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Inverse scattering for the Bethe-Peierls model

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Abstract. We consider the phased and phaseless inverse scattering problems for the Bethe-Peierls model. We give complete solutions of these problems including questions of uniqueness, nonuniqueness, reconstruction and characterization.

Keywords: inverse scattering, Schrödinger equation, exactly solvable models

MSC: 35R30, 35P25, 35J10, 81Q80

1. Introduction

We consider the three-dimensional Schrödinger equation

$$-\Delta\psi(x) = E\psi(x), \quad x \in \mathbb{R}^3 \setminus \{0\}, \quad E > 0, \quad (1)$$

where

$$\psi(x) = \frac{\psi_{-1}}{|x|} + \psi_0 + O(|x|), \quad x \rightarrow 0, \quad \psi_0 = -\alpha\psi_{-1}, \quad \alpha \in \mathbb{R}. \quad (2)$$

Actually, condition (2) is the Bethe-Peierls boundary condition; see [BP], [GN]. In [BP] model (1), (2) with some $\alpha > 0$ was used for describing the neutron-photon interaction. We say that model (1), (2) is the Bethe-Peierls model.

For the Bethe-Peierls model (1), (2) we consider the scattering solutions ψ^+ and the scattering amplitude f :

$$\psi^+(x, k) = e^{ikx} - 2\pi^2 \frac{e^{i\sqrt{E}|x|}}{|x|} f, \quad (3)$$

$$f = \frac{1}{2\pi^2} \frac{1}{\alpha + i\sqrt{E}}, \quad (4)$$

where $x \in \mathbb{R}^3$, $k \in \mathbb{R}^3$, $k^2 = E$, $\sqrt{E} > 0$; see [AGHH], [GN] and references therein.

In particular, it is well-known that f satisfies the unitarity condition:

$$f - \bar{f} = -\pi i \sqrt{E} \int_{\mathbb{S}^2} f \bar{f} d\omega, \quad (5)$$

where \mathbb{S}^2 is the unit sphere in \mathbb{R}^3 .

For model (1), (2) we also consider the differential scattering cross section $\sigma = |f|^2$.

The problem of inverse scattering (with phase information) for model (1), (2) consists in finding α from f .

The problem of inverse scattering without phase information for model (1), (2) consists in finding α from $\sigma = |f|^2$.

These problems including questions of uniqueness, nonuniqueness, reconstruction and characterization are solved in the next section.

2. Inverse scattering

We define the circle S_E in the complex plane \mathbb{C} :

$$\begin{aligned} S_E = \{ \zeta \in \mathbb{C} : \zeta - \bar{\zeta} = -4i\pi^2\sqrt{E}\zeta\bar{\zeta} \} = \\ \{ \zeta \in \mathbb{C} : \left| \zeta + \frac{i}{4\pi^2\sqrt{E}} \right| = \frac{1}{4\pi^2\sqrt{E}} \}, \quad \sqrt{E} > 0. \end{aligned} \quad (6)$$

Theorem 1. *Let $E > 0$ be fixed (with $\sqrt{E} > 0$). Let f be the scattering amplitude of (3), (4). Then $f \in S_E$ for any $\alpha \in \mathbb{R} \cup \{\infty\}$. Conversely, for any $f \in S_E$ there exists the unique $\alpha \in \mathbb{R} \cup \{\infty\}$ such that formula (4) holds, and this α is given by the formula:*

$$\alpha = \frac{\operatorname{Re} f}{2\pi^2|f|^2}. \quad (7)$$

Proof of Theorem 1. For the scattering amplitude f of (3), (4), the fact that $f \in S_E$ follows from the unitarity condition (5). This can be also verified by a direct computation using (4).

