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# A Boundary Element Formulation based on the Convolution Quadrature Method for the Quasi-static Behaviour Analysis of the Unsaturated Soils

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**Abstract.** This paper aims at obtaining an advanced formulation of the time-domain Boundary Element Method (BEM) for two-dimensional consolidation analysis of unsaturated soil. Unlike the usual time-domain BEM the present formulation applies a Convolution Quadrature developed by Lubich [1,2] which requires only the Laplace-domain instead of the time-domain fundamental solutions.

## Introduction

In compacted fills or in arid climate areas where soils are submitted to wetting-drying cycles such as groundwater recharge, surface runoff and evapo-transpiration, fine-grained soils are not saturated with water, and contain some air. Due to capillary effects and soil-clay adsorption, the pore water is no more positive, and is submitted to suction. Prediction and simulation of unsaturated soil behaviour are of great importance in making critical decisions that affect many facets of engineering design and construction and, therefore, have been the issue of growing concern for several decades.

From the mechanical point of view, an unsaturated porous medium can be represented as a three-phase (gas, liquid, and solid), or three-component (water, dry air, and solid) system in which two phases can be classified as fluids (i.e. liquid and gas). The liquid phase is considered to be pure water containing dissolved air and the gas phase is assumed to be a binary mixture of water vapor and 'dry' air in a non-isothermal case.

In this paper first of all, the set of fully coupled governing differential equations of a porous medium saturated by two compressible fluids (water and air) subjected to quasi-static loadings is obtained. These phenomenal formulations are presented based on the experimental observations and with respect to the poromechanics theory within the framework of the suction-based mathematical model presented by [3,4]. In this model, the effect of deformations on the suction distribution in the soil skeleton and the inverse effect are included in the formulation via a suction-dependent formulation of state surfaces of void ratio and degree of saturation. The linear constitutive law is assumed. The mechanical and hydraulic properties of porous media are assumed to be suction dependent. In this formulation, the solid skeleton displacements  $u_i$ , water pressure  $p_w$  and air pressure  $p_a$  are presumed to be independent variables.

Secondly, the Boundary Integral Equation (BIE) is developed directly from those equations via the use of the weighted residuals method for the first time in a way that permits an easy discretization and implementation in a numerical code. The associated fundamental solution obtained by [5] (in both Laplace transform and time domains) is used in the BIE. Since the corresponding time-domain fundamental solution do not have a simple mathematical structure, the convolution integral appeared in the BIE seems too difficult, even impossible, to be performed analytically. For this reason, a new approach so-called "Operational Quadrature Methods" developed by [1,2] can be used to evaluate the convolution integrals. In this formulation, the convolution integral is numerically approximated by a quadrature formula whose weights are determined by the Laplace transform of the fundamental solution and a linear multistep method

[6]. The resulting BEM time domain formulation represents the first of its kind for two-dimensional consolidation problems.

## Governing Equations

Governing differential equations consist of mass conservation equations of liquid and gaseous phases, the equilibrium equation of skeleton associated with water and air flow equations and constitutive relation. The assumption of infinitesimal transformation and incompressibility of solid matrix is considered.

**Solid Skeleton.** The equilibrium equation and the constitutive law for the soil's solid skeleton including the effects of suction are written [3]:

$$(\sigma_{ij} - \delta_{ij} p_a)_{,j} + p_{a,i} + f_i = 0 \quad (1)$$

$$(\sigma_{ij} - \delta_{ij} p_a) = (\lambda \delta_{ij} \varepsilon_{kk} + 2\mu \varepsilon_{ij}) - F_{ij}^s (p_a - p_w) \quad (2)$$

where  $\mu, \lambda$  are Lamé coefficients,  $p_{a=w,a}$  is the water or air pressure,  $\delta_{ij}$  is the Kronecker delta and  $F_{ij}^s$  is the suction modulus matrix

$$F^s = D.D^{suc-1} \quad (3)$$

In which  $D^{suc}$  is a vector obtained from the state surface of void ratio ( $e$ ) which is a function of the independent variables of  $(\sigma - p_a)$  and  $(p_a - p_w)$ .

$$D^{suc-1} = \partial e / ((1+e) \partial (p_a - p_w)) \quad (4)$$

The elasticity matrix ( $D$ ) can be presented by using the bulk modulus and the tangent modulus

$$D = D(K_0, E_t) = D(\sigma - p_a, p_a - p_w) \quad (5)$$

Where  $E_t$  is tangent elastic modulus which can be evaluated as

$$E_t = E_l + E_s \quad (6)$$

$E_l$  is the elastic modulus in absence of suction and

$$E_s = m_s (p_a - p_w) \quad (7)$$

$m_s$  being a constant,  $E_s$  represents the effect of suction on the elastic modulus.

