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A UNIFIED APPROACH TO SOLVING SOME INVERSE PROBLEMS FOR EVOLUTION EQUATIONS BY USING OBSERVABILITY INEQUALITIES

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Abstract. We survey some of our recent results on inverse problems for evolution equations. The goal is to provide a unified approach to solve various types of evolution equations. The inverse problems we consider consist in determining unknown coefficients from boundary measurements by varying initial conditions. Based on observability inequalities and a special choice of initial conditions, we provide uniqueness and stability estimates for the recovery of volume and boundary lower order coefficients in wave and heat equations. Some of the results presented here are slightly improved from their original versions.

Contents

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1. Introduction

Inverse coefficient problems for evolution equations have been a very active area in mathematical and numerical research over the last decades, driven by numerous applications. They are intrinsically difficult to solve: this fact is due in part to their very mathematical structure and to the fact that generally only partial data is available [17]. We survey in this paper some of our recent results on inverse problems for evolution equations concerning heat and wave equations. In [2] the authors proposed a general method to deal with inverse source problems for evolution equations. Starting from the ideas in [2], we developed an approach based on observability inequalities and a spectral decomposition to solve some inverse coefficients problems in evolution equations [3, 4, 5]. However the approach is older than that. Inverse coefficient problems in heat and wave equations using control techniques have been studied by a large community of people (see for instance [29, 21, 15, 16, 25, 24, 8, 6] and the references therein). It would be impossible to present here all the relevant results that have been proved in this research direction. We will be mainly focusing on the results that are closely connected to the considered inverse coefficient problems in heat and wave equations.

The measurements are made on a sub-boundary by varying initial conditions. The key idea in our analysis consists in reducing the inverse coefficients problems to inverse source problems. This is achieved by using a spectral decomposition and unique continuation property of eigenfunctions.

For simplicity convenience we limited ourselves to initial boundary value problems for wave and heat equations. But our analysis can be extended to other types of evolution equations such as dynamical Schrödinger equation.

The main ingredient in our approach is observability inequalities. We point out that the wave and the heat equations have different observability properties. We know that, under some appropriate conditions, the wave equation is exactly observable, while the heat equation is only final time observable [28, 30]. We refer to Section 2 for details. In Section 3, we establish weighted interpolation inequalities involving the eigenfunctions of Laplace-Beltrami operator that are useful in the analysis of the stability issue of the studied inverse coefficient problems. These inequalities have been obtained by quantifying the unique continuation property for the Laplace-Beltrami operator through weighted energy estimates with the aid of Carleman type inequalities. We present an abstract framework for the inverse source problem in Section 4. Based on the introduced observability inequalities we provide uniqueness and stability inequalities of the recovery of volume and boundary lower order coefficients in wave and heat equations from boundary measurements in respectively Sections 5 and 6.

2. Observability inequalities

We collect in this section various observability inequalities that are necessary to the analysis of the inverse problems we want to tackle in this text. Since most of these results are well recorded in the literature we limited ourselves to give their precise statement and provide the references where the proofs can be found.

2.1. Wave and heat equations in a Riemannian manifold.

Let $n \geqslant 2$ be an integer and consider $M = (M, g)$ a compact n-dimensional Riemannian manifold with boundary. By a manifold with boundary we mean a C^∞ manifold and its

boundary is C^∞ manifold of dimension $n-1$. Throughout, we adopt the Einstein convention summation for repeated indices. If in any term the same index name appears twice, as both an upper and a lower index, that term is assumed to be summed from 1 to n.

In local coordinates system $x = (x^1, \dots, x^n)$,

$$g = g_{ij} dx^i \otimes dx^j.$$

Let $(\partial_1, \dots, \partial_n)$ be the dual basis of $(x^1, \dots, x^n)$. For two vector fields $X = X^i \partial_i$ and $Y = Y^j \partial_j$ over M, set

$$\langle X, Y \rangle = g_{ij} X^i Y^j$$

and $|X| = \sqrt{\langle X, X \rangle}$.

Recall that the gradient of $u \in C^\infty(M)$ is the vector field given by

$$\nabla u = g^{ij} \partial_i u \partial_j$$

and the Laplace-Beltrami operator is the operator acting as follows

$$\Delta u = \frac{1}{\sqrt{\det g}} \partial_i \left(\sqrt{\det g}\, g^{ij} \partial_j u \right),$$

where (g^{ij}) denote the inverse of the metric (g_{ij}).

We are first concerned with observability inequalities for the wave equation. Consider then the following initial-boundary value problem, abbreviated to IBVP's in the sequel, for the wave equation:

$$\begin{cases} \partial_t^2 u - \Delta u + q(x) u + a(x) \partial_t u = 0 & \text{in } M \times (0, \tau), \\ u = 0 & \text{on } \partial M \times (0, \tau), \\ u(\cdot, 0) = u_0, \ \partial_t u(\cdot, 0) = u_1. \end{cases} \tag{2.1}$$

The usual energy space for the wave equation is given by

$$\mathcal{H} = H_0^1(M) \oplus L^2(M).$$

According to [10, sections 5 and 6, Chapter XVIII] or [7, Chapter 2]), for any $q, a \in L^\infty(M)$, $\tau > 0$ and $(u_0, u_1) \in \mathcal{H}_0$, the IBVP (2.1) has a unique solution

$$u = u(q, a, (u_0, u_1)) \in C([0, \tau], H_0^1(M))$$

so that $\partial_t u \in C([0, \tau], L^2(M))$. If in addition

$$\|q\|_\infty + \|a\|_\infty \leqslant \aleph,$$

for some constant $\aleph > 0$, then by the energy estimate

$$\|u\|_{C([0,\tau], H_0^1(M))} + \|\partial_t u\|_{C([0,\tau], L^2(M))} \leqslant C \|(u_0, u_1)\|_{\mathcal{H}_0}, \tag{2.2}$$

holds with $C = C(\aleph) > 0$ is a nondecreasing function.

Denote by ν the unit normal vector field pointing inward M and set $\partial_\nu u = \langle \nabla u, \nu \rangle$. From [7, Lemma 2.4.1] $\partial_\nu u \in L^2(\partial M \times (0, \tau))$ and

$$\|\partial_\nu u\|_{L^2(\partial M \times (0,\tau))} \leqslant c_M \left(\|(u_0, u_1)\|_{\mathcal{H}_0} + \|qu + a \partial_t u\|_{L^1((0,\tau), L^2(M))} \right), \tag{2.3}$$

where c_M is a constant depending only on M.

In light of (2.2), (2.3) yields

$$\|\partial_\nu u\|_{L^2(\partial M \times (0,\tau))} \leqslant C \|(u_0, u_1)\|_{\mathcal{H}}, \tag{2.4}$$

with a constant C of the same form as in (2.2).

Let Γ be a non empty open subset of ∂M and $\tau > 0$ so that (Γ, τ) geometrically control M. This means that every generalized geodesic traveling at speed one in

M meets Γ in a non-diffractive point at a time $t \in (0,\tau)$ (we refer to [19] for more details).

Fix $(q_0,a_0) \in L^\infty(M)\times L^\infty(M)$. In light of [19, theorem page 169] (which remains valid for the wave operator plus an operator involving space derivatives of first order) and bearing in mind that controllability is equivalent to observability we can state the following inequality

$$2\kappa_0\|(u_0,u_1)\|_{\mathcal{H}} \leqslant \|\partial_\nu u^0\|_{L^2(\Gamma\times(0,\tau))}, \tag{2.5}$$

for some constant $\kappa_0>0$, where we set $u^0=u(a_0,q_0,(u_0,u_1))$ for $(u_0,u_1)\in\mathcal{H}$.

By a perturbation argument, there exists $\beta>0$, depending on (q_0,a_0) and κ_0, so that, for any $(q,a)=(q_0,a_0)+(\tilde q,\tilde a)$, with $(\tilde q,\tilde a)\in L^\infty(M)\times L^\infty(M)$ satisfying $\|(\tilde q,\tilde a)\|_{L^\infty(M)\times L^\infty(M)}\leqslant\beta$, we have

$$\kappa_0\|(u_0,u_1)\|_{\mathcal{H}} \leqslant \|\partial_\nu u\|_{L^2(\Gamma\times(0,\tau))}.$$

Here $\kappa_0>0$ is the same as in previous inequality and $u=u(a,q,(u_0,u_1))$.

