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► **To cite this version:**

Julien Bensmail, Martin Merker, Carsten Thomassen. Decomposing graphs into a constant number of locally irregular subgraphs. *European Journal of Combinatorics*, 2017, 60, pp.124 - 134. 10.1016/j.ejc.2016.09.011 . hal-01629938

HAL Id: hal-01629938

<https://hal.science/hal-01629938>

Submitted on 7 Nov 2017

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Decomposing graphs into a constant number of locally irregular subgraphs

Julien Bensmail*

Martin Merker*

Carsten Thomassen*

Department of Applied Mathematics and Computer Science
Technical University of Denmark
DK-2800 Lyngby, Denmark

September 21, 2016

Abstract

A graph is *locally irregular* if no two adjacent vertices have the same degree. The *irregular chromatic index* $\chi'_{\text{irr}}(G)$ of a graph G is the smallest number of locally irregular subgraphs needed to edge-decompose G . Not all graphs have such a decomposition, but Baudon, Bensmail, Przybyło, and Woźniak conjectured that if G can be decomposed into locally irregular subgraphs, then $\chi'_{\text{irr}}(G) \leq 3$. In support of this conjecture, Przybyło showed that $\chi'_{\text{irr}}(G) \leq 3$ holds whenever G has minimum degree at least 10^{10} .

Here we prove that every bipartite graph G which is not an odd length path satisfies $\chi'_{\text{irr}}(G) \leq 10$. This is the first general constant upper bound on the irregular chromatic index of bipartite graphs. Combining this result with Przybyło's result, we show that $\chi'_{\text{irr}}(G) \leq 328$ for every graph G which admits a decomposition into locally irregular subgraphs. Finally, we show that $\chi'_{\text{irr}}(G) \leq 2$ for every 16-edge-connected bipartite graph G .

1 Introduction

A graph G is *locally irregular* if any two of its adjacent vertices have distinct degrees. An edge-weighting of G is called *neighbour-sum-distinguishing*, if for every two adjacent vertices of G the sums of their incident weights are distinct. The least number k for which G admits a neighbour-sum-distinguishing edge-weighting using weights $1, 2, \dots, k$ is denoted $\chi'_{\Sigma}(G)$.

Karoński, Łuczak, and Thomason [5] made the following conjecture.

Conjecture 1.1 (1-2-3 Conjecture [5]). *For every graph G with no component isomorphic to K_2 , we have $\chi'_{\Sigma}(G) \leq 3$.*

This conjecture is equivalent to stating that a graph can be made locally irregular by replacing some of its edges by two or three parallel edges. Although the 1-2-3 Conjecture has received considerable attention in the last decade, it is still an open question. The best result so far was shown by Kalkowski, Karoński, and Pfender [4] who proved $\chi'_{\Sigma}(G) \leq 5$

*The authors were supported by ERC Advanced Grant GRACOL, project no. 320812.

Email addresses: julien.bensmail.phd@gmail.com (Julien Bensmail), martin.merker@uni-hamburg.de (Martin Merker), ctho@dtu.dk (Carsten Thomassen).

whenever G has no component isomorphic to K_2 . For more details, we refer the reader to the survey by Seamone [8] on the 1-2-3 Conjecture and related problems.

If a graph G is regular, then G admits a neighbour-sum-distinguishing 2-edge-weighting if and only if G can be edge-decomposed into two locally irregular subgraphs. Motivated by this connection, Baudon, Bensmail, Przybyło, and Woźniak [1] asked the more general question when a graph can be edge-decomposed into locally irregular subgraphs, and how many locally irregular subgraphs are needed. From now on, all graphs we consider are simple and finite. A decomposition into locally irregular subgraphs can be regarded as an improper edge-colouring where each colour class induces a locally irregular graph. We call such an edge-colouring *locally irregular*. If G admits a locally irregular edge-colouring, then we call G *decomposable*. For every decomposable graph G , we define the *irregular chromatic index of G* , denoted by $\chi'_{\text{irr}}(G)$, as the least number of colours in a locally irregular edge-colouring of G . If G is not decomposable, then $\chi'_{\text{irr}}(G)$ is not defined and we call G *exceptional*. The following conjecture has a similar flavour to the 1-2-3 Conjecture.

Conjecture 1.2 ([1]). *For every decomposable graph G , we have $\chi'_{\text{irr}}(G) \leq 3$.*

Every connected graph of even size can be decomposed into paths of length 2 and is thus decomposable. Hence, all exceptional graphs have odd size and a complete characterisation of exceptional graphs was given by Baudon, Bensmail, Przybyło, and Woźniak [1]. To state this characterisation, we first need to define a family \mathcal{T} of graphs. The definition is recursive:

- The triangle K_3 belongs to \mathcal{T} .
- Every other graph in \mathcal{T} can be constructed by 1) taking an auxiliary graph F being either a path of even length or a path of odd length with a triangle glued to one of its ends, then 2) choosing a graph $G \in \mathcal{T}$ containing a triangle with at least one vertex, say v , of degree 2 in G , and finally 3) identifying v with a vertex of degree 1 of F .

In other words, the graphs in \mathcal{T} are obtained by connecting a collection of triangles in a tree-like fashion, using paths with certain lengths, depending on what elements these paths connect. Let us point out that all graphs in \mathcal{T} have maximum degree 3, have odd size, and all of their cycles are triangles.

