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► **To cite this version:**

Thierry Coquand, Stefan Neuwirth. An introduction to Lorenzen's "Algebraic and logistic investigations on free lattices" (1951). 2017. hal-01629913v1

**HAL Id: hal-01629913**

**<https://hal.science/hal-01629913v1>**

Preprint submitted on 7 Nov 2017 (v1), last revised 20 Sep 2023 (v2)

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# An introduction to Lorenzen’s “Algebraic and logistic investigations on free lattices” (1951)

Thierry Coquand

Stefan Neuwirth

Lorenzen’s “Algebraische und logistische Untersuchungen über freie Verbände” appeared in 1951 in *The journal of symbolic logic*. These “Investigations” have immediately been recognised as a landmark in the history of infinitary proof theory, but their approach and method of proof have not been incorporated into the corpus of proof theory.<sup>1</sup> We propose a translation (Lorenzen 2017) and this introduction with the intent of giving a new impetus to their reception. We also propose a translation of a preliminary manuscript as an appendix (see section 2).

The “Investigations” are best known for providing a constructive proof of consistency for ramified type theory without axiom of reducibility. They do so by showing that it is a part of a trivially consistent “inductive calculus” that describes our knowledge of arithmetic without detour. The proof resorts only to the inductive definition of formulas and theorems.

They propose furthermore a definition of a semilattice, of a distributive lattice, of a pseudocomplemented semilattice, and of a countably complete boolean lattice as deductive calculuses, and show how to present them for constructing the respective free object over a given preordered set. They illustrate that lattice theory is a bridge between algebra and logic.

The preliminary manuscript contains already the main ideas and applies them to a constructive proof of consistency for elementary number theory.

**1. The beginnings.** In 1938, Paul Lorenzen defends his Ph.D. thesis under the supervision of Helmut Hasse at Göttingen, an “Abstract foundation of the multiplicative ideal theory”, i.e. a foundation of divisibility theory upon the theory of cancellative monoids. He is in a process of becoming more and more aware that lattice theory is the right framework for his research. Lorenzen (1939a, footnote on page 536) thinks of understanding a system of ideals as a lattice, with a reference to Köthe 1937; in the definition of a semilattice-ordered monoid on page 544, he credits Dedekind’s two seminal articles of 1897 and 1900 for developing the concept

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<sup>1</sup>More precisely, Lorenzen proves the admissibility of cut by double induction, on the cut formula and on the complexity of the derivations, without using any ordinal assignment, contrary to the presentation of cut elimination in most standard texts on proof theory.

of lattice. On 6 July 1938 he reports to Hasse: “Momentarily, I am at making a lattice-theoretic excerpt for Köthe.”<sup>2</sup> He also reviews several articles on this subject for the *Zentralblatt*, [Birkhoff 1938](#) to start with, then [Klein 1939](#) and [George 1939](#) which both introduce semilattices, [Whitman 1941](#) which studies free lattices. He also knows about the representation theorem for boolean algebras in [Stone 1936](#) and he discusses the axioms for the arithmetic of real numbers in [Tarski 1937](#) with Heinrich Scholz.<sup>3</sup>

In 1939, he becomes assistant to Wolfgang Krull at Bonn. During World War II, he serves first as a soldier and then, from 1942 on, as a teacher at the naval college Wesermünde. He devotes his “off-duty evenings all alone on my own”<sup>4</sup> to mathematics with the goal of habilitating. On 25 April 1944, he writes to his advisor that “[...] it became clear to me—about 4 years ago—that an ideal system is nothing but a semilattice.”<sup>5</sup>

He will later recall a talk by Gerhard Gentzen on the consistency of elementary number theory in 1937 or 1938 as a trigger for his discovery that the reformulation of ideal theory in lattice-theoretic terms reveals that his “algebraic works [...] were concerned with a problem that had *formally* the same structure as the problem of consistency of the classical calculus of logic”;<sup>6</sup> compare also his letter to Eckart [Menzler-Trott \(2001, page 260\)](#). This explains the title of Lorenzen’s article.

In his letter of 13 March 1944 he announces: “Subsequently to an algebraic investigation of orthocomplemented semilattices, I am now trying to get out the connection of these questions with the consistency of classical logic. [...] actually I am much more interested into the algebraic side of proof theory than into the purely logical.”<sup>7</sup> The concept of “orthocomplementation” (see [Lorenzen 2017, page 5](#) for the definition; the terminology might be adapted from [Stone 1936](#), where it has a

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<sup>2</sup>„Augenblicklich bin ich dabei, für Köthe einen verbandstheoretischen Exzerpt zu machen.“ (Helmut-Hasse-Nachlass, Niedersächsische Staats- und Universitätsbibliothek Göttingen, Cod. Ms. H. Hasse 1:1022.)

<sup>3</sup>See the collection of documents grouped together by Scholz under the title “*Paul Lorenzen*: Gruppentheoretische Charakterisierung der reellen Zahlen” and deposited at the Bibliothek des Fachbereichs Mathematik und Informatik of the Westfälische Wilhelms-Universität Münster, as well as several letters filed in its Scholz-Archiv, the earliest dated 7 April 1944, <http://www.uni-muenster.de/IVV5WS/ScholzWiki/doku.php?id=scans:blogs:hb-01-1040>, accessed 14 March 2017.

<sup>4</sup>„ganz allein auf mich gestellt – [...] die dienstfreien Abende“ (carbon copy of a letter to Krull, 13 March 1944, Paul-Lorenzen-Nachlass, Philosophisches Archiv, Universität Konstanz, PL 1-1-131.)

<sup>5</sup>„mir vor etwa 4 Jahren – [...] klar wurde, daß ein Idealsystem nichts anderes als ein Halbverband ist.“ (Carbon copy of a letter to Krull, PL 1-1-132.)

<sup>6</sup>„[...] meine algebraischen Arbeiten [...] mit einem Problem beschäftigt waren, das *formal* die gleiche Struktur hatte wie das Problem der Widerspruchsfreiheit des klassischen Logikkalküls“ (letter to Carl Friedrich [Gethmann 1991, page 76](#)).

<sup>7</sup>„Im Anschluß an eine algebraische Untersuchung über orthokomplementäre Halbverbände versuche ich jetzt, den Zusammenhang dieser Fragen mit der Widerspruchsfreiheit der klassischen Logik herauszubekommen. [...] ich selber eigentlich viel mehr an der algebraischen Seite der Beweistheorie interessiert bin als an der rein logischen.“ (PL 1-1-131.)

Hilbert space background; today one says “pseudocomplementation”) must have been motivated by logical negation from the beginning. On the one hand, such lattices correspond to the calculus of derivations considered by [Gentzen \(1936, section IV\)](#), who shows that a given derivation can be transformed into a derivation “in which the connectives  $\vee$ ,  $\exists$  and  $\supset$  no longer occur” and provides a proof of consistency for this calculus (see section 2 below). On the other hand, note that Lorenzen reviewed [Ogasawara 1939](#) for the *Zentralblatt*.

**2. The 1944 manuscript.** The result of this investigation can be found in “Ein halbordnungstheoretischer Widerspruchsfreiheitsbeweis”<sup>8</sup>, for which we provide a translation on pages 18-25.

This manuscript renews the relationship between logic and lattice theory: whereas boolean algebras were originally conceived for modeling the classical calculus of propositions, and Heyting algebras for the intuitionistic one, here logic comes at the rescue of lattice theory for studying countably complete pseudocomplemented semilattices. They are described as deductive calculuses of their own, without any reference to a larger formal framework:<sup>9</sup> this conception dates back to the “systems of sentences” of [Hertz \(1922, 1923\)](#). The rules of the calculus construct the free countably complete pseudocomplemented semilattice over a given preordered set by taking the inequalities in the set as axioms, by defining inductively formal meets and formal negations, and by introducing inequalities between the formal elements. One of the introduction rules, stating that

$$\text{if } c \leq a_1, c \leq a_2, \dots, \text{ then } c \leq \bigwedge M, \text{ where } M = \{a_1, a_2, \dots\}$$

(rule  $c$  on page 20, compare rule [3.9] in [Lorenzen 2017](#), page 12), stands out: it has an infinity of premisses. It is in fact an instance of Hilbert’s  $\omega$ -rule (see [Sundholm 1983](#), [Feferman 1986](#)).