$K_0$  is the bulk modulus of an open system and evaluated from the surface state of void ratio

$$K_0^{-1} = (1+e) \partial e / \partial (\sigma - p_a) \quad (8)$$

**Continuity and transfer equations for water.** A combination of generalized Darcy's law for water transfer and conservation law for water mass, leads to the general equation for water transfer. The water velocity,  $u_w$ , is defined as

$$u_w = -K_w \nabla (p_w / \gamma_w + z) \quad (9)$$

where  $\gamma_w$  is water unit weight.  $K_w = \alpha_w 10^{\alpha_w e} [(S_r - S_{ru}) / (1 - S_{ru})]^{3.5}$  is the water permeability in which  $S_{ru}$  is residual degree of saturation. The mass conservation law for water unit volume is written as

$$\partial (\rho_w n S_r) / \partial t + \text{div}(\rho_w u_w) = 0 \quad (10)$$

**Continuity and transfer equations for air.** Considering the generalized Darcy's law, the air flow equation can be given as:

$$u_a = -K_a \nabla (p_a / \gamma_a + z) \quad (11)$$

where  $K_a = D_a \gamma_a [e(1 - S_{ru})]^{E_a} / \mu_a$  is the air permeability in which  $\mu_a$  and  $e$  are air dynamic viscosity and void ration, respectively.

Applying the mass conservation law for air, the air transfer equation will be

$$\partial (\rho_a n (1 - S_r)) / \partial t + \text{div}(\rho_a u_a) = 0 \quad (12)$$

in which  $\rho_a$  is air density and  $n$  stands for porosity.

**Summery of the field equations in Laplace transformed domain.** By introducing (2) into (1), (9) into (10), (11) into (12), and by applying the Laplace transformation in order to eliminate the time variable of the partial differential equations, we can write compactly the transformed coupled differential equation system into the following matrix form:

$$\tilde{\mathbf{B}} [\tilde{u}_i \quad \tilde{p}_w \quad \tilde{p}_a]^T + [\tilde{f}_i \quad 0 \quad 0]^T = \mathbf{0} \quad (13)$$

with the not self-adjoint operator  $\tilde{\mathbf{B}}$  :

$$\tilde{\mathbf{B}} = \begin{bmatrix} (\mu\Delta)\delta_{ij} + (\lambda + \mu)\partial_i\partial_j & F^s\partial_i & (1-F^s)\partial_i \\ s\rho_w\hat{S}_r\partial_j & -\rho_w k'_w\Delta - s\rho_w\hat{n}g_1 & s\rho_w\hat{n}g_1 \\ s\rho_a(1-\hat{S}_r)\partial_j & s\rho_a\hat{n}g_1 & -\rho_a k'_a\Delta - s\rho_a\hat{n}g_1 \end{bmatrix} \quad (14)$$

where  $s$  is the Laplace parameter,  $k'_w = k_w / \gamma_w$  and  $k'_a = k_a / \gamma_a$ .

In equation (14),  $i$  and  $j$  vary from one to four. The partial derivative  $(\cdot)_{,i}$  is denoted by  $\partial_i$  and  $\Delta = \partial_{ii}$  is the Laplacian operator. Note the operators  $\tilde{\mathbf{B}}$  in (14) are not self adjoint. Therefore, for the deduction of fundamental solutions the adjoint operator to  $\tilde{\mathbf{B}}$  has to be used:

$$\tilde{\mathbf{B}}^* = \begin{bmatrix} (\mu\Delta)\delta_{ij} + (\lambda + \mu)\partial_i\partial_j & -s\rho_w\hat{S}_r\partial_j & -s\rho_a(1-\hat{S}_r)\partial_j \\ -F^s\partial_i & -\rho_w k'_w\Delta - s\rho_w\hat{n}g_1 & s\rho_a\hat{n}g_1 \\ -(1-F^s)\partial_i & s\rho_w\hat{n}g_1 & -\rho_a k'_a\Delta - s\rho_a\hat{n}g_1 \end{bmatrix} \quad (15)$$

