(\mathcal{U}_t^s)'_s,e_i\big)\big]\right)\ dt,\\ \circledast\big(\mathsf{v}_t^s,(\mathcal{U}_t^s)'_s,e_i\big) &= \nabla^{(2)}\big(\mathsf{v}_t^s,(\mathcal{U}_t^s)'_0\big)H(\mathcal{U}_t^s,e_i)+R\big(H(\mathcal{U}_t^s,e_i),(\mathcal{U}_t^s)'_s\big)\mathsf{v}_t^s+\nabla(\mathsf{v}_t^s)H\big(\mathcal{U}_t^s,K_{\mathrm{V},\alpha^s}(t)e_i\big). \end{aligned}$$

5.3. **The existence of $\mathbf{F}_y^s$.** In this part, we prove the existence of the mapping $\mathrm{y}\mapsto\mathbf{F}_y^s(\mathrm{y})$. By Lemma 5.11, it suffices to solve $d\alpha_t^s(\mathrm{w})/ds=\underline{\Upsilon}_{\mathrm{V},\alpha^s}(t)$ in $\mathcal{A}$ with $\alpha^0=B$. We will do this using the classical Picard method as in [**Hs1**, Theorem 3.1]. In the meanwhile, we will also show the existence of the differential processes of $\mathcal{U}^s$ and DF_{t_1,t_2}^s in s. The tool we will use to obtain a continuous version of a two-parameter process is Kolmogorov's criterion.

Lemma 5.15. *(cf.* [**Kun2**, Theorem 1.4.1]*) Let* $\{\mathcal{Y}_t^s(\mathrm{w})\}_{t\in[0,T],s\in[-s_0,s_0]}$ *be a one-parameter of random processes on a complete manifold. Suppose there are positive constants* $\flat,\flat_1,\flat_2$*, with* $\flat_1,\flat_2>2$*, and* $\mathtt{C}_0(\flat)$ *such that for all* $t,t'\in[0,T]$ *and* $s,s'\in[-s_0,s_0]$*,*

$$\mathbb{E}\left[\left|\mathcal{Y}_t^s-\mathcal{Y}_{t'}^{s'}\right|^{\flat}\right]\leqslant\mathtt{C}_0(\flat)\left(|t-t'|^{\flat_1}+|s-s'|^{\flat_2}\right),$$

then $\mathcal{Y}_t^s$ *has a continuous modification with respect to the parameter* (t,s)*.*

Besides Burkholder's inequality (Lemma 4.7), another useful tool to estimate the L^q-norm of stochastic integrals is Gronwall's lemma:

Lemma 5.16. *(cf.* [**Elw**, p. 13]*) Let* ϕ,ϕ_1 *be real valued Lebesgue integrable functions on the interval* $[0,s]$ *such that for some* $\mathtt{C}>0$*,*

$$\phi(\jmath)\leqslant\phi_1(\jmath)+\mathtt{C}\int_0^{\jmath}\phi(\jmath')\ d\jmath',\ \ \forall\jmath\in[0,s].$$

Then

$$\phi(\jmath)\leqslant\phi_1(\jmath)+\mathtt{C}\int_0^{\jmath}e^{\mathtt{C}(\jmath-\jmath')}\phi_1(\jmath')\ d\jmath',\ \textit{for almost all}\ \jmath\in[0,s].$$

We are in a situation to state the existence theorem of the maps $\mathbf{F}_y^s$.

Theorem 5.17. *Let* V *be a bounded smooth vector field on* $\widetilde{M}$ *and let* F^s *be the flow map it generates. For* $y\in\widetilde{M}$*, let* $(\mathcal{U}_0^s\in\mathcal{O}^{\widetilde{g}}_{F^s y}(\widetilde{M}))_{s\in\mathbb{R}_+}$ *with* $d\mathcal{U}_0^s/ds=H(\mathcal{U}_0^s,(\mathcal{U}_0^s)^{-1}\mathrm{V}(F^s y))$ *be a fixed horizontal lift of the smooth curve* $(F^s y)_{s\in\mathbb{R}_+}$*.*

i) *There exists a unique family of stochastic processes* $\alpha^s \in \mathcal{A}$ *such that for almost all* w, $s \mapsto \alpha^s(\mathrm{w})$ *is differentiable with*

$$\alpha_t^s(\mathrm{w}) = \mathrm{w} + \int_0^s \underline{\Upsilon}_{\mathrm{V},\alpha^\jmath}(t,\mathrm{w})\ d\jmath,\ \forall t \in [0,T]. \tag{5.44}$$

The process $\underline{\Upsilon}_{\mathrm{V},\alpha^s}(t)$ *has a continuous modification in the parameter* (t,s).

ii) *Let* $\mathcal{U}^s \in \mathcal{O}^{\tilde{g}}(\widetilde{M})$ *be a horizontal lift of* y^s *with initial* $\mathcal{U}_0^s$. *There exists a one-parameter of* $\mathcal{F}_t$*-adapted stochastic processes* $(Y_t^s)_{t\in[0,T]}$ *with* $Y_t^s(\mathrm{w}) \in T_{\mathcal{U}_t^s(\mathrm{w})}(\mathcal{O}^{\tilde{g}}(\widetilde{M}))$ *for almost all* w, *which satisfies*

$$\nabla_{\frac{D}{\partial s}} \mathcal{U}_t^s(\mathrm{w}) = Y_t^s(\mathrm{w}),\ \forall t \in [0,T].$$

The process Y_t^s *has a continuous modification in the parameter* (t,s).

iii) *Let* $\mathrm{y}^s = \mathcal{I}_{\mathcal{U}_0^s}(\alpha^s)$. *Then* $s \mapsto \mathrm{y}^s(\mathrm{w})$ *is differentiable for almost all* w *with*

$$\nabla_{\frac{D}{\partial s}} \mathrm{y}_t^s(\mathrm{w}) = \Upsilon_{\mathrm{V},\mathrm{y}^s}(t,\mathrm{w}),\ \forall t \in [0,T]. \tag{5.45}$$

The process $\Upsilon_{\mathrm{V},\mathrm{y}^s}(t)$ *has a continuous modification in the parameter* (t,s).

iv) *For almost all* w, $(DF_{t_1,t}^s)_{0\leqslant t_1<t\leqslant T}$ *are* C^1 *in* s. *For* $\mathsf{v}_{t_1}^s \in T_{\mathcal{U}_{t_1}^s}\mathcal{F}(\widetilde{M})$ C^1 *in* s, *the process* $\mathsf{v}_t^s = [DF_{t_1,t}^s]\mathsf{v}_{t_1}^s$ *is differentiable in* s *and the differential process* v_t^s *has a continuous modification in the parameter* (t,s).

Proof. For simplicity, we will use C to denote a constant depending on $\|g\|_{C^3}$ and the norm bound of V and use $C(\cdot)$ to indicate the extra coefficients it depends on, for instance, $C(s_0,T)$ means C also depends on s_0, T. These constants C may vary from line to line.

We first show i). For any $s_0 \in \mathbb{R}^+$, we use (5.34), (5.35) for Picard's iteration and show the iteration converges to a one-parameter processes α^s $(s \leqslant s_0)$ in norm $\|\cdot\|_{\infty,T}$. Let $\mathsf{g}^{s,0} = 0, O^{s,0} = \mathrm{Id}$, $\alpha^{s,0} = B$ and let $\mathcal{U}^{s,0}$ be the horizontal lift of $\mathcal{I}_{\mathcal{U}_0^s}(B)$ in $\mathcal{O}^{\tilde{g}}(\widetilde{M})$ with $\mathcal{U}_0^{s,0} = \mathcal{U}_0^s$. Assume $\mathsf{g}^{s,n-1}, O^{s,n-1}$ and $\alpha^{s,n-1}$ are obtained for some $n \in \mathbb{N}$. We write $\mathcal{U}^{s,n-1}$ for the horizontal development of $\alpha^{s,n-1}$ in $\mathcal{O}^{\tilde{g}}(\widetilde{M})$ with $\mathcal{U}_0^{s,n-1} = \mathcal{U}_0^s$ and put

$$\begin{aligned} K_{\mathrm{V},\alpha^{s,n-1}}(t) = &\int_0^t (\mathcal{U}_\tau^{s,n-1})^{-1} R\left(\mathcal{U}_\tau^{s,n-1} d\alpha_\tau^{s,n-1}, \mathcal{U}_\tau^{s,n-1}(\mathcal{U}_0^s)^{-1}[\mathtt{s}(\tau)\mathrm{V}(F^s y)]\right)\mathcal{U}_\tau^{s,n-1} \\ &+ \int_0^t (\mathcal{U}_\tau^{s,n-1})^{-1}(\nabla(\mathcal{U}_\tau^{s,n-1}e_i)R)\left(\mathcal{U}_\tau^{s,n-1}e_i, \mathcal{U}_\tau^{s,n-1}(\mathcal{U}_0^s)^{-1}[\mathtt{s}(\tau)\mathrm{V}(F^s y)]\right)\mathcal{U}_\tau^{s,n-1} d\tau, \end{aligned}$$

$$\mathrm{R}_{\mathrm{V},\alpha^{s,n-1}}(t) = (\mathcal{U}_0^s)^{-1}[\mathtt{s}'(t)\mathrm{V}(F^s y)] - \mathrm{Ric}\left(\mathcal{U}_t^{s,n-1}(\mathcal{U}_0^s)^{-1}[\mathtt{s}(t)\mathrm{V}(F^s y)]\right).$$

Then define $\mathsf{g}^{s,n}, O^{s,n}, \alpha^{s,n}$ as the processes determined by the following SDEs:

$$\begin{cases} O_t^{s,n} = \mathrm{Id} - \int_0^s K_{\mathrm{V},\alpha^{\jmath,n-1}}(t) O_t^{\jmath,n}\ d\jmath, \\ \mathsf{g}_t^{s,n} = O_t^{s,n}\int_0^s [O_t^{\jmath,n}]^{-1}\mathrm{R}_{\mathrm{V},\alpha^{\jmath,n-1}}(t)\ d\jmath, \\ \alpha_t^{s,n} = \int_0^t O_\tau^{s,n}\ dB_\tau + \int_0^t \mathsf{g}_\tau^{s,n}\ d\tau. \end{cases}$$

When $n = 1$, the definitions of $\mathfrak{g}^{s,0}, O^{s,0}, \mathfrak{g}^{s,1}, O^{s,1}$ show that

$$O_t^{s,1} - O_t^{s,0} = -\int_0^s K_{\mathrm{V},\alpha^{\jmath,0}}(t) O_t^{\jmath,1}\, d\jmath, \quad \mathfrak{g}_t^{s,1} - \mathfrak{g}_t^{s,0} = O_t^{s,1} \int_0^s [O_t^{\jmath,1}]^{-1} \mathrm{R}_{\mathrm{V},\alpha^{\jmath,0}}(t)\, d\jmath.$$

Abbreviate $\|\cdot\|_{\infty,T}$ as $\|\cdot\|$. There is some C such that

$$\begin{aligned}
\|K_{\mathrm{V},\alpha^{\jmath,0}}\|^2 &\leqslant 2\mathbb{E} \sup_{t\in[0,T]} \left| \int_0^t (\mathfrak{V}_\tau^{s,0})^{-1} R\left(\mathfrak{V}_\tau^{s,0} dB_\tau, \mathfrak{V}_\tau^{s,0} (\mathfrak{V}_0^s)^{-1} [\mathfrak{s}(\tau) \mathrm{V}(F^s y)]\right) \mathfrak{V}_\tau^{s,0} \right|^2 + CT^2, \\
&\leqslant 4\mathbb{E} \left| \int_0^T (\mathfrak{V}_\tau^{s,0})^{-1} R\left(\mathfrak{V}_\tau^{s,0} dB_\tau, \mathfrak{V}_\tau^{s,0} (\mathfrak{V}_0^s)^{-1} [\mathfrak{s}(\tau) \mathrm{V}(F^s y)]\right) \mathfrak{V}_\tau^{s,0} \right|^2 + CT^2, \\
&\leqslant 4\mathbb{E} \int_0^T \left| (\mathfrak{V}_\tau^{s,0})^{-1} R\left(\mathfrak{V}_\tau^{s,0} e_i, \mathfrak{V}_\tau^{s,0} (\mathfrak{V}_0^s)^{-1} [\mathfrak{s}(\tau) \mathrm{V}(F^s y)]\right) \mathfrak{V}_\tau^{s,0} \right|^2 d\tau + CT^2 \\
&\leqslant C(T + T^2),
\end{aligned}$$

where the second inequality holds by Doob's inequality of sub-martingales and the third inequality holds by Lemma 4.7. Hence there is some $C(T)$ such that

$$\|O^{s,1} - O^{s,0}\| \leqslant \int_0^s \|K_{\mathrm{V},\alpha^{\jmath,0}}\|\, ds \leqslant C(T)s.$$

There also exists some C such that $\|\mathfrak{g}^{s,1} - \mathfrak{g}^{s,0}\| \leqslant Cs$ since $\mathrm{R}_{\mathrm{V},\alpha^{\jmath,0}}$ is bounded. So we obtain some $C_0(T)$ such that

$$\|\alpha^{s,1} - \alpha^{s,0}\| \leqslant C_0(T)s.$$

If we can further find some constant $C_1(T)$ such that

$$\|\alpha^{s,n} - \alpha^{s,n-1}\| \leqslant C_1(T) \int_0^s \|\alpha^{\jmath,n-1} - \alpha^{\jmath,n-2}\|\, d\jmath, \tag{5.46}$$

we will obtain

$$\|\alpha^{s,n} - \alpha^{s,n-1}\| \leqslant \frac{1}{n!}(C_0(T) + C_1(T))^n s^n,$$

which will imply the existence of the limits

$$\mathfrak{g}^s = \lim_{n\to+\infty} \mathfrak{g}^{s,n}, \; O^s = \lim_{n\to+\infty} O^{s,n}.$$

Then $\alpha_t^s = \int_0^t O_\tau^s\, dB_\tau + \int_0^t \mathfrak{g}_\tau^s\, d\tau$ will be our desired process for i) by Lemma 5.11.

For (5.46), let us analyze $\|O^{s,n} - O^{s,n-1}\|$ and $\|\mathfrak{g}^{s,n} - \mathfrak{g}^{s,n-1}\|$. Since each $O^{s,n}$ is $\mathcal{O}(\mathbb{R}^m)$ valued and is invertible, we have

$$\begin{aligned}
O^{s,n} - O^{s,n-1} &= O^{s,n}\left(\mathrm{Id} - (O^{s,n})^{-1} O^{s,n-1}\right) \\
&= -O^{s,n} \int_0^s \frac{d}{ds}\left[(O^{s,n})^{-1} O^{s,n-1}\right]\Big|_{s=\jmath}\, d\jmath \\
&= -O^{s,n} \int_0^s \left(\frac{d}{ds}\left[(O^{s,n})^{-1}\right]\Big|_{s=\jmath} O^{\jmath,n-1} + (O^{\jmath,n})^{-1} \frac{d}{ds}\left[O^{s,n-1}\right]\Big|_{s=\jmath} \right) d\jmath.
\end{aligned}$$

By the inductive defining equations of $O^{s,n-1}$, $O^{s,n}$, we obtain

$$\frac{d}{ds}\left[O^{s,n-1}\right]\Big|_{s=\jmath} = -K_{\mathrm{V},\alpha^{\jmath,n-2}}O^{\jmath,n-1},$$
$$\frac{d}{ds}\left[(O^{s,n})^{-1}\right]\Big|_{s=\jmath} = -(O^{\jmath,n})^{-1}\,\frac{d}{ds}\left[O^{s,n}\right]\Big|_{s=\jmath}(O^{\jmath,n})^{-1} = (O^{\jmath,n})^{-1}K_{\mathrm{V},\alpha^{\jmath,n-1}}.$$

Hence

$$O^{s,n} - O^{s,n-1} = -O^{s,n}\int_0^s (O^{\jmath,n})^{-1}\left(K_{\mathrm{V},\alpha^{\jmath,n-1}} - K_{\mathrm{V},\alpha^{\jmath,n-2}}\right)O^{\jmath,n-1}d\jmath.$$

Using (5.33), we conclude that there are some constants C, C' such that

$$\|O^{s,n} - O^{s,n-1}\| \leqslant C\int_0^s \left\|K_{\mathrm{V},\alpha^{\jmath,n-1}} - K_{\mathrm{V},\alpha^{\jmath,n-2}}\right\|\, d\jmath \leqslant C'\int_0^s \|\alpha^{\jmath,n-1} - \alpha^{\jmath,n-2}\|\, d\jmath. \tag{5.47}$$

For $\|\mathfrak{g}^{s,n} - \mathfrak{g}^{s,n-1}\|$, we can use the inductive definitions of $\mathfrak{g}^{s,n}$ and $\mathfrak{g}^{s,n-1}$ to compute that

$$\begin{aligned}\mathfrak{g}_t^{s,n} - \mathfrak{g}_t^{s,n-1} &= \left(O_t^{s,n} - O_t^{s,n-1}\right)\int_0^s [O_t^{\jmath,n}]^{-1}\mathrm{R}_{\mathrm{V},\alpha^{\jmath,n-1}}(t)\, d\jmath\\ &\quad + O_t^{s,n-1}\int_0^s [O_t^{\jmath,n}]^{-1}\left(O_t^{\jmath,n-1} - O_t^{\jmath,n}\right)[O_t^{\jmath,n-1}]^{-1}\mathrm{R}_{\mathrm{V},\alpha^{\jmath,n-1}}(t)\, d\jmath\\ &\quad + O_t^{s,n-1}\int_0^s [O_t^{\jmath,n-1}]^{-1}\left(\mathrm{R}_{\mathrm{V},\alpha^{\jmath,n-1}}(t) - \mathrm{R}_{\mathrm{V},\alpha^{\jmath,n-2}}(t)\right)\, d\jmath\\ &=: (\mathrm{a})_t + (\mathrm{b})_t + (\mathrm{c})_t.\end{aligned}$$

Hence

$$\|\mathfrak{g}^{s,n} - \mathfrak{g}^{s,n-1}\| \leqslant \|(\mathrm{a})\| + \|(\mathrm{b})\| + \|(\mathrm{c})\|.$$

Since V is bounded on $\widetilde{M}$ and $\mathfrak{s}$ is C^1 on $[0,T]$, $\mathrm{R}_{\mathrm{V},\alpha^{\jmath,n-1}}(t)$ is also bounded. So there are some constants C, C' such that

$$\|(\mathrm{a})\| \leqslant Cs_0\left\|O^{s,n} - O^{s,n-1}\right\| \leqslant C's_0\int_0^s \|\alpha^{\jmath,n-1} - \alpha^{\jmath,n-2}\|\, d\jmath,$$
$$\|(\mathrm{b})\| \leqslant C\int_0^s \left\|O^{\jmath,n-1} - O^{\jmath,n}\right\|\, d\jmath \leqslant C'\int_0^s \|\alpha^{\jmath,n-1} - \alpha^{\jmath,n-2}\|\, d\jmath.$$

For (c), we also have

$$\|(\mathrm{c})\| \leqslant C\int_0^s \left\|\mathrm{R}_{\mathrm{V},\alpha^{\jmath,n-1}} - \mathrm{R}_{\mathrm{V},\alpha^{\jmath,n-2}}\right\|\, d\jmath \leqslant C'\int_0^s \left\|\mathfrak{V}^{\jmath,n-1} - \mathfrak{V}^{\jmath,n-2}\right\|\, d\jmath,$$

where the last norm is measured using the distance function on $\mathcal{O}^{\tilde{g}}(\widetilde{M})$. Recall that

$$\mathfrak{V}_t^{\jmath,n} = \mathfrak{V}_0^{\jmath,n} + \int_0^t \sum_{i=1}^m H(\mathfrak{V}_\tau^{\jmath,n}, e_i)\circ d\alpha_\tau^{\jmath,n;i},\ 0 \leqslant t \leqslant T, \tag{5.48}$$

where $\alpha^{\jmath,n;i}$, $i = 1,\cdots,m$, denotes the i-th component of $\alpha^{\jmath,n}$. Embedding $\mathcal{O}^{\tilde{g}}(\widetilde{M})$ into some higher dimensional Euclidean space $\mathbb{R}^l$ and extending all $H(\cdot, e_i)$ to a small

tube neighborhood of $\mathcal{O}^{\tilde{g}}(\widetilde{M})$, we can consider (5.48) as a Euclidean SDE and compare $\left\|\mathfrak{V}^{\jmath,n-1}-\mathfrak{V}^{\jmath,n-2}\right\|$ with $\|\alpha^{\jmath,n-1}-\alpha^{\jmath,n-2}\|$ using the Euclidean norm. By (5.48),

$$\begin{aligned}\mathfrak{V}_t^{\jmath,n-1}-\mathfrak{V}_t^{\jmath,n-2}&=\int_0^t\left(H(\mathfrak{V}_\tau^{\jmath,n-1},e_i)-H(\mathfrak{V}_\tau^{\jmath,n-2},e_i)\right)\circ(O_\tau^{\jmath,n-1}dB_\tau)^i\\&\quad+\int_0^t H(\mathfrak{V}_\tau^{\jmath,n-2},e_i)\circ((O_\tau^{\jmath,n-1}-O_\tau^{\jmath,n-2})dB_\tau)^i\\&\quad+\int_0^t\left(H(\mathfrak{V}_\tau^{\jmath,n-1},e_i)-H(\mathfrak{V}_\tau^{\jmath,n-2},e_i)\right)\mathfrak{g}_\tau^{\jmath,n-1;i}d\tau\\&\quad+\int_0^t H(\mathfrak{V}_\tau^{\jmath,n-2},e_i)(\mathfrak{g}_\tau^{\jmath,n-1;i}-\mathfrak{g}_\tau^{\jmath,n-2;i})d\tau\\&=:(\mathrm{I})_t+(\mathrm{II})_t+(\mathrm{III})_t+(\mathrm{IV})_t.\end{aligned}$$

Using (4.22), we obtain, for $\widetilde{t}\in[0,T]$,

$$\mathbb{E}\sup_{t\in[0,\widetilde{t}]}\left|\mathfrak{V}_t^{\jmath,n-1}-\mathfrak{V}_t^{\jmath,n-2}\right|^2\leqslant 4\mathbb{E}\left(\sup_{t\in[0,\widetilde{t}]}|(\mathrm{I})_t|^2+\sup_{t\in[0,\widetilde{t}]}|(\mathrm{II})_t|^2+\sup_{t\in[0,\widetilde{t}]}|(\mathrm{III})_t|^2+\sup_{t\in[0,\widetilde{t}]}|(\mathrm{IV})_t|^2\right).$$

For $(\mathrm{I})_t$, we can consider its Itô form and then apply Doob's inequality of sub-martingales and Lemma 4.7, which gives

$$\mathbb{E}\sup_{t\in[0,\widetilde{t}]}|(\mathrm{I})_t|^2\leqslant C\mathbb{E}\int_0^{\widetilde{t}}\left|\mathfrak{V}_\tau^{\jmath,n-1}-\mathfrak{V}_\tau^{\jmath,n-2}\right|^2\,d\tau\leqslant C\int_0^{\widetilde{t}}\mathbb{E}\sup_{t\in[0,\tau]}\left|\mathfrak{V}_t^{\jmath,n-1}-\mathfrak{V}_t^{\jmath,n-2}\right|^2\,d\tau$$

The same argument shows there is some C such that

$$\mathbb{E}\sup_{t\in[0,\widetilde{t}]}|(\mathrm{II})_t|^2\leqslant C\mathbb{E}\int_0^{\widetilde{t}}\left|O_\tau^{\jmath,n-1}-O_\tau^{\jmath,n-2}\right|^2\,d\tau\leqslant CT\left\|\alpha^{\jmath,n-1}-\alpha^{\jmath,n-2}\right\|^2.$$

Note that $\|H(\cdot,e_i)\|$ and $|\mathfrak{g}^{s,n-1}|$ (for all s and n) are bounded. Hence

$$\mathbb{E}\sup_{t\in[0,\widetilde{t}]}|(\mathrm{III})_t|^2\leqslant C\widetilde{t}\mathbb{E}\int_0^{\widetilde{t}}\left|\mathfrak{V}_\tau^{\jmath,n-1}-\mathfrak{V}_\tau^{\jmath,n-2}\right|^2\,d\tau\leqslant CT\int_0^{\widetilde{t}}\mathbb{E}\sup_{t\in[0,\tau]}\left|\mathfrak{V}_t^{\jmath,n-1}-\mathfrak{V}_t^{\jmath,n-2}\right|^2\,d\tau.$$

For $(\mathrm{IV})_t$, we have

$$\mathbb{E}\sup_{t\in[0,\widetilde{t}]}|(\mathrm{IV})_t|^2\leqslant C\widetilde{t}^2\left\|\mathfrak{g}^{\jmath,n-1}-\mathfrak{g}^{\jmath,n-2}\right\|\leqslant CT^2\left\|\alpha^{\jmath,n-1}-\alpha^{\jmath,n-2}\right\|^2.$$

Altogether, there are some constants $C_2(T),C_3(T)$ such that

$$\mathbb{E}\sup_{t\in[0,\widetilde{t}]}\left|\mathfrak{V}_t^{\jmath,n-1}-\mathfrak{V}_t^{\jmath,n-2}\right|^2\leqslant C_2(T)\left\|\alpha^{\jmath,n-1}-\alpha^{\jmath,n-2}\right\|^2+C_3(T)\int_0^{\widetilde{t}}\mathbb{E}\sup_{t\in[0,\tau]}\left|\mathfrak{V}_t^{\jmath,n-1}-\mathfrak{V}_t^{\jmath,n-2}\right|^2d\tau.$$

Applying Lemma 5.16, we obtain some constant $C(T)$ independent of $\jmath$ such that

$$\left\|\mathfrak{V}^{\jmath,n-1}-\mathfrak{V}^{\jmath,n-2}\right\|\leqslant C(T)\left\|\alpha^{\jmath,n-1}-\alpha^{\jmath,n-2}\right\|.$$

So,

$$\|(\mathrm{c})\| \leqslant C\int_0^s \left\|\mho^{\jmath,n-1} - \mho^{\jmath,n-2}\right\| \, d\jmath \leqslant C(T)\int_0^s \left\|\alpha^{\jmath,n-1} - \alpha^{\jmath,n-2}\right\| \, d\jmath.$$

Putting together the estimations of $\|(\mathrm{a})\|, \|(\mathrm{b})\|$ and $\|(\mathrm{c})\|$, we conclude that

$$\left\|\mathfrak{g}^{s,n} - \mathfrak{g}^{s,n-1}\right\| \leqslant C(T)\int_0^s \left\|\alpha^{\jmath,n-1} - \alpha^{\jmath,n-2}\right\| \, d\jmath.$$

This and (5.47) imply (5.46). Hence the limit

$$\lim_{n\to\infty} \alpha^{s,n} = \int_0^{\cdot} O_\tau^s \, d\tau + \int_0^{\cdot} \mathfrak{g}_\tau^s \, d\tau =: \alpha^s$$

exists and α^s satisfies the equation (5.44) by i) of Lemma 5.11.

The $\{\alpha^s\}$ obtained by the above iteration is the the unique parameter of processes in $\mathcal{A}$ satisfying (5.44). Assume $\{\widetilde{\alpha}^s\} \subset \mathcal{A}$ is another parameter of processes solving (5.44). Then, by using (5.34), (5.35) for α^s and $\widetilde{\alpha}^s$, respectively, the above argument shows that (5.46) holds true by replacing $\alpha^{s,n}$, $\alpha^{s,n-1}$ by α^s, $\widetilde{\alpha}^s$, respectively, for all s, i.e.,

$$\|\alpha^s - \widetilde{\alpha}^s\| \leqslant C(T)\int_0^s \|\alpha^\jmath - \widetilde{\alpha}^\jmath\| \, d\jmath.$$

This implies $\alpha^s = \widetilde{\alpha}^s$ by Gronwall's lemma.

We proceed to show

$$\mathbb{E}\left|\underline{\Upsilon}_{\mathrm{V},\alpha^{s'}}(t') - \underline{\Upsilon}_{\mathrm{V},\alpha^s}(t)\right|^\flat \leqslant C(\flat, s_0, T)\left(|t'-t|^{\frac{1}{2}\flat} + |s'-s|^\flat\right) \tag{5.49}$$

for any $\flat > 4$, $t, t' \in [0,T]$ and $s, s' \in [-s_0, s_0]$. This, by applying Lemma 5.15, will imply that $\underline{\Upsilon}_{\mathrm{V},\alpha^s}(t)$ has a continuous modification in the parameter (t,s). Without loss of generality, we assume $t < t'$. Using (5.32) and (4.22), we compute that

$$\begin{aligned}
&\mathbb{E}\left|\underline{\Upsilon}_{\mathrm{V},\alpha^{s'}}(t') - \underline{\Upsilon}_{\mathrm{V},\alpha^s}(t)\right|^\flat \\
&\leqslant 5^{\flat-1}\Bigg(\mathbb{E}\Big|\int_t^{t'} \mathrm{R}_{\mathrm{V},\alpha^{s'}}(\tau)\, d\tau\Big|^\flat + \mathbb{E}\Big|\int_t^{t'} \langle K_{\mathrm{V},\alpha^{s'}}(\tau), d\alpha_\tau^{s'}\rangle\Big|^\flat + \mathbb{E}\Big|\int_0^t (\mathrm{R}_{\mathrm{V},\alpha^{s'}} - \mathrm{R}_{\mathrm{V},\alpha^s})(\tau)\, d\tau\Big|^\flat \\
&\qquad\qquad - \mathbb{E}\Big|\int_0^t \langle (K_{\mathrm{V},\alpha^{s'}} - K_{\mathrm{V},\alpha^s})(\tau), d\alpha_\tau^{s'}\rangle\Big|^\flat + \mathbb{E}\Big|\int_0^t \langle K_{\mathrm{V},\alpha^s}(\tau), d(\alpha_\tau^{s'} - \alpha_\tau^s)\rangle\Big|^\flat\Bigg) \\
&=: 5^{\flat-1}\big((\mathrm{d}) + (\mathrm{e}) + (\mathrm{f}) + (\mathrm{g}) + (\mathrm{h})\big).
\end{aligned}$$

To conclude (5.49), we will show

$$(\mathrm{d}), (\mathrm{e}) \leqslant C(\flat, s_0, T)|t'-t|^{\frac{1}{2}\flat} \text{ and } (\mathrm{f}), (\mathrm{g}), (\mathrm{h}) \leqslant C(\flat, s_0, T)|s'-s|^\flat. \tag{5.50}$$

Clearly,

$$(\mathrm{d}) \leqslant C^\flat \left|t'-t\right|^\flat \leqslant (CT)^\flat |t'-t|^{\frac{1}{2}\flat}$$

for some C which bounds $|\mathrm{R}_{\mathrm{V},\alpha^\jmath}|$. For (e), we have

$$2^{1-\mathfrak{b}}(\mathrm{e}) \leqslant \mathbb{E}\Big|\int_t^{t'} \langle K_{\mathrm{V},\alpha^{s'}}(\tau), O_\tau^{s'} dB_\tau\rangle\Big|^{\mathfrak{b}} + \mathbb{E}\Big|\int_t^{t'} \langle K_{\mathrm{V},\alpha^{s'}}(\tau), \mathfrak{g}_\tau^{s'}\rangle\, d\tau\Big|^{\mathfrak{b}} =: (\mathrm{e})_1 + (\mathrm{e})_2.$$

By Lemma 4.7, with the constant $\mathtt{C}_1(\mathfrak{b})$ there,

$$(\mathrm{e})_1 \leqslant \mathtt{C}_1(\mathfrak{b})\mathbb{E}\Big|\int_{\mathrm{t}}^{\mathrm{t}'} |\mathrm{K}_{\mathrm{V},\alpha^{s'}}(\tau)|^2\, \mathrm{d}\tau\Big|^{\frac{1}{2}\mathfrak{b}} \leqslant \mathtt{C}_1(\mathfrak{b})\Big(\int_{\mathrm{t}}^{\mathrm{t}'} \mathbb{E}(|\mathrm{K}_{\mathrm{V},\alpha^{s'}}(\tau)|^{\mathfrak{b}})\, \mathrm{d}\tau\Big)\cdot|\mathrm{t}'-\mathrm{t}|^{\frac{1}{2}\mathfrak{b}-1}.$$

Using (4.22) and Lemma 4.7, it is easy to deduce that

$$\begin{aligned}\mathbb{E}(|K_{\mathrm{V},\alpha^{s'}}(\tau)|^{\mathfrak{b}}) \leqslant 3^{\mathfrak{b}-1}\Bigg(&\mathbb{E}\bigg|\int_0^\tau (\mathcal{U}_{\tau'}^{s'})^{-1} R\Big(\mathcal{U}_{\tau'}^{s'} O_{\tau'}^{s'} dB_{\tau'}, \mathcal{U}_{\tau'}^{s'}(\mathcal{U}_0^{s'})^{-1}[\mathtt{s}(\tau')\mathrm{V}(F^{s'}y)]\Big)\,\mathcal{U}_{\tau'}^{s'}\bigg|^{\mathfrak{b}}\\ &+\mathbb{E}\bigg|\int_0^\tau (\mathcal{U}_{\tau'}^{s'})^{-1} R\Big(\mathcal{U}_{\tau'}^{s'}\mathfrak{g}_{\tau'}^{s'} d\tau', \mathcal{U}_{\tau'}^{s'}(\mathcal{U}_0^{s'})^{-1}[\mathtt{s}(\tau')\mathrm{V}(F^{s'}y)]\Big)\,\mathcal{U}_{\tau'}^{s'}\bigg|^{\mathfrak{b}}\\ &+\mathbb{E}\bigg|\int_0^\tau (\mathcal{U}_{\tau'}^{s'})^{-1}(\nabla(\mathcal{U}_{\tau'}^{s'}e_i)R)\big(\mathcal{U}_{\tau'}^{s'}e_i, \mathcal{U}_{\tau'}^{s'}(\mathcal{U}_0^{s'})^{-1}[\mathtt{s}(\tau')\mathrm{V}(F^{s'}y)]\big)\mathcal{U}_{\tau'}^{s'}\, d\tau'\bigg|^{\mathfrak{b}}\Bigg)\\ &\leqslant 3^{\mathfrak{b}-1}C(\tau^{\frac{1}{2}\mathfrak{b}}+\tau^{\mathfrak{b}}).\end{aligned} \tag{5.51}$$

Hence there is a constant $C(\mathfrak{b},T)$ such that

$$(\mathrm{e})_1 \leqslant C(\mathfrak{b},T)|t'-t|^{\frac{1}{2}\mathfrak{b}}.$$

Since $|\mathfrak{g}_\tau^s|$ is bounded by some constant depending on s_0 and $\sup|\mathrm{V}|$, using Hölder's inequality and the estimation in (5.51), we obtain

$$(\mathrm{e})_2 \leqslant \Big(\int_t^{t'} \mathbb{E}(|K_{\mathrm{V},\alpha^{s'}}(\tau)|^{\mathfrak{b}})\, d\tau\Big)\cdot|t'-t|^{\mathfrak{b}-1} \leqslant C(\mathfrak{b},T)|t'-t|^{\mathfrak{b}}.$$

Thus,

$$(\mathrm{e}) \leqslant 2^{\mathfrak{b}-1}\left((\mathrm{e})_1+(\mathrm{e})_2\right) \leqslant 2^{\mathfrak{b}-1}(T^{\frac{1}{2}\mathfrak{b}}+1)C(\mathfrak{b},T)|t'-t|^{\frac{1}{2}\mathfrak{b}} = C(\mathfrak{b},T)|t'-t|^{\frac{1}{2}\mathfrak{b}}.$$

Using Hölder's inequality and Burkholder's inequality, the conclusion in (5.50) for (f), (g) and (h) can be reduced to showing

$$\mathbb{E}\big|(\mathrm{R}_{\mathrm{V},\alpha^{s'}}-\mathrm{R}_{\mathrm{V},\alpha^s})(\tau)\big|^{\mathfrak{b}},\ \mathbb{E}\big|(K_{\mathrm{V},\alpha^{s'}}-K_{\mathrm{V},\alpha^s})(\tau)\big|^{\mathfrak{b}},\ \mathbb{E}\big|\alpha_\tau^{s'}-\alpha_\tau^s\big|^{\mathfrak{b}} \leqslant C(\mathfrak{b},s_0,T)|s'-s|^{\mathfrak{b}},$$

which can be further reduced to verifying

$$\mathbb{E}\big|O_\tau^{s'}-O_\tau^s\big|^{\mathfrak{b}},\ \mathbb{E}\big|\mathfrak{g}_\tau^{s'}-\mathfrak{g}_\tau^s\big|^{\mathfrak{b}},\ \mathbb{E}\big|\mathcal{U}_\tau^{s'}-\mathcal{U}_\tau^s\big|^{\mathfrak{b}} \leqslant C(\mathfrak{b},s_0,T)|s'-s|^{\mathfrak{b}}.$$

By (5.34) and (5.51), there is some constant $C(\mathfrak{b},T)$ such that

$$\begin{aligned}\mathbb{E}\big|O_\tau^{s'}-O_\tau^s\big|^{\mathfrak{b}} = \mathbb{E}\bigg|\int_s^{s'} K_{\mathrm{V},\alpha^\jmath}(\tau)O_\tau^\jmath\, d\jmath\bigg|^{\mathfrak{b}} &\leqslant \bigg|\int_s^{s'}\mathbb{E}(|K_{\mathrm{V},\alpha^{s'}}(\tau)|^{\mathfrak{b}})\, d\tau\bigg|\cdot|s'-s|^{\mathfrak{b}-1}\\ &\leqslant C(\mathfrak{b},T)|s'-s|^{\mathfrak{b}}.\end{aligned} \tag{5.52}$$

Using (5.35), (5.52) and Hölder's inequality, we obtain some constant $C(\flat, s_0, T)$ such that

$$\mathbb{E}\big|\mathfrak{g}_\tau^{s'} - \mathfrak{g}_\tau^{s}\big|^\flat \leqslant 2^{\flat-1}\left(\mathbb{E}\big(|O_\tau^{s'} - O_\tau^s|^\flat \cdot \Big|\int_0^{s'} [O_\tau^\jmath]^{-1}\mathrm{R}_{\mathrm{V},\alpha^\jmath}(\tau)\ d\jmath\Big|^\flat\big) + \mathbb{E}\left|\int_s^{s'} [O_\tau^\jmath]^{-1}\mathrm{R}_{\mathrm{V},\alpha^\jmath}(\tau)\ d\jmath\right|^\flat\right)$$

$$\leqslant C(\flat, s_0, T)|s' - s|^\flat. \tag{5.53}$$

Recall that each $\mathcal{U}^s$ satisfies the SDE

$$\mathcal{U}_\tau^s(\mathrm{w}) = \mathcal{U}_0^s + \int_0^\tau \sum_{i=1}^m H(\mathcal{U}_{\tau'}^s(\mathrm{w}), e_i) \circ d\alpha_{\tau'}^{s;i}(\mathrm{w}),\ \forall \tau \in [0,T].$$

As before, we can treat it as a Euclidean SDE. Hence,

$$\begin{aligned}\mathbb{E}\big|\mathcal{U}_\tau^{s'} - \mathcal{U}_\tau^s\big|^\flat \leqslant 3^{\flat-1}\Bigg(&\big|\mathcal{U}_0^{s'} - \mathcal{U}_0^s\big|^\flat + \mathbb{E}\Big|\int_0^\tau \sum_{i=1}^m H(\mathcal{U}_{\tau'}^s, e_i) \circ d\big(\alpha_{\tau'}^{s',i} - \alpha_{\tau'}^{s,i}\big)\Big|^\flat \\ &+\mathbb{E}\Big|\int_0^\tau \sum_{i=1}^m \big(H(\mathcal{U}_{\tau'}^{s'}, e_i) - H(\mathcal{U}_{\tau'}^s, e_i)\big) \circ d\alpha_{\tau'}^{s',i}\Big|^\flat\Bigg) \\ =:\ & 3^{\flat-1}\big(\mathrm{(i)} + \mathrm{(j)} + \mathrm{(k)}\big).\end{aligned}$$

Clearly, $\mathrm{(i)} \leqslant C|s' - s|^\flat$ for some C depending on $\sup|\mathrm{V}|$ and $\widetilde{g}$. For (j), we have

$$\begin{aligned}\mathrm{(j)} &\leqslant \mathbb{E}\Big|\int_0^\tau \sum_{i=1}^m H(\mathcal{U}_{\tau'}^s, e_i) \circ (O_{\tau'}^{s'} - O_{\tau'}^s)dB_{\tau'})^i\Big|^\flat + \mathbb{E}\Big|\int_0^\tau \sum_{i=1}^m H(\mathcal{U}_{\tau'}^s, e_i)(\mathfrak{g}_{\tau'}^{s'} - \mathfrak{g}_{\tau'}^s)^i d\tau'\Big|^\flat \\ &=: \mathrm{(j)}_1 + \mathrm{(j)}_2.\end{aligned}$$

where we use the superscript i to denote the i-th component of a vector. For $\mathrm{(j)}_1$, we can transfer the integral into Itô's form. Note that all $H(\cdot, e_i)$ are C^1 vector fields on $\mathcal{O}^{\widetilde{g}}(\widetilde{M})$ with bounded first order differentials. Hence, using Lemma 4.7, Hölder's inequality and (5.52), we can conclude that there is some constant $C(\flat, T)$ such that

$$\begin{aligned}\mathrm{(j)}_1 &\leqslant C(\flat, T)\big(\mathbb{E}\big(\int_0^\tau |O_{\tau'}^{s'} - O_{\tau'}^s|^2\ d\tau'\big)^{\frac{\flat}{2}} + \mathbb{E}\big(\int_0^\tau |O_{\tau'}^{s'} - O_{\tau'}^s|^2\ d\tau'\big)^\flat\big) \\ &\leqslant C(\flat, T)(T^{\frac{1}{2}\flat-1} + T^{\flat-1})\int_0^\tau \big(\mathbb{E}|O_{\tau'}^{s'} - O_{\tau'}^s|^{2\flat} + \mathbb{E}|O_{\tau'}^{s'} - O_{\tau'}^s|^\flat\big)d\tau' \\ &\leqslant C(\flat, T)(2s_0)^\flat|s' - s|^\flat.\end{aligned}$$

For $\mathrm{(j)}_2$, we can use Hölder's inequality and (5.53) to conclude that

$$\mathrm{(j)}_2 \leqslant C(\flat, s_0, T)T^\flat|s' - s|^\flat.$$

For (k), the same argument as for (j) gives some constant $C(\flat, s_0, T)$ such that

$$\mathrm{(k)} \leqslant C(\flat, s_0, T)\int_0^\tau \mathbb{E}\big|\mathcal{U}_{\tau'}^{s'} - \mathcal{U}_{\tau'}^s\big|^\flat\ d\tau'.$$

Altogether, there is some constant $C(\flat, s_0, T)$ such that

$$\mathbb{E}\big|\mho_\tau^{s'} - \mho_\tau^{s}\big|^\flat \leqslant C(\flat, s_0, T)\left(|s'-s|^\flat + \int_0^\tau \mathbb{E}\big|\mho_{\tau'}^{s'} - \mho_{\tau'}^{s}\big|^\flat \, d\tau'\right).$$

The inequality also holds for $\sup_{\widetilde{\tau}\in[0,\tau]}\mathbb{E}\big|\mho_{\widetilde{\tau}}^{s'} - \mho_{\widetilde{\tau}}^{s}\big|^\flat$. Hence we can apply Lemma 5.16 to conclude that there is some constant $C(\flat, s_0, T)$ such that

$$\mathbb{E}\big|\mho_\tau^{s'} - \mho_\tau^{s}\big|^\flat \leqslant C(\flat, s_0, T)|s'-s|^\flat. \tag{5.54}$$

This finish the proof of (5.50) and hence (5.49) holds true. By Lemma 5.15, we can obtain a continuous modification of $\underline{\Upsilon}_{\mathrm{V},\alpha^s}(t)$ in the parameter (t,s).

Let $\alpha^s, \mho^s$ be as above. By (5.54) and Lemma 5.15, $\mho^s$ has a version such that $s \mapsto \mho^s(\mathrm{w})$ is continuous. By Lemma 5.11, to show $\mho^s$ is differentiable in s, it suffices to show (5.36) is uniquely solvable with $Y_0^s = (\mho_0^s)'_s$. Let $Y^{s,0} \equiv 0$ and let $Y^{s,n}$ $(n \geqslant 1)$ be such that

$$\begin{cases} d\theta(Y_t^{s,n}) &= \varpi\big(Y_t^{s,n-1}\big) \circ d\alpha_t^s + \circ d\underline{\Upsilon}_{\mathrm{V},\alpha^s}, \\ d\varpi(Y_t^{s,n}) &= (\mho_t^s)^{-1} R\big(\mho_t^s \circ d\alpha_t^s, \mho_t^s \theta(Y_t^{s,n-1})\big)\mho_t^s. \end{cases} \tag{5.55}$$

For a $\mathbb{R}^m \times \mathcal{F}(\mathbb{R}^m)$ valued process $(\mathfrak{v}, \mathfrak{V})_{t\in[0,T]}$, let

$$\|(\mathfrak{v},\mathfrak{V})\| := \sqrt{\|\mathfrak{v}\|^2 + \|\mathfrak{V}\|^2}, \text{ where } \|\mathfrak{v}\|^2 = \mathbb{E}\sup_{t\in[0,T]}|\mathfrak{v}_t|^2, \|\mathfrak{V}\|^2 = \mathbb{E}\sup_{t\in[0,T]}|\mathfrak{V}_t|^2.$$

We show the sequence $(\theta,\varpi)(Y^{s,n})$ converges in norm $\|\cdot\|$. Clearly,

$$\big\|(\theta,\varpi)\big(Y^{s,1}\big) - (\theta,\varpi)\big(Y^{s,0}\big)\big\| \leqslant CT\|\alpha^s\|. \tag{5.56}$$

We continue to estimate $\|(\theta,\varpi)\big(Y^{s,n}\big) - (\theta,\varpi)\big(Y^{s,n-1}\big)\|$, $n \geqslant 2$. By (5.55),

$$\begin{cases} d\big(\theta(Y_t^{s,n}) - \theta(Y_t^{s,n-1})\big) &= \big(\varpi(Y_t^{s,n}) - \varpi(Y_t^{s,n-1})\big) \circ d\alpha_t^s, \\ d\big(\varpi(Y_t^{s,n}) - \theta(Y_t^{s,n-1})\big) &= (\mho_t^s)^{-1} R\big(\mho_t^s \circ d\alpha_t^s, \mho_t^s\big(\theta(Y_t^{s,n}) - \theta(Y_t^{s,n-1})\big)\big)\mho_t^s. \end{cases}$$

Following the above discussion on $\big\|\mho^{\jmath,n-1} - \mho^{\jmath,n-2}\big\|$, we can use Doob's inequality of submartingale and Lemma 4.7 to conclude that

$$\begin{aligned} &\mathbb{E}\sup_{t\in[0,\widetilde{t}]}\big|(\theta,\varpi)\big(Y^{s,n}\big) - (\theta,\varpi)\big(Y^{s,n-1}\big)\big|^2 \\ &\quad\leqslant C(s_0,T)\int_0^{\widetilde{t}}\mathbb{E}\sup_{t\in[0,\tau]}\big|(\theta,\varpi)\big(Y^{s,n-1}\big) - (\theta,\varpi)\big(Y^{s,n-2}\big)\big|^2 \, d\tau. \end{aligned} \tag{5.57}$$

Iterating this inequality for n steps, which, together with (5.56), imply

$$\mathbb{E}\sup_{t\in[0,\widetilde{t}]}\big|(\theta,\varpi)\big(Y^{s,n}\big) - (\theta,\varpi)\big(Y^{s,n-1}\big)\big|^2 \leqslant \frac{1}{n!}(CT + C(s_0,T))^n \widetilde{t}^n.$$

In particular, when $\widetilde{t} = T$, this is

$$\big\|(\theta,\varpi)\big(Y^{s,n}\big) - (\theta,\varpi)\big(Y^{s,n-1}\big)\big\| \leqslant \frac{1}{n!}(CT + C(s_0,T))^n T^n.$$

Hence $(\theta,\varpi)(Y^{s,n})$ converges in $\|\cdot\|$ with some limit $(\theta,\varpi)(Y^s)$ which solves (5.36).

Such a solution Y^s is unique. Assume $\mathcal{Y}^s$ is another solution to (5.36) with $\mathcal{Y}^s_0 = (\mho^s_0)'_s$. Then the same argument as for (5.57) shows that

$$\mathbb{E}\sup_{t\in[0,\tilde{t}]}\left|(\theta,\varpi)(Y^s_t)-(\theta,\varpi)(\mathcal{Y}^s_t)\right|^2 \leqslant C(s_0,T)\int_0^{\tilde{t}}\mathbb{E}\sup_{t\in[0,\tau]}\left|(\theta,\varpi)(Y^s_t)-(\theta,\varpi)(\mathcal{Y}^s_t)\right|^2 d\tau,$$

from which we can conclude $Y^s = \mathcal{Y}^s$ by Gronwall's Lemma.

By Corollary 5.12, the solution Y^s to (5.36) is actually given by (5.39). Hence, to show the process Y^s_t has a continuous modification in the parameter (t,s), it suffices to show both $\mho^s_t$ and $\varpi(Y^s_t)$ have a (t,s)-continuous version. Let $\flat > 4$, $t,t' \in [0,T]$ with $t<t'$ and $s,s' \in [-s_0,s_0]$. Using (5.54) and applying Burkholder's inequality and Hölder's inequality to the difference $\mho^s_{t'} - \mho^s_t$, we obtain

$$2^{1-\flat}\mathbb{E}\left|\mho^{s'}_{t'}-\mho^s_t\right|^\flat \leqslant \mathbb{E}\left|\mho^{s'}_{t'}-\mho^s_{t'}\right|^\flat + \mathbb{E}\left|\mho^s_{t'}-\mho^s_t\right|^\flat \leqslant C(\flat,s_0,T)\big(|s'-s|^\flat + |t'-t|^{\frac{1}{2}\flat}\big).$$

So Lemma 5.15 applies and shows that there is a version of $\mho^s_t$ which is continuous in the parameter (t,s). Since $\varpi(Y^s_0)=0$, by (5.39),

$$\begin{aligned}\varpi(Y^s_t) = &\int_0^t (\mho^s_\tau)^{-1}R\big(\mho^s_\tau d\alpha^s_\tau, \mathtt{s}(\tau)\mho^s_\tau(\mho^s_0)^{-1}V(F^s y)\big)\mho^s_\tau\\ &+\int_0^t (\mho^s_\tau)^{-1}(\nabla(\mho^s_\tau e_i)R)\left(\mho^s_\tau e_i, \mathtt{s}(\tau)\mho^s_\tau(\mho^s_0)^{-1}V(F^s y)\right)\mho^s_\tau\, d\tau.\end{aligned}$$

Again, by Burkholder's inequality and Hölder's inequality, it is easy to deduce that

$$\begin{aligned}&\mathbb{E}\left|\varpi(Y^{s'}_{t'})-\varpi(Y^s_{t'})\right|^\flat\\ &\leqslant C(s_0,b,T)\big(\mathbb{E}\Big|\int_0^{t'}|\alpha^{s'}_\tau-\alpha^s_\tau|^2\,d\tau\Big|^{\frac{\flat}{2}} + \mathbb{E}\Big|\int_0^{t'}|\mho^{s'}_\tau-\mho^s_\tau|^2\,d\tau\Big|^{\frac{\flat}{2}} + \mathbb{E}\int_0^{t'}|\mho^{s'}_\tau-\mho^s_\tau|^\flat\,d\tau\big)\\ &\leqslant C(s_0,b,T)\int_0^{t'}\left(\mathbb{E}|\alpha^{s'}_\tau-\alpha^s_\tau|^\flat + \mathbb{E}|\mho^{s'}_\tau-\mho^s_\tau|^\flat\right)d\tau\\ &\leqslant C(\flat,s_0,T)|s'-s|^\flat\end{aligned}$$

and

$$\mathbb{E}\left|\varpi(Y^s_{t'})-\varpi(Y^s_t)\right|^\flat \leqslant C(s_0,b,T)\left(|t'-t|^{\frac{\flat}{2}}+|t'-t|^\flat\right) \leqslant C(s_0,b,T)|t'-t|^{\frac{\flat}{2}}.$$

Hence

$$\begin{aligned}2^{1-\flat}\mathbb{E}\left|\varpi(Y^{s'}_{t'})-\varpi(Y^s_t)\right|^\flat &\leqslant \mathbb{E}\left|\varpi(Y^{s'}_{t'})-\varpi(Y^s_{t'})\right|^\flat + \mathbb{E}\left|\varpi(Y^s_{t'})-\varpi(Y^s_t)\right|^\flat\\ &\leqslant C(\flat,s_0,T)\big(|s'-s|^\flat+|t'-t|^{\frac{1}{2}\flat}\big),\end{aligned}$$

which implies that $\varpi(Y^s_t)$ has a (t,s)-continuous modification by Lemma 5.15.