The proof that the calculus thus defined is a countably complete pseudocomplemented semilattice turns out to be, as Lorenzen realises a posteriori, an instance of the strategy of [Gentzen](#)’s dissertation (1935, IV, §3) for proving the consistency of elementary number theory without complete induction: the introduction rules introduce inequalities for formal elements of increasing complexity, i.e. no inequality can result from a detour; then the corresponding elimination rules are shown to hold by an induction on the complexity of the introduced inequality, and at last transitivity of the preorder, i.e. the cut rule, is established by the same method. In Lorenzen’s later terminology, one would say that these rules are shown to be “admissible” and can be considered as resulting from an “inversion principle”. Note that the inductions used here are the ones accurately described by Jacques

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<sup>8</sup>“A preorder-theoretic proof of consistency”, Oskar-Becker-Nachlass, Philosophisches Archiv, Universität Konstanz, OB 5-3b-5.

<sup>9</sup>In contradistinction to the “consequence relation” of [Tarski \(1930\)](#) which presupposes set theory.

Herbrand (1930, pages 4–5) after having been emphasised by David Hilbert (1928, page 76), to be called “formula induction” and “theorem induction” by Lorenzen.<sup>10</sup> the first proceeds along the construction of formulas starting from prime formulas through rules; the second proceeds along the construction of theorems starting from prime theorems through deduction rules.

In other words, Lorenzen starts with a preordered set  $P$ , constructs the free countably complete pseudocomplemented semilattice  $K$  over  $P$  and emphasises conservativity, i.e. that no more inequalities come to hold among elements of  $P$  viewed as a subset of  $K$  than the ones that have been holding before.<sup>11</sup>

Then the consistency of elementary number theory with complete induction is established in § 3 by constructing the free countably complete pseudocomplemented semilattice over its “prime formulas”, i.e. the numerical formulas, viewed as a set preordered by material implication.

There are common points and differences with respect to the strategy developed by Gentzen for proving the consistency of elementary number theory with complete induction. In his first proof, submitted in August 1935, withdrawn and finally published posthumously by Bernays in 1974 (after its translation in Szabo 1969), he defines a concept of reduction process for a sequent and shows that such a process may be specified for every derivable sequent but not for the contradictory sequent  $\rightarrow 1 = 2$ . Let us emphasise two aspects of this concept.

- If the succedent of the sequent has the form  $\forall x F(x)$ , the next step of the reduction process consists in replacing it by  $F(n)$ , where  $n$  is a number to be chosen freely.
- A reduction process is defined as the specification of a sequence of steps for all possible free choices, with the requirement that the reduction terminates for every such choice.

In a letter to Bernays of 4 November 1935,<sup>12</sup> Gentzen visualises a reduction process as a tree whose every branch terminates.

The proof that a reduction process may be specified for every derivable sequent is by theorem induction. For this, a lemma is needed, claiming that if reduction processes are known for two sequents  $\Gamma \rightarrow D$  and  $D, \Delta \rightarrow C$ , then a reduction process may be specified for their cut sequent  $\Gamma, \Delta \rightarrow C$ . The proof goes by induction on the construction of the cut formula  $D$  and traces the claim back to the same claim with the same cut formula, but with the sequent  $D, \Delta \rightarrow C$  replaced by a sequent  $D, \Delta^* \rightarrow C^*$  resulting from it after one or more reduction

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<sup>10</sup>See Lorenzen 1939b for his interest in the foundation of inductive definitions.

<sup>11</sup>This is exactly the approach of Skolem (1921, §2) for constructing the free lattice over a preordered set, in the course of studying the decision problem for lattices.

<sup>12</sup>Paul-Bernays-Nachlass, ETH Zürich, Hs. 975:1652, translated by von Plato (2017, pages 241–244).

steps and the cut sequent replaced by  $\Gamma, \Delta^* \rightarrow C^*$ . By definition of the reduction process, this tracing back must terminate eventually.

This last kind of argument may be considered as an infinite descent in the reduction process. In his letter to Bernays, Gentzen seems to indicate that this infinite descent justifies an induction on the reduction process. As analysed by William W. Tait (2015), this would be an instance of the Bar theorem. But in his following letter, dated 11 December 1935,<sup>13</sup> he writes that “[his] proof is not satisfactory” and announces another proof, to be submitted in February 1936: in it, he defines the concept of reduction process for a derivation, associates inductively an ordinal to every derivation, and shows that a reduction process may be specified for every derivation by an induction on the ordinal.

Let us compare this strategy with Lorenzen’s.

- The free choice is subsumed in a deduction rule, an  $\omega$ -rule as described above.<sup>14</sup>
- Number theory is constructed as the cut-free derivations starting from the numerical formulas, so that it is trivially consistent, and the cut rule is shown to be admissible: if derivations are known for two sequents  $A \rightarrow B$  and  $B \rightarrow C$ , then a cut-free derivation may be specified for their cut sequent  $A \rightarrow C$  by a formula induction on the cut formula  $B$  nested with several instances of a theorem induction.

In this way, Lorenzen fulfils the endeavour expressed by Tait (2015): “the gap in Gentzen’s argument is filled, not by the Bar Theorem, but by taking as the basic notion that of a [cut-free] deduction tree in the first place rather than that of a reduction tree.”

This draft might be the one that he sends to Wilhelm Ackermann, Gentzen, Hans Hermes and Heinrich Scholz,<sup>15</sup> and for which he gets a dissuasive answer from Gentzen, dated 12 September 1944: “The consistency of number theory cannot be proven so simply.”<sup>16</sup>

**3. The 1945 manuscript.** In a letter dated 11 December 1945, Heinrich Scholz submits Lorenzen’s manuscript “Die Widerspruchsfreiheit der klassischen Logik mit verzweigter Typentheorie”<sup>17</sup> to Paul Bernays (Hs. 975:4111). It begins

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<sup>13</sup>Hs. 975:1653, translated by von Plato (2017, page 244).

<sup>14</sup>Compare Bernays’ suggestion in his letter to Gentzen of 9 May 1938, Hs. 975:1661, translated by von Plato (2017, pages 254–255).

<sup>15</sup>See the letters to Scholz dated 13 May 1944, <http://www.uni-muenster.de/IVV5WS/Scholz/Wiki/doku.php?id=scans:blogs:hb-01-1036>, and 2 June 1944, PL 1-1-138, the postcard to Hasse dated 25 July 1945, Cod. Ms. H. Hasse 1:1022, and Ackermann’s letter dated 11 November 1946, PL 1-1-125.

<sup>16</sup>The letter is reproduced in Menzler-Trott 2001, page 372.

<sup>17</sup>“The consistency of classical logic with ramified type theory”. A version of this manuscript can be found in Niedersächsische Staats- und Universitätsbibliothek Göttingen, Cod. Ms. G. Köthe M 10.

as follows.