Theorem 1.3 ([1]). *A connected graph is exceptional, if and only if it is (1) a path of odd length, (2) a cycle of odd length, or (3) a member of \mathcal{T} .*

The number 3 in Conjecture 1.2 cannot be decreased to 2, since $\chi'_{\text{irr}}(G) = 3$ if G is a complete graph or a cycle with length congruent to 2 modulo 4. Baudon, Bensmail, Przybyło, and Woźniak [1] verified Conjecture 1.2 for several classes of graphs such as trees, complete graphs, and regular graphs with degree at least 10^7 . Baudon, Bensmail, and Sopena [2] showed that determining the irregular chromatic index of a graph is NP-complete in general, and that, although infinitely many trees have irregular chromatic index 3, the same problem for trees can be solved in linear time. More recently, Przybyło [7] gave further evidence for Conjecture 1.2 by verifying it for graphs of large minimum degree.

Theorem 1.4 ([7]). *If a graph G has minimum degree at least 10^{10} , then $\chi'_{\text{irr}}(G) \leq 3$.*

Despite this result, Conjecture 1.2 is still wide open, even in much weaker forms. Until now it was not known whether there exists a constant c such that $\chi'_{\text{irr}}(G) \leq c$ holds for every decomposable graph G . This was also an open problem when restricted to bipartite graphs, see [1, 2, 3, 7].

In this paper we show that $\chi'_{\text{irr}}(G) \leq 328$ for every decomposable graph G , hence providing the first constant upper bound on the irregular chromatic index. The proof consists of the following three main steps.

- First, we show in Section 2 that we can restrict our attention to connected graphs of even size. Notice that every connected graph of even size can be decomposed into paths of length 2 and is thus decomposable. We show that every connected decomposable graph G of odd size contains a locally irregular subgraph H such that all components of $G - E(H)$ have even size.
- In Section 3, we investigate connected bipartite graphs G of even size and show that $\chi'_{\text{irr}}(G) \leq 9$ holds in this case.
- Finally, in Section 4 we decompose a connected graph G of even size into a graph H of minimum degree 10^{10} and a $(2 \cdot 10^{10} + 2)$ -degenerate graph D in which every component has even size. We use Theorem 1.4 to decompose H into three locally irregular subgraphs, and we further decompose D into 36 bipartite graphs with components of even size. By using our result for bipartite graphs, this results in a decomposition of G into $3 + 9 \cdot 36 = 327$ locally irregular subgraphs.

We have proved a constant upper bound on the irregular chromatic index, but there is still a significant gap between our result and the conjectured value 3. We conclude this article by showing in Section 5 that $\chi'_{\text{irr}}(G) \leq 2$ for every 16-edge-connected bipartite graph G .

2 Reduction to graphs of even size

In this section we show that the weakening of Conjecture 1.2, where 3 is replaced by a larger number, can be reduced to connected graphs of even size. More precisely, given a decomposable graph G with odd size, we can always remove a locally irregular subgraph H from G , so that all components of $G - E(H)$ have even size.

Lemma 2.1. *Let G be a connected graph of odd size. For every vertex $v \in V(G)$ there exists an edge e incident with v such that every component of $G - e$ has even size.*

Proof. Let $E(v)$ denote the set of edges incident with v . If $e \in E(v)$ is not a cut-edge, then $G - e$ is connected and of even size. We may thus assume that all edges in $E(v)$ are cut-edges. For every $e \in E(v)$, let H_e denote the component of $G - e$ not containing v . Now

$$E(G) = \bigcup_{e \in E(v)} E(H_e) \cup \{e\}.$$

Since $|E(G)|$ is odd, there exists $e \in E(v)$ for which $|E(H_e) \cup \{e\}|$ is odd. Thus, H_e is of even size, and so is the other component of $G - e$. \square

Lemma 2.2. *Let G be a connected graph of even size. For every vertex $v \in V(G)$ there exists a path P of length 2 containing v such that every component of $G - E(P)$ has even size.*

Proof. Let e be an edge incident with v . Then $G - e$ has precisely one component of odd size, and e is incident with a vertex u of that component, possibly $u = v$. By Lemma 2.1 we can delete an edge f incident with u so that every component of $G - \{e, f\}$ has even size. Since e and f are incident, they form a path P of length 2. \square

Theorem 2.3. *Let G be a connected graph of odd size. If G is decomposable, then G contains a locally irregular subgraph H such that every component of $G - E(H)$ has even size.*

Proof. We show that we can choose H to be isomorphic to $K_{1,3}$ or to $K_{1,3}$ in which two edges are subdivided once. Assume that G is a graph for which we cannot delete one of these two graphs so that every component in the resulting graph is of even size. If G has maximum degree at most 2 and odd size, then G is exceptional. We can thus assume that G has maximum degree at least 3. Notice that every vertex v of degree at least 3 in G must be a cutvertex, since otherwise we can delete a claw (i.e. a subgraph isomorphic to $K_{1,3}$) centred at v .