Now we have shown i) and ii). Hence we can use Lemma 5.11 to conclude that $\mathrm{y}^s = \mathcal{I}_{\mho^s_0}(\alpha^s)$ is differentiable in s and satisfies (5.45). The differential process

$$\Upsilon_{\mathrm{V},\mathrm{y}^s}(t,\mathrm{w}) = \mathtt{s}(t)\mho^s_t(\mho^s_0)^{-1}V(F^s y)$$

has a (t,s)-continuous version since $\mathcal{U}_t^s$ does.

Finally, by Lemma 5.14, for iv), it suffices to show for $\mathsf{v}_{t_1}^s \in T_{\mathcal{U}_{t_1}^s}\mathcal{F}(\widetilde{M})$ C^1 in s, (5.43) is uniquely solvable with initial $(\mathsf{v}_{t_1}^s)'_s$. Again, this can be done by Picard's iteration method. Let $\upsilon_t^{s,0} = (\theta,\varpi)^{-1}_{\mathcal{U}_t^s}(\theta,\varpi)_{\mathcal{U}_{t_1}^s}(\mathsf{v}_{t_1}^s)'_s$. For $n \geqslant 1$, let $\upsilon_t^{s,n}$ with initial $(\mathsf{v}_{t_1}^s)'_s$ be such that

$$\begin{cases} d\big(\theta(\upsilon_t^{s,n})\big) &= \varpi\big(\upsilon_t^{s,n-1}\big)d\alpha_t^s + \mathrm{Ric}(\mathcal{U}_t^s\theta(\upsilon_t^{s,n-1}))\ dt + \theta\left(\circledast_{\mathrm{I}}\big(\mathsf{v}_t^s,(\mathcal{U}_t^s)'_s\big)\right) + \circledast_{\mathrm{A}}^{\theta}\big(\mathsf{v}_t^s,(\mathcal{U}_t^s)'_s\big),\\ d\big(\varpi(\upsilon_t^{s,n})\big) &= (\mathcal{U}_t^s)^{-1}R\big(\mathcal{U}_t^s d\alpha_t^s, \mathcal{U}_t^s\theta(\upsilon_t^{s,n-1})\big)\mathcal{U}_t^s + (\mathcal{U}_t^s)^{-1}R\big(\mathcal{U}_t^s e_i, \mathcal{U}_t^s\varpi(\upsilon_t^{s,n-1})e_i\big)\mathcal{U}_t^s\ dt\\ &\quad +(\mathcal{U}_t^s)^{-1}(\nabla(\mathcal{U}_t^s e_i)R)\big(\mathcal{U}_t^s e_i, \mathcal{U}_t^s\theta(\upsilon_t^{s,n-1})\big)\mathcal{U}_t^s\ dt\\ &\quad +\varpi\left(\circledast_{\mathrm{I}}\big(\mathsf{v}_t^s,(\mathcal{U}_t^s)'_s\big)\right) + \circledast_{\mathrm{A}}^{\varpi}\big(\mathsf{v}_t^s,(\mathcal{U}_t^s)'_s\big), \end{cases}$$

where v_t^s, $\circledast_{\mathrm{I}}\big(\mathsf{v}_t^s,(\mathcal{U}_t^s)'_s\big)$, $\circledast_{\mathrm{A}}^{\theta}\big(\mathsf{v}_t^s,(\mathcal{U}_t^s)'_s\big)$, $\circledast_{\mathrm{A}}^{\varpi}\big(\mathsf{v}_t^s,(\mathcal{U}_t^s)'_s\big)$ are as in Lemma 5.14. We will show $(\theta,\varpi)(\upsilon_t^{s,n})$ converges in norm $\|\cdot\|$, where, for any $\mathbb{R}^m\times\mathcal{F}(\mathbb{R}^m)$ valued process $(\mathfrak{v},\mathfrak{Q})_{t\in[t_1,t_2]}$,

$$\|(\mathfrak{v},\mathfrak{Q})\| := \sqrt{\|\mathfrak{v}\|^2+\|\mathfrak{Q}\|^2},\ \|\mathfrak{v}\|^2 = \mathbb{E}\sup_{t\in[t_1,t_2]}|\mathfrak{v}_t|^2,\ \|\mathfrak{Q}\|^2 = \mathbb{E}\sup_{t\in[t_1,t_2]}|\mathfrak{Q}_t|^2.$$

Clearly, we have

$$\begin{aligned}\big\|(\theta,\varpi)(\upsilon^{s,1}) - (\theta,\varpi)(\upsilon^{s,0})\big\| &< C(s_0,T) + \left\|\int_{t_1}^t \theta\left(\circledast_{\mathrm{I}}\big(\mathsf{v}_\tau^s,(\mathcal{U}_\tau^s)'_s\big)\right)\right\| + \left\|\int_{t_1}^t \varpi\left(\circledast_{\mathrm{I}}\big(\mathsf{v}_\tau^s,(\mathcal{U}_\tau^s)'_s\big)\right)\right\|\\ &\quad + \left\|\int_{t_1}^t \circledast_{\mathrm{A}}^{\theta}\big(\mathsf{v}_\tau^s,(\mathcal{U}_\tau^s)'_s\big)\right\| + \left\|\int_{t_1}^t \circledast_{\mathrm{A}}^{\varpi}\big(\mathsf{v}_\tau^s,(\mathcal{U}_\tau^s)'_s\big)\right\|\\ &=: C(s_0,T) + (\mathrm{A})_1 + (\mathrm{A})_2 + (\mathrm{A})_3 + (\mathrm{A})_4.\end{aligned}$$

Using Doob's inequality of submartingale, Lemma 4.7 and Hölder's inequality, we see from the expressions of $\circledast_{\mathrm{I}}\big(\mathsf{v}_t^s,(\mathcal{U}_t^s)'_s\big)$, $\circledast_{\mathrm{A}}^{\theta}\big(\mathsf{v}_t^s,(\mathcal{U}_t^s)'_s\big)$, $\circledast_{\mathrm{A}}^{\varpi}\big(\mathsf{v}_t^s,(\mathcal{U}_t^s)'_s\big)$ that

$$(\mathrm{A})_i \leqslant C(s_0,T)\big(\,\|(\theta,\varpi)(\mathsf{v}_t^s)\| + \big\||(\theta,\varpi)(\mathsf{v}_t^s)|^2\big\|\cdot\big\||(\mathcal{U}_t^s)'_s|^2\big\|\,\big),\ i=1,\ 2,\ 3,\ 4.$$

Note that the process v_t^s satisfies the SDE

$$\begin{cases} d\big(\theta(\mathsf{v}_t^s)\big) &= \varpi\big(\mathsf{v}_t^s\big)d\alpha_t^s + \mathrm{Ric}(\mathcal{U}_t^s\theta(\mathsf{v}_t^s))\ dt,\\ d\big(\varpi(\mathsf{v}_t^s)\big) &= (\mathcal{U}_t^s)^{-1}R\big(\mathcal{U}_t^s d\alpha_t^s, \mathcal{U}_t^s\theta(\mathsf{v}_t^s)\big)\mathcal{U}_t^s + (\mathcal{U}_t^s)^{-1}R\big(\mathcal{U}_t^s e_i, \mathcal{U}_t^s\varpi(\mathsf{v}_t^s)e_i\big)\mathcal{U}_t^s\ dt\\ &\quad +(\mathcal{U}_t^s)^{-1}(\nabla(\mathcal{U}_t^s e_i)R)\left(\mathcal{U}_t^s e_i, \mathcal{U}_t^s\theta(\mathsf{v}_t^s)\right)\mathcal{U}_t^s\ dt. \end{cases} \tag{5.58}$$

So, using Doob's inequality of sub-martingales and Lemma 4.7, we compute that

$$\mathbb{E}\sup_{t\in[t_1,\widetilde{t}]}\big|(\theta,\varpi)(\mathsf{v}_t^s)\big|^\flat \leqslant C(s_0,T)\int_{t_1}^{\widetilde{t}}\mathbb{E}\sup_{t\in[t_1,\tau]}\big|(\theta,\varpi)(\mathsf{v}_t^s)\big|^\flat\ d\tau,\ \flat = 2,\ 4,$$

which, by Gronwall's lemma, implies

$$\|(\theta,\varpi)(\mathsf{v}_t^s)\|, \big\||(\theta,\varpi)(\mathsf{v}_t^s)|^2\big\| \leqslant C(s_0,T). \tag{5.59}$$

With a similar computation, we conclude from (5.39) that $\big\||(\mathcal{U}_t^s)'_s|^2\big\|$ is also bounded by constant $C(s_0,T)$. So,

$$\big\|(\theta,\varpi)(\upsilon^{s,1}) - (\theta,\varpi)(\upsilon^{s,0})\big\| \leqslant C(s_0,T). \tag{5.60}$$

For $n \geqslant 2$, the difference $(\theta,\varpi)\big(\upsilon^{s,n}\big) - (\theta,\varpi)\big(\upsilon^{s,n-1}\big)$ satisfies the SDE

$$\begin{cases} d\big(\theta(\upsilon_t^{s,n}) - \theta(\upsilon_t^{s,n-1})\big) = \big(\varpi(\upsilon_t^{s,n-1}) - \varpi(\upsilon_t^{s,n-2})\big) d\alpha_t^s + \mathrm{Ric}\big(\mathfrak{V}_t^s\big(\theta(\upsilon_t^{s,n-1}) - \theta(\upsilon_t^{s,n-2})\big)\big) dt, \\ d\big(\varpi(\upsilon_t^{s,n}) - \varpi(\upsilon_t^{s,n-1})\big) = (\mathfrak{V}_t^s)^{-1} R\big(\mathfrak{V}_t^s d\alpha_t^s, \mathfrak{V}_t^s\big(\theta(\upsilon_t^{s,n-1}) - \theta(\upsilon_t^{s,n-2})\big)\big)\mathfrak{V}_t^s \\ \qquad\qquad + (\mathfrak{V}_t^s)^{-1} R\big(\mathfrak{V}_t^s e_i, \mathfrak{V}_t^s\big(\varpi(\upsilon_t^{s,n-1}) - \varpi(\upsilon_t^{s,n-2})\big) e_i\big)\mathfrak{V}_t^s \, dt \\ \qquad\qquad + (\mathfrak{V}_t^s)^{-1} (\nabla(\mathfrak{V}_t^s e_i) R) \Big(\mathfrak{V}_t^s e_i, \mathfrak{V}_t^s\big(\theta(\upsilon_t^{s,n-1}) - \theta(\upsilon_t^{s,n-2})\big)\Big) \mathfrak{V}_t^s \, dt. \end{cases}$$

As before, we can use Doob's inequality of sub-martingales and Lemma 4.7 to obtain

$$\begin{aligned} &\mathbb{E} \sup_{t\in[t_1,\widetilde{t}]} \big|(\theta,\varpi)\big(\upsilon_t^{s,n}\big) - (\theta,\varpi)\big(\upsilon_t^{s,n-1}\big)\big|^2 \\ &\qquad \leqslant C(s_0,T) \int_{t_1}^{\widetilde{t}} \mathbb{E} \sup_{t\in[0,\tau]} \big|(\theta,\varpi)\big(\upsilon_t^{s,n-1}\big) - (\theta,\varpi)\big(\upsilon_t^{s,n-2}\big)\big|^2 \, d\tau. \end{aligned} \tag{5.61}$$

Iterate this inequality for n steps and then let $\widetilde{t} = t_2$. This, together with (5.60), implies

$$\big\|(\theta,\varpi)\big(\upsilon^{s,n}\big) - (\theta,\varpi)\big(\upsilon^{s,n-1}\big)\big\| \leqslant \frac{1}{n!} C(s_0,T)^n T^n.$$

Hence $(\theta,\varpi)(\upsilon^{s,n})$ converges in $\|\cdot\|$ with some limit $(\theta,\varpi)(\upsilon^s)$ which solves (5.43). We can also use (5.61) and Gronwall's Lemma to conclude the uniqueness of such υ_t^s.

For the existence of a continuous version of υ_t^s in the (t,s) parameter, we use Lemma 5.15. Let $\flat > 4$, $t, t' \in [t_1, T]$ with $t < t'$ and $s, s' \in [-s_0, s_0]$. Using (5.58) and Lemma 4.7, we deduce that

$$\mathbb{E}\big|(\theta,\varpi)(\upsilon_{t'}^{s'}) - (\theta,\varpi)(\upsilon_{t'}^{s})\big|^\flat \leqslant C(\flat,s_0,T)\big(|s'-s|^\flat + \int_{t_1}^{t'} \mathbb{E}\big|(\theta,\varpi)(\upsilon_\tau^{s'}) - (\theta,\varpi)(\upsilon_\tau^{s})\big|^\flat \, d\tau\big),$$

which, by Gronwall's lemma, implies

$$\mathbb{E}\big|(\theta,\varpi)(\upsilon_{t'}^{s'}) - (\theta,\varpi)(\upsilon_{t'}^{s})\big|^\flat \leqslant C(\flat,s_0,T)|s'-s|^\flat.$$

Similarly, it is true that

$$\begin{aligned} &\mathbb{E}\big|(\theta,\varpi)(\upsilon_{t'}^{s}) - (\theta,\varpi)(\upsilon_{t}^{s})\big|^\flat \\ &\leqslant C(\flat,s_0,T)\big(\big\||(\theta,\varpi)\upsilon^s|^{\frac{1}{2}\flat}\big\| + \big\||(\theta,\varpi)\mathsf{v}|^{\frac{1}{2}\flat}\big\| + \big\||(\theta,\varpi)\mathsf{v}|^{\frac{1}{2}\flat}\big\| \cdot \big\||(\mathfrak{V}^s)'_s|^{\frac{1}{2}\flat}\big\|\big)\big(|t'-t|^{\frac{1}{2}\flat} + |t'-t|^\flat\big) \\ &\leqslant C(\flat,s_0,T)|t'-t|^{\frac{1}{2}\flat}, \end{aligned}$$

where, to obtain the last inequality, we first show $\||(\mathfrak{V}^s)'_s|^{\frac{1}{2}\flat}\| < C(\flat,s_0,T)$ by (5.39) and then argue as for (5.59) to show $\||\mathsf{v}|^{\frac{1}{2}\flat}\|$, $\big\||\upsilon^s|^{\frac{1}{2}\flat}\big\|$ is also bounded by some $C(\flat,s_0,T)$. Thus,

$$\begin{aligned} 2^{1-\flat}\mathbb{E}\big|(\theta,\varpi)(\upsilon_{t'}^{s'}) - (\theta,\varpi)(\upsilon_{t}^{s})\big|^\flat \leqslant &\mathbb{E}\big|(\theta,\varpi)(\upsilon_{t'}^{s'}) - (\theta,\varpi)(\upsilon_{t'}^{s})\big| + \mathbb{E}\big|(\theta,\varpi)(\upsilon_{t'}^{s'}) - (\theta,\varpi)(\upsilon_{t'}^{s})\big|^\flat \\ \leqslant &C(\flat,s_0,T)\Big(|s'-s|^\flat + |t'-t|^{\frac{1}{2}\flat}\Big). \end{aligned}$$

By Lemma 5.15, there is a continuous modification of $(\theta,\varpi)(\upsilon_t^s)$ in the (t,s) parameter. Note that $(\theta,\varpi)_{\mathfrak{V}_t^s}$ varies continuously with respect to $\mathfrak{V}_t^s$, which is also continuous in the (t,s) parameter. So, we can also obtain a (t,s)-continuous version of the process υ_t^s. □

Remark 5.18. As we will see in the proof of Proposition 5.19, for almost all w,

$$\alpha^{s_1} \circ \alpha^{s_2}(\mathrm{w}) = \alpha^{s_1+s_2}(\mathrm{w}), \text{ for all } s_1, s_2 \in \mathbb{R}.$$

Hence, intuitively, $(\alpha^s)_{s\in\mathbb{R}}$ introduced a one-parameter of 'flow' maps on Brownian paths starting from the $o \in \mathbb{R}^m$. Consequently, $(\mathbf{F}^s)_{s\in\mathbb{R}}$ also behave like a one-parameter of 'flow' maps which satisfy the cocycle property $\mathbf{F}^{s_1} \circ \mathbf{F}^{s_2} = \mathbf{F}^{s_1+s_2}$ for any $s_1, s_2 \in \mathbb{R}$.

5.4. **Quasi-invariance property of $\mathbf{F}^s_y$.** Let $\mathrm{y}^s = \mathbf{F}^s_y\mathrm{y}$ be as in Theorem 5.17. We continue to study its distribution using the classical Cameron-Martin-Girsanov formula.

Let $(\mathrm{y}_t, \mathfrak{V}_t)$ be the stochastic process pair which defines the Brownian motion on $(\widetilde{M}, \widetilde{g})$ starting from y up to time T, i.e., $\mathrm{y}_t = \pi(\mathfrak{V}_t)$ and $\mathfrak{V}_t \in \mathcal{O}^{\widetilde{g}}(\widetilde{M})$ solves the Stranovich SDE

$$d\mathfrak{V}_t = \sum_{i=1}^m H(\mathfrak{V}_t, e_i) \circ dB^i_t(\mathrm{w}), \ \forall t \in [0,T].$$

By an abuse of notation, we continue to use $\mathbb{P}_y$ to denote the Brownian distribution in $C_y([0,T], \widetilde{M})$ (i.e., the distribution of $(\mathrm{y}_t)_{t\in[0,T]}$) and use Q to denote the distribution of $(B_t)_{t\in[0,T]}$ in $C_o([0,T], \mathbb{R}^m)$. Using the Itô map, we have the relation

$$B = (\mathcal{I}_{\mathfrak{V}_0})^{-1}(\mathrm{y}) \text{ and } \mathbb{P}_y = \mathrm{Q} \circ (\mathcal{I}_{\mathfrak{V}_0})^{-1}.$$

Similarly, let $\mathbb{P}_{F^s y}$ denote the Brownian motion distribution on $C_{F^s y}([0,T], \widetilde{M})$. Then

$$\mathbb{P}_{F^s y} = \mathrm{Q} \circ (\mathcal{I}_{\mathfrak{V}^s_0})^{-1}.$$

Let y^s and α^s be the one-parameter of stochastic processes on $\widetilde{M}$ and in $\mathbb{R}^m$, respectively, that we obtained in Theorem 5.17. They are related by the identity

$$\alpha^s = (\mathcal{I}_{\mathfrak{V}^s_0})^{-1}(\mathrm{y}^s).$$

Let $\mathbb{P}^s, \mathrm{Q}^s$ be the distributions of y^s, α^s, respectively, where $\mathrm{Q}^0 \equiv \mathrm{Q}$. Then

$$\mathbb{P}^s = \mathrm{Q}^s \circ (\mathcal{I}_{\mathfrak{V}^s_0})^{-1}. \tag{5.62}$$

To compare $\mathbb{P}^s$ with $\mathbb{P}_{F^s y}$, it suffices to compare Q^s with Q^0, which can be understood by a simple application of the Euclidean Cameron-Martin-Girsanov formula.

Proposition 5.19. *The distribution Q^s is equivalent to Q^0 with*

$$\frac{d\mathrm{Q}^s}{d\mathrm{Q}^0}(\mathrm{w}) = e^{\left\{\frac{1}{2}\int_0^T \langle \mathbf{g}^s_t(\alpha^{-s}(\mathrm{w})),\ dB_t(\mathrm{w})\rangle - \frac{1}{4}\int_0^T |\mathbf{g}^s_t(\alpha^{-s}(\mathrm{w}))|^2\ dt\right\}}. \tag{5.63}$$

Consequently, the distribution $\mathbb{P}^s$ is equivalent to the Brownian distribution $\mathbb{P}_{F^s y}$ with

$$\frac{d\mathbb{P}^s}{d\mathbb{P}_{F^s y}}(\beta) = \frac{d\mathrm{Q}^s}{d\mathrm{Q}^0}\left((\mathcal{I}_{\mathfrak{V}^s_0})^{-1}(\beta)\right), \ \beta \in C_{F^s y}([0,T], \widetilde{M}). \tag{5.64}$$

Proof. We follow the proof of [**Hs3**, Theorem 3.5]. Clearly, (5.64) follows from (5.63) by using the identity (5.62). For (5.63), recall that Q^s is the distribution of $(\alpha_t^s)_{t\in[0,T]}$, where

$$\alpha_t^s(\mathrm{w}) = \int_0^t O_\tau^s(\mathrm{w})\ dB_\tau(\mathrm{w}) + \int_0^t \mathrm{g}_\tau^s(\mathrm{w})\ d\tau.$$

The process $\int_0^t O_\tau^s\ dB_\tau$ has the same Brownian distribution as B_t since O^s are orthogonal frames and the distribution of a Euclidean Brownian motion is invariant under orthogonal transfers. So α^s only differs from a Brownian motion by a drift term $\int_0^t \mathrm{g}_\tau^s\ d\tau$. Let

$$\mathrm{M}_t^s(\mathrm{w}) := e^{\left\{-\frac{1}{2}\int_0^t \langle \mathrm{g}_\tau^s(\mathrm{w}), O_\tau^s(\mathrm{w}) dB_\tau(\mathrm{w})\rangle - \frac{1}{4}\int_0^t |\mathrm{g}_\tau^s(\mathrm{w})|^2\ d\tau\right\}}$$

and consider a new distribution $\widetilde{\mathrm{Q}}^s$ on $C_0([0,T],\mathbb{R}^m)$ which is given by

$$\frac{d\widetilde{\mathrm{Q}}^s}{d\mathrm{Q}}(\mathrm{w}) = \mathrm{M}_T^s(\mathrm{w}).$$

Since $|\mathrm{g}^s|$ is bounded from above by a multiple of $s\cdot\sup|\mathrm{V}|$, the Novikov's condition is satisfied. Hence the classical Carmeron-Martin-Girsanov Theorem says that the distribution of α^s under $\widetilde{\mathrm{Q}}^s$ is the same as Q, i.e., for any measurable subset A of $C_0([0,T],\mathbb{R}^m)$,

$$\mathrm{Q}\left(\{\mathrm{w}\in A\}\right) = \widetilde{\mathrm{Q}}^s\left(\{\alpha^s(\mathrm{w})\in A\}\right),$$

which, by a change of variable, gives

$$\mathrm{Q}\left(\{\mathrm{w}\in A\}\right) = \mathrm{Q}^s\left(\{\mathrm{M}^s(\alpha^{-s}(\mathrm{w})):\ \mathrm{w}\in A\}\right).$$

Since A is arbitrary, this means Q and Q^s are equivalent and

$$\frac{d\mathrm{Q}^s}{d\mathrm{Q}}(\mathrm{w}) = \frac{1}{\mathrm{M}_T^s(\alpha^{-s}(\mathrm{w}))}. \tag{5.65}$$

Note that the process M_t^s satisfies the equation

$$d\mathrm{M}_t^s = -\frac{1}{2}\mathrm{M}_t^s\left\langle \mathrm{g}_t^s(\mathrm{w}), O_t^s(\mathrm{w}) dB_t(\mathrm{w})\right\rangle.$$

So, by Itô's formula,

$$-d\ln \mathrm{M}_t^s(\alpha^{-s}(\mathrm{w})) = \frac{1}{2}\langle \mathrm{g}_t^s(\alpha^{-s}(\mathrm{w})), O_t^s(\alpha^{-s}(\mathrm{w}))\ d\alpha_t^{-s}(\mathrm{w})\rangle + \frac{1}{4}|\mathrm{g}_t^s(\alpha^{-s}(\mathrm{w}))|^2\ dt, \tag{5.66}$$

where the second term of the right hand side of (5.66) has coefficient 1/4 since α_t^{-s} has variance $2t$. On the other hand, we have

$$\alpha^s\circ\alpha^{-s}(\mathrm{w}) = \mathrm{w} = B(\mathrm{w}),\ \text{for almost all w}. \tag{5.67}$$

(Because of (5.65), the composition $\alpha^{s_1}\circ\alpha^{s_2}$, $s_1, s_2\in\mathbb{R}$, is well-defined and has a continuous version in the parameter (s_1,s_2) using Kolmogorov's criterion as in Theorem 5.17. So, by the uniqueness of the α^s family and its continuous in s, we must have $\alpha^{s_1}\circ\alpha^{s_2} = \alpha^{s_1+s_2}$. In particular, (5.67) holds true.) Now, from (5.67), we deduce

$$O_t^s(\alpha^{-s}(\mathrm{w}))\ d\alpha^{-s}(\mathrm{w}) + \mathrm{g}_t^s(\alpha^{-s}(\mathrm{w}))\ dt = dB_t(\mathrm{w}).$$

So, (5.66) is also of the form

$$-d\ln \mathrm{M}_t^s(\alpha^{-s}(\mathrm{w})) = \frac{1}{2}\langle \mathfrak{g}_t^s(\alpha^{-s}(\mathrm{w})), dB_t(\mathrm{w})\rangle - \frac{1}{4}|\mathfrak{g}_t^s(\alpha^{-s}(\mathrm{w}))|^2\ dt$$

and hence

$$\frac{1}{\mathrm{M}_T^s(\alpha^{-s}(\mathrm{w}))} = e^{\left\{\frac{1}{2}\int_0^T\langle \mathfrak{g}_t^s(\alpha^{-s}(\mathrm{w})),\ dB_t(\mathrm{w})\rangle - \frac{1}{4}\int_0^T |\mathfrak{g}_t^s(\alpha^{-s}(\mathrm{w}))|^2\ dt\right\}}.$$

□

Proposition 5.20. *The probability $\mathbb{P}_{F^s y}\circ \mathbf{F}_y^s$ is absolutely continuous with respect to $\mathbb{P}_y$ and the Radon-Nikodyn derivative $d\mathbb{P}_{F^s y}\circ\mathbf{F}_y^s/d\mathbb{P}_y$ conditioned on $\mathrm{y}_T = x$ is L^q integrable for every $q \geqslant 1$, locally uniformly in the s parameter. Moreover, $d\mathbb{P}_{F^s y}\circ\mathbf{F}_y^s/d\mathbb{P}_y$ is differentiable in s with differential $\overline{\mathcal{E}_T^s}(d\mathbb{P}_{F^s y}\circ\mathbf{F}_y^s/d\mathbb{P}_y)$, where $\overline{\mathcal{E}_T^s}$ conditioned on $\mathrm{y}_T = x$ is also L^q integrable for every $q \geqslant 1$, locally uniformly in the s parameter.*

Proof. For $\mathbb{P}_y$ almost all path β, let $\mathrm{w} = \mathcal{I}_{\mathcal{O}_0}^{-1}(\beta)$. Then $\mathcal{I}_{\mathcal{O}_0^s}^{-1}\left(\mathbf{F}_y^s(\beta)\right) = \alpha^s(\mathrm{w})$. As a corollary of Proposition 5.19, we have $\mathbb{P}_{F^s y}\circ\mathbf{F}_y^s$ is equivalent to $\mathbb{P}_y$ with

$$\frac{d\mathbb{P}_{F^s y}\circ\mathbf{F}_y^s}{d\mathbb{P}_y}(\beta) = \frac{d\mathbb{P}_{F^s y}}{d\mathbb{P}_y\circ(\mathbf{F}_y^s)^{-1}}\circ\mathbf{F}_y^s(\beta) = \frac{d\mathbb{P}_{F^s y}}{d\mathbb{P}^s}\circ\mathbf{F}_y^s(\beta) = \frac{d\mathrm{Q}^0}{d\mathrm{Q}^s}\big(\alpha^s(\mathrm{w})\big).$$

Note that $d\alpha^s(\mathrm{w}) = O^s(\mathrm{w})dB(\mathrm{w}) + g^s(\mathrm{w})dt$. So, by (5.63) and (5.35), we have

$$\frac{d\mathbb{P}_{F^s y}\circ\mathbf{F}_y^s}{d\mathbb{P}_y}(\beta) = e^{\left\{-\frac{1}{2}\int_0^T\langle \overline{\mathfrak{g}}_\tau^s(\mathrm{w}),\ dB_\tau(\mathrm{w})\rangle + \frac{1}{4}\int_0^T |\overline{\mathfrak{g}}_\tau^s(\mathrm{w})|^2\ d\tau\right\}},$$

where

$$\overline{\mathfrak{g}}_t^s(\mathrm{w}) := \int_0^s [O_\tau^\jmath]^{-1}\left\{(\mathcal{O}_0^\jmath)^{-1}\left[\mathfrak{s}'(\tau)\mathrm{V}(F^\jmath y)\right] - \mathrm{Ric}\left(\mathcal{O}_\tau^\jmath(\mathcal{O}_0^\jmath)^{-1}[\mathfrak{s}(\tau)\mathrm{V}(F^\jmath y)]\right)\right\}\ d\jmath.$$

Put

$$\mathcal{E}_t^s(\mathrm{w}) := e^{\left\{-\frac{1}{2}\int_0^t\langle \overline{\mathfrak{g}}_\tau^s(\mathrm{w}),\ dB_\tau(\mathrm{w})\rangle + \frac{1}{4}\int_0^t |\overline{\mathfrak{g}}_\tau^s(\mathrm{w})|^2\ d\tau\right\}},\ \forall t\in[0,T].$$

For $q \geqslant 1$, we estimate $\mathbb{E}_{\mathbb{P}^*_{y,x,T}}|\mathcal{E}_T^s(\mathrm{w})|^q$. Let b_t be the Brownian motion with respect to the bridge distribution (from y to x in time T) as in Lemma 4.16 such that

$$dB_\tau(\mathrm{w}) = db_\tau(\mathrm{w}) + 2\mathcal{O}_\tau^{-1}\nabla\ln p(T-\tau,\mathrm{y}_\tau,x)\ d\tau.$$

Then conditioned on $\mathrm{y}_T = x$, $|\mathcal{E}_T^s(\mathrm{w})|^q$ has the same distribution as

$$e^{\{-\frac{1}{2}q\int_0^T\langle \overline{\mathfrak{g}}_\tau^s(\mathrm{w}),\ db_\tau(\mathrm{w})\rangle + \frac{1}{4}q\int_0^T |\overline{\mathfrak{g}}_\tau^s(\mathrm{w})|^2\ d\tau - q\int_0^T\langle \overline{\mathfrak{g}}_\tau^s(\mathrm{w}),\mathcal{O}_\tau^{-1}\nabla\ln p(T-\tau,\mathrm{y}_\tau,x)\rangle\ d\tau\}}.$$

So, by Hölder's inequality and the Cameron-Martin-Girsanov Theorem,

$$\begin{aligned}\mathbb{E}_{\mathbb{P}^*_{y,x,T}}|\mathcal{E}_T^s(\mathrm{w})|^q \leqslant& \sqrt{p(T,x,y)}\left[\mathbb{E}_{\mathbb{P}_{y,x,T}} e^{\{-\int_0^T q\langle \overline{\mathfrak{g}}_\tau^s(\mathrm{w}),\ db_\tau(\mathrm{w})\rangle - q^2\int_0^T |\overline{\mathfrak{g}}_\tau^s(\mathrm{w})|^2\ d\tau\}}\right]^{\frac{1}{2}}\\ &\cdot\left[\mathbb{E}_{\mathbb{P}^*_{y,x,T}} e^{\{-2q\int_0^T\langle \overline{\mathfrak{g}}_\tau^s,\mathcal{O}_\tau^{-1}\nabla\ln p(T-\tau,\mathrm{y}_\tau,x)\rangle\ d\tau + (\frac{1}{2}q+\frac{3}{2}q^2)\int_0^T |\overline{\mathfrak{g}}_\tau^s|^2\ d\tau\}}\right]^{\frac{1}{2}}\\ \leqslant& \sqrt{p(T,x,y)}\left[\mathbb{E}_{\mathbb{P}^*_{y,x,T}} e^{\{-2q\int_0^T\langle \overline{\mathfrak{g}}_\tau^s,\mathcal{O}_\tau^{-1}\nabla\ln p(T-\tau,\mathrm{y}_\tau,x)\rangle\ d\tau + (\frac{1}{2}q+\frac{3}{2}q^2)\int_0^T |\overline{\mathfrak{g}}_\tau^s|^2\ d\tau\}}\right]^{\frac{1}{2}}.\end{aligned} \tag{5.68}$$

Let us continue to use C to denote a constant depending on $\|g\|_{C^3}$ and the norm bound of V and use $C(\cdot)$ to indicate the extra coefficients it depends on. By our choice of $\mathbf{s}$ (see (5.24)), for $s \in [-s_0, s_0]$, $|\overline{\mathrm{g}}^s_\tau(\mathrm{w})| \leqslant C(s_0, T)$ for some $C(s_0, T)$. Report this in (5.68) and then use (4.36) and (4.37). We obtain some $C(q, s_0, T), \widetilde{C}(q, s_0, T)$ such that

$$\mathbb{E}_{\mathbb{P}^*_{y,x,T}} |\mathcal{E}^s_T(\mathrm{w})|^q \leqslant C(q, s_0, T) \left[\mathbb{E}_{\mathbb{P}^*_{y,x,T}} e^{\{\widetilde{C}(q,s_0,T) \int_0^T \|\nabla \ln p(T-\tau, \mathrm{y}_\tau, x)\| \, d\tau} \right]^{\frac{1}{2}},$$

which, by (4.38), shows that $d\mathbb{P}_{F^s y} \circ \mathbf{F}^s / d\mathbb{P}_y$ conditioned on $\mathrm{y}_T = x$ is L^q integrable for $q \geqslant 1$, locally uniformly in the s parameter.

Note that $\mathcal{E}^s_t(\mathrm{w})$ satisfies the SDE

$$d\mathcal{E}^s_t(\mathrm{w}) = \mathcal{E}^s_t(\mathrm{w}) \left(-\frac{1}{2} \langle \overline{\mathrm{g}}^s_t(\mathrm{w}), \, dB_t(\mathrm{w}) \rangle + \frac{1}{2} |\overline{\mathrm{g}}^s_t(\mathrm{w})|^2 \, dt \right).$$

The differential process $(\mathcal{E}^s_t(\mathrm{w}))'_s = (d\mathcal{E}^J_t(\mathrm{w})/dJ)|_{J=s}$ exists and satisfies the Itô SDE

$$\begin{aligned} d(\mathcal{E}^s_t)'_s(\mathrm{w}) =& (\mathcal{E}^s_t)'_s(\mathrm{w}) \left(-\frac{1}{2} \langle \overline{\mathrm{g}}^s_t(\mathrm{w}), \, dB_t(\mathrm{w}) \rangle + \frac{1}{2} |\overline{\mathrm{g}}^s_t(\mathrm{w})|^2 \, dt \right) \\ &+ \mathcal{E}^s_t(\mathrm{w}) \left(-\frac{1}{2} \langle (\overline{\mathrm{g}}^s_t)'_s(\mathrm{w}), dB_\tau(\mathrm{w}) \rangle + \langle (\overline{\mathrm{g}}^s_t)'_s(\mathrm{w}), \overline{\mathrm{g}}^s_t(\mathrm{w}) \rangle \, d\tau \right). \end{aligned}$$

Hence the Radon-Nikodyn derivative $d\mathbb{P}_{F^s y} \circ \mathbf{F}^s_y / d\mathbb{P}_y$ is differentiable in s with differential $(\mathcal{E}^s_T)'_s(\mathrm{w})$, which, by using stochastic Duhamel principle (or Itô's formula), is

$$\begin{aligned} (\mathcal{E}^s_T)'_s =& \ \mathcal{E}^s_T \cdot \left((\mathcal{E}^s_0)'_s - \frac{1}{2} \int_0^T \langle (\overline{\mathrm{g}}^s_t)'_s(\mathrm{w}), dB_\tau(\mathrm{w}) \rangle + \frac{1}{2} \int_0^T \langle (\overline{\mathrm{g}}^s_t)'_s(\mathrm{w}), \overline{\mathrm{g}}^s_t(\mathrm{w}) \rangle \, d\tau \right) \\ =& \ \mathcal{E}^s_T \cdot \left(-\frac{1}{2} \int_0^T \langle (\overline{\mathrm{g}}^s_t)'_s(\mathrm{w}), dB_t(\mathrm{w}) \rangle + \frac{1}{2} \int_0^T \langle (\overline{\mathrm{g}}^s_t)'_s(\mathrm{w}), \overline{\mathrm{g}}^s_t(\mathrm{w}) \rangle \, dt \right) \\ =:& \ \mathcal{E}^s_T \cdot \overline{\mathcal{E}}^s_T. \end{aligned}$$

Conditioned on $\mathrm{y}_T = x$, $\overline{\mathcal{E}}^s_T$ has the same distribution as

$$-\frac{1}{2} \int_0^T \langle (\overline{\mathrm{g}}^s_\tau)'_s, db_\tau \rangle + \frac{1}{2} \int_0^T \langle (\overline{\mathrm{g}}^s_\tau)'_s, \overline{\mathrm{g}}^s_\tau \rangle \, d\tau - \int_0^T \left\langle (\overline{\mathrm{g}}^s_\tau)'_s, \mathfrak{V}^{-1}_\tau \nabla \ln p(T-\tau, \mathrm{y}_\tau, x) \right\rangle d\tau,$$

where both $|\overline{\mathrm{g}}^s_t|$ and $|(\overline{\mathrm{g}}^s_t)'_s|$ are bounded by some constant $C(s_0, T)$. Hence, by Hölder's inequality and (4.22), we compute that

$$\begin{aligned} \left(\mathbb{E}_{\mathbb{P}_{y,x,T}} |\overline{\mathcal{E}}^s_T|^q \right)^2 \leqslant 3^{2q-1} \Bigg(& \mathbb{E}_{\mathbb{P}_{y,x,T}} \left| \int_0^T \langle (\overline{\mathrm{g}}^s_\tau)'_s, db_\tau \rangle \right|^{2q} + (C(s_0, T))^{2q} \\ & + \mathbb{E}_{\mathbb{P}_{y,x,T}} \left| \int_0^T \left\langle (\overline{\mathrm{g}}^s_\tau)'_s, \mathfrak{V}^{-1}_\tau \nabla \ln p(T-\tau, \mathrm{y}_\tau, x) \right\rangle d\tau \right|^{2q} \Bigg) \\ =: 3^{2q-1} & \left(\overline{(\mathrm{I})} + \overline{(\mathrm{II})} + \overline{(\mathrm{III})} \right). \end{aligned}$$

Since b is a Brownian motion with respect to $\mathbb{P}_{y,x,T}$, by Lemma 4.7,

$$\overline{(\mathrm{I})} \leqslant \mathtt{C}_1(2q) \int_0^T \mathbb{E}_{\mathbb{P}_{y,x,T}} |(\overline{\mathfrak{g}}_\tau^s)'_s|^{2q} \, d\tau \leqslant C(q, s_0, T),$$

where $\mathtt{C}_1$ is from (4.21). Using $|(\overline{\mathfrak{g}}_\tau^s)'_s \mathrm{w})| \leqslant C(s_0, T)$ and (4.38), we obtain

$$\begin{aligned}
\overline{(\mathrm{III})} \leqslant & C(s_0,T)^{2q} \mathbb{E}_{\mathbb{P}_{y,x,T}} \Big(\int_0^T \|\nabla \ln p(T-\tau, \mathrm{y}_\tau, x)\| \, d\tau \Big)^{2q} \\
\leqslant & C(s_0,T)^{2q} \mathbb{E}_{\mathbb{P}_{y,x,T}} e^{\left\{2q \int_0^T \|\nabla \ln p(T-\tau, \mathrm{y}_\tau, x)\| \, d\tau\right\}} \\
\leqslant & C(s_0,T)^{2q} (p(T,x,y))^{-1} e^{c'(1+d(x,y))}.
\end{aligned}$$

Putting all the estimations on $\overline{(\mathrm{I})}$, $\overline{(\mathrm{II})}$, $\overline{(\mathrm{III})}$ together, we conclude that $\overline{\mathcal{E}_T^s}$ conditioned on $\mathrm{y}_T = x$ is L^q integrable for $q \geqslant 1$, locally uniformly in the s parameter. □

Consider the distribution of $\overline{\mathbb{P}}_x$ on $C_x([0,T], \widetilde{M})$. Let (x, u) be the stochastic pair which defines the Brownian motion on $(\widetilde{M}, \tilde{g})$ which starts from x. The distribution of $(\mathrm{x}_t)_{t\in[0,T]}$ is independent of the choice of u_0. Hence $\mathbb{P}_{(x,\mathrm{u}_0)}$, which is the distribution of $(\mathrm{x}_t)_{t\in[0,T]}$ with a initial frame u_0, coincides with $\mathbb{P}_x$ on $C_x([0,T], \widetilde{M})$ and $\mathbb{P}_{(x,\mathrm{u}_0),y,T} := \mathbb{E}_{\mathbb{P}_{(x,\mathrm{u}_0)}} \left(\cdot \middle| \mathrm{x}_T = y\right)$ coincides with $\mathbb{P}_{x,y,T}$ on $C_{x,y}([0,T], \widetilde{M})$. This means

$$\begin{aligned}
\overline{\mathbb{P}}_x &= \int\int \mathbb{P}_{(x,\mathrm{u}_0),y,T} \cdot p(T,x,y) \, d\mathrm{Vol}(y) \, d\mathrm{Vol}(\mathrm{u}_0) \left(= \int\int \mathbb{P}_{x,y,T} \cdot p(T,x,y) \, d\mathrm{Vol}(\mathrm{u}_0) \, d\mathrm{Vol}(y) \right) \\
&= \int\int \mathbb{P}_{(x,\mathrm{u}_0),y,T} \cdot p(T,x,y) \, d\mathrm{Vol}(\mathrm{u}_0) \, d\mathrm{Vol}(y),
\end{aligned}$$

where $d\mathrm{Vol}(\mathrm{u}_0)$ is the uniform distribution on $\mathcal{O}_x^{\tilde{g}}(\widetilde{M})$. For any $y \in \widetilde{M}$, the Brownian bridge process connecting x and y in time T has the following symmetric property.

Lemma 5.21. *Let $(X_t, U_t)_{t\in[0,T]}$ be the pair of stochastic processes for Brownian bridge from x to y in time T.*

i) *Under $\mathbb{P}_{x,y,T}$, the process $(X_{T-t})_{t\in[0,T]}$ has the law $\mathbb{P}_{y,x,T}$.*
ii) *If U_0 is chosen randomly with the uniform distribution in $\mathcal{O}_x^{\tilde{g}}(\widetilde{M})$, then U_T is also uniformly distributed in $\mathcal{O}_y^{\tilde{g}}(\widetilde{M})$.*

Proof. i) is [**Hs3**, Proposition 5.4.3]. (It is true since by (4.40), the finite margin of X, or the joint density function of $X_{t_1}, \cdots, X_{t_n}$, $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = T$, is given by

$$\frac{1}{p(T,x,y)} \prod_{i=0}^{n} p(t_{i+1} - t_i, x_i, x_{i+1}), \text{ where } x_0 = x, x_{n+1} = y,$$

which is the same as the joint density function of $\widetilde{X}_{T-t_n}, \cdots, \widetilde{X}_{T-t_1}$ of the bridge $\widetilde{X}$ from y to x in time T.) For ii), we consider (4.41). Note that the distribution of the $\mathbb{R}^m$-Brownian

motion b_t is invariant under rotations. So if $(U_t)_{t\in[0,T]}$ solves (4.41) with initial frame U_0, then for $\widetilde{U}_0 = U_0\upsilon$ with $\upsilon \in \mathcal{O}(\mathbb{R}^m)$, $(\widetilde{U}_t = U_t\upsilon)_{t\in[0,T]}$ solves (4.41). This implies ii). □

Let $\mathbf{F}^s_y$ be as in Theorem 5.17. It induces a map from $C_{y,x}([0,T],\widetilde{M})$ to $C_{F^s y,x}([0,T],\widetilde{M})$. We define $\mathbf{F}^s$ on $C_x([0,T],\widetilde{M})$ conditioned on the value of β_T by letting

$$\mathbf{F}^s(\beta) := \mathbf{F}^s_{\beta_T}(\beta).$$

By Lemma 5.21, a uniform random choice of u_0 at x will result in a uniform distribution of u_T at y for the Brownian bridge connecting x and y in time T. Therefore, to analyze $\overline{\mathbb{P}}_x \circ \mathbf{F}^s$, we can choose the initial $\mathcal{O}_0 \in \mathcal{O}^{\tilde{g}}_{\beta_T}(\widetilde{M})$ with a uniform distribution to define $\mathbf{F}^s_{\beta_T}$.

Lemma 5.22. *For $\overline{\mathbb{P}}_x$ almost all $\beta \in C_x([0,T],\widetilde{M})$,*

$$\frac{d\overline{\mathbb{P}}_x \circ \mathbf{F}^s}{d\overline{\mathbb{P}}_x}(\beta) = \frac{d\mathbb{P}_{F^s\beta_T} \circ \mathbf{F}^s}{d\mathbb{P}_{\beta_T}}(\beta) \cdot \frac{d\mathrm{Vol}(F^s\beta_T)}{d\mathrm{Vol}(\beta_T)}. \tag{5.69}$$

Proof. Lemma 5.21 implies that the distribution of u_T is uniform if u_0 is. So if we disintegrate $\overline{\mathbb{P}}_x$ according to the value of $(\mathrm{x}_T, \mathrm{u}_T)$, we obtain

$$\overline{\mathbb{P}}_x = \int\int \mathbb{P}_{(x,\mathrm{u}_0),y,T} \cdot p(T,x,y) d\mathrm{Vol}(\mathrm{u}_0)\, d\mathrm{Vol}(y) = \int\int \overline{\mathbb{P}}_{x,(y,\mathcal{O}_0),T} \cdot p(T,x,y)\, d\mathrm{Vol}(\mathcal{O}_0)\, d\mathrm{Vol}(y),$$

where $d\mathrm{Vol}(\mathcal{O}_0)$ is the uniform probability on $\mathcal{O}^{\tilde{g}}_y(\widetilde{M})$. For any measurable subset $A \subset C_x([0,T],\widetilde{M})$, by the change of variable formula,

$$\begin{aligned}\overline{\mathbb{P}}_x\left(\mathbf{F}^s(A)\right) &= \int\int \overline{\mathbb{P}}_{x,(F^s y, F^s\mathcal{O}_0),T}\left(\mathbf{F}^s(A)\right) \cdot p(T,x,F^s y)\, d\mathrm{Vol}(F^s\mathcal{O}_0)\, d\mathrm{Vol}(F^s y) \\ &= \int\int \overline{\mathbb{P}}_{x,(F^s y, F^s\mathcal{O}_0),T}\left(\mathbf{F}^s(A)\right) \cdot p(T,x,F^s y) \frac{d\mathrm{Vol}\circ F^s}{d\mathrm{Vol}}(y)\, d\mathrm{Vol}(\mathcal{O}_0)\, d\mathrm{Vol}(y).\end{aligned}$$

By Lemma 5.21, the distribution of $\overline{\mathbb{P}}_{x,(F^s y,F^s\mathcal{O}_0),T}$ on $C_{x,F^s y}([0,T],\widetilde{M}) \equiv C_{F^s y,x}([0,T],\widetilde{M})$ can be identified with that of $\mathbb{P}_{(F^s y,F^s\mathcal{O}_0),x,T}$, the Brownian bridge from $F^s y$ to x in time T with the initial frame $F^s\mathcal{O}_0 \in \mathcal{O}^{\tilde{g}}_{F^s y}(\widetilde{M})$. Hence

$$\overline{\mathbb{P}}_x(\mathbf{F}^s(A)) = \int\int \mathbb{P}_{(F^s y,F^s\mathcal{O}_0),x,T}(\mathbf{F}^s(A)) \cdot p(T,F^s y,x) \frac{d\mathrm{Vol}\circ F^s}{d\mathrm{Vol}}(y)\, d\mathrm{Vol}(\mathcal{O}_0)\, d\mathrm{Vol}(y). \tag{5.70}$$

The absolute continuity of $\overline{\mathbb{P}}_x \circ \mathbf{F}^s$ with respect to $\overline{\mathbb{P}}_x$ will follow if $\mathbb{P}_{(F^s y,F^s\mathcal{O}_0),x,T} \circ \mathbf{F}^s$ is absolutely continuous with respect to $\mathbb{P}_{(y,\mathcal{O}_0),x,T}$ and the Radon-Nikodym derivative $d\mathbb{P}_{(F^s y,F^s\mathcal{O}_0),x,T} \circ \mathbf{F}^s / d\mathbb{P}_{(y,\mathcal{O}_0),x,T}$ is integrable. Since the bridge process from y to x in time T is just the conditional process of y on $\mathrm{y}_T = x$, Lemma 5.19 implies that $\mathbb{P}_{(F^s y,F^s\mathcal{O}_0),x,T} \circ \mathbf{F}^s$ is absolutely continuous with respect to $\mathbb{P}_{(y,\mathcal{O}_0),x,T}$.