The proof of consistency undertaken in the sequel originated as an application of a purely algebraic theorem of existence about “free” complete boolean lattices. In the present work, though, I limit myself exclusively to the logistic application and use no algebraic conceptions whatsoever.<sup>18</sup>

The choice of stripping away lattice theory may be motivated by targeting a public of logicians. In this way, the strategy of [Gentzen \(1935, IV, §3\)](#) regains its original form: the deductive calculus of ramified type theory with the axioms of comprehension, extensionality and infinity, but without the axiom of reducibility, is compared to an inductive calculus that proceeds “without detour”; w.r.t. Gentzen’s calculus, it features an induction rule (compare rule [4] in [Lorenzen 2017](#), page 18) which is an instance of the  $\omega$ -rule. Lorenzen emphasises that

This proof uses as auxiliary means only formula inductions vs. theorem inductions, i.e. the fact that the concept of formula and the concept of theorem is defined inductively. The harmlessness of these auxiliary means seems to me to be even more perspicuous than the harmlessness of explicit transfinite inductions.<sup>19</sup>

Compare the first and the second kind of induction, respectively, in [Herbrand \(1930, page 5\)](#). They establish that the deductive calculus is a part of the inductive calculus: in [Lorenzen \(2017, section 7 on pages 20–25\)](#),

- (1a) the “logical axiom”  $c \preceq c$  is proved by formula induction;
- (1b) the axiom of comprehension follows from the construction of a  $\lambda$ -calculus and a rule of constants;
- (1c) the axiom of extensionality results from a formula induction with the help of two auxiliary rules proved by theorem induction;
- (1d) the axiom of infinity follows from the properties of the order on numbers;
- (2a) the cut rule is proved by a formula induction on the cut formula: if it is a numerical formula, a double theorem induction on the premisses is used;

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<sup>18</sup>„Der im folgenden durchgeführte Widerspruchsfreiheitsbeweis ist als eine Anwendung eines rein algebraischen Existenztheorems über “freie” vollständige Boole’sche Verbände entstanden. In dieser Arbeit beschränke ich mich jedoch ausschließlich auf die logistische Anwendung und benutze keinerlei algebraische Begriffsbildungen.“

<sup>19</sup>„Dieser Beweis benutzt als Hilfsmittel nur Formelinduktionen bzw. Satzinduktionen, d. h. die Tatsache, daß der Formelbegriff und der Satzbegriff induktiv definiert ist. Die Unbedenklichkeit dieser Hilfsmittel scheint mir noch einleuchtender zu sein, als die Unbedenklichkeit expliziter transfiniter Induktionen.“

the only difficulties in the induction step result from the copresence of constants and free and bound variables in rules like [3d] on page 18; as usual, contraction plays an important rôle.

Bernays is able to appreciate its content on the spot and replies with detailed comments to Scholz on 24 April 1946 (carbon copy, Hs. 975:4112). On 17 April 1946, Lorenzen writes directly to Bernays (Hs. 975:2947); he gets an answer on 22 May 1946 with the following appreciation.

It seems to me that your argumentation accomplishes in effect the desired and that thereby at the same time also a new, methodically more transparent proof of consistency for the number-theoretic formalism, as well as for Gentzen's subformula theorem<sup>20</sup> is provided.

In the circumstance that all this is included in your result shows at the same time the methodical superiority of your method of proof with respect to a proof (that probably did not come to your knowledge) that F. B. Fitch [...] gave in 1938, and that also bears on the comparison of the deductive formalism with a system of formulas which is not delimited in a purely operative way; namely, this delimitation is carried out there according to a definition of truth in which the "tertium non datur" (indeed only with respect to the species of natural numbers) is made use of.<sup>22</sup> By determining your system of comparison according to the idea of a generalisation of Gentzen's thought of "deduction without detour", you gain the possibility of applying the constructive proof-theoretic view also in the case of your "inductive calculus", i.e. of such an inference system that does not comply with the recursiveness conditions that the customary formalisms fulfil.<sup>23</sup>

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<sup>20</sup>In the letter of 24 April 1946, Bernays writes more precisely to Scholz "that one also gets a proof for the main theorem of Gentzen's 'Investigations into logical deduction' out of it, if on the one hand one omits the higher axioms [(1b, c, d) in Lorenzen 2017, page 15] in the deductive calculus, on the other hand one retains from the rules of the inductive calculus (for determining the concept of theorem) only [[2], [3a–d] in Lorenzen 2017, page 18], while one takes also the formula pairs  $[c \leq c]$  as initial theorems for this calculus."<sup>21</sup>

<sup>21</sup>„dass man aus ihm auch einen Beweis für den Hauptsatz von Gentzen's ‚Untersuchungen über das logische Schliessen‘ erhält, indem man einerseits beim deduktiven Kalkül die höheren Axiome 1.) b), c), d) weglässt, andererseits von den Regeln des induktiven Kalküls (zur Bestimmung des Satzbegriffes) nur 2) a)–d) beibehält, während man als Ausgangssätze auch für diesen Kalkül die Formelpaare  $c \leq c$  nimmt.“

<sup>22</sup>See Fitch 1938 and the review Bernays 1939.

<sup>23</sup>„Es scheint mir, dass Ihre Beweisführung in der Tat das Gewünschte leistet und dass damit zugleich auch ein neuer, methodisch durchsichtigerer Wf.-Beweis für den zahlentheoretischen Formalismus wie auch ein solcher für Gentzen's Teilformelsatz geliefert wird.

„In dem Umstande, dass alles dies in Ihrem Ergebnis eingeschlossen ist, zeigt sich zugleich die methodische Überlegenheit Ihres Beweisverfahrens gegenüber einem (Ihnen wohl nicht zur Kenntnis gelangten) Beweis, den F. B. Fitch 1938 für die Widerspruchsfreiheit der verzweigten Typentheorie gegeben hat (im Journal of symb. logic, vol. 3, S. 140-149), und der auch auf dem



Lorenzen learns about Fitch’s proof of consistency only by this letter. In his answer (dated 7 June 1946, Hs. 975:2949), he explains the lattice-theoretic background of his proof and encloses a manuscript, “Über das Reduzibilitätsaxiom”,<sup>24</sup> which is a preliminary version of the last section of the published article, in which the axiom of reducibility is shown to be independent; in fact [Fitch \(1939\)](#) proves this in his own framework. However, Bernays seems to already have received this manuscript with Scholz’s letter of 11 December 1945 (see his letter of 24 April 1946).

**4. The 1947 manuscript.** By a letter dated 21 February 1947, Lorenzen writes to Bernays:

After a revision of my proof of consistency according to your previous remarks and after addition of an algebraic part, I would like to allow myself to ask you for your intercession for a publication in the Journal of symbolic logic.<sup>25</sup>

This new draft tries to make a synthesis of “Ein halbordnungstheoretischer Widerspruchsfreiheitsbeweis” and “Die Widerspruchsfreiheit der klassischen Logik mit verzweigter Typentheorie”, but it is rather a juxtaposition of the two: the seams remain apparent. However, the introduction now takes into account the added algebraic part. In its first paragraph (see [Lorenzen 2017](#)), it emphasises that lattice theory is relevant for ideal theory.

In the new algebraic part, the construction of free semilattices and free distributive lattices stems in fact from ideal theory. Theorems 1–4 in section 2 ([Lorenzen 2017](#), page 4) introduce a semilattice as a “single statement entailment relation” and construct the free semilattice over a preordered set. This approach is paralleled in [Lorenzen 1952](#) by the definition of a system of ideals for an arbitrary preordered set  $M$  on which a monoid  $G$  acts by order-preserving operators  $x$ : it is a relation satisfying items 1–4 of theorem 1 and furthermore

$$\text{if } a_1, \dots, a_n \vdash b, \text{ then } xa_1, \dots, xa_n \vdash xb.$$

In the same way, theorems 5–8 provide the description of a distributive lattice as a deductive system that has been called since [Scott \(1971\)](#) an “entailment relation”.