## Fundamental Solutions

Here, the fundamental solution associated with the operator (15) is derived in the Laplace transform domain. Mathematically spoken a fundamental solution is a solution of the equation  $\mathbf{B}\mathbf{G} + \mathbf{I}\delta(x-y)\delta(t-\tau) = \mathbf{0}$  where the matrix of fundamental solutions is denoted by  $\mathbf{G}$ , the identity matrix by  $\mathbf{I}$  and the matrix differential operator by  $\mathbf{B}$ . These solutions can be used in a time-dependent convolution quadrature-based BE formulation which needs only Laplace transformed fundamental solutions. In this study, because the operator type of the governing equations is an elliptical operator, the explicit 2D Laplace transform domain fundamental solution can be derived by using the method of Kupradze et al. [7] or Hörmander [8]. An overview of this method is found in the original work by [8] and more exemplary in References [5,9,10,11,12,13]. The components of fundamental solution tensor are obtained as follows:

$$\tilde{G}_{ij} = \frac{1}{2\pi(\lambda + 2\mu)(\lambda_2^2 - \lambda_1^2)} \left\{ \left[ \left( \frac{2r_i r_j - \delta_{ij}}{r} \right) (\lambda_2 K_1(\lambda_2 r) - \lambda_1 K_1(\lambda_1 r)) + r_i r_j (\lambda_2^2 K_0(\lambda_2 r) - \lambda_1^2 K_0(\lambda_1 r)) \right] + \left[ \frac{g_1 n (k'_a + k'_w)}{k'_w k'_a} .s \left[ \left( \frac{2r_i r_j - \delta_{ij}}{r} \right) \left( \frac{K_1(\lambda_2 r)}{\lambda_2} - \frac{K_1(\lambda_1 r)}{\lambda_1} \right) + r_i r_j (K_0(\lambda_2 r) - K_0(\lambda_1 r)) \right] \right] \right\} \quad (16a)$$

$$\tilde{G}_{i3} = \frac{r_i}{2\pi(\lambda + 2\mu)k'_w(\lambda_2^2 - \lambda_1^2)} \left( \frac{g_1 n}{k'_a} .s^2 \left( \frac{K_1(\lambda_2 r)}{\lambda_2} - \frac{K_1(\lambda_1 r)}{\lambda_1} \right) + S_r .s (\lambda_2 K_1(\lambda_2 r) - \lambda_1 K_1(\lambda_1 r)) \right) \quad (16b)$$

$$\tilde{G}_{i4} = \frac{r_i}{2\pi(\lambda + 2\mu)k'_a(\lambda_2^2 - \lambda_1^2)} \left( \frac{g_1 n}{k'_w} .s^2 \left( \frac{K_1(\lambda_2 r)}{\lambda_2} - \frac{K_1(\lambda_1 r)}{\lambda_1} \right) + (1 - S_r) .s (\lambda_2 K_1(\lambda_2 r) - \lambda_1 K_1(\lambda_1 r)) \right) \quad (16c)$$

$$\tilde{G}_{3j} = \frac{r_j}{2\pi(\lambda + 2\mu)\rho_w k'_w(\lambda_2^2 - \lambda_1^2)} \left( \frac{g_1 n}{k'_a} .s \left( \frac{K_1(\lambda_2 r)}{\lambda_2} - \frac{K_1(\lambda_1 r)}{\lambda_1} \right) + F^s (\lambda_2 K_1(\lambda_2 r) - \lambda_1 K_1(\lambda_1 r)) \right) \quad (16d)$$

$$\tilde{G}_{33} = -\frac{1}{2\pi\rho_w k'_w(\lambda_2^2 - \lambda_1^2)} \left( (\lambda_2^2 K_0(\lambda_2 r) - \lambda_1^2 K_0(\lambda_1 r)) + \left( \frac{(1-F^s)(1-S_r)}{(\lambda + 2\mu)k'_a} + \frac{g_1 n}{k'_a} \right) .s (K_0(\lambda_2 r) - K_0(\lambda_1 r)) \right) \quad (16e)$$

$$\tilde{G}_{34} = \frac{1}{2\pi\rho_w k'_w k'_a(\lambda_2^2 - \lambda_1^2)} \left( \frac{F^s(1-S_r)}{(\lambda + 2\mu)} - g_1 n \right) .s (K_0(\lambda_2 r) - K_0(\lambda_1 r)) \quad (16f)$$