As to (5.69), we see that for any measurable set $\widetilde{A} \subset C_y([0,T], \widetilde{M})$,

(5.71)
$$\mathbb{P}_{F^s y} \circ \mathbf{F}^s(\widetilde{A}) = \mathbb{P}_y \left(\chi_{\widetilde{A}} \cdot \frac{d\mathbb{P}_{F^s y} \circ \mathbf{F}^s}{d\mathbb{P}_y} \right) = \int \mathbb{P}_{y,z,T} \left(\chi_{\widetilde{A}_z} \cdot \frac{d\mathbb{P}_{F^s y} \circ \mathbf{F}^s}{d\mathbb{P}_y} \right) p(T,y,z)\ d\mathrm{Vol}(z),$$

where $\widetilde{A}_z$ is the collection of elements $w \in \widetilde{A}$ with $w_T = z$. On the other hand,

$$\begin{aligned} \mathbb{P}_{F^s y} \circ \mathbf{F}^s(\widetilde{A}) &= \int \mathbb{P}_{F^s y, z, T} \circ \mathbf{F}^s(\chi_{\widetilde{A}_z}) \cdot p(F^s y, z, T)\ d\mathrm{Vol}(z) \\ &= \int \mathbb{P}_{y,z,T} \left(\chi_{\widetilde{A}_z} \cdot \frac{d\mathbb{P}_{F^s y, z, T} \circ \mathbf{F}^s}{d\mathbb{P}_{y,z,T}} \right) p(T, F^s y, z)\ d\mathrm{Vol}(z). \end{aligned} \tag{5.72}$$

Since $\widetilde{A}$ is arbitrary, we conclude from (5.71) and (5.72) that

$$\frac{d\mathbb{P}_{F^s y, z, T} \circ \mathbf{F}^s}{d\mathbb{P}_{y,z,T}} = \frac{d\mathbb{P}_{F^s y} \circ \mathbf{F}^s}{d\mathbb{P}_y} \cdot \frac{p(T,y,z)}{p(T, F^s y, z)}.$$

Reporting this in (5.72) and (5.70) shows (5.69) for $\overline{\mathbb{P}}_x$ almost all β with $\beta_T = y \in \widetilde{M}$. □

An immediate corollary of Proposition 5.20 and Lemma 5.22 is

Proposition 5.23. *The probability $\overline{\mathbb{P}}_x \circ \mathbf{F}^s$ is absolutely continuous with respect to $\overline{\mathbb{P}}_x$ and the Radon-Nikodyn derivative $d\overline{\mathbb{P}}_x \circ \mathbf{F}^s / d\overline{\mathbb{P}}_x$ conditioned on $\mathrm{x}_T = y$ is L^q integrable for every $q \geqslant 1$, locally uniformly in the s parameter. The differential of $d\overline{\mathbb{P}}_x \circ \mathbf{F}^s / d\overline{\mathbb{P}}_x$ in s exists and is of the form $\overleftarrow{\mathcal{E}^s_T} \cdot (d\overline{\mathbb{P}}_x \circ \mathbf{F}^s / d\overline{\mathbb{P}}_x)$, where $\overleftarrow{\mathcal{E}^s_T}$ conditioned on $\mathrm{x}_T = y$ is square integrable, locally uniformly in the s parameter.*

Using (5.69) and the proof of Proposition 5.20, we can deduce that $\overleftarrow{\mathcal{E}^s_T}$ differs from $\overline{\mathcal{E}^s_T}$ by the differential of $d\mathrm{Vol}(F^s y)/d\mathrm{Vol}(y)$ in the s parameter, where $\overline{\mathcal{E}^s_T}$ can be understood as a backward stochastic integral on the bridge paths from x to y in time T.

5.5. **The extended map $\mathbf{F}^s$.** In order to show the properties iii), iv) of $\mathbf{F}^s$ in Section 5.1, we need to clarify $(D\pi(\lfloor \mathrm{u}_T \rceil^\lambda)^{(1)}_\lambda) \circ \mathbf{F}^s$ for $\Phi^1_\lambda \circ \mathbf{F}^s$, where Φ^1_λ is as in (5.14). We will achieve this by extending $\mathbf{F}^s$ to the process $(\lfloor \mathrm{u}_T \rceil^\lambda)^{(1)}_\lambda$ and letting

$$\left(D\pi(\lfloor \mathrm{u}_T \rceil^\lambda)^{(1)}_\lambda \right) \circ \mathbf{F}^s := D\pi \left((\lfloor \mathrm{u}_T \rceil^\lambda)^{(1)}_\lambda \circ \mathbf{F}^s \right).$$

The rough idea is that the maps $\mathbf{F}^s$ on orbits extend naturally to their tangent maps for the parallel transportations and hence can be defined for the objects they make.

We first deal with $(\lfloor \mathrm{u}_T \rceil^\lambda)^{(1)}_0 \circ \mathbf{F}^s$. Let $\lambda \mapsto \lfloor \mathrm{u}_0 \rceil^\lambda \in \mathcal{O}^{\tilde{g}^\lambda}(\widetilde{M})$ be C^{k-2} in $\mathcal{F}(\widetilde{M})$ and let $(\lfloor \mathrm{u}_t \rceil^\lambda \in \mathcal{O}^{\tilde{g}^\lambda}(\widetilde{M}))_{t \in [0,T]}$ with initials $\lfloor \mathrm{u}_0 \rceil^\lambda$ be the unique solution to

$$d\lfloor \mathrm{u}_t \rceil^\lambda = \sum_{i=1}^m H^\lambda(\lfloor \mathrm{u}_t \rceil^\lambda, e_i) \circ dB^i_t(w),\ \forall t \in [0,T]. \tag{5.73}$$

By Lemma 4.18, there is a version of $\{\lfloor \mathrm{u}_t \rceil^\lambda\}$ such that $\lambda \mapsto \lfloor \mathrm{u}_t \rceil^\lambda(w)$ is C^{k-2} in λ for almost all w. By Lemma 4.21, the differential process $(\lfloor \mathrm{u}_t \rceil^\lambda)_0^{(1)}$ is given by

(5.74)
$$(\lfloor \mathrm{u}_T \rceil^\lambda)_0^{(1)} = \left[D\overrightarrow{F}_{0,T}(\mathrm{u}_0, w)\right](\lfloor \mathrm{u}_0 \rceil^\lambda)_0^{(1)} + \int_0^T \left[D\overrightarrow{F}_{t,T}(\mathrm{u}_t, w)\right](H^\lambda)_0^{(1)}(\mathrm{u}_t, e_i) \circ dB_t^i(w),$$

where $\mathrm{u} = \lfloor \mathrm{u} \rceil^0$ and $\{D\overrightarrow{F}_{\underline{t},\bar{t}}\}_{0\leqslant \underline{t}<\bar{t}\leqslant T}$ are the tangent maps of the flow maps $\{\overrightarrow{F}_{\underline{t},\bar{t}}\}_{0\leqslant \underline{t}<\bar{t}\leqslant T}$ associated to (5.73) at $\lambda = 0$ (the arrow is to indicate the time is recorded starting from x). By Lemma 4.4 (see also Lemma 5.13), the $\{D\overrightarrow{F}_{t,T}\}$ are determined by the paths $(\mathrm{x}_\tau(w) = \pi(\mathrm{u}_t(w)))_{\tau\in[0,T]}$ (or its anti-development in $\mathbb{R}^m$). Hence (5.74) shows that $(\lfloor \mathrm{u}_T \rceil^\lambda)_0^{(1)}(w)$ are objects completely determined by $(\mathrm{x}_\tau(w))_{\tau\in[0,T]}$, $(\lfloor \mathrm{u}_0 \rceil^\lambda)_0^{(1)}$ and $(H^\lambda)_0^{(1)}$.

By symmetry of the Brownian motion, we can describe the distribution of $(\lfloor \mathrm{u}_T \rceil^\lambda)_0^{(1)}$ conditioned on $\mathrm{x}_T = y$ using $(\mathrm{y}_t, \mathfrak{V}_t)_{t\in[0,T]}$, which is the stochastic pair defining the Brownian motion on $(\widetilde{M}, \widetilde{g})$ starting from y. The two path spaces $C_{y,x}([0,T], \widetilde{M})$ and $C_{x,y}([0,T], \widetilde{M})$ can be identified. Moreover, the distribution of y conditioned on $\mathrm{y}_T = x$ coincides with x conditioned on $\mathrm{x}_T = y$. This means for almost all such path $(\mathrm{y}_\tau)_{\tau\in[0,T]}(\mathrm{w}) =: \beta$, it is associated with a path $(\mathrm{x}_t)_{t\in[0,T]}(w) = (\beta_{T-\tau})_{\tau\in[0,T]} =: \overrightarrow{\beta}$. So the stochastic parallel transportation of u_t along $\overrightarrow{\beta}$ is well-defined and is given by

$$\mathrm{u}_t = \mathfrak{V}_{T-t}(\mathfrak{V}_T)^{-1}\mathrm{u}_0.$$

For any element $\mathrm{X} \in T_{\mathrm{u}_t}\mathcal{F}(\widetilde{M})$, let

$$(\theta, \varpi)_{\mathrm{u}_t}^{-1}\mathrm{X} =: (\mathrm{X}^1, \mathrm{X}^2).$$

Note that the orthonormal frames u_t and $\mathfrak{V}_{T-t}$ have the same footpoint $\mathrm{x}_t(\omega) = \mathrm{y}_{T-t}(\mathrm{w})$. Hence X also naturally corresponds to an element $\mathrm{Y}(\mathrm{X}) = \mathrm{Y}$ in $T_{\mathfrak{V}_{T-t}}\mathcal{F}(\widetilde{M})$ with

$$(\theta, \varpi)_{\mathfrak{V}_{T-t}}^{-1}\mathrm{Y} := \left(\mathfrak{V}_{T-t}^{-1}\mathrm{u}_t(\mathrm{X}^1), Ad(\mathfrak{V}_{T-t}^{-1}\mathrm{u}_t)(\mathrm{X}^2)\right).$$

We see that X and Y(X) are just the same vector expressed in different frame charts. Denote by Y this map which sends tangents $\mathrm{X} \in T_{\mathrm{u}_\tau}\mathcal{F}(\widetilde{M})$ to $\mathrm{Y}(\mathrm{X}) \in T_{\mathfrak{V}_{T-\tau}}\mathcal{F}(\widetilde{M})$ for any $\tau \in [0,T]$. Let $(F_{t_1,t_2})_{0\leqslant t_1<t_2\leqslant T}$ and $(DF_{t_1,t_2})_{0\leqslant t_1<t_2\leqslant T}$ be the invertible stochastic flow maps and tangent maps associated to y (cf. (5.40)). The following is true.

Lemma 5.24. *Let β, X, Y be introduced as above. Then for almost all β, we have*

$$\mathrm{Y}\left(D\overrightarrow{F}_{t,T}(\mathrm{u}_t, w)\mathrm{X}\right) = D(F_{0,T-t}(\mathfrak{V}_0, \mathrm{w}))^{-1}(\mathrm{Y}(\mathrm{X})) = \left[DF_{0,T-t}(\mathfrak{V}_0, \mathrm{w})\right]^{-1}(\mathrm{Y}(\mathrm{X})). \tag{5.75}$$

Proof. By Corollary 4.2, for almost all w, the maps $F_{0,T-t}(\cdot, \mathrm{w})$ are C^{k-2} diffeomorphisms. So for almost all w, the tangent maps $D(F_{0,T-t}(\mathfrak{V}_0, \mathrm{w}))^{-1}$ and $[DF_{0,T-t}(\mathfrak{V}_0, \mathrm{w})]^{-1}$ exist and are equal. For (5.75), it suffices to verify the first equality.

Write $(\theta, \varpi)_{\mathrm{u}_t}^{-1}\mathrm{X} =: (\mathrm{X}_t^1, \mathrm{X}_t^2)$ and let

$$(\mathrm{X}_\tau^1, \mathrm{X}_\tau^2) := (\theta, \varpi)_{\mathrm{u}_\tau}^{-1}D\overrightarrow{F}_{t,\tau}(\mathrm{u}_t, w)\mathrm{X}, \ \forall \tau \in [t,T].$$

It is true by Lemma 5.13 that

$$d\mathrm{X}^1_\tau = \mathrm{X}^2_\tau \circ dB_\tau(w), \tag{5.76}$$

$$d\mathrm{X}^2_\tau = (\mathrm{u}_\tau)^{-1} R\left(\mathrm{u}_\tau \circ dB_\tau(w), \mathrm{u}_\tau \mathrm{X}^1_\tau\right) \mathrm{u}_\tau. \tag{5.77}$$

Let

$$(\mathrm{Y}^1_\tau, \mathrm{Y}^2_\tau) := (\theta, \varpi)^{-1}_{\mathfrak{V}_{T-\tau}} \mathrm{Y}\left((\theta, \varpi)_{\mathrm{u}_\tau}(\mathrm{X}^1_\tau, \mathrm{X}^2_\tau)\right).$$

Note that $(\mathfrak{V}_{T-\tau})^{-1}\mathrm{u}_\tau \equiv (\mathfrak{V}_T)^{-1}\mathrm{u}_0$. So (5.76) gives

$$\begin{aligned} d\mathrm{Y}^1_\tau = (\mathfrak{V}_{T-\tau})^{-1}\mathrm{u}_\tau d\mathrm{X}^1_\tau &= (\mathfrak{V}_{T-\tau})^{-1}\mathrm{u}_\tau \mathrm{X}^2_\tau \circ dB_\tau(w) \\ &= -(\mathfrak{V}_{T-\tau})^{-1}\mathrm{u}_\tau \mathrm{X}^2_\tau \left((\mathfrak{V}_{T-\tau})^{-1}\mathrm{u}_\tau\right)^{-1} \circ (\mathfrak{V}_{T-\tau})^{-1}\mathrm{u}_\tau d\overrightarrow{B}_{T-\tau}(\mathrm{w}) \\ &= -\mathrm{Y}^2_\tau \circ d\overrightarrow{B}_{T-\tau}(\mathrm{w}), \end{aligned}$$

where $\circ\overrightarrow{B}_t(\mathrm{w})$ denote the backward Stranovich integral. Similarly, using (5.77), we obtain

$$\begin{aligned} d\mathrm{Y}^2_\tau &= ((\mathfrak{V}_{T-\tau})^{-1}\mathrm{u}_\tau) d\mathrm{X}^2_\tau ((\mathfrak{V}_{T-\tau})^{-1}\mathrm{u}_\tau)^{-1} \\ &= (\mathfrak{V}_{T-\tau})^{-1} R\left(\mathrm{u}_\tau \circ dB_\tau(w), \mathrm{u}_\tau \mathrm{X}^1_\tau\right) \mathfrak{V}_{T-\tau} \\ &= -(\mathfrak{V}_{T-\tau})^{-1} R\left(\mathfrak{V}_{T-\tau} \circ d\overrightarrow{B}_{T-\tau}(\mathrm{w}), \mathfrak{V}_{T-\tau}\mathrm{Y}^1_\tau\right) \mathfrak{V}_{T-\tau}. \end{aligned}$$

Altogether, we have

$$\begin{aligned} d\mathrm{Y}^1_\tau &= -\mathrm{Y}^2_\tau \circ d\overrightarrow{B}_{T-\tau}(\mathrm{w}), \\ d\mathrm{Y}^2_\tau &= -(\mathfrak{V}_{T-\tau})^{-1} R\left(\mathfrak{V}_{T-\tau} \circ d\overrightarrow{B}_{T-\tau}(\mathrm{w}), \mathfrak{V}_{T-\tau}\mathrm{Y}^1_\tau\right) \mathfrak{V}_{T-\tau} \end{aligned}$$

and the solution $(\mathrm{Y}^1_T, \mathrm{Y}^2_T)$ is exactly $(\theta, \varpi)_{\mathfrak{V}_0}\left(D(F_{0,T-t}(\mathfrak{V}_0, \mathrm{w}))^{-1}(\mathrm{Y}(\mathrm{X}))\right)$. □

As a corollary of (5.74) and Lemma 5.24, we have

Corollary 5.25. *Conditioned on $\mathrm{x}_T = y$, the distribution of $(\lfloor \mathrm{u}_T \rceil^\lambda)^{(1)}_0$ given by (5.74) is the same as, conditioned on $\mathrm{y}_T = x$, the distribution of*

$$\overline{(\lfloor \mathrm{u}_T \rceil^\lambda)^{(1)}_0} := [DF_{0,T}(\mathfrak{V}_0, \mathrm{w})]^{-1}(\lfloor \mathrm{u}_0 \rceil^\lambda)^{(1)}_0 - \int_0^T [DF_{0,t}(\mathfrak{V}_0, \mathrm{w})]^{-1}(H^\lambda)^{(1)}_0(\mathfrak{V}_t, e_i) \circ d\overrightarrow{B}^i_t(\mathrm{w}),$$

where $\circ d\overrightarrow{B}_t(\mathrm{w})$ is the backward Stratonovich infinitesimal.

Proof. Consider the mapping

$$(H^\lambda)^{(1)}_0(\mathrm{u}_t, \cdot) = (H^\lambda)^{(1)}_0(\mathfrak{V}_{T-t}(\mathfrak{V}_T)^{-1}\mathrm{u}_0, \cdot)$$

from $T_o\mathbb{R}^m$ to $T_{\mathrm{u}_t}\mathcal{F}(\widetilde{M})$. We have

$$(H^\lambda)^{(1)}_0(\mathrm{u}_t, e_i) \circ dB^i_t(w) = (H^\lambda)^{(1)}_0(\mathrm{u}_t, \circ dB_t(w))$$

and its correspondence at $T_{\mathfrak{V}_{T-t}}\mathcal{F}(\widetilde{M})$ is $-(H^\lambda)^{(1)}_0(\mathfrak{V}_{T-t}, \circ d\overrightarrow{B}_{T-t}(\mathrm{w}))$. So, by Lemma 5.24,

$$\left[D\overrightarrow{F}_{t,T}(\mathrm{u}_t, w)\right](H^\lambda)^{(1)}_0(\mathrm{u}_t, \circ dB_t(w)) = -[DF_{0,T-t}(\mathfrak{V}_0, \mathrm{w})]^{-1}(H^\lambda)^{(1)}_0(\mathfrak{V}_{T-t}, \circ d\overrightarrow{B}_{T-t}(\mathrm{w}))$$

and the conclusion follows by taking the integral with respect to t on $[0,T]$. □

Let α^s, y^s and $\mathfrak{V}^s$ be the processes obtained in Theorem 5.17. Let $(F^s_{t_1,t_2})_{0\leqslant t_1<t_2\leqslant T}$ be the parallel transportation stochastic flow of y^s and let $[DF^s_{t_1,t_2}(\mathfrak{V}^s_{t_1},\mathrm{w})]$ be the associated tangent maps. By Proposition 4.1, $[DF^s_{0,t}(\mathfrak{V}^s_t,\mathrm{w})]$ is invertible for almost all w. Hence the inverse maps $[DF^s_{0,t}(\mathfrak{V}^s_0,\mathrm{w})]^{-1}$ are well-defined. Corollary 5.25 shows the distribution of $(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_0(w)$ is the same as $\overline{(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_0}(\mathrm{w})$. We define

$$(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_0(w)\circ\mathbf{F}^s := \overline{(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_0}(\mathrm{w})\circ\mathbf{F}^s = \overline{(\lfloor \mathrm{u}^s_T\rceil^\lambda)^{(1)}_0}(\mathrm{w}),$$

where

$$\overline{(\lfloor \mathrm{u}^s_T\rceil^\lambda)^{(1)}_0} := [DF^s_{0,T}(\mathfrak{V}^s_0,\mathrm{w})]^{-1}(\lfloor \mathrm{u}_0\rceil^\lambda)^{(1)}_0 - \int_0^T [DF^s_{0,t}(\mathfrak{V}^s_0,\mathrm{w})]^{-1}(H^\lambda)^{(1)}_0(\mathfrak{V}^s_t,e_i)\circ d\overrightarrow{\alpha}^{s,i}_t(\mathrm{w}).$$

So the differentiability of $(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_0\circ\mathbf{F}^s$ in s will follow from the differentiability of $\overline{(\lfloor \mathrm{u}^s_T\rceil^\lambda)^{(1)}_0}$ in s, which is intuitively true by the differentiability of (in s) of α^s_t, $\mathfrak{V}^s_t$ and $[DF^s_{0,t}(\mathfrak{V}^s_0,\cdot)]^{-1}$. We will justify this and formulate $((\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_0\circ\mathbf{F}^s)'_s$ in the remaining part of this subsection.

Lemma 5.26. *Let $\alpha^s_t, O^s_t, \mathfrak{g}^s_t$, $\underline{\Upsilon}_{\mathrm{V},\alpha^s}$ and $\mathfrak{V}^s_t$ be as in Theorem 5.17. Fix $T_0>0$. For any $s_0>0$, $q\geqslant 1$ and $T>T_0$, there are constants $\underline{c}_{\mathrm{A}}$ (which depend on $s_0,q,\mathtt{s}$ and $\|g^0\|_{C^3}$) and c_{A} (which depends on q,T,T_0 and $\|g^0\|_{C^3}$) such that*

$$\sup_{s\in[-s_0,s_0]}\mathbb{E}_{\mathbb{P}^*_{y,x,T}}\sup_{t\in[0,T]}|\mathrm{A}|^q < \underline{c}_{\mathrm{A}}e^{c_{\mathrm{A}}(1+d_{\tilde{g}^\lambda}(x,y))}, \tag{5.78}$$

where $\mathrm{A}=\alpha^s_t,\ (O^s_t)'_s,\ (\mathfrak{g}^s_t)'_s,\ \underline{\Upsilon}_{\mathrm{V},\alpha^s},\ (\mathfrak{V}^s_t)'_s$ *or* $(\theta,\varpi)\big((\mathfrak{V}^s_t)'_s\big)$.

Proof. By our construction, $\alpha^s_t=\int_0^t O^s_\tau dB_\tau+\mathfrak{g}^s_\tau\,d\tau$, where $O^s\in\mathcal{O}(\mathbb{R}^m)$ and $|\mathfrak{g}^s|\leqslant \mathtt{c}s_0\sup|\mathrm{V}|$ for some $\mathtt{c}$ that bounds $\sup_{t\in[0,T]}\{|\mathtt{s}|,|\mathtt{s}'|\}$, $\sup\{\|\mathrm{Ric}\|\}$. So,

$$\begin{aligned}2^{1-q}\mathbb{E}_{\mathbb{P}^*_{y,x,T}}\sup_{t\in[0,T]}|\alpha^s_t|^q \leqslant&\mathbb{E}_{\mathbb{P}^*_{y,x,T}}\sup_{t\in[0,T]}\left\|\int_0^t O^s_\tau dB_\tau\right\|^q+\underline{c}_0T_0^{-m}(\mathtt{c}s_0T\sup|\mathrm{V}|)^qe^{c_0(1+T)}\\ =:&(\mathrm{I})+\underline{c}_0T_0^{-m}(\mathtt{c}s_0T\sup|\mathrm{V}|)^qe^{c_0(1+T)},\end{aligned}$$

where $\underline{c}_0,c_0$ are from (4.36). Let b be the Brownian motion in Lemma 4.16 for $\mathbb{P}_{y,x,T}$, i.e.,

$$dB_\tau = db_\tau + 2(\mathfrak{V}^0_\tau)^{-1}\nabla\ln p(T-\tau,\mathrm{y}^0_\tau,x)\,d\tau. \tag{5.79}$$

Then,

$$\begin{aligned}(\mathrm{I})&=\mathbb{E}_{\mathbb{P}^*_{y,x,T}}\sup_{t\in[0,T]}\left\|\int_0^t O^s_\tau\,db_\tau+2O^s_\tau(\mathfrak{V}^0_\tau)^{-1}\nabla\ln p(T-\tau,\mathrm{y}^0_\tau,x)\,d\tau\right\|^q\\ &\leqslant 2^{q-1}\mathbb{E}_{\mathbb{P}^*_{y,x,T}}\sup_{t\in[0,T]}\left\|\int_0^t O^s_\tau\,db_\tau\right\|^q+2^{2q-1}\mathbb{E}_{\mathbb{P}^*_{y,x,T}}\left|\int_0^T\big\|\nabla\ln p(T-t,\mathrm{y}^0_t,x)\big\|\,dt\right|^q\\ &=:(\mathrm{I})_1+(\mathrm{I})_2.\end{aligned}$$

For $(\mathrm{I})_1$, by successively using Doob's inequality of submartingale, Hölder's inequality and Burkholder's inequality, we obtain

$$\begin{aligned}
2^{1-q}(\mathrm{I})_1 &\leqslant \mathtt{C}(q)p(T,x,y)\mathbb{E}_{\mathbb{P}_{y,x,T}}\Big\|\int_0^T O_\tau^s\ db_\tau\Big\|^q \\
&\leqslant \mathtt{C}(q)p(T,x,y)\left(\mathbb{E}_{\mathbb{P}_{y,x,T}}\Big\|\int_0^T O_\tau^s\ db_\tau\Big\|^{2q}\right)^{\frac{1}{2}} \\
&\leqslant \ \underline{c}_0 T_0^{-m}\mathtt{C}(q)\mathtt{C}_1(q)\sqrt{T}^q e^{c_0(1+T)},
\end{aligned}$$

where $\mathtt{C}(q)=(q/q-1)^q$ and $\mathtt{C}_1(\cdot)$ is as in Lemma 4.7. For $(\mathrm{I})_2$, by Proposition 4.15,

$$2^{1-2q}(\mathrm{I})_2 \leqslant \mathbb{E}^*_{y,x,T}\left(e^{q\int_0^T\|\nabla\ln p(T-t,\mathrm{y}_t^0,x)\|\ dt}\right) < e^{c(1+d(x,y))},$$

where c is as in (4.38). Putting the estimations together, we obtain (5.78) for $\mathrm{A}=\alpha_t^s$.

Next, we consider (5.78) for $(O_t^s)'_s$, $(\mathtt{g}_t^s)'_s$. By (5.34) and (5.35), we have

$$\begin{aligned}
(O_t^s)'_s &= -K_{\mathrm{V},\alpha^s}(\tau)O_t^s, \\
(\mathtt{g}_t^s)'_s &= -K_{\mathrm{V},\alpha^s}(\tau)\mathtt{g}_t^s + (\mathcal{U}_0^s)^{-1}[\mathtt{s}'(t)\mathrm{V}(F^s y)] - \mathrm{Ric}\left(\mathcal{U}_t^s(\mathcal{U}_0^s)^{-1}[\mathtt{s}(t)\mathrm{V}(F^s y)]\right),
\end{aligned}$$

where

$$\begin{aligned}
K_{\mathrm{V},\alpha^s}(t) = &\int_0^t(\mathcal{U}_\tau^s)^{-1}R\left(\mathcal{U}_\tau^s d\alpha_\tau^s, \mathcal{U}_\tau^s(\mathcal{U}_0^s)^{-1}[\mathtt{s}(\tau)\mathrm{V}(F^s y)]\right)\mathcal{U}_\tau^s \\
&+\int_0^t(\mathcal{U}_\tau^s)^{-1}(\nabla(\mathcal{U}_\tau^s e_i)R)\big(\mathcal{U}_\tau^s e_i, \mathcal{U}_\tau^s(\mathcal{U}_0^s)^{-1}[\mathtt{s}(\tau)\mathrm{V}(F^s y)]\big)\mathcal{U}_\tau^s d\tau.
\end{aligned}$$

Since $O^s\in\mathcal{O}(\mathbb{R}^m)$, $|\mathtt{g}^s|\leqslant \mathtt{c}s_0\sup|\mathrm{V}|$ and all $|\mathtt{s}|,|\mathtt{s}'|,|\mathrm{V}|$ are uniformly bounded, it is clear that (5.78) holds for $(O_t^s)'_s,(\mathtt{g}_t^s)'_s$ if it holds for $K_{\mathrm{V},\alpha^s}(t)$. Using (5.79) and (4.22), we obtain

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}^*_{y,x,T}}\sup_{t\in[0,T]}|K_{\mathrm{V},\alpha^s}(t)|^q \\
&\leqslant\ 3^{q-1}\mathbb{E}_{\mathbb{P}^*_{y,x,T}}\sup_{t\in[0,T]}\left|\int_0^t(\mathcal{U}_\tau^s)^{-1}R\left(\mathcal{U}_\tau^s O_\tau^s db_\tau^s, \mathcal{U}_\tau^s(\mathcal{U}_0^s)^{-1}[\mathtt{s}(\tau)\mathrm{V}(F^s y)]\right)\mathcal{U}_\tau^s\right|^q \\
&+3^{q-1}(2\sup\|R\|\sup|\mathrm{V}|)^q\mathbb{E}_{\mathbb{P}^*{}_{y,x,T}}\left|\int_0^T\|\nabla\ln p(T-t,\mathrm{y}_t^0,x)\|\ dt\right|^q \\
&+3^{q-1}(\sup\|\nabla R\|\sup|\mathrm{V}|s_0T)^q,
\end{aligned}$$

which has the same bound type as in (5.78) by a computation similar to the one for (I).

To verify (5.78) for $\underline{\Upsilon}_{\mathrm{V},\alpha^s}$, it suffices to check it for $\mathsf{K}_t:=\int_0^t\langle K_{\mathrm{V},\alpha^s}(\tau),d\alpha_\tau^s\rangle$ since

$$\underline{\Upsilon}_{\mathrm{V},\alpha^s}(t)=\int_0^t(\mathcal{U}_0^s)^{-1}[\mathtt{s}'(\tau)\mathrm{V}(F^s y)]-\mathrm{Ric}\left(\mathcal{U}_\tau^s(\mathcal{U}_0^s)^{-1}[\mathtt{s}(\tau)\mathrm{V}(F^s y)]\right)-\mathsf{K}_t.$$

By (5.79) and (4.22),

$$\begin{aligned}3^{1-q}\mathbb{E}_{\mathbb{P}^*_{y,x,T}}\sup_{t\in[0,T]}|\mathsf{K}_t|^q \leqslant\ & \mathbb{E}_{\mathbb{P}^*_{y,x,T}}\sup_{t\in[0,T]}\left|\int_0^t\langle K_{\mathrm{V},\alpha^s}(\tau),O^s_\tau db_\tau\rangle\right|^q\\ &+\mathbb{E}_{\mathbb{P}^*_{y,x,T}}\sup_{t\in[0,T]}\left|\int_0^t\langle K_{\mathrm{V},\alpha^s}(\tau),2\mathfrak{V}^{-1}_\tau\nabla\ln p(T-\tau,\mathrm{y}^0_\tau,x)\ d\tau\rangle\right|^q\\ &+\mathbb{E}_{\mathbb{P}^*_{y,x,T}}\sup_{t\in[0,T]}\left|\int_0^t\langle K_{\mathrm{V},\alpha^s}(\tau),\mathfrak{g}^s_\tau\ d\tau\rangle\right|^q\\ =:\ &(\mathrm{II})_1+(\mathrm{II})_2+(\mathrm{II})_3.\end{aligned}$$

For $(\mathrm{II})_1$, it is routine to apply successively Hölder's inequality, Doob's inequality of submartingale and Burkholder's inequality, which gives

$$\begin{aligned}((\mathrm{II})_1)^2\leqslant& p(T,x,y)\mathbb{E}_{\mathbb{P}^*_{y,x,T}}\sup_{t\in[0,T]}\left|\int_0^t\langle K_{\mathrm{V},\alpha^s}(\tau),O^s_\tau db_\tau\rangle\right|^{2q}\\ \leqslant&\mathtt{C}(2q)p(T,x,y)\mathbb{E}_{\mathbb{P}_{y,x,T}}\left|\int_0^T\langle K_{\mathrm{V},\alpha^s}(\tau),O^s_\tau db_\tau\rangle\right|^{2q}\\ \leqslant&\mathtt{C}(2q)\mathtt{C}_1(2q)p(T,x,y)\mathbb{E}_{\mathbb{P}_{y,x,T}}\left|\int_0^T\|\langle K_{\mathrm{V},\alpha^s}(\tau),O^s_\tau\rangle\|^2\ d\tau\right|^q\\ \leqslant&\mathtt{C}(2q)\mathtt{C}_1(2q)T^q\mathbb{E}_{\mathbb{P}^*_{y,x,T}}\sup_{\tau\in[0,T]}|K_{\mathrm{V},\alpha^s}(\tau)|^{2q}.\end{aligned}$$

For $(\mathrm{II})_2$, it is true that

$$((\mathrm{II})_2)^2\leqslant 2^{2q}\mathbb{E}_{\mathbb{P}^*_{y,x,T}}\sup_{\tau\in[0,T]}|K_{\mathrm{V},\alpha^s}(\tau)|^{2q}\cdot\mathbb{E}_{\mathbb{P}^*_{y,x,T}}\left|\int_0^T\|\nabla\ln p(T-\tau,\mathrm{y}^0_t,x)\|\ d\tau\right|^{2q}.$$

For $(\mathrm{II})_3$, a routine calculation shows

$$(\mathrm{II})_3\leqslant(cs_0\sup|\mathrm{V}|)^qT^q\mathbb{E}_{\mathbb{P}^*_{y,x,T}}\sup_{\tau\in[0,T]}|K_{\mathrm{V},\alpha^s}(\tau)|^q.$$

Putting the estimations on $(\mathrm{II})_1,(\mathrm{II})_2,(\mathrm{II})_3$ together, we conclude from Proposition 4.15 and the estimation for K_{V,α^s} that (5.78) also holds true for K_t. This shows (5.78) for $\underline{\Upsilon}_{\mathrm{V},\alpha^s}$.

Finally, to check (5.78) for $(\mathfrak{V}^s_t)'_s$, $(\theta,\varpi)\big((\mathfrak{V}^s_t)'_s\big)$, it suffices to consider the latter, which holds true by the above conclusion for K_{V,α^s} since, by (5.39),

$$\theta(Y^s_t)=\mathfrak{s}(t)(\mathfrak{V}^s_0)^{-1}V(F^sy),\ \varpi(Y^s_t)=K_{\mathrm{V},\alpha^s}(t).$$

□

Lemma 5.27. *Let α^s be as in Theorem 5.17. For $\underline{t},t$, $0\leqslant\underline{t}<t\leqslant T$, we abbreviate*

$$\begin{aligned}[DF^s_{\underline{t},t}(\mathfrak{V}^s_{\underline{t}},\mathrm{w})]&:=[DF^{\alpha^s}_{\underline{t},t}(\mathfrak{V}^s_{\underline{t}},\mathrm{w})],\\ [\widetilde{(DF^s_{\underline{t},t})}(\mathfrak{V}^s_0,\mathrm{w})]&:=(\theta,\omega)_{\mathfrak{V}^s_t}[DF^s_{\underline{t},t}(\mathfrak{V}^s_{\underline{t}},\mathrm{w})](\theta,\omega)^{-1}_{\mathfrak{V}^s_{\underline{t}}}.\end{aligned}$$

Let $T_0 > 0$. For any $s_0 > 0$, $q \geqslant 1$ and $T > T_0$, there are constants $\underline{c}_{\mathrm{F}}$ (which depends on $s_0, q, \mathtt{s}$ and $\|g^0\|_{C^2}$) and c_{F} (which depends on $s_0, q, \mathtt{s}$, T, T_0 and $\|g^0\|_{C^3}$) such that

$$\sup_{s\in[-s_0,s_0]} \mathbb{E}^*_{\mathbb{P}^*_{y,x,T}} \sup_{0\leqslant \underline{t}<t\leqslant T} \left\|[DF^s_{\underline{t},t}(\mathfrak{U}^s_{\underline{t}}, \mathrm{w})]^{-1}\right\|^q, \ \sup_{0\leqslant \underline{t}<t\leqslant T} \left\|[(\widetilde{DF^s_{\underline{t},t}})(\mathfrak{U}^s_{\underline{t}}, \mathrm{w})]^{-1}\right\|^q$$
$$< \underline{c}_{\mathrm{F}} e^{c_{\mathrm{F}}(1+d_{\tilde{g}^\lambda}(x,y))}. \tag{5.80}$$

Proof. For (5.80), it suffices to consider the second estimation. Let $s \in [-s_0, s_0]$ and $\underline{t}, t \in [0, T]$ with $\underline{t} < t$. For $(\mathtt{v}_0, \mathtt{Q}_0) \in T_o\mathcal{F}(\mathbb{R}^m)$, let

$$(\mathtt{v}_{t-\tau}, \mathtt{Q}_{t-\tau}) := [(\widetilde{DF^s_{\tau,t}})(\mathfrak{U}^s_0, \mathrm{w})]^{-1}(\mathtt{v}_0, \mathtt{Q}_0), \ \forall \tau \in [\underline{t}, t].$$

Then Lemma 5.13 shows that $\mathrm{z}_\tau := (\mathtt{v}_\tau, \mathtt{Q}_\tau)$ satisfies the Itô form SDE

$$d\mathrm{z}_{t-\tau}(\mathrm{w}) = \sum_{j=1}^m \left(-\mathbf{M}_j(\mathfrak{U}^s_\tau)\mathrm{z}_{t-\tau}(\mathrm{w}) \ d\overleftarrow{\alpha}^{s,j}_\tau(\mathrm{w}) + [\mathbf{M}_j(\mathfrak{U}^s_\tau)]^2\mathrm{z}_{t-\tau}(\mathrm{w}) \ d\tau\right) + \mathbf{N}(\mathfrak{U}^s_\tau)\mathrm{z}_{t-\tau}(\mathrm{w}) \ d\tau,$$

where $\mathbf{M}_j, \mathbf{N}$ are given in (4.27), (4.28). In terms of the multiplicative stochastic integral,

$$\mathrm{z}_{t-\underline{t}} = e^{\left\{\int_{\underline{t}}^t \mathbf{M}_j(\mathfrak{U}^s_\tau) \ d\overleftarrow{\alpha}^{s,j}_\tau(\mathrm{w}) - \mathbf{N}(\mathfrak{U}^s_\tau) \ d\tau\right\}}\mathrm{z}_0.$$

Note that $|\mathtt{g}^s| \leqslant \mathtt{c}s_0 \sup|\mathrm{V}|$ for some $\mathtt{c}$ that bounds $\sup_{t\in[0,T]}\{|\mathtt{s}(t)|, |\mathtt{s}'(t)|\}$, $\sup\{\|\mathrm{Ric}\|\}$. Hence, there is some $\mathtt{c}'$ which depends on $\mathtt{c}$ and $\sup\{\|R\|, \|\nabla R\|\}$ such that

$$\left\|[(\widetilde{DF^s_{\underline{t},t}})(\mathfrak{U}^s_{\underline{t}}, \mathrm{w})]^{-1}\right\| \leqslant e^{\mathtt{c}s_0 T \sup|\mathrm{V}|} \cdot e^{\left\{\left\|\int_{\underline{t}}^t \mathbf{M}_j(\mathfrak{U}^s_\tau) \ d(O^s_\tau d\overleftarrow{b}_\tau)^j - \mathbf{N}(\mathfrak{U}^s_\tau) \ d\tau\right\|\right\}}.$$

The remaining estimation for (5.80) can be done by following the proof of Proposition 4.17. □

Lemma 5.28. *Let α^s be as in Theorem 5.17. Then $((DF^s_{0,t})^{-1})_{t\in[0,T]}, ((\widetilde{DF^s_{0,t}})^{-1})_{t\in[0,T]}$ are C^1 in the s parameter. Let $T_0 > 0$. For any $s_0 > 0$, $q \geqslant 1$ and $T > T_0$, there exist $\underline{c}'_{\mathrm{F}}$ (which depends on $s_0, q, \mathtt{s}$ and $\|g^0\|_{C^3}$) and c'_{F} (which depends on $s_0, q, \mathtt{s}, T, T_0$ and $\|g^0\|_{C^3}$) such that*

$$\sup_{s\in[-s_0,s_0]} \overline{\mathbb{E}}^*_{\overline{\mathbb{P}}^*_{y,x,T}} \sup_{t\in[0,T]} \|([[(DF^s_{0,t})(\mathfrak{U}^s_0, \mathrm{w})]^{-1})'_s\|^q, \sup_{t\in[0,T]} \left\|([(\widetilde{DF^s_{0,t}})(\mathfrak{U}^s_0, \mathrm{w})]^{-1})'_s\right\|^q$$
$$< \underline{c}'_{\mathrm{F}} e^{c'_{\mathrm{F}}(1+d_{\tilde{g}^\lambda}(x,y))}. \tag{5.81}$$

Proof. The C^1 regularity of $s \mapsto (DF^s_{0,t})^{-1}$ follows from that of $s \mapsto (\widetilde{DF^s_{0,t}})^{-1}$ since

$$[DF^s_{0,t}(\mathfrak{U}^s_0, \mathrm{w})]^{-1} = (\theta, \omega)^{-1}_{\mathfrak{U}^s_0}[(\widetilde{DF^s_{0,t}})(\mathfrak{U}^s_0, \mathrm{w})]^{-1}(\theta, \omega)_{\mathfrak{U}^s_t}$$

and $s \mapsto (\theta, \omega)^{-1}_{\mathfrak{U}^s_t}$ is C^1. By Theorem 5.17, $(\widetilde{DF^s_{0,t}})(\mathfrak{U}^s_0, \mathrm{w})$ is C^1 in s for almost all w. Hence $[(\widetilde{DF^s_{0,t}})(\mathfrak{U}^s_0, \mathrm{w})]^{-1}$ is also C^1 in s by the identity

$$[(\widetilde{DF^s_{0,t}})(\mathfrak{U}^s_0, \mathrm{w})]^{-1} \circ (\widetilde{DF^s_{0,t}})(\mathfrak{U}^s_0, \mathrm{w}) = \mathrm{Id}.$$

For (5.81), it suffices to consider the second estimation. For $\mathrm{z}_0 \in T_o\mathcal{F}(\mathbb{R}^m)$, let

$$\mathrm{z}^s_{t-\tau} := [(\widetilde{DF^s_{0,t-\tau}})(\mho^s_0, \mathrm{w})]^{-1}\mathrm{z}_0,\ \forall \tau \in [0,t],\ \forall s \in [-s_0, s_0].$$

It satisfies the SDE

$$d\mathrm{z}^s_{t-\tau}(\mathrm{w}) = \sum_{j=1}^m \big(-\mathbf{M}_j(\mho^s_\tau)\mathrm{z}^s_{t-\tau}(\mathrm{w})\ d\overleftarrow{\alpha}^{s,j}_\tau(\mathrm{w}) + [\mathbf{M}_j(\mho^s_\tau)]^2\mathrm{z}^s_{t-\tau}(\mathrm{w})\ d\tau\big) + \mathbf{N}(\mho^s_\tau)\mathrm{z}^s_{t-\tau}(\mathrm{w})\ d\tau,$$

where $(\mathbf{M}_j)_{1\leqslant j\leqslant m}, \mathbf{N}$ are given in (4.27), (4.28). For $(\mathrm{z}^s_t)'_s := (d\mathrm{z}^s_t/ds)|_s$, its SDE is

$$\begin{aligned} &d(\mathrm{z}^s_{t-\tau})'_s(\mathrm{w}) \\ &= \sum_{j=1}^m \big(-\mathbf{M}_j(\mho^s_\tau)(\mathrm{z}^s_{t-\tau})'_s(\mathrm{w})\ d\overleftarrow{\alpha}^{s,j}_\tau(\mathrm{w}) + [\mathbf{M}_j(\mho^s_\tau)]^2(\mathrm{z}^s_{t-\tau})'_s(\mathrm{w})\ d\tau\big) + \mathbf{N}(\mho^s_\tau)(\mathrm{z}^s_{t-\tau})'_s(\mathrm{w})\ d\tau \\ &\quad + \sum_{j=1}^m - \big(\mathbf{M}_j(\mho^s_\tau)\ d\overleftarrow{\alpha}^{s,j}_\tau(\mathrm{w})\big)'_s\,\mathrm{z}^s_{t-\tau} + \big(\sum_{j=1}^m[\mathbf{M}_j(\mho^s_\tau)]^2 + \mathbf{N}(\mho^s_\tau)\big)'_s\ \mathrm{z}^s_{t-\tau}\ d\tau. \end{aligned}$$

Let $O^s = ((O^s)^j_l)_{j,l\leqslant m}$, $\mathsf{g} = (\mathsf{g}^{s,j})_{j\leqslant m}$. They are differentiable in s by Theorem 5.17. Let

$$\big(\mathbf{A}^{(1)}_l\big)^s_\tau := \sum_{j=1}^m \Big(\big(\mathbf{M}_j(\mho^s_\tau)\big)'_s(O^s_\tau)^j_l + \mathbf{M}_j(\mho^s_\tau)((O^s_\tau)^j_l)'_s\Big), \forall l \leqslant m,$$

$$\big(\mathbf{A}^{(2)}\big)^s_\tau := \sum_{j=1}^m \Big(\mathbf{M}_j(\mho^s_\tau)(\mathsf{g}^{s,j}_\tau)'_s + \big([\mathbf{M}_j(\mho^s_\tau)]^2\big)'_s\Big) + \big(\mathbf{N}(\mho^s_\tau)\big)'_s - 2\sum_{l,j=1}^m \mathbf{M}_j(\mho^s_\tau)(O^s_\tau)^j_l\big(\mathbf{A}^{(1)}_l\big)^s_\tau.$$

By Duhamel's principle, we have

$$\begin{aligned} (\mathrm{z}^s_t)'_s &= \bigg[\big[(\widetilde{DF^s_{0,t}})(\mho^s_0, \mathrm{w})\big]^{-1}\int_0^t \big[(\widetilde{DF^s_{\tau,t}})(\mho^s_\tau, \mathrm{w})\big]\big(\mathbf{A}^{(1)}_l\big)^s_\tau(\mathrm{w})\big[(\widetilde{DF^s_{\tau,t}})(\mho^s_\tau, \mathrm{w})\big]^{-1}d\overleftarrow{B}^l_\tau\bigg]\mathrm{z}_0 \\ &\quad + \bigg[\big[(\widetilde{DF^s_{0,t}})(\mho^s_0, \mathrm{w})\big]^{-1}\int_0^t \big[(\widetilde{DF^s_{\tau,t}})(\mho^s_\tau, \mathrm{w})\big]\big(\mathbf{A}^{(2)}\big)^s_\tau(\mathrm{w})\big[(\widetilde{DF^s_{\tau,t}})(\mho^s_\tau, \mathrm{w})\big]^{-1}d\tau\bigg]\mathrm{z}_0. \end{aligned}$$

This means

$$\begin{aligned} \big(\big[(\widetilde{DF^s_{0,t}})(\mho^s_0, \mathrm{w})\big]^{-1}\big)'_s &= \int_0^t \big[(\widetilde{DF^s_{0,\tau}})(\mho^s_0, \mathrm{w})\big]^{-1}\big(\mathbf{A}^{(1)}_l\big)^s_\tau(\mathrm{w})\big[(\widetilde{DF^s_{\tau,t}})(\mho^s_\tau, \mathrm{w})\big]^{-1}d\overleftarrow{B}^l_\tau \\ &\quad + \int_0^t \big[(\widetilde{DF^s_{0,\tau}})(\mho^s_0, \mathrm{w})\big]^{-1}\big(\mathbf{A}^{(2)}\big)^s_\tau(\mathrm{w})\big[(\widetilde{DF^s_{\tau,t}})(\mho^s_\tau, \mathrm{w})\big]^{-1}d\tau \\ &=: (\mathbf{I})^s_t + (\mathbf{II})^s_t. \end{aligned}$$

For (5.81), it suffices to show the same bound type is valid for

$$(\mathbf{I}) := \sup_{s\in[-s_0,s_0]} \overline{\mathbb{E}}_{\overline{\mathbb{P}}^*_{y,x,T}} \sup_{t\in[0,T]} \|(\mathbf{I})^s_t\|^q,\ (\mathbf{II}) := \sup_{s\in[-s_0,s_0]} \overline{\mathbb{E}}_{\overline{\mathbb{P}}^*_{y,x,T}} \sup_{t\in[0,T]} \|(\mathbf{II})^s_t\|^q.$$

This will follow from Lemma 5.27 and Proposition 4.15. Clearly,

$$\overline{\mathbb{E}}_{\overline{\mathbb{P}}^*_{y,x,T}} \sup_{\tau\in[0,T]} \Big\|\big(\mathbf{A}^{(i)}_l\big)^s_\tau(\mathrm{w})\Big\|^q \leqslant c_{\mathbf{M}}\overline{\mathbb{E}}_{\overline{\mathbb{P}}^*_{y,x,T}} \max\big\{\sup_{t\in[0,T]}\big\|(\mho^s_t)'_s\big\|^{2q}, \sup_{t\in[0,T]}\big\|(O^s_t)'_s\big\|^{2q}, \sup_{t\in[0,T]}\big\|(\mathsf{g}^s_t)'_s\big\|^q\big\},$$

where $c_{\mathbf{M}}$ depends on the norm bounds of $\{\mathbf{M}_j\}$ and their differentials. Hence by Lemma 5.26, there are constants $\underline{c}_{\mathbf{A}}$ (which depends on $s_0, q, \mathtt{s}$ and $\|g^0\|_{C^3}$) and $c_{\mathbf{A}}$ (which depends on q, T, T_0 and $\|g^0\|_{C^3}$) such that

$$\overline{\mathbb{E}}_{\overline{\mathbb{P}}^*_{y,x,T}} \sup_{\tau\in[0,T]} \left\| \big(\mathbf{A}_l^{(i)}\big)_\tau^s(\mathrm{w}) \right\|^q \leqslant \underline{c}_{\mathbf{A}} e^{c_{\mathbf{A}}(1+d_{\tilde{g}^\lambda}(x,y))}. \tag{5.82}$$

Let

$$(\mathbf{III})_t^s := \int_0^t \Big[(\widetilde{DF_{0,\tau}^s})(\mathcal{U}_0^s,\mathrm{w})\Big]^{-1} \big(\mathbf{A}_l^{(1)}\big)_\tau^s(\mathrm{w}) \Big[(\widetilde{DF_{\tau,t}^s})(\mathcal{U}_\tau^s,\mathrm{w})\Big]^{-1} d\overleftarrow{b}_\tau^l,$$

$$(\mathbf{IV})_t^s := \int_0^t \Big[(\widetilde{DF_{0,\tau}^s})(\mathcal{U}_0^s,\mathrm{w})\Big]^{-1} \big(\mathbf{A}_l^{(1)}\big)_\tau^s(\mathrm{w}) \Big[(\widetilde{DF_{\tau,t}^s})(\mathcal{U}_\tau^s,\mathrm{w})\Big]^{-1} (\mathcal{U}_\tau^{-1}\nabla \ln p(T-\tau, \mathrm{y}_\tau^0, x))^l \, d\tau,$$

where b_τ is the Brownian motion in Lemma 4.16 for $\mathbb{P}_{y,x,T}$. Then

$$\overline{\mathbb{E}}_{\overline{\mathbb{P}}^*_{y,x,T}} \sup_{t\in[0,T]} \|(\mathbf{I})_t^s\|^q \leqslant 2^{q-1}\overline{\mathbb{E}}_{\overline{\mathbb{P}}^*_{y,x,T}} \sup_{t\in[0,T]} \|(\mathbf{III})_t^s\|^q + 2^{2q-1}\overline{\mathbb{E}}_{\overline{\mathbb{P}}^*_{y,x,T}} \sup_{t\in[0,T]} \|(\mathbf{IV})_t^s\|^q.$$

As usual, we can use Hölder's inequality and Doob's maximal inequality of sub-martingales to deduce that

$$\big(\overline{\mathbb{E}}_{\overline{\mathbb{P}}^*_{y,x,T}} \sup_{t\in[0,T]} \|(\mathbf{III})_t^s\|^q\big)^2 \leqslant p(T,x,y)\big(\frac{2q}{2q-1}\big)^{2q}\overline{\mathbb{E}}_{\overline{\mathbb{P}}^*_{y,x,T}} \|(\mathbf{III})_T^s\|^{2q}.$$

Let $\mathtt{C}_1(\cdot)$ be the constant function in Lemma 4.7. We continue to compute that

$$\begin{aligned}
&\overline{\mathbb{E}}_{\overline{\mathbb{P}}^*_{y,x,T}} \|(\mathbf{III})_T^s\|^{2q} \\
&\leqslant \mathtt{C}_1(2q)\overline{\mathbb{E}}_{\overline{\mathbb{P}}^*_{y,x,T}} \left| \int_0^T \left\| \big[(\widetilde{DF_{0,\tau}^s})(\mathcal{U}_0^s,\mathrm{w})\big]^{-1} \big(\mathbf{A}_l^{(1)}\big)_\tau^s(\mathrm{w}) \big[(\widetilde{DF_{\tau,T}^s})(\mathcal{U}_\tau^s,\mathrm{w})\big]^{-1} \right\|^2 d\tau \right|^q \\
&\leqslant \mathtt{C}_1(2q)T^q \left(\overline{\mathbb{E}}_{\overline{\mathbb{P}}^*_{y,x,T}} \sup_{0\leqslant \underline{t}<t\leqslant T} \big\|[DF_{\underline{t},t}^s(\mathcal{U}_{\underline{t}}^s,\mathrm{w})]^{-1}\big\|^{8q} \cdot \overline{\mathbb{E}}_{\overline{\mathbb{P}}^*_{y,x,T}} \sup_{\tau\in[0,T]} \left\| \big(\mathbf{A}_l^{(1)}\big)_\tau^s(\mathrm{w}) \right\|^{4q} \right)^{\frac{1}{2}},
\end{aligned}$$

which has the same type of bound as in (5.81) by Lemma 5.27 and (5.82). Similarly,

$$\begin{aligned}
\left(\overline{\mathbb{E}}_{\mathbb{P}^*_{y,x,T}} \sup_{t\in[0,T]} \|(\mathbf{IV})_t^s\|^q\right)^3 \leqslant &\overline{\mathbb{E}}_{\mathbb{P}^*_{y,x,T}} \sup_{0\leqslant \underline{t}<t\leqslant T} \big\|[DF_{\underline{t},t}^s(\mathcal{U}_{\underline{t}}^s,\mathrm{w})]^{-1}\big\|^{6q} \cdot \overline{\mathbb{E}}_{\mathbb{P}^*_{y,x,T}} \sup_{\tau\in[0,T]} \left\| \big(\mathbf{A}_l^{(1)}\big)_\tau^s(\mathrm{w}) \right\|^{3q} \\
&\cdot \overline{\mathbb{E}}_{\mathbb{P}^*_{y,x,T}} \left| \int_0^T \|\nabla \ln p(T-\tau, \mathrm{y}_\tau^0, x)\| \, d\tau \right|^{3q},
\end{aligned}$$

which also has the same type of bound as in (5.81) by Proposition 4.15, Lemma 5.27 and (5.82). Altogether, the same type of bound as in (5.81) is valid for (**I**). This is also true for (**II**) by Lemma 5.27 and (5.82) since

$$\left(\overline{\mathbb{E}}_{\mathbb{P}^*_{y,x,T}} \|(\mathbf{II})_T^s\|^q\right)^2 \leqslant T^{2q}\overline{\mathbb{E}}_{\mathbb{P}^*_{y,x,T}} \sup_{0\leqslant \underline{t}<t\leqslant T} \big\|[(\widetilde{DF_{\underline{t},t}^s})(\mathcal{U}_{\underline{t}}^s,\mathrm{w})]^{-1}\big\|^{2q} \overline{\mathbb{E}}_{\mathbb{P}^*_{y,x,T}} \sup_{\tau\in[0,T]} \left\| \big(\mathbf{A}_l^{(2)}\big)_\tau^s(\mathrm{w}) \right\|^{2q}.$$

□

With Lemmas 5.26-5.28, we can deduce the differentiability of $(D\pi(\lfloor \mathrm{u}_T \rceil^\lambda)_0^{(1)}) \circ \mathbf{F}^s$ in s.