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Vergleich des deduktiven Formalismus mit einem Formelsystem beruht, das auf eine nicht rein operative Art abgegrenzt ist; diese Abgrenzung erfolgt nämlich dort im Sinne einer Wahrheitsdefinition, wobei von dem ‚tertium non datur‘ (allerdings nur demjenigen in Bezug auf die Gattung der natürlichen Zahlen) Gebrauch gemacht wird. Indem Sie Ihr Vergleichssystem gemäss der Idee einer Verallgemeinerung von Gentzen’s Gedanken der ‚umweglosen Herleitung‘ bestimmen, gewinnen Sie die Möglichkeit, die konstruktive beweistheoretische Betrachtung auch im Falle Ihres ‚induktiven Kalküls‘ anzuwenden, d. h. eines solchen Folgerungssystems, welches nicht den durch die üblichen Formalismen erfüllten Rekursivitätsbedingungen genügt.“ (Hs. 975:2948.)

<sup>24</sup>“On the axiom of reducibility”, Hs. 974:149.

<sup>25</sup> „nach einer Überarbeitung meines Wf.beweises nach Ihren wertvollen Bemerkungen und nach Hinzufügung eines algebraischen Teiles möchte ich mir erlauben, Sie um Ihre Fürsprache zu bitten für eine Veröffentlichung im Journal of symbolic logic“ (Hs. 975:2949.)

This description strikes Bernays as new to him (letter of 3 April 1947, PL 1-1-118). His theorem 7 (Lorenzen 2017, page 5) corresponds to theorem 1 in Cederquist and Coquand 2000. This construction is used in Lorenzen 1953 for embedding a preordered group endowed with a system of ideals into a lattice group containing this system. Compare Coquand, Lombardi, and Neuwirth 2017.

Section 3 deals with (finite) pseudocomplemented semilattices very much like in his 1944 manuscript.<sup>26</sup> He emphasises that the decision problem has a positive answer.<sup>27</sup>

In section 4, Lorenzen shows how to apply the construction of the free pseudocomplemented semilattice to a simple intuitionistic logic calculus. He does not develop the notion of a countably complete pseudocomplemented semilattice as in the 1944 version, but that of a countably complete boolean lattice,<sup>28</sup> for which cut may also be established. Here the  $\omega$ -rules are [3.9] and [3.10] on page 12. He only sketches the application to consistency, which goes along the same lines, with one significant difference: in the deductive system associated to the former setting, he is able to prove contraction (see lemma (8) on page 8), whereas he has to put it into the definition in the latter setting (he provides a counterexample on page 12). This should be put in relation

- with the rôle of contraction, especially for steps 13.5 1–13.5 3 in Gentzen’s proofs of consistency (1936, 1974),
- with the calculus of P. S. Novikoff (1943), in which contraction may be proved,
- and with the calculus defined by Per Martin-Löf (1970, § 30) for Borel sets, where the problem of contraction is eluded by “identify[ing] sequents which are equal considered as finite sets”.

Then he shows how to deduce consistency for the logic of ramified type by a repeated construction of free countably complete boolean lattices, starting from a calculus without free variables, along the hierarchy of types.

**5. Toward publication.** At the end of his letter of 21 February 1947, Lorenzen asks:

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<sup>26</sup>The theory of pseudocomplemented semilattices continues to develop: one can find an account of it by Grätzer (2011, pages 99–101) and Chajda, Halaš, and Kühr (2007, chapter 3).

<sup>27</sup>The existence of the free pseudocomplemented semilattice over a preordered set seems to be unknown in the literature, which considers only the case where the preorder is trivial; in the latter case, the decision problem was solved by Tamura (1974).

<sup>28</sup>In the second paragraph of the introduction, he addresses complete boolean algebras over a preordered set as studied by MacNeille (1937). The question about the existence of the free complete boolean algebra is usually attributed to Rieger (1951) and has led to the works of Gaifman (1964) and Hales (1964).

I beg once again to ask you for your advice—namely, it is not clear to me whether I rightly call the logic used here “finite” logic.<sup>29</sup>

Bernays provides the following answer in his letter of 3 April 1947:

When it comes to the methodical standpoint and to the terminology to be used in relation, then it seems advisable to me to keep with the mode chosen by Mr Gentzen, that one speaks of “finite” reflections only in the narrower sense, i.e. relating to considerations that may be formalised in the framework of recursive number theory (possibly with extension of the domain of functions to arbitrary computable functions), that one uses in contrast the expression “constructive” for the appropriate extension of the standpoint of the intuitive self-evidence; by the way, this is employed also by many an American logician in the corresponding sense.

Your proof of consistency cannot, I deem, be a finite one in the narrower sense. Of course, this would conflict with the Gödel theorem. Actually, a nonfinite element of your reflection lies in the induction rule of the inductive calculus, which contains indeed a premiss of a more general form.<sup>30</sup>

In other words, the  $\omega$ -rule does not fit into a formal system, and this explains why Gödel’s theorem does not apply here. But Hilbert (1931, page 491) termed the  $\omega$ -rule a “finite deduction rule” and this is probably why Lorenzen qualifies his deductions as “finite”. More precisely, he answers on 4 May 1947:

Your proposal to call the means of proof not “finite” but “constructive” acted on me as a sort of redemption. I was sticking so far to the word finite only to emphasise that these are hilbertian ideas that I am trying to pursue.<sup>31</sup>

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<sup>29</sup> „Ich bitte noch einmal Sie um Ihren Rat fragen zu dürfen – es ist mir nämlich nicht klar, ob ich die hier benutzte Logik mit Recht ‚finite‘ Logik nenne.“

<sup>30</sup> „Was den methodischen Standpunkt und die in Bezug darauf zu verwendende Terminologie betrifft, so erscheint es mir als empfehlenswert, den von Herrn Gentzen gewählten Modus beizubehalten, dass man von ‚finiten‘ Betrachtungen nur im engeren Sinne spricht, d. h. mit Bezug auf Überlegungen, die sich im Rahmen der rekursiven Zahlentheorie (eventuell mit Erweiterung des Funktionenbereiches auf beliebige berechenbare Funktionen) formalisieren lassen, dass man dagegen für die sachgemässe Erweiterung des Standpunktes der anschaulichen Evidenz den Ausdruck ‚konstruktiv‘ verwendet; dieser wird übrigens auch von manchen amerikanischen Logikern im entsprechenden Sinn gebraucht.“

„Ihr Wf-Beweis kann, so meine ich, kein finiter in dem genannten engeren Sinne sein. Das würde doch dem Gödelschen Theorem widerstreiten. Tatsächlich liegt, so viel ich sehe, ein nicht-finites Element Ihrer Betrachtung in der Induktionsregel des induktiven Kalküls, welche ja eine Prämissen von allgemeiner Form enthält.“

<sup>31</sup> „Ihr Vorschlag, die Beweismittel nicht ‚finit‘, sondern ‚konstruktiv‘ zu nennen, hat wie eine Art Erlösung auf mich gewirkt. Ich habe bisher an dem Wort finit nur festgehalten, um zu betonen, dass es Hilbertsche Ideen sind, die ich fortzuführen versuche.“ (Hs. 975:2953.)

Lorenzen prepares another final draft that is very close to the published version.<sup>32</sup> Bernays sends a first series of comments on 1 September 1947 (PL 1-1-112) and a second series of comments (on a version including the final section on the axiom of reducibility) on 6 February 1949 (PL 1-1-107); the article is submitted to *The journal of symbolic logic* soon afterwards<sup>33</sup> and published as Lorenzen 1951 with date of reception 17 March 1950. In fact, from 1947 on, Lorenzen is already mostly occupied by his project of layers of language which will lead to his operative logic.