$$\tilde{G}_{4j} = \frac{r_j}{2\pi(\lambda + 2\mu)\rho_a k'_a(\lambda_2^2 - \lambda_1^2)} \left( (1-F^s)(\lambda_2 K_1(\lambda_2 r) - \lambda_1 K_1(\lambda_1 r)) + \frac{g_1 n}{k'_w} .s \left( \frac{K_1(\lambda_2 r)}{\lambda_2} - \frac{K_1(\lambda_1 r)}{\lambda_1} \right) \right) \quad (16g)$$

$$\tilde{G}_{43} = \frac{1}{2\pi\rho_a k'_w k'_a(\lambda_2^2 - \lambda_1^2)} \left( \frac{(1-F^s)S_r}{(\lambda + 2\mu)} - g_1 n \right) \frac{1}{(\lambda_2^2 - \lambda_1^2)} .s (K_0(\lambda_2 r) - K_0(\lambda_1 r)) \quad (16h)$$

$$\tilde{G}_{44} = -\frac{1}{2\pi\rho_a k'_a(\lambda_2^2 - \lambda_1^2)} \left( (\lambda_2^2 K_0(\lambda_2 r) - \lambda_1^2 K_0(\lambda_1 r)) + \left( \frac{F^s S_r}{(\lambda + 2\mu)k'_w} + \frac{g_1 n}{k'_w} \right) .s (K_0(\lambda_2 r) - K_0(\lambda_1 r)) \right) \quad (16i)$$

in which  $K_0(\lambda_i r)$  is the modified Bessel function of the second kind of order zero with the argument  $r = |x - \xi|$  which denoted the distance between a load point and an observation point. In the above Laplace transform domain fundamental solutions, i.e. eqs (27),  $\tilde{G}_{ij}$  is the displacement of the solid skeleton in the  $i^{\text{th}}$  direction due to unit force in the  $j^{\text{th}}$  direction. Whereas  $\tilde{G}_{i3}$  and  $\tilde{G}_{i4}$  are the displacement of the solid skeleton in the  $i^{\text{th}}$  direction due to a unit rate of water and air injection, respectively. Also,  $\tilde{G}_{3j}$  and  $\tilde{G}_{4j}$  are, respectively, the water pressure and air pressure due to the unit force applied in the  $j^{\text{th}}$  direction.  $\tilde{G}_{33}$  and  $\tilde{G}_{34}$  are the water pressure due to a unit rate of water and air injection, respectively. Also,  $\tilde{G}_{43}$  and  $\tilde{G}_{44}$  are the air pressure due to a unit rate of water and air injection, respectively.

### Boundary Element Formulation

To the authors' knowledge, the boundary integral equations for quasi-static unsaturated poroelasticity have not yet been obtained. The boundary integral equations for this problem will be derived by taking the fundamental solution as the weighted function and using the method of weighted residuals, which is essentially an integration by parts technique. In this method, the integral equation is derived directly by equating the inner product of eq (13) and the matrix of the adjoint fundamental solutions  $\tilde{\mathbf{G}}^*$  implying that

$$\tilde{\mathbf{B}}^* \tilde{\mathbf{G}}^* + \mathbf{I} \delta(x - \xi) = 0 \quad (17)$$

to a null vector, i.e.

$$\int_{\Omega} \tilde{\mathbf{B}} \begin{bmatrix} \tilde{u}_\alpha \\ \tilde{p}_w \\ \tilde{p}_a \end{bmatrix} \tilde{\mathbf{G}}^* d\Omega = 0 \quad \text{with} \quad \tilde{\mathbf{G}}^* = \begin{bmatrix} \tilde{G}_{\alpha\beta}^* & \tilde{G}_{\alpha 3}^* & \tilde{G}_{\alpha 4}^* \\ \tilde{G}_{3\beta}^* & \tilde{G}_{33}^* & \tilde{G}_{34}^* \\ \tilde{G}_{4\beta}^* & \tilde{G}_{43}^* & \tilde{G}_{44}^* \end{bmatrix} = \begin{bmatrix} \tilde{U}_{\alpha\beta}^{s*} & \tilde{U}_\alpha^{w*} & \tilde{U}_\alpha^{a*} \\ \tilde{P}_{w\beta}^{s*} & \tilde{P}_w^{w*} & \tilde{P}_w^{a*} \\ \tilde{P}_{a\beta}^{s*} & \tilde{P}_a^{w*} & \tilde{P}_a^{a*} \end{bmatrix} \quad (18)$$