Proposition 5.29. *Fix $T_0 > 0$. For any $q \geqslant 1$ and $T > T_0$, there are $\underline{c}_{\mathbf{F}}$ (depending on $s_0, q, \mathtt{s}$, $\|g^0\|_{C^2}$ and $\|\mathcal{X}^0\|_{C^1}$) and $c_{\mathbf{F}}$ (depending on $s_0, q, \mathtt{s}, T, T_0$ and $\|g^0\|_{C^3}$) such that*

$$\sup_{s\in[-s_0,s_0]} \overline{\mathbb{E}}_{\overline{\mathbb{P}}^*_{y,x,T}} \left\| (D\pi(\lfloor \mathrm{u}_T \rceil^\lambda)_0^{(1)}) \circ \mathbf{F}^s \right\|^q \leqslant \underline{c}_{\mathbf{F}} e^{c_{\mathbf{F}}(1+d_{\tilde{g}}(x,y))}. \tag{5.83}$$

The one-parameter of processes $\{(D\pi(\lfloor \mathrm{u}_T \rceil^\lambda)_0^{(1)}) \circ \mathbf{F}^s\}$ is differentiable in s. Let

$$\nabla^s_{T,\mathrm{V},\mathtt{s}} D\pi(\lfloor \mathrm{u}_T \rceil^\lambda)_0^{(1)} := \Big((D\pi(\lfloor \mathrm{u}_T \rceil^\lambda)_0^{(1)}) \circ \mathbf{F}^s \Big)'_s.$$

For any $q \geqslant 1$ and $T > T_0$, there are $\underline{c}'_{\mathbf{F}}$ (depending on $s_0, q, \mathtt{s}$, $\|g^0\|_{C^3}$ and $\|\mathcal{X}^0\|_{C^2}$) and $c'_{\mathbf{F}}$ (depending on $s_0, q, \mathtt{s}, T, T_0$ and $\|g^0\|_{C^3}$) such that

$$\sup_{s\in[-s_0,s_0]} \overline{\mathbb{E}}_{\overline{\mathbb{P}}^*_{y,x,T}} \left\| \nabla^s_{T,\mathrm{V},\mathtt{s}} D\pi(\lfloor \mathrm{u}_T \rceil^\lambda)_0^{(1)} \right\|^q \leqslant \underline{c}'_{\mathbf{F}} e^{c'_{\mathbf{F}}(1+d_{\tilde{g}}(x,y))}. \tag{5.84}$$

Proof. Recall that

$$(D\pi(\lfloor \mathrm{u}_T \rceil^\lambda)_0^{(1)}) \circ \mathbf{F}^s = D\pi\big((\lfloor \mathrm{u}_T \rceil^\lambda)_0^{(1)} \circ \mathbf{F}^s\big) = D\pi\big(\overline{(\lfloor \mathrm{u}^s_T \rceil^\lambda)_0^{(1)}}(\mathrm{w})\big),$$

where

$$\overline{(\lfloor \mathrm{u}^s_T \rceil^\lambda)_0^{(1)}}(\mathrm{w}) = [DF^s_{0,T}(\mathcal{U}^s_0, \mathrm{w})]^{-1} (\lfloor \mathrm{u}_0 \rceil^\lambda)_0^{(1)} - \int_0^T [DF^s_{0,t}(\mathcal{U}^s_0, \mathrm{w})]^{-1} (H^\lambda)_0^{(1)}(\mathcal{U}^s_t, e_i) \circ d\overleftarrow{\alpha}^{s,i}_t(\mathrm{w}).$$

Let

$$\widetilde{(\lfloor \mathrm{u}^s_T \rceil^\lambda)_0^{(1)}}(\mathrm{w}) := (\theta,\omega)_{\mathcal{U}^s_0}\left(\overline{(\lfloor \mathrm{u}^s_T \rceil^\lambda)_0^{(1)}}(\mathrm{w})\right), \quad \widetilde{(\lfloor \mathrm{u}^s_0 \rceil^\lambda)_0^{(1)}} := (\theta,\omega)_{\mathcal{U}^s_T}\left((\lfloor \mathrm{u}_0 \rceil^\lambda)_0^{(1)}\right).$$

It is easy to obtain the following Itô form expression:

$$\begin{aligned}\widetilde{(\lfloor \mathrm{u}^s_T \rceil^\lambda)_0^{(1)}}(\mathrm{w}) =& [\widetilde{(DF^s_{0,T})}(\mathcal{U}^s_0, \mathrm{w})]^{-1} \widetilde{(\lfloor \mathrm{u}^s_0 \rceil^\lambda)_0^{(1)}} \\ &- \int_0^T [\widetilde{(DF^s_{0,t})}(\mathcal{U}^s_0, \mathrm{w})]^{-1} \Big(\varpi\big((H^\lambda)_0^{(1)}(\mathcal{U}^s_t, e_i)\big) e_i \, dt, \varpi\big((H^\lambda)_0^{(1)}(\mathcal{U}^s_t, d\overleftarrow{\alpha}^s_t(\mathrm{w}))\big) \Big).\end{aligned}$$

For Proposition 5.29, it is equivalent to show the differentiability of $s \mapsto \widetilde{(\lfloor \mathrm{u}^s_T \rceil^\lambda)_0^{(1)}}(\mathrm{w})$ and estimate the conditional L^q integrals of its differential process and itself.

The estimation in (5.83) is valid since

$$\sup_{s\in[-s_0,s_0]} \overline{\mathbb{E}}_{\overline{\mathbb{P}}^*_{y,x,T}} \left\| (D\pi(\lfloor \mathrm{u}_T \rceil^\lambda)_0^{(1)}) \circ \mathbf{F}^s \right\|^q \leqslant \sup_{s\in[-s_0,s_0]} \overline{\mathbb{E}}_{\overline{\mathbb{P}}^*_{y,x,T}} \left\| \widetilde{(\lfloor \mathrm{u}^s_T \rceil^\lambda)_0^{(1)}}(\mathrm{w}) \right\|^q,$$

where the second term has a bound in (5.83) by following the argument of (4.50) in Proposition 4.28 and using Lemma 5.26 and Lemma 5.27.

The processes $\alpha^s, \mathcal{U}^s$ and $\widetilde{[DF^s_{0,t}}(\mathcal{U}^s_0, \mathrm{w})]^{-1}$ are all differentiable in s by Theorem 5.17. Lemmas 5.26-5.28 show that α^s_t, $\underline{\Upsilon}_{\mathrm{V},\alpha^s}$, $(\mathcal{U}^s_t)'_s$, $[\widetilde{(DF^s_{0,t})}(\mathcal{U}^s_0, \mathrm{w})]^{-1}$ and $([\widetilde{(DF^s_{0,t})}(\mathcal{U}^s_0, \mathrm{w})]^{-1})'_s$

all have bounded sup L^q ($q \geqslant 1$) norm with respect to $\overline{\mathbb{P}}_{y,x,T}$. Hence $s \mapsto (\widetilde{\lfloor \mathrm{u}_T^s \rceil^\lambda})_0^{(1)}(\mathrm{w})$ is also differentiable in s and the differential is

$$\begin{aligned}
&\big((\widetilde{\lfloor \mathrm{u}_T^s \rceil^\lambda})_0^{(1)}(\mathrm{w})\big)'_s \\
&\quad = \big([(\widetilde{DF_{0,T}^s})(\mathcal{U}_0^s, \mathrm{w})]^{-1}\big)'_s (\widetilde{\lfloor \mathrm{u}_0^s \rceil^\lambda})_0^{(1)} \\
&\qquad - \int_0^T \big([(\widetilde{DF_{0,t}^s})(\mathcal{U}_0^s, \mathrm{w})]^{-1}\big)'_s \Big(\varpi\big((H^\lambda)_0^{(1)}(\mathcal{U}_t^s, e_i)\big)e_i \ dt, \ \varpi\big((H^\lambda)_0^{(1)}(\mathcal{U}_t^s, d\overleftarrow{\alpha}_t^s(\mathrm{w}))\big)\Big) \\
&\qquad - \int_0^T [(\widetilde{DF_{0,t}^s})(\mathcal{U}_0^s, \mathrm{w})]^{-1} \Big(\varpi\big((H^\lambda)_0^{(1)}(\mathcal{U}_t^s, e_i)\big)'_s e_i \ dt, \\
&\qquad\qquad\qquad \varpi\big((H^\lambda)_0^{(1)}(\mathcal{U}_t^s, \cdot)\big)'_s d\overleftarrow{\alpha}_t^s(\mathrm{w}) + \varpi\big((H^\lambda)_0^{(1)}(\mathcal{U}_t^s, d\underline{\Upsilon}_{\mathrm{V},\alpha^s})\big)\Big) \\
&\quad =: \ \mathrm{I}(s) + \mathrm{II}(s) + \mathrm{III}(s).
\end{aligned}$$

This process has a continuous version in s by Kolmogorov's criterion (or by continuity of α^s, $\mathcal{U}^s$, $\underline{\Upsilon}_{\mathrm{V},\alpha^s}$, $(\mathcal{U}_t^s)'_s$ and $[(\widetilde{DF_{0,t}^s})(\mathcal{U}_0^s, \mathrm{w})]^{-1}$ in s using Theorem 5.17).

For (5.84), we do the conditional L^q estimations for $\mathrm{I}(s), \mathrm{II}(s)$ and $\mathrm{III}(s)$. Clearly,

$$\overline{\mathbb{E}}_{\overline{\mathbb{P}}^*_{y,x,T}} |\mathrm{I}(s)|^q \leqslant \overline{\mathbb{E}}_{\overline{\mathbb{P}}^*_{y,x,T}} \sup_{t\in[0,T]} \Big\|([(\widetilde{DF_{0,t}^s})(\mathcal{U}_0^s, \mathrm{w})]^{-1})'_s\Big\|^q \cdot \big\|(\lfloor \mathrm{u}_0 \rceil^\lambda)_0^{(1)}\big\|^q,$$

which, by (5.81), has a bound as in (5.84). Put

$$\begin{aligned}
\mathrm{II}_1(s) &:= -\int_0^T \big([(\widetilde{DF_{0,t}^s})(\mathcal{U}_0^s, \mathrm{w})]^{-1}\big)'_s \Big(\varpi\big((H^\lambda)_0^{(1)}(\mathcal{U}_t^s, e_i)\big)e_i \ dt, \varpi\big((H^\lambda)_0^{(1)}(\mathcal{U}_t^s, \mathfrak{g}_t^s(\mathrm{w})dt)\big)\Big), \\
\mathrm{II}_2(s) &:= -\int_0^T \big([(\widetilde{DF_{0,t}^s})(\mathcal{U}_0^s, \mathrm{w})]^{-1}\big)'_s \Big(0, \varpi\big((H^\lambda)_0^{(1)}(\mathcal{U}_t^s, O_t^s d\overleftarrow{B}_t(\mathrm{w}))\big)\Big).
\end{aligned}$$

For $\mathrm{II}(s)$, we have

$$\overline{\mathbb{E}}_{\overline{\mathbb{P}}^*_{y,x,T}} |\mathrm{II}(s)|^q \leqslant 2^{q-1}\Big(\overline{\mathbb{E}}_{\overline{\mathbb{P}}^*_{y,x,T}} |\mathrm{II}_1(s)|^q + \overline{\mathbb{E}}_{\overline{\mathbb{P}}^*_{y,x,T}} |\mathrm{II}_2(s)|^q\Big).$$

As before, we can use Hölder's inequality, Doob's inequality of submartingales and Burkholder's inequality to obtain some $C(q,T)$ depending on $s_0, q, \mathtt{s}, T$, $\|g^0\|_{C^3}$ and $\|\mathcal{X}^0\|_{C^2}$ such that

$$\begin{aligned}
\overline{\mathbb{E}}_{\overline{\mathbb{P}}^*_{y,x,T}} |\mathrm{II}(s)|^q \leqslant & C(q,T)T_0^{-m}\Bigg(\overline{\mathbb{E}}_{\overline{\mathbb{P}}^*_{y,x,T}} \sup_{t\in[0,T]} \Big\|([(\widetilde{DF_{0,t}^s})(\mathcal{U}_0^s, \mathrm{w})]^{-1})'_s\Big\|^q + \\
&\big(\overline{\mathbb{E}}_{\overline{\mathbb{P}}^*_{y,x,T}} \sup_{t\in[0,T]} \Big\|([(\widetilde{DF_{0,t}^s})(\mathcal{U}_0^s, \mathrm{w})]^{-1})'_s\Big\|^{2q} \overline{\mathbb{E}}_{\overline{\mathbb{P}}^*_{y,x,T}} e^{\{2q\int_0^T \|\nabla \ln p(T-\tau, \mathrm{y}_\tau^0, x)\| \ d\tau\}}\big)^{\frac{1}{2}}\Bigg),
\end{aligned}$$

which has a bound as in (5.84) by Lemma 5.28 and Proposition 4.15. The same argument applies to $\mathrm{III}(s)$ and we obtain some $C'(q,T)$ depending on $s_0, q, \mathtt{s}, T$, $\|g^0\|_{C^3}$ and $\|\mathcal{X}^0\|_{C^2}$

such that

$$\begin{aligned}\overline{\mathbb{E}}_{\overline{\mathbb{P}}^*_{y,x,T}}|\mathrm{III}(s)|^q \leqslant & C'(q,T)T_0^{-m}\big(\overline{\mathbb{E}}_{\overline{\mathbb{P}}^*_{y,x,T}}\sup_{t\in[0,T]}\Big\|[(\widetilde{DF^s_{0,t}})(\mathfrak{V}^s_0,\mathrm{w})]^{-1}\Big\|^{2q}\big)^{\frac{1}{2}}\cdot\\ &\Big\{1+\big(\overline{\mathbb{E}}_{\overline{\mathbb{P}}^*_{y,x,T}}e^{\{2q\int_0^T\|\nabla\ln p(T-\tau,\mathrm{y}^0_\tau,x)\|\,d\tau\}}\big)^{\frac{1}{2}}\cdot\\ &\bigg(\big(\overline{\mathbb{E}}_{\overline{\mathbb{P}}^*_{y,x,T}}\int_0^T\Big|\big(\varpi\big((H^\lambda)^{(1)}_0(\mathfrak{V}^s_t,e_i)\big)e_i\big)'_s\Big|^{2q}dt\big)^{\frac{1}{2}}+\big(\overline{\mathbb{E}}_{\overline{\mathbb{P}}^*_{y,x,T}}\int_0^T|K_{\mathrm{V},\alpha^s}|^{2q}\,dt\big)^{\frac{1}{2}}\\ &+\big(\overline{\mathbb{E}}_{\overline{\mathbb{P}}^*_{y,x,T}}\int_0^T\Big|\big(\varpi(H^\lambda)^{(1)}_0(\mathfrak{V}^s_t,\cdot)\big)'_s\mathfrak{g}^s_t\Big|^{2q}dt\big)^{\frac{1}{2}}\bigg)\Big\},\end{aligned}$$

which also has a bound as in (5.84) by Lemma 5.26, Lemma 5.27 and Proposition 4.15. □

We can define $\big(D\pi(\lfloor\mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda\big)\circ\lfloor\mathbf{F}^s\rceil^\lambda$ for all λ. Let V, F^s and $\mathfrak{s}$ be as in Section 5.2. For $y\in\widetilde{M}$, let $(\lfloor\mathrm{y}_t\rceil^\lambda(\mathrm{w}),\lfloor\mathfrak{V}_t\rceil^\lambda(\mathrm{w}))_{t\in[0,T]}$ be the stochastic pair in $(\widetilde{M},\mathcal{O}^{\widetilde{g}^\lambda}(\widetilde{M}))$ which defines the $\widetilde{g}^\lambda$-Brownian motion on $\widetilde{M}$ starting from y. Following Theorem 5.17, we can extend the flow map F^s on y to be a map $\lfloor\mathbf{F}^s_y\rceil^\lambda$ on paths $(\lfloor\mathrm{y}_t\rceil^\lambda(\mathrm{w}))_{t\in[0,T]}$ so that

$$\lfloor\mathrm{y}^s_t\rceil^\lambda(\mathrm{w}):=\Big(\lfloor\mathbf{F}^s_y\rceil^\lambda(\lfloor\mathrm{y}\rceil^\lambda_{[0,T]}(\mathrm{w}))\Big)(t),\ \forall t\in[0,T],$$

and its horizontal lift $\big(\lfloor\mathfrak{V}^s_t\rceil^\lambda(\mathrm{w})\big)_{t\in[0,T]}$ with $(\lfloor\mathfrak{V}^s_0\rceil^\lambda)'_s=0$ are such that

$$\frac{d}{ds}\big(\lfloor\mathrm{y}^s_t\rceil^\lambda(\mathrm{w})\big)=\Upsilon_{\mathrm{V},\lfloor\mathrm{y}^s\rceil^\lambda}(t)=\mathfrak{s}(t)\lfloor\mathfrak{V}^s_t\rceil^\lambda(\lfloor\mathfrak{V}^s_0\rceil^\lambda)^{-1}\mathrm{V}(F^s(y)).$$

Accordingly, we denote by $\lfloor\alpha^s_t\rceil^\lambda$ the anti-development of $\lfloor\mathrm{y}^s_t\rceil^\lambda$ and let $\big(\lfloor F^s_{\underline{t},\bar{t}}\rceil^\lambda\big)_{0<\underline{t}<\bar{t}<T}$ be the stochastic flow map corresponding to the SDE

$$d\overline{\beta}_t=H^\lambda(\overline{\beta}_t,\circ d\lfloor\alpha^s_t\rceil^\lambda(\mathrm{w}))$$

with tangent maps $(D\lfloor F^s_{\underline{t},\bar{t}}\rceil^\lambda)_{0<\underline{t}<\bar{t}<T}$. We will omit the upper-script at $s=0$. For $x\in\widetilde{M}$, we define $\lfloor\mathbf{F}^s\rceil^\lambda$ on $C_x([0,T],\widetilde{M})$ conditioned on the value of β_T, i.e.,

$$\lfloor\mathbf{F}^s\rceil^\lambda(\beta):=\lfloor\mathbf{F}^s_{\beta_T}\rceil^\lambda(\beta),\ \forall\beta\in C_x([0,T],\widetilde{M}).$$

Let $(\lfloor\mathrm{x}_t\rceil^\lambda(w),\lfloor\mathrm{u}_t\rceil^\lambda(w))_{t\in[0,T]}$ be the stochastic pair in $(\widetilde{M},\mathcal{O}^{\widetilde{g}^\lambda}(\widetilde{M}))$ which defines the $\widetilde{g}^\lambda$-Brownian motion on $\widetilde{M}$ starting from x. The correspondence rule in Corollary 5.25 shows that conditioned on $\lfloor\mathrm{x}_T\rceil^\lambda=y$, the distribution of $(\lfloor\mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda(w)$ is the same as, conditioned on $\lfloor\mathrm{y}_T\rceil^\lambda=x$, the distribution of

$$\begin{aligned}\overline{(\lfloor\mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda}(\mathrm{w}):=&[D\lfloor F_{0,T}\rceil^\lambda(\lfloor\mathfrak{V}_0\rceil^\lambda,\mathrm{w})]^{-1}(\lfloor\mathrm{u}_0\rceil^\lambda)^{(1)}_\lambda\\ &-\int_0^T[D\lfloor F_{0,t}\rceil^\lambda(\lfloor\mathfrak{V}_0\rceil^\lambda,\mathrm{w})]^{-1}(H^\lambda)^{(1)}_\lambda\big(\lfloor\mathfrak{V}_t\rceil^\lambda,\circ dB_t(\mathrm{w})\big),\end{aligned}$$

where $\circ d\overrightarrow{B}_t(\mathrm{w})$ is the backward Stratonovich infinitesimal. Then we define

$$(D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda)\circ\lfloor \mathbf{F}^s\rceil^\lambda := D\pi\big((\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda\circ\lfloor \mathbf{F}^s\rceil^\lambda\big) = D\pi\big(\overline{(\lfloor \mathrm{u}^s_T\rceil^\lambda)^{(1)}_\lambda}(\mathrm{w})\big),$$

where

$$\begin{aligned}\overline{(\lfloor \mathrm{u}^s_T\rceil^\lambda)^{(1)}_\lambda}(\mathrm{w}) &= [D\lfloor F^s_{0,T}\rceil^\lambda(\lfloor \mathfrak{V}^s_0\rceil^\lambda,\mathrm{w})]^{-1}(\lfloor \mathrm{u}_0\rceil^\lambda)^{(1)}_\lambda\\ &\quad-\int_0^T[D\lfloor F^s_{0,t}\rceil^\lambda(\lfloor \mathfrak{V}^s_0\rceil^\lambda,\mathrm{w})]^{-1}(H^\lambda)^{(1)}_\lambda\big(\lfloor \mathfrak{V}^s_t\rceil^\lambda,\circ d\lfloor \alpha^s_t\rceil^\lambda(\mathrm{w})\big).\end{aligned}\tag{5.85}$$

The proof of Proposition 5.29 works for $\lfloor \mathbf{F}^s\rceil^\lambda$, which gives the following.

Proposition 5.30. *For each λ, the one-parameter of processes $\{(D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda)\circ\lfloor \mathbf{F}^s\rceil^\lambda\}$ is differentiable in s. Moreover, $(D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda)\circ\lfloor \mathbf{F}^s\rceil^\lambda$ and the differential stochastic process*

$$\nabla^{s,\lambda}_{T,\mathrm{V},\mathfrak{s}}D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda := \Big((D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda)\circ\lfloor \mathbf{F}^s\rceil^\lambda\Big)'_s$$

conditioned on $\lfloor \mathrm{x}_T\rceil^\lambda = y$ are L^q ($q\geqslant 1$) integrable, locally uniformly in the s parameter.

For later use, we list and reformulate some differentials related to $\nabla_{T,\mathrm{V},\mathfrak{s}}D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda$. The upper-scripts λ in ∇^λ, R^λ, Ric^λ, θ^λ, ϖ^λ and $(\theta,\varpi)^\lambda$ are to indicate the metric $\widetilde{g}^\lambda$ used.

Lemma 5.31. *We have the following for almost all* w *and for all $t\in[0,T]$.*

i)

$$\begin{aligned}&(\lfloor \alpha^s_t\rceil^\lambda)'_0 = \underline{\Upsilon}^\lambda_{\mathrm{V},B}(t)\\ &:= \int_0^t\Big((\lfloor \mathfrak{V}_0\rceil^\lambda)^{-1}(\mathfrak{s}'(\tau)\mathrm{V}(y)) - \mathrm{Ric}^\lambda\big(\Upsilon_{\mathrm{V},\lfloor y\rceil^\lambda}(\tau)\big)\Big)\ d\tau - \int_0^t\langle K^\lambda_{\mathrm{V},B}(\tau),dB_\tau\rangle,\\ &\text{where } \Upsilon_{\mathrm{V},\lfloor y\rceil^\lambda}(\tau) := \mathfrak{s}(\tau)\lfloor \mathfrak{V}_\tau\rceil^\lambda(\lfloor \mathfrak{V}_0\rceil^\lambda)^{-1}\mathrm{V}(y) \text{ and}\\ &K^\lambda_{\mathrm{V},B}(\tau) = \int_0^\tau(\lfloor \mathfrak{V}_{\widetilde{\tau}}\rceil^\lambda)^{-1}R^\lambda\Big(\lfloor \mathfrak{V}_{\widetilde{\tau}}\rceil^\lambda dB_{\widetilde{\tau}},\Upsilon_{\mathrm{V},\lfloor y\rceil^\lambda}(\widetilde{\tau})\Big)\lfloor \mathfrak{V}_{\widetilde{\tau}}\rceil^\lambda\\ &\qquad+\int_0^\tau(\lfloor \mathfrak{V}_{\widetilde{\tau}}\rceil)^{-1}(\nabla^\lambda(\lfloor \mathfrak{V}_{\widetilde{\tau}}\rceil^\lambda e_i)R^\lambda)\Big(\lfloor \mathfrak{V}_{\widetilde{\tau}}\rceil^\lambda e_i,\Upsilon_{\mathrm{V},\lfloor y\rceil^\lambda}(\widetilde{\tau})\Big)\lfloor \mathfrak{V}_{\widetilde{\tau}}\rceil^\lambda\ d\widetilde{\tau}.\end{aligned}$$

ii)

$$(\theta,\varpi)^\lambda_{\lfloor \mathfrak{V}_t\rceil^\lambda}(\lfloor \mathfrak{V}^s_t\rceil^\lambda)'_0 = \big(\mathfrak{s}(t)(\lfloor \mathfrak{V}_0\rceil^\lambda)^{-1}\mathrm{V}(y),K^\lambda_{\mathrm{V},B}(t)\big).$$

iii) *For $s\mapsto \mathsf{v}^s_t\in HT_{\lfloor \mathfrak{V}^s_t\rceil^\lambda}\mathcal{F}(\widetilde{M})$, let $\mathsf{v}^s_\tau := [D\lfloor F^s_{\tau,t}\rceil^\lambda(\lfloor \mathfrak{V}^s_\tau\rceil^\lambda,\mathrm{w})]^{-1}\mathsf{v}^s_t, \tau\in[0,t]$. Then*

$$\big(\mathsf{v}^s_0\big)'_0 = \int_0^t\big[D\lfloor F_{0,\tau}\rceil^\lambda\big(\lfloor \mathfrak{V}_0\rceil^\lambda,\mathrm{w}\big)\big]^{-1}\Big(\circledast^\lambda\big(\mathsf{v}_\tau,(\lfloor \mathfrak{V}^s_\tau\rceil^\lambda)'_0\big)\Big),$$

where

$$\begin{aligned}\circledast^\lambda\big(\mathsf{v}_\tau,(\lfloor \mathfrak{V}^s_\tau\rceil^\lambda)'_0\big) =&\nabla^{(2),\lambda}\big(\mathsf{v}_\tau,(\lfloor \mathfrak{V}^s_\tau\rceil^\lambda)'_0\big)H^\lambda(\lfloor \mathfrak{V}_\tau\rceil^\lambda,\circ dB_\tau) + \nabla^\lambda(\mathsf{v}_\tau)H^\lambda\big(\lfloor \mathfrak{V}_\tau\rceil^\lambda,\circ d\Upsilon^\lambda_{\mathrm{V},B}(\tau)\big)\\ &+R^\lambda\big(H^\lambda(\lfloor \mathfrak{V}_\tau\rceil^\lambda,\circ dB_\tau),(\lfloor \mathfrak{V}^s_\tau\rceil^\lambda)'_0\big)\mathsf{v}_\tau.\end{aligned}\tag{5.86}$$

The Itô form of $\big(\mathsf{v}_0^s\big)_0'$ *in* (θ,ϖ)*-chart is*

$$(\theta,\varpi)^\lambda_{\lfloor\mathfrak{V}_0\rceil^\lambda}\big(\mathsf{v}_0^s\big)_0' = \int_0^t \Big[\widetilde{D\lfloor F_{0,\tau}\rceil^\lambda}\big(\lfloor\mathfrak{V}_0\rceil^\lambda,\mathrm{w}\big)\Big]^{-1}\Big\{(\theta,\varpi)^\lambda \circledast^\lambda_{\mathrm{I}} (\mathsf{v}_\tau,(\lfloor\mathfrak{V}_\tau^s\rceil^\lambda)_0') + \big(\widetilde{\circledast}^{\theta,\lambda}_{\mathrm{A}}(\mathsf{v}_\tau,(\lfloor\mathfrak{V}_\tau^s\rceil^\lambda)_0'),\widetilde{\circledast}^{\varpi,\lambda}_{\mathrm{A}}(\mathsf{v}_\tau,(\lfloor\mathfrak{V}_\tau^s\rceil^\lambda)_0')\big)\Big\},$$

where $\circledast^\lambda_{\mathrm{I}}(\mathsf{v}_\tau,(\lfloor\mathfrak{V}_\tau^s\rceil^\lambda)_0')$ *is* $\circledast^\lambda(\mathsf{v}_\tau,(\lfloor\mathfrak{V}_\tau^s\rceil^\lambda)_0')$ *with the Stratonovich infinitesimals* $\circ dB_\tau$, $\circ d\Upsilon^\lambda_{\mathrm{V},B}(\tau)$ *replaced by the Itô infinitesimals* $dB_\tau, d\Upsilon^\lambda_{\mathrm{V},B}(\tau)$,

$$\begin{aligned}
\widetilde{\circledast}^{\theta,\lambda}_{\mathrm{A}}\big(\mathsf{v}_\tau,(\mathfrak{V}_\tau^s)_0'\big) &= \varpi^\lambda\Big(\circledast\big(\mathsf{v}_\tau,(\lfloor\mathfrak{V}_\tau^s\rceil^\lambda)_0',e_i\big)\Big)\,e_i\;d\tau+\theta^\lambda\left(\Big[H(\mathfrak{V}_\tau,e_i),\circledast^\lambda\big(\mathsf{v}_\tau,(\lfloor\mathfrak{V}_\tau^s\rceil^\lambda)_0',e_i\big)\Big]\right)d\tau,\\
\widetilde{\circledast}^{\varpi,\lambda}_{\mathrm{A}}\big(\mathsf{v}_\tau,(\lfloor\mathfrak{V}_\tau^s\rceil^\lambda)_0'\big) &= (\lfloor\mathfrak{V}_\tau\rceil^\lambda)^{-1}R^\lambda\Big(\lfloor\mathfrak{V}_\tau\rceil^\lambda e_i,\lfloor\mathfrak{V}_\tau\rceil^\lambda\theta^\lambda(\circledast^\lambda(\mathsf{v}_\tau,(\lfloor\mathfrak{V}_\tau^s\rceil^\lambda)_0',e_i))\Big)\lfloor\mathfrak{V}_\tau\rceil^\lambda d\tau\\
&\quad+\varpi^\lambda\left(\Big[H^\lambda(\lfloor\mathfrak{V}_\tau\rceil^\lambda,e_i),\circledast^\lambda\big(\mathsf{v}_\tau,(\lfloor\mathfrak{V}_\tau^s\rceil^\lambda)_0',e_i\big)\Big]\right)\;d\tau,\ \textit{and}\\
\circledast^\lambda\left(\mathsf{v}_\tau,(\lfloor\mathfrak{V}_\tau^s\rceil^\lambda)_0',e_i\right) &= \nabla^{(2),\lambda}\big(\mathsf{v}_\tau,(\lfloor\mathfrak{V}_\tau^s\rceil^\lambda)_0'\big)H^\lambda(\lfloor\mathfrak{V}_\tau\rceil^\lambda,e_i)+R^\lambda\big(H^\lambda(\lfloor\mathfrak{V}_\tau\rceil^\lambda,e_i),(\lfloor\mathfrak{V}_\tau^s\rceil^\lambda)_0'\big)\mathsf{v}_\tau\\
&\quad+\nabla^\lambda(\mathsf{v}_\tau)H^\lambda\big(\lfloor\mathfrak{V}_\tau\rceil^\lambda,K^\lambda_{\mathrm{V},B}e_i\big).
\end{aligned}$$

Proof. Without loss of generality, we can consider the case $\lambda=0$. The i), ii) are straight forward consequences of Theorem 5.17 reporting $\alpha^0=B$ in the formulas in Lemma 5.12 and Corollary 5.10. For iii), a comparison of the SDEs (5.41), (5.43) in Lemma 5.14 with that of the tangent maps $[DF_{0,\tau}]^{-1}$, $[\widetilde{DF_{0,\tau}}]^{-1}$ shows that we can use Duhamel's principle to formulate $(\mathsf{v}_0^s)_0'$, $(\theta,\varpi)_{\mathfrak{V}_0}(\mathsf{v}_0^s)_0'$ as above. □

Proposition 5.32. *With all the notations as above, then*

$$\begin{aligned}
\left(\overline{(\lfloor\mathrm{u}_T^s\rceil^\lambda)^{(1)}_\lambda}(\mathrm{w})\right)_0' &= \int_0^T[D\lfloor F_{0,\tau}\rceil^\lambda(\lfloor\mathfrak{V}_0\rceil^\lambda,\mathrm{w})]^{-1}\Big\{\circledast^\lambda\big((\lfloor\mathfrak{V}_\tau\rceil^\lambda)^{(1)}_\lambda,(\lfloor\mathfrak{V}_\tau^s\rceil^\lambda)_0'\big)\\
&\quad-\big(\nabla^\lambda((\lfloor\mathfrak{V}_\tau^s\rceil^\lambda)_0')(H^\lambda)^{(1)}_\lambda(\lfloor\mathfrak{V}_\tau\rceil^\lambda,\cdot)\big)\circ dB_\tau-(H^\lambda)^{(1)}_\lambda\big(\lfloor\mathfrak{V}_\tau\rceil^\lambda,\circ d\underline{\Upsilon}^\lambda_{\mathrm{V},B}(\tau)\big)\Big\},
\end{aligned}\tag{5.87}$$

where $\circledast^\lambda\big((\lfloor\mathfrak{V}_\tau\rceil^\lambda)^{(1)}_\lambda,(\mathfrak{V}_\tau^s)_0'\big)$ *is as in (5.86) replacing* v_τ *by* $(\lfloor\mathfrak{V}_\tau\rceil^\lambda)^{(1)}_\lambda$ *and*

$$\begin{aligned}
(\lfloor\mathfrak{V}_\tau\rceil^\lambda)^{(1)}_\lambda(\mathrm{w}) :=& \big[D\lfloor F_{\tau,T}\rceil^\lambda(\lfloor\mathfrak{V}_0\rceil^\lambda,\mathrm{w})\big]^{-1}(\lfloor\mathrm{u}_0\rceil^\lambda)^{(1)}_\lambda\\
&-\int_\tau^T\big[D\lfloor F_{\tau,t}\rceil^\lambda(\lfloor\mathfrak{V}_\tau\rceil^\lambda,\mathrm{w})\big]^{-1}(H^\lambda)^{(1)}_\lambda\big(\lfloor\mathfrak{V}_t\rceil^\lambda,\circ dB_t(\mathrm{w})\big).
\end{aligned}$$

In $(\theta,\varpi)^\lambda$-chart, we have the Itô integral expression

$$\begin{aligned}
&\Big((\theta,\omega)^\lambda_{\mathfrak{O}^s_0}\big(\overline{(\lfloor \mathrm{u}^s_T\rceil^\lambda)^{(1)}_0}(\mathrm{w})\big)\Big)'_0\\
&=\int_0^T\Big[\widetilde{D\lfloor F_{0,\tau}\rceil^\lambda}\big(\lfloor\mathfrak{O}_0\rceil^\lambda,\mathrm{w}\big)\Big]^{-1}\Big\{(\theta,\varpi)^\lambda\circledast^\lambda_{\mathrm{I}}\big((\lfloor\mathfrak{O}_\tau\rceil^\lambda)^{(1)}_\lambda,(\lfloor\mathfrak{O}^s_\tau\rceil^\lambda)'_0\big)\\
&\quad+\Big(\widetilde{\circledast}^{\theta,\lambda}_{\mathrm{A}}((\lfloor\mathfrak{O}_\tau\rceil^\lambda)^{(1)}_\lambda,(\lfloor\mathfrak{O}^s_\tau\rceil^\lambda)'_0),\widetilde{\circledast}^{\varpi,\lambda}_{\mathrm{A}}((\lfloor\mathfrak{O}_\tau\rceil^\lambda)^{(1)}_\lambda,(\lfloor\mathfrak{O}^s_\tau\rceil^\lambda)'_0)\Big)\\
&\quad+\Big(\varpi^\lambda\big(\nabla^\lambda((\lfloor\mathfrak{O}^s_\tau\rceil^\lambda)'_0)(H^\lambda)^{(1)}_\lambda(\lfloor\mathfrak{O}_\tau\rceil^\lambda,e_i)\big)e_i\;d\tau,\\
&\qquad\varpi^\lambda\big(\nabla^\lambda((\lfloor\mathfrak{O}^s_\tau\rceil^\lambda)'_0)(H^\lambda)^{(1)}_\lambda(\lfloor\mathfrak{O}_\tau\rceil^\lambda,dB_\tau)\big)+\varpi^\lambda\big((H^\lambda)^{(1)}_\lambda(\lfloor\mathfrak{O}_\tau\rceil^\lambda,d\underline{\Upsilon}^\lambda_{\mathrm{V},B}(\tau))\big)\Big)\Big\}.
\end{aligned}$$

Proof. Differentiating (5.85), we obtain

$$\begin{aligned}
\Big(\overline{(\lfloor \mathrm{u}^s_T\rceil^\lambda)^{(1)}_\lambda}(\mathrm{w})\Big)'_0=&\big([D\lfloor F^s_{0,T}\rceil^\lambda(\lfloor\mathfrak{O}^s_0\rceil^\lambda,\mathrm{w})]^{-1}\big)'_0(\lfloor \mathrm{u}_0\rceil^\lambda)^{(1)}_\lambda\\
&-\int_0^T\big([D\lfloor F^s_{0,t}\rceil^\lambda(\lfloor\mathfrak{O}^s_0\rceil^\lambda,\mathrm{w})]^{-1}\big)'_0(H^\lambda)^{(1)}_\lambda\big(\lfloor\mathfrak{O}_t\rceil^\lambda,\circ dB_t(\mathrm{w})\big)\\
&-\int_0^T[D\lfloor F_{0,t}\rceil^\lambda(\lfloor\mathfrak{O}_0\rceil^\lambda,\mathrm{w})]^{-1}\big(\nabla^\lambda((\lfloor\mathfrak{O}^s_t\rceil^\lambda)'_0)(H^\lambda)^{(1)}_\lambda(\lfloor\mathfrak{O}_t\rceil^\lambda,\cdot)\big)\circ dB_t(\mathrm{w})\\
&-\int_0^T[D\lfloor F_{0,t}\rceil^\lambda(\lfloor\mathfrak{O}_0\rceil^\lambda,\mathrm{w})]^{-1}(H^\lambda)^{(1)}_\lambda\big(\lfloor\mathfrak{O}_t\rceil^\lambda,\circ d\Upsilon^\lambda_{\mathrm{V},B}(t,\mathrm{w})\big)\\
=:&\ \mathrm{(I)}+\mathrm{(II)}+\mathrm{(III)}+\mathrm{(IV)}.
\end{aligned}$$

By iii) of Lemma 5.31, we have

$$\mathrm{(I)}=\int_0^T\big[D\lfloor F_{0,\tau}\rceil^\lambda(\lfloor\mathfrak{O}_0\rceil^\lambda,\mathrm{w})\big]^{-1}\circledast^\lambda\Big(\big[D\lfloor F_{\tau,T}\rceil^\lambda(\lfloor\mathfrak{O}_0\rceil^\lambda,\mathrm{w})\big]^{-1}(\lfloor \mathrm{u}_0\rceil^\lambda)^{(1)}_\lambda,(\lfloor\mathfrak{O}^s_\tau\rceil^\lambda)'_0\Big).$$

For $0\leqslant\tau<t\leqslant T$, put

$$\mathsf{v}_{\tau,t}:=\big[D\lfloor F_{\tau,t}\rceil^\lambda(\lfloor\mathfrak{O}_\tau\rceil^\lambda,\mathrm{w})\big]^{-1}(H^\lambda)^{(1)}_\lambda\big(\lfloor\mathfrak{O}_t\rceil^\lambda,\circ dB_t(\mathrm{w})\big).$$

We continue to compute that

$$\begin{aligned}
\mathrm{(II)}&=-\int_0^T\int_0^t\big[D\lfloor F_{0,\tau}\rceil^\lambda(\lfloor\mathfrak{O}_0\rceil^\lambda,\mathrm{w})\big]^{-1}\circledast^\lambda\big(\mathsf{v}_{\tau,t},(\lfloor\mathfrak{O}^s_\tau\rceil^\lambda)'_0\big)\\
&=-\int_0^T\big[D\lfloor F_{0,\tau}\rceil^\lambda(\lfloor\mathfrak{O}_0\rceil^\lambda,\mathrm{w})\big]^{-1}\circledast^\lambda\Big(\int_\tau^T\mathsf{v}_{\tau,t},(\lfloor\mathfrak{O}^s_\tau\rceil^\lambda)'_0\Big).
\end{aligned}$$

Altogether, we obtain

$$\mathrm{(I)}+\mathrm{(II)}=\int_0^T[D\lfloor F_{0,\tau}\rceil^\lambda(\lfloor\mathfrak{O}_0\rceil^\lambda,\mathrm{w})]^{-1}\circledast^\lambda\big((\lfloor\mathfrak{O}_\tau\rceil^\lambda)^{(1)}_\lambda,(\lfloor\mathfrak{O}^s_\tau\rceil^\lambda)'_0\big).$$

Hence (5.87) holds true. The Itô form integral expression of $\big((\theta,\omega)_{\mho_0^s}\big(\overline{(\lfloor \mathrm{u}_T^s\rceil^\lambda)_0^{(1)}}(\mathrm{w})\big)\big)_0'$ can be obtained using the Itô form in iii) of Lemma 5.31. □

As a corollary of Proposition 5.32, we can further express the differential

$$((\lfloor \mathrm{u}_T\rceil^\lambda)_\lambda^{(1)}\circ\lfloor \mathbf{F}^s\rceil^\lambda)_0' =: \Big((\lfloor \mathrm{u}_T^s\rceil^\lambda)_\lambda^{(1)}(w)\Big)_0'$$

using $(\lfloor \mathrm{u}_t\rceil^\lambda(w))_{t\in[0,T]}$ and the tangent maps $\{[D\lfloor \overrightarrow{F}_{\underline{t},\bar{t}}\rceil^\lambda(\lfloor \mathrm{u}_{\underline{t}}\rceil^\lambda,w)]\}_{0\leqslant\underline{t}<\bar{t}\leqslant T}$ of the flow maps $\{\lfloor \overrightarrow{F}_{\underline{t},\bar{t}}\rceil^\lambda(\lfloor \mathrm{u}_{\underline{t}}\rceil^\lambda,w)\}_{0\leqslant\underline{t}<\bar{t}\leqslant T}$ associated to (5.73). We only give the Stratonovich form. Let

$$\begin{aligned}
&K_{\mathrm{V},B}^{\lambda,\mathrm{u}}(t,w):=\\
&\int_0^t(\lfloor \mathrm{u}_{T-\widetilde{\tau}}\rceil^\lambda)^{-1}R^\lambda\Big((\lfloor \mathrm{u}_{T-\widetilde{\tau}}\rceil^\lambda)^{-1}dB_{T-\widetilde{\tau}}(w),\lfloor \mathrm{u}_{T-\widetilde{\tau}}\rceil^\lambda(\lfloor \mathrm{u}_T\rceil^\lambda)^{-1}[\mathtt{s}(\widetilde{\tau})\mathrm{V}(y)]\Big)\lfloor \mathrm{u}_{T-\widetilde{\tau}}\rceil^\lambda\\
&+\int_0^t(\lfloor \mathrm{u}_{T-\widetilde{\tau}}\rceil^\lambda)^{-1}(\nabla^\lambda(\lfloor \mathrm{u}_{T-\widetilde{\tau}}\rceil^\lambda e_i)R^\lambda)\Big(\lfloor \mathrm{u}_{T-\widetilde{\tau}}\rceil^\lambda e_i,\lfloor \mathrm{u}_{T-\widetilde{\tau}}\rceil^\lambda(\lfloor \mathrm{u}_T\rceil^\lambda)^{-1}[\mathtt{s}(\widetilde{\tau})\mathrm{V}(y)]\Big)\lfloor \mathrm{u}_{T-\widetilde{\tau}}\rceil^\lambda\,d\widetilde{\tau},\\
&\underline{\Upsilon}_{\mathrm{V},B}^{\lambda,\mathrm{u}}(\tau):=\\
&\int_0^{T-\tau}\big((\lfloor \mathrm{u}_T\rceil^\lambda)^{-1}(\mathtt{s}'(t)\mathrm{V}(y))-\mathrm{Ric}^\lambda\big(\lfloor \mathrm{u}_{T-t}\rceil^\lambda(\lfloor \mathrm{u}_0\rceil^\lambda)^{-1}[\mathtt{s}(t)\mathrm{V}(y)]\big)\big)dt-\int_0^{T-\tau}\langle K_{\mathrm{V},B}^{\lambda,\mathrm{u}}(t,w),dB_t\rangle.
\end{aligned}$$

Then $\underline{\Upsilon}_{\mathrm{V},B}^{\lambda,\mathrm{u}}(\tau)$ corresponds to $\underline{\Upsilon}_{\mathrm{V},B}^{\lambda}(T-\tau)$ and they have the same distribution. Put

$$(\lfloor \mathrm{u}_{\tau,T}^s\rceil^\lambda)_0' := (\theta,\varpi)_{\lfloor \mathrm{u}_\tau\rceil^\lambda}^{-1}\Big(\mathtt{s}(T-\tau)(\lfloor \mathrm{u}_T\rceil^\lambda)^{-1}\mathrm{V}(y),K_{\mathrm{V},B}^{\lambda,\mathrm{u}}(T-\tau)\Big).$$

Corollary 5.33. *With all the notations as above,*

$$\begin{aligned}
&\Big((\lfloor \mathrm{u}_T^s\rceil^\lambda)_\lambda^{(1)}(w)\Big)_0'\\
&\quad=\int_0^T\big[D\lfloor \overrightarrow{F}_{\tau,T}\rceil^\lambda(\lfloor \mathrm{u}_\tau\rceil^\lambda,w)\big]\Big\{\circledast^{\lambda,\mathrm{u}}\big((\lfloor \mathrm{u}_\tau\rceil^\lambda)_\lambda^{(1)},(\lfloor \mathrm{u}_{\tau,T}^s\rceil^\lambda)_0'\big)\\
&\qquad+\big(\nabla^\lambda((\lfloor \mathrm{u}_{\tau,T}^s\rceil^\lambda)_0')(H^\lambda)_\lambda^{(1)}(\lfloor \mathrm{u}_\tau\rceil^\lambda,\cdot)\big)\circ dB_\tau(w)+(H^\lambda)_\lambda^{(1)}\big(\lfloor \mathrm{u}_\tau\rceil^\lambda,\circ d\Upsilon_{\mathrm{V},B}^{\lambda,\mathrm{u}}(\tau)\big)\Big\},
\end{aligned}$$

where

$$\begin{aligned}
\circledast^{\lambda,\mathrm{u}}\big((\lfloor \mathrm{u}_\tau\rceil^\lambda)_\lambda^{(1)},(\lfloor \mathrm{u}_{\tau,T}^s\rceil^\lambda)_0'\big)=&-\nabla^{\lambda,(2)}\big((\lfloor \mathrm{u}_\tau\rceil^\lambda)_\lambda^{(1)},(\lfloor \mathrm{u}_{\tau,T}^s\rceil^\lambda)_0'\big)H^\lambda\big(\lfloor \mathrm{u}_\tau\rceil^\lambda,\circ dB_\tau(w)\big)\\
&-\nabla^\lambda((\lfloor \mathrm{u}_\tau\rceil^\lambda)_\lambda^{(1)})H^\lambda\big(\lfloor \mathrm{u}_\tau\rceil^\lambda,\circ d\underline{\Upsilon}_{\mathrm{V},B}^{\lambda,\mathrm{u}}(\tau)\big)\\
&-R^\lambda\big(H^\lambda(\lfloor \mathrm{u}_\tau\rceil^\lambda,\circ dB_\tau(w)),(\lfloor \mathrm{u}_{\tau,T}^s\rceil^\lambda)_0'\big)(\lfloor \mathrm{u}_\tau\rceil^\lambda)_\lambda^{(1)}.
\end{aligned}$$

In $(\theta,\varpi)^\lambda$-chart, we have the Itô integral expression

$$\begin{aligned}
&\Big((\theta,\varpi)(\lfloor \mathrm{u}_T^s\rceil^\lambda)_\lambda^{(1)}(w)\Big)_0' \\
&= \int_0^T \big[D\lfloor\widetilde{\overrightarrow{F}_{\tau,T}}\rceil^\lambda(\lfloor \mathrm{u}_\tau\rceil^\lambda, w)\big]\Big\{(\theta,\varpi)^\lambda \circledast_{\mathrm{I}}^{\lambda,\mathrm{u}} \big((\lfloor \mathrm{u}_\tau\rceil^\lambda)_\lambda^{(1)}, (\lfloor \mathrm{u}_{\tau,T}^s\rceil^\lambda)_0'\big) \\
&\quad + \Big(\widetilde{\circledast}_{\mathrm{A}}^{\theta,\lambda}\big((\lfloor \mathrm{u}_\tau\rceil^\lambda)_\lambda^{(1)}, (\lfloor \mathrm{u}_{\tau,T}^s\rceil^\lambda)_0'\big), \widetilde{\circledast}_{\mathrm{A}}^{\varpi,\lambda}\big((\lfloor \mathrm{u}_\tau\rceil^\lambda)_\lambda^{(1)}, (\lfloor \mathrm{u}_{\tau,T}^s\rceil^\lambda)_0'\big)\Big) \\
&\quad + \Big(\varpi^\lambda\big(\nabla^\lambda((\lfloor \mathrm{u}_{\tau,T}^s\rceil^\lambda)_0')(H^\lambda)_\lambda^{(1)}(\lfloor \mathrm{u}_\tau\rceil^\lambda, e_i)\big)e_i\, d\tau, \\
&\qquad \varpi^\lambda\big(\nabla^\lambda((\lfloor \mathrm{u}_{\tau,T}^s\rceil^\lambda)_0')(H^\lambda)_\lambda^{(1)}(\lfloor \mathrm{u}_\tau\rceil^\lambda, dB_\tau)\big) + \varpi^\lambda\big((H^\lambda)_\lambda^{(1)}\big(\lfloor \mathrm{u}_\tau\rceil^\lambda, d\underline{\Upsilon}_{\mathrm{V},B}^{\lambda,\mathrm{u}}(\tau)\big)\big)\Big)\Big\}.
\end{aligned}$$

Proof. Note that $\big((\lfloor \mathrm{u}_T^s\rceil^\lambda)_\lambda^{(1)}(w)\big)_0'$ conditioned on $\lfloor \mathrm{x}_T\rceil^\lambda = y$ is the same as $\big(\overline{(\lfloor \mathrm{u}_T^s\rceil^\lambda)_\lambda^{(1)}}(\mathrm{w})\big)_0'$ conditioned on $\lfloor \mathrm{y}_T\rceil^\lambda = x$. The formulas follow by Proposition 5.32 using the correspondence between

$$\big[D\lfloor\overrightarrow{F}_{\underline{t},\bar{t}}\rceil^\lambda(\lfloor \mathrm{u}_{\underline{t}}\rceil^\lambda, w)\big] \text{ and } \big[D\lfloor F_{T-\bar{t},T-\underline{t}}\rceil^\lambda(\lfloor \mathfrak{O}_{T-\bar{t}}\rceil^\lambda, \mathrm{w})\big]^{-1}.$$

□

5.6. **The differential of $\lambda \mapsto p^\lambda(T,x,\cdot)$.** We will show Theorem 5.1 in two steps, namely, the $k=3$ and $k>3$ cases. We begin with the $k=3$ case. As we sketched in Section 5.1, the strategy is to show $\mathrm{z}_T^{\lambda,1}$ defined in (5.15) is a C^1 vector field, then derive a conditional path-wise formula of $\mathrm{Div}^\lambda \mathrm{z}_T^{\lambda,1}(y)$ and use it to give the estimation in (5.2).