**6. Reception.** For early accounts of the manuscripts, see Lorenzen 1948, Köthe 1948, Schmidt 1950, § 11.

On 1 May 1950, Kurt Schütte writes to Lorenzen in order to acknowledge the latter's priority in implementing the  $\omega$ -rule into proofs of consistency.<sup>34</sup>

[...] I came to know that you had provided already before a proof of consistency for a still more general domain, and had arrived at the following result: the cut-eliminability, that in Gentzen had only been carried out in pure logic, may also be transferred to mathematical formalisms, if instead of the inference of complete induction more general schemes of inference with infinitely many premisses are drawn on by extending the concept of derivation so that it may contain infinitely many formulas. This insight gained by you, that appears to me exceptionally important for fundamental research, I have now taken up.<sup>35</sup>

In fact, the reception of the logistic part of Lorenzen's article takes mostly place through the articles Schütte 1951, 1952 and the book Schütte 1960.<sup>36</sup> In his book, Schütte coins the expression "semi-formal system" for a calculus with an  $\omega$ -rule. This detour may have contributed to proof theory continuing to focus on measures of complexity by ordinal numbers; Tait (1968) provides a very clear presentation

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<sup>32</sup>Two pages of this draft may be found in the file OB 5-3b-5; Cod. Ms. G. Köthe M 10 contains an excerpt of Part I.

<sup>33</sup>See the letter of 27 April 1949 to Alonzo Church, in which Lorenzen thanks him for acknowledging receipt of the manuscript, says a few words on its history, and proposes Bernays as a referee (Alonzo Church Papers, box 26 folder 4, Manuscripts division, Department of rare books and special collections, Princeton university library.)

<sup>34</sup>Both are not aware of the work of Novikoff (1939, 1943) in this respect. See Grigori Mints (1991, 1.2).

<sup>35</sup>„[...] erfuhr ich, daß Sie schon vorher einen Widerspruchsfreiheitsbeweis für einen noch allgemeineren Bereich erbracht hatten und dabei zu folgendem Ergebnis gekommen waren: Die Schnitt-Eliminierbarkeit, die bei Gentzen nur in der reinen Logik durchgeführt wurde, läßt sich auch auf mathematische Formalismen übertragen, wenn statt des Schlusses der vollständigen Induktion allgemeinere Schlußschemata mit unendlich vielen Prämissen herangezogen werden, indem der Begriff der Herleitung so erweitert wird, dass er unendlich viele Formeln enthalten darf. Diese von Ihnen gewonnene Erkenntnis, die mir außerordentlich wichtig für die Grundlagenforschung zu sein scheint, habe ich nun aufgegriffen.“ (PL 1-1-45.)

<sup>36</sup>See § 18 and chapter IX. Note that Schütte 1977 is not providing a translation, as the author abandons the treatment of ramified type theory in this second edition; in doing so, he forgets about Lorenzen's contributions to proof theory but for a spurious presence in the bibliography.

of Schütte’s approach. The fact that Lorenzen does not resort to ordinals in his proof of consistency should be considered as a feature of his approach.

Köthe gives a lecture on proofs of consistency up to Lorenzen’s in Fall 1947 at Mainz (see Cod. Ms. G. Köthe G 3); the preparation of the lecture gives rise to a correspondence between the two.<sup>37</sup> Hermes gives a course on constructive mathematics centred on Lorenzen in Fall 1951 at Münster.<sup>38</sup>

Hao Wang (1951) writes the review for *The journal of symbolic logic* and tries to compare Lorenzen’s approach with Fitch’s; see Coquand (2014) for a discussion of this review. Wang (1954, page 252) provides a more accurate comparison.

Lorenzen (1955) expands on the rôle of lattices in mathematics. Lorenzen (1958, 1987) provide a proof of Gentzen’s subformula theorem by the method of his article. Lorenzen (1962, §7) returns to the subject of proofs of consistency.

Haskell B. Curry (1963, Chapter 4, Theorem B9) follows Lorenzen in characterising a distributive lattice as a lattice satisfying cut.

Manfred E. Szabo (1969, pages 12–13) writes on the relationship of Gentzen’s work with Lorenzen’s article.

Its philosophical significance is addressed by Matthias Wille (2016).

**7. Conclusion.** Lorenzen’s article is remarkable for its metamathematical standpoint. A mathematical object is presented as a construction described by rules. A claim on the object is established by an induction that expresses the very meaning of the construction.

Also our certitudes admit such a metamathematical presentation; they have the additional feature that the construction of a certitude proceeds as accumulatively (“without detour”, i.e. cut) as the construction of the formulas appearing in the certitude.

In number theory and for the free countably complete boolean lattice, the construction of a certitude uses an  $\omega$ -rule that is stronger than the rule of complete induction but requires infinitely many premisses, so that a certitude corresponds to a well-formed tree.

Lorenzen’s standpoint holds equally well for a logical calculus and for a lattice: “logical calculuses are semilattices or lattices” (Lorenzen 2017, page 9). The consistency of a logical calculus is recognised as a consequence of the existence of the free semilattice or lattice over its certitudes.

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<sup>37</sup>See Köthe’s letter of 3 September 1947 (PL 1-1-113) and Cod. Ms. G. Köthe M 10.

<sup>38</sup>See Scholz’s letter to Bernays dated 29 January 1952, Hs. 975:4165.

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A preorder-theoretic proof of consistency.

The dissertation of G. Gentzen contains a proof of consistency of elementary number theory without complete induction that relies on the following basic thought: every deducible sequent must also be deducible without detour, so that during the deduction only those connectives are being introduced that are absolutely necessary, i.e. those that are contained in the sequent itself. In the proof of consistency of number theory with complete induction, this basic thought steps back with regard to others. I wish however to show in the following that it alone suffices to obtain also this consistency.

Without knowledge of the dissertation of Gentzen I have arrived at this possibility on the basis of a semilattice-theoretic question. This question is: how may a preordered set be embedded into an orthocomplemented complete semilattice? In general several such embeddings are possible – but among the possible embeddings one is distinguished, i.e. the one which may be mapped homomorphically into every other. The existence of this distinguished embedding will be proved in § 2.

In order to obtain from this in § 3 the sought-after proof of consistency, now just a translation of the semilattice-theoretic proof into the logistic language is necessary. For the calculus that we consider and to which the usual calculuses may be reduced is contained in the distinguished embedding of the preordered set of the number-theoretic prime formulas.

§ 1. A set  $M$  is called preordered if a binary relation  $\leq$  is defined in  $M$  so that for the elements  $a, b, \dots$  of  $M$  holds:

$$\begin{aligned} & a \leq a \\ & a \leq b, b \leq c \quad \Rightarrow \quad a \leq c. \end{aligned}$$

If  $a \leq b$  and  $b \leq a$  holds, then write we  $a \equiv b$ .

If  $a \leq x$  holds for every  $x \in M$ , then we write  $a \leq \cdot$ . We write as well  $\leq a$  if  $x \leq a$  holds for every  $x$ . ( $\leq$  means thus that  $x \leq y$  holds for every  $x, y \in M$ .)

A preordered set  $M$  is called semilattice if to every  $a, b \in M$  there is a  $c \in M$  so that for every  $x \in M$  holds

$$x \leq a, x \leq b \quad \iff \quad x \leq c.$$

$c$  is called the conjunction of  $a$  and  $b$ :  $c \equiv a \wedge b$ .

A semilattice  $M$  is called orthocomplemented if to every  $a \in M$  there is a  $b \in M$  so that for every  $x \in M$  holds

$$a \wedge x \leq \quad \iff \quad x \leq b.$$

$b$  is called the orthocomplement of  $a$ :  $b \equiv \bar{a}$ .

A semilattice  $M$  is called  $\omega$ -complete if to every countable sequence  $M = a_1, a_2, \dots$  in  $M$  there is a  $c \in M$  so that for every  $x \in M$  holds

$$(\text{for every } n: x \leq a_n) \iff x \leq c.$$

$c$  is called the conjunction of the elements of  $M$ :  $c \equiv \bigwedge_n a_n \equiv \bigwedge_M$ .

If  $M$  and  $M'$  are preordered sets, then  $M$  is called a part of  $M'$  if  $M$  is a subset of  $M'$  and for every  $a, b \in M$   $a \leq b$  holds in  $M'$  exactly if  $a \leq b$  holds in  $M$ .