where the integration is performed over a domain  $\Omega$  with boundary  $\Gamma$  and vanishing body forces and sources are assumed. After integrating by parts twice over the domain according to the theory of Green's formula and using partial integration, the operator  $\tilde{\mathbf{B}}$  is transformed from acting on the vector of unknowns  $[\tilde{u}_\alpha \ \tilde{p}_w \ \tilde{p}_a]^T$  to the matrix of fundamental solutions  $\tilde{\mathbf{G}}^*$ . This yields the following system of integral equations in index notation as

$$\begin{aligned} & \int_{\Gamma} [(\lambda \tilde{u}_{k,k} - F_s (\tilde{p}_a - \tilde{p}_w) + \tilde{p}_a) n_\beta \delta_{\alpha\beta} + \mu (\tilde{u}_{\beta,\alpha} + \tilde{u}_{\alpha,\beta}) n_\beta] \tilde{G}_{\alpha j}^* d\Gamma \\ & - \int_{\Gamma} \tilde{u}_\alpha [(\lambda \tilde{G}_{kj,k}^*) n_\beta \delta_{\alpha\beta} + \mu (\tilde{G}_{\alpha j,\beta}^* + \tilde{G}_{\beta j,\alpha}^*) n_\beta] d\Gamma \\ & - \rho_a k'_a \int_{\Gamma} (\tilde{p}_{a,n} \tilde{G}_{4j}^* - \tilde{p}_a \tilde{G}_{4j,n}^*) d\Gamma \\ & + \rho_w k'_w \int_{\Gamma} (\tilde{p}_w \tilde{G}_{3j,n}^* - \tilde{p}_{w,n} \tilde{G}_{3j}^*) d\Gamma + s \rho_a (1 - \hat{S}_r) \int_{\Gamma} \tilde{G}_{4j}^* \tilde{u}_\alpha n_\beta \delta_{\alpha\beta} d\Gamma + s \rho_w \hat{S}_r \int_{\Gamma} \tilde{G}_{3j}^* \tilde{u}_\alpha n_\beta \delta_{\alpha\beta} d\Gamma \\ & + \int_{\Omega} (\tilde{u}_i B_{im}^*) \tilde{G}_{mj}^* d\Omega = 0 \end{aligned} \quad (19)$$

By substituting Eq (17) into (19) and using the property of Dirac's delta function  $\delta(x - \xi)$ , we reach the transformed quasi-static unsaturated poroelastic boundary integral representation for the transformed internal displacements and pressures given in matrix form, i.e.,

$$c\mathbf{I} \begin{bmatrix} u_\alpha \\ p_w \\ p_a \end{bmatrix} = \int_{\Gamma} \begin{bmatrix} \tilde{U}_{\alpha\beta}^S & -\tilde{P}_\alpha^{wS} & -\tilde{P}_\alpha^{aS} \\ \tilde{U}_\beta^W & -\tilde{P}^{wW} & -\tilde{P}^{aW} \\ \tilde{U}_\beta^A & -\tilde{P}^{wA} & -\tilde{P}^{gA} \end{bmatrix} \begin{bmatrix} \tilde{t}_\alpha \\ \tilde{q}_w \\ \tilde{q}_a \end{bmatrix} d\Gamma - \int_{\Gamma} \begin{bmatrix} \tilde{T}_{\alpha\beta}^S & \tilde{Q}_\alpha^{wS} & \tilde{Q}_\alpha^{aS} \\ \tilde{T}_\beta^W & \tilde{Q}^{wW} & \tilde{Q}^{aW} \\ \tilde{T}_\beta^A & \tilde{Q}^{wA} & \tilde{Q}^{aA} \end{bmatrix} \begin{bmatrix} \tilde{u}_\alpha \\ \tilde{p}_w \\ \tilde{p}_a \end{bmatrix} d\Gamma \quad (20)$$

where the traction vector, the normal water flux and the normal gas flux are respectively

$$\tilde{t}_\alpha = \sigma_{\alpha\beta} n_\beta = (\lambda \tilde{u}_{k,k} - F_s (\tilde{p}_a - \tilde{p}_w) + \tilde{p}_a) n_\beta \delta_{\alpha\beta} + \mu (\tilde{u}_{\beta,\alpha} + \tilde{u}_{\alpha,\beta}) n_\beta \quad (21a)$$