Conversely, for fixed $f \in S_E$ we consider (4) as an equation with respect to $\alpha \in \mathbb{R} \cup \{\infty\}$. One can see that this equation is uniquely solvable and that the solution is given by the formula

$$\alpha = \frac{1}{2\pi^2 f} - i\sqrt{E}. \quad (8)$$

In addition, a direct computation also shows that

$$\alpha = \bar{\alpha} \quad \text{if } f \in S_E. \quad (9)$$

Finally, formulas (8), (9) imply that

$$\alpha = \frac{\alpha + \bar{\alpha}}{2} = \frac{\operatorname{Re} f}{2\pi^2|f|^2}. \quad (10)$$

Theorem 1 is proved.

Remark 1. Theorem 1 gives a complete solution of the inverse scattering problem with phase information for model (1), (2) at fixed energy $E > 0$, including uniqueness, reconstruction and characterization.

Next, we have that

$$\sigma = \frac{1}{4\pi^4} \frac{1}{\alpha^2 + E}, \quad (11)$$

where $\sigma = |f|^2$ is the differential scattering cross section for model (1), (2).

In connection with σ of (11) we consider the interval

$$I_E = [0, \frac{1}{4\pi^4 E}], \quad E > 0. \quad (12)$$

Theorem 2. *Let $E > 0$ be fixed. Let σ be the differential scattering cross section of (11). Then $\sigma \in I_E$ for any $\alpha \in \mathbb{R} \cup \{\infty\}$. Conversely, for any $\sigma \in I_E$ there exists the unique $\alpha^2 \in [0, +\infty]$ such that formula (11) holds; in addition, the values of α for this α^2 are given by the formula*

$$\alpha = \pm \sqrt{\frac{1}{4\pi^4 \sigma} - E}. \quad (13)$$

The simplest proof of Theorem 2 consists in direct computations proceeding from formula (11).

Theorem 2 also follows from Theorem 1 and formula (5).

Remark 2. Theorem 2 gives a complete solution of the inverse scattering problem without phase information for model (1), (2) at fixed energy $E > 0$, including uniqueness, nonuniqueness, reconstruction and characterization. In particular, the solution is unique if $\alpha \geq 0$.

Finally, using formula (11) we also obtain the following example of nonuniqueness for phaseless inverse scattering from the differential scattering cross section σ given for all positive energies E .

Example 1. *Let σ_+ and σ_- be the differential scattering cross sections for model (1), (2) with $\alpha = \alpha_+$ and with $\alpha = \alpha_-$, respectively, where $\alpha_+ > 0$, $\alpha_- = -\alpha_+$. Then*

$$\sigma_+ \equiv \sigma_- \quad \text{for all } E > 0. \quad (14)$$

3. Additional remarks

Remark 3. Finding α from f or from $\sigma = |f|^2$ is not considered in [BP]. The work [BP] refers to other physical approaches for determining α .

Remark 4. Two-dimensional analogs of Theorems 1 and 2 were given in [AN]. However, the two-dimensional analog of Example 1 does not valid in the framework of considerations of [AN].

Remark 5. Analog of Theorems 1 and 2 and Example 1 are valid for the case of the one-dimensional Schrödinger equation on the whole line with the potential $v(x) = \varepsilon\delta(x)$, where $\delta = \delta(x)$, $x \in \mathbb{R}$, is the Dirac function, $\varepsilon \in \mathbb{R}$.

Remark 6. The property that $f \in S_E$ (see Theorem 1) can be considered as a relationship between the amplitude and phase of a signal scattered by a point inhomogeneity at $\{0\}$. In the framework of acoustic scattering such a relationship was obtained in [BM], [BBMR] in the form:

$$\begin{aligned} \sin \varphi &= \frac{-|\beta|}{(4\pi/\sqrt{E})}, \quad \text{where} \\ \beta &= |\beta| \exp(i\varphi), \quad \beta = (2\pi)^3 f. \end{aligned} \tag{15}$$

However, in [BM], [BBMR] this relation is not yet related with the unitarity property (5) of the scattering amplitude f .

Note that a two-dimensional analog of Remark 6 was given in [AN].

Remark 7. In connection with recent results on acoustic scattering by a quasi-point inhomogeneity of a medium, we refer to [D].

Remark 8. Formulas (3), (4), (8) remain valid for $\alpha \in \mathbb{C}$ in model (1), (2).

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