If  $M$  and  $M'$  are preordered sets, we understand by a mapping of  $M$  into  $M'$  an assignment that to every  $a \in M$  assigns an  $a' \in M'$  so that holds

$$a \equiv b \implies a' \equiv b'.$$

If  $M$  and  $M'$  are orthocomplemented  $\omega$ -complete semilattices, we understand by a homomorphism of  $M$  into  $M'$  a mapping  $\rightarrow$  of  $M$  into  $M'$ , so that for every  $a, b \in M$  and  $a', b' \in M'$  with  $a \rightarrow a'$  and  $b \rightarrow b'$  holds

$$\begin{aligned} a \wedge b &\rightarrow a' \wedge b' \\ \bar{a} &\rightarrow \bar{a}'. \end{aligned}$$

Moreover, for every sequence  $M = a_1, a_2, \dots$  in  $M$  and  $M' = a'_1, a'_2, \dots$  in  $M'$  with  $a_n \rightarrow a'_n$  is to hold

$$\bigwedge_M \rightarrow \bigwedge_{M'}.$$

We want to prove now that to every preordered set  $P$  there is an orthocomplemented  $\omega$ -complete semilattice  $K$  so that

- 1)  $P$  is a part of  $K$ ,
- 2)  $K$  may be mapped homomorphically into every orthocomplemented  $\omega$ -complete semilattice that contains  $P$  as part.

If  $K'$  were a further orthocomplemented  $\omega$ -complete semilattice that fulfils conditions 1) and 2), then there would be an assignment by which  $K$  would be mapped homomorphically into  $K'$  and  $K'$  into  $K$ , i.e.  $K$  and  $K'$  would be isomorphic.  $K$  is thus determined uniquely up to isomorphy by conditions 1) and 2). We call  $K$  the distinguished orthocomplemented  $\omega$ -complete semilattice over  $P$ .

**§ 2. Theorem:** Over every preordered set there is the distinguished orthocomplemented  $\omega$ -complete semilattice.

We construct for the preordered set  $P$  a set  $K$  in the following way:

- 1) Let  $K$  contain the elements of  $P$ . (These we call the prime elements of  $K$ .)
- 2) Let  $K$  contain with finitely many elements  $a_1, a_2, \dots, a_n$  also the combination formed out of these as element. (These we designate by  $a_1 \wedge a_2 \wedge \dots \wedge a_n$ .)

3) Let  $K$  contain with every element  $a$  also an element  $\bar{a}$ .

4) Let  $K$  contain with every countable sequence  $M$  also an element  $\bigwedge_M$ .

Every element of  $K$  may thus be written uniquely as combination  $a_1 \wedge a_2 \wedge \dots \wedge a_n$  of prime elements and elements of the form  $\bar{a}$  or  $\bigwedge_M$ .

We define a relation  $\leq$  in  $K$  in the following way:

- 1) For prime elements  $p, q$  let  $p \leq q$  hold in  $K$  if  $p \leq q$  holds in  $P$ . (These relations we call the basic relations.)
- 2) Every relation  $\leq$  that may be deduced from the basic relations with help of the following rules is to hold in  $K$ :

$$\begin{array}{ll}
 a) \frac{c \leq a \quad c \leq b}{c \leq a \wedge b} & d) \frac{a \leq c}{a \wedge b \leq c} \\
 b) \frac{a \wedge c \leq}{c \leq \bar{a}} & e) \frac{a \leq b}{a \wedge \bar{b} \leq c} \\
 c) \frac{c \leq a_1 \quad \dots \quad c \leq a_n \quad \dots}{c \leq \bigwedge_M} & f) \frac{a_n \wedge b \leq c}{\bigwedge_M \wedge b \leq c} \\
 & (M = a_1, a_2, \dots) \\
 & g) \frac{a \wedge a \wedge b \leq c}{a \wedge b \leq c} .
 \end{array}$$

We call the relations above the line the premisses of the relation below the line.

We have now to show first that  $K$  is an orthocomplemented  $\omega$ -complete semilattice w.r.t. the relation  $\leq$ . For this we must prove

$$\begin{array}{ll}
 \alpha) & a \leq a \\
 \beta) & a \leq b, b \leq c \Rightarrow a \leq c \\
 \gamma) & c \leq a \wedge b \Rightarrow c \leq a \\
 \delta) & c \leq \bar{a} \Rightarrow a \wedge c \leq \\
 \varepsilon) & c \leq \bigwedge_M \Rightarrow c \leq a_n \quad (M = a_1, a_2, \dots)
 \end{array}$$

These properties together with a), b), and c) express in fact that  $K$  is an orthocomplemented  $\omega$ -complete semilattice.

$\alpha)$  holds for prime elements. If  $\alpha)$  holds for  $a$  and  $b$ , then also for  $a \wedge b$  because of

$$\frac{\frac{a \leq a}{a \wedge b \leq a} \quad \frac{b \leq b}{a \wedge b \leq b}}{a \wedge b \leq a \wedge b} .$$

If  $\alpha)$  holds for every  $a_n \in M$ , then also for  $\bigwedge_M$  because of

$$\frac{\frac{a_1 \leq a_1}{\bigwedge_M \leq a_1} \quad \dots \quad \frac{a_n \leq a_n}{\bigwedge_M \leq a_n} \quad \dots}{\bigwedge_M \leq \bigwedge_M} .$$

If  $\alpha)$  holds for  $a$ , then also for  $\bar{a}$ , because of

$$\frac{a \leq a}{a \wedge \bar{a} \leq a} \\ \bar{a} \leq \bar{a} .$$

Hereby  $\alpha)$  is proved in general.

As  $\beta)$  is the most difficult to prove, we take first  $\gamma)$ .

In order to prove  $\gamma)$ , we have to show that if  $c \leq a \wedge b$  is deducible, then also  $c \leq a$  must always be deducible.

We lead the proof indirectly by a transfinite induction. Let  $c \leq a \wedge b$  be deducible, but not  $c \leq a$ . Then the last step of the deduction of  $c \leq a \wedge b$  cannot be  $\frac{c \leq a \quad c \leq b}{c \leq a \wedge b}$ , likewise not  $\frac{c_1 \leq c_2}{c_1 \wedge \bar{c}_2 \leq a \wedge b}$  ( $c = c_1 \wedge \bar{c}_2$ ), as then  $\frac{c_1 \leq c_2}{c_1 \wedge \bar{c}_2 \leq a}$  would be deducible at once.

For the last step remain only the possibilities

$$\frac{\frac{c_1 \leq a \wedge b}{c_1 \wedge c_2 \leq a \wedge b} \quad \frac{c_1 \wedge c_1 \wedge c_2 \leq a \wedge b}{c_1 \wedge c_2 \leq a \wedge b} \quad (c = c_1 \wedge c_2)}{\frac{c_1 \wedge c' \leq a \wedge b \quad \dots \quad c_n \wedge c' \leq a \wedge b \quad \dots}{\bigwedge_M \wedge c' \leq a \wedge b} \quad \left( \begin{array}{l} M = a_1, a_2, \dots \\ c = \bigwedge_M \wedge c' . \end{array} \right)}$$

Here must now  $c_1 \leq a$  resp.  $c_1 \wedge c_1 \wedge c_2 \leq a$  resp. for at least one  $n$   $c_n \wedge c' \leq a$  not be deducible, as otherwise at once  $c \leq a$  would be deducible. In the deduction of  $c \leq a \wedge b$  the claim  $\gamma)$  would thus already be false for a premiss. If in the deduction of a relation I go over to a premiss, of this again to a premiss, etc., then I am after finitely many steps at a basic relation. We would thus obtain a basic relation, for which the claim  $\gamma)$  would be false. But as this is impossible,  $\gamma)$  is thereby proved.

We call the induction that we have undertaken here a premiss induction.

With help of premiss inductions, the proof for  $\delta)$  and  $\varepsilon)$  proceeds just as simply as for  $\gamma)$ , so that I am not going into this any further.

