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# Gain-scheduled static output feedback control for saturated LPV systems with bounded parameter variations <sup>\*</sup>

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## Abstract

This paper presents a new design method for gain-scheduled static output feedback (SOF) controllers of saturated LPV (linear parameter varying) systems. Our solution is based on a special SOF scheme in conjunction with specific congruence transformations. The regional design is established through an effective treatment of nonlinear effects introduced by the saturations in the closed loop. Using parameter-dependent Lyapunov functions, the control design is reformulated as a parameter-dependent LMI optimization with a single line search parameter. Then, by an *equivalent* polytopic transformation, tractable design conditions are derived for constrained LPV systems with a broad class of parametric dependencies. In particular, it is proved that the new method *generalizes* some well-known results based on linear matrix equalities while reducing the design conservatism. Moreover, explicit rank constraints on state-space system matrices are not required. A physically motivated example is given for illustration purposes.

**Keywords:** LPV systems, gain-scheduled output control, input saturation, parameter-dependent Lyapunov functions, domain of attraction, linear matrix inequality (LMI).

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## 1. Introduction

Static output feedback (SOF) control has received considerable attention. The main reason is its practical and theoretical importance: (i) SOF represents a simple closed-loop control that can be reliably realized in practice; (ii) many designs of dynamic controllers can be reformulated as SOF control problems involving augmented plants [1]. However, SOF still remains one of the most challenging topics in control theory [2]. Due to its non-convex characterization, existing results in the SOF literature are often too restrictive [3]. Up to now, a great deal of efforts has been devoted to develop numerical tractable solutions for SOF designs, especially through LMI (linear matrix inequality) formulations, see [2, 4, 5] and references therein. For instance, conditions for the existence of SOF solutions were presented in [3, 6], in which matrix-equality constraints are required for the convexification procedure. A state coordinate transformation approach was proposed in [5] for linear polytopic uncertain systems. To design  $\mathcal{H}_2$  and  $\mathcal{H}_\infty$  SOF controllers, slack variables with a lower-triangular structure were introduced in [7] while the conditions in [8] were based on a linear parameter dependent approach. Two-step methods for designing SOF controllers have been proposed in [9–11]. These methods imply suboptimal design procedures since the SOF solution in the second step strongly depends on the state feedback gains obtained in the first step.

Plant uncertainty and actuator saturation are practically encountered in all real-world applications. LPV (linear parameter-

varying) control has been widely applied to address the robustness issue in various engineering areas, including aerospace, automotive, robotics, etc., see [12] for a recent survey. Control design of saturated LPV systems has been also investigated, see for instance [13–16]. Note that most existing works focus on either state feedback or full-order dynamic output feedback involving anti-windup (AW) mechanisms. Unfortunately, all the states are not always available in practice for the state feedback case and the full-order dynamic output feedback control incurs more computational/hardware overheads than a SOF approach [17]. Moreover, the AW paradigm may lead to excessively small operating regions due to the separated design of nominal *unconstrained* controllers and AW compensators [16].

It is important to stress that there is a serious lack of literature on robust SOF control dealing with saturation nonlinearities, which motivates our new solution for this control issue. The contributions of the paper can be summarized as follows.

- Using a special SOF scheme and specific congruence transformations for the convexification, the design of robust SOF controllers for saturated LPV systems can be reformulated in LMI framework with a single line search parameter.
- We demonstrate, with theoretical arguments and numerical illustration, that the new method precisely includes (in terms of design conservatism) some well-known results based on *matrix equality* constraints, for instance [3, 6]. In addition, our method does not require any rank restriction on the state-space matrices. These constraints (matrix equality and rank restriction) are hard to be satisfied for general LPV systems.
- To reduce further the conservatism, a parameter-dependent Lyapunov function (PDLF) and a generalized sector condition are used for theoretical developments. PDLF based ap-

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proaches can exploit *finite* bounds of the rates of the parameter variation and provide less conservative results than quadratic approaches assuming arbitrary parameter variation, see for example [18].

The paper is organized as follows. The control problem is defined in Section 2 and solved in Section 3, where design algorithms are proposed. An illustrative example is shown in Section 4, and conclusions are drawn in Section 5.

