On oriented cliques with respect to push operation

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Abstract

An oriented graph is a directed graph without any directed cycle of length at most 2. An oriented clique is an oriented graph whose non-adjacent vertices are connected by a directed 2-path. To push a vertex v of a directed graph $\overrightarrow{G}$ is to change the orientations of all the arcs incident to v. A push clique is an oriented clique that remains an oriented clique even if one pushes any set of vertices of it. We show that it is NP-complete to decide if an undirected graph is the underlying graph of a push clique or not. We also prove that a planar push clique can have at most 8 vertices and provide an exhaustive list of planar push cliques.

Keywords: oriented graphs, oriented cliques, push operation, planar graphs.

1. Introduction and main results

An *oriented graph* $\overrightarrow{G}$ is a directed graph with vertex set $V(\overrightarrow{G})$ and arc set $A(\overrightarrow{G})$ having no directed cycle of length 1 or 2 with set of vertices. We denote by G the underlying graph of $\overrightarrow{G}$. An *orientation* of G is an oriented graph obtained by assigning each edge uv of G one of the two possible orientations, namely, $\overrightarrow{uv}$ or $\overrightarrow{vu}$.

An *oriented k-coloring* of an oriented graph $\overrightarrow{G}$ is a mapping f from the vertex set $V(\overrightarrow{G})$ to a set of k colors such that (i) $f(u) \neq f(v)$ whenever u and v are adjacent and, (ii) if $\overrightarrow{uv}$ and $\overrightarrow{wx}$ are two arcs in $\overrightarrow{G}$, then $f(u) = f(x)$ implies $f(v) \neq f(w)$. The *oriented chromatic number* $\chi_o(\overrightarrow{G})$ of $\overrightarrow{G}$ is the smallest integer k for which $\overrightarrow{G}$ admits an oriented k-coloring. Oriented coloring is a well studied topic (see the latest survey [1] for details).

To *push* a vertex v of a directed graph $\overrightarrow{G}$ is to change the orientations of all the arcs (that is, to replace an arc $\overrightarrow{xy}$ by $\overrightarrow{yx}$) incident to v. The study of push operation was introduced by Mosesain [2] and studied in [3, 4, 5, 6, 7, 8, 9].

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Two orientations $\overrightarrow{G}$ and $\overrightarrow{G}'$ of G are in a *push relation* if one can obtain $\overrightarrow{G}'$ by pushing a set of vertices of $\overrightarrow{G}$. The *pushable chromatic number* $\chi_p(\overrightarrow{G})$, introduced by Klostermeyer and MacGillivray [10], of an oriented graph $\overrightarrow{G}$ is the minimum oriented chromatic number taken over all oriented graphs that are in push relation with $\overrightarrow{G}$. Following the work of Klostermeyer and MacGillivray [10], the pushable chromatic number of planar graphs was studied by Sen [11].

An oriented clique, introduced by Klostermeyer and MacGillivray [12], is an oriented graph $\overrightarrow{C}$ with $\chi_o(\overrightarrow{C}) = |V(\overrightarrow{C})|$. In fact, an oriented clique is characterized as an oriented graph in which each pair of non-adjacent vertices are connected by a directed 2-path. Due to this characterization oriented cliques can be viewed as natural objects. Moreover, they play a significant role in studying oriented coloring as pointed out in [13]. An undirected simple graph is called an *underlying oriented clique* if it is the underlying graph of an oriented clique.

Let $\overrightarrow{C}$ be an oriented clique such that each oriented graph in a push relation with $\overrightarrow{C}$ is also an oriented clique. We are interested in such oriented cliques. Observe that $\overrightarrow{C}$ is such an oriented clique if and only if $\chi_p(\overrightarrow{C}) = |V(\overrightarrow{C})|$. Thus we define the following notion: an oriented graph $\overrightarrow{C}$ is a *push clique* if $\chi_p(\overrightarrow{C}) = |V(\overrightarrow{C})|$. Also an undirected simple graph is an *underlying push clique* if it is the underlying graph of a push clique.

Given an undirected simple graph it is NP-hard to determine if it is an underlying oriented clique [14]. We prove an analogous result for underlying push cliques.

Theorem 1.1. *It is NP-complete to decide whether a given graph is an underlying push clique.*

Oriented cliques of planar and outerplanar graphs are studied in details, see [13]. Settling a conjecture of Klostermeyer and MacGillivray [12], it is proved in [13] that a planar oriented clique can have at most 15 vertices. Note that there exists a planar oriented clique on 15 vertices which implies that the above mentioned bound is tight. Here, we exhibit all planar push cliques, in particular proving that any such clique has at most 8 vertices.

Theorem 1.2. *A planar push clique can have at most 8 vertices. Moreover, there exists a planar push clique on 8 vertices.*

Klostermeyer and MacGillivray showed that an outerplanar oriented clique can have at most 7 vertices and any outerplanar oriented clique must have a particular oriented clique as a spanning subgraph [12]. Later this result was extended by providing an explicit list of eleven outerplanar graphs and proving that any outerplanar underlying oriented clique must have one of the eleven outerplanar graphs as its spanning subgraph [13]. In the same article the following question was asked: "Characterize the set L of graphs such that any planar graph is an underlying oriented clique if and only if it contains one of the

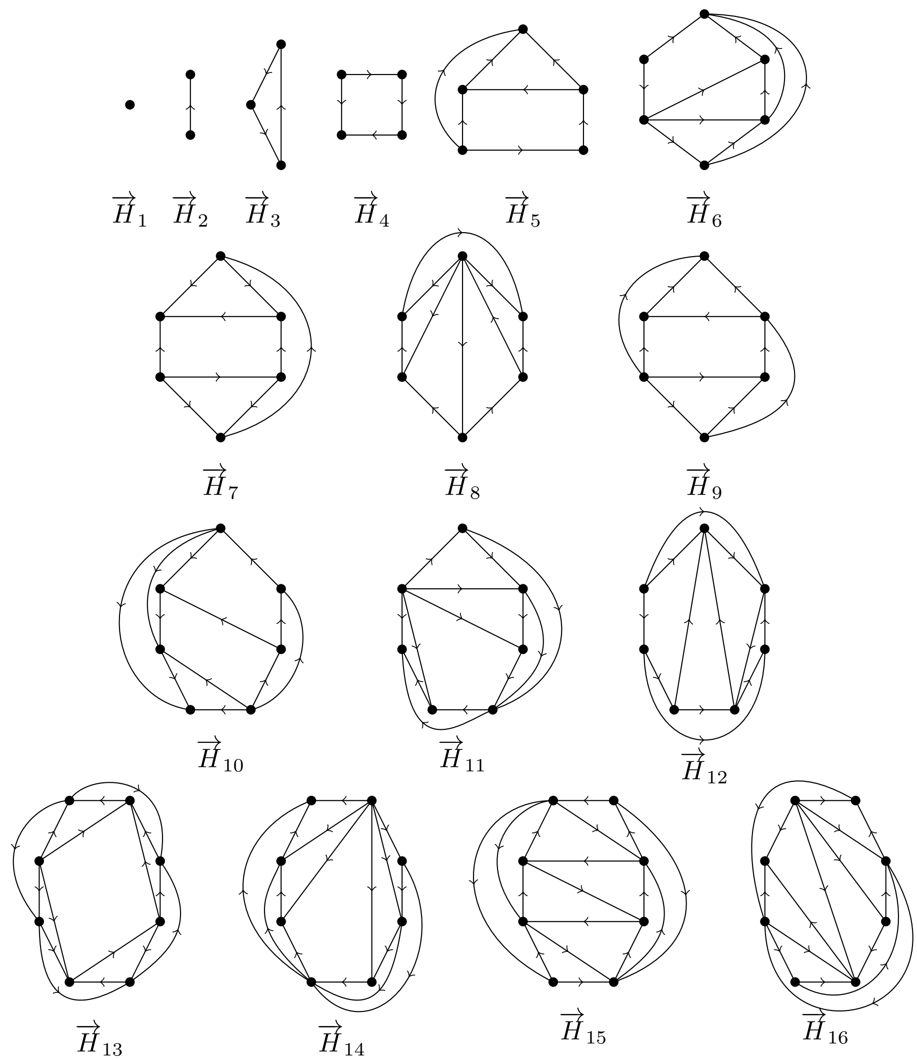

Figure 1: A list of planar push cliques whose underlying graphs $H_1, H_2, \cdots, H_{16}$ is an exhaustive list of edge minimal planar underlying push cliques.

graphs from L as a spanning subgraph." Here we answer an analogous version of this question for planar underlying push cliques.

Theorem 1.3. *An undirected planar graph is an underlying push clique if and only if it contains an underlying graph of one of the 16 planar graphs depicted in Figure 1 as a spanning subgraph.*

In Section 2 we introduce some basic definitions and notations. The proofs of Theorems 1.1, 1.2 and 1.3 are given in Sections 3, 4 and 5, respectively. Theorem 1.2 was published in EuroComb 2013 [15].