Lemma 5.34. *Let $\lambda \in (-1,1) \mapsto g^\lambda \in \mathcal{M}^3(M)$ be a C^3 curve. Let $x \in \widetilde{M}$, $T \in \mathbb{R}_+$. The map $\overline{\Phi}_\lambda^1 :\ Y \mapsto \overline{\Phi}_\lambda^1(Y)$ defined in (5.13) is a locally bounded C^1 functional on C^k bounded vector fields Y on $\widetilde{M}$. Consequently, $\{\mathrm{z}_T^{\lambda,1}(y)\}$ is a C^1 vector field on $\widetilde{M}$.*

Proof. Recall that

$$\overline{\Phi}_\lambda^1(Y)(y) = \overline{\mathbb{E}}\left(\big\langle Y(\lfloor \mathrm{x}_T\rceil^\lambda(w)), D\pi(\mathrm{u}_T\rceil^\lambda)_\lambda^{(1)}(w)\big\rangle_\lambda \Big| \lfloor \mathrm{x}_T\rceil^\lambda(w) = y\right) = \big\langle Y(y), \mathrm{z}_T^{\lambda,1}(y)\big\rangle.$$

Hence,

$$\big\|\overline{\Phi}_\lambda^1(\cdot)(y)\big\| \leqslant \big\|\mathrm{z}_T^{\lambda,1}(y)\big\| \leqslant \overline{\mathbb{E}}_{\mathbb{P}_{x,y,T}^\lambda}\big\|(\lfloor \mathrm{u}_T\rceil^\lambda)_\lambda^{(1)}(w)\big\| = \frac{1}{p(T,x,y)}\overline{\mathbb{E}}_{\mathbb{P}_{x,y,T}^{\lambda,*}}\big\|(\lfloor \mathrm{u}_T\rceil^\lambda)_\lambda^{(1)}(w)\big\|.$$

By Proposition 4.28, there are $\underline{c}_\Phi^\lambda$ (depending on $\|g^\lambda\|_{C^2}$ and $\|\mathcal{X}^\lambda\|_{C^1}$) and c_Φ^λ (depending on T, T_0 and $\|g^\lambda\|_{C^3}$) such that

$$\big\|\overline{\Phi}_\lambda^1(\cdot)(y)\big\| \leqslant \frac{1}{p(T,x,y)}\overline{\mathbb{E}}_{\overline{\mathbb{P}}_{y,x,T}^{\lambda,*}}\big\|\overline{(\lfloor \mathrm{u}_T\rceil^\lambda)_\lambda^{(1)}}(\mathrm{w})\big\| \leqslant \frac{1}{p(T,x,y)}\underline{c}_\Phi^\lambda e^{c_\Phi^\lambda(1+d_{\tilde{g}^\lambda}(x,y))},$$

where the last term is locally uniformly bounded in the y-coordinate. This shows the map $Y \mapsto \overline{\Phi}_\lambda^1(Y)$ is locally bounded.

To show $\overline{\Phi}^1_\lambda$ is C^1, it suffices to show for any flow F^s generated by a smooth bounded vector field V on $\widetilde{M}$, $s \mapsto \overline{\Phi}^1_\lambda(Y)(F^s y), y \in \widetilde{M}$, is differentiable at $s=0$ and the differential

$$(\overline{\Phi}^1_\lambda(Y)(F^s y))'_0 := \left.\frac{d}{ds}\overline{\Phi}^1_\lambda(Y)(F^s y)\right|_{s=0}$$

varies continuously in y. Let $\lfloor \mathbf{F}^s \rceil^\lambda$ be introduced in Section 5.5, which extends F^s to Brownian paths starting from x up to time T using the auxiliary function $\mathtt{s}$. By Proposition 5.23, $\overline{\mathbb{P}}^\lambda_x \circ \lfloor \mathbf{F}^s \rceil^\lambda$ is absolutely continuous with respect to $\overline{\mathbb{P}}^\lambda_x$. So the change of variable comparison in Section 5.1 works, which gives (5.20), i.e.,

$$\overline{\Phi}^1_\lambda(Y)(F^s y) = \overline{\mathbb{E}}_{\overline{\mathbb{P}}^\lambda_{x,y,T}}\left(\Phi^1_\lambda(Y,w)\circ\lfloor \mathbf{F}^s\rceil^\lambda \cdot \frac{d\overline{\mathbb{P}}^\lambda_x \circ \lfloor \mathbf{F}^s\rceil^\lambda}{d\overline{\mathbb{P}}^\lambda_x}\right)\frac{p^\lambda(T,x,y)}{p^\lambda(t,x,F^s y)}\frac{d\mathrm{Vol}^\lambda}{d\mathrm{Vol}^\lambda\circ F^s}(y),$$

where

$$\Phi^1_\lambda(Y,w) = \left\langle Y(\lfloor \mathrm{x}_T\rceil^\lambda(w)), D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda(w)\right\rangle_\lambda.$$

By Proposition 5.30, the process $\Phi^1_\lambda(Y,w)\circ\lfloor \mathbf{F}^s\rceil^\lambda$ is differentiable in s with

(5.88)

$$(\Phi^1_\lambda\circ\lfloor \mathbf{F}^s\rceil^\lambda)'_s = \left\langle \nabla_{\mathrm{V}(\lfloor \mathrm{x}^s_T\rceil^\lambda)}Y(\lfloor \mathrm{x}^s_T\rceil^\lambda), D\pi(\lfloor \mathrm{u}^s_T\rceil^\lambda)^{(1)}_\lambda\right\rangle_\lambda + \left\langle Y(\lfloor \mathrm{x}^s_T\rceil^\lambda), \nabla^{s,\lambda}_{T,\mathrm{V},\mathtt{s}} D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda\right\rangle_\lambda$$

and this differential is L^q integrable conditioned on $\mathrm{x}_T = y$ for every $q \geqslant 1$, locally uniformly in the s parameter. By Lemma 5.22, for $\beta \in C_{x,y}([0,T],\widetilde{M})$,

$$\frac{d\overline{\mathbb{P}}^\lambda_x \circ \lfloor \mathbf{F}^s\rceil^\lambda}{d\overline{\mathbb{P}}^\lambda_x}(\beta) = \frac{d\overline{\mathbb{P}}^\lambda_{F^s y,x,T}\circ\lfloor \mathbf{F}^s\rceil^\lambda}{d\overline{\mathbb{P}}^\lambda_{y,x,T}}(\beta)\cdot\frac{p^\lambda(T,x,F^s y)}{p^\lambda(T,x,y)}\cdot\frac{d\mathrm{Vol}^\lambda\circ F^s}{d\mathrm{Vol}^\lambda}(y).$$

By Proposition 5.20, $d\overline{\mathbb{P}}^\lambda_{F^s y,x,T}\circ\lfloor \mathbf{F}^s\rceil^\lambda / d\overline{\mathbb{P}}^\lambda_{y,x,T}$ is differentiable in s with

$$\left(d\overline{\mathbb{P}}^\lambda_{F^s y,x,T}\circ\lfloor \mathbf{F}^s\rceil^\lambda / d\overline{\mathbb{P}}^\lambda_{y,x,T}\right)'_s = \left(d\overline{\mathbb{P}}^\lambda_{F^s y,x,T}\circ\lfloor \mathbf{F}^s\rceil^\lambda / d\overline{\mathbb{P}}^\lambda_{y,x,T}\right)\cdot\overline{\mathcal{E}}^s_T,$$

where

$$\overline{\mathcal{E}}^s_T = -\frac{1}{2}\int_0^T \left\langle (\overline{\mathrm{g}}^s_t)'_s(\mathrm{w}), dB_t(\mathrm{w})\right\rangle + \frac{1}{2}\int_0^T \left\langle (\overline{\mathrm{g}}^s_t)'_s(\mathrm{w}), \overline{\mathrm{g}}^s_t(\mathrm{w})\right\rangle dt,$$

and both $\left(d\overline{\mathbb{P}}^\lambda_{F^s y,x,T}\circ\lfloor \mathbf{F}^s\rceil^\lambda / d\overline{\mathbb{P}}^\lambda_{y,x,T}\right)$ and $\overline{\mathcal{E}}^s_T$ are L^q integrable for all $q \geqslant 1$, locally uniformly in the s parameter. Using Hölder's inequality, we conclude that

$$\left(\Phi^1_\lambda\circ\lfloor \mathbf{F}^s\rceil^\lambda \cdot \left(d\overline{\mathbb{P}}^\lambda_x\circ\lfloor \mathbf{F}^s\rceil^\lambda / d\overline{\mathbb{P}}^\lambda_x\right)\right)'_s$$

is also L^q integrable for every $q \geqslant 1$, locally uniformly in the s parameter. This allows us to take the differential in s under the expectation sign of the expression of $\overline{\Phi}^1_\lambda(Y)(F^s y)$. In particular, this shows $s \mapsto \overline{\Phi}^1_\lambda(Y)(F^s y)$ is differentiable at $s=0$.

Let us derive a formula for $(\overline{\Phi}^1_\lambda(Y)(F^s y))'_0$. Note that $\lfloor \overline{\mathrm{g}}^0\rceil^\lambda \equiv 0$ and

$$(\lfloor \overline{\mathrm{g}}^s_t\rceil^\lambda)'_0(\mathrm{w}) = (\lfloor \mathcal{U}_0\rceil^\lambda)^{-1}\left[\mathtt{s}'(t)\mathrm{V}(y)\right] - \mathrm{Ric}^\lambda\left(\lfloor \mathcal{U}_t\rceil^\lambda(\lfloor \mathcal{U}_0\rceil^\lambda)^{-1}[\mathtt{s}(t)\mathrm{V}(y)]\right).$$

Using the correspondence between $\lfloor \mathfrak{O}_t \rceil^\lambda(\mathrm{w})$ conditioned on $\mathrm{y}_T^\lambda = x$ and $\lfloor \mathrm{u}_{T-t} \rceil^\lambda(w)$ conditioned on $\mathrm{x}_T^\lambda = y$, we have the distribution of $\overline{\mathcal{E}}_T^0$ under $\overline{\mathbb{P}}_{y,x,T}^\lambda$ is the same as

$$\overleftarrow{\mathcal{E}}_{T,\mathrm{V},\mathtt{s}} = -\frac{1}{2}\int_0^T \left\langle \mathtt{s}'(T-t)(\lfloor \mathrm{u}_T \rceil^\lambda)^{-1}\mathrm{V}(\lfloor \mathrm{x}_T \rceil^\lambda) - \mathrm{Ric}(\lfloor \mathrm{u}_t \rceil^\lambda(\lfloor \mathrm{u}_T \rceil^\lambda)^{-1}\mathtt{s}(T-t)\mathrm{V}(\lfloor \mathrm{x}_T \rceil^\lambda)), d\overleftarrow{B}_t \right\rangle.$$

under $\overline{\mathbb{P}}_{x,y,T}^\lambda$, where $\mathtt{s}$ is given in Section 5.2. So, by (5.20) and (5.88), we have

$$\begin{aligned}(\overline{\Phi}_\lambda^1(Y)(F^s y))_0' = \overline{\mathbb{E}}_{\overline{\mathbb{P}}_{x,y,T}^\lambda}\Big(&\left\langle \nabla_{\mathrm{V}(\lfloor \mathrm{x}_T \rceil^\lambda)}Y, D\pi(\lfloor \mathrm{u}_T \rceil^\lambda)_\lambda^{(1)} \right\rangle_\lambda + \left\langle Y(\lfloor \mathrm{x}_T \rceil^\lambda), \nabla_{T,\mathrm{V},\mathtt{s}}^\lambda D\pi(\lfloor \mathrm{u}_T \rceil^\lambda)_\lambda^{(1)} \right\rangle_\lambda \\ &+ \left\langle Y(\lfloor \mathrm{x}_T \rceil^\lambda), D\pi(\lfloor \mathrm{u}_T \rceil^\lambda)_\lambda^{(1)} \right\rangle_\lambda \overleftarrow{\mathcal{E}}_{T,\mathrm{V},\mathtt{s}}\Big) \\ =: \overline{\mathbb{E}}_{\overline{\mathbb{P}}_{x,y,T}^\lambda}&\left(\Psi_\lambda^1(Y,\mathrm{V})(w)\right), \end{aligned} \tag{5.89}$$

where we omit the upper-script 0 of x, u and $\nabla_{T,\mathrm{V},\mathtt{s}}$ at $s = 0$ for simplicity.

To show $(\overline{\Phi}_\lambda^1(Y)(F^s y))_0'$ is continuous in y, we compare it with its value at nearby points. Choose another smooth bounded vector field W on $\widetilde{M}$ and let ${}^rF$ be the flow it generates, where we use the left upper script to indicate the parameter associated with W. As before, we can extend ${}^rF$ to be a one-parameter of maps $\lfloor {}^r\mathbf{F} \rceil^\lambda = \{\lfloor {}^r\mathbf{F}_y \rceil^\lambda\}$ on $\tilde{g}^\lambda$-Brownian paths starting from x up to time T. Let $\lfloor {}^r\alpha \rceil^\lambda, \lfloor {}^rO \rceil^\lambda, \lfloor {}^r\mathfrak{g} \rceil^\lambda, \lfloor {}^r\overline{\mathfrak{g}} \rceil^\lambda, \lfloor {}^r\mathrm{y} \rceil^\lambda, \lfloor {}^r\mathfrak{O} \rceil^\lambda, (\lfloor {}^r\mathfrak{O} \rceil^\lambda)_r'$ denote the corresponding stochastic processes of $\lfloor {}^r\mathbf{F} \rceil^\lambda$ in Theorem 5.17. Then a change of variable argument for (5.89) with $\lfloor {}^r\mathbf{F} \rceil^\lambda$ shows that for $z = {}^rF(y)$,

$$(\overline{\Phi}_\lambda^1(Y)(F^s z))_0' = \overline{\mathbb{E}}_{\overline{\mathbb{P}}_{x,y,T}^\lambda}\left(\Psi_\lambda^1(Y,\mathrm{V}) \circ \lfloor {}^r\mathbf{F} \rceil^\lambda \cdot \frac{d\overline{\mathbb{P}}_x^\lambda \circ \lfloor {}^r\mathbf{F} \rceil^\lambda}{d\overline{\mathbb{P}}_x^\lambda}\right)\frac{p^\lambda(T,x,y)}{p^\lambda(T,x,z)}\frac{d\mathrm{Vol}^\lambda}{d\mathrm{Vol}^\lambda \circ {}^rF}(y).$$

Since p^λ and Vol^λ are continuous in y, for continuity of $(\overline{\Phi}_\lambda^1(Y)(F^s y))_0'$ in y, it remains to show the conditional expectation of the following difference tends to 0 as r goes to 0:

$$\begin{aligned}&\Psi_\lambda^1(Y,\mathrm{V}) \circ \lfloor {}^r\mathbf{F} \rceil^\lambda \cdot \frac{d\overline{\mathbb{P}}_x^\lambda \circ \lfloor {}^r\mathbf{F} \rceil^\lambda}{d\overline{\mathbb{P}}_x^\lambda} - \Psi_\lambda^1(Y,\mathrm{V}) \\ &= \left(\Psi_\lambda^1(Y,\mathrm{V}) \circ \lfloor {}^r\mathbf{F} \rceil^\lambda - \Psi_\lambda^1(Y,\mathrm{V})\right)\frac{d\overline{\mathbb{P}}_x^\lambda \circ \lfloor {}^r\mathbf{F} \rceil^\lambda}{d\overline{\mathbb{P}}_x^\lambda} + \left(\frac{d\overline{\mathbb{P}}_x^\lambda \circ \lfloor {}^r\mathbf{F} \rceil^\lambda}{d\overline{\mathbb{P}}_x^\lambda} - 1\right)\Psi_\lambda^1(Y,\mathrm{V}) \\ &=: {}^r(\mathrm{I})_1 \cdot {}^r(\mathrm{I})_2 + {}^r(\mathrm{II})_1 \cdot (\mathrm{II})_2. \end{aligned}$$

For this, it suffices to show

$$\lim_{r\to 0}\overline{\mathbb{E}}_{\overline{\mathbb{P}}_{x,y,T}^\lambda}\left|{}^r(\mathrm{I})_1\right|^2 = 0 \text{ and } \lim_{r\to 0}\overline{\mathbb{E}}_{\overline{\mathbb{P}}_{x,y,T}^\lambda}\left|{}^r(\mathrm{II})_1\right|^2 = 0 \tag{5.90}$$

since $\overline{\mathbb{E}}_{\overline{\mathbb{P}}_{x,y,T}^\lambda}|{}^r(\mathrm{I})_2|^2$ is locally uniformly bounded in r by Proposition 5.23 and $\overline{\mathbb{E}}_{\overline{\mathbb{P}}_{x,y,T}^\lambda}|(\mathrm{II})_2|^2$ is bounded by using Proposition 5.20 and Proposition 5.29.

Note that $|Y|, |\nabla_{\mathrm{V}}Y|$ are locally bounded at y and the difference between $Y(z)$ and $Y(y)$, $\nabla_{\mathrm{V}}Y(z)$ and $\nabla_{\mathrm{V}}Y(y)$ under parallel transportation along $(\imath \mapsto {}^\imath F(y))_{\imath\in[0,r]}$ is bounded by

a multiple of r. Using this, (5.89) and a standard split argument by Hölder's inequality, we see that to conclude the first property in (5.90), it suffices to show

$$\begin{aligned}
{}^r(\mathrm{III}) &:= \overline{\mathbb{E}}_{\overline{\mathbb{P}}^\lambda_{x,y,T}}\left(\left|\overleftarrow{\mathcal{E}}_{T,\mathrm{V},\mathtt{s}}\circ\lfloor{}^r\mathbf{F}\rceil^\lambda - \overleftarrow{\mathcal{E}}_{T,\mathrm{V},\mathtt{s}}\right|^4\right)\to 0,\ r\to 0,\\
{}^r(\mathrm{IV}) &:= \overline{\mathbb{E}}_{\overline{\mathbb{P}}^\lambda_{x,y,T}}\left(\left|D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda\circ\lfloor{}^r\mathbf{F}\rceil^\lambda - D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda\right|^4\right)\to 0,\ r\to 0,\\
{}^r(\mathrm{V}) &:= \overline{\mathbb{E}}_{\overline{\mathbb{P}}^\lambda_{x,y,T}}\left(\left|\nabla_{T,\mathrm{V},\mathtt{s}}D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda\circ\lfloor{}^r\mathbf{F}\rceil^\lambda - \nabla^\lambda_{T,\mathrm{V},\mathtt{s}}D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda\right|^2\right)\to 0,\ r\to 0.
\end{aligned}$$

Let

$$\begin{aligned}
\overline{\mathcal{E}}^0_T\circ\lfloor{}^r\mathbf{F}\rceil^\lambda &= -\frac{1}{2}\int_0^T\Big\langle(\lfloor{}^r\mathfrak{V}_0\rceil^\lambda)^{-1}\big[\mathtt{s}'(t)\mathrm{V}(y)\big] - \mathrm{Ric}^\lambda\big(\lfloor{}^r\mathfrak{V}_t\rceil^\lambda(\lfloor{}^r\mathfrak{V}_0\rceil^\lambda)^{-1}[\mathtt{s}(t)\mathrm{V}(y)]\big),\ d\overleftarrow{\lfloor{}^r\alpha_t\rceil^\lambda}\Big\rangle\\
&=: -\frac{1}{2}\int_0^T\big\langle{}^r(\lfloor\overline{\mathrm{g}}^s_t\rceil^\lambda)'_0(\mathrm{w}),\ d\overleftarrow{\lfloor{}^r\alpha_t\rceil^\lambda}\big)\big\rangle.
\end{aligned}$$

For ${}^r(\mathrm{III})$, we have

$$\begin{aligned}
2\cdot{}^r(\mathrm{III}) &= 2\cdot\overline{\mathbb{E}}_{\overline{\mathbb{P}}^\lambda_{y,x,T}}\left(\left|\overline{\mathcal{E}}^0_T\circ\lfloor{}^r\mathbf{F}\rceil^\lambda - \overline{\mathcal{E}}^0_T\right|^4\right)\\
&\leqslant \overline{\mathbb{E}}_{\overline{\mathbb{P}}^\lambda_{y,x,T}}\left|\int_0^T\Big\langle{}^r(\lfloor\overline{\mathrm{g}}^s_t\rceil^\lambda)'_0(\mathrm{w}),\ \big(\overleftarrow{\lfloor{}^rO_t\rceil^\lambda}-\mathrm{Id}\big)dB_t(\mathrm{w}) + \overleftarrow{\lfloor{}^r\mathrm{g}_t\rceil^\lambda}\ dt\Big\rangle\right|^4\\
&\quad+\overline{\mathbb{E}}_{\overline{\mathbb{P}}^\lambda_{y,x,T}}\left|\int_0^T\big\langle{}^r(\lfloor\overline{\mathrm{g}}^s_t\rceil^\lambda)'_0(\mathrm{w}) - {}^0(\lfloor\overline{\mathrm{g}}^s_t\rceil^\lambda)'_0(\mathrm{w}),\ dB_t(\mathrm{w})\big\rangle\right|^4\\
&=: {}^r(\mathrm{III})_1 + {}^r(\mathrm{III})_2.
\end{aligned}$$

For ${}^r(\mathrm{III})_1$, the usual argument using Lemma 4.16 and Burkholder's inequality shows

$$\begin{aligned}
3^{-3}\cdot{}^r(\mathrm{III})_1 \leqslant{}& \overline{\mathbb{E}}_{\overline{\mathbb{P}}^\lambda_{y,x,T}}\left|\int_0^T\big\|\overleftarrow{\lfloor{}^rO_t\rceil^\lambda}-\mathrm{Id}\big\|^2\big\|{}^r(\lfloor\overline{\mathrm{g}}^s_t\rceil^\lambda)'_0(\mathrm{w})\big\|^2\ dt\right|^2\\
&+\overline{\mathbb{E}}_{\overline{\mathbb{P}}^\lambda_{y,x,T}}\left|\int_0^T 2\big\|{}^r(\lfloor\overline{\mathrm{g}}^s_t\rceil^\lambda)'_0(\mathrm{w})\big\|\big\|\overleftarrow{\lfloor{}^rO_t\rceil^\lambda}-\mathrm{Id}\big\|\big\|\nabla^\lambda\ln p^\lambda(T-t,\lfloor \mathrm{y}_t\rceil^\lambda(\mathrm{w}),x)\big\|\ dt\right|^4\\
&+\overline{\mathbb{E}}_{\overline{\mathbb{P}}^\lambda_{y,x,T}}\left|\int_0^T\big\|{}^r(\lfloor\overline{\mathrm{g}}^s_t\rceil^\lambda)'_0(\mathrm{w})\big\|\big\|\overleftarrow{\lfloor{}^r\mathrm{g}_t\rceil^\lambda}\big\|\ dt\right|^4.
\end{aligned}$$

Note that there is some constant C which depends on $\|g^\lambda\|_C, \mathtt{s}$ and $\sup\{\|\mathrm{V}\|\}$ such that

$$\big|\overleftarrow{\lfloor{}^r\mathrm{g}_t\rceil^\lambda}(\mathrm{w})\big|,\ \big|{}^r(\lfloor\overline{\mathrm{g}}^s_t\rceil^\lambda)'_0(\mathrm{w})\big|\leqslant Cr.$$

Hence

$$\begin{aligned}
3^{-3}\cdot{}^r(\mathrm{III})_1 \leqslant{}& (Cr)^8T^4 + (Cr)^4T^2\overline{\mathbb{E}}_{\overline{\mathbb{P}}^\lambda_{y,x,T}}\sup_{t\in[0,T]}\big\|\overleftarrow{\lfloor{}^rO_t\rceil^\lambda}-\mathrm{Id}\big\|^4\\
&+(2Cr)^4\left(\overline{\mathbb{E}}_{\overline{\mathbb{P}}^\lambda_{y,x,T}}\sup_{t\in[0,T]}\big\|\overleftarrow{\lfloor{}^rO_t\rceil^\lambda}-\mathrm{Id}\big\|^8\cdot\overline{\mathbb{E}}_{\overline{\mathbb{P}}^\lambda_{y,x,T}}e^{8\int_0^T\left\|\nabla^\lambda\ln p^\lambda(T-t,\lfloor \mathrm{y}_t\rceil^\lambda,x)\right\|\ dt}\right)^{\frac{1}{2}}.
\end{aligned}$$

By Lemma 5.26 and Lemma 5.28, for any $q \geqslant 1$, there is some $C_1(q,T)$ such that

$$\overline{\mathbb{E}}_{\overline{\mathbb{P}}^\lambda_{y,x,T}} \sup_{t\in[0,T]} \left\| \lfloor {}^r\alpha_t \rceil^\lambda - \lfloor \alpha_t \rceil^\lambda \right\|^q \leqslant C_1(q,T) r^q.$$

Using this and (4.38), we conclude that ${}^r(\mathrm{III})_1 \to 0$ as $r \to 0$. Similarly, using (4.41), Burkholder's inequality and (4.38), we obtain some C_2 depending on $T, d(x,y)$ such that

$$\begin{aligned}
{}^r(\mathrm{III})_2 &\leqslant \overline{\mathbb{E}}_{\overline{\mathbb{P}}^\lambda_{y,x,T}} \left(\int_0^T \left\| {}^r(\lfloor \overline{\mathfrak{g}}^s_t \rceil^\lambda)'_0 - (\lfloor \overline{\mathfrak{g}}^s_t \rceil^\lambda)'_0 \right\|^2 dt \right)^2 \\
&\quad + \overline{\mathbb{E}}_{\overline{\mathbb{P}}^\lambda_{y,x,T}} \left| \int_0^T \left\langle \left({}^r(\lfloor \overline{\mathfrak{g}}^s_t \rceil^\lambda)'_0 - (\lfloor \overline{\mathfrak{g}}^s_t \rceil^\lambda)'_0 \right), 2(\lfloor \mathfrak{V}_t \rceil^\lambda)^{-1} \nabla^\lambda \ln p^\lambda(T-t, \lfloor \mathrm{y}_t \rceil^\lambda, x) \right\rangle_\lambda dt \right|^4 \\
&\leqslant C_2 \overline{\mathbb{E}}_{\overline{\mathbb{P}}^\lambda_{y,x,T}} \left(\sup_{t\in[0,T]} \left\| {}^r(\lfloor \overline{\mathfrak{g}}^s_t \rceil^\lambda)'_0 - (\lfloor \overline{\mathfrak{g}}^s_t \rceil^\lambda)'_0 \right\|^4 + \sup_{t\in[0,T]} \left\| {}^r(\lfloor \overline{\mathfrak{g}}^s_t \rceil^\lambda)'_0 - (\lfloor \overline{\mathfrak{g}}^s_t \rceil^\lambda)'_0 \right\|^8 \right).
\end{aligned}$$

The argument in Lemma 5.26 shows there is some C_3 depending on $\|g^\lambda\|_{C^3}$ such that

$$\begin{aligned}
\overline{\mathbb{E}}_{\overline{\mathbb{P}}^\lambda_{y,x,T}} \sup_{t\in[0,T]} \left\| {}^r(\lfloor \overline{\mathfrak{g}}^s_t \rceil^\lambda)'_0 - (\lfloor \overline{\mathfrak{g}}^s_t \rceil^\lambda)'_0 \right\|^q &\leqslant C_3 \cdot \overline{\mathbb{E}}_{\overline{\mathbb{P}}^\lambda_{y,x,T}} \sup_{t\in[0,T]} \left\| \lfloor {}^r\mathfrak{V}_t \rceil^\lambda - \lfloor \mathfrak{V}_t \rceil^\lambda \right\|^q \\
&\leqslant C_3 r^q \sup_{\imath\in[-r_0,r_0]} \overline{\mathbb{E}}_{\overline{\mathbb{P}}^\lambda_{y,x,T}} \sup_{t\in[0,T]} \left\| (\lfloor {}^r\mathfrak{V}_t \rceil^\lambda)'_\imath \right\|^q.
\end{aligned}$$

This immediately implies that $\lim_{r\to 0} {}^r(\mathrm{III})_2 = 0$. For ${}^r(\mathrm{IV})$, we have

$${}^r(\mathrm{IV}) \leqslant \mathrm{Const.} \cdot \overline{\mathbb{E}}_{\overline{\mathbb{P}}^\lambda_{y,x,T}} \left(\left| \overline{(\lfloor {}^r\mathrm{u}_T \rceil^\lambda)^{(1)}_0} - \overline{(\lfloor \mathrm{u}_T \rceil^\lambda)^{(1)}_0} \right|^4 \right).$$

Note that

$$\begin{aligned}
(\theta,\omega)^\lambda_{\lfloor {}^r\mathfrak{V}_0 \rceil^\lambda} \overline{(\lfloor {}^r\mathrm{u}_T \rceil^\lambda)^{(1)}_\lambda} &= [D\widetilde{\lfloor {}^rF_{0,T} \rceil^\lambda}(\lfloor {}^r\mathfrak{V}_0 \rceil^\lambda, \mathrm{w})]^{-1} (\widetilde{\lfloor \mathrm{u}_0 \rceil^\lambda})^{(1)}_\lambda + \int_0^T [D\widetilde{\lfloor {}^rF_{0,t} \rceil^\lambda}(\lfloor {}^r\mathfrak{V}_0 \rceil^\lambda, \mathrm{w})]^{-1} \\
&\qquad \left(\varpi\big((H^\lambda)^{(1)}_\lambda (\lfloor {}^r\mathfrak{V}_t \rceil^\lambda, e_i) \big) e_i \, d\tau, \varpi\big((H^\lambda)^{(1)}_\lambda (\lfloor {}^r\mathfrak{V}_t \rceil^\lambda, d\overleftarrow{\lfloor {}^r\alpha \rceil^\lambda_t}) \big) \right).
\end{aligned}$$

By Lemma 5.26 and Lemma 5.27, for any $q \geqslant 1$,

$$\sup_{r\in[-r_0,r_0]} \overline{\mathbb{E}}_{\overline{\mathbb{P}}^\lambda_{y,x,T}} \sup_{t\in[0,T]} \left\| \lfloor {}^r\alpha \rceil^\lambda_t \right\|^q, \ \left\| [D\widetilde{\lfloor {}^rF_{0,t} \rceil^\lambda}(\lfloor {}^r\mathfrak{V}_0 \rceil^\lambda, \mathrm{w})]^{-1} \right\|^q < {}^r\underline{c} e^{{}^r c(1+d_{\tilde{g}^\lambda}(x,y))},$$

where ${}^r\underline{c}$ depends on $r_0, q, \mathtt{s}$ and $\|g^\lambda\|_{C^2}$, and ${}^rc$ depends on $r_0, q, \mathtt{s}$, T, T_0 and $\|g^\lambda\|_{C^3}$. Moreover, by Lemma 5.26 and Lemma 5.28,

$$\begin{aligned}
&\overline{\mathbb{E}}_{\overline{\mathbb{P}}^\lambda_{y,x,T}} \sup_{t\in[0,T]} \left\| \lfloor {}^r\alpha_t \rceil^\lambda - \lfloor \alpha_t \rceil^\lambda \right\|^q \leqslant C_4 r^q, \\
&\overline{\mathbb{E}}_{\overline{\mathbb{P}}^\lambda_{y,x,T}} \sup_{t\in[0,T]} \left\| [D\widetilde{\lfloor {}^rF_{0,t} \rceil^\lambda}(\lfloor {}^r\mathfrak{V}_0 \rceil^\lambda, \mathrm{w})]^{-1} - [D\widetilde{\lfloor F_{0,t} \rceil^\lambda}(\lfloor \mathfrak{V}_0 \rceil^\lambda, \mathrm{w})]^{-1} \right\|^q \leqslant C_5 r^q,
\end{aligned}$$

where the constants C_4, C_5 depend on $\|g^\lambda\|_{C^2}$. Again, a standard split argument using these estimations and Hölder's inequality gives $\lim_{r\to 0} {}^r(\mathrm{IV}) = 0$. To conclude that $\lim_{r\to 0} {}^r(\mathrm{V}) = 0$, we see from (5.87) that it suffices to show for any $q > 1$,

$$\overline{\mathbb{E}}_{\overline{\mathbb{P}}^\lambda_{y,x,T}} \sup_{t\in[0,T]} \left\| \mathbf{A}_t \circ \lfloor {}^r\mathbf{F} \rceil^\lambda - \mathbf{A}_t \right\|^q \leqslant C_{\mathbf{A}} r^q$$

for some $C_{\mathbf{A}}$ depending on $\|g^\lambda\|_{C^3}, \|\mathcal{X}\|_{C^2}$, T and $d(x,y)$, where $\mathbf{A}_t = (\lfloor \mathfrak{g}^s_t \rceil^\lambda)'_0$, $(\lfloor O^s_t \rceil^\lambda)'_0$, $(\lfloor \mathfrak{V}^s_t \rceil^\lambda)'_0$, $(\lfloor \mathfrak{V}_t \rceil^\lambda)^{(1)}_\lambda$ or $[\widetilde{(DF_{0,t})}(\lfloor \mathfrak{V}_0 \rceil^\lambda, \mathrm{w})]^{-1}$. Using Lemma 5.11, this can be be reduced to the cases that $\mathbf{A}_t = \lfloor \alpha \rceil^\lambda_t$, $\lfloor \mathfrak{V}_t \rceil^\lambda$ or $[\widetilde{D\lfloor F_{0,t} \rceil^\lambda}(\lfloor \mathfrak{V}_0 \rceil^\lambda, \mathrm{w})]^{-1}$, which were shown as above.

Let C' be a bound of $|d\mathrm{Vol}^\lambda \circ {}^rF / d\mathrm{Vol}^\lambda(y)|$ for $r \in [-r_0, r_0]$. By using (5.69), we obtain

$$\begin{aligned}\overline{\mathbb{E}}_{\overline{\mathbb{P}}^\lambda_{y,x,T}} \left| \frac{d\overline{\mathbb{P}}^\lambda_x \circ \lfloor {}^r\mathbf{F} \rceil^\lambda}{d\overline{\mathbb{P}}^\lambda_x} - 1 \right|^2 &\leqslant 2C' \left(\overline{\mathbb{E}}_{\overline{\mathbb{P}}^\lambda_{y,x,T}} \left| \frac{d\mathbb{P}^\lambda_z \circ \lfloor {}^r\mathbf{F} \rceil^\lambda}{d\mathbb{P}^\lambda_y} - 1 \right|^2 + \overline{\mathbb{E}}_{\overline{\mathbb{P}}^\lambda_{y,x,T}} \left| \frac{d\mathrm{Vol}^\lambda \circ {}^rF}{d\mathrm{Vol}^\lambda}(y) - 1 \right|^2 \right) \\ &=: C' \left({}^r(\mathrm{VI}) + {}^r(\mathrm{VII}) \right).\end{aligned}$$

Clearly, ${}^r(\mathrm{VII}) \to 0$ as $r \to 0$. For the second property in (5.90), it remains to show ${}^r(\mathrm{VI}) \to 0$ as $r \to 0$. Following the proof of Proposition 5.20, we obtain

$$\frac{d\mathbb{P}^\lambda_z \circ \lfloor {}^r\mathbf{F} \rceil^\lambda}{d\mathbb{P}^\lambda_y} = e^{\left\{ -\frac{1}{2}\int_0^T \langle \lfloor {}^r\overline{\mathfrak{g}}_\tau \rceil^\lambda(\mathrm{w}),\, dB_\tau(\mathrm{w}) \rangle + \frac{1}{4}\int_0^T |\lfloor {}^r\overline{\mathfrak{g}}_\tau \rceil^\lambda(\mathrm{w})|^2 \, d\tau \right\}} =: {}^r\mathcal{E}_T(\mathrm{w}),$$

and

$$\begin{aligned}({}^r\mathcal{E}_T(\mathrm{w}))'_r &= {}^r\mathcal{E}_T(\mathrm{w}) \cdot \left(-\frac{1}{2}\int_0^T \langle (\lfloor {}^r\overline{\mathfrak{g}}_t \rceil^\lambda)'_r(\mathrm{w}), dB_t(\mathrm{w}) \rangle + \frac{1}{2}\int_0^T \langle (\lfloor {}^r\overline{\mathfrak{g}}_t \rceil^\lambda)'_r(\mathrm{w}), \lfloor {}^r\overline{\mathfrak{g}}_t \rceil^\lambda(\mathrm{w}) \rangle \, dt \right) \\ &=: {}^r\mathcal{E}_T(\mathrm{w}) \cdot \overline{{}^r\mathcal{E}_T}(\mathrm{w}).\end{aligned}$$

The usual argument using Lemma 4.16 and Burkholder's inequality shows that for every $q \geqslant 1$, $\overline{\mathbb{E}}_{\overline{\mathbb{P}}^\lambda_{y,x,T}} \left| \overline{{}^r\mathcal{E}_T}(\mathrm{w}) \right|^q$ is locally uniformly bounded in r. Hence ${}^r(\mathrm{VI}) \to 0$ as $r \to 0$.

Altogether, we have shown the map $Y \mapsto \overline{\Phi}^1_\lambda(Y)$ is a C^1 locally bounded functional on C^k vector fields Y on $\widetilde{M}$. Hence there exists some C^1 vector field $\widetilde{\mathrm{z}^{\lambda,1}_T}$ on $\widetilde{M}$ such that

$$\overline{\Phi}^1_\lambda(Y)(y) = \left\langle Y(y), \widetilde{\mathrm{z}^{\lambda,1}_T}(y) \right\rangle_\lambda.$$

This shows $\widetilde{\mathrm{z}^{\lambda,1}_t}(y) \equiv \mathrm{z}^{\lambda,1}_T(y)$. Thus $\{\mathrm{z}^{\lambda,1}_T(y)\}$ forms a C^1 vector field on $\widetilde{M}$ as claimed. □

Lemma 5.35. *Let $\lambda \in (-1,1) \mapsto g^\lambda \in \mathcal{M}^3(M)$ be a C^3 curve. Let $x \in \widetilde{M}$, $T \in \mathbb{R}_+$. For any smooth bounded vector field V on $\widetilde{M}$, let $\mathtt{s}$, $\nabla^\lambda_{T,\mathrm{V},\mathtt{s}}$, $\overleftarrow{\mathcal{E}}_{T,\mathrm{V},\mathtt{s}}$ be as above, then*

(5.91)
$$\nabla^\lambda_{\mathrm{V}(y)} \mathrm{z}^{\lambda,1}_T(y) = \overline{\mathbb{E}} \left(\nabla^\lambda_{T,\mathrm{V},\mathtt{s}} D\pi(\lfloor \mathrm{u}_T \rceil^\lambda)^{(1)}_\lambda(w) + D\pi(\lfloor \mathrm{u}_T \rceil^\lambda)^{(1)}_\lambda(w) \overleftarrow{\mathcal{E}}_{T,\mathrm{V},\mathtt{s}}(w) \middle| \lfloor \mathrm{x}_T \rceil^\lambda(w) = y \right).$$

As a consequence,

$$\begin{aligned}&\mathrm{Div}^\lambda \mathrm{z}_T^{\lambda,1}(y)\\&= \overline{\mathbb{E}}\left(\mathrm{tr}\left(\mathrm{V}\mapsto \nabla^\lambda_{T,\mathrm{V},\mathtt{s}} D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda\right) - \big\langle D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda,\ \frac{1}{2}\lfloor \mathrm{u}_T\rceil^\lambda \int_0^T \mathtt{s}'(T-\tau) d\overleftarrow{B}_\tau\big\rangle_\lambda \right.\\&\qquad \left. + \big\langle D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda, \frac{1}{2}\int_0^T \mathtt{s}(T-\tau)\lfloor \mathrm{u}_T\rceil^\lambda (\lfloor \mathrm{u}_\tau\rceil^\lambda)^{-1}(\mathrm{Ric}^\lambda_{\lfloor \mathrm{u}_\tau\rceil^\lambda})^{-1} d\overleftarrow{B}_\tau\big\rangle_\lambda \,\middle|\, \lfloor \mathrm{x}_T\rceil^\lambda(w) = y\right).\end{aligned}$$

Proof. Let Y be a C^k bounded vector field on $\widetilde{M}$. By Lemma 5.34,

$$\overline{\Phi}^1_\lambda(Y)(y) = \big\langle Y(y), \mathrm{z}_T^{\lambda,1}(y)\big\rangle_\lambda, \tag{5.92}$$

where all the variables $\overline{\Phi}^1_\lambda(Y), Y, \mathrm{z}_T^{\lambda,1}$ are C^1 in y. Hence

$$\nabla^\lambda_{\mathrm{V}}(\overline{\Phi}^1_\lambda(Y))(y) = \big\langle \nabla^\lambda_{\mathrm{V}} Y(y), \mathrm{z}_T^{\lambda,1}(y)\big\rangle_\lambda + \big\langle Y(y), \nabla^\lambda_{\mathrm{V}} \mathrm{z}_T^{\lambda,1}(y)\big\rangle_\lambda. \tag{5.93}$$

Let F^s be the flow map generated by a smooth vector field V. Then

$$\nabla^\lambda_{\mathrm{V}}(\overline{\Phi}^1_\lambda(Y))(y) = \Big(\overline{\Phi}^1_\lambda(Y)(F^s y)\Big)'_0.$$

It was shown in Lemma 5.34 that

$$\begin{aligned}\Big(\overline{\Phi}^1_\lambda(Y)(F^s y)\Big)'_0 = \overline{\mathbb{E}}\Big(&\big\langle \nabla^\lambda_{\mathrm{V}(\lfloor \mathrm{x}_T\rceil^\lambda)} Y, D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda\big\rangle_\lambda + \big\langle Y(\lfloor \mathrm{x}_T\rceil^\lambda), \nabla^\lambda_{T,\mathrm{V},\mathtt{s}} D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda\big\rangle_\lambda\\&+\big\langle Y(\lfloor \mathrm{x}_T\rceil^\lambda), D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda\big\rangle_\lambda \overleftarrow{\mathcal{E}}_{T,\mathrm{V},\mathtt{s}} \,\Big|\, \lfloor \mathrm{x}_T\rceil^\lambda(w) = y\Big).\end{aligned} \tag{5.94}$$

Applying (5.92) for the C^{k-1} vector field $\nabla^\lambda Y$ (instead of Y) gives

$$\overline{\mathbb{E}}\left(\big\langle \nabla^\lambda_{\mathrm{V}(\lfloor \mathrm{x}_T\rceil^\lambda)} Y, D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda\big\rangle_\lambda \,\middle|\, \lfloor \mathrm{x}_T\rceil^\lambda(w) = y\right) = \big\langle \nabla^\lambda_{\mathrm{V}} Y(y), \mathrm{z}_T^{\lambda,1}(y)\big\rangle_\lambda.$$

Report this in (5.94) and then compare it with (5.93). We obtain

$$\begin{aligned}&\big\langle Y(y), \nabla^\lambda_{\mathrm{V}} \mathrm{z}_t^{\lambda,1}(y)\big\rangle_\lambda\\&= \overline{\mathbb{E}}\left(\big\langle Y(\lfloor \mathrm{x}_T\rceil^\lambda), \nabla^\lambda_{T,\mathrm{V},\mathtt{s}} D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda\big\rangle_\lambda + \big\langle Y(\lfloor \mathrm{x}_T\rceil^\lambda), D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda\big\rangle_\lambda \overleftarrow{\mathcal{E}}_{T,\mathrm{V},\mathtt{s}} \,\middle|\, \lfloor \mathrm{x}_T\rceil^\lambda(w) = y\right)\\&= \left\langle Y(y),\ \overline{\mathbb{E}}\left(\nabla^\lambda_{T,\mathrm{V},\mathtt{s}} D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda(w) + D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda(w) \overleftarrow{\mathcal{E}}_{T,\mathrm{V},\mathtt{s}}(w) \,\middle|\, \lfloor \mathrm{x}_T\rceil^\lambda(w) = y\right)\right\rangle_\lambda.\end{aligned}$$

This implies (5.91) since Y is arbitrary.

The divergence $(\mathrm{Div}^\lambda \mathrm{z}_T^{\lambda,1}(y))$ is just the trace of the mapping $\mathrm{V}(y)\mapsto \nabla^\lambda_{\mathrm{V}(y)} \mathrm{z}_T^{\lambda,1}(y)$. Put

$$\begin{aligned}\overleftarrow{\mathcal{E}}^1_{T,\mathrm{V},\mathtt{s}} &= -\frac{1}{2}\int_0^T \big\langle \mathtt{s}'(T-\tau)(\lfloor \mathrm{u}_T\rceil^\lambda)^{-1}\mathrm{V}(\lfloor \mathrm{x}_T\rceil^\lambda), d\overleftarrow{B}_\tau\big\rangle,\\ \overleftarrow{\mathcal{E}}^2_{T,\mathrm{V},\mathtt{s}} &= \frac{1}{2}\int_0^T \big\langle \mathrm{Ric}^\lambda_{\lfloor \mathrm{u}_\tau\rceil^\lambda}(\lfloor \mathrm{u}_\tau\rceil^\lambda(\lfloor \mathrm{u}_T\rceil^\lambda)^{-1}\mathtt{s}(T-\tau)\mathrm{V}(\lfloor \mathrm{x}_T\rceil^\lambda)), d\overleftarrow{B}_\tau\big\rangle.\end{aligned}$$

Then

$$(\mathrm{Div}^\lambda \mathrm{z}_T^{\lambda,1}(y)) = \overline{\mathbb{E}}\left(\mathrm{tr}\left(\mathrm{V}\mapsto \nabla^\lambda_{T,\mathrm{V},\mathfrak{s}} D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda(w)\right) + \sum_{i=1}^2 \mathrm{tr}\left(\mathrm{V}\mapsto D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda(w)\overleftarrow{\mathcal{E}}^i_{T,\mathrm{V},\mathfrak{s}}(w)\right)\middle| \lfloor \mathrm{x}_T\rceil^\lambda(w)=y\right).$$

Take $\mathrm{V}_1,\cdots,\mathrm{V}_m$ to be orthogonal at y in the metric $\widetilde{g}^\lambda$. We obtain

$$\begin{aligned}\mathrm{tr}\left(\mathrm{V}\mapsto D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda(w)\overleftarrow{\mathcal{E}}^1_{T,\mathrm{V},\mathfrak{s}}(w)\right) &= \sum_{j=1}^m \left\langle D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda(w),\mathrm{V}_i\right\rangle_\lambda \cdot \overleftarrow{\mathcal{E}}^1_{T,\mathrm{V}_i,\mathfrak{s}}(w)\\ &= \left\langle D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda(w), -\frac{1}{2}\mathrm{u}_T\int_0^T \mathfrak{s}'(T-\tau)d\overleftarrow{B}_\tau\right\rangle_\lambda,\end{aligned}$$

Note that $\lfloor \mathrm{u}_\tau\rceil^\lambda(\lfloor \mathrm{u}_T\rceil^\lambda)^{-1}$ is the backward parallel transportation along $\lfloor \mathrm{x}\rceil^\lambda_{[\tau,T]}(w)$ which preserves the inner-product. Using (4.12), we obtain

$$\overleftarrow{\mathcal{E}}^2_{T,\mathrm{V},\mathfrak{s}} = \frac{1}{2}\int_0^T \left\langle \mathrm{V}(\lfloor \mathrm{x}_T\rceil^\lambda), \mathfrak{s}(T-\tau)\lfloor \mathrm{u}_T\rceil^\lambda(\lfloor \mathrm{u}_\tau\rceil^\lambda)^{-1}\mathrm{Ric}^{-1}_{\lfloor \mathrm{u}_\tau\rceil^\lambda} d\overleftarrow{B}_\tau\right\rangle_\lambda$$

and

$$\begin{aligned}&\mathrm{tr}\left(\mathrm{V}\mapsto D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda(w)\overleftarrow{\mathcal{E}}^2_{T,\mathrm{V},\mathfrak{s}}(w)\right)\\ &\quad= \sum_{j=1}^m \left\langle D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda(w),\mathrm{V}_i\right\rangle \cdot \overleftarrow{\mathcal{E}}^2_{T,\mathrm{V}_i,\mathfrak{s}}(w)\\ &\quad= \left\langle D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda(w), \tfrac{1}{2}\int_0^T \mathfrak{s}(T-\tau)\lfloor \mathrm{u}_T\rceil^\lambda(\lfloor \mathrm{u}_\tau\rceil^\lambda)^{-1}\mathrm{Ric}^{-1}_{\lfloor \mathrm{u}_\tau\rceil^\lambda} d\overleftarrow{B}_\tau\right\rangle_\lambda.\end{aligned}$$

□

Proof of Theorem 5.1 ($k=3$). Let $x\in\widetilde{M}$ and $T\in\mathbb{R}_+$. Let $(\lfloor \mathrm{x}_t\rceil^\lambda, \lfloor \mathrm{u}_t\rceil^\lambda)_{t\in\mathbb{R}_+}$ be the stochastic process pair which defines the Brownian motion on $(\widetilde{M},\widetilde{g}^\lambda)$ starting from x. By Lemma 4.18 and Proposition 4.28, it is true that for any $f\in C_c^\infty(\widetilde{M})$,

$$\left(\int_{\widetilde{M}} f(y)p^\lambda(T,x,y)\ d\mathrm{Vol}^\lambda(y)\right)^{(1)}_\lambda = \int_{\widetilde{M}}\left\langle \nabla^\lambda_y f(y), \mathrm{z}_T^{\lambda,1}(y)\cdot p^\lambda(T,x,y)\right\rangle_\lambda\ d\mathrm{Vol}^\lambda(y).$$

Since $\{\mathrm{z}_T^{\lambda,1}(y)\}$ is a C^1 vector field on $\widetilde{M}$ by Lemma 5.34, the classical integration by parts argument in Section 5.1 shows that

$$\begin{aligned}&\left(\int_{\widetilde{M}} f(y)p^\lambda(T,x,y)\ d\mathrm{Vol}^\lambda(y)\right)^{(1)}_\lambda\\ &= -\int_{\widetilde{M}} f(y)\left((\mathrm{Div}^\lambda \mathrm{z}_T^{\lambda,1}(y))\cdot p^\lambda(T,x,y) + \left\langle \mathrm{z}_T^{\lambda,1}(y), \nabla^\lambda p^\lambda(T,x,y)\right\rangle_\lambda\right)\ d\mathrm{Vol}^\lambda(y)\\ &= \int_{\widetilde{M}} f(y)\phi^1_\lambda(T,x,y)p^\lambda(T,x,y)\ d\mathrm{Vol}^\lambda(y).\end{aligned}$$

The function $\phi^1_\lambda(T,x,y)$ is continuous in y, uniformly in λ (see Lemma 5.34). Hence its continuity in λ follows from the continuity in λ of $\big(\int_{\widetilde{M}} f(y)p^\lambda(T,x,y)\ d\mathrm{Vol}^\lambda(y)\big)^{(1)}_\lambda$ for any $f\in C^\infty_c(\widetilde{M})$, which is true by (5.10) and the convergence in λ of $\lfloor \mathrm{x}_T\rceil^\lambda(w)$ and $\lfloor \mathrm{u}_T\rceil^\lambda(w)$ in the L^q-norm for every $q\geqslant 1$. So the first part argument in the proof of Lemma 5.5 works, which shows that $\lambda\mapsto p^\lambda(T,x,\cdot)$ is C^1, the differential $(p^\lambda)^{(1)}_\lambda(T,x,y)$ is continuous in y and

$$(p^\lambda)^{(1)}_\lambda(T,x,y)\cdot\rho^\lambda(y)+p^\lambda(T,x,y)\cdot(\rho^\lambda)^{(1)}_\lambda(y)=\phi^1_\lambda(T,x,y)p^\lambda(T,x,y)\rho^\lambda(y).$$

This gives (5.1) since ρ^λ is non-zero for $\mathcal{V}_g$ small.