First, suppose that G contains a cycle C . Let V_C denote the vertices of C with degree at least 3. For every $v \in V_C$, let $E_C(v)$ denote the two edges of C that are incident with v . If $G - E_C(v)$ is connected, then we can use Lemma 2.1 to delete one more edge at v so that every component in the resulting graph has even size. We may thus assume that $G - E_C(v)$ is disconnected. Let $G_C(v)$ denote the component of $G - E_C(v)$ containing v . If $|E(G_C(v))|$ is odd, then we can again use Lemma 2.1 to delete one more edge at v to reach the desired conclusion. Thus we may assume that $|E(G_C(v))|$ is even for all $v \in V_C$. By Lemma 2.2, there exists a path P_v of length 2 in $G_C(v)$ incident with v such that every component of $G_C(v) - E(P_v)$ has even size. If v is the middle vertex of P_v , then P_v together with one of the two edges in $E_C(v)$ forms a claw whose removal leaves a graph with only components of even size. Thus, we may assume that v is an endvertex of P_v . If C has length at least 4, then let P_C be a path of length 3 in C in which v has degree 2. The graph $P_v \cup P_C$ is locally irregular and it is easy to see that every component of $G - E(P_v) - E(P_C)$ has even size.

Thus we may assume that all cycles of G have length 3. Suppose two cycles C_1, C_2 have a vertex v in common. Choose an edge e_i incident with v in C_i for $i \in \{1, 2\}$. Now $G - \{e_1, e_2\}$ is connected, so we can apply Lemma 2.1 to delete one more edge at v so that every component has even size.

So far, we have shown that triangles are the only cycles in G and that any two triangles are disjoint. Now we show that there exists no induced claw in G . Suppose for a contradiction that v is a vertex of degree at least 3 which is the center of an induced claw in G . If v is contained in a triangle, then we assume that the degree of v is at least 4. Since any two triangles are disjoint, there exists at most one edge between the neighbours of v . By Lemma 2.1, we can delete an edge uv so that every component of $G' = G - uv$ has even size. By our choice of v , there exists two neighbours u_1 and u_2 of v such that $\{u, u_1, u_2\}$ is an independent set in G . Let G_1 denote the component of odd size in $G' - u_1v$. If G_1 contains v , then we can delete a third edge e at v by Lemma 2.1 such that all components of $G' - u_1v - e$ have even size. Thus, we can assume that G_1 contains u_1 but not v . Similarly, we may assume that the odd component G_2 of $G' - u_2v$ contains u_2 but not v . Now we can apply Lemma 2.1 to delete an edge e_i incident with u_i in G_i such that every component of $G_i - e_i$ has even size for $i \in \{1, 2\}$. Thus, every component of $G' - e_1 - e_2 - u_1v - u_2v$ has even size. Since G_1, G_2 are distinct components of $G - v$, the graph we removed is

isomorphic to $K_{1,3}$ in which two edges are subdivided once. This contradicts our choice of G , implying that G has no induced claw.

Thus we may assume that the maximum degree in G is 3 and that every vertex of degree 3 is contained in a triangle. Since there are no other cycles, this implies that the contraction of all triangles results in a tree of maximum degree 3. All that remains to show is that the parities of the path lengths are the same as for the exceptional graphs. Let P be a path joining a vertex of degree 1 in G with a triangle C . Let v be the common vertex of P and C . Now $P = G_C(v)$ and since $|E(G_C(v))|$ is even, the length of P is even. Finally, let P be a path joining two different triangles C_1 and C_2 . If v_1 and v_2 denote the endvertices of P , then

$$|E(G)| = |E(G_{C_1}(v_1))| + |E(G_{C_2}(v_2))| - |E(P)|.$$

Since $|E(G_{C_1}(v_1))|$ and $|E(G_{C_2}(v_2))|$ are even and $|E(G)|$ is odd, we get that $|E(P)|$ must also be odd. This shows that G is exceptional. \square

3 Locally irregular decompositions of bipartite graphs

We now focus on the irregular chromatic index of bipartite graphs. Recall that the only bipartite exceptional graphs are odd length paths. In Corollary 3.10 we show $\chi'_{\text{irr}}(G) \leq 10$ for every decomposable bipartite graph G , which is the first constant upper bound on χ'_{irr} for bipartite graphs.

If all vertices in one partition class of the bipartite graph G have even degree, while the vertices in the other partition class have odd degree, then G is locally irregular. The idea of the proof is to remove some well-behaved subgraphs from G to obtain a graph which is very close to this structure. These well-behaved subgraphs include a particular kind of forest, which is defined as follows.

Definition 3.1. *We say a forest is **balanced** if it has a bipartition such that all vertices in one of the partition classes have even degree.*

Since a balanced forest cannot contain an odd length path as a component, it follows from [1] that $\chi'_{\text{irr}}(F) \leq 3$ for every balanced forest F . The characterisation of trees T with $\chi'_{\text{irr}}(T) \leq 2$ by Baudon, Bensmail, and Sopena [2] implies that $\chi'_{\text{irr}}(F) \leq 2$ holds for balanced forests F . For the sake of completeness, we present a short proof of this special case.

Lemma 3.2. *If F is a balanced forest, then F has a 2-colouring of the edges, such that each colour induces a locally irregular graph, and, for each vertex v in the partition class with no odd degree vertex, all edges incident with v have the same colour. In particular, $\chi'_{\text{irr}}(F) \leq 2$.*

Proof. The proof is by induction on the number of edges of F . Clearly, we may assume that F is connected. Let A and B be the partition classes of F , where all vertices in B have even degree. We may assume that some vertex in A has even degree since otherwise we can give all edges of F the same colour. Let v be a vertex in A of even degree q . We delete v but keep the edges incident with v and let them go to q new vertices v_1, v_2, \dots, v_q each of degree 1. In other words, we split F into q new trees T_1, T_2, \dots, T_q such that the union of their edges is the edge set of F . Each of the trees T_1, T_2, \dots, T_q is balanced and has therefore a colouring of its edges in colours red and blue satisfying the conclusion of

Lemma 3.2. This also gives a colouring of the edges of F in colours red and blue. By switchings colours in some of the T_i , if necessary, we can ensure that the red degree of v is 1. This shows that also F satisfies the conclusion of Lemma 3.2. \square

Apart from balanced forests we shall also delete a subgraph which is the union of a path and an induced cycle. The following lemma gives an upper bound on the irregular chromatic index in this case.