$$\tilde{q}_w = \rho_w k'_w \tilde{p}_{w,n} \quad (21b)$$

$$\tilde{q}_a = \rho_a k'_a \tilde{p}_{a,n} \quad (21c)$$

Also the  $\tilde{\mathbf{T}}^S$ ,  $\tilde{\mathbf{Q}}^{wS}$  and  $\tilde{\mathbf{Q}}^{aS}$  in eq (20) can be interpreted as the adjoint terms to the traction vector  $\tilde{t}_\alpha$ , the water flux  $\tilde{q}_w$  and the air flux  $\tilde{q}_a$  as follows

$$T_{\alpha\beta}^S = [(\lambda \tilde{U}_{k\beta,k}^S + s \rho_w \hat{S}_r \tilde{P}_\beta^{wS} + s \rho_a (1 - \hat{S}_r) \tilde{P}_\beta^{aS}) \delta_{\alpha l} + \mu (\tilde{U}_{\alpha\beta,l}^S + \tilde{U}_{l\beta,\alpha}^S)] n_l \quad (22a)$$

$$T_\alpha^W = [(\lambda \tilde{U}_{k,k}^W + s \rho_w \hat{S}_r \tilde{P}^{wW} + s \rho_a (1 - \hat{S}_r) \tilde{P}^{aW}) \delta_{\alpha l} + \mu (\tilde{U}_{\alpha,l}^W + \tilde{U}_{l,\alpha}^W)] n_l \quad (22b)$$

$$T_{\alpha}^A = [(\lambda \tilde{U}_{k,k}^A + s \rho_w \hat{S}_r \tilde{P}^{wA} + s \rho_a (1 - \hat{S}_r) \tilde{P}^{aA}) \delta_{\alpha l} + \mu (\tilde{U}_{\alpha,l}^A + \tilde{U}_{l,\alpha}^A)] n_l \quad (22c)$$

$$\tilde{Q}_{\alpha}^{wS} = \rho_w k'_w P_{\alpha,n}^{wS} \quad (22d)$$

$$\tilde{Q}^{wW} = \rho_w k'_w P_{\alpha,n}^{wW} \quad (22e)$$

$$\tilde{Q}^{wA} = \rho_w k'_w P_{\alpha,n}^{wA} \quad (22f)$$

$$\tilde{Q}_{\alpha}^{aS} = \rho_a k'_a P_{\alpha,n}^{aS} \quad (22g)$$

$$\tilde{Q}^{aW} = \rho_a k'_a P_{\alpha,n}^{aW} \quad (22h)$$

$$\tilde{Q}^{aA} = \rho_a k'_a P_{\alpha,n}^{aA} \quad (22i)$$

The coefficient  $c_{ij}$  in eq (20) has a value  $\delta_{ij}$  for points inside  $\Omega$  and zero outside  $\Omega$ . The value of  $c_{ij}$  for points on the boundary  $\Gamma$  is determined from the Cauchy principal value of the integrals. It is equal to  $0.5\delta_{ij}$  for points on  $\Gamma$  where the boundary is smooth.

Eq (20) can be compacted in index notation for the 2-D case as follows

$$c(\xi) \mathbf{I} \tilde{u}_j(\xi; s) = \int_{\Gamma} (\tilde{G}_{ij}(x, \xi; s) \tilde{t}_i(x; s) - \tilde{F}_{ij}(x, \xi; s) \tilde{u}_i(x; s)) d\Gamma \quad (23)$$

where  $\tilde{t}_i = [\tilde{t}_{\alpha} \quad \tilde{q}_w \quad \tilde{q}_a]^T$ ,  $\tilde{u}_i = [\tilde{u}_{\alpha} \quad \tilde{p}_w \quad \tilde{p}_a]^T$  and also

$$\tilde{G}_{ij} = \begin{bmatrix} \tilde{U}_{\alpha\beta}^S & -\tilde{P}_{\alpha}^{wS} & -\tilde{P}_{\alpha}^{aS} \\ \tilde{U}_{\beta}^W & -\tilde{P}^{wW} & -\tilde{P}^{aW} \\ \tilde{U}_{\beta}^A & -\tilde{P}^{wA} & -\tilde{P}^{aA} \end{bmatrix}; \quad \tilde{F}_{ij} = \begin{bmatrix} \tilde{T}_{\alpha\beta}^S & \tilde{Q}_{\alpha}^{wS} & \tilde{Q}_{\alpha}^{aS} \\ \tilde{T}_{\beta}^W & \tilde{Q}^{wW} & \tilde{Q}^{aW} \\ \tilde{T}_{\beta}^A & \tilde{Q}^{wA} & \tilde{Q}^{aA} \end{bmatrix} \quad (24)$$