Next, we show (5.2) with $l=0$. For this, it suffices to show the same type of bound holds for the L^q-norm of $\phi^1_\lambda(T,x,y)$. Note that, by Lemma 5.34, $\mathrm{z}^{\lambda,1}_T(y)$ is such that

$$\big\langle \mathrm{z}^{\lambda,1}_T(y),\nabla^\lambda\ln p^\lambda(T,x,y)\big\rangle_\lambda=\overline{\mathbb{E}}\left(\big\langle D\pi(\mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda(w),\nabla^\lambda\ln p^\lambda(T,x,\lfloor \mathrm{x}_T\rceil^\lambda(w))\big\rangle_\lambda\Big|\ \lfloor \mathrm{x}_T\rceil^\lambda=y\right).$$

Using this and the formula of $\mathrm{Div}^\lambda \mathrm{z}^{\lambda,1}_T$ in Lemma 5.35, we obtain

$$\phi^1_\lambda(T,x,y)=\overline{\mathbb{E}}\left(\widetilde{\phi}^1_\lambda(T,x,w)\Big|\ \lfloor \mathrm{x}_T\rceil^\lambda(w)=y\right), \tag{5.95}$$

where

$$\begin{aligned}\widetilde{\phi}^1_\lambda(T,x,w)=&-\mathrm{tr}\big(\mathrm{V}\mapsto\nabla^\lambda_{T,\mathrm{V},\mathtt{s}}D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda\big)+\big\langle D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda,\frac{1}{2}\lfloor \mathrm{u}_T\rceil^\lambda\int_0^T\mathtt{s}'(T-\tau)d\overleftarrow{B}_\tau\big\rangle_\lambda\\ &-\big\langle D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda,\frac{1}{2}\int_0^T\mathtt{s}(T-\tau)\lfloor \mathrm{u}_T\rceil^\lambda(\lfloor \mathrm{u}_\tau\rceil^\lambda)^{-1}\mathrm{Ric}^{-1}_{\lfloor \mathrm{u}_\tau\rceil^\lambda}d\overleftarrow{B}_\tau\big\rangle_\lambda\\ &-\big\langle D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda,\nabla^\lambda\ln p^\lambda(T,x,\lfloor \mathrm{x}_T\rceil^\lambda)\big\rangle_\lambda\\ =:&\ \mathrm{(I)}(T,x,w)+\mathrm{(II)}(T,x,w)+\mathrm{(III)}(T,x,w)+\mathrm{(IV)}(T,x,w).\end{aligned} \tag{5.96}$$

So,

$$\begin{aligned}\big\|\phi^1_\lambda(T,x,\cdot)\big\|^q_{L^q}=&\int_{\widetilde{M}}\Big|\overline{\mathbb{E}}\big(\widetilde{\phi}^1_\lambda(T,x,w)\big|\lfloor \mathrm{x}_T\rceil^\lambda(w)=y\big)\Big|^q p^\lambda(T,x,y)\ d\mathrm{Vol}^\lambda(y)\\ \leqslant&\int_{\widetilde{M}}\overline{\mathbb{E}}\big(\|\widetilde{\phi}^1_\lambda(T,x,w)\|^q\big|\lfloor \mathrm{x}_T\rceil^\lambda(w)=y\big)p^\lambda(T,x,y)\ d\mathrm{Vol}^\lambda(y)\\ \leqslant&\ 4^{q-1}\left(\overline{\mathbb{E}}\|\mathrm{(I)}\|^q+\overline{\mathbb{E}}\|\mathrm{(II)}\|^q+\overline{\mathbb{E}}\|\mathrm{(III)}\|^q+\overline{\mathbb{E}}\|\mathrm{(IV)}\|^q\right).\end{aligned}$$

Hence we will obtain (5.2) with $l=0$ if (I), (II), (III) and (IV) all have the same type of L^q bounds. This actually follows from Proposition 4.8 and Proposition 4.28. For (IV), it is true by (4.49) and (4.37) since

$$\big(\overline{\mathbb{E}}\,\|\mathrm{(IV)}\|^q\big)^2\leqslant\ \overline{\mathbb{E}}\left\|(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda\right\|^{2q}\cdot\overline{\mathbb{E}}\left\|\nabla^\lambda\ln p^\lambda(T,x,\lfloor \mathrm{x}_T\rceil^\lambda)\right\|^{2q}.$$

Using Hölder's inequality, we obtain

$$\big(\overline{\mathbb{E}}\,\|\mathrm{(III)}\|^q\big)^2\leqslant\ \overline{\mathbb{E}}\left\|(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda\right\|^{2q}\cdot\overline{\mathbb{E}}\left\|\frac{1}{2}\int_0^T\mathtt{s}(T-\tau)\lfloor \mathrm{u}_T\rceil^\lambda(\lfloor \mathrm{u}_\tau\rceil^\lambda)^{-1}\mathrm{Ric}^{-1}_{\lfloor \mathrm{u}_\tau\rceil^\lambda}d\overleftarrow{B}_\tau\right\|^{2q}.$$

Using Proposition 4.28 and Lemma 4.7, it is easy to show that $\overline{\mathbb{E}}\,\|(\mathrm{III})\|^q$ has the same bound type in (5.2) with $l = 0$. The term (II) can be handled in the same way. For (I), it suffices to estimate the L^q-norm of $\big((\lfloor \mathrm{u}_T^s \rceil^\lambda)_\lambda^{(1)}(w)\big)_0'$ for V with norm 1. Split the Itô integral of $\big((\theta,\varpi)(\lfloor \mathrm{u}_T^s \rceil^\lambda)_\lambda^{(1)}(w)\big)_0'$ in Corollary 5.33 with infinitesimal increments dB_τ and $d\tau$, respectively, as

$$\overline{(\mathrm{I})} := \Big((\theta,\varpi)(\lfloor \mathrm{u}_T^s \rceil^\lambda)_\lambda^{(1)}(w)\Big)_0' = \int_0^T \big[D\lfloor \widetilde{\overrightarrow{F}_{\tau,T}} \rceil^\lambda(\lfloor \mathrm{u}_\tau \rceil^\lambda, w)\big]\left((\mathrm{I})_1(\tau,w)dB_\tau + (\mathrm{I})_2(\tau,w)d\tau\right).$$

Then it is standard to use Burkholder's inequality and Hölder's inequality to deduce that

$$\begin{aligned} 2^{1-p}\mathbb{E}\left\|\overline{(\mathrm{I})}\right\|^p \leqslant\ & \left(\overline{\mathbb{E}}\left|\int_0^T \left\|\big[D\lfloor \widetilde{\overrightarrow{F}_{\tau,T}} \rceil^\lambda(\lfloor \mathrm{u}_\tau \rceil^\lambda, w)\big](\mathrm{I})_1(\tau,w)\right\|^2 d\tau\right|^q\right)^{\frac{1}{2}} \\ & + \mathbb{E}\left\|\int_0^T \big[D\lfloor \widetilde{\overrightarrow{F}_{\tau,T}} \rceil^\lambda(\lfloor \mathrm{u}_\tau \rceil^\lambda, w)\big](\mathrm{I})_2(\tau,w)\ d\tau\right\|^q. \end{aligned}$$

Using Corollary 5.33 and (4.48), we can continue to estimate $\overline{\mathbb{E}}\,\|(\mathrm{I})_1\|^{4q}$, $\overline{\mathbb{E}}\,\|(\mathrm{I})_2\|^{2q}$ as in Proposition 5.29 and show they have same bound type in (5.2) with $l = 0$.

To complete the proof of i), we apply Lemma 5.5. It remains to show $(p^\lambda)_\lambda^{(1)}(T,x,y)$ is continuous in the (T,y)-coordinate, locally uniformly in λ, which is true if we have

1) the continuity of $y \mapsto (p^\lambda)_\lambda^{(1)}(T,x,y)$, locally uniformly in T and λ, and
2) the continuity of $T \mapsto (p^\lambda)_\lambda^{(1)}(T,x,y)$ for every x, y fixed, locally uniformly in λ.

For 1), it holds if the continuity of $y \mapsto (\ln p^\lambda)_\lambda^{(1)}(T,x,y)$ is locally uniform in T and λ, where the latter is true if $y \mapsto (\overline{\Phi}_\lambda^1(Y)(F^s y))_0'$ is continuous, locally uniformly in T and λ. Since all the bounds in Lemmas 5.26-5.28 are locally uniform in (y,T) and λ, the limits for continuity of $y \mapsto (\overline{\Phi}_\lambda^1(Y)(F^s y))_0'$ in proof of Lemma 5.34 are all locally uniform in T and λ.

We proceed to show 2). Simply denote by $(\mathrm{x}^\lambda, \mathrm{u}^\lambda)$ the stochastic pair which defines the Brownian motion starting from x. Then for any smooth function f on $\widetilde{M}$ with support contained in a small neighborhood of y,

$$f(\mathrm{x}_T^\lambda) = f(x) + \int_0^T \Delta^\lambda f(\mathrm{x}_t^\lambda)\ dt + \int_0^T H^\lambda(\mathrm{u}_t^\lambda, e_i)\big(\widetilde{f}(\mathrm{u}_t^\lambda)\big)\ dB^i.$$

Taking expectations on both sides shows

$$\mathbb{E}\big(f(\mathrm{x}_T^\lambda)\big) = \int_0^T \mathbb{E}\big(\Delta^\lambda f(\mathrm{x}_t^\lambda)\big)\ dt.$$

Hence for $T' > T$,

$$\mathbb{E}\big(f(\mathrm{x}_{T'}^\lambda)\big) - \mathbb{E}\big(f(\mathrm{x}_T^\lambda)\big) = \int_T^{T'} \mathbb{E}\big(\Delta^\lambda f(\mathrm{x}_t^\lambda)\big)\ dt.$$

Differentiating both sides in λ gives

$$\begin{aligned}
&\int_{\widetilde{M}} f(z)\left((p^\lambda)^{(1)}_\lambda(T',x,z)-(p^\lambda)^{(1)}_\lambda(T,x,z)\right)\,d\mathrm{Vol}^\lambda(z)\\
&=-\int_{\widetilde{M}} f(z)\left(p^\lambda(T',x,z)-p^\lambda(T,x,z)\right)(\ln\rho^\lambda)^{(1)}_\lambda(z)\,d\mathrm{Vol}^\lambda(z)+\int_T^{T'}\mathbb{E}\left(\left(\Delta^\lambda f\right)^{(1)}_\lambda(\mathrm{x}^\lambda_t)\right)\,dt\\
&\quad+\int_T^{T'}\int_{\widetilde{M}}\left(\Delta^\lambda f\right)(z)\phi^1_\lambda(t,x,z)p^\lambda(t,x,z)\,d\mathrm{Vol}^\lambda(z),
\end{aligned}$$

where, as $T'\to T$, the first term tends to zero since $p^\lambda(T,x,z)$ is continuous at $T>0$, locally uniformly in z, the second term tends to zero since $\mathbb{E}\big(\left(\Delta^\lambda f\right)^{(1)}_\lambda(\mathrm{x}^\lambda_t)\big)$ is uniformly bounded for t in a small neighborhood of T and the last term goes to zero as well by using that the bound in (5.2) with $l=0$ is locally uniform in t. In summary, we have

$$\lim_{T'\to T}\int_{\widetilde{M}} f(z)\left((p^\lambda)^{(1)}_\lambda(T',x,z)-(p^\lambda)^{(1)}_\lambda(T,x,z)\right)\,d\mathrm{Vol}^\lambda(z)=0.$$

Since $z\mapsto(p^\lambda)^{(1)}_\lambda(T,x,z)$ is continuous, locally uniformly in T and λ, and f is arbitrary, we must have $\lim_{T'\to T}(p^\lambda)^{(1)}_\lambda(T',x,y)=(p^\lambda)^{(1)}_\lambda(T,x,y)$, locally uniformly in λ. This shows 2).

Finally, we show iii). By symmetry, the mapping $x\mapsto(p^\lambda)^{(1)}_\lambda(T,x,y)$ is continuous for all T,y, locally uniformly in y. Therefore iii) holds for any bounded function with compact support. Fix $q\geqslant 1$. Any uniformly continuous and bounded $\widetilde{f}\in C(\widetilde{M})$ can be approximated by a sequence $\{\widetilde{f}_n\}_{n\in\mathbb{N}}$ of continuous functions on $\widetilde{M}$ with compact support in such a way that

$$\lim_{n\to\infty}\left\|\widetilde{f}(y)-\widetilde{f}_n(y)\right\|_q=0, \tag{5.97}$$

locally uniformly in x. Property iii) follows by using (5.97) and (5.2) with $l=0$. □

Proof of Theorem 5.1 ($k>3$). By Theorem 5.1 i) of the $k=3$ case and Lemma 5.5, we deduce Theorem 5.1 i). Hence $\nabla^{(l)}(\ln p^\lambda)^{(1)}_\lambda(T,x,\cdot)$, $l\leqslant k-3$, are well-defined. By taking the gradients of the identity (5.1), we obtain that $\nabla^{(l)}\phi^1_\lambda(T,x,\cdot)$, $l\leqslant k-3$, exist as well. For (5.2), it suffices to show the same type of L^q-norm bounds hold for $\nabla^{(l)}\phi^1_\lambda(T,x,\cdot)$, $l\leqslant k-3$.

The $l=0$ case was treated in the previous proof of Theorem 5.1 with $k=3$. We proceed to consider the $l=1$ case. Let W be a smooth bounded vector field on $\widetilde{M}$ and let ${}^rF$ be the flow it generates. Then

$$\nabla^\lambda_{W(y)}\phi^1_\lambda(T,x,\cdot)=\left.\frac{d}{dr}\right|_{r=0}\left(\phi^1_\lambda(T,x,{}^rF(y))\right).$$

We will look for a conditional expectation expression of $\nabla^\lambda_{W(y)}\phi^1_\lambda(T,x,\cdot)$ and use it to estimate $|\nabla\phi^1_\lambda(T,x,\cdot)|$. For this, we adopt the idea we used in analyzing the regularity of $\overline{\Phi}^1_\lambda(Y)$ (see Section 5.1). Let f be an arbitrary bounded measurable function on $\widetilde{M}$. By the definition of the conditional expectation and the change of variable formula under ${}^rF$,

$$\begin{aligned}\overline{\mathbb{E}}\left(\widetilde{\phi}^1_\lambda(T,x,w)f(\lfloor \mathrm{x}_T\rceil^\lambda(w))\right) &= \overline{\mathbb{E}}\left(\overline{\mathbb{E}}\big(\,\widetilde{\phi}^1_\lambda(T,x,w)\Big|\,\lfloor \mathrm{x}_T\rceil^\lambda(w)=y\big)f(y)\right)\\ &= \int \phi^1_\lambda(T,x,y)f(y)p^\lambda(T,x,y)d\mathrm{Vol}^\lambda(y)\\ &= \int \phi^1_\lambda(T,x,{}^rF(y))f({}^rF(y))p^\lambda(T,x,{}^rF(y))d\mathrm{Vol}^\lambda({}^rF(y)).\end{aligned}$$

Let $\lfloor {}^r\mathbf{F}\rceil^\lambda$ be the extension of ${}^rF$ to $C_x([0,T],\widetilde{M})$ constructed in the previous subsections. By Proposition 5.23, all probabilities $\overline{\mathbb{P}}^\lambda_x\circ\lfloor {}^r\mathbf{F}\rceil^\lambda$ are absolutely continuous with respect to $\overline{\mathbb{P}}^\lambda_x$. Hence, using the change of variable formula under $\lfloor {}^r\mathbf{F}\rceil^\lambda$, we obtain

$$\begin{aligned}\overline{\mathbb{E}}\left(\widetilde{\phi}^1_\lambda\cdot f(\lfloor \mathrm{x}_T\rceil^\lambda)\right) &= \overline{\mathbb{E}}\left(\widetilde{\phi}^1_\lambda\circ\lfloor {}^r\mathbf{F}\rceil^\lambda\cdot f\circ\lfloor {}^r\mathbf{F}\rceil^\lambda\cdot\frac{d\overline{\mathbb{P}}^\lambda_x\circ\lfloor {}^r\mathbf{F}\rceil^\lambda}{d\overline{\mathbb{P}}^\lambda_x}\right)\\ &= \int\overline{\mathbb{E}}\left(\widetilde{\phi}^1_\lambda\circ\lfloor {}^r\mathbf{F}\rceil^\lambda\cdot\frac{d\overline{\mathbb{P}}^\lambda_x\circ\lfloor {}^r\mathbf{F}\rceil^\lambda}{d\overline{\mathbb{P}}^\lambda_x}\,\middle|\,\lfloor \mathrm{x}_T\rceil^\lambda=y\right)f({}^rF(y))p^\lambda(T,x,y)d\mathrm{Vol}^\lambda(y).\end{aligned}$$

Since f is arbitrary, a comparison of the two expressions of $\overline{\mathbb{E}}\big(\widetilde{\phi}^1_\lambda\cdot f(\lfloor \mathrm{x}_T\rceil^\lambda)\big)$ shows

$$\phi^1_\lambda(T,x,{}^rF(y))=\overline{\mathbb{E}}\left(\widetilde{\phi}^1_\lambda\circ\lfloor {}^r\mathbf{F}\rceil^\lambda\cdot\frac{d\overline{\mathbb{P}}^\lambda_x\circ\lfloor {}^r\mathbf{F}\rceil^\lambda}{d\overline{\mathbb{P}}^\lambda_x}\,\middle|\,\lfloor \mathrm{x}_T\rceil^\lambda=y\right)\cdot\frac{p^\lambda(T,x,y)}{p^\lambda(T,x,{}^rF(y))}\cdot\frac{d\mathrm{Vol}^\lambda}{d\mathrm{Vol}^\lambda\circ{}^rF}(y).$$

So differentiating both sides in r at $r=0$ gives

$$\begin{aligned}\nabla^\lambda_{W(y)}\phi^1_\lambda(T,x,\cdot) = \phi^1_\lambda(T,x,y)\Big(\big(\ln p^\lambda(T,x,{}^rF(y))\big)'_0+\big(\ln\rho^\lambda({}^rF(y))\big)'_0\Big)\\ +\left(\overline{\mathbb{E}}\left(\widetilde{\phi}^1_\lambda\circ\lfloor {}^r\mathbf{F}\rceil^\lambda\cdot\frac{d\overline{\mathbb{P}}^\lambda_x\circ\lfloor {}^r\mathbf{F}\rceil^\lambda}{d\overline{\mathbb{P}}^\lambda_x}\,\middle|\,\lfloor \mathrm{x}_T\rceil^\lambda=y\right)\right)'_0.\end{aligned}$$

It was shown in Proposition 5.23 that $d\overline{\mathbb{P}}^\lambda_x\circ\lfloor {}^r\mathbf{F}\rceil^\lambda/d\overline{\mathbb{P}}^\lambda_x$ is differentiable in r with

$$\big(d\overline{\mathbb{P}}^\lambda_x\circ\lfloor {}^r\mathbf{F}\rceil^\lambda/d\overline{\mathbb{P}}^\lambda_x\big)'_r={}^r\overline{\mathcal{E}_t}\cdot\big(d\overline{\mathbb{P}}^\lambda_x\circ\lfloor {}^r\mathbf{F}\rceil^\lambda/d\overline{\mathbb{P}}^\lambda_x\big)$$

and both ${}^r\overline{\mathcal{E}_t}$ and $\big(d\overline{\mathbb{P}}^\lambda_x\circ\lfloor {}^r\mathbf{F}\rceil^\lambda/d\overline{\mathbb{P}}^\lambda_x\big)$ conditioned on $\mathrm{x}_T=y$ are L^q $(q\geqslant 1)$ integrable, locally uniformly in the r parameter. Using Hölder's inequality, if we can further show

★) $\widetilde{\phi}^1_\lambda\circ\lfloor {}^r\mathbf{F}\rceil^\lambda$ is also differentiable in r with both $\widetilde{\phi}^1_\lambda\circ\lfloor {}^r\mathbf{F}\rceil^\lambda$ and $(\widetilde{\phi}^1_\lambda\circ\lfloor {}^r\mathbf{F}\rceil^\lambda)'_r$ conditioned on $\mathrm{x}_T=y$ are L^2 integrable, locally uniformly in the r parameter,

we are allowed to take the differentiation under the expectation sign:

$$\begin{aligned}\left(\overline{\mathbb{E}}\left(\widetilde{\phi}^1_\lambda\circ\lfloor{}^r\mathbf{F}\rceil^\lambda\cdot\frac{d\overline{\mathbb{P}}^\lambda_x\circ\lfloor{}^r\mathbf{F}\rceil^\lambda}{d\overline{\mathbb{P}}^\lambda_x}\middle|\lfloor \mathrm{x}_T\rceil^\lambda=y\right)\right)'_0&=\overline{\mathbb{E}}\left(\left(\widetilde{\phi}^1_\lambda\circ\lfloor{}^r\mathbf{F}\rceil^\lambda\cdot\frac{d\overline{\mathbb{P}}^\lambda_x\circ\lfloor{}^r\mathbf{F}\rceil^\lambda}{d\overline{\mathbb{P}}^\lambda_x}\right)'_0\middle|\lfloor \mathrm{x}_T\rceil^\lambda=y\right)\\&=\overline{\mathbb{E}}\left(\left(\widetilde{\phi}^1_\lambda\circ\lfloor{}^r\mathbf{F}\rceil^\lambda\right)'_0+\widetilde{\phi}^1_\lambda\cdot{}^0\overline{\mathcal{E}_T}\middle|\lfloor \mathrm{x}_T\rceil^\lambda=y\right).\end{aligned}$$

Altogether, we will have

$$\begin{aligned}\nabla^\lambda_{W(y)}\phi^1_\lambda(T,x,\cdot)=\phi^1_\lambda(T,x,y)&\left(\nabla^\lambda_{W(y)}(\ln p^\lambda(T,x,\cdot))+\nabla^\lambda_{W(y)}(\ln\rho^\lambda)\right)\\&+\overline{\mathbb{E}}\left(\left(\widetilde{\phi}^1_\lambda\circ\lfloor{}^r\mathbf{F}\rceil^\lambda\right)'_0+\widetilde{\phi}^1_\lambda\cdot{}^0\overline{\mathcal{E}_T}\middle|\lfloor \mathrm{x}_T\rceil^\lambda=y\right)\end{aligned}\tag{5.98}$$

and we can use it to show that a L^q-norm bound as in (5.2) is valid for $\nabla^\lambda\phi^1_\lambda(T,x,\cdot)$.

We show ★) first. Consider the processes

$$\lfloor{}^r\mho_\tau\rceil^\lambda:=\lfloor\mho_\tau\rceil^\lambda\circ\lfloor{}^r\mathbf{F}\rceil^\lambda,\ \left[D\lfloor{}^rF_{\tau,t}\rceil^\lambda(\lfloor{}^r\mho_\tau\rceil^\lambda,\mathrm{w})\right]^{-1}:=\left[D\lfloor F_{\tau,t}\rceil^\lambda(\lfloor\mho_\tau\rceil^\lambda,\mathrm{w})\right]^{-1}\circ\lfloor{}^r\mathbf{F}\rceil^\lambda.$$

They are well-defined by Theorem 5.17 and the corresponding estimations in Lemmas 5.26-5.28 (for $\lfloor{}^r\mathbf{F}\rceil^\lambda$) are valid. Note that $D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda$, $\nabla^\lambda_{T,\mathrm{V},\mathfrak{s}}D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda$ can be expressed by stochastic integrals using $\lfloor\mho_\tau\rceil^\lambda$ and $\lfloor\mathbf{F}\rceil^\lambda$ (see Proposition 5.29 and Proposition 5.32). Their images under $\lfloor{}^r\mathbf{F}\rceil^\lambda$ can be defined by applying $\lfloor{}^r\mathbf{F}\rceil^\lambda$ to each components in the integrals. So $\widetilde{\phi}^1_\lambda\circ\lfloor{}^r\mathbf{F}\rceil^\lambda$ is well-defined. By using Lemma 4.14, Proposition 4.28 ii) and (5.96), it is easy to obtain

$$\overline{\mathbb{E}}_{\overline{\mathbb{P}}^{\lambda,*}_{x,y,T}}\left|\widetilde{\phi}^1_\lambda\circ\lfloor{}^r\mathbf{F}\rceil^\lambda\right|^2\leqslant\underline{\mathtt{c}}\left(\left(\frac{1}{T}d_{\tilde{g}^\lambda}(x,y)+\frac{1}{\sqrt{T}}\right)^2+1\right)e^{\mathtt{c}(1+d_{\tilde{g}^\lambda}(x,y))}$$

for some constants $\underline{\mathtt{c}}$ (depending on $\mathfrak{s},r_0$, $\|g^\lambda\|_{C^3}$ and $\|\mathcal{X}^\lambda\|_{C^2}$) and $\mathtt{c}$ (depending on T,T_0 and $\|g^\lambda\|_{C^3}$). By Propositions 4.15, 5.20 and 5.23, we may also assume $\underline{\mathtt{c}},\mathtt{c}$ are such that

$$\left(\overline{\mathbb{E}}_{\overline{\mathbb{P}}^{\lambda,*}_{x,y,T}}\left|{}^0\overline{\mathcal{E}_T}\right|^2\right)^{\frac{1}{2}}\leqslant\|W(y)\|\underline{\mathtt{c}}e^{\mathtt{c}(1+d_{\tilde{g}^\lambda}(x,y))}.$$

To justify (5.98), it remains to check the differentiability of $r\mapsto\mathrm{A}\circ\lfloor{}^r\mathbf{F}\rceil^\lambda$, for $\mathrm{A}=(\mathrm{I}),(\mathrm{II}),(\mathrm{III}),(\mathrm{IV})$ in (5.96) and show the differentials $\left(\mathrm{A}\circ\lfloor{}^r\mathbf{F}\rceil^\lambda\right)'_r$ are L^2 integrable, uniformly in the r parameter. We begin with $\mathrm{A}=(\mathrm{IV})$. By Proposition 5.30, $(D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda)\circ\lfloor{}^r\mathbf{F}\rceil^\lambda$ is differentiable in r. Let $r\in[-r_0,r_0]$. As usual, we write

$$\lfloor{}^r\mathrm{x}\rceil^\lambda:=\lfloor\mathrm{x}\rceil^\lambda\circ\lfloor{}^r\mathbf{F}\rceil^\lambda,\ \ \lfloor{}^r\mathrm{u}\rceil^\lambda:=\lfloor\mathrm{u}\rceil^\lambda\circ\lfloor{}^r\mathbf{F}\rceil^\lambda,\ \nabla^{r,\lambda}_{T,\mathrm{W},\mathfrak{s}}D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda:=\left((D\pi(\lfloor{}^r\mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda)\right)'_r.$$

Then $(\mathrm{IV})\circ\lfloor{}^r\mathbf{F}\rceil^\lambda$ is differentiable in r with differential

$$\begin{aligned}\left((\mathrm{IV})\circ\lfloor{}^r\mathbf{F}\rceil^\lambda\right)'_r=&-\left\langle\nabla^{r,\lambda}_{T,\mathrm{W},\mathfrak{s}}D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda,\ \nabla^\lambda\ln p^\lambda(T,x,\lfloor{}^r\mathrm{x}_T\rceil^\lambda)\right\rangle_\lambda\\&-\left\langle D\pi(\lfloor{}^r\mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda,\ \nabla^\lambda_{\mathrm{W}(\lfloor{}^r\mathrm{x}_T\rceil^\lambda)}\nabla^\lambda\ln p^\lambda(T,x,\lfloor{}^r\mathrm{x}_T\rceil^\lambda)\right\rangle_\lambda.\end{aligned}$$

By Proposition 4.28, we can obtain some $\underline{\mathtt{c}}'$ (depending on $\mathtt{s}, r_0$, $\|g^\lambda\|_{C^3}$ and $\|\mathcal{X}^\lambda\|_{C^1}$) and $\mathtt{c}'$ (depending on T, T_0 and $\|g^\lambda\|_{C^3}$) such that

$$\left(\overline{\mathbb{E}}_{\overline{\mathbb{P}}^{\lambda,*}_{x,y,T}}\left\|D\pi(\lfloor {}^r\mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda\right\|^2\right)^{\frac{1}{2}} \leqslant \|W(y)\|\underline{\mathtt{c}}' e^{\mathtt{c}'(1+d_{\tilde{g}^\lambda}(x,y))}.$$

By Proposition 5.29, we can obtain some $\underline{\mathtt{c}}''$ (depending on $\mathtt{s}, r_0$, $\|g^\lambda\|_{C^3}$ and $\|\mathcal{X}^\lambda\|_{C^2}$) and $\mathtt{c}''$ (depending on T, T_0 and $\|g^\lambda\|_{C^3}$) such that

$$\left(\overline{\mathbb{E}}_{\overline{\mathbb{P}}^{\lambda,*}_{x,y,T}}\left\|\nabla^{r,\lambda}_{T,\mathrm{W},\mathtt{s}} D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda\right\|^2\right)^{\frac{1}{2}} \leqslant \|W(y)\|\underline{\mathtt{c}}'' e^{\mathtt{c}''(1+d_{\tilde{g}^\lambda}(x,y))}.$$

Using (4.37), we further obtain

$$\begin{aligned}
&\left(\overline{\mathbb{E}}_{\overline{\mathbb{P}}^{\lambda,*}_{x,y,T}}\left|\left((\mathrm{IV})\circ\lfloor {}^r\mathbf{F}\rceil^\lambda\right)'_r\right|^2\right)^{\frac{1}{2}}\\
&\leqslant \left(\overline{\mathbb{E}}_{\overline{\mathbb{P}}^{\lambda,*}_{x,y,T}}\left\|\nabla^{r,\lambda}_{T,\mathrm{W},\mathtt{s}} D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda\right\|^2\right)^{\frac{1}{2}} \left\|\nabla^\lambda \ln p^\lambda(T,x,\lfloor {}^r F\rceil^\lambda(y))\right\|\\
&\quad + \left(\overline{\mathbb{E}}_{\overline{\mathbb{P}}^{\lambda,*}_{x,y,T}}\left\|D\pi(\lfloor {}^r\mathrm{u}_\mathrm{T}\rceil^\lambda)^{(1)}_\lambda\right\|^2\right)^{\frac{1}{2}} \left\|\nabla^\lambda_{W(\lfloor {}^r F\rceil^\lambda(y))}\nabla^\lambda \ln p^\lambda(T,x,\lfloor {}^r F\rceil^\lambda(y))\right\|\\
&\leqslant \|W(y)\|\underline{\mathtt{c}}''' e^{\mathtt{c}'''(1+d_{\tilde{g}^\lambda}(x,y))}\sum_{i=1}^{2}\big(\frac{1}{T}d_{\tilde{g}^\lambda}(x,\lfloor {}^r F\rceil^\lambda(y)) + \frac{1}{\sqrt{T}}\big)^i,
\end{aligned}$$

where $\underline{\mathtt{c}}'''$ (depending on $\mathtt{s}, r_0$, $\|g^\lambda\|_{C^3}$ and $\|\mathcal{X}^\lambda\|_{C^2}$) and $\mathtt{c}'''$ (depending on T, T_0 and $\|g^\lambda\|_{C^3}$) and this bound is finite and is uniform in r. For $\mathrm{A} = (\mathrm{II}), (\mathrm{III})$, the same argument shows the C^1 regularity of $r \mapsto \mathrm{A}\circ\lfloor {}^r\mathbf{F}\rceil^\lambda$ and

$$\left(\overline{\mathbb{E}}_{\overline{\mathbb{P}}^{\lambda,*}_{x,y,T}}\left|\left(\mathrm{A}\circ\lfloor {}^r\mathbf{F}\rceil^\lambda\right)'_r\right|^2\right)^{\frac{1}{2}} \leqslant \|W(y)\|\underline{c}_\mathrm{A} e^{c_\mathrm{A}(1+d_{\tilde{g}^\lambda}(x,y))}$$

for some $\underline{c}_\mathrm{A}$ (depending on $\mathtt{s}, r_0$, $\|g^\lambda\|_{C^3}$ and $\|\mathcal{X}^\lambda\|_{C^1}$) and c_A (depending on T, T_0 and $\|g^\lambda\|_{C^3}$). It remains to analyze $(\mathrm{I})\circ\lfloor {}^r\mathbf{F}\rceil^\lambda$. Recall that for any smooth bounded vector field V on $\widetilde{M}$,

$$\nabla^\lambda_{T,\mathrm{V},\mathtt{s}} D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda = D\pi\big(\overline{(\lfloor \mathrm{u}^s_T\rceil^\lambda)^{(1)}_\lambda}(\mathrm{w})\big)'_0,$$

where $\big(\overline{(\lfloor \mathrm{u}^s_T\rceil^\lambda)^{(1)}_\lambda}(\mathrm{w})\big)'_0$ was formulated in (5.87). Hence the regularity of $r \mapsto (\mathrm{I})\circ\lfloor {}^r\mathbf{F}\rceil^\lambda$ can be reduced to the regularity of each component of (5.87) under $\lfloor {}^r\mathbf{F}\rceil^\lambda$. Applying Theorem 5.17 to $\lfloor {}^r\mathbf{F}\rceil^\lambda$ shows $r \mapsto \lfloor {}^r\mho_\tau\rceil^\lambda, \big[D\lfloor {}^r F_{\tau,t}\rceil^\lambda(\lfloor {}^r\mho_\tau\rceil^\lambda, \mathrm{w})\big]^{-1}$ are C^1. Lemmas 5.26-5.28 also hold true for $\lfloor {}^r\mathbf{F}\rceil^\lambda$. Using these properties and the fact that $\lambda \mapsto g^\lambda$ is C^k in $\mathcal{M}^k(M)$ with $k \geqslant 4$, we can deduce the regularity of the components of (5.87) under $\lfloor {}^r\mathbf{F}\rceil^\lambda$. Moreover, by a routine computation using Lemmas 5.26-5.28, we can obtain some $\underline{c}_\mathrm{I}$ (depending on

$\mathtt{s}, r_0$, $\|g^\lambda\|_{C^4}$ and $\|\mathcal{X}^\lambda\|_{C^3}$) and c_{I} (depending on T, T_0 and $\|g^\lambda\|_{C^3}$) such that

$$\left(\overline{\mathbb{E}}_{\overline{\mathbb{P}}^{\lambda,*}_{x,y,T}}\left|\left((\mathrm{I})\circ\lfloor{}^r\mathbf{F}\rceil^\lambda\right)'_r\right|^2\right)^{\frac{1}{2}}\leqslant\|W(y)\|\underline{c}_{\mathrm{I}}e^{c_{\mathrm{I}}(1+d_{\tilde{g}^\lambda}(x,y))}.$$

Altogether, we have the differentiability of $\lambda\mapsto\widetilde{\phi}^1_\lambda\circ\lfloor{}^r\mathbf{F}\rceil^\lambda$ and also obtain some $\underline{c}$ (depending on $\mathtt{s}, r_0$, $\|g^\lambda\|_{C^4}$ and $\|\mathcal{X}^\lambda\|_{C^3}$) and c (depending on T, T_0 and $\|g^\lambda\|_{C^3}$) such that

$$\left(\overline{\mathbb{E}}_{\overline{\mathbb{P}}^{\lambda,*}_{x,y,T}}\left|\left(\widetilde{\phi}^1_\lambda\circ\lfloor{}^r\mathbf{F}\rceil^\lambda\right)'_0\right|^2\right)^{\frac{1}{2}}\leqslant\|W(y)\|\underline{c}e^{c(1+d_{\tilde{g}^\lambda}(x,y))}\left((\frac{1}{T}d_{\tilde{g}^\lambda}(x,y)+\frac{1}{\sqrt{T}})^2+1\right).$$

Now (5.98) holds true. Using Hölder's inequality, it is easy to deduce

$$\begin{aligned}4^{1-q}\left\|\nabla^\lambda_{W(y)}\phi^1_\lambda(T,x,\cdot)\right\|^q_{L^q}&\leqslant\|\phi^1_\lambda(T,x,\cdot)\|^q_{L^{2q}}\left(\left\|\nabla^\lambda_{W(y)}(\ln p^\lambda(T,x,\cdot))\right\|^q_{L^{2q}}+\left\|\nabla^\lambda_{W(y)}(\ln\rho^\lambda)\right\|^q_{L^{2q}}\right)\\&\quad+\left(\mathbb{E}\|\widetilde{\phi}^1_\lambda\|^{2q}\right)^{\frac{1}{2}}\left(\mathbb{E}\left\|{}^0\overline{\mathcal{E}_T}\right\|^{2q}\right)^{\frac{1}{2}}+\overline{\mathbb{E}}\|\left(\widetilde{\phi}^1_\lambda\circ\lfloor{}^r\mathbf{F}\rceil^\lambda\right)'_0\|^q\\&=:\mathrm{D}_1(q)+\mathrm{D}_2(q)+\mathrm{D}_3(q).\end{aligned}$$

By (4.37), (4.35), we see that, for the i-th covariant derivative $\nabla^{\lambda,(i)}\ln p^\lambda(t,x,\cdot)$ ($i\leqslant k-2$), there is $\underline{c}(i)$ (depending on q, T_0 and $\|g^\lambda\|_{C^{i+2}}$) and $c(i)$ (depending on $\|g^\lambda\|_{C^3}$) such that

$$\left\|\nabla^{\lambda,(i)}\ln p^\lambda(T,x,\cdot)\right\|^q_{L^q}\leqslant\underline{c}(i)e^{c(i)(1+T)}.\tag{5.99}$$

Using this and the L^q estimation of $\phi^1_\lambda(T,x,\cdot)$ in the proof of Theorem 5.1 for the $k=3$ case, we obtain $\mathrm{D}_1(q)\leqslant\underline{c}_\lambda(q)$, where $\underline{c}_\lambda(q)$ depends on q, T_0, T, $\|g^\lambda\|_{C^3}$ and $\|\mathcal{X}^\lambda\|_{C^2}$. With the L^{2q} estimations of $\widetilde{\phi}^1_\lambda(T,x,\cdot)$ and ${}^0\overline{\mathcal{E}_T}$ for Theorem 5.1 with $k=3$, we can also conclude that $\mathrm{D}_2(q)$ has the same type of bound as $\mathrm{D}_1(q)$. For $\mathrm{D}_3(q)$, we check the L^q-norms of $\left(\mathrm{A}\circ\lfloor{}^r\mathbf{F}\rceil^\lambda\right)'_0$ for $\mathrm{A}=(\mathrm{I}),(\mathrm{II}),(\mathrm{III})$ or (IV) in (5.96), respectively. Using Hölder's inequality, (5.99) and Proposition 4.28, it suffices to estimate the L^q-norms of

$$\left(\left(\lfloor{}^r\mathrm{u}_{\mathrm{T}}\rceil^\lambda\right)^{(1)}_\lambda\right)'_{r=0},\ \left(\left(\overline{\left(\lfloor\mathrm{u}^s_T\rceil^\lambda\right)^{(1)}_\lambda}(\mathrm{w})\right)'_0\circ\lfloor{}^r\mathbf{F}\rceil^\lambda\right)'_{r=0}.$$

This, by using Lemma 5.31 and Proposition 5.32, can be eventually reduced to a multiple of a constant depending on q, T_0, T, $\|g^\lambda\|_{C^4}$ and $\|\mathcal{X}^\lambda\|_{C^3}$ with a combination of some $L^{q'}$ norm estimations (with $q'\geqslant 1$ depending on q) of $\sup_{0\leqslant t\leqslant T}\|(\lfloor\mathrm{u}_t\rceil^\lambda)^{(1)}_\lambda\|$ and $\sup_{0\leqslant\underline{t}<\bar{t}\leqslant T}\left\|\left[D\widetilde{\lfloor\overrightarrow{F}_{\underline{t},\bar{t}}\rceil^\lambda}(\lfloor\mathrm{u}_{\underline{t}}\rceil^\lambda,w)\right]\right\|$. Hence, by Proposition 4.28, we conclude that $\mathrm{D}_3(q)$ has the same type of bound as $\mathrm{D}_1(q)$ with $\underline{c}_\lambda(q)$ depending on q, T_0, T, $\|g^\lambda\|_{C^4}$ and $\|\mathcal{X}^\lambda\|_{C^3}$.

For the L^q-norm estimation of $\nabla^{(2)}\phi^1_\lambda(T,x,\cdot)$, we continue to differentiate (5.98). Let W_2 be another smooth bounded vector field on $\widetilde{M}$. Then

$$\begin{aligned}\nabla^\lambda_{W_2(y)}\nabla^\lambda_{W(y)}\phi^1_\lambda(T,x,\cdot) =&\phi^1_\lambda(T,x,\cdot)\left(\nabla^\lambda_{W_2(y)}\nabla^\lambda_{W(y)}(\ln p^\lambda(T,x,\cdot))+\nabla^\lambda_{W_2(y)}\nabla^\lambda_{W(y)}(\ln\rho^\lambda)\right)\\&+\nabla^\lambda_{W_2(y)}\phi^1_\lambda(T,x,\cdot)\left(\nabla^\lambda_{W(y)}(\ln p^\lambda(T,x,\cdot))+\nabla^\lambda_{W(y)}(\ln\rho^\lambda)\right)\\&+\nabla^\lambda_{W_2(y)}\left(\overline{\mathbb{E}}\left(\left(\widetilde{\phi}^1_\lambda\circ\lfloor{}^r\mathbf{F}\rceil^\lambda\right)'_0+\widetilde{\phi}^1_\lambda\cdot{}^0\overline{\mathcal{E}_T}\middle|\lfloor \mathrm{x}_T\rceil^\lambda=y\right)\right)\\=:&(\mathbf{a})_y+(\mathbf{b})_y+(\mathbf{c})_y.\end{aligned}$$

Using the previous estimations of ϕ^1_λ, $\nabla^\lambda_W\phi^1_\lambda$, (5.99) and Hölder's inequality, we obtain

$$\left\|(\mathbf{a})_y\right\|_{L^q},\left\|(\mathbf{b})_y\right\|_{L^q}\leqslant\|W_2(y)\|\|W(y)\|\underline{c}_{\mathbf{a},\mathbf{b}}e^{c_{\mathbf{a},\mathbf{b}}T},$$

where $\underline{c}_{\mathbf{a},\mathbf{b}}$ depends on $q,\|g^\lambda\|_{C^4},\|\mathcal{X}^\lambda\|_{C^3}$ and $c_{\mathbf{a},\mathbf{b}}$ depends on q,T,T_0 and $\|g\|_{C^3}$. For $(\mathbf{c})_y$, we can follow the above argument for $\nabla^\lambda_{W(y)}\phi^1_\lambda(T,x,\cdot)$ to 'exchange' the differentiation $\nabla^\lambda_{W_2(y)}$ with the conditional expectation sign and obtain

$$\begin{aligned}(\mathbf{c})_y=&\overline{\mathbb{E}}\left(\left.\frac{d}{da}\right|_{a=0}\left(\left(\left(\widetilde{\phi}^1_\lambda\circ\lfloor{}^r\mathbf{F}\rceil^\lambda\right)'_0+\widetilde{\phi}^1_\lambda\cdot{}^0\overline{\mathcal{E}_T}\right)\circ\lfloor{}^a\mathbf{F}^{W_2}\rceil^\lambda\right)\middle|\lfloor \mathrm{x}_T\rceil^\lambda=y\right)\\&+\overline{\mathbb{E}}\left(\left(\left(\widetilde{\phi}^1_\lambda\circ\lfloor{}^r\mathbf{F}\rceil^\lambda\right)'_0+\widetilde{\phi}^1_\lambda\cdot{}^0\overline{\mathcal{E}_T}\right){}^0\overline{\mathcal{E}^{W_2}_T}\middle|\lfloor \mathrm{x}_T\rceil^\lambda=y\right),\end{aligned}$$

where $\lfloor{}^a\mathbf{F}^{W_2}\rceil^\lambda,{}^0\overline{\mathcal{E}^{W_2}_T}$ are the corresponding objects $\lfloor{}^a\mathbf{F}\rceil^\lambda,{}^0\overline{\mathcal{E}_T}$ for W_2. In addition to the terms involving a single differentiation of $\lfloor{}^a\mathbf{F}^{W_2}\rceil^\lambda$ or $\lfloor{}^r\mathbf{F}^{W_1}\rceil^\lambda$, we have the differentiation of $\left(\widetilde{\phi}^1_\lambda\circ\lfloor{}^r\mathbf{F}\rceil^\lambda\right)'_0$ under $\lfloor{}^a\mathbf{F}^{W_2}\rceil^\lambda$, which involves $\nabla^\lambda_{W_2(y)}\nabla^\lambda_{W(y)}\nabla^\lambda\ln p^\lambda(T,x,\cdot)$ and multi-stochastic integrals using the tangent maps $\left[D\lfloor F_{\tau,t}\rceil^\lambda(\lfloor\mathcal{V}_\tau\rceil^\lambda,\mathrm{w})\right]$ and geometric terms with bounds determined by $\|g^\lambda\|_{C^5}$ and $\|\mathcal{X}^\lambda\|_{C^4}$. So, a routine calculation as above using Proposition 4.28 gives

$$\left\|(\mathbf{c})_y\right\|_{L^q}\leqslant\|W_2(y)\|\|W(y)\|\underline{c}_{\mathbf{c}}e^{c_{\mathbf{c}}T},$$

where $\underline{c}_{\mathbf{c}}$ depends on $q,\|g^\lambda\|_{C^5},\|\mathcal{X}^\lambda\|_{C^4}$ and $c_{\mathbf{c}}$ depends on q,T,T_0 and $\|g\|_{C^3}$.

Continuing this argument, we can obtain the estimations in (5.2) for all $l\leqslant k-3$. We stop at $l=k-3$ step since $\nabla^{(l)}\phi^1_\lambda(T,x,\cdot)$ involves $\nabla^{(l+1)}(\ln p^\lambda)(T,x,\cdot)$ and the bound estimation in (4.37) is only valid for $\nabla^{(l)}(\ln p^\lambda)(T,x,\cdot)$, $l\leqslant k-2$, in general. □

In proving (1.3), we also obtain the following coarse estimation, which will be used in the inductive argument in the next section.

Corollary 5.36. *For all l, $0 \leqslant l \leqslant k-3$, there is $\underline{c}_{\lambda,(l,1)}$, depending on $\|g^\lambda\|_{C^{l+3}}$, $\|\mathcal{X}^\lambda\|_{C^{l+2}}$, and $c^{\lambda,(l,1)}$, depending on l, q, T, T_0 and $\|g^\lambda\|_{C^3}$, such that*

$$\begin{aligned}&\left|\nabla^{(l)}(\ln p^\lambda)^{(1)}_\lambda(T,x,y)\right|, \left|\nabla^{(l)}\phi^1_\lambda(T,x,y)\right| \\ &\quad\leqslant (p^\lambda(T,x,y))^{-1}\underline{c}_{\lambda,(l,1)}\left(\big(\frac{1}{T}d_{\tilde{g}^\lambda}(x,y)+\frac{1}{\sqrt{T}}\big)^{l+1}+1\right)\cdot e^{c^{\lambda,(l,1)}(1+d_{\tilde{g}^\lambda}(x,y))}.\end{aligned}$$

6. Higher order regularity of the heat kernels in metrics

To conclude Theorem 1.3 for all i, $2 \leqslant i \leqslant k-2$, we use an inductive argument based on the proof of Theorem 5.1 to identify the differentials $(p^\lambda)^{(i)}_\lambda(T,x,\cdot)$, $2 \leqslant i \leqslant k-2$, using the SDE theory in Section 4. The estimations in (1.3) and (1.4) will be obtained using the conditional stochastic expressions of $\{(\ln p^\lambda)^{(i)}_\lambda(T,x,\cdot)\}$. In the following, we first pick out the properties of $(p^\lambda)^{(i)}_\lambda(T,x,\cdot)$ necessary for an inductive argument, then verify these properties for the $i=2$ case and the $i>2$ case, respectively.