2. Preliminaries

For an (oriented) graph G every parameter we introduce below is denoted using G as a subscript. In order to simplify the notations, this subscript will be dropped whenever there is no chance of confusion.

The set of all adjacent vertices of a vertex v of an (oriented) graph G is called its set of *neighbors* and is denoted by $N_G(v)$. If there is an arc $\overrightarrow{uv}$, then u is an *in-neighbor* of v and v is an *out-neighbor* of u. The set of all in-neighbors and the set of all out-neighbors of v are denoted by $N^-_G(v)$ and $N^+_G(v)$, respectively. The *degree* of a vertex v of an (oriented) graph G, denoted by $d_G(v)$, is the number of neighbors of v in G. Naturally, the *in-degree* (resp. *out-degree*) of a vertex v of an oriented graph G, denoted by $d^-_G(v)$ (resp. $d^+_G(v)$), is the number of in-neighbors (resp. out-neighbors) of v in G. The *order* $|V(G)|$ of an (oriented) graph G is the cardinality of its set of vertices $V(G)$.

Two vertices u and v of an oriented graph *agree* on a third vertex w of that graph if $w \in N^\alpha(u) \cap N^\alpha(v)$ for some $\alpha \in \{+,-\}$. Two vertices u and v of an oriented graph *disagree* on a third vertex $w \in N(u) \cap N(v)$ if u and v do not agree on w.

A *k-cycle* is an undirected cycle having k vertices. Let $\overrightarrow{C}_4$ be an oriented 4-cycle with arcs ab, bc, cd, ad. A *special 4-cycle* is an oriented 4-cycle isomorphic to $\overrightarrow{C}_4$. Note that all the oriented graphs which are in push relation with a special 4-cycle are isomorphic to it. Notice that the non-adjacent vertices of a special 4-cycle always get different colors as they are always connected with a 2-dipath, no matter which vertex of the graph you push. This is, in fact, a necessary and sufficient condition for two non-adjacent vertices to receive two distinct colors under any oriented coloring with respect to any push relation.

Lemma 2.1. *An oriented graph $\overrightarrow{G}$ is a push clique if and only if any two non-adjacent vertices of $\overrightarrow{G}$ are part of a special 4-cycle.*

Due to Lemma 2.1 we know that a push clique is an oriented graph with each pair of non-adjacent vertices agreeing on at least one vertex and disagreeing on at least one vertex.

Corollary 2.2. *Each pair of non-adjacent vertices of a push clique must have at least two common neighbors.*

This implies the following observations.

Observation 2.3. *Each pair of non-adjacent vertices of an underlying push clique must have at least two common neighbors.*

Observation 2.4. *An underlying push clique has diameter at most 2.*

Also if an underlying push clique is not a complete graph, then a pair of non-adjacent vertices in it must have at least two common neighbors by Corollary 2.2.

Observation 2.5. *Any underlying push clique that is not a complete graph must contain a 4-cycle as a subgraph.*

A *dominating set* of a graph G is a set of vertices D such that every vertex of G is either in D or has a neighbor in D. The *domination number* $\gamma(G)$ of a graph G is the cardinality of its smallest dominating set. In this article, a dominating set and the domination number of an oriented graph $\overrightarrow{G}$ will correspond to a dominating set and the domination number of its underlying graph G.

3. On the proof of Theorem 1.1

Given an oriented graph one can check, in polynomial-time, if it is a push clique or not using the characterization given in Lemma 2.1.

Let G be a graph. Define G_* to be the graph obtained by adding a vertex v_* to G such that v_* is adjacent to each vertex of G. Then the following holds.

Lemma 3.1. *The graph G_* is an underlying push clique if and only if G is an underlying oriented clique.*

Proof. Assume that G_* is an underlying push clique. Let $\overrightarrow{G_*}$ be an orientation of G_* such that $\overrightarrow{G_*}$ is a push clique. Let $\overrightarrow{G_*}'$ be the orientation of G_* obtained by pushing all the in-neighbors of v_*. In $\overrightarrow{G_*}'$ the vertices of G are all out-neighbors of v_*. As $\overrightarrow{G_*}'$ is a push clique, each pair of non-adjacent vertices of $\overrightarrow{G_*}'$ must agree on at least one vertex and must disagree on at least one vertex. Note that any pair of non-adjacent vertices must be vertices of G and they agree on v_*. Thus, they must disagree on a vertex of G. Hence, the oriented graph induced by the vertices of G obtained from $\overrightarrow{G_*}'$ is an oriented clique. It follows that G is an underlying oriented clique.

On the other hand, assume that G is an underlying oriented clique. Let $\overrightarrow{G}$ be an orientation of G such that $\overrightarrow{G}$ is an oriented clique. Now consider a orientation $\overrightarrow{G_*}$ of G_* in which every vertex of G is an out-neighbor of v_* and the oriented graph induced by the vertices of G from $\overrightarrow{G_*}$ is isomorphic to $\overrightarrow{G}$. Thus $\overrightarrow{G_*}$ is a push clique due to Lemma 2.1. □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. It is known that determining if a graph is an underlying oriented clique is NP-hard [14]. Therefore, the proof follows from Lemma 3.1. □

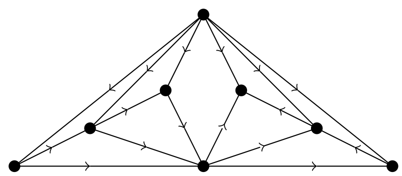

Figure 2: A planar push clique of order 8.

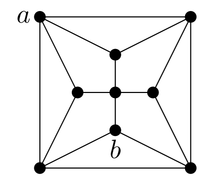

Figure 3: The unique diameter-2 planar graph of domination number 3 [16].

4. On the proof of Theorem 1.2

Note that the oriented planar graph of order 8 depicted in Figure 2 is a push clique due to Lemma 2.1. Thus:

Lemma 4.1. *There exists a planar push clique on 8 vertices.*

Now we will prove that a planar push clique cannot have more than 8 vertices.

Goddard and Henning [16] showed that every planar graph of diameter-2 has domination number at most 2 except for a particular graph on 9 vertices (see Figure 3). Observe that the vertices a and b of this graph have exactly one common neighbor. Therefore, it is not a push clique. Since a planar push clique must have diameter at most 2 (by Observation 2.4), by the result of Goddard and Henning, it also has domination number at most 2.

First we will handle the case when a planar push clique has domination number 1.

Lemma 4.2. *If $\overrightarrow{B}$ is a planar push clique of domination number 1, then $|V(\overrightarrow{B})| \leq 8$.*

Proof. Let $\overrightarrow{B}$ be a planar push clique of domination number 1. Let $\{v\}$ be a dominating set of B. Then the induced graph $B[V(B) \setminus \{v\}]$ is an outerplanar underlying oriented clique by Lemma 2.1. Moreover, $|V(\overrightarrow{B}) \setminus \{v\}| \leq 7$ due to Nandy, Sen and Sopena (see Lemma 2.2 and Theorem 2.3 of [13]). Thus, $\overrightarrow{B}$ has order at most 8. □

A *plane graph* is a planar graph with a specific planar embedding. Note that there exists a planar push clique of order at least 9 if and only if there exists a plane push clique of order at least 9. Therefore, to prove Theorem 1.2 it is enough to prove that any plane push clique of domination number 2 must have order at most 8. To be precise, we need to prove the following lemma.

Lemma 4.3. *If $\overrightarrow{G}$ is a plane push clique of domination number 2, then $|V(\overrightarrow{G})| \leq 8$.*

Before proving the lemma we perform some preprocessing by definining certain notions and proving some claims.

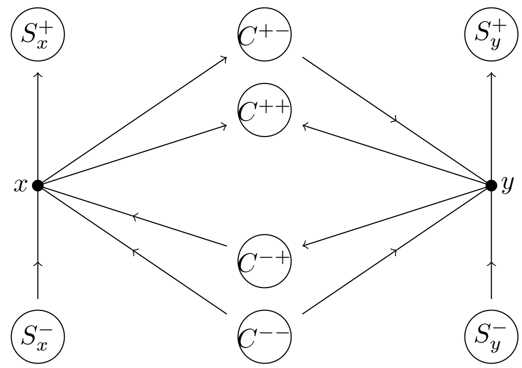

Figure 4: Neighbors of x and y in $\vec{G}$.

We will frequently use the following notation: $\{\alpha, \beta\} = \{+, -\}$. It means either $\alpha = +, \beta = -$ or $\alpha = -, \beta = +$ as these are the only two solutions for the above set-theoric equation.

Let $\overrightarrow{G}$ be a plane push clique with $|V(\overrightarrow{G})| > 8$. Assume that $\overrightarrow{G}$ is triangulated and has domination number 2. Define the partial order $\prec$ for the set of all dominating sets of order 2 of $\overrightarrow{G}$ as follows: for any two dominating sets $D = \{x, y\}$ and $D' = \{x', y'\}$ of order 2 of $\overrightarrow{G}$, $D' \prec D$ if and only if $|N_{\overrightarrow{G}}(x') \cap N_{\overrightarrow{G}}(y')| < |N_{\overrightarrow{G}}(x) \cap N_{\overrightarrow{G}}(y)|$.