Theorem 2.1. *Let $(q_0,a_0)\in L^\infty(M)\times L^\infty(M)$ and assume that (Γ,τ) geometrically control M. There exist $\kappa>0$ and $\beta>0$, only depending on Γ, M and (q_0,a_0), such that for any $(q,a)=(q_0,a_0)+(\tilde q,\tilde a)$ with $(\tilde q,\tilde a)$ satisfying*

$$\|(\tilde q,\tilde a)\|_{L^\infty(M)\times L^\infty(M)}\leqslant\beta$$

we have

$$\kappa\|(u_0,u_1)\|_{\mathcal{H}_0} \leqslant \|\partial_\nu u\|_{L^2(\Gamma\times(0,\tau))}, \tag{2.6}$$

where $u=u(q,a,(u_0,u_1))$.

Next, we examine the case where we do not assume that (Γ,τ) geometrically control M. Define

$$\mathbf{d}(\Gamma)=\sup\{d(x,\Gamma);\ x\in M\},$$

let

$$v=v(q,(u_0,v_0))=u(q,0,(u_0,u_1))$$

and set

$$\mathcal{H}_{-1}=L^2(M)\oplus H^{-1}(M).$$

In light of [18, Corollary 3.2] we have

Theorem 2.2. *Let $\aleph>0$. Under the assumption $\tau>2\mathbf{d}(\Gamma)$ there exist positive constants C, κ and ϵ_0 so that for any $q\in L^\infty(M)$ with $\|q\|_{L^\infty(M)}\leqslant\aleph$ we have*

$$C\|(u_0,u_1)\|_{\mathcal{H}_{-1}} \leqslant e^{\kappa\epsilon}\|\partial_\nu v\|_{L^2(\Gamma\times(0,\tau))}+\frac{1}{\epsilon}\|(u_0,u_1)\|_{\mathcal{H}},\quad (u_0,u_1)\in\mathcal{H},\ \epsilon\geqslant\epsilon_0. \tag{2.7}$$

Here $v=v(q,(u_0,v_0))$.

We now give an observability inequality for a parabolic equation. Consider then the IBVP

$$\begin{cases} \partial_t u-\Delta u+q(x)u=0 & \text{in } M\times(0,\tau),\\ u=0 & \text{on } \partial M\times(0,\tau),\\ u(\cdot,0)=u_0. \end{cases} \tag{2.8}$$

For $q\in L^\infty(M)$ let $A_q=\Delta-q$ with domain $D(A_q)=H_0^1(M)\cap H^2(M)$. As A_0 (that is A_q with $q=0$) is an m-dissipative operator we deduce form the well

established theory of continuous semigroups that A_q generates a strongly continuous semigroup e^{tA_q}. Therefore, for any $u_0 \in L^2(M)$, the IBVP has a unique solution

$$u = u(q, u_0) = e^{tA_q} u_0 \in C([0,\tau], L^2(M)) \cap C^1(]0,\tau], H^2(M) \cap H_0^1(M)).$$

The perturbation argument we used previously for the wave equation is in fact stated in general abstract setting [28, Proposition 6.3.3, page 189], which is also applicable for the heat equation. This together with [20, Corollary 4] yield the following final time observability inequality.

Theorem 2.3. *Let $\tau > 0$, Γ a non empty open subset of ∂M and $\aleph > 0$. There exists a constant $C > 0$ so that for any $q \in L^\infty(M)$ satisfying $\|q\|_{L^2(M)} \leqslant \aleph$ we have*

$$\|u(\cdot,\tau)\|_{L^2(M)} \leqslant C \|\partial_\nu u\|_{L^2(\Gamma\times(0,\tau))}, \tag{2.9}$$

where $u = u(q, u_0)$ with $u_0 \in L^2(M)$.

2.2. **The wave equation in a rectangular domain with boundary damping.** Consider on $\Omega = (0,1)\times(0,1)$ the IBVP

$$\begin{cases} \partial_t^2 u - \Delta u = 0 & \text{in } \Omega \times (0,\tau), \\ u = 0 & \text{on } \Gamma_0 \times (0,\tau), \\ \partial_\nu u + a\partial_t u = 0 & \text{on } \Gamma_1 \times (0,\tau), \\ u(\cdot,0) = u_0,\ \partial_t u(\cdot,0) = u_1. \end{cases} \tag{2.10}$$

Here

$$\Gamma_0 = ((0,1)\times\{1\}) \cup (\{1\}\times(0,1)),$$
$$\Gamma_1 = ((0,1)\times\{0\}) \cup (\{0\}\times(0,1))$$

and $\partial_\nu = \nu \cdot \nabla$ is the derivative along ν, the unit normal vector pointing outward of Ω. Note that ν is everywhere defined except at the vertices of Ω.

We identify in the sequel $a|_{(0,1)\times\{0\}}$ by $a_1 = a_1(x)$, $x \in (0,1)$ and $a|_{\{0\}\times(0,1)}$ by $a_2 = a_2(y)$, $y \in (0,1)$. In that case it is natural to identify a, defined on Γ_1, by the pair (a_1, a_2).

Fix $1/2 < \alpha \leqslant 1$ and let

$$\mathscr{A} = \{b = (b_1, b_2) \in C^\alpha([0,1]) \oplus C^\alpha([0,1]),\ b_1(0) = b_2(0),\ b_j \geqslant 0\}.$$

Let $V = \{u \in H^1(\Omega);\ u = 0 \text{ on } \Gamma_0\}$ and define on $V \oplus L^2(\Omega)$ the unbounded operator A_a, $a \in \mathscr{A}$, by

$$A_a = (w, \Delta v), \quad D(A_a) = \{(v,w) \in V \oplus V;\ \Delta v \in L^2(\Omega) \text{ and } \partial_\nu v = -aw \text{ on } \Gamma_1\}.$$

From [4] A_a generates a strongly continuous semigroup e^{tA_a}. Whence, for any $(u_0, u_1) \in D(A_a)$, the IBVP (2.10) has a solution $u = u(a, (u_0, u_1))$ so that

$$(u, \partial_t u) \in C([0,\tau], D(A_a)) \cap C^1([0,\tau], V \oplus L^2(\Omega)).$$

We proved in [4, Corollary 2.2] the following observability inequality

Theorem 2.4. *Fix $0 < \delta_0 < \delta_1$. Then there exist $\tau_0 > 0$ and $\kappa > 0$, depending only on δ_0 and δ_1, so that for any $\tau \geqslant \tau_0$ and $a \in \mathscr{A}$ satisfying $\delta_0 \leqslant a \leqslant \delta_1$ on Γ_1 we have*

$$\kappa \|(u_0, u_1)\|_{V\oplus L^2(\Omega)} \leqslant \|\partial_\nu u\|_{L^2(\Gamma_1\times(0,\tau))},$$

where $u = u(a, (u_0, u_1))$, with $(u_0, u_1) \in D(A_a)$.

It is worth noticing that Γ_1 satisfies the geometric control condition given in the multiplier method. We also point out that a special case was considered by the third author and Ren [27] in which the observation is made only on one side of Γ_1.

3. Weighted interpolation inequalities

We aim in the present section establishing two weighted interpolation inequalities. These inequalities will be useful in the proof of Hölder stability estimates for certain inverse problems we discuss in the coming sections.

As in the preceding section M is a compact n-dimensional Riemannian manifold with boundary.

Consider the Hardy's inequality

$$\int_M |\nabla f(x)|^2 dV \geqslant c \int_M \frac{|f(x)|^2}{d(x,\partial M)^2} dV, \quad f \in H_0^1(M), \tag{3.1}$$

for some constant $c > 0$, where dV is the volume form on M, d is the geodesic distance introduced previously and $d(\cdot, \partial M)$ is the distance to ∂M.

Define $r_x(v) = \inf\{|t|;\ \gamma_{x,v}(t) \notin M\}$, where $\gamma_{x,v}$ is the geodesic satisfying the initial condition $\gamma_{x,v} = x$ and $\dot\gamma_{x,v} = v$. It was observed in [26] that Hardy's inequality (3.1) holds for any open subset $\mathcal{O}$, of a complete Riemannian manifold, whenever $\mathcal{O}$ has the following uniform interior cone property: there are an angle $\alpha > 0$ and a constant $c_0 > 0$ so that, for any $x \in M$, there exists an α-angled cone $C_x \subset T_xM$ [1] with the property that $r_x(v) \leqslant c_0 d(x, \partial M)$, for all $v \in C_x$. The proof of this result follows the method by Davies [11, page 25] for the flat case. Since in our case M is a compact Riemannian manifold, it is obvious that it satisfies the uniform interior cone property. Then slight modifications of the proof in [26] show that Hardy's inequality is satisfied for any compact Riemannian manifold.

It is worth mentioning that Hardy's inequality holds for any bounded Lipschitz domain of $\mathbb{R}^n$ with constant $c \leqslant 1/4$, with equality if and only if Ω is convex.