Lemma 3.3. *Let G be a bipartite graph and let v be a vertex in G . If G is the edge-disjoint union of an induced cycle C through v and a path P starting at v , then $\chi'_{\text{irr}}(G) \leq 4$.*

Proof. If the length of P is 0, then $\chi'_{\text{irr}}(G) \leq 3$, so we may assume P has positive length. First suppose that P has odd length. Let e denote the edge of P incident with v . It is easy to see that $\chi'_{\text{irr}}(C + e) \leq 2$. Thus,

$$\chi'_{\text{irr}}(G) \leq \chi'_{\text{irr}}(C + e) + \chi'_{\text{irr}}(P - e) \leq 2 + 2 = 4.$$

Now suppose the length of P is even. If the length of C is divisible by 4, then

$$\chi'_{\text{irr}}(G) \leq \chi'_{\text{irr}}(C) + \chi'_{\text{irr}}(P) \leq 2 + 2 = 4.$$

We may therefore assume the length of C is congruent to 2 modulo 4. Let e denote the edge of P incident with v , and let f denote the edge incident with e on P . It is easy to check that if e , f and all edges of C incident to e or f are coloured 1, then this colouring can be extended to a locally irregular $\{1, 2\}$ -edge-colouring of $C + e + f$. Thus, we have

$$\chi'_{\text{irr}}(G) \leq \chi'_{\text{irr}}(C + e + f) + \chi'_{\text{irr}}(P - e - f) \leq 2 + 2 = 4.$$

\square

The following lemma is well-known.

Lemma 3.4. *Let G be a connected graph and let S be a set of vertices. If S is even, then there exists a collection of $\frac{|S|}{2}$ edge-disjoint paths in G such that each vertex in S is an endvertex of precisely one of them, and the union of these paths forms a forest.*

Proof. Take a spanning tree T of G , and choose a collection of paths in T having the vertices in S as endvertices, and for which the total length is minimal. \square

Corollary 3.5. *If G is a connected bipartite graph of even size with partition classes A and B , then there exists a balanced forest F with leaves in A such that in $G - E(F)$ all vertices in A have even degree.*

Proof. Notice that since G has even size, the number of vertices in A with odd degree is even. The statement follows by choosing S to be the set of odd-degree vertices in A , and F as the union of the paths given by Lemma 3.4. \square

Corollary 3.6. *Let G be a connected bipartite graph with partition classes A and B , and let v be a vertex in B . If all vertices in A have even degree, then there exists a balanced forest F with leaves in B such that in $G - E(F)$ all vertices in $B \setminus \{v\}$ have odd degree.*

Proof. Choose S as the set of even-degree vertices in B . If $|S|$ is odd, then we apply Lemma 3.4 to the set $S \cup \{v\}$ or $S \setminus \{v\}$, and if $|S|$ is even we apply Lemma 3.4 to the set S . The union of the paths is the desired balanced forest. \square

Lemma 3.7. *Let G be a bipartite graph with partition classes A and B , and let v be a vertex in B . If all vertices in A have even degree and all vertices in $B \setminus \{v\}$ have odd degree, then there exists a path P starting in v such that $G - E(P)$ is locally irregular.*

Proof. If v has odd degree, then we can choose P as a path of length 0. If v has even degree and G is not locally irregular, then v is adjacent to a vertex u_1 of the same degree. We choose the edge vu_1 as the first edge of P and define $G_1 = G - vu_1$. If G_1 is not locally irregular, then u_1 is adjacent to a neighbour u_2 of the same degree. In this case we extend P by the edge u_1u_2 and define $G_2 = G_1 - u_1u_2$. We continue like this, defining G_{i+1} if G_i is not locally irregular by deleting a conflict edge u_iu_{i+1} . We claim that this process stops with a locally irregular graph G_k and that the deleted edges form a path. Notice that if G_i is not locally irregular, then u_i is incident to a vertex u_{i+1} of the same degree. Moreover, the degree of u_i in G_i is $d(v) - i$, so the degrees $d(u_i)$ form a decreasing sequence. In particular, $u_i \neq u_j$ for $i \neq j$ and $u_i \neq v$ for all i . Thus, eventually the process stops with a locally irregular graph G_k and $G - E(G_k)$ is a path of length k . \square

Lemma 3.8. *Let G be a bipartite graph with partition classes A and B . If all vertices in A have even degree, then $\chi'_{\text{irr}}(G) \leq 7$.*

Proof. We may assume that G is connected. By Lemma 3.6, we can delete a balanced forest F with leaves in B such that in the resulting graph G' there is at most one vertex of even degree in B , say v . If v does not exist or if v is an isolated vertex in G' , then G' is locally irregular and $\chi'_{\text{irr}}(G) \leq \chi'_{\text{irr}}(F) + \chi'_{\text{irr}}(G') \leq 3$. Thus, we may assume that v exists. Notice that G' might consist of several components, but every component not containing v is locally irregular. Let H denote the component of G' containing v .