It remains only to show in addition  $\beta)$ . Instead of this we prove the stronger claim

$$\zeta) \quad a \leq b, b \wedge b \wedge \dots \wedge b \wedge c \leq d \quad \Rightarrow \quad a \wedge c \leq d$$

in order to be able to apply premiss inductions hereupon.

Let first  $b$ ,  $c$  and  $d$  be prime elements. Then  $\zeta$ ) holds for every basic relation  $a \leq b$ . We assume as induction hypothesis that  $\zeta$  holds for every premiss of  $a \leq b$ .

As  $b$  is a prime element, the last step of the deduction of  $a \leq b$  can only be:

$$\frac{a_1 \leq b}{a_1 \wedge a_2 \leq b} \quad \frac{a_1 \wedge a_1 \wedge a_2 \leq b}{a_1 \wedge a_2 \leq b} \quad (a = a_1 \wedge a_2)$$

$$\frac{a_1 \leq a_2}{a_1 \wedge \bar{a}_2 \leq b} \quad (a = a_1 \wedge \bar{a}_2) \quad \frac{a_n \wedge a' \leq b}{\bigwedge_M \wedge a' \leq b} \quad \left( \begin{array}{l} M = a_1, a_2, \dots \\ a = \bigwedge_M \wedge a'. \end{array} \right)$$

According to the induction hypothesis, then  $a_1 \wedge c \leq d$  resp.  $a_1 \wedge a_1 \wedge a_2 \wedge c \leq d$  resp.  $a_n \wedge a' \wedge c \leq d$  is deducible. In every case  $a \wedge c \leq d$  is at once deducible, as well from  $a_1 \leq a_2$  because of

$$\frac{\frac{a_1 \leq a_2}{a_1 \wedge \bar{a}_2 \leq d}}{a \wedge c \leq d} .$$

Thereby  $\zeta$ ) is proved for prime elements  $b$ ,  $c$  and  $d$ .

Now let only  $b$  still be a prime element. Then  $\zeta$ ) holds thus for arbitrary  $a$  and prime elements  $c$ ,  $d$ . A premiss induction results now in  $\zeta$ ) holding for every relation  $b \wedge b_1 \wedge \dots \wedge b_1 \wedge c \leq d$ . Every premiss of  $b \wedge b \wedge \dots \wedge b \wedge c \leq d$  has in fact again the form  $b \wedge \dots \wedge b \wedge c \leq d$ . Thereby  $\zeta$ ) is proved in general for prime elements  $b$ .

If  $\zeta$ ) holds for elements  $b_1$  and  $b_2$ , then obviously also for  $b_1 \wedge b_2$ . If  $\zeta$ ) holds for every  $b_n \in M$ , then also for  $b = \bigwedge_M$ . (Proof by premiss induction:  $\bigwedge_M \wedge \bigwedge_M \wedge \dots \wedge \bigwedge_M \wedge c \leq d$  can have the following premiss:  $b_n \wedge \bigwedge_M \wedge \dots \wedge \bigwedge_M \wedge c \leq d$ . According to induction hypothesis holds then  $a \leq \bigwedge_M$ ,  $b_n \wedge \bigwedge_M \wedge \dots \wedge \bigwedge_M \wedge c \leq d \Rightarrow b_n \wedge a \wedge c \leq d$ . But as  $\zeta$ ) is also assumed for  $b = b_n$ , and because of

$$a \leq \bigwedge_M \Rightarrow a \leq b_n,$$

also  $a \leq \bigwedge_M$ ,  $b_n \wedge a \wedge c \leq d \Rightarrow a \wedge a \wedge c \leq d$  holds. But from  $a \wedge a \wedge c \leq d$  may be deduced  $a \wedge c \leq d$ . Every other premiss of  $\bigwedge_M \wedge \bigwedge_M \wedge \dots \wedge \bigwedge_M \wedge c \leq d$  is trivial.)

If  $\zeta$ ) holds for  $b$ , then also for  $\bar{b}$ . (Proof by premiss induction:  $\bar{b} \wedge \bar{b} \wedge \dots \wedge \bar{b} \wedge c \leq d$  can have the following premiss:  $\bar{b} \wedge \dots \wedge \bar{b} \wedge c \leq b$ . Then holds according to induction hypothesis

$$a \leq \bar{b}, \bar{b} \wedge \dots \wedge \bar{b} \wedge c \leq b \Rightarrow a \wedge c \leq b.$$

As  $\zeta$ ) is also assumed for  $b$ , also holds

$$a \wedge c \leq b, a \wedge b \leq d \quad \Rightarrow \quad a \wedge a \wedge c \leq d.$$

Thus holds also  $a \leq \bar{b}, \bar{b} \wedge \dots \wedge \bar{b} \wedge c \leq b \Rightarrow a \wedge c \leq d$  because of  $a \leq \bar{b} \Rightarrow a \wedge b \leq d$ .  
Every other premiss is again trivial.)

Thus  $\zeta$ ) is valid in general. Thereby is proved that  $K$  is an orthocomplemented  $\omega$ -complete semilattice.

$P$  is a part of  $K$ , as

$$p \leq q \text{ in } P \quad \Longleftrightarrow \quad p \leq q \text{ in } K$$

holds. We have for this to convince ourselves that no relation  $p \leq q$  is deducible in  $K$  that is not already holding in  $P$ . But this goes without saying, as none of the rules except  $g$ ) actually yield relations  $p \leq q$  below the line. A deduction of a relation  $p \leq q$  can thus only use the rules  $d$ ) and  $g$ ). But with these only the basic relations are deducible.

For the proof of our theorem it remains now in addition to show that  $K$  may be mapped homomorphically into every other orthocomplemented  $\omega$ -complete semilattice  $K'$  that contains  $P$  as part. This mapping we define by

1) for prime elements  $p$  holds  $p \rightarrow p$ ,

2) moreover is to hold

$$\begin{aligned} a \rightarrow a', b \rightarrow b' &\Rightarrow a \wedge b \rightarrow a' \wedge b' \\ a \rightarrow a' &\Rightarrow \bar{a} \rightarrow \bar{a}' \\ a_n \rightarrow a'_n &\Rightarrow \bigwedge_M \rightarrow \bigwedge_{M'} \quad \left( \begin{array}{l} M = a_1, a_2, \dots \\ M' = a'_1, a'_2, \dots \end{array} \right) \end{aligned}$$

Hereby obviously a homomorphism is being defined, for with  $a \rightarrow a'$  and  $b \rightarrow b'$  always holds  $a \leq b \Rightarrow a' \leq b'$ .

Every deduction of  $a \leq b$  proves in fact at once also  $a' \leq b'$ , as the deduction steps  $a$ - $g$ ) are always correct in every orthocomplemented  $\omega$ -complete semilattice.

**§ 3.** In order to be able to prove the consistency of elementary number theory with complete induction from the theorem proved in § 2, we use the following formalisation. As prime formulas we take the signs for number-theoretic predicates  $A(\dots), B(\dots), \dots$  with the numbers  $1, 1', 1'', \dots$  as arguments, e.g.  $1 = 1''$ ,  $1 + 1 = 1'$ .

These prime formulas  $\mathfrak{P}, \mathfrak{Q}, \dots$  form a preordered set if we set  $\mathfrak{P} \rightarrow \mathfrak{Q}$  in case the predicate  $\mathfrak{P}$  implies the predicate  $\mathfrak{Q}$ . To the basic relations  $\mathfrak{P} \rightarrow \mathfrak{Q}$  we are also adding the relations of the form  $\rightarrow \mathfrak{P}, \mathfrak{P} \rightarrow, \rightarrow, \rightarrow$ , as far as they are contentually correct.



Over this preordered set  $P$  of the prime formulas we construct now as in § 2 the distinguished orthocomplemented  $\omega$ -complete semilattice. We use for this the logistic signs, thus  $\rightarrow$  instead of  $\leq$ ,  $\&$  instead of  $\wedge$ .