The following Hopf's maximum principle is a key ingredient in establishing our first weighted interpolation inequality.

Lemma 3.1. *Let $q \in C(M)$ and $u \in C^2(M) \cap H_0^1(M)$ satisfying $q \leqslant 0$ and $\Delta u + qu \leqslant 0$. If u is non identically equal to zero then $u > 0$ in M and $\partial_\nu u(y) = \langle \nabla u(y), \nu(y)\rangle > 0$ for any $y \in \partial M$.*

Proof. Similar to that of [12, Lemma 3.4, page 34 and Theorem 3.5, page 35]. The tangent ball in the classical Hopf's lemma is substitute by a tangent geodesic ball (see the construction in [23, Proof of Theorem 9.2, page 51]). □

Proposition 3.1. *Let $q \in C(M)$ and $u \in C^2(M) \cap H_0^1(M)$ satisfying $q \leqslant 0$ and $\Delta u + qu \leqslant 0$. If u is non identically equal to zero then*

$$u(x) \geqslant c_u d(x, \partial M), \quad x \in M,$$

where the constant c_u only depends on u and M.

Proof. Let $0 < \epsilon$ to be specified later. Let $x \in M$ so that $d(x, \partial M) \leqslant \epsilon$ and $y \in \partial M$ satisfying $d(x, \partial M) = d(x, y)$. Since M is complete there exist a unit speed minimizing geodesic $\gamma : [0, r] \to M$ such that $\gamma(0) = y$, $\gamma(r) = x$ and $\dot\gamma(0) = \nu(y)$, where we set $r = d(x, \partial M)$ (see for instance [22, page 150]).

Define $\phi(t) = u(\gamma(t))$. Then

$$\phi'(t) = du(\gamma(t))(\dot\gamma(t))$$
$$\phi''(t) = d^2u(\gamma(t))(\dot\gamma(t), \dot\gamma(t)) + du(\gamma(t))(\ddot\gamma(t)).$$

[1] Here $C_x = \{Y \in T_xM;\ \angle\langle X, Y\rangle < \alpha\}$, for some $X \in T_xM$.

Here $\dot\gamma(t) = \dot\gamma^i(t)\partial_i \in T_{\gamma(t)}$. Observe that by the geodesic equation

$$\ddot\gamma^k(t) = -\dot\gamma^i(t)\dot\gamma^j(t)\Gamma^k_{ij}(\gamma(t)),$$

where Γ^k_{ij} are the Christoffel symbols associated to the metric g.

We get by taking into account that $\phi'(0) = du(y)(\nu(y)) = \langle \nabla u(y), \nu(y)\rangle = \partial_\nu u(y)$

$$\phi(r) = r\partial_\nu u(y) + \frac{r^2}{2}\phi''(st),$$

for some $0 < s < 1$. Hence there exists $c > 0$ depending on u and M so that

$$\phi(r) \geqslant 2r\eta - cr^2 \geqslant r\eta + r(\eta - c\epsilon),$$

with $2\eta = \min_{y\in\Gamma} \partial_\nu u(y) > 0$ (by the compactness of ∂M and Lemma 3.1). Thus

$$\phi(r) \geqslant r\eta$$

provided that $\epsilon \leqslant \eta/c$. In other words we proved

$$u(x) = \phi(r) \geqslant r\eta = \eta d(x, \partial M). \tag{3.2}$$

On the other hand an elementary compactness argument yields, where $M^\epsilon = \{x \in M;\ d(x, \partial M) \geqslant \epsilon\}$,

$$u(x) \geqslant \min_{z\in M^\epsilon} u(z) \geqslant \frac{\min_{z\in M^\epsilon} u(z)}{\max_{z\in M^\epsilon} d(z, \partial M)} d(x, \partial M), \quad x \in M^\epsilon. \tag{3.3}$$

In light of (3.2) and (3.3) we end up getting

$$u(x) \geqslant c_u d(x, \partial M), \quad x \in M.$$

The proof is then complete. □

A consequence of Proposition 3.1 is the following corollary.

Corollary 3.1. *Let $q \in C(M)$, $q \leqslant 0$, and $u \in C^2(M) \cap H^1_0(M)$ non identically equal to zero satisfying $\Delta u + qu \leqslant 0$. There exists a constant C_u only depending on u and M so that we have*

$$\|f\|_{L^2(M)} \leqslant C_u \|fu\|^{\frac{1}{2}}_{L^2(M)} \|f\|^{\frac{1}{2}}_{H^2(M)}$$

for any $f \in H^2(M)$.

Proof. By Proposition 3.1 $u(x) \geqslant c_u d(x, \partial M)$. Therefore

$$\int_M f(x)^2 dV(x) \leqslant c_u^{-1} \int_M \frac{f(x)^2 u(x)^2}{d(x, \partial M)^2} dV(x).$$

Combined with Hardy's inequality (3.1) this estimate gives

$$\int_M f(x)^2 dV(x) \leqslant c_u^{-1} c \int_M |\nabla(fu)(x)|^2 dV(x). \tag{3.4}$$

But from usual interpolation inequalities we have

$$\|fu\|_{H^1(M)} \leqslant C \|fu\|^{\frac{1}{2}}_{L^2(M)} \|fu\|^{\frac{1}{2}}_{H^2(M)},$$

where the constant C only depends on M.

Whence (3.4) implies

$$\|f\|_{L^2(M)} \leqslant C_u \|fu\|^{\frac{1}{2}}_{L^2(M)} \|f\|^{\frac{1}{2}}_{H^2(M)},$$

which is the expected inequality □

Let $0 \leqslant q \in C^1(M)$ be fixed and consider the operator $A = -\Delta + q$ with domain $D(A) = H^2(M) \cap H_0^1(M)$. An extension of [12, Theorem 8.38, page 214] to a compact Riemannian manifold with boundary shows that the first eigenvalue of A, denoted by λ_1, is simple and has a positive eigenfunction. Let then $\phi_1 \in C^2(M)$ (by elliptic regularity) be the unique first eigenfunction satisfying $\phi_1 > 0$ and normalized by $\|\phi_1\|_{L^2(M)} = 1$. Since $\Delta\phi_1 - q\phi_1 = -\lambda_1\phi_1$ the Hopf's maximum principle is applicable to ϕ_1. Therefore a particular weight in the preceding corollary is obtained by taking $u = \phi_1$.

Corollary 3.2. *There exists a constant $c > 0$, depending on ϕ_1, so that we have*

$$\|f\|_{L^2(M)} \leqslant c\|f\phi_1\|_{L^2(M)}^{\frac{1}{2}}\|f\|_{H^2(M)}^{\frac{1}{2}} \tag{3.5}$$

for any $f \in H^2(M)$.

The second weighted interpolation inequality relies on the following proposition.

Proposition 3.2. *Let $p \in L^\infty(M)$ and $u \in W^{2,n}(M)$ satisfying $(\Delta + p)\varphi = 0$ in M and $\varphi^2 \in W^{2,n}(M)$. Then there exists $\delta = \delta(\varphi) > 0$ so that $|\varphi|^{-\delta} \in L^1(M)$.*

It is worth mentioning that in general $\delta < 1$ as soon as φ vanishes at some point $x_0 \in M$. Consider for instance in the flat case $\psi(x) \sim |x - x_0|^k$ near x_0 if x_0 is a zero of order k. It is then clear that $|\varphi|^{-\delta}$ is locally integrable in a neighborhood of x_0 if and only if $\delta k < n - 1$. In consequence $\delta < 1$ whenever $k \geqslant n - 1$.

Sketch of the proof. First step. Denote by B the unit ball of $\mathbb{R}^n$ and let $B_+ = B \cap \mathbb{R}^n_+$, with $\mathbb{R}^n_+ = \{x = (x', x_n) \in \mathbb{R}^n;\ x_n > 0\}$. Let L be a second order differential operator acting as follows

$$Lu = \partial_j(a_{ij}\partial_i u) + V \cdot \nabla u + du.$$

Assume that (a_{ij}) is a symmetric matrix with entries in $C^1(2\overline{B_+})$, $V \in L^\infty(2B_+)^n$ is real valued and $d \in L^\infty(2B_+)$ is complex valued. Suppose furthermore that

$$a_{ij}(x)\xi_j \cdot \xi_j \geqslant \kappa_0|\xi|^2, \quad x \in 2B_+,\ \xi \in \mathbb{R}^n,$$

for some $\kappa_0 > 0$.