Let $D = \{x, y\}$ be a maximal dominating set of order 2 of $\overrightarrow{G}$ with respect to $\prec$. Also for the rest of this proof, $t, t', \alpha, \overline{\alpha}, \beta, \overline{\beta}$ are variables satisfying $\{t, t'\} = \{x, y\}$ and $\{\alpha, \overline{\alpha}\} = \{\beta, \overline{\beta}\} = \{+, -\}$.

Now, we fix the following notations (see Figure 4):

$$
\begin{aligned}
&C = N_{\overrightarrow{G}}(x) \cap N_{\overrightarrow{G}}(y),\ C^{\alpha\beta} = N^{\alpha}_{\overrightarrow{G}}(x) \cap N^{\beta}_{\overrightarrow{G}}(y),\\
&C_t = N_{\overrightarrow{G}}(t) \cap C,\ C_t^{\alpha} = N^{\alpha}_{\overrightarrow{G}}(t) \cap C,\\
&S_t = N_{\overrightarrow{G}}(t) \setminus C,\ S_t^{\alpha} = S_t \cap N^{\alpha}_{\overrightarrow{G}}(t) \text{ and } S = S_x \cup S_y.
\end{aligned}
$$

In the above notations, C denotes the common neighbors of x and y and S denotes the neighbors of x or y that are not common neighbors. Also, the subscripts refer to x or y suggesting that particular set deals with only neighbors of the vertex/vertices mentioned in the subscript. The superscripts refer to the corresponding orientations.

If necessary, by pushing some vertices, we assume that the plane oriented graph $\overrightarrow{G}$ is such that $C_x = C_x^+$, $S_x = S_x^+$, $S_y = S_y^+$ and $|C_y^+| \geq |C_y^-|$. Note that, it is possible to obtain an alternative orientation of G from any orientation $\overrightarrow{G'}$ that satisfies the above conditions by pushing some vertices of $\overrightarrow{G'}$. That is why our assumption is valid.

Hence we have,

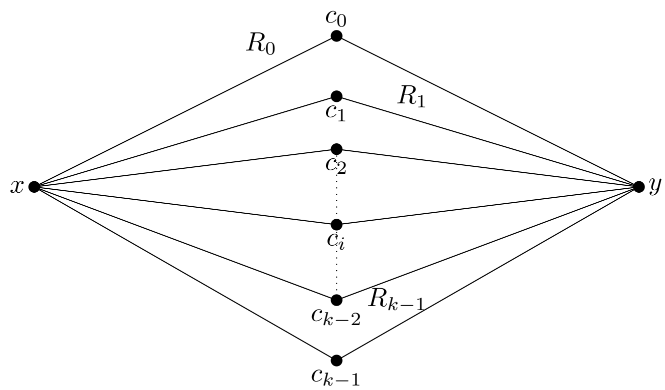

Figure 5: A planar embedding of H

$$9 \leq |V(\overrightarrow{G})| = |D| + |C| + |S|. \qquad (1)$$

Let $\overrightarrow{H}$ be the plane oriented graph obtained from $\overrightarrow{G}$ by deleting all the vertices in S and the arc of D and all the arcs between the vertices of C. Assume that the vertices $c_0, c_1, ..., c_{k-1}$ of C, which are also neighbors of x in $\overrightarrow{H}$, are cyclically arranged around x in a clockwise order and are all adjacent to y (see Figure 5).

Notice that H has k faces, namely the unbounded face F_0 and the faces F_i bounded by edges $xc_{i-1}, c_{i-1}y, yc_i, c_ix$ for $i \in \{1, ..., k-1\}$. Geometrically, H divides the plane into k connected components. The *region* R_i of $\overrightarrow{G}$ is the i^{th} connected component (corresponding to the face F_i) of the plane. The *boundary points* of a region R_i are c_{i-1} and c_i for $i \in \{1, ..., k-1\}$ and, c_0 and c_{k-1} for $i = 0$. Two regions are *adjacent* if their corresponding faces have at least one common edge (hence, a region is adjacent to itself as well).

Now for different possible values of $|C|$, we want to show that H cannot be extended to a plane push clique of order at least 9. Note that for extending H to $\overrightarrow{G}$ we can add new vertices only from S. Any vertex $v \in S$ will be inside one of the regions R_i. If there is at least one vertex of S in a region R_i, then R_i is *non-empty* and *empty* otherwise. In fact, when there is no chance of confusion, R_i might represent the set of vertices of S contained in the region R_i.

We first prove the following lower bound on $|C|$.

Claim 1: $|C| \geq 2$.
Proof of the claim. We know that x and y are either connected by two distinct 2-paths or by an arc. So, if x and y are non-adjacent, then we have $|C| \geq 2$. If x and y are adjacent, then the triangulation of $\overrightarrow{G}$ implies $|C| \geq 2$. ⋄

Next we will show that $|C| \neq 2$. Before that we will prove two more claims to facilitate the proof.

Claim 2: If $|C| = 2$ and both $S_x \cap R_i$ and $S_y \cap R_j$ are non-empty for some $i \neq j$, then $(S_x \cap R_i) \cup (S_y \cap R_j) \subseteq N(c_0) \cap N(c_1)$.
Proof of the claim. Suppose that both $S_x \cap R_0$ and $S_y \cap R_1$ are non-empty. Assume $u \in S_x \cap R_0$ and $v \in S_y \cap R_1$. Note that u and v are non-adjacent as they are separated by the cycle xc_0yc_1x. Thus to have at least two common neighbors due to Observation 2.3, u and v must be both adjacent to at least two vertices of xc_0yc_1x. Notice that y is non-adjacent to u as u belongs to S_x and x is non-adjacent to v as v belongs to S_y. Thus the two common neighbors of u and v must be c_0 and c_1. $\diamond$

Claim 3: If $|C| = 2$ and both $S_x \cap R_i$ and $S_y \cap R_j$ are non-empty for some $i \neq j$, then $|S_x \cap R_i|, |S_y \cap R_j| \leq 1$.
Proof of the claim. If $|S_x \cap R_i| \geq 2$, then there are at least two vertices in R_i that are adjacent to x, c_0 and c_1 due to Claim 2. This is not possible as G is a plane graph. Thus $|S_x \cap R_i| \leq 1$. Similarly, $|S_y \cap R_j| \leq 1$. $\diamond$

We are now ready to analyze the case $|C| = 2$.

Claim 4: If $|C| = 2$, then exactly two sets among the four sets $S_t \cap R_i$ for all $(t, i) \in \{x, y\} \times \{0, 1\}$ are non-empty.
Proof of the claim. We consider three cases.
Case (i): If $S_t = \emptyset$, then t' is a dominating vertex. This is not possible due to Lemma 4.2. Hence we do not have $S_t = \emptyset$ for any $t \in \{x, y\}$. $\circ$
Case (ii): If all the four sets $S_t \cap R_i \neq \emptyset$ for all $(t, i) \in \{x, y\} \times \{0, 1\}$, then $|S| \leq 4$ due to Claim 3. $\circ$
Case (iii): If there are exactly three non-empty sets $S_x \cap R_0$, $S_x \cap R_1$ and $S_y \cap R_0$ among the four sets $S_t \cap R_i$ for all $(t, i) \in \{x, y\} \times \{0, 1\}$, then by triangulation we have the edge c_0c_1 inside R_1. At least one vertex of $S_x \cap R_0$ must be adjacent to c_0 because of triangulation. Recall that, c_0 is adjacent to y and all the vertices of $S_y = S_y \cap R_0$ by Claim 2. Thus we have a dominating set $\{x, c_0\}$ with at least three common neighbors (c_1, a vertex from $S_x \cap R_0$ and a vertex from $S_x \cap R_1$) contradicting the maximality of D. $\circ$

Thus exactly two sets among the four sets $S_t \cap R_i$ for all $(t, i) \in \{x, y\} \times \{0, 1\}$ are non-empty. $\diamond$

Claim 5: $|C| \neq 2$.
Proof of the claim. Assume that the claim is false, that is, $|C| = 2$.

By Claim 4, exactly two sets among the four sets $S_t \cap R_i$ for all $(t, i) \in \{x, y\} \times \{0, 1\}$ are non-empty. Therefore, $S_x \cap R_i$ and $S_y \cap R_j$ are non-empty sets for some $i, j \in \{0, 1\}$ as $S_x, S_y \neq \emptyset$ by Case (i) of Claim 4.

If $i \neq j$, then $|S| \leq 2$ by Claim 3. Thus $i = j$ and without loss of generality we can assume that $S_x \cap R_1$ and $S_y \cap R_1$ are the two non-empty sets.