*Notation.* For a vector  $x$ ,  $x_i$  denotes its  $i$ th entry. For a matrix  $X$ ,  $X^\top$  denotes its transpose,  $X > 0$  means that  $X$  is positive definite,  $X_{(i)}$  denotes its  $i$ th row, and  $\text{He } X = X + X^\top$ .  $\text{diag}(X_1, X_2)$  denotes a block-diagonal matrix composed of  $X_1, X_2$ . For a matrix  $P > 0$ ,  $\mathcal{E}(P) = \{x \in \mathbb{R}^{n_x} : x^\top P x \leq 1\}$ .  $I$  denotes the identity matrix of appropriate dimension. For two integers  $k_1 < k_2$ ,  $I[k_1, k_2] = \{k_1, k_1 + 1, \dots, k_2\}$ . Arguments are omitted when their meaning is straightforward.

## 2. Problem Formulation

### 2.1. System Description

Consider a continuous-time saturated LPV system

$$\begin{aligned} \dot{x}(t) &= A(\theta(t))x(t) + B(\theta(t))\text{sat}(u(t)) \\ y(t) &= C(\theta(t))x(t) \end{aligned} \quad (1)$$

where  $x(t) \in \mathbb{R}^{n_x}$  is the state,  $u(t) \in \mathbb{R}^{n_u}$  is the control,  $y(t) \in \mathbb{R}^{n_y}$  is the measured output, and  $\theta(t) \in \mathbb{R}^p$  is the vector of *time-varying* parameters whose measurement is available in real time for gain scheduling control. The vector valued saturation function  $\text{sat}(\cdot) : \mathbb{R}^{n_u} \rightarrow \mathbb{R}^{n_u}$  is defined as  $\text{sat}(u_l) = \text{sign}(u_l) \min(|u_l|, \bar{u}_l)$ ,  $l \in I[1, n_u]$ , where  $\bar{u}_l > 0$  denotes the bound of the  $l$ th input. It is assumed that the parameter  $\theta(t) = [\theta_1(t) \dots \theta_p(t)]^\top$  and its unknown rate of variation  $\dot{\theta}(t)$  are smooth and respectively valued in the hypercubes

$$\begin{aligned} \Omega &= \{(\theta_1, \dots, \theta_p)^\top : \theta_j \in [\underline{\theta}_j, \bar{\theta}_j], j \in I[1, p]\} \\ \Omega_d &= \{(\dot{\theta}_1, \dots, \dot{\theta}_p)^\top : \dot{\theta}_j \in [\underline{v}_j, \bar{v}_j], j \in I[1, p]\} \end{aligned}$$

where  $\underline{\theta}_j \leq \bar{\theta}_j$  (respectively  $\underline{v}_j \leq \bar{v}_j$ ) are *known* lower and upper bounds on  $\theta_j$  (respectively  $\dot{\theta}_j$ ), for  $j \in I[1, p]$ . Here, these bounds are not required to be symmetric as in most of existing results in LPV control context. We assume that the time-varying matrices  $A(\cdot)$ ,  $B(\cdot)$  and  $C(\cdot)$  of (1) are continuous on the hypercube  $\Omega$ . Then, using the sector nonlinearity approach proposed in [19, Chapter 2], these state-space matrices can be equivalently represented by

$$\begin{aligned} A(\theta(t)) &= \sum_{i=1}^N \eta_i(\theta(t))A_i, \quad B(\theta(t)) = \sum_{i=1}^N \eta_i(\theta(t))B_i \\ C(\theta(t)) &= \sum_{i=1}^N \eta_i(\theta(t))C_i, \quad N = 2^p \end{aligned} \quad (2)$$

where the membership functions (MFs)  $\eta_i(\cdot)$  are continuously differentiable and belong to the simplex, defined as

$$\Xi_\theta = \left\{ \eta(\theta) \in \mathbb{R}^N : \sum_{i=1}^N \eta_i(\theta) = 1, \eta_i(\theta) \geq 0, \forall \theta \in \Omega \right\}.$$

Note that, since  $(\theta, \dot{\theta}) \in \Omega \times \Omega_d$ , one can easily compute the lower bound  $\phi_{i1}$  and the upper bound  $\phi_{i2}$  of  $\dot{\eta}_i(\cdot)$  as

$$\dot{\eta}_i(\theta) \in [\phi_{i1}, \phi_{i2}], \quad \phi_{i1} \leq \phi_{i2}, \quad i \in I[1, N]. \quad (3)$$

**Remark 1.** The sector nonlinearity approach [19] is used in this paper to derive an *exact* polytopic form of general LPV systems (1). The MFs capture the parameter nonlinearities, i.e. they can be a *nonlinear* function of components of  $\theta(t)$ . Hence, the new method can deal with a larger class of parametric dependencies than, e.g. linear, affine or rational.