Let $u \in W^{2,n}(2B_+) \cap C^0(2\overline{B_+})$ be a weak solution of $Lu = 0$ satisfying $u = 0$ on $\partial(2B_+) \cap \overline{\mathbb{R}^n_+}$ and $|u|^2 \in W^{2,n}(2B_+) \cap C^0(2\overline{B_+})$. From [1, Theorem 1.1, page 942] there exists a constant C, depending of u, so that the following doubling inequality at the boundary

$$\int_{B_{2r}\cap B_+} |u|^2 dx \leqslant C \int_{B_r \cap B_+} |u|^2 dx,$$

holds for any ball B_{2r} of radius $2r$ contained in $2B$.

On the other hand simple calculations yield, where $v = \Re u$ and $w = \Im u$,

$$\begin{aligned}\partial_j(a_{ij}\partial_i|u|^2) + 2V \cdot \nabla|u|^2 + 4(|\Re d| + |\Im d|)|u|^2 \\ \geqslant 2a_{ij}\partial_i v\partial_j v + 2a_{ij}\partial_i w\partial_j w \geqslant 0 \quad \text{in } 2B_+\end{aligned}$$

and $|u|^2 = 0$ on $\partial(2B_+) \cap \overline{\mathbb{R}^n_+}$.

Harnak's inequality at the boundary (see [12, Theorem 9.26, page 250]) then yields

$$\sup_{B_r\cap B_+} |u|^2 \leqslant \frac{C}{|B_{2r}|}\int_{B_{2r}\cap B_+} |u|^2 dx,$$

for any ball B_{2r} of radius $2r$ contained in $2B$.

Define $\tilde{u}$ by

$$\tilde{u}(x', x_n) = u(x', x_n) \quad \text{if } (x', x_n) \in 2B_+,$$
$$\tilde{u}(x', x_n) = u(x', -x_n) \quad \text{if } (x', -x_n) \in 2B_+.$$

Therefore $\tilde{u}$ belongs to $H^1(2B) \cap L^\infty(2B)$ and satisfies

$$\int_{B_{2r}} |\tilde{u}|^2 dx \leqslant C \int_{B_r} |\tilde{u}|^2 dx, \tag{3.6}$$

$$\sup_{B_r} |\tilde{u}|^2 \leqslant \frac{C}{|B_{2r}|} \int_{B_{2r}} |\tilde{u}|^2 dx, \tag{3.7}$$

for any ball B_{2r} of radius $2r$ contained in $2B$.

Inequalities (3.6) and (3.7) at hand we mimic the proof of [9, Theorem 4.2, page 1784] in order to obtain that $|\tilde{u}|^{-\delta} \in L^1(B)$, for some $\delta > 0$ depending on u. Whence $|u|^{-\delta} \in L^1(B_+)$.

Second step. As ∂M is compact there exists a finite cover (U_α) of ∂M and C^∞-diffeomorphisms $f_\alpha : U_\alpha \to 2B$ so that $f_\alpha(U_\alpha \cap M) = 2B_+$, $f_\alpha(U_\alpha \cap \partial M) = 2B \cap \overline{\mathbb{R}^n_+}$ and, for any $x \in \partial M$, $x \in V_\alpha = f_\alpha^{-1}(B)$, for some α. Then $u_\alpha = \varphi \circ f_\alpha^{-1}$ satisfies $L_\alpha u_\alpha = 0$ in $2B$ and $u = 0$ on $\partial(2B_+) \cap \overline{\mathbb{R}^n_+}$ for some $L = L_\alpha$ satisfying the conditions of the first step. Hence $|u_\alpha|^{-\delta_\alpha} \in L^1(B_+)$ and then $|\varphi|^{-\delta_\alpha} \in L^1(V_\alpha)$. Let V the union of V_α's. Since $u \in L^\infty(V)$ we get $|u|^{-\delta_0} \in L^1(V)$ with $\delta_0 = \min \delta_\alpha$. Next, let ϵ sufficiently small in such a way that $M \backslash M_\epsilon \subset V$, where $M_\epsilon = \{x \in M;\ \mathrm{dist}(x, \partial M) > \epsilon\}$. Proceeding as previously it is not hard to get that there exists δ_1 so that $|\varphi|^{-\delta_1} \in L^1(M_{\epsilon/2})$. Finally, as it is expected we obtain that $|\varphi|^{-\delta} \in L^1(M)$ with $\delta = \min(\delta_0, \delta_1)$. □

Lemma 3.2. *Let φ be as in Proposition 3.2. There exists a constant $C > 0$, depending on φ, so that we have*

$$\|f\|_{L^2(M)} \leqslant C \|f\|_{L^\infty(M)}^{\frac{2}{2+\delta}} \|f\varphi\|_{L^2(M)}^{\frac{\delta}{2+\delta}},$$

for any $f \in L^\infty(M)$

Proof. Let $\delta = \delta(\varphi)$ given as in the preceding proposition. Since φ belongs to $L^\infty(M)$, substituting δ by $\min(1, \delta)$ if necessary, we may assume that $\delta < 2$. We get by applying Cauchy-Schwarz's inequality

$$\int_M |f|^{\delta/2} dV \leqslant \| |f\varphi|^\delta \|_{L^1(M)}^{1/2} \| |\varphi|^{-\delta} \|_{L^1(M)}^{1/2}.$$

But by Hölder's inequality

$$\| |f\varphi|^\delta \|_{L^1(M)}^{1/2} \leqslant \mathrm{Vol}(M)^{(2-\delta)/4} \|f\varphi\|_{L^2(M)}^{\delta/2}.$$

Whence

$$\| |f|^{\delta/2} \|_{L^1(M)} \leqslant \mathrm{Vol}(M)^{(2-\delta)/4} \|f\varphi\|_{L^2(M)}^{\delta/2} \| |\varphi|^{-\delta} \|_{L^1(M)}^{1/2}. \tag{3.8}$$

On the other hand

$$\|f\|_{L^2(M)} \leqslant \|f\|_{L^\infty(M)}^{1-\delta/4} \| |f|^{\delta/2} \|_{L^1(M)}^{1/2}. \tag{3.9}$$

A combination of (3.8) and (3.9) yields

$$\|f\|_{L^2(M)} \leqslant C \|f\|_{L^\infty(M)}^{1-\delta/4} \|f\varphi\|_{L^2(M)}^{\delta/4},$$

which is the expected inequality. □

4. Inverse source problem: abstract framework

Let H be a Hilbert space and $A : D(A) \subset H \to H$ be the generator of continuous semigroup $T(t)$. An operator $\mathscr{C} \in \mathscr{B}(D(A), Y)$, Y is another Hilbert space which is identified with its dual space, is called an admissible observation for $T(t)$ if for some (and hence for all) $\tau > 0$ the operator $\Psi \in \mathscr{B}(D(A), L^2((0,\tau), Y))$ given by

$$(\Psi x)(t) = \mathscr{C}T(t)x, \quad t \in [0,\tau], \quad x \in D(A),$$

has a bounded extension to H.

We introduce the definition of exact observability for the system

$$z'(t) = Az(t), \quad z(0) = x, \tag{4.1}$$

$$y(t) = \mathscr{C}z(t), \tag{4.2}$$

where $\mathscr{C}$ is an admissible observation for $T(t)$. The pair $(A, \mathscr{C})$ is said exactly observable at time $\tau > 0$ if there is a constant κ such that the solution (z, y) of (4.1) and (4.2) satisfies

$$\int_0^\tau \|y(t)\|_Y^2 dt \geqslant \kappa^2 \|x\|_H^2, \quad x \in D(A).$$

Or equivalently

$$\int_0^\tau \|(\Psi x)(t)\|_Y^2 dt \geqslant \kappa^2 \|x\|_H^2, \quad x \in D(A). \tag{4.3}$$

Consider the Cauchy problem

$$z'(t) = Az(t) + \lambda(t)x, \quad z(0) = 0 \tag{4.4}$$

and set

$$y(t) = \mathscr{C}z(t), \quad t \in [0,\tau]. \tag{4.5}$$

By Duhamel's formula we have

$$y(t) = \int_0^t \lambda(t-s)\mathscr{C}T(s)x ds = \int_0^t \lambda(t-s)(\Psi x)(s) ds. \tag{4.6}$$