The time dependent boundary integral equation for the unsaturated soil is obtained by a transformation to time domain.

$$c(\xi) \mathbf{I} u_j(\xi; t) = \int_0^t \int_{\Gamma} (G_{ij}(t-\tau; x, \xi) t_i(\tau; x) - F_{ij}(t-\tau; x, \xi) u_i(\tau; x)) d\Gamma \quad (25)$$

Equation (25) is an exact represent of the quasi-static response of a multiphase porous medium, involving integrations over the surface as well as the time history. For the practical problem, suitable approximations are needed for both the spatial and temporal variations of field variables. As will be shown, temporal integrations of the time functions involved will be performed numerically using an operational convolution quadrature method (CQM), as like as the spatial integration which be evaluated using numerical techniques. The salient features of the temporal and spatial integrations are outlined below.

**Temporal integration.** Because of the complexity of the time-dependent fundamental solution presented by [5], the convolution quadrature method (CQM) (see References [1,2]) is used. In this formulation, the convolution integral is numerically approximated by a quadrature formula whose weights are determined by the Laplace transform of the fundamental solution and a linear multistep method [6]. By applying this method, the convolution integrals between the fundamental solutions and the nodal values in eq (25) are approximated by

$$G_{ij} \otimes t_i = \int_0^t G_{ij}(t-\tau; x, \xi) t_j(\tau; x) d\tau = \sum_{n=1}^N (\omega_{ij}^G)^{(N-n+1)} t_j^n \quad (26a)$$

$$F_{ij} \otimes u_i = \int_0^t F_{ij}(t-\tau; x, \xi) u_j(\tau; x) d\tau = \sum_{n=1}^N (\omega_{ij}^F)^{(N-n+1)} u_j^n \quad (26b)$$

within each time step, the field variable (displacement/pressure and traction/flux) is assumed to remain constant. In eqs (26)  $(\omega^G)^{N-n+1}$  and  $(\omega^F)^{N-n+1}$  are the influence function which are defined by

$$(\omega_{ij}^G)^m(x, \xi) = \frac{\Re^{-m}}{L} \sum_{l=0}^{L-1} \tilde{G}_{ij}(x, \xi; s_l) e^{-2\pi i \cdot ml/L} \quad (27a)$$

$$(\omega_{ij}^F)^m(x, \xi) = \frac{\Re^{-m}}{L} \sum_{l=0}^{L-1} \tilde{F}_{ij}(x, \xi; s_l) e^{-2\pi i \cdot ml/L} \quad (27b)$$

In eqs. (27)  $s_l$  is given by  $s_l = \gamma(\Re e^{-2\pi il/L}) / \Delta t$ .

By substituting eqs (27) into Equation (25), the time-convoluted boundary element equation is:

$$c_{ij} u_i^N(\xi) = \sum_{n=1}^N \left( \int_{\Gamma} (\omega_{ij}^G)^{(N-n+1)}(x, \xi) t_i^n(x) d\Gamma - \int_{\Gamma} (\omega_{ij}^F)^{(N-n+1)}(x, \xi) u_i^n(x) d\Gamma \right) \quad (28)$$

**Spatial integration.** Using isoparametric quadratic elements and assuming a quadratic variation over both geometry and field variables, the functions (displacements and tractions) at any point over an element can be expressed in terms of the nodal values as  $u_i = N_\alpha(\eta)\bar{U}_{i\alpha}^e$ ,  $t_i = N_\alpha(\eta)\bar{T}_{i\alpha}^e$ ,  $p_w = N_\alpha(\eta)\bar{P}_{w\alpha}^e$ ,  $q_w(\eta) = N_\alpha(\eta)\bar{Q}_{w\alpha}^e$ ,  $p_\alpha = N_\alpha(\eta)\bar{P}_{\alpha\alpha}^e$  and  $q_\alpha(\eta) = N_\alpha(\eta)\bar{Q}_{\alpha\alpha}^e$  where  $i=1,2$  for 2D and  $\alpha=1,2,3$  for a quadratic element and  $N_\alpha(\eta)$  are the shape functions in the local intrinsic coordinates ( $\eta$ ) of the element. Once the spatial discretization process described above has been accomplished, the nodal quantities can be brought outside the surface integrals of Eq (28), since now the integrands contain only known functions. Therefore, the discretized BE equation corresponding to point  $\xi$  can be written as