If there exists no cycle through v in H , then all edges incident with v are cut-edges. Let e be an edge incident with v , and let H_1 and H_2 denote the two components of $H - e$. We may assume that H_1 contains v . Notice that the degree of v in H_1 and in $H_2 + e$ is odd, while the degrees of its neighbours are even. It follows that both H_1 and $H_2 + e$ are locally irregular and hence

$$\chi'_{\text{irr}}(G) \leq \chi'_{\text{irr}}(F) + \chi'_{\text{irr}}(H_1) + \chi'_{\text{irr}}(H_2 + e) \leq 4.$$

Thus, we may assume that there exists a cycle going through v . Let C be a cycle through v of shortest length and set $H' = H - E(C)$. Since the parities of the degrees remain unchanged, the vertex v is still the only vertex in B that could have positive even degree in H' , while all vertices in A have even degree. By Lemma 3.7, there exists a path P in H' starting in v such that $H' - E(P)$ is locally irregular. Now $\chi'_{\text{irr}}(C \cup P) \leq 4$ by Lemma 3.3 and we have

$$\chi'_{\text{irr}}(G) \leq \chi'_{\text{irr}}(F) + \chi'_{\text{irr}}(H' - E(P)) + \chi'_{\text{irr}}(C \cup P) \leq 2 + 1 + 4 = 7.$$

\square

We are now ready for the main result of this section.

Theorem 3.9. *If G is a connected bipartite graph of even size, then $\chi'_{\text{irr}}(G) \leq 9$.*

Proof. By Lemma 3.5, we can delete a balanced forest F of G so that the degrees in A in the resulting graph G' are even. By Lemma 3.2 we have $\chi'_{\text{irr}}(F) \leq 2$, and $\chi'_{\text{irr}}(G') \leq 7$ follows from Lemma 3.8. Thus $\chi'_{\text{irr}}(G) \leq \chi'_{\text{irr}}(F) + \chi'_{\text{irr}}(G') \leq 2 + 7 = 9$. \square

Corollary 3.10. *If G is a connected bipartite graph and not an odd length path, then $\chi'_{\text{irr}}(G) \leq 10$.*

Proof. Since paths of odd lengths are the only exceptional bipartite graphs, this follows immediately from Theorems 2.3 and 3.9. \square

4 Locally irregular decompositions of degenerate graphs

Here we apply the result from the previous section by decomposing degenerate graphs into bipartite graphs of even size. We show that every connected d -degenerate graph of even size can be decomposed into at most $\lceil \log_2(d+1) \rceil + 1$ bipartite graphs whose components all have even size. The proof makes repeated use of the following easy lemma.

Lemma 4.1. *If G is a graph with a vertex v such that $G - v$ is bipartite, then there exists a set E of at most $\lfloor \frac{d(v)}{2} \rfloor$ edges incident with v such that $G - E$ is bipartite.*

Proof. Since $G - v$ is bipartite, there exists a partition class containing at most $\lfloor \frac{d(v)}{2} \rfloor$ neighbours of v . Deleting all edges in G from v to these vertices results in a bipartite graph. \square

Lemma 4.2. *Let d be an even natural number, $\ell \geq \lceil \log_2 d \rceil + 1$, and v a vertex of degree d in a graph G . If $G - v$ is the edge-disjoint union of ℓ bipartite graphs in which every component has even size, then so is G .*

Proof. Notice that it suffices to prove the statement for $\ell = \lceil \log_2 d \rceil + 1$. We use induction on d . In the case $d = 2$ we colour $G - v$ with colours 1 and 2 so that the monochromatic components are bipartite subgraphs of even size. Let u_1, u_2 be the neighbours of v in G . If u_1 and u_2 are not connected by an odd length path in colour 1, then colouring both vu_1 and vu_2 with colour 1 will keep all monochromatic components bipartite and of even size. Thus, we may assume that u_1 and u_2 are connected by a monochromatic path of odd length in each colour. Let $P = v_0v_1 \dots v_k$ be a monochromatic path of odd length from u_1 to u_2 in colour 2, so $v_0 = u_1$ and $v_k = u_2$. Suppose that for every $i \in \{0, \dots, k-1\}$ there exists an even length path in colour 1 from v_i to v_{i+1} . By concatenating them, we get a walk of even length from v_0 to v_k . Since there is also a path of odd length joining v_0 and v_k in colour 1, this contradicts the assumption that the subgraph in colour 1 is bipartite. Thus, there exists $i \in \{0, \dots, k-1\}$ for which there is no even length path in colour 1 from v_i to v_{i+1} . Choose i minimal with this property. We change the colour of v_iv_{i+1} to colour 1. By the choice of i , all monochromatic components in colour 1 are still bipartite. Now there exists precisely one monochromatic component of odd size in each colour. Notice that the monochromatic component of odd size in colour 1 is incident with both u_1 and u_2 , while the one in colour 2 is incident with at least one of u_1 and u_2 . Thus, we can colour one of the edges at v with colour 2 so that all monochromatic components in colour 2 are bipartite and of even size. Colouring the other edge at v with colour 1 yields the desired decomposition.