Let

$$H_\ell^1((0,\tau), Y) = \left\{u \in H^1((0,\tau), Y);\ u(0) = 0\right\}.$$

Define the operator $S : L^2((0,\tau), Y) \longrightarrow H_\ell^1((0,\tau), Y)$ by

$$(Sh)(t) = \int_0^t \lambda(t-s)h(s) ds. \tag{4.7}$$

If $E = S\Psi$ then (4.6) takes the form

$$y(t) = (Ex)(t).$$

Theorem 4.1. *Assume that $(A, \mathscr{C})$ is exactly observable for $\tau \geqslant \tau_0$, for some $\tau_0 > 0$. Let $\lambda \in H^1(0,\tau)$ satisfies $\lambda(0) \neq 0$. Then E is one-to-one from H onto $H_\ell^1((0,\tau), Y)$ and*

$$\frac{\kappa|\lambda(0)|}{\sqrt{2}} e^{-\tau \frac{\|\lambda'\|^2_{L^2(0,\tau)}}{|\lambda(0)|^2}} \|x\|_H \leqslant \|Ex\|_{H_\ell^1((0,\tau),Y)}, \quad x \in H. \tag{4.8}$$

Proof. Taking first the derivative with respect to t of both sides of the integral equation

$$\int_0^t \lambda(t-s)\varphi(s)ds = \psi(t)$$

we get the following Volterra integral equation of second kind

$$\lambda(0)\varphi(t) + \int_0^t \lambda'(t-s)\varphi(s)ds = \psi'(t).$$

Mimicking the proof of [14, Theorem 2, page 33] we obtain that this integral equation has a unique solution $\varphi \in L^2((0,\tau),Y)$ and

$$\begin{aligned}\|\varphi\|_{L^2((0,\tau),Y)} &\leqslant C\|\psi'\|_{L^2((0,\tau),Y)}\\ &\leqslant C\|\psi\|_{H^1_\ell((0,\tau),Y)}.\end{aligned}$$

Here $C = C(\lambda)$ is a constant.

For estimating the constant C above we first use the elementary convexity inequality $(a+b)^2 \leqslant 2(a^2+b^2)$ in order to get

$$\||\lambda(0)|\varphi(t)\|_Y^2 \leqslant 2\left(\int_0^t \frac{|\lambda'(t-s)}{|\lambda(0)|}\left[|\lambda(0)|\|\varphi(s)\|_Y\right]ds\right)^2 + 2\|\psi'(t)\|_Y^2.$$

Thus

$$|\lambda(0)|^2\|\varphi(t)\|_Y^2 \leqslant 2\frac{\|\lambda'\|^2_{L^2((0,\tau))}}{|\lambda(0)|^2}\int_0^t |\varphi(0)|^2\|\varphi(s)\|_Y^2 ds + 2\|\psi'(t)\|_Y^2$$

by the Cauchy-Schwarz's inequality. Therefore using Gronwall's lemma we obtain in a straightforward manner

$$\|\varphi\|_{L^2((0,\tau),Y)} \leqslant \frac{\sqrt{2}}{|\lambda(0)|}e^{\tau\frac{\|\lambda'\|^2_{L^2((0,\tau))}}{|\lambda(0)|^2}}\|\psi'\|_{L^2((0,\tau),Y)}$$

and then

$$\|\varphi\|_{L^2((0,\tau),Y)} \leqslant \frac{\sqrt{2}}{|\lambda(0)|}e^{\tau\frac{\|\lambda'\|^2_{L^2((0,\tau))}}{|\lambda(0)|^2}}\|S\varphi\|_{H^1_\ell((0,\tau),Y)}.$$

In light of (4.3) we end up getting

$$\|Ex\|_{H^1_\ell((0,\tau),Y)} \geqslant \frac{\kappa|\lambda(0)|}{\sqrt{2}}e^{-\tau\frac{\|\lambda'\|^2_{L^2((0,\tau))}}{|\lambda(0)|^2}}\|x\|_H.$$

This is the expected inequality. □

We shall need a variant of Theorem 4.1. If $(A,\mathscr{C})$ is as in Theorem 4.1 then, as in the preceding section, by the perturbation argument in [28, Proposition 6.3.3, page 189], there exist $\aleph > 0$ and $\kappa > 0$ such that for any $P \in \mathscr{B}(H)$ satisfying $\|P\| \leqslant \aleph$ we have that $(A+P,\mathscr{C})$ is exactly observable with $\kappa(P+A) \geqslant \kappa$.

Define E^P similarly to E by substituting in E A by $A+P$.

Theorem 4.2. *Assume that $(A,\mathscr{C})$ is exactly observable for $\tau \geqslant \tau_0$, for some $\tau_0 > 0$, and let $\lambda \in H^1(0,\tau)$ satisfies $\lambda(0) \neq 0$. There exist $\aleph > 0$ and $\kappa > 0$ so that for any $P \in \mathscr{B}(H)$ satisfying $\|P\| \leqslant \aleph$ we have that E^P is one-to-one from H onto $H^1_\ell((0,\tau),Y)$ and*

$$\frac{\kappa|\lambda(0)|}{\sqrt{2}}e^{-\tau\frac{\|\lambda'\|^2_{L^2(0,\tau)}}{|\lambda(0)|^2}}\|x\|_H \leqslant \|E^P x\|_{H^1_\ell((0,\tau),Y)}, \quad x \in H. \tag{4.9}$$

We will consider inverse source problems with singular sources. For this purpose we need to extend Theorem 4.1. Fix then ϱ in the resolvent set of A. Let H_1 be the space $D(A)$ equipped with the norm $\|x\|_1 = \|(\varrho - A)x\|$ and denote by H_{-1} the completion of H with respect to the norm $\|x\|_{-1} = \|(\varrho - A)^{-1}x\|$. As we observed in [28, Proposition 4.2, page 1644] and its proof, when $x \in H_{-1}$ (which is the dual space of H_1 with respect to the pivot space H) and $\lambda \in H^1(0,\tau)$, then according to the classical extrapolation theory of semigroups the Cauchy problem (4.1) has a unique solution $z \in C([0,\tau], H)$. In addition y given in (4.2) belongs to $L^2((0,\tau), Y)$.

If $x \in H$ we have by Duhamel's formula

$$y(t) = \int_0^t \lambda(t-s)\mathscr{C}T(s)x ds = \int_0^t \lambda(t-s)(\Psi x)(s) ds. \tag{4.10}$$

Let

$$H^1_\ell((0,\tau), Y) = \left\{u \in H^1((0,\tau), Y);\ u(0) = 0\right\}.$$

We define the operator $S : L^2((0,\tau), Y) \longrightarrow H^1_\ell((0,\tau), Y)$ by

$$(Sh)(t) = \int_0^t \lambda(t-s)h(s) ds. \tag{4.11}$$

Hence $E = S\Psi$ then (4.10) the form

$$y(t) = (Ex)(t).$$

Let $\mathcal{Z} = (\varrho - A^*)^{-1}(X + \mathscr{C}^*Y)$.

Theorem 4.3. *Assume that $(A, \mathscr{C})$ is exactly observable at time τ. Then*
(i) E is one-to-one from H onto $H^1_\ell((0,\tau), Y)$.
(ii) E is extended to an isomorphism, denoted by $\tilde{E}$, from $\mathcal{Z}'$ onto $L^2((0,\tau), Y)$.
(iii) There exists a constant $\tilde{\kappa}$, independent of λ, so that

$$\|x\|_{\mathcal{Z}'} \leqslant \tilde{\kappa}|\lambda(0)| e^{\frac{\|\lambda'\|^2_{L^2((0,\tau))}}{|\lambda(0)|^2}\tau} \|\tilde{E}x\|_{L^2((0,\tau),Y)}. \tag{4.12}$$

Proof. We give the proof of (ii) and (iii) and we note that (i) is contained in Theorem 4.1. We first observe that S^*, the adjoint of S, maps $L^2((0,\tau), Y)$ into $H^1_r((0,\tau), Y)$, where

$$H^1_r((0,\tau), Y) = \left\{u \in H^1((0,\tau), Y);\ u(\tau) = 0\right\}.$$

Moreover

$$S^*h(t) = \int_t^\tau \lambda(s-t)h(s) ds, \quad h \in L^2((0,\tau), Y).$$

Fix $h \in L^2((0,\tau), Y)$ and set $k = S^*h$. Then

$$k'(t) = \lambda(0)h(t) - \int_t^\tau \lambda'(s-t)h(s) ds.$$

Hence

$$\begin{aligned}|\lambda(0)\|h(t)\|^2 &\leqslant \left(\int_t^\tau \frac{|\lambda'(s-t)|}{|\lambda(0)|}[|\lambda(0)|\|h(s)\|]ds + \|k'(t)\|\right)^2\\ &\leqslant 2\left(\int_t^\tau \frac{|\lambda'(s-t)|}{|\lambda(0)|}[|\lambda(0)|\|h(s)\|]ds\right)^2 + 2\|k'(t)\|^2\\ &\leqslant 2\frac{\|\lambda'\|^2_{L^2((0,\tau))}}{|\lambda(0)|^2}\int_0^t [|\lambda(0)|\|h(s)\|]^2 ds + 2\|k'(t)\|^2.\end{aligned}$$

The last estimate is obtained by applying Cauchy-Schwarz's inequality.