Assume that $S_x = \{x_1, x_2, \ldots, x_{n_x}\}$ and $S_y = \{y_1, y_2, ..., y_{n_y}\}$. Suppose the vertices $x_1, x_2, ..., x_{n_x}$ are cyclically arranged around x in a clockwise order and the vertices $y_1, y_2, ..., y_{n_y}$ are cyclically arranged around y in an anti-clockwise order. Therefore we have the edges $c_0x_1, x_1x_2, \ldots, x_{n_x-1}x_{n_x}, x_{n_x}c_1$ and the edges $c_0y_1, y_1y_2, ..., y_{n_y-1}y_{n_y}, y_{n_y}c_1$ by triangulation. Furthermore, we can assume $n_x \geq n_y$ without loss of generality.

Assume $n_y = 1$. So, to have $|S| \geq 5$ we should have $n_x \geq 4$. Note that we cannot have the edge xy as otherwise $\{y_1, x\}$ would be a dominating set with at least three common neighbors $\{c_0, c_1, y\}$ contradicting the maximality of D. Hence we have the edge c_0c_1 inside R_1 by triangulation. Note that the vertex $x_2 \in S_x$ must be adjacent to either c_0 or c_1 or y_1 to have at least two common neighbors with y by Observation 2.3. This will create a dominating set $\{c_0, x\}$ or $\{c_1, x\}$ or $\{y_1, x\}$ with at least three common neighbors $\{x_1, x_2, c_1\}$ or $\{x_{n_x}, x_2, c_0\}$ or $\{x_2, c_0, c_1\}$ respectively. This will contradict the maximality of D. Therefore, $n_y \geq 2$.

Now assume that we have the edge x_2c_0. Then to have two distinct 2-paths connecting a vertex $w \in S_y$ and x_1 we need to have w adjacent to both x_2 and c_0. That means, each vertex of S_y is adjacent to both x_2 and c_0. But this is not possible keeping the graph plane as $n_y \geq 2$. So, there is no edge between c_0 and x_2. By similar arguments, we can show that every t_i is non-adjacent to c_0 for $i \in \{2, 3, ..., n_t\}$ and every t_j is non-adjacent to c_1 for $i \in \{1, 2, ..., n_t - 1\}$ for all $t \in \{x, y\}$. A similar argument also proves that the edge t_it_{i+k} for $1 \leq i < i + k \leq n_t$ does not exist unless $k = 1$ for any $t \in \{x, y\}$.

Now notice that $n_x \geq 3$ by Equation (1) and the assumption that $n_x \geq n_y$. By triangulation we must have the edge x_2y_i for some $i \in \{1, 2, ..., n_y\}$. Then to have two distinct 2-paths connecting x_1 and y_j for $j \in \{i + 1, ..., n_y\}$ and to have two distinct 2-paths connecting x_3 and y_l for $l \in \{1, ..., i - 1\}$ we must have every vertex of S_y adjacent to x_2.

If $n_y \geq 3$, then we cannot have two distinct 2-paths connecting the non-adjacent vertices x_1 and y_3. So we must have $n_y = 2$.

Now to have two distinct 2-paths connecting the non-adjacent vertices x_1 and y_2 we must have the edge x_1y_1. This creates the dominating set $\{x, y_1\}$ with at least three common neighbors $\{c_0, x_1, x_2\}$ contradicting the maximality of D. $\diamond$

We now prove upper bounds on $|C|$.

Claim 6: If $|C| \geq 3$, then any two non-empty regions must be adjacent.
Proof of the claim. If two non-empty regions R_i and R_j are not adjacent then they do not share any common boundary points. Thus a vertex of R_i and a vertex of R_j are non-adjacent and can have at most one common neighbor (either x or y). This is not possible due to Observation 2.3. $\diamond$

Claim 7: If $|C| \geq 3$, then both $S_x \cap R_i$ and $S_y \cap R_j$ cannot be non-empty for $i \neq j$.
Proof of the claim. Suppose both $S_x \cap R_i$ and $S_y \cap R_j$ are non-empty for some $i \neq j$. By Claim 5, R_i and R_j are adjacent. Note that a vertex of $S_x \cap R_i$ must have at least two common neighbors with a vertex of $S_y \cap R_j$ due to Observation 2.3. That is not possible as $|C| \geq 3$. $\diamond$

Claim 8: $|C| \leq 4$.
Proof of the claim. First assume that $S = \emptyset$. Then $|C| = k \geq 7$ and there are at least four vertices in C^{++} (recall our basic assumptions: $C = C_x^+$ and $|C_y^+| \geq |C_y^-|$). Among those four vertices, two must be such that they are

non-adjacent and the only common neighbors they have are x and y. Thus, those two non-adjacent vertices do not disagree on any vertex, a contradiction to Corollary 2.2.

Now assume that $5 \leq |C| = k \leq 6$. Then $S \neq \emptyset$. Without loss of generality assume that there exists a vertex $v \in S_x \cap R_1$. Note that it is not possible for v and c_3 to have any common neighbor other than x. Thus $|C| \leq 4$. $\diamond$

Thus the only possible values for $|C|$ are 3 and 4. First we show that it is not possible to have $|C| = 4$.

Claim 9: $|C| \neq 4$.
Proof of the claim. If $|C| = 4$, then $|S| \geq 3$ by Equation (1). Without loss of generality assume that there exists a vertex $v \in S_x \cap R_0$. Note that v must be adjacent to c_0 and c_3 in order to have at least two common neighbors with c_1 and c_2, respectively, by Observation 2.3. So $|S_x \cap R_0| \leq 1$ as G is a plane graph. Similarly,

$$|S_t \cap R_i| \leq 1 \text{ for all } (t, i) \in \{x, y\} \times \{0, 1, ..., k-1\}. \tag{2}$$

Also note that if we have a vertex $v \in S_x \cap R_0$, then it is not possible to have any vertex in $S_y \cap R_i$ for all $i \in \{1, 2, ..., k-1\}$ by Claim 7 and in $S_x \cap R_j$ for all $j \in \{2, ..., k-2\}$ by Claim 6. Thus, if $S_x \cap R_0$ is non-empty, then only $S_x \cap R_1$, $S_x \cap R_{k-1}$ and $S_y \cap R_0$ can be non-empty. Note that among these three sets, at most one can be non-empty due to Claims 6 and 7. Hence, at most two of the sets $S_t \cap R_i$ for all $(t, i) \in \{x, y\} \times \{0, 1, ...k-1\}$ can be non-empty. Then Equation (2) implies $|S| \leq 2$ and by Equation (1) we have

$$9 \leq |V(G)| \leq 2 + 4 + 2 = 8.$$

This is a contradiction. $\diamond$

Claim 10: The graph W (depicted in Figure 6) is not an underlying push clique.
Proof of the claim. Suppose W is an underlying push clique. Also let an orientation $\overrightarrow{W'}$ of W be a push clique. Push (if necessary) some of the vertices x, c_0, y_1, x_2 of $\overrightarrow{W'}$ to obtain the arcs $\overrightarrow{x_1x}, \overrightarrow{x_1c_0}, \overrightarrow{x_1y_1}$ and $\overrightarrow{x_1x_2}$. Furthermore, push (if necessary) some of the vertices c_1, c_2, y, y_2 to obtain the arcs $\overrightarrow{xc_2}, \overrightarrow{c_0y}, \overrightarrow{y_2y_1}$ and $\overrightarrow{c_1x_2}$. Let this so obtained orientation of W be $\overrightarrow{W}$. Note that $\overrightarrow{W}$ is also a push clique as it is in a push relation with $\overrightarrow{W'}$.

Note that the arc $\overrightarrow{c_2c_0}$ must belong to $\overrightarrow{W}$ by Corollary 2.2 as the non-adjacent vertices x_1 and c_2 have exactly two common neighbors x and c_0. Similarly, the pair $\{x_1, y\}$ of non-adjacent vertices implies the existence of the arc $\overrightarrow{yy_1}$, the pair $\{x_1, y_2\}$ of non-adjacent vertices implies the existence of the arc $\overrightarrow{x_2y_2}$ and the pair $\{x_1, c_1\}$ of non-adjacent vertices implies the existence of the arc $\overrightarrow{xc_1}$.

We now consider the following two cases.

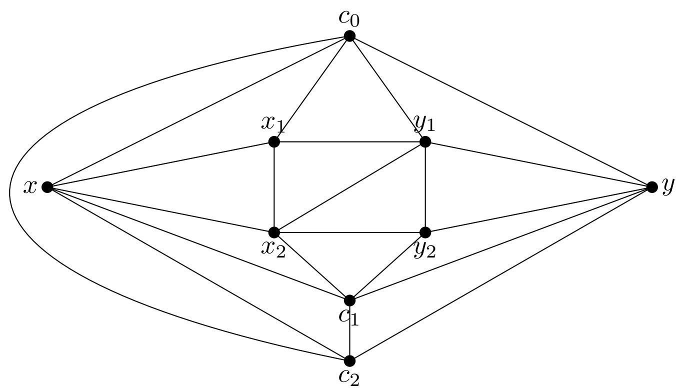

Figure 6: The graph W.