### 2.2. Problem Definition

Let us consider a gain-scheduled SOF controller

$$u(t) = K(\theta(t))X(\theta(t))^{-1}y(t) \quad (4)$$

where  $K(\theta) = \sum_{i=1}^N \eta_i(\theta)K_i$ ,  $K_i \in \mathbb{R}^{n_u \times n_y}$ , and  $X(\theta) = \sum_{i=1}^N \eta_i(\theta)X_i$ ,  $X_i \in \mathbb{R}^{n_y \times n_y}$ . From (1) and (4), the closed-loop LPV system can be rewritten as follows:

$$\dot{x}(t) = \mathbf{A}_{cl}x(t) - B(\theta(t))\psi(u(t)) \quad (5)$$

with  $\mathbf{A}_{cl} = A(\theta) + B(\theta)K(\theta)X(\theta)^{-1}C(\theta)$  and  $\psi(u) = u - \text{sat}(u)$ . For the control design of LPV system (5), we consider the following PDLF:

$$\mathbb{V}(x) = x^\top Q(\theta)^{-1}x \quad (6)$$

where  $Q(\theta) = \sum_{i=1}^N \eta_i(\theta)Q_i$  and  $Q_i > 0 \in \mathbb{R}^{n_x \times n_x}$ ,  $\forall i \in I[1, N]$ . The level set associated with  $\mathbb{V}(x)$  is defined as

$$\mathbf{L}_\mathbb{V} = \{x \in \mathbb{R}^{n_x} : x^\top Q(\theta)^{-1}x \leq 1, \text{ for } \forall \theta \in \Omega\}.$$

The set  $\mathbf{L}_\mathbb{V}$  is said to be *contractively invariant* if

$$\dot{\mathbb{V}}(x) = 2x^\top Q(\theta)^{-1}(\mathbf{A}_{cl}x - B(\theta)\psi(u)) + x^\top \dot{Q}(\theta)^{-1}x < 0$$

for all  $x \in \mathbf{L}_\mathbb{V} \setminus \{0\}$  and  $(\theta, \dot{\theta}) \in \Omega \times \Omega_d$ . Clearly, if  $\mathbf{L}_\mathbb{V}$  is contractively invariant, then for every initial state  $x(0) \in \mathbf{L}_\mathbb{V}$ , the state trajectory will converge to the origin and  $\mathbf{L}_\mathbb{V}$  is inside the domain of attraction, see [20, 21]. This paper proposes a constructive solution for the following control problem.

**Problem 1.** Determine the gain-scheduled matrices  $K(\theta)$  and  $X(\theta)$  in (4) and a contractively invariant set, as large as possible, for the closed loop (5) with  $\forall(\theta, \dot{\theta}) \in \Omega \times \Omega_d$ .

The following lemma is useful to deal with the dead-zone nonlinearity  $\psi(u)$ .

**Lemma 1.** Given diagonal matrices  $U_i > 0 \in \mathbb{R}^{n_u \times n_u}$ , matrices  $H_i \in \mathbb{R}^{n_u \times n_x}$ , for  $i \in I[1, N]$ , a vector-valued function  $\eta(\theta) \in \Xi_\theta$ . If  $x \in \mathcal{P}_u \subset \mathbb{R}^{n_x}$  with

$$\mathcal{P}_u = \left\{ x \in \mathbb{R}^{n_x} : \left| (H(\theta)Q(\theta)^{-1})_{(l)} x \right| \leq \bar{u}_l, l \in I[1, n_u] \right\},$$

then  $\psi(u)^\top U(\theta)^{-1} [u - \psi(u) + H(\theta)Q(\theta)^{-1}x] \geq 0$ , where  $U(\theta) = \sum_{i=1}^N \eta_i(\theta)U_i$  and  $H(\theta) = \sum_{i=1}^N \eta_i(\theta)H_i$ .

Lemma 1 presents a parameter-dependent (PD) version of the *generalized sector condition* in [21]. This powerful tool provides a *regional* characterization of the stability and performance properties of the nonlinear LPV system (5) by means of an extension of the absolute stability theory [20].

### 3. SOF Control for Saturated LPV Systems

The following theorem provides the conditions proving the existence of a SOF controller (4) that can asymptotically stabilize the nonlinear LPV system (1).