Then Gronwall's lemma yields

$$[|\lambda(0)|\|h(t)\|]^2 \leqslant 2e^{2\frac{\|\lambda'\|^2_{L^2((0,\tau))}}{|\lambda(0)|^2}\tau}\|k'(t)\|^2.$$

Therefore

$$\|h\|_{L^2((0,\tau),Y)} \leqslant \frac{\sqrt{2}}{|\lambda(0)|}e^{\frac{\|\lambda'\|^2_{L^2((0,\tau))}}{|\lambda(0)|^2}\tau}\|k'\|_{L^2((0,\tau),Y)}.$$

Whence

$$\|h\|_{L^2((0,\tau),Y)} \leqslant \frac{\sqrt{2}}{|\lambda(0)|}e^{\frac{\|\lambda'\|^2_{L^2((0,\tau))}}{|\lambda(0)|^2}\tau}\|S^*h\|_{H^1_r((0,\tau),Y)}. \tag{4.13}$$

The adjoint operator of S^*, acting as a bounded operator from $[H^1_r((0,\tau);Y)]'$ into $L^2((0,\tau);Y)$, gives an extension of S. We denote by $\tilde{S}$ this operator. By [28, Proposition 4.1, page 1644] $\tilde{S}$ defines an isomorphism from $[H_r((0,1);Y)]'$ onto $L^2((0,\tau);Y)$. In light of the identity

$$\|\tilde{S}\|_{\mathscr{B}([H^1_r((0,\tau);Y)]';L^2((0,\tau),Y))} = \|S^*\|_{\mathscr{B}(L^2((0,\tau);Y);H^1_r((0,\tau),Y))},$$

(4.13) implies

$$\frac{|\lambda(0)|}{\sqrt{2}}e^{-\frac{\|\lambda'\|^2_{L^2((0,\tau))}}{|\lambda(0)|^2}\tau} \leqslant \|\tilde{S}\|_{\mathscr{B}([H^1_r((0,\tau);Y)]';L^2((0,\tau);Y))}. \tag{4.14}$$

On the other hand according to [28, Proposition 2.13, page 1641] Ψ possesses a unique bounded extension, denoted by $\tilde{\Psi}$, from $\mathcal{Z}'$ into $[H^1_r((0,\tau);Y)]'$ and there exists a constant $c>0$ so that

$$\|\tilde{\Psi}\|_{\mathcal{B}(\mathcal{Z}';[H^1_r((0,\tau);Y)]')} \geqslant c. \tag{4.15}$$

The operator $\tilde{E} = \tilde{S}\tilde{\Psi}$ gives the unique extension of E to an isomorphism from $\mathcal{Z}'$ onto $L^2((0,\tau),Y)$.

We end up the proof by noting that (4.12) follows from (4.14) and (4.15). □

5. Inverse problems for evolution equations associated to Laplace-Beltrami operator

Throughout this section M is a compact n-dimensional Riemannian manifold with boundary, $\tau>0$ and Γ is a nonempty open subset of ∂M.

5.1. **Inverse source problem for the wave equation.** Consider the IBVP for the wave equation

$$\begin{cases} \partial_t^2 u - \Delta u + q(x)u + a(x)\partial_t u = g(t)f(x) & \text{in } M \times (0,\tau), \\ u = 0 & \text{on } \partial M \times (0,\tau), \\ u(\cdot,0) = 0,\ \partial_t u(\cdot,0) = 0. \end{cases} \tag{5.1}$$

Assume that (Γ,τ) geometrically control M. Fix $(q_0,a_0) \in L^\infty(M) \oplus L^\infty(M)$ and denote by 2κ the observability constant for (q_0,a_0). In light of Theorem 2.1 there exists a constant $\beta > 0$ only depending on Γ, M and (q_0,a_0) such that, for any $(q,a) = (q_0,a_0) + (\tilde{q},\tilde{a})$ with $(\tilde{q},\tilde{a}) \in L^\infty(M) \oplus L^\infty(M)$ satisfying

$$\|(\tilde{q},\tilde{a})\|_{L^\infty(M)\times L^\infty(M)} \leqslant \beta, \tag{5.2}$$

the observability contant for (q,a) is κ. We denote the set of couples $(q,a) \in L^\infty(M) \oplus L^\infty(M)$ of the form $(q,a) = (q_0,a_0) + (\tilde{q},\tilde{a})$, where $(\tilde{q},\tilde{a}) \in L^\infty(M) \oplus L^\infty(M)$ satisfies (5.2), by $\mathscr{D}$.

Let $f \in L^2(M)$ and $g \in H^1(0,\tau)$ with $g(0) \neq 0$. We have according to Theorem 4.1

$$\|f\|_{L^2(M)} \leqslant \frac{\sqrt{2}}{\kappa|g(0)|} e^{\tau \frac{\|g'\|^2_{L^2(0,\tau)}}{|g(0)|^2}} \|\partial_\nu u\|_{H^1((0,\tau),L^2(\Gamma))}, \tag{5.3}$$

where $u = u(q,a,f,g)$ denotes the solution of the IBVP (5.1).

An immediate consequence of this inequality is the following theorem.

Theorem 5.1. *Assume that (Γ,τ) geometrically control M. Let $g \in H^1(0,\tau)$ satisfying $g(0) \neq 0$. Then there exists a constant C, only depending on (q_0,a_0), κ, Γ, τ and g, so that for any $(q,a) \in \mathscr{D}$ we have*

$$\|f\|_{L^2(M)} \leqslant C\|\partial_\nu u\|_{H^1((0,\tau),L^2(\Gamma))}.$$

Here $u = u(q,a,f,g)$ denotes the solution of the IBVP (5.1).

Set for simplicity $v = v(q,f,g) = u(q,0,f,g)$. That is v is the solution of the IBVP

$$\begin{cases} \partial_t^2 u - \Delta u + q(x)u = g(t)f(x) & \text{in } M \times (0,\tau), \\ u = 0 & \text{on } \partial M \times (0,\tau), \\ u(\cdot,0) = 0, \quad \partial_t u(\cdot,0) = 0. \end{cases} \tag{5.4}$$

Using Duhamel's formula it is not hard to check that

$$v(x,t) = \int_0^t g(t-s)w(x,s)ds,$$

where $w = w(f)$ is the solution of the IBVP

$$\begin{cases} \partial_t^2 w - \Delta w + q(x)w = 0 & \text{in } M \times (0,\tau), \\ w = 0 & \text{on } \partial M \times (0,\tau), \\ w(\cdot,0) = f,\ \partial_t w(\cdot,0) = 0. \end{cases} \tag{5.5}$$

Let

$$H^1_\ell((0,\tau),L^2(\Gamma)) = \left\{u \in H^1((0,\tau),L^2(\Gamma));\ u(0) = 0\right\}$$

and define the operator $S : L^2(\Gamma \times (0,\tau)) \longrightarrow H^1_\ell((0,\tau),L^2(\Gamma))$ by

$$(Sh)(t) = \int_0^t g(t-s)h(s)ds.$$

We have seen in the proof of Theorem 4.1 that S is an isomorphism and

$$\|h\|_{L^2(\Gamma\times(0,\tau))} \leqslant \frac{\sqrt{2}}{\kappa|g(0)|} e^{\tau \frac{\|g'\|^2_{L^2((0,\tau))}}{|g(0)|^2}} \|Sh\|_{H^1((0,\tau),L^2(\Gamma))}.$$

Whence

$$\|\partial_\nu w\|_{L^2(\Gamma\times(0,\tau))} \leqslant \frac{\sqrt{2}}{\kappa|g(0)|} e^{\tau \frac{\|g'\|^2_{L^2((0,\tau))}}{|g(0)|^2}} \|\partial_\nu v\|_{H^1((0,\tau),L^2(\Gamma))}. \tag{5.6}$$

Let $\aleph > 0$, assume that $\mathbf{d}(\Gamma) < \infty$ and let $\tau > 2\mathbf{d}(\Gamma)$. From Theorem 2.2 there exist three constants C, κ and ϵ_0 so that for any $q \in L^\infty(M)$ with $\|q\|_{L^\infty(M)} \leqslant \aleph$ we have

$$C\|f\|_{L^2(M)} \leqslant e^{\kappa\epsilon}\|\partial_\nu w\|_{L^2(\Gamma\times(0,\tau))} + \frac{1}{\epsilon}\|f\|_{H^1_0(M)}, \quad \epsilon \geqslant \epsilon_0. \tag{5.7}$$

Now (5.6) in (5.7) yields

$$C\|f\|_{L^2(M)} \leqslant e^{\kappa\epsilon}\frac{\sqrt{2}}{\kappa|g(0)|} e^{\tau \frac{\|g'\|^2_{L^2((0,\tau))}}{|g(0)|^2}} \|\partial_\nu v\|_{H^1((0,\tau),L^2(\Gamma))} + \frac{1}{\epsilon}\|f\|_{H^1_0(M)} \tag{5.8}$$

for any $\epsilon \geqslant \epsilon_0$.