To the formulas belong thus the prime formulas, with  $\mathfrak{A}$  and  $\mathfrak{B}$  also  $\mathfrak{A} \& \mathfrak{B}$ , with  $\mathfrak{A}$  also  $\bar{\mathfrak{A}}$ . We restrict the conjunction of countable sequences to the sequences of the form  $\mathfrak{A}(1), \mathfrak{A}(1'), \dots$ . We designate this conjunction by  $(\mathfrak{r})\mathfrak{A}(\mathfrak{r})$ .

Moreover we introduce in addition free variables  $\mathfrak{a} = a, b, \dots$  by the following rule of inference:

if  $A(1), A(1'), \dots$  are deducible relations, then  
 $A(\mathfrak{a})$  is also to be deducible.

By this the proofs of § 2 are only modified unessentially. We obtain overall a calculus  $N$  with the following rules of inference

$$\begin{array}{ll}
 a) \frac{\mathfrak{C} \rightarrow \mathfrak{A} \quad \mathfrak{C} \rightarrow \mathfrak{B}}{\mathfrak{C} \rightarrow \mathfrak{A} \& \mathfrak{B}} & d) \frac{\mathfrak{A} \rightarrow \mathfrak{C}}{\mathfrak{A} \& \mathfrak{B} \rightarrow \mathfrak{C}} \\
 b) \frac{\mathfrak{A} \& \mathfrak{C} \rightarrow}{\mathfrak{C} \rightarrow \bar{\mathfrak{A}}} & e) \frac{\mathfrak{A} \rightarrow \mathfrak{B}}{\mathfrak{A} \& \bar{\mathfrak{B}} \rightarrow \mathfrak{C}} \\
 c) \frac{\mathfrak{C} \rightarrow \mathfrak{A}(1) \quad \dots \quad \mathfrak{C} \rightarrow \mathfrak{A}(n) \quad \dots}{\mathfrak{C} \rightarrow (\mathfrak{r})\mathfrak{A}(\mathfrak{r})} & f) \frac{\mathfrak{A}(n) \& \mathfrak{B} \rightarrow \mathfrak{C}}{(\mathfrak{r})\mathfrak{A}(\mathfrak{r}) \& \mathfrak{B} \rightarrow \mathfrak{C}} \\
 & g) \frac{\mathfrak{A} \& \mathfrak{A} \& \mathfrak{B} \rightarrow \mathfrak{C}}{\mathfrak{A} \& \mathfrak{B} \rightarrow \mathfrak{C}} \\
 h) \frac{\mathfrak{A} \& \mathfrak{B} \rightarrow \mathfrak{C}}{\mathfrak{B} \& \mathfrak{A} \rightarrow \mathfrak{C}} & i) \frac{\mathfrak{A} \& (\mathfrak{B} \& \mathfrak{C}) \rightarrow \mathfrak{D}}{(\mathfrak{A} \& \mathfrak{B}) \& \mathfrak{C} \rightarrow \mathfrak{D}} \\
 j) \frac{A(1) \quad \dots \quad A(n) \quad \dots}{A(\mathfrak{a})} & .
 \end{array}$$

The rules of inference  $h)$  and  $i)$  were dispensable in § 2, as we have introduced there  $a \wedge b \wedge c \wedge \dots$  at once as sign for the combination of  $a, b, c, \dots$ .

The proof in § 2 yields now the following result: the calculus  $N$  is consistent, e.g. the empty relation  $\rightarrow$  is not deducible, as only the contentually correct relations hold in  $P$  and  $P$  is a part of  $N$ . To the calculus  $N$  the following rules of inference can be added without increasing the set the deducible relations:

$$\begin{array}{ll}
 k) \frac{\mathfrak{A} \rightarrow \mathfrak{B} \quad \mathfrak{B} \rightarrow \mathfrak{C}}{\mathfrak{A} \rightarrow \mathfrak{C}} & \\
 l) \frac{\mathfrak{C} \rightarrow \mathfrak{A} \& \mathfrak{B}}{\mathfrak{C} \rightarrow \mathfrak{A}} & m) \frac{\mathfrak{C} \rightarrow \mathfrak{A} \& \mathfrak{B}}{\mathfrak{C} \rightarrow \mathfrak{B}} \\
 n) \frac{\mathfrak{C} \rightarrow \bar{\mathfrak{A}}}{\mathfrak{A} \& \mathfrak{C} \rightarrow} & o) \frac{\mathfrak{C} \rightarrow (\mathfrak{r})\mathfrak{A}(\mathfrak{r})}{\mathfrak{C} \rightarrow \mathfrak{A}(n)} .
 \end{array}$$

To the basic relations can be added  $\mathfrak{A} \rightarrow \mathfrak{A}$ .

This result from § 2 we can now supplement:

- 1) The rule of inference  $p) \frac{A(\mathfrak{a})}{A(n)}$  can also be added.

The proof is again being led by a transfinite premiss induction. If  $A(\mathfrak{a})$  is deducible in  $\mathbb{N}$  and if the last rule of inference of this deduction is not

$$\frac{A(1) \quad \cdots \quad A(n) \quad \cdots}{A(\mathfrak{a})},$$

then the premiss has the form  $A'(\mathfrak{a})$ . If we assume as induction hypothesis that for every premiss  $A'(\mathfrak{a})$  also  $A'(n)$  is deducible, then  $A(n)$  follows at once.

- 2) To the basic relations may be added  $\overline{\overline{\mathfrak{A}}} \rightarrow \mathfrak{A}$ .

For every prime formula  $\mathfrak{P}$  holds in fact always either  $\rightarrow \mathfrak{P}$  or  $\mathfrak{P} \rightarrow$ . Because of

$$\frac{\overline{\overline{\rightarrow \mathfrak{P}}}}{\overline{\overline{\mathfrak{P} \rightarrow \mathfrak{P}}}} \quad \frac{\overline{\overline{\mathfrak{P} \rightarrow}}}{\overline{\overline{\rightarrow \mathfrak{P}}}}, \quad \overline{\overline{\mathfrak{P} \rightarrow \mathfrak{P}}} \rightarrow \mathfrak{P} \text{ is thus always deducible for every prime formula.}$$

From this follows in general the deducibility of  $\overline{\overline{\mathfrak{A}}} \rightarrow \mathfrak{A}$  (cf. e.g. Hilbert-Bernays, *Grundlagen der Mathematik II*).

- 3) The complete induction

$$q) \frac{\mathfrak{A}(\mathfrak{a}) \rightarrow \mathfrak{A}(\mathfrak{a}')}{\mathfrak{A}(1) \rightarrow \mathfrak{A}(\mathfrak{b})}$$

can also be added to the rules of inference without increasing the set the deducible relations.

In fact, if  $\mathfrak{A}(\mathfrak{a}) \rightarrow \mathfrak{A}(\mathfrak{a}')$  is deducible, then also the relation  $\mathfrak{A}(n) \rightarrow \mathfrak{A}(n')$  for every number  $n$ .

For every number  $m$  follows therefrom at once  $\mathfrak{A}(1) \rightarrow \mathfrak{A}(m)$  by  $m$ -fold application of the rule of inference  $k)$ .

Because of  $\frac{\mathfrak{A}(1) \rightarrow \mathfrak{A}(1) \quad \cdots \quad \mathfrak{A}(1) \rightarrow \mathfrak{A}(m) \quad \cdots}{\mathfrak{A}(1) \rightarrow \mathfrak{A}(\mathfrak{b})}$  also  $\mathfrak{A}(1) \rightarrow \mathfrak{A}(\mathfrak{b})$  is thus deducible.

Thereby the consistency the elementary number theory is proved, as the overall admissible rules of inference define a calculus that obviously contains the classical calculus of predicates.