**Theorem 1.** Consider LPV system (1) with  $(\theta, \dot{\theta}) \in \Omega \times \Omega_d$ . If there exist a PD matrix  $Q(\theta) > 0 \in \mathbb{R}^{n_x \times n_x}$ , a PD diagonal matrix  $U(\theta) > 0 \in \mathbb{R}^{n_u \times n_u}$ , PD matrices  $H(\theta) \in \mathbb{R}^{n_u \times n_x}$ ,  $K(\theta) \in \mathbb{R}^{n_u \times n_y}$ ,  $X(\theta) \in \mathbb{R}^{n_u \times n_u}$ , and a scalar  $\epsilon > 0$  such that (7)-(8) hold for  $\forall(\theta, \dot{\theta}) \in \Omega \times \Omega_d$ . Then, the SOF controller (4) guarantees that  $\mathbf{L}_V$  is a contractively invariant set of (5).

$$\text{He} \begin{bmatrix} \Upsilon_1(\theta, \dot{\theta}) & -B(\theta)U(\theta) & \epsilon B(\theta)K(\theta) \\ \Upsilon_2(\theta) & -U(\theta) & \epsilon K(\theta) \\ \Upsilon_3(\theta) & 0 & -\epsilon X(\theta) \end{bmatrix} < 0 \quad (7)$$

$$\begin{bmatrix} Q(\theta) & H(\theta)^\top \\ H(\theta)_{(l)} & u_l^2 \end{bmatrix} \geq 0, \quad l \in I[1, n_u] \quad (8)$$

where  $\Upsilon_1(\theta, \dot{\theta}) = A(\theta)Q(\theta) + B(\theta)K(\theta)C(\theta) - \dot{Q}(\theta)/2$  and  $\Upsilon_2(\theta) = H(\theta) + K(\theta)C(\theta)$ ,  $\Upsilon_3(\theta) = C(\theta)Q(\theta) - X(\theta)C(\theta)$ .

*Proof.* Condition (7) implies that  $X(\theta) + X(\theta)^\top > 0$ , which ensures the nonsingularity of  $X(\theta)$ . Thus, the expression of the control law (4) is well-defined.

Pre- and postmultiplying (8) with  $\text{diag}(Q(\theta)^{-1}, I)$  yields  $\begin{bmatrix} Q(\theta)^{-1} & Q(\theta)^{-1}H(\theta)^\top \\ H(\theta)_{(l)}Q(\theta)^{-1} & u_l^2 \end{bmatrix} \geq 0, l \in I[1, n_u]$ . By applying Schur complement lemma [22] to this latter, it is easy to show that

$$x^\top Q(\theta)^{-1}x \geq \frac{1}{u_l^2} x^\top Q(\theta)^{-1}H(\theta)_{(l)}^\top H(\theta)_{(l)}Q(\theta)^{-1}x \quad (9)$$

for  $x \in \mathbb{R}^{n_x}$ , and  $l \in I[1, n_u]$ . For  $\forall x \in \mathbf{L}_V$ , it follows clearly from (9) that  $x \in \mathcal{P}_u$ , since  $\mathbb{V}(x) \leq 1$ .

Pre- and postmultiplying (7) with the parameter-dependent matrix  $\begin{bmatrix} I & 0 & B(\theta)K(\theta)X(\theta)^{-1} \\ 0 & I & K(\theta)X(\theta)^{-1} \end{bmatrix}$  and its transpose, we can prove that inequality (7) implies

$$\text{He} \begin{bmatrix} \Upsilon_4(\theta, \dot{\theta}) & -B(\theta)U(\theta) \\ \Upsilon_5(\theta) & -U(\theta) \end{bmatrix} < 0 \quad (10)$$

with  $\Upsilon_4(\theta, \dot{\theta}) = A(\theta)Q(\theta) + B(\theta)K(\theta)X(\theta)^{-1}C(\theta)Q(\theta) - \dot{Q}(\theta)/2$ , and  $\Upsilon_5(\theta) = H(\theta) + K(\theta)X(\theta)^{-1}C(\theta)Q(\theta)$ . Note that  $\dot{Q}(\theta)^{-1} = -Q(\theta)^{-1}\dot{Q}(\theta)Q(\theta)^{-1}$ , then pre- and postmultiplying (10) with  $\text{diag}(Q(\theta)^{-1}, U(\theta)^{-1})$  yields

$$\text{He} \begin{bmatrix} Q(\theta)^{-1}\mathbf{A}_{cl} + \dot{Q}(\theta)^{-1}/2 & -Q(\theta)^{-1}B(\theta) \\ \Upsilon_6(\theta) & -U(\theta)^{-1} \end{bmatrix} < 0 \quad (11)$$

where  $\Upsilon_6(\theta) = U(\theta)^{-1}(H(\theta)Q(\theta)^{-1} + K(\theta)X(\theta)^{-1}C(\theta))$ . Pre- and postmultiplying (11) with  $[x^\top \quad \psi(u)^\top]$  and its transpose together with the use of (4) and (5), we obtain (12) after simple but tedious algebraic manipulations

$$x^\top Q(\theta)^{-1}\dot{x} + \dot{x}^\top Q(\theta)^{-1}x + x^\top \dot{Q}(\theta)^{-1}x + 2\psi(u)^\top U(\theta)^{-1}[u - \psi(u) + H(\theta)Q(\theta)^{-1}x] < 0. \quad (12)$$