Case (i): If the arc $\overrightarrow{c_0y_1}$ is present in $\overrightarrow{W}$, then the non-adjacent pairs of vertices c_2, y_1 and y_2, c_0 imply the existence of the arcs $\overrightarrow{yc_2}$ and $\overrightarrow{yy_2}$, respectively, due to Corollary 2.2.

Moreover, for satisfying Corollary 2.2 for c_2 and y_2, the 2-path $c_2c_1y_2$ must be a 2-dipath (in either directions). That makes it impossible to satisfy Corollary 2.2 for the non-adjacent pairs of vertices x, y_2 and x_2, c_2. $\circ$

Case (ii): If the arc $\overrightarrow{y_1c_0}$ is present in $\overrightarrow{W}$, then the non-adjacent pairs of vertices c_2, y_1 and y_2, c_0 imply the existence of the arcs $\overrightarrow{c_2y}$ and $\overrightarrow{y_2y}$, respectively, due to Corollary 2.2.

The rest of the arguments are similar to Case (i). $\circ$

Therefore, W is not an underlying push clique. $\diamond$

Therefore, the only possible value for $|C|$ is 3. In the next claim we will show that $|C| = 3$ is actually not possible.

Claim 11: If $S_t = \emptyset$, then $|C| \neq 3$.

I do not understand why we may assume "$S_y = \emptyset$". But having assumed that, $|S_x| \geq |S_y| = 0$ is a given and need not to be stated.

Proof of the claim. We will prove this claim by contradiction. So, assume that $|C| = 3$. Also, without loss of generality, assume that $|S_x| \geq |S_y| = 0$. Hence we do not have the edge xy as otherwise x would dominate the whole graph. Now note that any two regions are adjacent for $|C| = 3$. The vertices from different regions must be adjacent to their unique common boundary point to have two distinct 2-paths connecting them.

Hence, if all regions are non-empty, then remove the vertices of each region adjacent to both the boundary points of that region. This implies

$$|S_x \cap R_i| \leq 1 \text{ for all } i \in \{0, 1, 2\}.$$

Which would imply $|S| \leq 3$, a contradiction to our assumption. Hence, it is not possible to have all the three regions non-empty when $S_y = \emptyset$.

If we have exactly two regions, say R_0 and R_1, non-empty, then every vertex of S_x must be adjacent to c_0 to create two distinct 2-paths connecting the vertices of $S_x \cap R_0$ and the vertices of $S_x \cap R_1$. This will create a dominating set $\{c_0, x\}$ with at least four common neighbors contradicting the maximality of D. Hence, we can have at most one region non-empty when $S_y = \emptyset$.

Now assume that exactly one region, say R_1, is non-empty. Then each vertex of S_x must be adjacent to either c_0 or c_1 to have two distinct 2-paths connecting it to c_2. Then, without loss of generality, we will have at least three vertices of S_x adjacent to c_0 by pigeonhole principle and triangulation. This will create a dominating set $\{c_0, x\}$ with at least four common neighbors (three vertices from S_x and c_2 because of triangulation) contradicting the maximality of D. $\diamond$

Claim 12: $|C| \neq 3$.
Proof of the claim. We prove this claim by contradiction. So, assume that $|C| = 3$. Also, without loss of generality, assume that $|S_x| \geq |S_y|$. Note that by Equation (1) we have $|S| \geq 4$. Hence $|S_x| \geq 2$.

By Claim 11 we know that $S_y \neq \emptyset$. Now assume, without loss of generality, that $S_x \cap R_0 \neq \emptyset$. This implies $S_y \cap R_1 = \emptyset$ and $S_y \cap R_2 = \emptyset$ by Claim 6. But we also know that $S_y \neq \emptyset$. Hence we must have $S_y \cap R_0 \neq \emptyset$. This implies $S_x \cap R_1 = \emptyset$ and $S_x \cap R_2 = \emptyset$ by Claim 7.

Now assume that $S_x = \{x_1, x_2, \ldots, x_{n_x}\}$ and $S_y = \{y_1, y_2, ..., y_{n_y}\}$. Suppose that the vertices $x_1, x_2, ..., x_{n_x}$ are cyclically arranged around x in a clockwise order and the vertices $y_1, y_2, ..., y_{n_y}$ are cyclically arranged around y in an anti-clockwise order. Thus we have the edges $c_0x_1, x_1x_2, .., x_{n_x-1}x_{n_x}$, $x_{n_x}c_1$ and the edges $c_0y_1, y_1y_2, ...,$ $y_{n_y-1}y_{n_y}, y_{n_y}c_1$ due to triangulation.

If we have the edge xy (say, inside region R_2), then each vertex of S_x must be adjacent to c_0 to have two distinct 2-paths connecting it to c_1 creating the dominating set $\{c_0, x\}$ with at least four common neighbors (the vertices of S_x) contradicting the maximality of D. Hence we do not have the edge xy. Therefore, we have the edges c_oc_1 and c_1c_2.

Each vertex of S is adjacent to either c_0 or c_2 to have two distinct 2-paths connecting it to c_1. This implies that $\{c_0, c_2\}$ is a dominating set with common neighbors x, y, c_1. If $n_y = 1$, then y_1 is also a common neighbor of c_0 and c_2, contradicting the maximality of D. Hence $n_y \geq 2$.

Suppose, $n_x \geq 3$. Now x_2 must be adjacent to either c_0 or c_2 for having two distinct 2-paths connecting x_2 and c_1. Without loss of generality assume that x_2 is adjacent to c_0. Now each vertex of S_y must be adjacent to both x_2 and c_0 to have two distinct 2-paths connecting it to x_1. But this contradicts the fact that $\overrightarrow{G}$ is a plane graph. Thus $n_x \leq 3$.

So we have $n_x \geq n_y$ by assumption, and $n_x \leq 2, n_y \geq 2$. Therefore, $n_x = n_y = 2$. Also note that c_0 and c_2 cannot have any common neighbor other than x, y, c_1 as otherwise this would contradict the maximality of D. So, x_1, y_1 are adjacent to c_2 and x_2, y_2 are adjacent to c_0. Due to triangulation, we can have only one other edge in G: either x_1y_2 or y_1x_2. Due to symmetry of the graph, we can assume without loss of generality, that G has the edge y_1x_2. Therefore, G is the graph W depicted in Figure 6 which is not an underlying push clique by Claim 10. ◇

The above claims prove that for no value of $|C|$ it is possible to have a planar push clique of domination number 2 and order at least 9.

Proof of Lemma 4.3. The proof follows from Claims 1, 5, 8, 9 and 12. □

Finally we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. The proof follows from Lemmas 4.1, 4.2 and 4.3. □

5. On the proof of Theorem 1.3

We want to show that any planar underlying push clique must contain the underlying graph of one of the graphs listed in Figure 1 as a spanning subgraph. Observe that each graph listed in Figure 1 is planar (we provided a planar drawing) and is a push clique (using the characterization given by Lemma 2.1).

Note that by adding edges to a push clique one obtains another push clique. So to prove our result it is enough to show that if $\overrightarrow{G}$ is a planar push clique on k vertices then its underlying graph G must contain one of the underlying graphs on k vertices listed in Figure 1 as its subgraph. From now on in this section whenever we mention a graph H_i for any $i \in \{1, 2, \cdots, 16\}$, we mean the underlying graph of the ith graph depicted in Figure 1. Moreover, G will refer to an underlying push clique in this section.

Before starting the analysis, we want to define the following terminology. Let u, v be a pair of vertices of an undirected simple graph G. We say that u *reaches* v if either u and v are adjacent or u and v have at least 2 common neighbors. Furthermore, by u *reaches* v *through* w, we will mean that w is a common neighbor of u and v. Also if each pair of vertices of G reaches each other, then we say that G is *reach-complete*.

First we will handle the case $|V(G)| \leq 4$.

Lemma 5.1. *If G is an underlying push clique having $|V(G)| \leq 4$, then G contains H_1, H_2, H_3 or H_4 as its spanning subgraph.*

Proof. The statement follows from Observation 2.5. □

We will also set the following conventions for this section. The graph $C_k = a_0a_1 \cdots a_{k-1}a_0$ denotes the k-cycle for all $k \geq 5$. For all $k \geq 5$, a *short chord* of C_k is a chord joining vertices at distance 2 and is denoted by $s_i = a_ia_{i+2}$ where the + operation is taken modulo k. For all $k \geq 6$, a *medium chord* of C_k is a chord joining vertices at distance 3 and is denoted by $m_i = a_ia_{i+3}$ where the + operation is taken modulo k. For all $k \geq 8$, a *long chord* of C_k is a chord joining vertices at distance 4 and is denoted by $l_i = a_ia_{i+4}$ where the + operation is taken modulo k.

Now we will consider the case $|V(G)| = 5$.

Lemma 5.2. *If G is an underlying push clique having $|V(G)| = 5$, then G contains H_5 as its spanning subgraph.*

Proof. If $|V(G)| = 5$, G cannot be a complete graph as G is planar. Thus G must contain a 4-cycle by Lemma 2.1. The fifth vertex of G must be adjacent to at least two vertices of the 4-cycle. Thus, G either contains a $K_{2,3}$ or must contain a 5-cycle.