Now suppose $d \geq 4$ and that the statement is true for all smaller even numbers. Set $d' = \frac{d}{2}$ if d is divisible by 4, and $d' = \frac{d}{2} + 1$ otherwise. Notice that d' is even and $\lceil \log_2 d \rceil = \lceil \log_2 d' \rceil + 1$. Let \mathcal{H} be the collection of $\lceil \log_2 d \rceil + 1$ bipartite graphs in $G - v$ with even component sizes. Choose $H \in \mathcal{H}$ and denote by G_H the graph we get by adding v and all its incident edges to H . By Lemma 4.1, there exists a set E of d' edges incident with v such that $G_H - E$ is bipartite. Since $d - d'$ is even, all components of $G_H - E$ have even size. We add

the edges in E to the union of the graphs in $\mathcal{H} \setminus \{H\}$ to obtain a graph G' . By the induction hypothesis, we can decompose G' into $\lceil \log_2 d' \rceil + 1$ bipartite graphs where every component has even size. Together with $G_H - E$, this is a collection of $\lceil \log_2 d' \rceil + 2 = \lceil \log_2 d \rceil + 1$ such graphs. \square

Notice that in general the bound $\lceil \log_2 d \rceil + 1$ cannot be decreased by more than 1. The complete graph K_{d+1} is d -degenerate and at least $\lceil \log_2(d+1) \rceil$ bipartite graphs are needed to decompose it. Moreover, we might need more bipartite graphs to achieve that all components have even size. For example, the complete graph K_4 can be decomposed into two bipartite graphs, but three bipartite graphs are necessary to achieve even component sizes.

Theorem 4.3. *Let $d \geq 1$ be a natural number. If G is a d -degenerate graph in which every component has even size, then G can be decomposed into $\lceil \log_2(d+1) \rceil + 1$ bipartite graphs in which all components have even size.*

Proof. Suppose not, and let G be a smallest counterexample. Clearly G is connected.

Claim. *If v is a cutvertex of G , then v is adjacent to precisely one vertex u of degree 1 and $G - u - v$ is connected.*

To prove the claim, suppose there exists a 1-separation $\{V_1, V_2\}$ of G with $V_1 \cap V_2 = \{v\}$ and $|V_1|, |V_2| \geq 3$. If $G[V_1]$ and $G[V_2]$ have even size, then we can decompose $G[V_1]$ and $G[V_2]$ by induction. If $G[V_1]$ and $G[V_2]$ have odd size, then we construct two new graphs H_1 and H_2 by adding a new vertex v_i to $G[V_i]$ together with the single edge vv_i . Since $|V_1|, |V_2| \geq 3$, both H_1 and H_2 are smaller than G so we can decompose them by induction. We think of the decomposition as an edge-colouring, and we permute colours so that the edges vv_i receive the same colour in both subgraphs. This corresponds to a colouring of G in which every monochromatic component is bipartite and of even size. This proves the claim.

In particular, every vertex is adjacent to at most one vertex of degree 1. Among all vertices of degree greater than 1, let v be one of minimal degree. Since G is d -degenerate, we have $d(v) \leq d+1$. Suppose first that $d(v)$ is even. Since G is a smallest counterexample, we can decompose $G - v$ into $\lceil \log_2(d+1) \rceil + 1$ bipartite graphs in which all components have even size. By Lemma 4.2, this gives rise to the desired decomposition of G .

We may thus assume that $d(v)$ is odd. Set $d' = \frac{1}{2}(d(v) - 1)$ if $d(v)$ is congruent to 1 modulo 4, and $d' = \frac{1}{2}(d(v) + 1)$ otherwise. Notice that d' is even and $\lceil \log_2(d+1) \rceil \geq \lceil \log_2 d' \rceil + 1$. Let u be a neighbour of v of degree greater than 1. If $G - v$ has an isolated vertex, then we let w denote that vertex. Otherwise we add an isolated vertex w . The graph $G - v + uw$ has even size, so we can decompose it as in the previous case. This gives us a decomposition of $G - v$ into $\lceil \log_2(d+1) \rceil + 1$ bipartite graphs in which all components are of even size, apart from one component of odd size which is incident with u . Let H be the bipartite subgraph of odd size, and let H_o be its component of odd size. Let G_H be the graph we get by adding v and all its incident edges to H . By Lemma 4.1, there exists a set E of precisely d' edges incident with v such that $G_H - E$ is bipartite. We may assume that E does not contain all edges that are incident with H_o . Since $d(v) - d'$ is odd, all components of $G_H - E$ have even size. We add the edges in E to $G - v - E(H)$ to obtain a graph G' . Notice that $G - v - E(H)$ is the union of $\lceil \log_2(d+1) \rceil$ bipartite graphs with components of even size. By Lemma 4.2, we can decompose G' into $\lceil \log_2(d+1) \rceil$ bipartite graphs where every component has even size. Together with $G_H - E$, this is a collection of $\lceil \log_2(d+1) \rceil + 1$ such graphs. \square

Now we can use our result on bipartite graphs to get an upper bound on the irregular chromatic index of d -degenerate graphs.

Corollary 4.4. *If G is a connected d -degenerate graph of even size, then*

$$\chi'_{\text{irr}}(G) \leq 9(\lceil \log_2(d+1) \rceil + 1).$$

Proof. This follows immediately from Theorems 3.9 and 4.3. \square

To get a constant upper bound for decomposable graphs in general, we combine Corollary 4.4 with Przybyło's result on graphs with large minimum degree. For this purpose, we need the following lemma.