By Lemma 1, it follows from (12) that  $\dot{\mathbb{V}}(x) < 0, \forall x \in \mathbf{L}_V, x \neq 0$ , and  $\forall(\theta, \dot{\theta}) \in \Omega \times \Omega_d$ , which concludes the proof.  $\square$

**Remark 2.** The control law (4) relies on the use of the extra variable  $X(\theta)$ , independent to the Lyapunov matrix  $Q(\theta)$ . With the special congruence transformation to obtain (10), this feature allows for an LMI formulation where all decision matrices can be parameter-dependent to reduce the conservatism.

**Remark 3.** It is important to note that the result in Theorem 1 is a generalization of that based on the  $W$ -problem in [3], which is formulated in Corollary 1.

**Corollary 1.** Consider LPV system (1) with  $C(\theta)$  of full row rank and  $(\theta, \dot{\theta}) \in \Omega \times \Omega_d$ . If there exist a PD matrix  $Q(\theta) > 0 \in \mathbb{R}^{n_x \times n_x}$ , a PD diagonal matrix  $U(\theta) > 0 \in \mathbb{R}^{n_u \times n_u}$ , and PD matrices  $H(\theta) \in \mathbb{R}^{n_u \times n_x}$ ,  $K(\theta) \in \mathbb{R}^{n_u \times n_y}$ ,  $X(\theta) \in \mathbb{R}^{n_u \times n_u}$  such that (8), (13) and (14) hold. Then, the SOF controller (4) guarantees that  $\mathbf{L}_V$  is a contractively invariant set of (5).

$$C(\theta)Q(\theta) = X(\theta)C(\theta), \quad \forall \theta \in \Omega \quad (13)$$

$$\text{He} \begin{bmatrix} \Upsilon_1(\theta, \dot{\theta}) & -B(\theta)U(\theta) \\ \Upsilon_2(\theta) & -U(\theta) \end{bmatrix} < 0, \quad \forall(\theta, \dot{\theta}) \in \Omega \times \Omega_d \quad (14)$$

Differently from Theorem 1, Corollary 1 requires explicitly that  $C(\theta)$  must be of full row rank for  $\forall \theta \in \Omega$ . Due to (13) and  $Q(\theta) > 0$ , this ensures the nonsingularity of  $X(\theta)$ . In case of state-feedback scheme, i.e.  $C(\theta) = I$ , it follows from (13) that  $Q(\theta) = X(\theta)$ . Thus, the extra variable  $X(\theta)$  vanishes in Corollary 1. In particular, it is not difficult to prove that the result of Theorem 1 is no more conservative than that of Corollary 1. Indeed, applying Schur complement lemma to (7) while imposing (13) yields

$$\text{He} \begin{bmatrix} \Upsilon_1(\theta, \dot{\theta}) & -B(\theta)U(\theta) \\ \Upsilon_2(\theta) & -U(\theta) \end{bmatrix} + \epsilon \begin{bmatrix} B(\theta)K(\theta) \\ K(\theta) \end{bmatrix} (X(\theta) + X(\theta)^\top)^{-1} \begin{bmatrix} B(\theta)K(\theta) \\ K(\theta) \end{bmatrix}^\top < 0,$$

which is equivalent to (14) for sufficiently small  $\epsilon > 0$ . Note that the linear equality (13) implies that  $C(\theta)Q(\theta)$  is close to commute, which is not possible for a general case [2], especially in LPV control context. We note also that the result presented in [6] leads to the same drawback.

**Remark 4.** Theorem 1 is not convenient for design purposes since (7) and (8) depend explicitly on  $\theta \in \Omega$  and  $\dot{\theta} \in \Omega_d$ . Based on the convex combination form of state-space matrices in (2), we derive in Theorem 2 numerically tractable design conditions for (1). To ease the presentation, we assume that  $C_i = C$ , for  $\forall i \in I[1, N]$ , i.e. constant output matrix. The case involving parameter-dependent output matrix will be discussed later.