Note that $K_{2,3}$ is not an underlying push clique. If we replace the partite set containing two vertices with an edge, then also the graph obtained is not an underlying push clique. But if we add an edge in the other partite set, a 5-cycle is created. Thus, G must contain a 5-cycle C_5.

Note that C_5 with two incident chords s_0 and s_2 is not an underlying push clique as s_1 does not reach s_3. A 5-cycle and a 5-cycle with a single chord is a subgraph of the above mentioned graph, thus are not underlying push cliques. Therefore, to obtain an underlying push clique from C_5 we need at least two non-incident short chords. The graph we get is H_5. $\square$

Nandy, Sen and Sopena [13] provided a list of edge-minimal outerplanar underlying oriented cliques (see Figure 2 of [13]). If a planar underlying push clique G has a dominating vertex v, then Lemma 2.1 implies that the graph $G[V(G) \setminus \{v\}]$ obtained by deleting the vertex v from G is an outerplanar underlying oriented clique. Thus it must contain one of the graphs depicted in Figure 2 of [13] as a spanning subgraph.

Let G_x be the graph obtained by adding a dominating vertex to the graph depicted in Figure 2(x) of [17] where $x \in \{a, b, \cdots, k\}$. By the previous arguments, the following observation holds:

Observation 5.3. *If $\sqsubseteq$ denotes spanning subgraph inclusion, then $H_2 \sqsubseteq G_a$, $H_3 \sqsubseteq G_b$, $H_4 \sqsubseteq G_c$, $H_5 \sqsubseteq G_d$, $H_5 \sqsubseteq G_e$, $H_8 \sqsubseteq G_f$, $H_9 \sqsubseteq G_g$, $H_6 \sqsubseteq G_h$, $H_{10} \sqsubseteq G_i$, $H_{11} \sqsubseteq G_j$, $H_{14} \sqsubseteq G_k$.*

Then we prove a general result to show that every planar underlying push clique with at least 6 vertices must have minimum degree at least 3 unless it contains H_6 as a spanning subgraph.

Lemma 5.4. *If G is an underlying planar push clique having $|V(G)| \geq 6$ and has minimum degree 2, then G contains H_6 as a spanning subgraph.*

Proof. Let G be an underlying planar push clique having $|V(G)| \geq 6$ with minimum degree 2. Let v be a vertex of G having exactly two neighbors x and y. Then each non-neighbor of v must be adjacent to both x and y, resulting in a $K_{2,|V(G)|-2}$. Given any orientation of G we can push x and y in such a way that we have the arcs $\overrightarrow{xv}$ and $\overrightarrow{vy}$. Thus, for being a push clique, x and y must agree on each non-neighbor of v by Corollary 2.2. Therefore, it is possible to push the non-neighbors of v (if necessary) to obtain an orientation of G such that all non-neighbors of v are out-neighbors of both x and y. Let this so-obtained orientation of G be $\overrightarrow{G}$. Now note that the only way for $\overrightarrow{G}$ to be a planar push clique is to have the non-neighbors of v induce a 2-dipath. In that case, $|V(G)| - 2 = 4$ and the underlying graph of $\overrightarrow{G}$ contains H_6 as a subgraph. $\square$

After this we prove a useful lemma.

Lemma 5.5. *If G is a planar underlying push clique having $|V(G)| = 6$, then G has a Hamiltonian cycle.*

Proof. If $|V(G)| = 6$, then G has minimum degree at least 2 due to Observation 2.3. If G has minimum degree 2, then G contains H_6 as a subgraph and hence has a Hamiltonian cycle. If G has minimum degree at least 3, then also G has a Hamiltonian cycle due to Dirac's Theorem [18]. □

After that we characterize all edge-minimal planar underlying push cliques having minimum degree at least 3 on 6 vertices.

Lemma 5.6. *If G is an underlying push clique having $|V(G)| = 6$ and the minimum degree of G is at least 3, then G contains H_6, H_7, H_8 or H_9 as its spanning subgraph.*

Proof. Now we will try to construct a Hamiltonian planar reach-complete graph G on 6 vertices, without any dominating vertex, with minimum degree 3, and not containing H_6, H_7, H_8 or H_9 as a subgraph. If such a graph G does not exist, then we are done due to Lemma 5.5, Observation 5.3, Lemma 5.4 and Observation 2.3. We show the non-existence of G through a case analysis.

Assume that G is C_6 having m medium chords and s short chords.

If $m = 0$, then we need to have $s \geq 4$ so that G does not have any vertex of degree at most 2. Furthermore, if $s \geq 4$, then either G has at least one vertex of degree 2 or $H_7 \sqsubseteq G$ or $H_9 \sqsubseteq G$.

If $m = 1$, then without loss of generality assume that the medium chord is m_0. Now to have $d(a_2), d(a_4) \geq 3$, we must either have s_0 and s_4 or have s_2. But the edges s_0 and s_4 create a dominating vertex a_0. Thus we must have the edge s_2. Similarly, to have $d(a_1), d(a_5) \geq 3$ and to avoid creating a dominating vertex a_3, we must have the edge s_1. This implies $H_7 \sqsubseteq G$.

If $m = 2$, then without loss of generality assume that the medium chords are m_0 and m_1. Note that to have $d(a_5) \geq 3$, without loss of generality we must have the edge s_5. Also to have $d(a_2) \geq 3$, we must either have the edge s_0 or s_2. In either case $H_7 \sqsubseteq G$.

If $m = 3$, then G is not planar as it contains a $K_{3,3}$. □

After that we continue with the following observation.

Observation 5.7. *Each edge of the graphs H_6, H_7, H_8, H_9 is part of a Hamiltonian cycle.*

Using Observation 5.7 we will show that every planar underlying push clique on 7 vertices must be Hamiltonian.

Lemma 5.8. *If G is a planar underlying push clique having $|V(G)| = 7$, then G has a Hamiltonian cycle.*

Proof. If $|V(G)| = 7$, then each vertex of G must have degree at least 3 by Lemma 5.4. If G has minimum degree 4 then G is Hamiltonian by Dirac's Theorem [18].

Assume the contrary, and let v be a degree 3 vertex of G. Delete the vertex v from G and add the edges among its neighbors to obtain a graph G'. Note that as G is an underlying planar push clique on 7 vertices, G' must be an underlying planar push clique on 6 vertices.

Thus, by Lemma 5.6 G' must contain one of H_6, H_7, H_8, H_9 as its spanning subgraph. If that spanning underlying push clique of G' contains one of the edges among the neighbors of v, then G is Hamiltonian using Observation 5.7.

As the graphs H_7, H_8, H_9 have independence number 2, we will be done if G' has one of these graphs as a spanning subgraph.

The graph H_6 has independence number 3 and has exactly one independent set of cardinality 3. If we add a vertex to the graph and make it adjacent to those three vertices, then a $K_{3,3}$ is created and thus the so-obtained graph is not planar.

Therefore, we can conclude that G is Hamiltonian. □

Using Lemma 5.8 we now prove the following:

Lemma 5.9. *If G is an underlying push clique having $|V(G)| = 7$, then G contains H_{10}, H_{11} or H_{12} as its spanning subgraph.*

Proof. We will try to construct a Hamiltonian planar reach-complete graph G on 7 vertices, without any dominating vertex, with minimum degree 3, not containing H_{10}, H_{11} or H_{12} as a subgraph. If such a graph G does not exist, then we are done due to Lemma 5.8, Observation 5.3, Lemma 5.4 and Observation 2.3. We will show that such a graph G does not exist through a case analysis.

Assume that G is C_7 having m medium chords and s short chords.

If $m = 0$, then to make G reach-complete we need to have all the short chords, thus a K_5-minor.

If $m = 1$, then without loss of generality assume that the medium chord is m_0. Now we must add s_2 and s_6 to make G reach-complete. Also we add s_5 without loss of generality to have $d(a_5) \geq 3$. This graph has a Hamiltonian cycle $a_0a_3a_4a_2a_1a_6a_5a_0$ with 2 medium chords a_0a_1 and a_4a_5. Thus this case gets reduced to the case $m \geq 2$.

If $m = 2$, then without loss of generality assume that one of the medium chords is m_0. The second medium chord can be chosen in three ways (up to symmetry):

- The second medium chord is m_4. Observe that it is not possible to make this graph reach-complete by adding short chords without creating a dominating vertex or a K_5-minor.

- The second medium chord is m_5. Thus we must have s_2 for a_2 to reach a_4 and s_4 for a_4 to reach a_6. This graph has a Hamiltonian cycle $a_0a_3a_4a_2a_1a_5$ a_6a_0 with 3 long chords a_0a_1, a_4a_5 and a_4a_6. Thus this case gets reduced to the case $m \geq 3$.