Let

$$\Psi(\rho) = |\ln \rho|^{-1} + \rho, \quad \rho > 0,$$

and $\Psi(0) = 0$. Then a standard minimization argument with respect to ϵ in (5.8) enables us establishing the following result.

Theorem 5.2. *Let $\aleph > 0$, $R > 0$ and $\tau > 2\mathbf{d}(\Gamma)$. Let $g \in H^1(0,\tau)$ satisfying $g(0) \neq 0$. Then there exists a constant $C > 0$, only depending on $\aleph$, R, Γ, τ and g, so that for any $q \in L^\infty(M)$ with $\|q\|_{L^\infty(M)} \leqslant \aleph$ and any $f \in H^1_0(M)$ satisfying $\|f\|_{H^1_0(M)} \leqslant R$ we have*

$$\|f\|_{L^2(M)} \leqslant C\Psi\left(\|\partial_\nu v\|_{H^1((0,\tau),L^2(\Gamma))}\right),$$

where $v = v(q,f,g)$ is the solution of the IBVP (5.4).

5.2. **Determining the potential and the damping coefficient in a wave equation.** Introduce the IBVP for the wave equation

$$\begin{cases} \partial_t^2 u - \Delta u + q(x)u + a(x)\partial_t u = 0 & \text{in } M\times(0,\tau), \\ u = 0 & \text{on } \partial M\times(0,\tau), \\ u(\cdot,0) = u_0, \quad \partial_t u(\cdot,0) = u_1. \end{cases} \tag{5.9}$$

Let $\aleph > 0$ and recall that $\mathcal{H}_0 = H^1_0(M) \oplus L^2(M)$. We have seen in Section 1 that, for any $(q,a) \in L^\infty(M) \oplus L^\infty(M)$, $\tau > 0$ and $(u_0,u_1) \in \mathcal{H}_0$, the IBVP (5.9) has a unique solution

$$u = u(q,a,(u_0,u_1)) \in C([0,\tau],H^1_0(M))$$

so that $\partial_t u \in C([0,\tau],L^2(M))$ and $\partial_\nu u \in L^2(\partial M\times(0,\tau))$. Moreover under the assumption

$$\|(q,a)\|_{L^\infty(M)\oplus L^\infty(M)} \leqslant \aleph$$

we have

$$\|u\|_{C([0,\tau],H^1_0(M))} + \|\partial_t u\|_{C([0,\tau],L^2(M))} \leqslant C\|(u_0,u_1)\|_{\mathcal{H}_0} \tag{5.10}$$

and

$$\|\partial_\nu u\|_{L^2(\partial M\times(0,\tau))} \leqslant C\|(u_0,u_1)\|_{\mathcal{H}_0}. \tag{5.11}$$

Here $C = C(\aleph)$ is a nondecreasing function.

Define the initial-to-boundary operator $\Lambda(q,a)$ as follows

$$\Lambda(q,a) : (u_0,u_1) \in \mathcal{H}_0 \mapsto \partial_\nu u(q,a,(u_0,u_1)) \in L^2(\Gamma\times(0,\tau)).$$

Let

$$\mathcal{H}_1 = (H_0^1(M)\cap H^2(M))\oplus H_0^1(M).$$

Observing that

$$\partial_t u(q,a,(u_0,u_1)) = u(q,a,(u_1,\Delta u_0 - qu_0 - au_1))$$

we easily obtain that $\Lambda(q,a)\in\mathscr{B}(\mathcal{H}_1, H^1((0,\tau),L^2(\Gamma)))$. Furthermore we get as a consequence of (5.11)

$$\|\Lambda(q,a)\|_{\mathscr{B}(\mathcal{H}_1,H^1((0,\tau),L^2(\Gamma)))}\leqslant C,$$

where the constant C is similar to that in (5.11).

Denote by $\mathscr{D}_0$ the set $\mathscr{D}$ in the case where $(q_0,a_0)=(q_0,0)$ with $q_0\geqslant 0$. Define then $\mathscr{D}_1(\aleph)$ as the subset of $\mathscr{D}_0$ consisting in couples $(q,a)\in H^2(M)\oplus H^2(M)$ satisfying

$$\|(q,a)\|_{H^2(M)\oplus H^2(M)}\leqslant\aleph.$$

It is then clear that $\mathscr{D}_1(\aleph)$ is nonempty provided that $\aleph\geqslant\aleph_0$, for some $\aleph_0=\aleph(\beta)$.

Theorem 5.3. *Assume that (Γ,τ) geometrically control M and let $\aleph\geqslant\aleph_0$. There exists a constant $C>0$, depending on $\aleph$ and q_0, so that for any $(q,a)\in\mathscr{D}_1(\aleph)$ we have*

$$\|q-q_0\|_{L^2(M)}+\|a-0\|_{L^2(M)}\leqslant C\|\Lambda(q,a)-\Lambda(q_0,0)\|^{1/2}_{\mathscr{B}(\mathcal{H}_1,H^1((0,\tau),L^2(\Gamma)))}.$$

Proof. Let $0\leqslant\phi_1$ be the first eigenfunction of the operator $-\Delta+q_0$ with domain $H^2(M)\cap H_0^1(M)$. This eigenfunction is normalized by $\|\phi_1\|_{L^2(M)}=1$. If

$$u_0 = u(q_0,0,(\phi_1,i\sqrt{\lambda_1}\phi_1)) = e^{i\sqrt{\lambda_1}\,t}\phi_1 \quad\text{and}\quad u = u(q,a,(\phi_1,i\sqrt{\lambda_1}\,\phi_1))$$

then $v=u-u_0$ is the solution of the following IBVP

(5.12)
$$\begin{cases} \partial_t^2 v-\Delta v+qv+a\partial_t v = -[(q-q_0)+i\sqrt{\lambda_1}a]e^{i\sqrt{\lambda_1}t}\phi_1 & \text{in } M\times(0,\tau),\\ v=0 & \text{on } \partial M\times(0,\tau),\\ v(\cdot,0)=0,\quad \partial_t v(\cdot,0)=0. \end{cases}$$

Bearing in mind that (Γ,τ) geometrically control M we get from Theorems 5.1

$$\|\phi_1(q-q_0)\|_{L^2(M)}+\|\phi_1 a\|_{L^2(M)}\leqslant C\|\partial_\nu v\|_{H^1((0,\tau),L^2(\partial M))}.$$

This inequality combined with Corollary 3.1 yields

$$\|q-q_0\|_{L^2(M)}+\|a-0\|_{L^2(M)}\leqslant C\|\partial_\nu v\|^{1/2}_{H^1((0,\tau),L^2(\Gamma))}$$

which gives in a straightforward manner the expected result. □

Denote the sequence of eigenvalues, counted according to their multiplicity, of $A=-\Delta$ with domain $H^2(M)\cap H_0^1(M)$ by $0<\lambda_1<\lambda_2\leqslant\ldots\leqslant\lambda_k\to\infty$.

Consider on $\mathcal{H}_0$ the operators

$$\mathcal{A}=\begin{pmatrix}0 & I\\ -A & 0\end{pmatrix},\quad D(\mathcal{A})=\mathcal{H}_1$$

and $\mathcal{A}(q,a) = \mathcal{A} + \mathcal{B}(q,a)$ with $D(\mathcal{A}(q,a)) = D(\mathcal{A})$, where

$$\mathcal{B}(q,a) = \begin{pmatrix} 0 & 0 \\ -q & -a \end{pmatrix} \in \mathscr{B}(\mathcal{H}_0).$$

From [28, Proposition 3.7.6, page 100] $\mathcal{A}$ is skew-adjoint operator with $0 \in \rho(\mathcal{A})$ and

$$\mathcal{A}^{-1} = \begin{pmatrix} 0 & -A^{-1} \\ I & 0 \end{pmatrix}.$$

We note that, since $\mathcal{A}^{-1} : \mathcal{H} \to \mathcal{H}_1$ is bounded and the embedding $\mathcal{H}_1 \hookrightarrow \mathcal{H}$ is compact, $\mathcal{A}^{-1} : \mathcal{H} \to \mathcal{H}$ is compact.