**Theorem 2.** Consider LPV system (1) with state-space matrices in (2) and  $(\theta, \dot{\theta}) \in \Omega \times \Omega_d$ . If there exist positive definite matrices  $Q_i \in \mathbb{R}^{n_x \times n_x}$ , diagonal positive definite matrices  $U_i \in \mathbb{R}^{n_u \times n_u}$ , matrices  $H_i \in \mathbb{R}^{n_u \times n_x}$ ,  $K_i \in \mathbb{R}^{n_u \times n_y}$ ,  $X_i \in \mathbb{R}^{n_u \times n_u}$ , for  $i \in I[1, N]$ , and a scalar  $\epsilon > 0$  such that (15), (16) and (17) hold. Then, the SOF controller (4) guarantees that the set  $\mathbf{L}_V$  is contractively invariant with respect to the

closed-loop system (5) for all admissible pairs  $(\theta, \dot{\theta}) \in \Omega \times \Omega_d$ .

$$\begin{bmatrix} Q_i & H_{i(l)}^\top \\ H_{i(l)} & \bar{u}_i^2 \end{bmatrix} \geq 0, \quad i \in I[1, N], l \in I[1, n_u] \quad (15)$$

$$\Psi_{ii}^{klm} < 0, \quad i, k, l \in I[1, N], m \in I[1, 2], k \neq l \quad (16)$$

$$\Psi_{ij}^{klm} + \Psi_{ji}^{klm} < 0, \quad i, j, k, l \in I[1, N], \\ m \in I[1, 2], i < j, k \neq l \quad (17)$$

where the quantity  $\Psi_{ij}^{klm}$  is defined as follows:

$$\Psi_{ij}^{klm} = \text{He} \begin{bmatrix} \Psi_{ij[11]}^{klm} & -B_i U_j & \epsilon B_i K_j \\ H_j + K_j C & -U_j & \epsilon K_j \\ C Q_j - X_j C & 0 & -\epsilon X_j \end{bmatrix} \quad (18)$$

with  $\Psi_{ij[11]}^{klm} = A_i Q_j + B_i K_j C - \phi_{km}(Q_k - Q_l)/2$ .

*Proof.* Multiplying (15) by  $\eta_i(\theta) > 0$  and summing up for  $\forall i \in I[1, N]$ , we obtain (8). Note that  $\sum_{i=1}^N \dot{\eta}_i(\theta) = 0$  since  $\eta(\theta) \in \Xi_\theta$ . Then, one has  $\dot{Q}(\theta) = \dot{\eta}_l(\theta) Q_l + \sum_{k=1, k \neq l}^N \dot{\eta}_k(\theta) Q_k = \sum_{k=1, k \neq l}^N \dot{\eta}_k(\theta) (Q_k - Q_l)$ . For any  $\dot{\eta}_k(\theta)$  such that (3), it follows that

$$\dot{\eta}_k(\theta) = \omega_{k1}(\theta) \phi_{k1} + \omega_{k2}(\theta) \phi_{k2}, \quad k \in I[1, N]$$

where  $\omega_{k1}(\theta) = \frac{\phi_{k2} - \dot{\eta}_k(\theta)}{\phi_{k2} - \phi_{k1}}$  and  $\omega_{k2}(\theta) = \frac{\dot{\eta}_k(\theta) - \phi_{k1}}{\phi_{k2} - \phi_{k1}}$ . It is clear that  $0 \leq \omega_{kl}(\theta) \leq 1$  and  $\sum_{l=1}^2 \omega_{kl}(\theta) = 1$ , for  $k \in I[1, N]$ . Thus, we can exactly represent the term  $\dot{Q}(\theta)$  as

$$\dot{Q}(\theta) = \sum_{k=1}^N \sum_{\substack{m=1 \\ k \neq l}}^2 \omega_{km}(\theta) \phi_{km} (Q_k - Q_l). \quad (19)$$

From (18) and (19), conditions (16)-(17) imply that

$$\Upsilon_{ii}(\theta, \dot{\theta}) < 0, \quad i \in I[1, N] \quad (20)$$

$$\Upsilon_{ij}(\theta, \dot{\theta}) + \Upsilon_{ji}(\theta, \dot{\theta}) < 0, \quad i, j \in I[1, N], i < j \quad (21)$$

with

$$\Upsilon_{ij}(\theta, \dot{\theta}) = \text{He} \begin{bmatrix} \Upsilon_{ij[11]}(\theta, \dot{\theta}) & -B_i U_j & \epsilon B_i K_j \\ H_j + K_j C & -U_j & \epsilon K_j \\ C Q_j - X_j C & 0 & -\epsilon X_j \end{bmatrix} \quad (22)$$