6.1. A sketch of the proof for Theorem 1.3 with $i \geqslant 2$.

Lemma 6.1. *The* i) *of Theorem 1.3 holds true if there are locally absolutely integrable functions $\{\phi^i_\lambda(T,x,y)\}_{x\in\widetilde{M},T\in\mathbb{R}_+,i\leqslant k-2}$ on $\widetilde{M}$, which are continuous in the λ-parameter and are continuous in the (T,y)-parameter, locally uniformly in λ, such that for any $f\in C^\infty_c(\widetilde{M})$,*

$$\left(\int_{\widetilde{M}} f(y)p^\lambda(T,x,y)\ d\mathrm{Vol}^\lambda(y)\right)^{(i)}_\lambda = \int_{\widetilde{M}} f(y)\phi^i_\lambda(T,x,y)p^\lambda(t,x,y)\ d\mathrm{Vol}^\lambda(y). \tag{6.1}$$

Proof. Assume (6.1) holds true. We show the differentials $(p^\lambda)^{(i)}_\lambda(T,x,\cdot)$, $i=1,\cdots,k-2$, exist as continuous functions on $\widetilde{M}$ and satisfy

$$\sum_{i=0}^{j}\binom{j}{i}(p^\lambda)^{(i)}_\lambda(T,x,y)(\rho^\lambda)^{(j-i)}(y) = \phi^j_\lambda(T,x,y)p^\lambda(T,x,y)\rho^\lambda(y),\ j=1,\cdots,k-2. \tag{6.2}$$

The $j=1$ case was handled in Lemma 5.5 and we know that $(p^\lambda)^{(1)}_\lambda(T,x,\cdot)$ is a continuous function on $\mathbb{R}_+\times\widetilde{M}$. Assume $(p^\lambda)^{(i)}_\lambda(T,x,\cdot)$, $i\leqslant j_0<k-2$, exist, are continuous, and satisfy (6.2) for $j\leqslant j_0$. Using this, a comparison of (6.1) for $i=j_0$ and j_0+1 gives

$$\begin{aligned}&\int_{\widetilde{M}}\left(\phi^{j_0}_\lambda(T,x,y)p^\lambda(T,x,y)\rho^\lambda(y)-\phi^{j_0}_0(T,x,y)p^0(T,x,y)\rho^0(y)\right)f(y)\ d\mathrm{Vol}^0(y)\\ &=\int_{\widetilde{M}}\left(\int_0^\lambda \phi^{j_0+1}_{\tilde{\lambda}}(T,x,y)p^{\tilde{\lambda}}(T,x,y)\rho^{\tilde{\lambda}}(y)d\tilde{\lambda}\right)f(y)\ d\mathrm{Vol}^0(y),\ \forall f\in C^\infty_c(\widetilde{M}).\end{aligned}$$

Since both sides are continuous functions in y-variable, we must have

$$\phi_\lambda^{j_0}(T,x,y)p^\lambda(T,x,y)\rho^\lambda(y)-\phi_0^{j_0}(T,x,y)p^0(T,x,y)\rho^0(y)\\ =\int_0^\lambda \phi_{\tilde\lambda}^{j_0+1}(T,x,y)p^{\tilde\lambda}(T,x,y)\rho^{\tilde\lambda}(y)d\tilde\lambda.$$

Consequently,

$$\left(\sum_{i=0}^{j_0}\binom{j_0}{i}(p^\lambda)_\lambda^{(i)}(T,x,y)(\rho^\lambda)_\lambda^{(j_0-i)}(y)\right)_\lambda^{(1)}=\phi_\lambda^{j_0+1}(T,x,y)p^\lambda(T,x,y)\rho^\lambda(y),$$

which implies that $(p^\lambda)_\lambda^{(j_0+1)}(T,x,y)$ exists for every y and satisfies (6.2). Then we can conclude from this and the inductive assumption on the continuity of $(p^\lambda)_\lambda^{(i)}(T,x,\cdot)$, $i=1,\cdots,j_0$, that $(p^\lambda)_\lambda^{(j_0+1)}(T,x,\cdot)$ is also a continuous function on $\mathbb{R}_+\times\widetilde{M}$.

Now, the differentials $(p^\lambda)_\lambda^{(i)}(T,x,\cdot)$, $i=1,\cdots,k-2$, exist as continuous functions on $\mathbb{R}_+\times\widetilde{M}$ and hence their weak derivatives in (T,y) of any order are well-defined. Taking the differential of the heat equations $\mathrm{L}^\lambda p^\lambda=0$ in λ gives the following identities in distribution:

$$\mathrm{L}^{\lambda,\mathrm{w}}(p^\lambda)_\lambda^{(i)}(T,x,\cdot)+\sum_{j=1}^{i}\binom{i}{j}(\mathrm{L}^\lambda)^{(j),\mathrm{w}}(p^\lambda)_\lambda^{(i-j)}(T,x,\cdot)=0,\quad i=1,\cdots,k-2,$$

where $(\mathrm{L}^\lambda)^{(j),\mathrm{w}}$ is the weak derivative of the j-th differential operator $(L^\lambda)_\lambda^{(j)}$. We can use Lemma 5.4 and Lemma 5.2 inductively to improve the regularity of $(p^\lambda)_\lambda^{(i)}(T,x,\cdot)$. Shrinking the neighborhood $\mathcal{V}_g$ of g if necessary, we may assume there is $\iota>0$ such that $p^\lambda\in C^{k,\iota}$ for all λ. Since it is a local problem, for $(T,y)\in\mathbb{R}_+\times\widetilde{M}$, we can also restrict ourselves to a bounded domain $\mathcal{D}$ containing (T,y). By Lemma 5.5, there is some domain $\mathcal{D}_1\subset\mathcal{D}$ such that $p^\lambda(T,x,\cdot),(p^\lambda)_\lambda^{(1)}(T,x,\cdot)\in C^{k,\iota}(\mathcal{D}_1)$. Assume for all $i\leqslant j_0<k-2$ there are domains $\mathcal{D}_i$ containing (T,y) such that $|(p^\lambda)_\lambda^{(i)}(T,x,\cdot)|_{0,2+\iota}<\infty$ on $\mathcal{D}_i$ and $(p^\lambda)_\lambda^{(i)}(T,x,\cdot)\in C^{k,\iota}(\mathcal{D}_i)$. Then

$$\begin{aligned}\mathrm{L}^{\lambda,\mathrm{w}}(p^\lambda)_\lambda^{(j_0+1)}(T,x,\cdot)&=-\sum_{j=1}^{j_0+1}\binom{j_0+1}{j}(\mathrm{L}^\lambda)^{(j),\mathrm{w}}(p^\lambda)_\lambda^{(j_0+1-j)}(T,x,\cdot)\\ &=-\sum_{j=1}^{j_0+1}\binom{j_0+1}{j}(\mathrm{L}^\lambda)_\lambda^{(j)}(p^\lambda)_\lambda^{(j_0+1-j)}(T,x,\cdot).\end{aligned}\tag{6.3}$$

Shrinking $\mathcal{D}_{j_0}$ to $\mathcal{D}_{j_0+1}$ if necessary, we can deduce from $|(p^\lambda)_\lambda^{(i)}(T,x,\cdot)|_{0,2+\iota}<\infty$ on $\mathcal{D}_i$ that $|(\mathrm{L}^\lambda)^{(j)}(p^\lambda)_\lambda^{(j_0+1-j)}(T,x,\cdot)|_{2,\iota}$ is finite for all $j\leqslant j_0+1$ on $\mathcal{D}_{j_0+1}$. Since $(p^\lambda)_\lambda^{(j_0+1)}(T,x,\cdot)$ is continuous, Lemma 5.4 shows that (6.3) holds in the usual sense. Then we can apply Lemma 5.3 to conclude that $|(p^\lambda)_\lambda^{(j_0+1)}(T,x,\cdot)|_{0,2+\iota}<\infty$ on $\mathcal{D}_{j_0+1}$ and apply Lemma 5.2 to conclude $(p^\lambda)_\lambda^{(j_0+1)}(T,x,\cdot)\in C^{k,\iota}(\mathcal{D}_{j_0+1})$. Accordingly, the continuity of $\lambda\mapsto(p^\lambda)_\lambda^{(1)}(T,x,\cdot)$

in $C(\widetilde{M})$ can be improved to be the continuity in $C^{k,\iota}(\widetilde{M})$ by using the parabolic differential equation (6.3), Lemma 5.2 and Lemma 5.3. □

The ϕ^1_λ satisfying (6.1) was identified in Theorem 5.1. We continue to pick up a candidate ϕ^2_λ for (6.1). Let $\widetilde{\phi}^1_\lambda(T,x,\cdot)$ be as in (5.96) such that (5.95) holds. Then, for any $f \in C^\infty_c(\widetilde{M})$,

$$\left(\int_{\widetilde{M}} f(y)p^\lambda(T,x,y)\ d\mathrm{Vol}^\lambda(y)\right)^{(1)}_\lambda = \overline{\mathbb{E}}\left(f(\lfloor \mathrm{x}_T\rceil^\lambda(w))\widetilde{\phi}^1_\lambda(T,x,w)\right). \tag{6.4}$$

If we can show $\lambda \mapsto \widetilde{\phi}^1_\lambda$ is differentiable and both $\widetilde{\phi}^1_\lambda$ and the differential $(\widetilde{\phi}^1_\lambda)^{(1)}_\lambda$ are L^q integrable for some $q \geqslant 1$, we are allowed to differentiate under the expectation sign of the right hand side term of (6.4). This will give

$$\begin{aligned}
&\left(\int_{\widetilde{M}} f(y)p^\lambda(T,x,y)\ d\mathrm{Vol}^\lambda(y)\right)^{(2)}_\lambda \\
&\qquad = \int_{\widetilde{M}} f(y)\overline{\mathbb{E}}\left((\widetilde{\phi}^1_\lambda)^{(1)}_\lambda(T,x,w)\big|\lfloor \mathrm{x}_T\rceil^\lambda(w)=y\right)p^\lambda(T,x,y)\ d\mathrm{Vol}^\lambda(y) \\
&\qquad\quad + \overline{\mathbb{E}}\left(\left\langle \nabla^\lambda_{\lfloor \mathrm{x}_T\rceil^\lambda(w)} f(\lfloor \mathrm{x}_T\rceil^\lambda(w)),\ \widetilde{\phi}^1_\lambda(T,x,w)\cdot D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda(w)\right\rangle_\lambda\right).
\end{aligned} \tag{6.5}$$

We can deal with the last expectation term in (6.5) as we did for ϕ^1_λ in Section 5. Define

$$\Phi^2_\lambda(Y,w) := \left\langle Y(\lfloor \mathrm{x}_T\rceil^\lambda(w)),\ \widetilde{\phi}^1_\lambda(T,x,w)\cdot D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda(w)\right\rangle_\lambda,$$

where Y is any C^k bounded vector field on $\widetilde{M}$, and consider the linear functional

$$\overline{\Phi}^2_\lambda :\ Y \mapsto \overline{\mathbb{E}}\left(\Phi^2_\lambda(Y,w)\big|\lfloor \mathrm{x}_T\rceil^\lambda(w)=y\right).$$

If we can show $\overline{\Phi}^2_\lambda$ is such that $\overline{\Phi}^2_\lambda(Y)$ is C^1 in y variable, we can conclude that

$$\overline{\mathbb{E}}\left(\widetilde{\phi}^1_\lambda(T,x,w)\cdot D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda(w)\Big|\lfloor \mathrm{x}_T\rceil^\lambda(w)=y\right) =: \mathrm{z}^{\lambda,2}_T(y) \tag{6.6}$$

is a C^1 vector field on $\widetilde{M}$ and satisfies

$$\overline{\Phi}^2_\lambda(\nabla f)(y) = \left\langle \nabla^\lambda_y f(y), \mathrm{z}^{\lambda,2}_T(y)\right\rangle_\lambda.$$

Using the classical integration by parts formula, we obtain

$$\begin{aligned}
&\overline{\mathbb{E}}\left(\left\langle \nabla^\lambda_{\lfloor \mathrm{x}_T\rceil^\lambda(w)} f(\lfloor \mathrm{x}_T\rceil^\lambda(w)), \widetilde{\phi}^1_\lambda(T,x,w)\cdot D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda(w)\right\rangle_\lambda\right) \\
&= \int_{\widetilde{M}} \left\langle \nabla^\lambda_y f(y), p^\lambda(T,x,y)\mathrm{z}^{\lambda,2}_T(y)\right\rangle_\lambda\ d\mathrm{Vol}^\lambda(y) \\
&= -\int_{\widetilde{M}} f(y)\left(\mathrm{Div}^\lambda \mathrm{z}^{\lambda,2}_T(y) + \left\langle \mathrm{z}^{\lambda,2}_T(y), \nabla^\lambda \ln p^\lambda(T,x,y)\right\rangle_\lambda\right)p^\lambda(T,x,y)\ d\mathrm{Vol}^\lambda(y).
\end{aligned}$$

Therefore, a candidate of $\phi^2_\lambda(T,x,\cdot)$ for (6.1) is

$$\begin{aligned}\phi^2_\lambda(T,x,y) := \overline{\mathbb{E}}\left((\widetilde{\phi}^1_\lambda)^{(1)}_\lambda(T,x,w)\big|\lfloor \mathrm{x}_T\rceil^\lambda(w)=y\right)\\ -\left(\mathrm{Div}^\lambda \mathrm{z}^{\lambda,2}_T(y)+\left\langle \mathrm{z}^{\lambda,2}_T(y),\nabla^\lambda \ln p^\lambda(T,x,y)\right\rangle_\lambda\right).\end{aligned} \tag{6.7}$$

Once we show $\phi^2_\lambda(T,x,y)$ fulfills the continuity requirement of Lemma 6.1, we can conclude the second order differentiability of $\lambda\mapsto p^\lambda(T,x,)$ in $C^{k,\iota}$ for some $\iota>0$. It follows that

$$(\ln p^\lambda)^{(2)}_\lambda(T,x,y)=\phi^2_\lambda(T,x,y)-(\phi^1_\lambda)^2(T,x,y)-(\ln\rho^\lambda)^{(2)}_\lambda(y).$$

Note that the gradients estimations of ϕ^1_λ were already handled in Theorem 5.1. Hence the gradients estimations of $(\ln p^\lambda)^{(2)}_\lambda$ can be reduced to that of ϕ^2_λ, which can be analyzed following the proof of Theorem 5.1, if we can find some controllable $\widetilde{\phi}^2_\lambda(T,x,\cdot)$ such that

$$\phi^2_\lambda(T,x,y)=\overline{\mathbb{E}}\left(\widetilde{\phi}^2_\lambda(T,x,w)\Big|\lfloor \mathrm{x}_T\rceil^\lambda(w)=y\right).$$

We will follow this steam line of discussion to find all the candidates ϕ^i_λ for (6.1). Put $\widetilde{\phi}^0_\lambda(T,x,w)\equiv 1$ and let $\widetilde{\phi}^1_\lambda(T,x,w)$ be as in (5.96). For i, $2\leqslant i\leqslant k-2$, define

$$\begin{aligned}\widetilde{\phi}^i_\lambda(T,x,w) \;:=\; &\big(\widetilde{\phi}^{i-1}_\lambda(T,x,w)\big)^{(1)}_\lambda-\left\langle\nabla^\lambda_{T,\mathbf{s}}\widetilde{\phi}^{i-1}_\lambda(T,x,w),D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda(w)\right\rangle\\ &+\widetilde{\phi}^{i-1}_\lambda(T,x,w)\widetilde{\phi}^1_\lambda(T,x,w),\end{aligned} \tag{6.8}$$

where the 'path-wise gradient' $\nabla^\lambda_{T,\mathbf{s}}\widetilde{\phi}^{i-1}_\lambda(T,x,w)$ will be specified later. We will show each

$$\phi^i_\lambda(T,x,y):=\overline{\mathbb{E}}\left(\widetilde{\phi}^i_\lambda(T,x,w)\Big|\lfloor \mathrm{x}_T\rceil^\lambda(w)=y\right) \tag{6.9}$$

fulfills all the requirements in Lemma 6.1. The stochastic expression (6.9) will be used for two purposes: one is for the gradient estimations of $\phi^i_\lambda(T,x,\cdot)$ and $(\ln p^\lambda)^{(i)}_\lambda(T,x,\cdot)$; the other is for obtaining $\phi^{i+1}_\lambda(T,x,y)$ as we exposed above for $\phi^2_\lambda(T,x,y)$ (see (6.7)).

Let us highlight the necessary steps to undergo an inductive argument for Theorem 1.3. Assume for all $i<j\leqslant k-2$, the ϕ^i_λ defined in (6.9) are such that Lemma 6.1 holds true, $(p^\lambda)^{(i)}_\lambda(T,x,\cdot)\in C^{k,\iota}(\widetilde{M})$, is continuous in λ and (1.3) holds for $(\ln p^\lambda)^{(i)}_\lambda(T,x,\cdot)$. We also assume the following coarse pointwise estimation holds true for all $i<j$.

0) For all l, $0\leqslant l\leqslant k-2-i$, there is $\underline{c}_{\lambda,(l,i)}$ depending on $\|g^\lambda\|_{C^{l+i+2}}$, $\|\mathcal{X}^\lambda\|_{C^{l+i+1}}$ and $c^{\lambda,(l,i)}$ depending on (l,i), q, T,T_0 and $\|g^\lambda\|_{C^3}$ such that

$$\begin{aligned}&\left|\nabla^{(l)}(\ln p^\lambda)^{(i)}_\lambda(T,x,y)\right|,\left|\nabla^{(l)}\phi^i_\lambda(T,x,y)\right|\\ &\qquad\leqslant (p^\lambda(T,x,y))^{-i}\underline{c}_{\lambda,(l,i)}\left(\Big(\frac{1}{T}d_{\widetilde{g}^\lambda}(x,y)+\frac{1}{\sqrt{T}}\Big)^{l+i}+1\right)\cdot e^{c^{\lambda,(l,i)}(1+d_{\widetilde{g}^\lambda}(x,y))}.\end{aligned} \tag{6.10}$$

For the existence of $(p^\lambda)^{(j)}_\lambda(T,x,\cdot)$, the very first step is find some measurable candidate satisfying (6.1), which can be done once we show the following.

i) The function $\widetilde{\phi}_\lambda^{j-1}(T,x,w)$ is differentiable in λ for almost all $w \in \Theta_+$ and both $\widetilde{\phi}_\lambda^{j-1}(T,x,w)$ and $\big(\widetilde{\phi}_\lambda^{j-1}\big)_\lambda^{(1)}(T,x,w)$ are L^q integrable in w for all $q \geqslant 1$.

ii) For any C^k bounded vector field Y on $\widetilde{M}$, let

$$\Phi_\lambda^j(Y,w) := \big\langle Y(\lfloor \mathrm{x}_T \rceil^\lambda(w)), \widetilde{\phi}_\lambda^{j-1}(T,x,w) D\pi(\lfloor \mathrm{u}_T \rceil^\lambda)_\lambda^{(1)}(w) \big\rangle_\lambda.$$

Then the linear functional

$$\overline{\Phi}_\lambda^j : \ Y \mapsto \overline{\mathbb{E}}\left(\Phi_\lambda^j(Y,w) \Big| \lfloor \mathrm{x}_T \rceil^\lambda(w) = y \right)$$

is bounded with $\overline{\Phi}_\lambda^j(Y)(y)$ varying C^1 in the y-coordinate.

Claim 6.2. *Assume* **i), ii)** *are true. Then (6.1) hold with some* $\overline{\phi}_\lambda(T,x,\cdot)$ *for* $i = j$.

Proof. By the inductive assumption,

$$\left(\int_{\widetilde{M}} f(y) p^\lambda(T,x,y) \, d\mathrm{Vol}^\lambda(y) \right)_\lambda^{(j-1)} = \ \overline{\mathbb{E}}\left(f(\lfloor \mathrm{x}_T \rceil^\lambda) \widetilde{\phi}_\lambda^{j-1}(T,x,w) \right). \tag{6.11}$$

If **i)** is true, we can differentiate under the expectation sign of (6.11). This gives

$$\left(\int_{\widetilde{M}} f(y) p^\lambda(T,x,y) \, d\mathrm{Vol}^\lambda(y) \right)_\lambda^{(j)} = \ \overline{\mathbb{E}}\left(f(\lfloor \mathrm{x}_T \rceil^\lambda(w)) (\widetilde{\phi}_\lambda^{j-1})_\lambda^{(1)}(T,x,w) \right) + \overline{\mathbb{E}}\left(\Phi_\lambda^j(\nabla f, w) \right).$$

The property **ii)** implies

$$\mathrm{z}_T^{\lambda,j}(y) := \ \overline{\mathbb{E}}\left(\widetilde{\phi}_\lambda^{j-1}(T,x,w) \cdot D\pi(\lfloor \mathrm{u}_T \rceil^\lambda)_\lambda^{(1)}(w) \Big| \lfloor \mathrm{x}_T \rceil^\lambda(w) = y \right) \tag{6.12}$$

is a C^1 vector field on $\widetilde{M}$ such that

$$\overline{\Phi}_\lambda^j(Y)(y) = \big\langle Y(y), \mathrm{z}_T^{\lambda,j}(y) \big\rangle_\lambda.$$

In particular, we have

$$\begin{aligned} \overline{\mathbb{E}}\left(\Phi_\lambda^j(\nabla f, w) \right) &= \int_{\widetilde{M}} \big\langle \nabla_y^\lambda f(y), p^\lambda(T,x,y) \mathrm{z}_T^{\lambda,j}(y) \big\rangle_\lambda \, d\mathrm{Vol}^\lambda(y) \\ &= -\int_{\widetilde{M}} f(y) \Big(\mathrm{Div}^\lambda \mathrm{z}_T^{\lambda,j}(y) + \big\langle \mathrm{z}_T^{\lambda,j}(y), \nabla^\lambda \ln p^\lambda(T,x,y) \big\rangle_\lambda \Big) p^\lambda(T,x,y) \, d\mathrm{Vol}^\lambda(y). \end{aligned}$$

This means a measurable candidate ϕ_λ^j for (6.1) at $i = j$ is

$$\begin{aligned} \overline{\phi}_\lambda(T,x,y) := \overline{\mathbb{E}}\left((\widetilde{\phi}_\lambda^{j-1})_\lambda^{(1)}(T,x,w) \Big| \lfloor \mathrm{x}_T \rceil^\lambda(w) = y \right) \\ - \left(\mathrm{Div}^\lambda \mathrm{z}_T^{\lambda,j}(y) + \big\langle \mathrm{z}_T^{\lambda,j}(y), \nabla^\lambda \ln p^\lambda(T,x,y) \big\rangle_\lambda \right). \end{aligned} \tag{6.13}$$

□

We will show $\overline{\phi}_\lambda(T,x,\cdot)$ given in Claim 6.2 coincides with $\phi_\lambda^j(T,x,\cdot)$ defined by (6.9). For any smooth bounded vector field V on $\widetilde{M}$, let F^s be the flow it generates. As we did for $\overline{\Phi}_\lambda^1$ in Section 5 (see Lemma 5.34), we will prove the following in verifying **ii)**.

iii) For any $y \in \widetilde{M}$, $s \mapsto \overline{\Phi}^j_\lambda(Y)(F^s y)$ is differentiable at $s=0$ and the differential $(\overline{\Phi}^j_\lambda(Y)(F^s y))'_0$ varies continuously in y. Moreover,

$$\begin{aligned}
&(\overline{\Phi}^j_\lambda(Y)(F^s y))'_0\\
&= \overline{\Phi}^j_\lambda(\nabla^\lambda_{\mathrm{V}} Y)(y) + \overline{\mathbb{E}}\Big(\big\langle Y(\lfloor \mathrm{x}_T\rceil^\lambda(w)), \nabla^\lambda_{T,\mathrm{V},\mathfrak{s}}\big(\widetilde{\phi}^{j-1}_\lambda(T,x,w) D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda(w)\big)\big\rangle_\lambda\\
&\quad + \big\langle Y(\lfloor \mathrm{x}_T\rceil^\lambda(w)), \big(\widetilde{\phi}^{j-1}_\lambda(T,x,w) D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda(w)\big)\big\rangle_\lambda \overleftarrow{\mathcal{E}}_{T,\mathrm{V},\mathfrak{s}}(w)\Big| \lfloor \mathrm{x}_T\rceil^\lambda(w) = y\Big),
\end{aligned}$$

where the path-wise differential $\nabla^\lambda_{T,\mathrm{V},\mathfrak{s}}\big(\widetilde{\phi}^{i-1}_\lambda(T,x,w) D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda(w)\big)$ will be clarified later and it satisfies

$$\begin{aligned}
\nabla^\lambda_{T,\mathrm{V},\mathfrak{s}}\big(\widetilde{\phi}^{i-1}_\lambda(T,x,w) D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda(w)\big) &= \big(\nabla^\lambda_{T,\mathrm{V},\mathfrak{s}}\widetilde{\phi}^{i-1}_\lambda(T,x,w)\big)\cdot D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda(w)\\
&\quad + \widetilde{\phi}^{i-1}_\lambda(T,x,w)\cdot\big(\nabla^\lambda_{T,\mathrm{V},\mathfrak{s}} D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda(w)\big).
\end{aligned}$$

Claim 6.3. *Assume* **i)-iii)** *are true. Then* $\overline{\phi}_\lambda(T,x,\cdot) = \phi^j_\lambda(T,x,\cdot)$.

Proof. By **ii)**, both $\overline{\Phi}^j_\lambda(Y)(y)$ and $\mathrm{z}^{\lambda,j}_T(y)$ vary C^1 in y. Hence

$$(\overline{\Phi}^j_\lambda(Y)(F^s y))'_0 = \nabla^\lambda_{\mathrm{V}}(\overline{\Phi}^j_\lambda(Y))(y) = \big\langle \nabla^\lambda_{\mathrm{V}} Y(y), \mathrm{z}^{\lambda,j}_T(y)\big\rangle_\lambda + \big\langle Y(y), \nabla^\lambda_{\mathrm{V}}\mathrm{z}^{\lambda,j}_T(y)\big\rangle_\lambda.$$

Comparing this with the expression of $\big(\overline{\Phi}^j_\lambda(Y)(F^s y)\big)'_0$ in **iii)** gives

$$\begin{aligned}
\nabla^\lambda_{\mathrm{V}(y)}\mathrm{z}^{\lambda,j}_T(y) = \overline{\mathbb{E}}\Big(&\nabla^\lambda_{T,\mathrm{V},\mathfrak{s}}\big(\widetilde{\phi}^{j-1}_\lambda(T,x,w)\cdot D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda(w)\big)\\
&+\big(\widetilde{\phi}^{j-1}_\lambda(T,x,w)\cdot D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda(w)\big)\overleftarrow{\mathcal{E}}_{T,\mathrm{V},\mathfrak{s}}(w)\Big|\lfloor \mathrm{x}_T\rceil^\lambda(w) = y\Big).
\end{aligned}\tag{6.14}$$

Following the argument in the proof of Lemma 5.35 and then using (5.96), we obtain

$$\begin{aligned}
&\mathrm{Div}^\lambda \mathrm{z}^{\lambda,j}_T(y)\\
&= \overline{\mathbb{E}}\Big(\mathrm{tr}\big(\mathrm{V}\mapsto \nabla^\lambda_{T,\mathrm{V},\mathfrak{s}}\big(\widetilde{\phi}^{j-1}_\lambda(T,x,w)\cdot D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda(w)\big)\big)\\
&\quad + \widetilde{\phi}^{j-1}_\lambda(T,x,w)\cdot\big\langle D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda(w), \frac{1}{2}\int_0^T \mathfrak{s}(T-\tau)\lfloor \mathrm{u}_T\rceil^\lambda(\lfloor \mathrm{u}_\tau\rceil^\lambda)^{-1}\mathrm{Ric}^{-1}_{\lfloor \mathrm{u}_\tau\rceil^\lambda} d\overleftarrow{B}_\tau\big\rangle_\lambda\\
&\quad + \widetilde{\phi}^{j-1}_\lambda(T,x,w)\cdot\big\langle D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda(w), -\frac{1}{2}\lfloor \mathrm{u}_T\rceil^\lambda\int_0^T \mathfrak{s}'(T-\tau) d\overleftarrow{B}_\tau\big\rangle_\lambda \ \Big|\lfloor \mathrm{x}_T\rceil^\lambda(w) = y\Big)\\
&= \overline{\mathbb{E}}\Big(\big\langle \nabla^\lambda_{T,\mathfrak{s}}\widetilde{\phi}^{j-1}_\lambda(T,x,w), D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda(w)\big\rangle\Big|\lfloor \mathrm{x}_T\rceil^\lambda(w) = y\Big)\\
&- \overline{\mathbb{E}}\Big(\widetilde{\phi}^{j-1}_\lambda(T,x,w)\big(\widetilde{\phi}^1_\lambda(T,x,w) + \big\langle D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda, \nabla^\lambda \ln p^\lambda(T,x,\lfloor \mathrm{x}_T\rceil^\lambda)\big\rangle_\lambda\big)\Big|\lfloor \mathrm{x}_T\rceil^\lambda(w) = y\Big).
\end{aligned}$$

Note that

$$\begin{aligned}
&\langle \mathrm{z}^{\lambda,j}_T(y), \nabla^\lambda \ln p^\lambda(T,x,y)\rangle_\lambda\\
&\quad = \overline{\mathbb{E}}\Big(\widetilde{\phi}^{j-1}_\lambda(T,x,w)\big\langle D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda(w), \nabla^\lambda \ln p^\lambda(T,x,\lfloor \mathrm{x}_T\rceil^\lambda(w))\big\rangle_\lambda\Big|\lfloor \mathrm{x}_T\rceil^\lambda(w) = y\Big).
\end{aligned}$$

Reporting these two expressions in (6.13), we obtain

$$\begin{aligned}\overline{\phi}_\lambda(T,x,y) &= \overline{\mathbb{E}}\Big(\big(\widetilde{\phi}_\lambda^{j-1}(T,x,w)\big)_\lambda^{(1)} - \big\langle \nabla^\lambda_{T,\mathrm{s}}\widetilde{\phi}_\lambda^{j-1}(T,x,w), D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)_\lambda^{(1)}(w)\big\rangle \\ &\qquad + \widetilde{\phi}_\lambda^{j-1}(T,x,w)\widetilde{\phi}_\lambda^{1}(T,x,w)\Big|\, \lfloor \mathrm{x}_T\rceil^\lambda(w) = y\Big) \\ &= \phi_\lambda^j(T,x,y).\end{aligned}$$

□

To study the continuity of $\phi_\lambda^j(T,x,y)$ in (T,y), we first show the following.

iv) For all $x \in \widetilde{M}$, $T \in \mathbb{R}_+$,

$$y \mapsto \overline{\mathbb{E}}\Big((\widetilde{\phi}_\lambda^{j-1})_\lambda^{(1)}(T,x,w)\Big|\, \lfloor \mathrm{x}_T\rceil^\lambda(w) = y\Big)$$

is continuous, locally uniformly in T and λ. Moreover, there exist $\underline{c}_{\lambda,j}$ (depending on $\|g^\lambda\|_{C^{j+2}}, \|\mathcal{X}^\lambda\|_{C^{j+1}}$) and $\widetilde{c}^{\lambda,j}$ depending on j, T, T_0 and $\|g^\lambda\|_{C^3}$ such that

$$\begin{aligned}&\Big|\overline{\mathbb{E}}_{\overline{\mathbb{P}}^\lambda_{x,y,T}}\Big((\widetilde{\phi}_\lambda^{j-1})_\lambda^{(1)}(T,x,w)\Big)\Big| \\ &\qquad \leqslant (p^\lambda(T,x,y))^{-j}\underline{c}_{\lambda,j}\left(\big(\frac{1}{T}d_{\tilde{g}^\lambda}(x,y) + \frac{1}{\sqrt{T}}\big)^j + 1\right)\cdot e^{\widetilde{c}^{\lambda,j}(1+d_{\tilde{g}^\lambda}(x,y))}.\end{aligned}$$

v) For all $x \in \widetilde{M}$, $T \in \mathbb{R}_+$, the mappings $y \mapsto \mathrm{z}_T^{\lambda,j}(y), y \mapsto \mathrm{Div}^\lambda \mathrm{z}_T^{\lambda,j}(y)$ are continuous, locally uniformly in T and λ. Moreover, there are $\underline{c}'_{\lambda,j}$ (depending on $\|g^\lambda\|_{C^{j+2}}, \|\mathcal{X}^\lambda\|_{C^{j+1}}$) and $\overline{c}^{\lambda,j}$ depending on j, T, T_0 and $\|g^\lambda\|_{C^3}$ such that

$$\begin{aligned}&\big|\mathrm{z}_T^{\lambda,j}(y)\big|,\ \big|\mathrm{Div}^\lambda \mathrm{z}_T^{\lambda,j}(y)\big| \\ &\qquad \leqslant (p(T,x,y))^{-j}\underline{c}'_{\lambda,j}\left(\big(\frac{1}{T}d_{\tilde{g}^\lambda}(x,y) + \frac{1}{\sqrt{T}}\big)^j + 1\right)\cdot e^{\overline{c}^{\lambda,j}(1+d_{\tilde{g}^\lambda}(x,y))}.\end{aligned}$$

vi) There is $\underline{c}_{\lambda,(0,j)}(q)$ depending on j, q, T, T_0, $\|g^\lambda\|_{C^{j+2}}$ and $\|\mathcal{X}^\lambda\|_{C^{j+1}}$ such that

$$\Big\|\phi_\lambda^j(T,x,\cdot)\Big\|_{L^q} \leqslant \underline{c}_{\lambda,(0,j)}(q),\ \forall q \geqslant 1. \tag{6.15}$$

Claim 6.4. *Assume* **i)-vi)**. *For every $x \in \widetilde{M}$, $(T,y) \mapsto \phi_\lambda^j(T,x,y)$ is continuous, locally uniformly in λ.*

Proof. To conclude the continuity of $\phi_\lambda^j(T,x,y)$ in (T,y), we verify

1) For all $x \in \widetilde{M}$, $T \in \mathbb{R}_+$, $y \mapsto \phi_\lambda^j(T,x,y)$ is continuous, locally uniformly in T and λ.
2) For each x, y fixed, $T \mapsto \phi_\lambda^j(T,x,y)$ is continuous, locally uniformly in λ.

By the inductive assumption, all $(p^\lambda)^{(i)}(T,x,y)$, $i<j$, exist, are continuous in (T,y) and satisfy the bound estimation (1.3). By **i)-iii),**

$$\begin{aligned}\phi^j_\lambda(T,x,y) &= \overline{\mathbb{E}}\left((\widetilde{\phi}^{j-1}_\lambda)^{(1)}_\lambda(T,x,w)\Big|\,\lfloor \mathrm{x}_T\rceil^\lambda(w)=y\right)\\ &\quad-\left(\mathrm{Div}^\lambda \mathrm{z}^{\lambda,j}_T(y)+\langle \mathrm{z}^{\lambda,j}_T(y),\nabla^\lambda \ln p^\lambda(T,x,y)\rangle_\lambda\right)\end{aligned}\tag{6.16}$$

satisfies (6.1). By **iv)** and **v)**, for each $x\in\widetilde{M}$, we have that the mapping $y\mapsto\phi^j_\lambda(T,x,y)$ is continuous, locally uniformly in T.

For 2), we follow the proof of Theorem 5.1 for the $k=3$ case. Simply denote by $(\mathrm{x}^\lambda,\mathrm{u}^\lambda)$ the stochastic pair which defines the Brownian motion on $(\widetilde{M},\widetilde{g}^\lambda)$ starting from x. Then, for any $f\in C^\infty_c(\widetilde{M})$ with support contained in a small neighborhood of y and $T'>T$,

$$\mathbb{E}\big(f(\mathrm{x}^\lambda_{T'})\big)-\mathbb{E}\big(f(\mathrm{x}^\lambda_T)\big)=\int_T^{T'}\mathbb{E}\big(\Delta^\lambda f(\mathrm{x}^\lambda_t)\big)\,dt.$$

Take the j-th differential in λ of both sides and use (6.1). We obtain

$$\begin{aligned}&\int_{\widetilde{M}} f(z)\left(\phi^j_\lambda(T',x,z)-\phi^j_\lambda(T,x,z)\right)p^\lambda(T',x,z)\,d\mathrm{Vol}^\lambda(z)\\ &+\int_{\widetilde{M}} f(z)\phi^j_\lambda(T,x,z)\left(p^\lambda(T',x,z)-p^\lambda(T,x,z)\right)\,d\mathrm{Vol}^\lambda(z)\\ &=\int_T^{T'}\int_{\widetilde{M}}\sum_{i=0}^{j}\binom{j}{i}(\Delta^\lambda f(z))^{(i)}_\lambda\left(p^\lambda(t,x,z)\rho^\lambda(z)\right)^{(j-i)}_\lambda\,d\mathrm{Vol}^0(z)\,dt.\end{aligned}$$

Using (6.15), we deduce that

$$\lim_{T'\to T}\int_{\widetilde{M}} f(z)\left(\phi^j_\lambda(T',x,z)-\phi^j_\lambda(T,x,z)\right)p^\lambda(T',x,z)\,d\mathrm{Vol}^\lambda(z)=0.$$

Since $\phi^j_\lambda(T,x,z)$ is continuous in z and λ, locally uniformly in T, and f is arbitrary, we must have $\lim_{T'\to T}\phi^j_\lambda(T',x,y)=\phi^j_\lambda(T,x,y)$, locally uniformly in λ. This shows 2) and finishes the proof of Claim 6.4. □

Claim 6.5. *Assume* **i)-vi)**. *Then for any $x\in\widetilde{M}$, $T\in\mathbb{R}_+$, $\lambda\mapsto p^\lambda(T,x,\cdot)$ is C^j in $C^{k,\iota}(\widetilde{M})$ for some $\iota>0$. The differential $(p^\lambda)^{(j)}_\lambda(T,x,y)$ satisfies the equation*

$$(p^\lambda)^{(j)}_\lambda(T,x,y)=\phi^j_\lambda(T,x,y)p^\lambda(T,x,y)-(\rho^\lambda(y))^{-1}\sum_{i=0}^{j-1}\binom{j}{i}(p^\lambda)^{(i)}_\lambda(T,x,y)(\rho^\lambda)^{(j-i)}_\lambda(y).\tag{6.17}$$

Consequently, $\phi^j_\lambda(T,x,\cdot)\in C^{k,\iota}(\widetilde{M})$ as well.

Proof. The function $\phi^j_\lambda(T,x,y)$ is continuous in y, uniformly in λ by using **iv), v)** and (6.16). So, it is continuous in λ if for any $f \in C^\infty_c(\widetilde{M})$, we have the continuity of

$$\lambda \mapsto \left(\int_{\widetilde{M}} f(y) p^\lambda(T,x,y)\, d\mathrm{Vol}^\lambda(y)\right)^{(j)}_\lambda =: A_j(\lambda,T,x). \tag{6.18}$$

Note that

$$A_1(\lambda,T,x) = \mathbb{E}\left(\left\langle \nabla^\lambda_{\lfloor \mathrm{x}_T\rfloor^\lambda(w)}(f\circ\pi)(\lfloor \mathrm{u}_T\rfloor^\lambda(w)), (\lfloor \mathrm{u}_T\rfloor^\lambda)^{(1)}_\lambda(w)\right\rangle_\lambda\right).$$

Differentiating $A_1(\lambda,T,x)$ in λ for $j-1$ times, we get a similar expression $A_j(\lambda,T,x)$ of a combination of inner products involving $\{\nabla^{(i)}f\}_{i\leqslant j}$, $\{(\lfloor \mathrm{x}_T\rfloor^\lambda)^{(i)}\}_{i\leqslant j}$ and $\{(\lfloor \mathrm{u}_T\rfloor^\lambda)^{(i)}_\lambda\}_{i\leqslant j}$. Following Proposition 4.28 i), we can derive the L^q $(q\geqslant 1)$ convergence of $(\lfloor \mathrm{x}_T\rfloor^\lambda)^{(i)}$ and $(\lfloor \mathrm{u}_T\rfloor^\lambda)^{(i)}_\lambda$ in λ. As a consequence, we obtain the continuity of $\lambda\mapsto A_j(\lambda,T,x)$.

Now, by **vi)**, the continuity of $\phi^j_\lambda(T,x,y)$ in λ and (T,y) and the induction assumption, we can apply Lemma 6.1 to conclude that $\lambda\mapsto p^\lambda(T,x,\cdot)$ is C^j in $C^{k,\iota}(\widetilde{M})$ for some $\iota>0$. The equation (6.17) holds by comparison and hence $\phi^j_\lambda(T,x,\cdot)\in C^{k,\iota}(\widetilde{M})$. □

With **i)-vi)**, the gradients $\{\nabla^{(l)}(\ln p^\lambda)^{(j)}_\lambda(T,x,y)\}_{1\leqslant l\leqslant k-2-j}$ are well-defined. To conclude Theorem 1.3 ii) by induction, it remains to show (1.3) for $i=j$. With the identity (6.17) and **vi)**, it remains to show the following.

vii) For all l, $1\leqslant l\leqslant k-2-j$, $q\geqslant 1$, there is $\underline{c}_{\lambda,(l,j)}(q)$ which depends on (l,j), q, T, T_0, $\|g^\lambda\|_{C^{l+j+2}}$ and $\|\mathcal{X}^\lambda\|_{C^{l+j+1}}$ such that

$$\left\|\nabla^{(l)}\phi^j_\lambda(T,x,\cdot)\right\|_{L^q} \leqslant \underline{c}_{\lambda,(l,j)}(q). \tag{6.19}$$

For (6.19), we will use (6.9) to formulate $\nabla^{(l)}\nabla_{W_1,W_2,\cdots,W_j}\phi^1_\lambda$ (for any smooth bounded vector fields $W_1,W_2,\cdots,W_j$) as some conditional expectation and use it for evaluations as in the proof of Theorem 5.1. For this, we need the bounds control on $\phi^j_\lambda(T,x,y)$ from **iv), v)**. Note that in showing **iv), v)**, we need a bound control of $\{|\nabla^{(l)}(\ln p^\lambda)^{(i)}_\lambda(T,x,\cdot)|\}_{1\leqslant l\leqslant j-i}$. So, to continue the inductive argument, we also need to verify **0)** at $i=j$, which can be obtained in showing **vi)** and **vii)**.

Theorem 1.3 iii) will follow from ii). Indeed, for $i=1$, (1.4) is true by Theorem 5.1 for the $k=3$ case. For $i\geqslant 2$, by (6.17),

$$\frac{(p^\lambda)^{(i)}(T,x,y)}{p^\lambda(T,x,y)} = \phi^i_\lambda(T,x,y) - (\rho^\lambda(y))^{-1}\sum_{j=0}^{i-1}\binom{i}{j}\frac{(p^\lambda)^{(j)}(T,x,y)}{p^\lambda(T,x,y)}(\rho^\lambda)^{(i-j)}(y).$$

So an inductive argument using (6.19) and (1.3) will conclude (1.4) for all $i\leqslant k-2$.

Finally, consider Theorem 1.3 iv). By symmetry, the mapping $x\mapsto (p^\lambda)^{(i)}(T,x,y)$ is continuous for all T,y, locally uniformly in y. We conclude using (5.97) and (1.4) as in the proof of Theorem 5.1 iii).

In summary, to carry out the above inductive argument for Theorem 1.3, all we need to do is to verify the properties **i)-vii)** at each step. We first consider **i)**, followed by **iv)** and then check **ii), iii), v), vi)** and **vii)**. The ideas to show these properties at each step are similar. So we only check them for the $j = 2$ case in details and indicate the necessary modifications to make them work for the general case.

6.2. **Proofs of the properties concerning ϕ^j_λ.** Let $\lambda \in (-1,1) \mapsto g^\lambda \in \mathcal{M}^k(M)$ be a C^k curve ($k \geqslant 4$). Assume all the properties **i)-vii)** in Section 1.3 hold true for $\widetilde{\phi}^i_\lambda, \phi^i_\lambda$ and $(p^\lambda)^{(i)}_\lambda$, $i < j \leqslant k-2$. We continue to verify the conclusions for $\widetilde{\phi}^j_\lambda, \phi^j_\lambda$ and $(p^\lambda)^{(j)}_\lambda$.

Proof of properties **i)** *and* **iv)** *in Section 6.1.* We first show **i)** and the estimation in **iv)**. We begin with the case $j = 2$. For **i)**, it suffices to consider the differentiability and L^q integrability of each term in (5.96). We add an upper-script λ to (I), (II), (III) and (IV) in (5.96) to indicate their dependence on λ.

For $(\mathrm{IV})^\lambda$, it is differentiable in λ by Lemma 4.18 and Theorem 5.1 for the $k = 3$ case. Denote the differential by $\big((\mathrm{IV})^\lambda\big)^{(1)}_\lambda$. Then

$$\begin{aligned}\big((\mathrm{IV})^\lambda\big)^{(1)}_\lambda = & -\big\langle D\pi(\lfloor \mathrm{u}_T \rceil^\lambda)^{(2)}_\lambda, \nabla^\lambda \ln p^\lambda(T,x,\lfloor \mathrm{x}_T \rceil^\lambda)\big\rangle_\lambda \\ & -\big\langle D\pi(\lfloor \mathrm{u}_T \rceil^\lambda)^{(1)}, \nabla^\lambda (\ln p^\lambda)^{(1)}_\lambda(T,x,\lfloor \mathrm{x}_T \rceil^\lambda)\big\rangle_\lambda \\ & -\big\langle D\pi(\lfloor \mathrm{u}_T \rceil^\lambda)^{(1)}, \nabla^\lambda_{(\lfloor \mathrm{x}_T \rceil^\lambda)^{(1)}_\lambda} \nabla^\lambda \ln p^\lambda(T,x,\lfloor \mathrm{x}_T \rceil^\lambda)\big\rangle_\lambda \\ =: & \ (\mathrm{IV})^\lambda_1 + (\mathrm{IV})^\lambda_2 + (\mathrm{IV})^\lambda_3.\end{aligned}$$

By an abuse of notation, we use $\underline{c}(q)$ to denote a constant depending on $T, T_0, q, \|g^\lambda\|_{C^4}$ and $\|\mathcal{X}\|_{C^3}$, which may vary from line to line. Using i) of Proposition 4.28 and Lemma 4.14, we obtain $\underline{c}(q)$ such that

$$\big(\overline{\mathbb{E}}\big|(\mathrm{IV})^\lambda_1\big|^q\big)^2 \leqslant \overline{\mathbb{E}}\big\|(\lfloor \mathrm{u}_T \rceil^\lambda)^{(2)}_\lambda\big\|^{2q} \cdot \overline{\mathbb{E}}\big\|\nabla^\lambda \ln p^\lambda(T,x,\lfloor \mathrm{x}_T \rceil^\lambda)\big\|^{2q} \leqslant \underline{c}(q).$$

Similarly, by i) of Proposition 4.28 and Lemma 4.14, we can derive that

$$\big(\overline{\mathbb{E}}\big|(\mathrm{IV})^\lambda_3\big|^q\big)^3 \leqslant \overline{\mathbb{E}}\big\|(\lfloor \mathrm{u}_T \rceil^\lambda)^{(1)}_\lambda\big\|^{4q} \cdot \overline{\mathbb{E}}\big\|\nabla^{\lambda,(2)} \ln p^\lambda(T,x,\lfloor \mathrm{x}_T \rceil^\lambda)\big\|^q \leqslant \underline{c}(q).$$

Using i) of Proposition 4.28 and (5.2), we obtain

$$\big(\overline{\mathbb{E}}\big|(\mathrm{IV})^\lambda_2\big|^q\big)^2 \leqslant \overline{\mathbb{E}}\big\|(\lfloor \mathrm{u}_T \rceil^\lambda)^{(1)}_\lambda\big\|^{2q} \cdot \overline{\mathbb{E}}\big\|\nabla^\lambda (\ln p^\lambda)^{(1)}_\lambda(T,x,\lfloor \mathrm{x}_T \rceil^\lambda)\big\|^{2q} \leqslant \underline{c}(q).$$

As for the conditional expectations, by ii) of Proposition 4.28 and Lemma 4.14, we obtain

$$\begin{aligned}\overline{\mathbb{E}}_{\mathbb{P}^{\lambda,*}_{x,y,T}}\Big|(\mathrm{IV})^\lambda_1\Big| + \overline{\mathbb{E}}_{\mathbb{P}^{\lambda,*}_{x,y,T}}\Big|(\mathrm{IV})^\lambda_3\Big| \leqslant & \ \overline{\mathbb{E}}_{\mathbb{P}^{\lambda,*}_{x,y,T}}\big\|(\lfloor \mathrm{u}_T \rceil^\lambda)^{(2)}_\lambda\big\| \cdot \big\|\nabla^\lambda \ln p^\lambda(T,x,y)\big\| \\ & + \overline{\mathbb{E}}_{\mathbb{P}^{\lambda,*}_{x,y,T}}\big\|(\lfloor \mathrm{u}_T \rceil^\lambda)^{(1)}_\lambda\big\| \cdot \big\|\nabla^{\lambda,(2)} \ln p^\lambda(T,x,y)\big\| \\ \leqslant & \ \underline{c}\left(\big(\frac{1}{T} d_{\tilde{g}^\lambda}(x,y) + \frac{1}{\sqrt{T}}\big)^2 + 1\right) e^{c(1+d_{\tilde{g}^\lambda}(x,y))}.\end{aligned}$$

By Corollary 5.36, for some different $\underline{c}, c$,

$$p^\lambda(T,x,y)\|\nabla^\lambda(\ln p^\lambda)^{(1)}_\lambda(T,x,y)\| \leqslant \underline{c}\left(\big(\frac{1}{T}d_{\tilde{g}^\lambda}(x,y)+\frac{1}{\sqrt{T}}\big)^2+1\right)e^{c(1+d_{\tilde{g}^\lambda}(x,y))}.$$

Using this and ii) of Proposition 4.28, we conclude that the same type of bound is valid for the term $p^\lambda(T,x,y)\overline{\mathbb{E}}_{\mathbb{P}^{\lambda,*}_{x,y,T}}\big|(\mathrm{IV})^\lambda_2\big|$.