- The second medium chord is m_1. Without loss of generality we may add the short chord s_0 for having $d(a_2) \geq 3$. Observe that there are exactly 4 ways to make this graph reach-complete by adding short chords without creating a dominating vertex or a K_5-minor: (i) by adding s_0, s_2, s_3 and s_6 implying $H_{12} \sqsubseteq G$, (ii) by adding s_0, s_2, s_4 and s_5 implying $H_{11} \sqsubseteq G$, (iii) by adding s_0, s_3, s_4 and s_5 implying $H_{10} \sqsubseteq G$, (iv) by adding s_0, s_3, s_5 and s_6 implying $H_{12} \sqsubseteq G$.

If $m = 3$, then three non incident medium chords will create a $K_{3,3}$-minor. Thus we can assume that G has two incident medium chords m_0 and m_4. Without loss of generality the choice of the third chord gives us 3 subcases.

- The third medium chord is m_1. Observe that it is not possible to make this graph reach-complete by adding short chords without creating a dominating vertex or a K_5-minor.
- The third medium chord is m_2. Observe that there are exactly 2 ways to make this graph reach-complete by adding short chords without creating a dominating vertex or a K_5-minor: (i) by adding s_2 and s_6 implying $H_{12} \sqsubseteq G$, (ii) by adding s_4 and s_6 implying $H_{12} \sqsubseteq G$.
- The third medium chord is m_5. Observe that there are exactly 2 ways to make this graph reach-complete by adding short chords without creating a dominating vertex or a K_5-minor: (i) by adding s_0, s_1 and s_2 implying $H_{10} \sqsubseteq G$, (ii) by adding s_1, s_2 and s_6 implying $H_{11} \sqsubseteq G$.

If $m = 4$, then three non-incident medium chords will create a $K_{3,3}$-minor. Thus barring those cases, without loss of generality we have two cases.

- Suppose the four medium chords are m_0, m_1, m_4 and m_5. To have $d(a_2)$, $d(a_6) \geq 3$ we must add a matching of size 2 in the set $\{a_1a_6, a_4a_6, a_0a_2, a_4a_2\}$ of short chords. Note that s_5 implies a dominating vertex a_0 and the edge s_1 implies a dominating vertex a_1. Suppose that we have s_6 and s_0. Then a_3 must reach a_5 by being adjacent s_3, creating a K_5-minor.
- Suppose the four medium chords are m_0, m_2, m_5, m_6. To have $d(a_1) \geq 3$ without loss of generality we have the edge s_6. Also if we add either s_1 or s_4, then $H_{12} \sqsubseteq G$. Thus a_1 reaches a_4 through a_2 implying the edge s_2 and $H_{10} \sqsubseteq G$.

If $l \geq 5$, then G is not planar as it contains a $K_{3,3}$. □

Finally we turn our attention to the case $|V(G)| = 8$. We start with two important observations.

Observation 5.10. *Each edge of the graphs H_{10}, H_{12} is part of a Hamiltonian cycle.*

Before stating the next observation, recall the graph H_{11} depicted in Figure 1. Note that H_{11} has exactly 2 vertices of degree 5 and one vertex of degree 4. Without loss of generality call the two vertices of degree 5 as x and z. Furthermore, call the vertex of degree 4 as y. Based on these names, we have the following:

Observation 5.11. *Each edge, except xy and yz, of the graph H_{11} is part of a Hamiltonian cycle.*

Using the above observations we will show that every planar underlying push clique on 8 vertices must be Hamiltonian.

Lemma 5.12. *If G is a planar underlying push clique having $|V(G)| = 8$, then G has a Hamiltonian cycle.*

Proof. If $|V(G)| = 8$, then each vertex of G must have degree at least 3 by Lemma 5.4. If G has minimum degree 4 then G is Hamiltonian by Dirac's Theorem [18].

Otherwise, let v be a degree 3 vertex of G. Delete the vertex v from G and add the edges among its neighbors to obtain the graph G'. Note that as G is an underlying planar push clique on 8 vertices, G' must be an underlying planar push clique on 7 vertices.

Thus, by Lemma 5.9 G' must contain one of H_{10}, H_{11} or H_{12} as its spanning subgraph. If that spanning underlying push clique of G' contains one of the edges, other than xy or yz of H_{11}, among the neighbors of v, then G is Hamiltonian using Observation 5.7.

The graphs H_{10}, H_{12} has independence number 3 and has exactly one independent set of cardinality 3. If we add a vertex to the graph and make it adjacent to those three vertices, then a $K_{3,3}$ is created and thus the so-obtained graph is not planar. Thus we will be done if G' contains one of H_{10}, H_{12} as a spanning subgraph.

The graph H_{11} has independence number 3 and has exactly two independent sets of cardinality 3. If we add a vertex to the graph and make it adjacent to the vertices of one of its independent sets of cardinality 3, then a $K_{3,3}$-minor is created and thus the so-obtained graph is not planar.

Now assume that G' contains H_{11} as a spanning subgraph. If $N(v)$ contains an edge of H_{11} other than xy and yz, then by Observation 5.11 we are done. Otherwise, we must have $N(v) = \{x, y, z\}$ as x and z are adjacent to all vertices from $V(H_{11}) \setminus \{x, z\}$. In that case, we have a $K_{3,3}$ in G'. Thus $N(v) = \{x, y, z\}$ is not possible.

Therefore, we can conclude that G is Hamiltonian. □

Now we prove another result regarding the two graphs A and B depicted in Figure 7.

Lemma 5.13. *The two graphs A and B (depicted in Figure 7) are not underlying push cliques.*

Proof. Note that A and B are both planar graphs. Also A is a subgraph of B.

The vertex a (see Figure 7) of B is a degree-3 vertex whose neighbors are pairwise adjacent. Therefore, if B is an underlying push clique, then the graph B' obtained by deleting the vertex a from B is also an underlying push clique.

However, B' is a planar graph on 7 vertices. Thus due to Lemma 5.9 B' must contain H_{10}, H_{11} or H_{12} as a spanning subgraph. This is not possible as

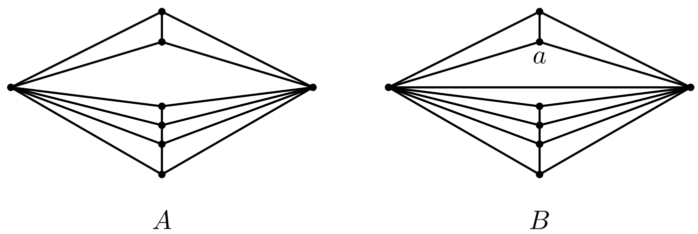

Figure 7: Two graphs: A and B.

B' contains a vertex of degree 2 while the minimum degree vertex of H_{10}, H_{11} and H_{12} is at least 3.

Therefore, B is not an underlying push clique. As A is a subgraph of B, the graph A is also not an underlying push clique. $\square$

Using Lemma 5.12 we will prove the following:

Lemma 5.14. *If G is an underlying push clique having $|V(G)| = 8$, then G contains H_{13}, H_{14}, H_{15} or H_{16} as its spanning subgraph.*

Proof. Now we will try to construct a Hamiltonian planar reach-complete graph G on 8 vertices, without any dominating vertex, with minimum degree 3, not containing H_{10}, H_{11} or H_{12} as a subgraph, and non-isomorphic to A or B. If such a graph G does not exist, then we are done due to Lemma 5.12, Observation 5.3, Lemma 5.4 and Observation 2.3. We will show that such a graph G does not exist through a case analysis.

Assume that G is C_8 having l long chords, m medium chords and s short chords. Let $X^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q}$ be the graph obtained by adding $l_{i_1}, l_{i_2}, \cdots, l_{i_p}$ and $m_{j_1}, m_{j_2}, \cdots, m_{j_q}$ to C_8 where $i_1 < i_2 < \cdots < i_p$ and $j_1 < j_2 < \cdots < j_q$. Moreover $\hat{X}^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q}$ will denote a reach-complete graph obtained from $X^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q}$ by adding short chords having no dominating vertex. Note that $\hat{X}^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q}$ may not be unique for a particular $X^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q}$. In particular, X is nothing but the graph C_8. Similarly, if X does not have a superscript/subscript, then it denotes a graph without any long/medium chords, respectively.

We want to do an exhaustive case analysis with respect to the number of long and medium chords in G and consider only the cases which are distinct up to reflectional/rotational symmetry (denoted by $\cong$). The first long/medium chord can be assumed to be l_0/m_0 without loss of generality. First we will list out the distinct cases to be considered in the following.

For $l + m \leq 1$, the cases to consider are X, X^0, X_0. From this we will recursively build the other cases by adding a medium or a long chord and present the cases in Table 1.

In Table 1 the GRAY entries contain a $K_{3,3}$-minor and thus are not planar. On the other hand, let $\mathbb{B}$ be the set of the **BOLDFACE** entries of the table.