and  $\Upsilon_{ij[11]}(\theta, \dot{\theta}) = A_i Q_j + B_i K_j C - \dot{Q}(\theta)/2$ . From (20), (21) and (22), it follows that

$$\begin{aligned} & \sum_{i=1}^N \eta_i(\theta)^2 \Upsilon_{ii}(\cdot) + \sum_{i=1}^N \sum_{i < j}^N \eta_i(\theta) \eta_j(\theta) (\Upsilon_{ij}(\cdot) + \Upsilon_{ji}(\cdot)) \\ &= \sum_{i=1}^N \sum_{j=1}^N \eta_i(\theta) \eta_j(\theta) \Upsilon_{ij}(\cdot) < 0, \end{aligned}$$

which is exactly (7). Following the arguments of Theorem 1, we can conclude the proof.  $\square$

Note that similar reasoning can be applied to generalize the result of Theorem 2 to deal with a PD output matrix  $C(\theta)$  as well. This is accomplished by including the matrices  $C_i$ ,  $i \in I[1, N]$ , accordingly in conditions (16)-(17).

**Remark 5.** In Theorem 2, the information on both  $\theta(t)$  and  $\dot{\theta}(t)$  is explicitly considered in the control design by exploiting the bounds  $\phi_{kl}$ , for  $k \in I[1, N]$  and  $l \in I[1, 2]$ , given in (3). This enables the use of the PDLF (6) to reduce the design conservatism. Indeed, if conditions (16)-(17) are feasible for arbitrarily large values of  $\phi_{kl}$ , then the only possible solution is such that  $Q_1 \approx \dots \approx Q_N$  to minimize the effect of the terms  $\phi_{km}(Q_k - Q_l)$  involved in (18). Moreover, if one imposes  $Q_i = Q > 0$ , for  $\forall i \in I[1, N]$ , in (6), then the quadratic Lyapunov function  $V(x) = x^\top Q^{-1} x$  is recovered. These discussions mean that the design conditions of Theorem 2 include precisely the quadratic results.

**Remark 6.** We prove in Theorems 1 and 2 that  $\mathbf{L}_V$  is a contractively invariant set of (5). Due to its dependency to  $\theta \in \Omega$ ,  $\mathbf{L}_V$  is time-varying by nature. For convenience, the time-independent set  $\mathcal{E}_V = \bigcap_{i \in I[1, N]} \mathcal{E}(Q_i^{-1}) \subset \mathbf{L}_V$  can be used to characterize the domain of attraction.

In the light of the results in Theorems 1 and 2, the following theorem provides a solution for Problem 1.

**Theorem 3.** Consider LPV system (1) with state-space matrices in (2) and  $(\theta, \dot{\theta}) \in \Omega \times \Omega_d$ . If there exist positive definite matrices  $Q_i$ ,  $\bar{Q} \in \mathbb{R}^{n_x \times n_x}$ , diagonal positive definite matrices  $U_i \in \mathbb{R}^{n_u \times n_u}$ , matrices  $H_i \in \mathbb{R}^{n_u \times n_x}$ ,  $K_i \in \mathbb{R}^{n_u \times n_y}$ ,  $X_i \in \mathbb{R}^{n_u \times n_u}$ , for  $i \in I[1, N]$ , and a scalar  $\epsilon > 0$  such that the optimization (23) is feasible.

$$\begin{aligned} & \max_{\xi_i, i \in I[1, N]} \log \det(\bar{Q}) \\ & \text{s.t. (15)-(17) and } Q_i \geq \bar{Q}, \forall i \in I[1, N] \end{aligned} \quad (23)$$

with  $\xi_i = (\epsilon, Q_i, \bar{Q}, U_i, H_i, K_i, X_i)$ . Then, the SOF controller (4) solves Problem 1 with the guaranteed contractively invariant set  $\mathcal{E}_V$  of the nonlinear LPV system (5).