Lemma 4.5. *Let d be a natural number. If G is a connected graph of even size, then G can be decomposed into two graphs D and H such that D is $2d$ -degenerate, every component of D has even size, and the minimum degree of H is at least $d - 1$.*

Proof. Start with two graphs D and H having the same vertex set as G , and $E(D) = \emptyset$ and $E(H) = E(G)$. As long as H has a vertex v with degree at most $2d$, remove all edges of H incident with v and add them to D , and delete the isolated vertex v from H . Once this process stops, the graph D is $2d$ -degenerate and H has minimum degree at least $2d + 1$. Every component C of D with odd size intersects H ; let $v(C)$ be a vertex in the intersection. Notice that $v(C) \neq v(C')$ for different components C and C' of D . We choose an almost-balanced orientation of H , i.e. an orientation where the out-degree and in-degree at every vertex differ by at most 1. For each component C of odd size, we choose an out-edge $e(C)$ at $v(C)$ in H . We remove $e(C)$ from H and add it to D . Since every vertex in H might lose all of its in-edges but at most one out-edge, the minimum degree in H remains at least $d - 1$. The edges we add to D in this step induce a 2-degenerate subgraph, so D will still be $2d$ -degenerate. Moreover, every component of odd size gains an edge and possibly gets joined to other components of even size. In any case, all components of D now have even size. \square

Now we are ready for the proof of our main result.

Theorem 4.6. *If G is a decomposable graph, then $\chi'_{\text{irr}}(G) \leq 328$.*

Proof. By Theorem 2.3 it suffices to show that $\chi'_{\text{irr}}(G) \leq 327$ holds for connected graphs G of even size. By Lemma 4.5, we can decompose G into two graphs D and H so that D is $(2 \cdot 10^{10} + 2)$ -degenerate, every component of D has even size, and the minimum degree of H is at least 10^{10} . By Theorem 1.4, we have $\chi'_{\text{irr}}(H) \leq 3$ and by Corollary 4.4 we have

$$\chi'_{\text{irr}}(D) \leq 9(\lceil \log_2(2 \cdot 10^{10} + 3) \rceil + 1) = 324.$$

Hence, $\chi'_{\text{irr}}(G) \leq \chi'_{\text{irr}}(H) + \chi'_{\text{irr}}(D) \leq 3 + 324 = 327$. \square

5 Decomposing highly edge-connected bipartite graphs

Our bound on the irregular chromatic index of decomposable graphs in Theorem 4.6, depends partly on the irregular chromatic index of bipartite graphs. In particular, decreasing our bound in Theorem 3.9 from 9 down to 3 (as suggested by Conjecture 1.2) would already

yield an improvement on the constant in Theorem 4.6. The restriction of Conjecture 1.2 to bipartite graphs, however, appears to be a surprisingly non-trivial problem.

Another interesting question concerning bipartite graphs and locally irregular decompositions is about whether the family of bipartite graphs with irregular chromatic index at most 2 admits an “easy” characterisation. It is legitimate to raise this question, as trees admit such a characterisation, see [2]. We also note that a similar study of bipartite graphs G satisfying $\chi'_\Sigma(G) \leq 2$ was recently conducted by Thomassen, Wu, and Zhang [11], resulting in such a characterisation. So far all bipartite graphs G with irregular chromatic index 3 we know have minimum degree 1 or 2. It could be the case that minimum degree 3 already suffices to push the irregular chromatic index down to 2 for bipartite graphs. Notice that d -regular bipartite graphs G with $d \geq 3$ do indeed satisfy $\chi'_{\text{irr}}(G) = 2$, as was shown in [1].

Question 5.1. *Does there exist a bipartite graph G with minimum degree at least 3 and $\chi'_{\text{irr}}(G) > 2$?*

In this section we prove that 16-edge-connected bipartite graphs have irregular chromatic index at most 2. Our main tool is the following result on factors modulo k in bipartite graphs, due to Thomassen [10]. That result is based on the proof of the weak version of Jaeger’s Circular Flow Conjecture by Thomassen [9]. His proof gave a quadratic bound on the edge-connectivity, which was improved to a linear bound by Lovász, Thomassen, Wu, and Zhang [6].

Theorem 5.2 ([10]). *Let k be a natural number, and let G be a $(3k - 2)$ -edge-connected bipartite graph with partition classes A and B . If $f : V(G) \rightarrow \mathbb{Z}$ is a function satisfying*

$$\sum_{v \in A} f(v) \equiv \sum_{v \in B} f(v) \pmod{k},$$

then G has a spanning subgraph H with $d(v, H) \equiv f(v) \pmod{k}$ for every $v \in V(G)$.

Theorem 1 in [9] assumes edge-connectivity $3k - 3$. But this holds only for k odd. For k even it should be $3k - 2$, see page 11 in [6], and below we shall apply Theorem 5.2 for $k = 6$.

Theorem 5.3. *For every 16-edge-connected bipartite graph G , we have $\chi'_{\text{irr}}(G) \leq 2$.*

Proof. Let A and B denote the partition classes of G . Since G is 16-edge-connected, we have $|A|, |B| \geq 16$.