By Lemma 4.18, $\lambda\mapsto(\mathrm{III})^\lambda$ is also differentiable in λ. Its differential is given by

$$\begin{aligned}\big((\mathrm{III})^\lambda\big)^{(1)}_\lambda &= \big\langle D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(2)}_\lambda,\ \frac{1}{2}\int_0^T \mathfrak{s}(T-\tau)\lfloor \mathrm{u}_T\rceil^\lambda(\lfloor \mathrm{u}_\tau\rceil^\lambda)^{-1}\mathrm{Ric}^{-1}_{\lfloor \mathrm{u}_\tau\rceil^\lambda}d\overleftarrow{B}_\tau\big\rangle_\lambda\\ &\quad+\big\langle D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda,\ \frac{1}{2}\int_0^T \mathfrak{s}(T-\tau)\big(\lfloor \mathrm{u}_T\rceil^\lambda(\lfloor \mathrm{u}_\tau\rceil^\lambda)^{-1}\mathrm{Ric}^{-1}_{\lfloor \mathrm{u}_\tau\rceil^\lambda}\big)^{(1)}_\lambda d\overleftarrow{B}_\tau\big\rangle_\lambda\\ &\quad+\big\langle D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda,\ \frac{1}{2}\int_0^T \mathfrak{s}(T-\tau)\lfloor \mathrm{u}_T\rceil^\lambda(\lfloor \mathrm{u}_\tau\rceil^\lambda)^{-1}\mathrm{Ric}^{-1}_{\lfloor \mathrm{u}_\tau\rceil^\lambda}d\overleftarrow{B}_\tau\big\rangle^{(1)}_\lambda\\ &=: (\mathrm{III})^\lambda_1+(\mathrm{III})^\lambda_2+(\mathrm{III})^\lambda_3,\end{aligned}$$

where the last term denotes the differential of the inner product. Then it is standard to estimate the expectation of $\big((\mathrm{III})^\lambda\big)^{(1)}_\lambda$ using Hölder's inequality, Burkholder's inequality and i) of Proposition 4.28, which gives

$$\Big(\overline{\mathbb{E}}\big|\big((\mathrm{III})^\lambda\big)^{(1)}_\lambda\big|^q\Big)^2\leqslant\underline{c}(T^q+T^{2q})\Big(\overline{\mathbb{E}}\|(\lfloor \mathrm{u}_T\rceil^\lambda)^{(2)}_\lambda\|^{2q}+\overline{\mathbb{E}}\|(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda\|^{2q}+\overline{\mathbb{E}}\|(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda\|^{4q}\Big)\leqslant\underline{c}(q).$$

For the corresponding conditional expectation estimation, we use (4.43), Hölder's inequality and Burkholder's inequality as before. It is easy to deduce that

$$\begin{aligned}\Big(\overline{\mathbb{E}}_{\mathbb{P}^\lambda_{x,y,T}}\big|\big((\mathrm{III})^\lambda\big)^{(1)}_\lambda\big|\Big)^2 &\leqslant \Big(\overline{\mathbb{E}}_{\mathbb{P}^\lambda_{x,y,T}}\|(\lfloor \mathrm{u}_T\rceil^\lambda)^{(2)}_\lambda\|^2+\overline{\mathbb{E}}_{\mathbb{P}^\lambda_{x,y,T}}\|(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda\|^2+\overline{\mathbb{E}}_{\mathbb{P}^\lambda_{x,y,T}}\|(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda\|^4\Big)\\ &\quad\cdot\underline{c}\left(T+\overline{\mathbb{E}}_{\mathbb{P}^\lambda_{x,y,T}}\Big|\int_0^T\big\|2\nabla^\lambda\ln p(T-\tau,\lfloor \mathrm{x}_\tau\rceil^\lambda,y)\big\|\,d\tau\Big|^2\right).\end{aligned}$$

So by ii) of Proposition 4.28 and Proposition 4.15, we have

$$\overline{\mathbb{E}}_{\mathbb{P}^{\lambda,*}_{x,y,T}}\big|\big((\mathrm{III})^\lambda\big)^{(1)}_\lambda\big|\leqslant\underline{c}e^{c(1+d_{\tilde{g}^\lambda}(x,y))}.$$

For $(\mathrm{II})^\lambda$, the same argument gives

$$\overline{\mathbb{E}}\big|\big((\mathrm{II})^\lambda\big)^{(1)}_\lambda\big|^q\leqslant\underline{c}(q),\ \overline{\mathbb{E}}_{\mathbb{P}^{\lambda,*}_{x,y,T}}\big|\big((\mathrm{II})^\lambda\big)^{(1)}_\lambda\big|\leqslant\underline{c}e^{c(1+d_{\tilde{g}^\lambda}(x,y))}.$$

For $(\mathrm{I})^\lambda$, we can check the differentiability of $\nabla^\lambda_{T,\mathrm{V},\mathfrak{s}}D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda$ term-by-term using its expression in Corollary 5.33. The estimation can be done as above using Proposition 4.28.

By the inductive construction, $\widetilde{\phi}^{j-1}_\lambda(T,x,w)$ involves the mixed differentials of order j in λ and in $\nabla^\lambda_{T,\mathfrak{s}}$ of $\lfloor u_T\rceil^\lambda$ and can be expressed by a multi-stochastic integral involving a mixture of differential processes $\{\big(\lfloor \mathrm{u}_t\rceil^\lambda\big)^{(j')}_\lambda\}_{j'\leqslant j-1}$, $\{\big[D^{(1)}\lfloor F_{\underline{t},\bar{t}}\rceil^\lambda(\lfloor \mathrm{u}_{\underline{t}}\rceil^\lambda,w)\big]^{(i)}_\lambda\}_{i\leqslant j-2}$ and

$\{\nabla^{\lambda,(l)}(\ln p^\lambda)^{(i)}_\lambda(T,x,\lfloor \mathrm{x}_T\rceil^\lambda)\}_{l+i\leqslant j-1,i\leqslant j-2}$. So, by Lemma 4.18 and Proposition 4.28, we have the differentiability of $\lambda\mapsto\widetilde{\phi}^{j-1}_\lambda(T,x,w)$ and the derivative $(\widetilde{\phi}^{j-1}_\lambda)^{(1)}_\lambda(T,x,w)$ involves $\{\big(\lfloor \mathrm{u}_t\rceil^\lambda\big)^{(j')}_\lambda\}_{j'\leqslant j}$, $\{\big[D^{(1)}\lfloor F_{\underline{t},\bar{t}}\rceil^\lambda(\lfloor \mathrm{u}_{\underline{t}}\rceil^\lambda,w)\big]^{(i)}_\lambda\}_{i\leqslant j-1}$ and $\{\nabla^{\lambda,(l)}(\ln p^\lambda)^{(i)}_\lambda(T,x,\lfloor \mathrm{x}_T\rceil^\lambda)\}_{l+i\leqslant j,i\leqslant j-1}$. The estimations in **i)** and **iv)** will follow from a repeated application of Proposition 4.28 to the multiple stochastic integral as in the $j=2$ case. The bound estimation in **iv)** contains $\big(T^{-1}d_{\widetilde{g}^\lambda}(x,y)+(\sqrt{T})^{-1}\big)^j$ since the formula of $(\widetilde{\phi}^{j-1}_\lambda)^{(1)}_\lambda(T,x,w)$ contains the terms $\nabla^{\lambda,(j)}\ln p^\lambda(T,x,\lfloor \mathrm{x}_T\rceil^\lambda)$, $\nabla^{\lambda,(j-1)}(\ln p^\lambda)^{(1)}_\lambda(T,x,\lfloor \mathrm{x}_T\rceil^\lambda)$.

As to the continuity and its uniformity in T and λ of the map

$$y\mapsto \Psi^j(y):=\overline{\mathbb{E}}\left((\widetilde{\phi}^{j-1}_\lambda)^{(1)}_\lambda(T,x,w)\Big|\lfloor \mathrm{x}_T\rceil^\lambda(w)=y\right),$$

we compare $\Psi^j(y)$ with $\Psi^j({}^rF(y))$, where ${}^rF$ is the flow map generated by a bounded smooth vector field W on $\widetilde{M}$. Let $\lfloor {}^r\mathbf{F}\rceil^\lambda$ be the flow as in Section 5 which extends ${}^rF$ to the $\widetilde{g}^\lambda$-Brownian paths. Then, as in the proof of Theorem 5.1, we obtain

$$\Psi^j({}^rF(y))=\overline{\mathbb{E}}\left((\widetilde{\phi}^{j-1}_\lambda)^{(1)}_\lambda\circ\lfloor {}^r\mathbf{F}\rceil^\lambda\cdot\frac{d\overline{\mathbb{P}}^\lambda_x\circ\lfloor {}^r\mathbf{F}\rceil^\lambda}{d\overline{\mathbb{P}}^\lambda_x}\middle|\lfloor \mathrm{x}_T\rceil^\lambda=y\right)\cdot\frac{p^\lambda(T,x,y)}{p^\lambda(T,x,{}^rF(y))}\cdot\frac{d\mathrm{Vol}^\lambda}{d\mathrm{Vol}^\lambda\circ{}^rF}(y).$$

In the proof of Lemma 5.34, we obtained the local uniform boundedness in (T,y) and λ of

$$\overline{\mathbb{E}}_{\overline{\mathbb{P}}^\lambda_{x,y,T}}\|d\overline{\mathbb{P}}^\lambda_x\circ\lfloor {}^r\mathbf{F}\rceil^\lambda/d\overline{\mathbb{P}}^\lambda_x\|^q$$

and the local uniformity in (T,y) and λ of the convergence of

$$\overline{\mathbb{E}}_{\overline{\mathbb{P}}^\lambda_{x,y,T}}\left\|d\overline{\mathbb{P}}^\lambda_x\circ\lfloor {}^r\mathbf{F}\rceil^\lambda/d\overline{\mathbb{P}}^\lambda_x-1\right\|^2\to 0,\ as\ r\to 0.$$

Following the estimation for **iv)**, we obtain the local uniform boundedness in (T,y) and λ of

$$\overline{\mathbb{E}}_{\overline{\mathbb{P}}^\lambda_{x,y,T}}\|(\widetilde{\phi}^{j-1}_\lambda)^{(1)}_\lambda(T,x,w)\circ\lfloor {}^r\mathbf{F}\rceil^\lambda\|^2.$$

So for **iv)**, it remains to show the local uniform convergence in (T,y) and λ of

$$\overline{\mathbb{E}}_{\overline{\mathbb{P}}^\lambda_{x,y,T}}\left\|(\widetilde{\phi}^{j-1}_\lambda)^{(1)}_\lambda(T,x,w)\circ\lfloor {}^r\mathbf{F}\rceil^\lambda-(\widetilde{\phi}^{j-1}_\lambda)^{(1)}_\lambda(T,x,w)\right\|^2\to 0,\ \text{as } r\to 0.$$

This, by using ii) of Proposition 4.28, can be reduced to showing the local uniformity in (T,y) and λ of the convergence of

$$\overline{\mathbb{E}}_{\overline{\mathbb{P}}^\lambda_{x,y,T}}\left\|\mathbf{A}\circ\lfloor {}^r\mathbf{F}\rceil^\lambda-\mathbf{A}\right\|^2\to 0,\ \text{as } r\to 0,$$

for elements $\lfloor \mathrm{u}_t\rceil^\lambda$, $\{\big(\lfloor \mathrm{u}_t\rceil^\lambda\big)^{(j')}_\lambda\}_{j'\leqslant j}$, $\{\big[D^{(1)}\lfloor F_{\underline{t},\bar{t}}\rceil^\lambda(\lfloor \mathrm{u}_{\underline{t}}\rceil^\lambda,w)\big]^{(i)}_\lambda\}_{i\leqslant j-1}$ that appear in the expression of $(\widetilde{\phi}^{j-1}_\lambda)^{(1)}_\lambda$, which is true since they can be further reduced to the $\mathbf{A}$ appearing in Lemma 5.34 by the construction of $\lfloor {}^r\mathbf{F}\rceil^\lambda$. □

Proof of properties **ii)**, **iii)** *and* **v)** *in Section 6.1.* Using (4.50) and the inductive assumption on the boundedness of $\overline{\mathbb{E}}_{\overline{\mathbb{P}}^\lambda_{x,y,T}} \|\widetilde{\phi}^{j-1}_\lambda(T,x,w)\|^q$, we deduce that $\overline{\Phi}^j_\lambda : Y \mapsto \overline{\Phi}^j_\lambda(Y)$, where

$$\overline{\Phi}^j_\lambda(Y)(y) := \overline{\mathbb{E}}\left(\left\langle Y(\lfloor \mathrm{x}_T \rceil^\lambda(w)), \widetilde{\phi}^{j-1}_\lambda(T,x,w)\cdot D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda(w)\right\rangle_\lambda \Big| \lfloor \mathrm{x}_T\rceil^\lambda(w) = y\right)$$

is a locally bounded functional on C^k bounded vector fields Y on $\widetilde{M}$.

To show $\overline{\Phi}^j_\lambda(Y)$ is C^1, we follow the argument in the proof of Lemma 5.34. Let F^s be generated by a smooth bounded vector field V on $\widetilde{M}$ Let $\lfloor \mathbf{F}^s\rceil^\lambda$ be constructed as in Section 5.5, which extends F^s to Brownian paths starting from x up to time T using the auxiliary function $\mathtt{s}$. Then the change of variable comparison in Section 5.1 gives

$$\overline{\Phi}^j_\lambda(Y)(F^s y) = \overline{\mathbb{E}}_{\overline{\mathbb{P}}^\lambda_{x,y,T}}\left(\Phi^j_\lambda(Y,w)\circ\lfloor\mathbf{F}^s\rceil^\lambda \cdot \frac{d\overline{\mathbb{P}}^\lambda_x \circ \lfloor \mathbf{F}^s\rceil^\lambda}{d\overline{\mathbb{P}}^\lambda_x}\right)\frac{p^\lambda(T,x,y)}{p^\lambda(t,x,F^s y)}\frac{d\mathrm{Vol}^\lambda}{d\mathrm{Vol}^\lambda\circ F^s}(y),$$

where

$$\Phi^j_\lambda(Y,w) = \left\langle Y(\lfloor \mathrm{x}_T\rceil^\lambda(w)), \widetilde{\phi}^{j-1}_\lambda(T,x,w)\cdot D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda(w)\right\rangle_\lambda.$$

The process $\Phi^j_\lambda(Y,w)\circ\lfloor\mathbf{F}^s\rceil^\lambda$ is differentiable in s with

$$\begin{aligned}(\Phi^j_\lambda\circ\lfloor\mathbf{F}^s\rceil^\lambda)'_s = &\left\langle \nabla_{\mathrm{V}(\lfloor \mathrm{x}^s_T\rceil^\lambda)} Y(\lfloor \mathrm{x}^s_T\rceil^\lambda), \widetilde{\phi}^{j-1}_\lambda\circ\lfloor\mathbf{F}^s\rceil^\lambda D\pi(\lfloor \mathrm{u}^s_T\rceil^\lambda)^{(1)}_\lambda\right\rangle_\lambda\\ &+\left\langle Y(\lfloor \mathrm{x}^s_T\rceil^\lambda), \left(\nabla^{s,\lambda}_{T,\mathrm{V},\mathtt{s}}\widetilde{\phi}^{j-1}_\lambda\right)\circ\lfloor\mathbf{F}^s\rceil^\lambda D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda\right\rangle_\lambda\\ &+\left\langle Y(\lfloor \mathrm{x}^s_T\rceil^\lambda), \widetilde{\phi}^{j-1}_\lambda\circ\lfloor\mathbf{F}^s\rceil^\lambda \nabla^{s,\lambda}_{T,\mathrm{V},\mathtt{s}} D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda\right\rangle_\lambda\end{aligned}$$

and this differential is L^q integrable conditioned on $\mathrm{x}_T = y$, uniformly in s, for all $q \geqslant 1$. Using this and Proposition 5.20, we can conclude that $\overline{\Phi}^j_\lambda(Y)(F^s y)$ is differentiable in s. Following the proof of Lemma 5.34 (see (5.89)), we obtain

$$\begin{aligned}(\overline{\Phi}^j_\lambda(Y)(F^s y))'_0 = &\overline{\mathbb{E}}_{\overline{\mathbb{P}}^\lambda_{x,y,T}}\Big(\left\langle \nabla_{\mathrm{V}(\lfloor \mathrm{x}_T\rceil^\lambda)} Y, \widetilde{\phi}^{j-1}_\lambda\cdot D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda(w)\right\rangle_\lambda\\ &\qquad+\left\langle Y(\lfloor \mathrm{x}_T\rceil^\lambda), \nabla^\lambda_{T,\mathrm{V},\mathtt{s}}\left(\widetilde{\phi}^{j-1}_\lambda\cdot D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda(w)\right)\right\rangle_\lambda\\ &\qquad+\left\langle Y(\lfloor \mathrm{x}_T\rceil^\lambda), \widetilde{\phi}^{j-1}_\lambda\cdot D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda(w)\right\rangle_\lambda \overleftarrow{\mathcal{E}}_{T,\mathrm{V},\mathtt{s}}\Big)\\ =: &\,\overline{\mathbb{E}}_{\overline{\mathbb{P}}^\lambda_{x,y,T}}\left(\Psi^j_\lambda(Y,\mathrm{V})(w)\right).\end{aligned} \tag{6.20}$$

To show $y \mapsto (\overline{\Phi}^j_\lambda(Y)(F^s y))'_0$ is continuous, we compare (6.20) with its value at nearby points. Choose another smooth bounded vector field W on $\widetilde{M}$ and let ${}^rF$ be the flow it generates and let $\lfloor {}^r\mathbf{F}\rceil^\lambda$ be its extension to $\widetilde{g}^\lambda$-Brownian paths starting from x up to time T. A change of variable argument in Section 5.1 for $\lfloor {}^r\mathbf{F}\rceil^\lambda$ shows that for $z = {}^rF(y)$,

$$(\overline{\Phi}^j_\lambda(Y)(F^s z))'_0 = \overline{\mathbb{E}}_{\overline{\mathbb{P}}^\lambda_{x,y,T}}\left(\Psi^j_\lambda(Y,\mathrm{V})\circ\lfloor{}^r\mathbf{F}\rceil^\lambda\cdot\frac{d\overline{\mathbb{P}}^\lambda_x\circ\lfloor{}^r\mathbf{F}\rceil^\lambda}{d\overline{\mathbb{P}}^\lambda_x}\right)\frac{p^\lambda(T,x,y)}{p^\lambda(T,x,z)}\frac{d\mathrm{Vol}^\lambda}{d\mathrm{Vol}^\lambda\circ{}^rF}(y).$$

We can show the local uniform convergence (in (y,T) and λ) the convergence of) of $(\overline{\Phi}^j_\lambda(Y)(F^s z))'_0$ to $(\overline{\Phi}^j_\lambda(Y)(F^s y))'_0$ as $r \to 0$ exactly as we did in the previous proofs of properties **i)** and **iv)**.

As to the estimations in **v)**, $\left|z^{\lambda,j}_T(y)\right|$ can be estimated using the conditional L^q expectations of $\widetilde{\phi}^{j-1}_\lambda(T,x,w)$, $(\lfloor \mathrm{u}_T \rceil^\lambda)^{(1)}_\lambda(w)$, respectively. By (6.20), we have the formula in **iii)**. By Claim 6.3, we obtain the formula of $\nabla^\lambda_{\mathrm{V}} \mathrm{z}^{\lambda,j}_t(y)$ in (6.14). We can use them and Proposition 4.28 to give the desired estimation of $\mathrm{Div}^\lambda \mathrm{z}^{\lambda,j}_t(y)$. □

Proof of properties **vi)** *and* **vii)** *in Section 6.1.* The $j=1$ case was considered in Theorem 5.1. When $j=2$, since we have (6.1) for $i=1,2$, so, for all $f \in C^\infty_c(\widetilde{M})$,

$$\int_{\widetilde{M}} f(y)\phi^2_\lambda(T,x,y)p^\lambda(T,x,y)\ d\mathrm{Vol}^\lambda(y) = \left(\int_{\widetilde{M}} f(y)\phi^1_\lambda(T,x,y)p^\lambda(T,x,y)\ d\mathrm{Vol}^\lambda(y)\right)^{(1)}_\lambda.$$

This implies

$$\begin{aligned}\phi^2_\lambda(T,x,y) &= (\phi^1_\lambda)^{(1)}_\lambda(T,x,y) + \phi^1_\lambda(T,x,y)\cdot\Big(\ln(p^\lambda(T,x,y)\rho^\lambda(y))\Big)^{(1)}_\lambda \\ &= (\phi^1_\lambda)^{(1)}_\lambda(T,x,y) + \big(\phi^1_\lambda(T,x,y)\big)^2.\end{aligned}$$

Hence,

$$(\ln p^\lambda)^{(2)}_\lambda(T,x,y) = \phi^2_\lambda(T,x,y) - (\phi^1_\lambda)^2(T,x,y) - (\ln\rho^\lambda)^{(2)}_\lambda(y)$$

and

$$\nabla(\ln p^\lambda)^{(2)}_\lambda(T,x,y) = \nabla\phi^2_\lambda(T,x,y) - 2\phi^1_\lambda(T,x,y)\nabla\phi^1_\lambda(T,x,y) - \nabla(\ln\rho^\lambda)^{(2)}_\lambda(y).$$

This, together with **vi)**, **vii)** in $j=1$ case, shows the estimation for $|\nabla^{(l)}(\ln p^\lambda)^{(2)}_\lambda(T,x,y)|$ in (6.10) holds true if the same type of estimation is valid for $|\nabla^{(l)}\phi^2_\lambda(T,x,y)|$. By **i)-v)**, Claims 6.2-6.3 apply. We have

$$\phi^2_\lambda(T,x,y) = \overline{\mathbb{E}}\left(\widetilde{\phi}^2_\lambda(T,x,w)\Big|\lfloor \mathrm{x}_T\rceil^\lambda(w) = y\right), \tag{6.21}$$

where

$$\widetilde{\phi}^2_\lambda(T,x,w) = \big(\widetilde{\phi}^1_\lambda\big)^{(1)}_\lambda(T,x,w) - \big\langle \nabla^\lambda_{T,\mathbf{s}}\widetilde{\phi}^1_\lambda(T,x,w), D\pi(\lfloor \mathrm{u}_T\rceil^\lambda)^{(1)}_\lambda(w)\big\rangle + \widetilde{\phi}^1_\lambda(T,x,w)\widetilde{\phi}^1_\lambda(T,x,w).$$

We can use (6.21) to derive the conditional expectation expressions of $\nabla^{(l)}\phi^2_\lambda(T,x,y)$ as in the proof of Theorem 5.1. Using this and Proposition 4.28, we can derive the desired estimations of $\nabla^{(l)}\phi^2_\lambda(T,x,y)$ and its L^q-norm.

For $j \geqslant 3$, with **i)-v)**, we have the identity

$$\begin{aligned}&\nabla^{(l)}(\ln p^\lambda)^{(j)}_\lambda(T,x,y) \\ &\quad= \nabla^{(l)}\phi^j_\lambda(T,x,y) - \sum_{i=1}^{j-1}\nabla^{(l)}\big(\phi^i_\lambda\cdot\phi^1_\lambda\big)^{(j-i-1)}_\lambda(T,x,y) - \nabla^{(l)}\big(\ln\rho^\lambda\big)^{(j)}_\lambda(y).\end{aligned} \tag{6.22}$$

By Theorem 5.1 i),

$$\phi^1_\lambda(T,x,y)=\Big(\ln(p^\lambda(T,x,y)\rho^\lambda(T,x,y))\Big)^{(1)}_\lambda=(\ln p^\lambda)^{(1)}_\lambda(T,x,y)+(\ln\rho^\lambda)^{(1)}_\lambda(y).$$

By (6.1), for all i, $i\leqslant j$, and all $f\in C^\infty_c(\widetilde{M})$,

$$\int_{\widetilde{M}}f(y)\phi^i_\lambda(T,x,y)p^\lambda(T,x,y)\ d\mathrm{Vol}^\lambda(y)=\bigg(\int_{\widetilde{M}}f(y)\phi^{i-1}_\lambda(T,x,y)p^\lambda(T,x,y)\ d\mathrm{Vol}^\lambda(y)\bigg)^{(1)}_\lambda.$$

Since f is arbitrary, we must have

$$\begin{aligned}\phi^i_\lambda(T,x,y)&=(\phi^{i-1}_\lambda)^{(1)}_\lambda(T,x,y)+\phi^{i-1}_\lambda(T,x,y)\cdot\Big(\ln(p^\lambda(T,x,y)\rho^\lambda(T,x,y))\Big)^{(1)}_\lambda\\&=(\phi^{i-1}_\lambda)^{(1)}_\lambda(T,x,y)+\phi^{i-1}_\lambda(T,x,y)\cdot\phi^1_\lambda(T,x,y).\end{aligned}$$

Using this relationship inductively, we obtain

$$\phi^j_\lambda(T,x,y)=(\phi^1_\lambda(T,x,y))^{(j-1)}_\lambda+\sum_{i=1}^{j-1}\big(\phi^i_\lambda\cdot\phi^1_\lambda\big)^{(j-i-1)}_\lambda(T,x,y),$$

which implies

$$(\ln p^\lambda)^{(j)}_\lambda(T,x,y)=\phi^j_\lambda(T,x,y)-\sum_{i=1}^{j-1}\big(\phi^i_\lambda\cdot\phi^1_\lambda\big)^{(j-i-1)}_\lambda(T,x,y)-\big(\ln\rho^\lambda\big)^{(j)}_\lambda(y).$$

A differentiation of this equation gives (6.22). By induction, we see that the differentials $(\phi^i_\lambda)^{(r)}_\lambda(T,x,y)$ for $r\leqslant j-i-1$ only consist of $(\ln p^\lambda)^{(s)}_\lambda(T,x,y)$, $\big(\ln\rho^\lambda\big)^{(s)}_\lambda(y)$ up to order $s=i+r\leqslant j-1$. By the inductive assumption on the gradient estimations of $(\ln p^\lambda)^{(s)}_\lambda(T,x,y)$, $s\leqslant j-1$, to obtain **vii)** for $\big|\nabla^{(l)}(\ln p^\lambda)^{(j)}_\lambda(T,x,y)\big|$, it suffices to give the estimation for $|\nabla^{(l)}\phi^j_\lambda(T,x,y)|$. By **i)-v)**, Claims 6.2-6.3 apply and we have

$$\phi^j_\lambda(T,x,y):=\overline{\mathbb{E}}\left(\widetilde{\phi}^j_\lambda(T,x,w)\Big|\lfloor\mathrm{x}_T\rceil^\lambda(w)=y\right),$$

where $\widetilde{\phi}^j_\lambda(T,x,w)$ is defined inductively using (6.8). As in the proof of Theorem 5.1, we can further obtain $\widetilde{\phi}^{j,(l)}_\lambda(T,x,w)$ such that

$$\nabla^{(l)}\phi^j_\lambda(T,x,y)=\overline{\mathbb{E}}\left(\widetilde{\phi}^{j,(l)}_\lambda(T,x,w)\Big|\lfloor\mathrm{x}_T\rceil^\lambda(w)=y\right).$$

The term $\widetilde{\phi}^{j,(l)}_\lambda(T,x,w)$ involves the derivatives of $\widetilde{\phi}^j_\lambda(T,x,w)$ under flow ${}^r\mathbf{F}$ up to the j-th order and can be formulated as a stochastic integral using $\mathtt{s},\mathtt{s}'$, $\lfloor\mathrm{u}\rceil^\lambda$, $\{\big(\lfloor\mathrm{u}_t\rceil^\lambda\big)^{(j')}_\lambda\}_{j'\leqslant j}$, $\{\big[D^{(1)}\lfloor F_{\underline{t},\bar{t}}\rceil^\lambda(\lfloor\mathrm{u}_{\underline{t}}\rceil^\lambda,w)\big]^{(i)}_\lambda\}_{i\leqslant j-1}$ and $\{\nabla^{\lambda,(l)}(\ln p^\lambda)^{(i)}_\lambda(T,x,\lfloor\mathrm{x}_T\rceil^\lambda)\}_{l+i\leqslant j,i\leqslant j-1}$. So we can use this and ii) of Proposition 4.28 to derive the desired bound of $|\nabla^{(l)}\phi^j_\lambda(T,x,y)|$ as in Theorem 5.1. As to **vi)**, it is equivalent to estimate $\overline{\mathbb{E}}(\|\widetilde{\phi}^{j,(l)}_\lambda(T,x,w)\|^q)$, which can be handled using i) of Proposition 4.28 and the inductive assumption on (1.3) for $i<j$. □

7. Regularity of the entropy

The analog of formula (1.1) for the entropy involves the Martin kernel of the Brownian motion on $(\widetilde{M}, \tilde{g}^\lambda)$ for $g^\lambda \in \Re^k(M)$. Recall that the Green function on $(\widetilde{M}, \tilde{g}^\lambda)$ is given by

$$\mathbf{G}^\lambda(x,y) := \int_0^\infty p^\lambda(t,x,y)\, dt, \text{ for } x,y \in \widetilde{M},$$

and it can be associated with a "Green metric" on $\widetilde{M}$ ([**LS1**]) by letting

$$d_{\mathbf{G}^\lambda}(x,y) := \begin{cases} -\ln(\mathtt{c}_0 \mathbf{G}^\lambda(x,y)), & \text{if } d_{\tilde{g}^\lambda}(x,y) > 1, \\ -\ln \mathtt{c}_0, & \text{otherwise,} \end{cases}$$

where $\mathtt{c}_0$ can be chosen to be independent of λ for g^λ in a small neighborhood of g^0. By Anderson and Schoen [**AS**] (see also [**Anc**]), the Martin kernel for $(\widetilde{M}, \tilde{g}^\lambda)$ is defined by

$$k^\lambda(x,y,\xi) := \lim_{z\to\xi} k^\lambda(x,y,z), \text{ where } k^\lambda(x,y,z) := \frac{\mathbf{G}^\lambda(y,z)}{\mathbf{G}^\lambda(x,z)}. \tag{7.1}$$

Hence, the logarithm of the Martin kernel is an analog of the Busemann function using the Green metric since

$$\ln k^\lambda(x,y,\xi) := \lim_{z\to\xi} \left(d_{\mathbf{G}^\lambda}(x,z) - d_{\mathbf{G}^\lambda}(y,z)\right), \text{ for } x,y \in \widetilde{M},\ \xi \in \partial\widetilde{M}.$$

Moreover, it is known that the entropy satisfies the following formula ([**K1**])

$$h^\lambda = -\int \Delta_y^\lambda \ln k^\lambda(x,y,\xi)\Big|_{y=x}\, d\mathbf{m}^\lambda(x,\xi).$$

For $x, y \in \widetilde{M}$ fixed, the function $k^\lambda(x,y,\xi)$ is a continuous version of the Radon-Nikodyn derivative $(d\tilde{\mathbf{m}}_y/d\tilde{\mathbf{m}}_x)(\xi)$; the gradient $\nabla_y^\lambda k^\lambda(x,y,\xi)|_{y=x}$ is a G-equivariant stable vector field that depends Hölder continuously on ξ with the Hölder exponent uniformly in λ for g^λ in a small neighborhood of g^0 ([**AS**], see also [**Ha**]). Furthermore, we have the following.

Lemma 7.1. *Let M be a closed connected smooth manifold. For each $g \in \Re^k(M)$ ($k \geqslant 3$), there exist a neighborhood $\mathcal{V}_g''$ of g in $\Re^k(M)$ and $\mathtt{b}''$, $\mathtt{b}'' > 0$, such that for any C^k curve $\lambda \in (-1,1) \mapsto g^\lambda \in \mathcal{V}_g''$ with $g^0 = g$, the second order differentials of $k^\lambda(x,y,\xi)$ in y at $y = x$ are Hölder continuous in ξ with exponent $\mathtt{b}''$; for $\mathtt{b} < \mathtt{b}''\varkappa$, where $\varkappa$ is as in (3.1), we have*

$$\Delta_y^\lambda \ln k^0(x,y,\xi)\big|_{y=x},\ (\Delta_y^\lambda)'_\lambda \ln k^0(x,y,\xi)\big|_{y=x} \in \mathcal{H}^0_{\mathtt{b}}. \tag{7.2}$$

Proof. The second part follows from [**AS**, Theorem 6.2] and the first part. We show the first part by following the proof of [**Ha**, Lemma 3.2].

Let $x \in \widetilde{M}$ and let $B(x,\delta)$ be a small neighborhood around x with a positive radius δ. For $\mathbf{v} = (x',\zeta) \in S\widetilde{M}_{\tilde{g}}$ with $x' \in B(x,2\delta)\backslash B(x,\delta)$ and $\rho, 0 < \rho \leqslant \pi/2$, let

$$\mathrm{C}^\lambda(\mathbf{v},\rho) := \left\{ z \in \widetilde{M} :\ \angle_{x'}^{\tilde{g}^\lambda}(\mathbf{v}, \dot\gamma_{x',z}^{\tilde{g}^\lambda}(0)) < \rho \right\}, \quad \mathrm{C}(\mathbf{v},\rho) := \mathrm{C}^0(\mathbf{v},\rho)$$

be the open cone of vertex x', axis $\mathbf{v}$ and angle ρ, where $\angle^{\widetilde{g}^\lambda}_{x'}(\cdot,\cdot)$ is the $\widetilde{g}^\lambda$ angle function in $S_{x'}\widetilde{M}$ and $\dot\gamma^{\widetilde{g}^\lambda}_{x',z}(0)$ is the initial tangent vector of the $\widetilde{g}^\lambda$ unit speed geodesic $\gamma^{\widetilde{g}^\lambda}_{x',z}$ from x' to z.

There exists a neighborhood $\mathcal{V}_g$ of g in $\mathfrak{R}^k(M)$ such that if $g^\lambda \in \mathcal{V}_g$, then for all $\mathbf{v}, \mathrm{C}(\mathbf{v},\pi/6) \subset \mathrm{C}^\lambda(\mathbf{v},\pi/4) \subset \mathrm{C}(\mathbf{v},\pi/3)$ and for all $x \in \widetilde{M}, B(x,\delta/4) \subset B_{g^\lambda}(x,\delta/2) \subset B(x,\delta)$.[3] Let $\{x_{s,t}\}_{|s|,|t|<1}$, with $x_{0,0} = x$, be any C^2 two parameter family of points in $B(x,\delta/4)$. For $\mathrm{C}(\mathbf{v},\pi/2)$ apart from $B(x,\delta)$ and $z \in \mathrm{C}(\mathbf{v},\pi/2)$, let

$$\varphi_{s,t}(z) := \frac{1}{st}\Big(k^\lambda(x,x_{s,t},z) - k^\lambda(x,x_{0,t},z) - k^\lambda(x,x_{s,0},z) + k^\lambda(x,x_{0,0},z)\Big).$$

To conclude the first part of Lemma 7.1, it suffices to show for $\mathcal{V}_g$ small, there is some C, $C>0$, independent of s,t,x,z and g^λ such that

$$|\varphi_{s,t}(z)| \leqslant C. \tag{7.3}$$

This is because (7.3) implies that, for $z \in \mathrm{C}^\lambda(\mathbf{v},\pi/4)$,

$$\varphi_{s,t}(z) + C = \frac{\frac{1}{st}\big(\mathbf{G}^\lambda(x_{s,t},z) - \mathbf{G}^\lambda(x_{0,t},z) - \mathbf{G}^\lambda(x_{s,0},z) + \mathbf{G}^\lambda(x_{0,0},z)\big) + C\mathbf{G}^\lambda(x,z)}{\mathbf{G}^\lambda(x,z)}$$

is the quotient of two positive harmonic functions in $\mathrm{C}^\lambda(\mathbf{v},\pi/4)$ which vanish at the infinity boundary $\partial\widetilde{M}$. Hence, by using [**AS**, Theorem 6.2], for $\mathcal{V}_g$ small, we obtain two positive numbers $C', \mathtt{b}''$, independent of s,t,x and g^λ, such that

$$\big|(\varphi_{s,t}(z)+C) - (\varphi_{s,t}(z')+C)\big| \leqslant C' e^{-\mathtt{b}''\varkappa(z|z')^\lambda_x},\ \forall z,z' \in \mathrm{C}^\lambda(\mathbf{v},\pi/4). \tag{7.4}$$

Let $\xi,\eta \in \partial\widetilde{M}$ be points lying in the closure of $\mathrm{C}(\mathbf{v},\pi/6)$. Letting $z\to\xi, z'\to\eta$ and then letting $s,t\to 0$ in (7.4), the first part statement of Lemma 7.1 follows by using (7.1).

It remains to show (7.3), or, equivalently,

$$\left|\frac{1}{st}\Big(\mathbf{G}^\lambda(x_{s,t},z) - \mathbf{G}^\lambda(x_{0,t},z) - \mathbf{G}^\lambda(x_{s,0},z) + \mathbf{G}^\lambda(x_{0,0},z)\Big)\right| \leqslant C\mathbf{G}^\lambda(x,z). \tag{7.5}$$

Since $\mathbf{G}^\lambda(\cdot,z)$ is harmonic in $B_{g^\lambda}(x,\delta/2)$, by the Harnack inequality ([**AS**]) and the infinitesimal Harnack inequality of Cheng-Yau ([**CY**]), for $\mathcal{V}_g$ small, there is some C'', $C''>0$, independent of s,t,x,z and g^λ such that

$$\mathbf{G}^\lambda(y,z),\ \big\|\nabla^\lambda_y \mathbf{G}^\lambda(y,z)\big\|_{\widetilde{g}^\lambda} \leqslant C''\mathbf{G}^\lambda(x,z),\ \forall y \in B_{g^\lambda}(x,\delta/2),\ z \in \mathrm{C}^\lambda(\mathbf{v},\pi/4).$$

[3]There is a neighborhood $\mathcal{V}_g$ of g in $\mathfrak{R}^3(M)$ and a number r such that for $\widetilde{g},\widetilde{g}' \in \mathcal{V}_g, \tau \geqslant r$,

$$\angle^{\widetilde{g}'}_{x'}(\dot\gamma^{\widetilde{g}'}_{x',\gamma^{\widetilde{g}'}_{x,\xi(\tau)}}, \dot\gamma^{\widetilde{g}}_{x',\gamma^{\widetilde{g}}_{x,\xi(\tau)}}) < \angle^{\widetilde{g}'}_{x'}(\dot\gamma^{\widetilde{g}'}_{x',\gamma^{\widetilde{g}'}_{x,\xi(r)}}, \dot\gamma^{\widetilde{g}}_{x',\gamma^{\widetilde{g}}_{x,\xi(r)}}) + \pi/100.$$

It suffices then to control the angles on $B(x,r+2\delta)$.

To continue, we consider $L_W\big|_y \mathbf{G}^\lambda(y,z)$, where W is any smooth bounded vector field on $\widetilde{M}$, $L_W\big|_y$ is the Lie derivative in W evaluated at y. Then, in the distribution sense,

$$\Delta^\lambda L_W\big|_y \mathbf{G}^\lambda(y,z) = L_W\big|_y \Delta^\lambda \mathbf{G}^\lambda(y,z) + \big[L_W, \Delta^\lambda\big]\mathbf{G}^\lambda(y,z) = \big[L_W, \Delta^\lambda\big]\mathbf{G}^\lambda(y,z),$$

where the last commutator term is a linear combination of the contractions $R^\lambda * \nabla^\lambda \mathbf{G}^\lambda(\cdot,z)$, $\nabla^\lambda R^\lambda * \mathbf{G}^\lambda(\cdot,z)$ evaluated at $W \in T_y\widetilde{M}$. Since $L_W\big|_y \mathbf{G}^\lambda(y,z)$ is C^1 in y, it must be a real solution function f to the equation

$$\Delta^\lambda f(y) = \big[L_W, \Delta^\lambda\big]\mathbf{G}^\lambda(y,z).$$

Hence the classical estimation property of elliptic equation (cf. [**Fr**]) shows that there is some positive C''' depending on the geometry, which can be chosen to be independent of x, z, g^λ for $\mathcal{V}_g$ small, such that

$$\begin{aligned}\big\|\nabla^\lambda_y L_W\big|_y \mathbf{G}^\lambda(y,z)\big\|_{\tilde{g}^\lambda} &\leqslant C'''\big(\sup_{y\in B(x,\delta)} \big\|\nabla^\lambda_y \mathbf{G}^\lambda(y,z)\big\|_{\tilde{g}^\lambda} + \sup_{y\in B(x,\delta)} \mathbf{G}^\lambda(y,z)\big)\\ &\leqslant 2C'''C''\mathbf{G}^\lambda(x,z).\end{aligned}$$

This shows (7.5) since W can be arbitrary. □

Proof of Theorem 1.6. Let $\mathcal{V}''_g$, $\mathfrak{b}''$ be as in Lemma 7.1. Let $\mathfrak{b} < \mathfrak{b}''\varkappa$, $\mathcal{V}_g$ and $\mathcal{H}^0_{\mathfrak{b}}$ be such that Theorem 1.2 holds true. Let $\lambda \in (-1,1) \mapsto g^\lambda \in \mathcal{V}''_g \cap \mathcal{V}_g$ with $g^0 = g$ be a C^3 curve. We omit the index 0 for h^0, k^0, p^0, $\overline{Z}^0$, Δ^0, Div^0, ∇^0, $\langle\cdot,\cdot\rangle_0$, Vol^0 and $\mathbf{m}^0$ at g^0.

We study the differentiability of h^λ by writing, as in [**LS1**],

$$\frac{1}{\lambda}(h^\lambda - h) = \frac{1}{\lambda}(h^\lambda - h^{\lambda,0}) + \frac{1}{\lambda}(h^{\lambda,0} - h) =: (\mathrm{I})_\lambda + (\mathrm{II})_\lambda,$$

where

$$h^{\lambda,0} = -\int \Delta^\lambda_y \ln k(x,y,\xi)\Big|_{y=x}\, d\mathbf{m}^\lambda(x,\xi).$$

Then, by (7.2) and Theorem 1.2,

$$\lim_{\lambda\to 0} (\mathrm{II})_\lambda = -\int (\Delta^\lambda_y)'_0 \ln k(x,y,\xi)\big|_{y=x}\, d\mathbf{m}(x,\xi) - \int \Delta_y \ln k(x,y,\xi)\big|_{y=x}\, d(\mathbf{m}^\lambda)'_0(x,\xi).$$

Using u_1 for the function such that

$$\Delta u_1 = -\Delta_y \ln k(x,y,\xi)\big|_{y=x} - h, \ \text{(see [\textbf{LS1}, (5.7)])},$$

we obtain, as in Section 3.3,

$$\mathcal{K} := \lim_{\lambda\to 0} (\mathrm{II})_\lambda = \int \left(-\frac{1}{2}\langle \nabla \mathrm{trace}\mathcal{X}, \overline{Z} + \nabla u_1\rangle + \mathrm{Div}\big(\mathcal{X}(\overline{Z} + \nabla u_1)\big)\right)\, d\mathbf{m}.$$

Clearly, $\mathcal{K}$ is linear on $\mathcal{X} \in C^k(S^2T^*)$. When g is locally symmetric, $u_1 \equiv 0$ and $\overline{Z} = \ell\overline{X}$. Hence,

$$\mathcal{K} = \ell \int \left(-\frac{1}{2}\langle \nabla \mathrm{trace}\mathcal{X}, \overline{X}\rangle - \ell\mathcal{X}(\overline{X}, \overline{X}) \right) \, d\mathbf{m},$$

which vanishes (see Remark 3.12).

We will now show $\lim_{\lambda\to 0} (\mathrm{I})_\lambda = 0$, which will complete the proof of Theorem 1.6. Following [**LS1**, Proposition 2.4], we obtain $h^{\lambda,0} = \inf_{s>0}\{\overline{h}^{\lambda,0}(s)\}$, where

$$\overline{h}^{\lambda,0}(s) := \lim_{t\to\infty} -\frac{1}{t} \int (\ln p(st, x, y)) p^\lambda(t, x, y) \, d\mathrm{Vol}^\lambda(y).$$

Then, for all $\lambda > 0$,

$$\begin{aligned}
(\mathrm{I})_\lambda &= \frac{1}{\lambda} \sup_{s>0} \lim_{t\to\infty} -\frac{1}{t} \int \ln \frac{p^\lambda(t, x, y)}{p(st, x, y)} p^\lambda(t, x, y) \, d\mathrm{Vol}^\lambda(y) \\
&= \frac{1}{\lambda} \sup_{s>0} \lim_{t\to\infty} -\frac{1}{t} \int \ln \frac{p^\lambda(t, x, y)\rho^\lambda(y)}{p(st, x, y)} p^\lambda(t, x, y) \, d\mathrm{Vol}^\lambda(y) \\
&\leqslant \frac{1}{\lambda} \sup_{s>0} \lim_{t\to\infty} \frac{1}{t} \left(1 - \int p(st, x, y) \, d\mathrm{Vol}(y) \right) \leqslant 0,
\end{aligned}$$

where the third inequality holds since $-\ln a \leqslant a^{-1} - 1$ for all $a > 0$. On the other hand,

$$\begin{aligned}
(\mathrm{I})_\lambda &\geqslant \frac{1}{\lambda} \left(h^\lambda - \overline{h}^{\lambda,0}(1) \right) \\
&= \frac{1}{\lambda} \lim_{t\to\infty} -\frac{1}{t} \int \ln \frac{p^\lambda(t, x, y)}{p(t, x, y)} p^\lambda(t, x, y) \, d\mathrm{Vol}^\lambda(y) \\
&= \frac{1}{\lambda} \lim_{n\in\mathbb{N}, n\to\infty} \frac{1}{n} \int \ln \frac{p(n, x, y)}{p^\lambda(n, x, y)\rho^\lambda(y)} p^\lambda(n, x, y) \, d\mathrm{Vol}^\lambda(y).
\end{aligned}$$

To estimate $(\mathrm{I})_\lambda$, consider the stationary Markov chain on the space $\underline{\Omega} = \widetilde{M}^{\mathbb{N}\cup\{0\}}$ with transition probability $p^\lambda(1, x, y) \, d\mathrm{Vol}^\lambda(y)$ and the process $Y_0(\underline{w}) = 1$ and, for $n \geqslant 1$,

$$Y_n(\underline{w}) := \frac{p(1, \underline{w}_0, \underline{w}_1)}{p^\lambda(1, \underline{w}_0, \underline{w}_1)\rho^\lambda(\underline{w}_1)} \cdot \frac{p(1, \underline{w}_1, \underline{w}_2)}{p^\lambda(1, \underline{w}_1, \underline{w}_2)\rho^\lambda(\underline{w}_2)} \cdots \frac{p(1, \underline{w}_{n-1}, \underline{w}_n)}{p^\lambda(1, \underline{w}_{n-1}, \underline{w}_n)\rho^\lambda(\underline{w}_n)}.$$

Observe that

$$\frac{p(n, x, y)}{p^\lambda(n, x, y)\rho^\lambda(y)} = \mathbb{E}_{\mathbb{P}^\lambda_x} \left(Y_n(\underline{w}) \mid \underline{w}_n = y \right).$$

So we may write

$$\begin{aligned}
(\mathrm{I})_\lambda &\geqslant \frac{1}{\lambda} \lim_{n\in\mathbb{N},n\to\infty} \frac{1}{n}\mathbb{E}_{\mathbb{P}_x^\lambda}\left(\ln \mathbb{E}_{\mathbb{P}_x^\lambda}\left(Y_n(\underline{w}) \mid \underline{w}_n = y\right)\right) \\
&\geqslant \frac{1}{\lambda} \lim_{n\in\mathbb{N},n\to\infty} \frac{1}{n}\mathbb{E}_{\mathbb{P}_x^\lambda}\left(\ln Y_n(\underline{w})\right) \\
&= \frac{1}{\lambda} \lim_{n\in\mathbb{N},n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \mathbb{E}_{\mathbb{P}_x^\lambda}\left(\ln \frac{Y_{i+1}(\underline{w})}{Y_i(\underline{w})}\right) \\
&= \frac{1}{\lambda} \lim_{n\in\mathbb{N},n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \mathbb{E}_{\mathbb{P}_x^\lambda}\left(\mathbb{E}_{\mathbb{P}_{\underline{w}_i}^\lambda}\left(\ln \frac{Y_{i+1}(\underline{w})}{Y_i(\underline{w})}\right)\right).
\end{aligned}$$

Set $\underline{w}_i = y$ and let $(\lfloor \mathrm{y}_t \rceil^\lambda, \lfloor \mathcal{V}_t \rceil^\lambda)_{t\geqslant 0}$ be the stochastic pair in $\widetilde{M} \times \mathcal{O}^{\tilde{g}^\lambda}(\widetilde{M})$ that defines the $\tilde{g}^\lambda$ Brownian motion on $\widetilde{M}$ starting from y. Then,

$$\mathbb{E}_{\mathbb{P}_{\underline{w}_i}^\lambda}\left(\ln \frac{Y_{i+1}(\underline{w})}{Y_i(\underline{w})}\right) = \mathbb{E}_{\mathrm{Q}}\left(\ln \frac{p(1,y,\lfloor \mathrm{y}_1 \rceil^\lambda(\mathrm{w}))}{p^\lambda(1,y,\lfloor \mathrm{y}_1 \rceil^\lambda(\mathrm{w}))\rho^\lambda(\lfloor \mathrm{y}_1 \rceil^\lambda(\mathrm{w}))}\right) =: \mathbb{E}_{\mathrm{Q}}\left((\mathrm{III})_{\lambda,y}\right),$$

which is L^1 integrable in y. Hence the ergodic theorem applied to the g^λ Brownian motion on M (see Proposition 2.2) gives that

$$\lim_{n\in\mathbb{N},n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \mathbb{E}_{\mathbb{P}_x^\lambda}\left(\mathbb{E}_{\mathbb{P}_{\underline{w}_i}^\lambda}\left(\ln \frac{Y_{i+1}(\underline{w})}{Y_i(\underline{w})}\right)\right) = \mathbb{E}_{\mathbb{P}^\lambda}\mathbb{E}_{\mathrm{Q}}\left((\mathrm{III})_{\lambda,\underline{w}_0}\right).$$

Since

$$\begin{aligned}
\mathbb{E}_{\mathrm{Q}}\left((\mathrm{III})_{\lambda,y}\right)'_\lambda &= \mathbb{E}_{\mathrm{Q}}\left(-(\ln p^\lambda)'_\lambda(1,y,\lfloor \mathrm{y}_1 \rceil^\lambda(\mathrm{w})) - (\ln \rho^\lambda)'_\lambda(\lfloor \mathrm{y}_1 \rceil^\lambda(\mathrm{w}))\right) \\
&\quad + \mathbb{E}_{\mathrm{Q}}\left(\left\langle \nabla_z \ln \frac{p(1,y,z)}{p^\lambda(1,y,z)\rho^\lambda(z)}\Big|_{z=\lfloor \mathrm{y}_1 \rceil^\lambda(\mathrm{w})}, D\pi\big(\lfloor \mathcal{V}_1 \rceil^\lambda\big)^{(1)}_\lambda(\mathrm{w})\right\rangle\right),
\end{aligned}$$

we conclude that $((\mathrm{III})_{\lambda,y})'_\lambda$ is L^1 integrable, uniformly in λ and y, by using Theorem 1.3 ii), Lemma 4.14 and Proposition 4.28 i) for $(\lfloor \mathrm{y}_1 \rceil^\lambda, \lfloor \mathcal{V}_1 \rceil^\lambda)$. Moreover,

$$\left(\mathbb{E}_{\mathrm{Q}}\left((\mathrm{III})_{\lambda,y}\right)\right)'_0 = \mathbb{E}_{\mathrm{Q}}\left(-(\ln p^\lambda)'_0(1,y,\lfloor \mathrm{y}_1 \rceil^0) - (\ln \rho^\lambda)'_0(\lfloor \mathrm{y}_1 \rceil^0)\right) = 0$$

by taking the differential in λ of $\int p^\lambda(1,y,z)\, d\mathrm{Vol}^\lambda(z) \equiv 1$.

Hence,

$$\lim_{\lambda\to 0+0} \frac{1}{\lambda}\mathbb{E}_{\mathbb{P}^\lambda}\mathbb{E}_{\mathrm{Q}}\left((\mathrm{III})_{\lambda,\underline{w}_0}\right) = \mathbb{E}_{\mathbb{P}}\left(\mathbb{E}_{\mathrm{Q}}\left((\mathrm{III})_{\lambda,\underline{w}_0}\right)\right)'_0 = 0,$$

where the first equality holds since we are integrating a function that depends only on $\underline{w}_0$. Consequently, we obtain $\lim_{\lambda\to 0+0}(\mathrm{I})_\lambda = 0$. In the same way, we show $\lim_{\lambda\to 0-0}(\mathrm{I})_\lambda = 0$. Thus, $\lim_{\lambda\to 0}(\mathrm{I})_\lambda = 0$. □

Remark 7.2. Note that for all λ, $h^\lambda \leqslant (v^\lambda)^2$ by [**Gu**] and [**K1**]. As in Corollary 3.10, we can also use [**BCG**], [**KKPW**] and the C^1 differentiability of $\lambda \mapsto h^\lambda$ for any C^3 curve $\lambda \mapsto g^\lambda \in \Re^3(M)$ to conclude that $(h^\lambda)'_0 = 0$ at locally symmetric g^0.

In proving Theorem 1.6, we obtain the following formula.

Theorem 7.3. *Let M be a closed connected smooth manifold and let $g \in \Re^3(M)$. For any C^3 curve $\lambda \in (-1,1) \mapsto g^\lambda \in \Re^3(M)$ with $g^0 = g$ and constant volume,*

$$(h^\lambda)'_0 = \int \left(-\frac{1}{2}\langle \nabla \mathrm{trace}\mathcal{X}, \overline{Z} + \nabla u_1 \rangle + \mathrm{Div}\big(\mathcal{X}(\overline{Z} + \nabla u_1)\big) \right) d\mathbf{m}.$$

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