$$c_{ij} u_i^N(\xi) = \sum_{n=1}^N \sum_{e=1}^E \left[ \sum_{\alpha=1}^3 \left\{ \int_{\Gamma_e} (\omega_{ij}^G)^{(N-n+1)} N_\alpha(\eta) d\Gamma_e(\eta) \right\} \cdot (\bar{T}_{j\alpha}^e)^n - \sum_{\alpha=1}^3 \left\{ \int_{\Gamma_e} (\omega_{ij}^F)^{(N-n+1)} N_\alpha(\eta) d\Gamma_e(\eta) \right\} \cdot (\bar{U}_{j\alpha}^e)^n \right] \quad (29)$$

where  $\Gamma_e$  is the surface of the  $e$ th boundary element,  $E$  is the total number of boundary elements,  $(\bar{U}_\alpha^e)^n$  and  $(\bar{T}_\alpha^e)^n$  represent the nodal values at the node  $\alpha$  of element  $e$  at the moment  $t_n = n\Delta t$  of  $\mathbf{U}$  and  $\mathbf{T}$ .

The integrals which have to be evaluated over the isoparametric element, can be written in intrinsic coordinates. Then

$$c_{ij} u_i^N(\xi) = \sum_{n=1}^N \sum_{e=1}^E \left[ \sum_{\alpha=1}^3 \left\{ \int_{-1}^1 (\omega_{ij}^G)^{(N-n+1)} N_\alpha(\eta) |J(\eta)| d\eta \right\} \cdot (\bar{T}_{j\alpha}^e)^n - \sum_{\alpha=1}^3 \left\{ \int_{-1}^1 (\omega_{ij}^F)^{(N-n+1)} N_\alpha(\eta) |J(\eta)| d\eta \right\} \cdot (\bar{U}_{j\alpha}^e)^n \right] \quad (30)$$

where  $J(\eta)$  is the Jacobian of transformation.

The usual point collocations scheme, i.e. by allowing point  $\xi$  to coincide sequentially with all the nodal points of the boundary, is used to establish a set of integral equations in order to obtain unknown boundary values. Also, writing systematically at each global boundary node produces a system of algebraic equations containing the generalized displacement and traction at all collocation points at time steps  $N$  of the form:

$$\mathbf{c} \cdot \mathbf{u}^N(\xi) = \sum_{e=1}^E \sum_{\alpha=1}^3 \left( (\bar{T}_\alpha^e)^N \Delta \mathbf{G}_\alpha^{e(1)} - (\bar{U}_\alpha^e)^N \Delta \mathbf{F}_\alpha^{e(1)} \right) + \sum_{n=1}^{N-1} \sum_{e=1}^E \sum_{\alpha=1}^3 \left( (\bar{T}_\alpha^e)^n \Delta \mathbf{G}_\alpha^{e(N-n+1)} - (\bar{U}_\alpha^e)^n \Delta \mathbf{F}_\alpha^{e(N-n+1)} \right) \quad (31)$$

where  $\Delta \mathbf{G}_{i\alpha}^{e(N-n+1)} = \int_{-1}^1 (\omega_{ij}^G)^{(N-n+1)} N_\alpha(\eta) |J(\eta)| d\eta$  and  $\Delta \mathbf{F}_{i\alpha}^{e(N-n+1)} = \int_{-1}^1 (\omega_{ij}^F)^{(N-n+1)} N_\alpha(\eta) |J(\eta)| d\eta$ .

Then, for each time steps it is sufficient to obtain  $\Delta \mathbf{G}_\alpha^e$  and  $\Delta \mathbf{F}_\alpha^e$  only for the current time steps.

## Conclusion

In this paper, an advanced formulation of the time-domain Boundary Element Method (BEM) for two-dimensional consolidation analysis of unsaturated soil is obtained. Unlike the usual time-domain BEM the present formulation applies a Convolution Quadrature developed by [1,2] which requires only the Laplace-domain instead of the time-domain fundamental solutions.

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