Suppose first that every vertex has at most one non-neighbour in the other partition class. Let us assume that $|A| \geq |B|$. If $|A| - |B| \geq 2$, then G is already locally irregular. If $|A| - |B| \leq 1$, then let H be a subgraph of G consisting of two vertices in B and all edges incident with one of these vertices. Clearly H is locally irregular. Let H' denote the subgraph $G - E(H)$. For every $v \in A$, we have $d(v, H') \leq |B| - 2 \leq |A| - 2$. For every $v \in B$, we have $d(v, H') \geq |A| - 1$ (or $d(v, H') = 0$), so H' is locally irregular and $\chi'_{\text{irr}}(G) \leq 2$.

Now suppose that G contains a vertex u with at least two non-neighbours in the other partition class. We may assume $u \in A$. We denote by A_0 and B_0 the subsets of A and B consisting of the vertices of even degree, and by A_1 and B_1 , respectively, the subsets consisting of the vertices of odd degree. For any function $g : V(G) \rightarrow \mathbb{Z}$, we define $\sigma_A(g) = \sum_{v \in A} g(v)$ and $\sigma_B(g) = \sum_{v \in B} g(v)$. If G is $(3k - 2)$ -edge-connected and $\sigma_A(g) \equiv \sigma_B(g) \pmod{k}$, then, by Theorem 5.2, there exists a subgraph H of G with $d(v, H) \equiv g(v) \pmod{k}$ for all $v \in V(G)$. We apply this for $k = 6$. We assign to each vertex $v \in V(G)$ an integer $f(v)$ in the following way:

- for every $v \in A_0$, we set $f(v) = 0$;
- for every $v \in A_1$, we set $f(v) = 1$;
- for every $v \in B_0$, we set $f(v) = 3$;
- for every $v \in B_1$, we set $f(v) = 2$.

If $\sigma_A(f) \equiv \sigma_B(f) \pmod{6}$, then we apply Theorem 5.2 to find a subgraph H of G with $d(v, H) \equiv f(v) \pmod{6}$ for all $v \in V(G)$. Since the degrees of the vertices in A have different residues modulo 6 than the degrees of the vertices in B , the graph H is locally irregular. Let H' denote the subgraph $G - E(H)$. The vertices in A in H' have even degrees, while the vertices in B have odd degrees. Thus, also H' is locally irregular and $\chi'_{\text{irr}}(G) \leq 2$.

We may therefore assume $\sigma_A(f) \not\equiv \sigma_B(f) \pmod{6}$. First, suppose $\sigma_A(f) \equiv \sigma_B(f) \pmod{2}$. Then $\sigma_A(f) \equiv \sigma_B(f) + 2 \pmod{6}$ or $\sigma_A(f) \equiv \sigma_B(f) + 4 \pmod{6}$. Let x and y be two different vertices in B . We define two new functions f_1 and f_2 by setting $f_1(x) = f_2(x) = f(x) + 2$, $f_2(y) = f(y) + 2$ and setting $f_1(v) = f(v)$ for all $v \in V(G) \setminus \{x\}$ and $f_2(v) = f(v)$ for all $v \in V(G) \setminus \{x, y\}$. Now $\sigma_A(f_1) = \sigma_A(f_2) = \sigma_A(f)$ and $\sigma_B(f_2) = \sigma_B(f_1) + 2 = \sigma_B(f) + 4$. Thus, one of the functions f_1 or f_2 satisfies the condition in Theorem 5.2 and the same argument as above yields a decomposition into two locally irregular subgraphs.

Finally, suppose $\sigma_A(f) \not\equiv \sigma_B(f) \pmod{2}$. We define a new function g by setting $g(u) = 1 - f(u)$, where u is the special vertex with at least two non-neighbours in B . We set $g(v) = f(v)$ for all $v \in A$ and all $v \in B$ that are non-neighbours of u . If $v \in B$ is a neighbour of u with $d(v) - f(v) \not\equiv d(u) - g(u) \pmod{6}$, then we also set $g(v) = f(v)$. In the case that $v \in B$ is a neighbour of u with $d(v) - f(v) \equiv d(u) - g(u) \pmod{6}$, we set $g(v) = f(v) + 2$. Now we have $\sigma_A(g) \equiv \sigma_A(f) + 1 \pmod{2}$ and $\sigma_B(g) \equiv \sigma_B(f) \pmod{2}$, so $\sigma_A(g) \equiv \sigma_B(g) \pmod{2}$.

If $\sigma_A(g) \equiv \sigma_B(g) \pmod{6}$, then we use Theorem 5.2 for the function g . Let H be the subgraph of G with $d(v, H) \equiv g(v) \pmod{6}$ for all $v \in V(G)$. Notice that $g(v) \in \{0, 1\}$ for $v \in A$ and $g(v) \in \{2, 3, 4, 5\}$ for $v \in B$, so H is locally irregular. Let H' denote the subgraph $G - E(H)$. Notice that in H' all vertices in B have odd degree, while u is the only vertex in A of odd degree. Notice that for every neighbour v of u we have $d(v, G) - g(v) \not\equiv d(u, G) - g(u) \pmod{6}$ by definition of g , so u is not adjacent to a vertex of the same degree in H' . Thus, also H' is locally irregular.

If $\sigma_A(g) \not\equiv \sigma_B(g) \pmod{6}$, then we define functions g_1 and g_2 as before by adding 2 to the values of one or two vertices in B , and use Theorem 5.2 for the function g_1 or g_2 . This time we only choose vertices that are non-neighbours of u , to make sure the value at a vertex never increases by more than 2 compared to its original f -value. As before, this yields a decomposition of G into two locally irregular subgraphs. \square

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