$l:m$	Add a chord	Cases
$0:2$	medium to X_0	$\boldsymbol{X_{01}}$ ($\cong \boldsymbol{X_{07}}$), X_{02} ($\cong X_{06}$), $\boldsymbol{X_{03}}$ ($\cong \boldsymbol{X_{05}}$), X_{04}.
$0:3$	medium to case $(0,2)$	X_{012} ($\cong X_{017}$), X_{013} ($\cong X_{016}, X_{023}, X_{027}$), $\boldsymbol{X_{014}}$ ($\cong \boldsymbol{X_{015}, X_{034}, X_{037}, X_{045}, X_{047}}$), $\boldsymbol{X_{024}}$ ($\cong \boldsymbol{X_{026}, X_{046}}$), $\boldsymbol{X_{025}}$ ($\cong \boldsymbol{X_{035}, X_{036}}$).
$0:4$	medium to case $(0,3)$	X_{0134}, $\boldsymbol{X_{0135}}$ ($\cong \boldsymbol{X_{0146}, X_{0245}, X_{0247}, X_{0256}}$), $\boldsymbol{X_{0136}}$ ($\cong \boldsymbol{X_{0235}, X_{0257}}$), X_{0137}, X_{0145}, X_{0147}, X_{0246}.
$1:1$	long to X_0	X_0^0, $\boldsymbol{X_0^1}$.
$1:2$	long to case $(0,2)$	$\boldsymbol{X_{01}^0}$, $\boldsymbol{X_{01}^1}$ ($\cong \boldsymbol{X_{01}^3}$), X_{01}^2, $\boldsymbol{X_{02}^0}$ ($\cong \boldsymbol{X_{02}^1}$), $\boldsymbol{X_{02}^2}$ ($\cong \boldsymbol{X_{02}^3}$), $\boldsymbol{X_{03}^0}$ ($\cong \boldsymbol{X_{03}^2}$), X_{03}^1, $\boldsymbol{X_{03}^3}$, $\boldsymbol{X_{04}^0}$ ($\cong \boldsymbol{X_{04}^3}$), $\boldsymbol{X_{04}^1}$ ($\cong \boldsymbol{X_{04}^2}$).
$1:3$	long to case $(0,3)$	$\boldsymbol{X_{013}^0}$, $\boldsymbol{X_{013}^1}$, X_{013}^2, $\boldsymbol{X_{013}^3}$, $\boldsymbol{X_{014}^0}$, $\boldsymbol{X_{014}^1}$, X_{014}^2, $\boldsymbol{X_{014}^3}$, $\boldsymbol{X_{024}^0}$ ($\cong \boldsymbol{X_{024}^3}$), $\boldsymbol{X_{024}^1}$ ($\cong \boldsymbol{X_{024}^2}$), $\boldsymbol{X_{025}^0}$ ($\cong \boldsymbol{X_{025}^1}$), $\boldsymbol{X_{025}^2}$ ($\cong \boldsymbol{X_{025}^3}$).
$1:4$	long to case $(0,4)$	$\boldsymbol{X_{0134}^0}$ ($\cong \boldsymbol{X_{0134}^3}$), X_{0134}^1 ($\cong X_{0134}^2$), $\boldsymbol{X_{0135}^0}$, $\boldsymbol{X_{0135}^1}$, X_{0135}^2, $\boldsymbol{X_{0135}^3}$, X_{0136}^0, X_{0136}^1 ($\cong X_{0136}^3$), X_{0136}^2, X_{0145}^0, X_{0145}^1 ($\cong X_{0145}^3$), X_{0145}^2, X_{0246}^0 ($\cong X_{0246}^1, X_{0246}^2, X_{0246}^3$).
$2:0$	long to X^0	X^{01} ($\cong X^{03}$), $\boldsymbol{X^{02}}$.
$2:1$	long to case $(1,1)$	$\boldsymbol{X_0^{01}}$, $\boldsymbol{X_0^{02}}$ ($\cong \boldsymbol{X_0^{13}}$), $\boldsymbol{X_0^{03}}$, X_0^{12}.
$2:2$	long to case $(1,2)$	$\boldsymbol{X_{01}^{01}}$, X_{01}^{02}, $\boldsymbol{X_{01}^{03}}$, X_{01}^{12}, $\boldsymbol{X_{01}^{13}}$, X_{02}^{01}, $\boldsymbol{X_{02}^{02}}$, X_{02}^{03}, X_{02}^{12}, X_{02}^{23}, $\boldsymbol{X_{03}^{01}}$, X_{03}^{02}, $\boldsymbol{X_{03}^{03}}$ ($\cong \boldsymbol{X_{03}^{23}}$), X_{03}^{12}, X_{03}^{13}, $\boldsymbol{X_{04}^{01}}$, $\boldsymbol{X_{04}^{02}}$ ($\cong \boldsymbol{X_{04}^{13}}$), $\boldsymbol{X_{04}^{03}}$, X_{04}^{12}.
$2:3$	long to case $(1,3)$	X_{013}^{01}, X_{013}^{02}, $\boldsymbol{X_{013}^{03}}$, X_{013}^{12}, X_{013}^{13}, X_{013}^{23}, $\boldsymbol{X_{014}^{01}}$, X_{014}^{02}, $\boldsymbol{X_{014}^{03}}$, X_{014}^{12}, $\boldsymbol{X_{014}^{13}}$, X_{014}^{23}, X_{024}^{01}, X_{024}^{02} ($\cong X_{024}^{13}$), X_{024}^{03}, X_{024}^{12}, X_{025}^{01}, $\boldsymbol{X_{025}^{02}}$, X_{025}^{03}, X_{025}^{12}, X_{025}^{23}.
$2:4$	long to case $(1,4)$	X_{0134}^{01}, X_{0134}^{02}, X_{0134}^{03}, X_{0135}^{01}, X_{0135}^{02}, X_{0135}^{03}, X_{0135}^{12}, X_{0135}^{13}, X_{0135}^{23}, X_{0136}^{01}, X_{0136}^{12}, X_{0136}^{13}, X_{0145}^{01} ($\cong X_{0145}^{03}$), X_{0145}^{12}, X_{0145}^{13}, X_{0145}^{02}, X_{0246}^{01}, X_{0246}^{02}, X_{0246}^{03}.

Table 1: Exhaustiveness of the case analysis of the graph $X_{j_1 j_2 \cdots j_q}^{i_1 i_2 \cdots i_p}$ obtained by adding long and medium chords.

Observe that, if $Y \in \mathbb{B}$, then $\hat{Y}$ contains either a dominating vertex, or a K_5-minor, or a $K_{3,3}$-minor.

For any of the other entries, say Z, $\hat{Z}$ can be obtained in number of ways. We have observed that for a particular Z, its associated $\hat{Z}$'s can contain a number of structures proving our result. We have clubbed the same types together and have described the remaining cases below:

- Each of $\hat{X}, \hat{X}_0, \hat{X}_{02}, \hat{X}_{04}, \hat{X}_{013}, \hat{X}^0, \hat{X}_0^0, \hat{X}_{03}^0, \hat{X}^{01}, \hat{X}_{03}^{13}, \hat{X}_{024}^{02}, \hat{X}_{0135}^{13}$ contain a K_5-minor or H_{13}.

- Each of $\hat{X}_{0145}, \hat{X}^{0}_{0145}, \hat{X}^{1}_{0145}, \hat{X}^{03}_{0134}, \hat{X}^{13}_{0136}, \hat{X}^{01}_{0145}, \hat{X}^{13}_{0145}$ contain a K_5-minor or H_{14}.
- Each of $\hat{X}_{0134}, \hat{X}^{1}_{0136}, \hat{X}^{23}_{02}, \hat{X}^{13}_{013}$ contain a K_5-minor or H_{13} or H_{15}.
- Each of $\hat{X}_{0246}, \hat{X}^{0}_{0246}, \hat{X}^{02}_{0246}$ contain a K_5-minor or H_{13} or H_{16}.
- Each of $\hat{X}^{01}_{02}, \hat{X}^{01}_{024}$ is either isomorphic to A, or contain a K_5-minor or H_{14}.
- Each of $\hat{X}^{01}_{025}, \hat{X}^{03}_{0135}$ is either isomorphic to B, or contain a K_5-minor or H_{14}.
- The graph $\hat{X}^{01}_{0246}$ is either isomorphic to A, or contains K_5-minor, H_{13}, H_{14} or H_{15}.

If $m \geq 5$ or $l \geq 3$, G contains a $K_{3,3}$-minor. □

Proof of Theorem 1.3. The proof directly follows from Lemmas 5.1, 5.2, 5.6, 5.9 and 5.14. □

6. Conclusions

We listed all minimal planar underlying push cliques upto spanning subgraph inclusion. One can notice that there are 4 distinct minimal planar underlying push cliques of maximum order (eight vertices). This result is unlike the case of planar underlying oriented clique where there is a unique planar underlying oriented clique of maximum order [13].

Moreover, we would like to report that there are total 55 non-isomorphic planar underlying push cliques (1 on 1 vertex, 1 on 2 vertices, 1 on 3 vertices, 3 on 4 vertices, 4 on 5 vertices, 11 on 6 vertices, 21 on 7 vertices and 13 on 8 vertices). See the lists in [19] for details.

The planar oriented cliques were instrumental in improving the bound of oriented chromatic number of planar graphs. Thus we hope that our list will help studies related to pushable chromatic number of oriented planar graphs.

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