*Proof.* The inequalities  $Q_i \geq \bar{Q}$ ,  $\forall i \in I[1, N]$ , imply clearly that  $\mathcal{E}(\bar{Q}^{-1}) \subseteq \mathcal{E}_V$ . Then, the proof is a direct consequence of Theorem 2 and Remark 6.  $\square$

#### 4. Illustrative Example

Consider the control design of the lateral axis dynamics for an L-1011 aircraft adapted from [9]. The state vector  $x = [r \ \beta \ \varphi \ \psi]^\top$  is composed of the yaw rate  $r$ , the sideslip angle  $\beta$ , the bank angle  $\varphi$  and the roll rate  $\psi$ . The control input  $u = \delta_a$  is the aileron deflection with magnitude constraint  $\bar{u} = 5$ . The system matrices are given as follows:

$$\begin{aligned} A(\theta) &= \begin{bmatrix} -2.980 & \theta(t) & 0 & -0.034 \\ -\theta(t) & -0.210 & 0.035 & -0.001 \\ 0 & 0 & 0 & 1.000 \\ 0.390 & -1.350 - 3\theta(t) & 0 & -1.890 \end{bmatrix}, \\ B(\theta) &= \begin{bmatrix} -0.032 \\ 0 \\ 0 \\ -\theta(t) \end{bmatrix}, \quad C(\theta) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

For comparison purposes, we consider Problem 1 with

$$\Omega = \{\theta : -0.57 \leq \theta \leq 2.43\}, \quad \Omega_d = \{\dot{\theta} : -v \leq \dot{\theta} \leq v\},$$

for  $v \geq 0$ . Obviously, this affine LPV system can be *equivalently* represented in the form (2) with  $N = 2$ . The details on the corresponding MFs and linear subsystems are not given here due to the space restriction. All LMI computations were done with YALMIP toolbox and SeDuMi solver [23]. The line search for  $\epsilon$  is performed with 100 points linearly gridded over a logarithmic scale in the interval  $[10^{-5}, 10^5]$ . The aim is to compute the maximum  $\log \det(\bar{Q})$  in (23) with different values of  $v$  for two cases: (i) using directly Theorem 3, (ii) using Theorem 3 while imposing  $Q_1 = Q_2 = Q$  (i.e. quadratic approach). Note that the results of the quadratic case are *independent* of the values of  $v$ . Note also that LMI-based design conditions derived from Corollary 1 are *infeasible* for this example for any  $v \geq 0$ . Figure 1 (left) reports the results obtained with the proposed method. These results confirm the interests of considering the PDLF (6) to improve the control performance, in terms of maximizing the domain of attraction.

For illustration, we consider the case  $v = 1$  where the numerical solver takes 41.908 seconds to provide a solution of the optimization (23). Figure 1 (right) depicts a projection of different ellipsoids obtained from the optimization (23). The dashdot blue lines denote 50 ellipsoids  $\mathcal{E}(Q(\theta)^{-1})$ , corresponding to 50 values of  $\theta$  uniformly distributed in  $\Omega$ , which are bounded by  $\mathcal{E}(Q_1^{-1})$  and  $\mathcal{E}(Q_2^{-1})$ . In addition, the domain of attraction provided by quadratic design conditions is precisely included in that of the PDLF approach.

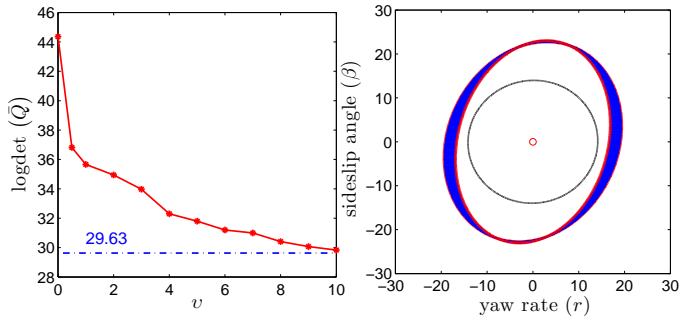


Figure 1: Left: upper bound of  $\log \det(\bar{Q})$  versus  $v$  obtained with Theorem 3 (solid red line) and the quadratic approach (dashdot blue line). Right: projection of different sets obtained from the optimization (23) with  $v = 1$ :  $\mathcal{E}(Q_1^{-1})$  and  $\mathcal{E}(Q_2^{-1})$  (solid red lines),  $\mathcal{E}(Q(\theta)^{-1})$  with  $\theta \in \Omega$  (dashdot blue lines), and  $\mathcal{E}(Q^{-1})$  given by the quadratic approach (dotted black line).

## 5. Conclusions

A new solution for robust SOF control of saturated LPV systems is proposed. Using a PDLF and an exact polytopic transformation, LMI-based design conditions are derived for a large class of constrained LPV systems with general parametric dependencies. The convexification is based on a special congruence transformation which is specific to the proposed SOF structure. As a consequence, the new method requires neither explicit matrix rank constraints nor linear matrix equalities,

which are often hardly tractable. An illustrative example puts in evidence the interests of the proposed method.

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