Also, from [28, Proposition 3.7.6, page 100] $\mathcal{A}$ is diagonalizable and its spectrum consists in the sequence $(i\sqrt{\lambda_k})$.

Introduce the bounded operator

$$\mathcal{C}(q,a) = (i\mathcal{A}^{-1})(-i\mathcal{B}(q,a))(i\mathcal{A}^{-1}).$$

Let $s_k(\mathcal{C}(q,a))$, $k \geqslant 1$, denote the singular values of $\mathcal{C}(q,a)$, that is the eigenvalues of $[\mathcal{C}(q,a)^*\mathcal{C}(q,a)]^{\frac{1}{2}}$. In light of [13, formulas (2.2) and (2.3), page 27] we have

$$s_k(\mathcal{C}(q,a)) \leqslant \|\mathcal{B}(q,a)\| s_k(i\mathcal{A}^{-1})^2 = \|\mathcal{B}(q,a)\| \lambda_k^{-1},$$

where $\|\mathcal{B}(q,a)\|$ denote the norm of $\mathcal{B}(q,a)$ in $\mathscr{B}(\mathcal{H})$.

On the other hand referring to Weyl's asymptotic formula we get $\lambda_k = O(k^{2/n})$. Hence $\mathcal{C}_{q,a}$ belongs to the Shatten class $\mathcal{S}_p$ for any $p > n/2$, that is

$$\sum_{k \geqslant 1} [s_k(\mathcal{C}(q,a))]^p < \infty.$$

We get by applying [13, Theorem 10.1, page 276] that the spectrum of $\mathcal{A}(q,a)$ consists in a sequence of eigenvalues $(\mu_k(q,a))$, counted according to their multiplicity, and the corresponding eigenfunctions $(\phi_k(q,a))$ form a Riesz basis of $\mathcal{H}$.

Fix (q,a) and k. Set $\mu = \mu_k(q,a)$ and $\phi = \phi_k(q,a) = (\varphi, \psi) \in \mathcal{H}_1$ be an eigenfunction associated to μ. Then it is straightforward to check that $\psi = \mu\varphi$ and $(-\Delta + q + a\mu + \mu^2)\varphi = 0$ in M. Since $-\Delta\varphi = f$ in M with $f = -(q + a\mu + \mu^2)\varphi$ we can use iteratively [12, Corollary 7.11, page 158] (Sobelev embedding theorem) together with [12, Theorem 9.15, page 241] in order to obtain that $\varphi \in W^{2,p}(M)$ for any $1 < p < \infty$. In particular $\varphi, |\varphi|^2 \in W^{2,n}(M) \cap C^0(M)$. In other words φ satisfies the assumption of Proposition 3.2.

Set, for $(q,a), (\tilde{q},\tilde{a}) \in \mathscr{D}$,

$$u = u(q,a,\phi) \quad \text{and} \quad \tilde{u} = u(\tilde{q},\tilde{a},\phi).$$

Then similarly to Theorem 5.3 we prove, where $v = \tilde{u} - u$,

$$\|\varphi(\tilde{q} - q)\|_{L^2(M)} + \|\varphi(\tilde{a} - a)\|_{L^2(M)} \leqslant C\|\partial_\nu v\|_{H^1((0,\tau),L^2(\partial M))}.$$

This and Lemma 3.1 yield

Theorem 5.4. *Assume that (Γ,τ) geometrically control M and fix $(q,a) \in \mathscr{D}$. Then there exists two constants $C > 0$ and $\alpha > 0$, depending of (q,a), so that for any $(\tilde{q},\tilde{a}) \in \mathscr{D}$ we have*

$$\|\tilde{q} - q\|_{L^2(M)} + \|\tilde{a} - a\|_{L^2(M)} \leqslant C\|\Lambda(\tilde{q},\tilde{a}) - \Lambda(q,a)\|^{\alpha}_{\mathscr{B}(\mathcal{H}_1,H^1((0,\tau),L^2(\Gamma)))}.$$

5.3. **Determining the potential in a wave equation without geometric control condition.** Consider the IBVP

$$\begin{cases} \partial_t^2 u - \Delta u + q(x)u = 0 & \text{in } M \times (0,\tau), \\ u = 0 & \text{on } \partial M \times (0,\tau), \\ u(\cdot,0) = u_0, \quad \partial_t u(\cdot,0) = 0. \end{cases} \tag{5.13}$$

From the preceding subsection the initial-to-boundary mapping

$$\Lambda(q) : u_0 \in H_0^1(M) \cap H^2(M) \mapsto \partial_\nu u \in H^1((0,\tau), L^2(\Gamma)),$$

where $u = u(q,u_0)$ is the solution on the IBVP, defines a bounded operator. Moreover for any $\aleph > 0$ there exists a constant $C > 0$, depending of $\aleph$, so that for any $q \in L^\infty(M)$ satisfying $\|q\|_{L^\infty(M)} \leqslant \aleph$ we have

$$\|\Lambda(q)\|_{\mathscr{B}(H_0^1(M)\cap H^2(M), H^1((0,\tau),L^2(\Gamma)))} \leqslant C.$$

Theorem 5.5. *Let $\aleph > 0$ and suppose that $\tau > 2\mathbf{d}(\Gamma)$. There exists a constant $C > 0$ so that for any $0 \leqslant q \in L^\infty(M)$, $\tilde{q} \in L^\infty(M)$ satisfying $q - \tilde{q} \in W^{1,\infty}(M)$ and*

$$\|q\|_{L^\infty(M)} \leqslant \aleph, \quad \|\tilde{q}\|_{L^\infty(M)} \leqslant \aleph, \quad \|q - \tilde{q}\|_{W^{1,\infty}(M)} \leqslant \aleph$$

we have

$$\|q - \tilde{q}\|_{L^2(M)} \leqslant C\Phi\left(\|\Lambda(q) - \Lambda(\tilde{q})\|_{\mathscr{B}(H_0^1(M)\cap H^2(M), H^1((0,\tau),L^2(\Gamma)))}\right),$$

with $\Phi(\rho) = |\ln \rho|^{-1/(n+3)} + \rho$, $\rho > 0$, and $\Phi(0) = 0$.

Proof. Let $0 \leqslant q \in L^\infty(M)$ satisfying $\|q\|_{L^\infty(M)} \leqslant \aleph$. Denote by $0 < \lambda_1 \leqslant \lambda_2 \ldots \leqslant \lambda_k \ldots$ the sequence of eigenvalues of the operator $-\Delta + q$ with domain $H_0^1(M) \cap H^2(M)$. Let (ϕ_k) an orthonormal basis of $L^2(M)$ consisting in eigenfunctions, each ϕ_k is an eigenvalue for λ_k. Note that according to the usual elliptic regularity we have $\phi_k \in C^\infty(M)$ for each k.

By the Weyl's asymptotic formula and the min-max principle there exists a constant $\varkappa > 1$, depending on $\aleph$ but not in q, so that

$$\varkappa^{-1} k^{2/n} \leqslant \lambda_k \leqslant \varkappa k^{2/n}. \tag{5.14}$$

Set, for $\tilde{q} \in L^\infty(M)$ satisfying $\|\tilde{q}\|_{L^\infty(M)} \leqslant \aleph$,

$$u = u(q,\phi_k) = \cos(\lambda_k t)\phi_k \quad \text{and} \quad \tilde{u} = u(\tilde{q},\phi_k).$$

Then $v = \tilde{u} - u$ is the solution of the IBVP, where $g_k(t) = \cos(\sqrt{\lambda_k}t)$,

$$\begin{cases} \partial_t^2 u - \Delta u + \tilde{q}u = (\tilde{q} - q)\phi_k g_k(t) & \text{in } M \times (0,\tau), \\ u = 0 & \text{on } \partial M \times (0,\tau), \\ u(\cdot,0) = 0, \quad \partial_t u(\cdot,0) = 0. \end{cases} \tag{5.15}$$

We have $\|g_k'\|^2_{L^2(0,\tau)} \leqslant \lambda_k \tau$. Hence

$$\|g_k'\|^2_{L^2(0,\tau)} \leqslant \varkappa \tau k^{2/n} \tag{5.16}$$

by (5.14).

In the rest of this proof C and c denote generic constant only depending of M, $\aleph$, Γ and τ. From (5.8) we have

$$C\|(\tilde{q} - q)\phi_k\|_{L^2(M)} \leqslant e^{\kappa\epsilon} e^{ck^{2/n}} \|\partial_\nu v\|_{H^1((0,\tau),L^2(\Gamma))} + \frac{1}{\epsilon}\|(\tilde{q} - q)\phi_k\|_{H_0^1(M)} \tag{5.17}$$

for any $\epsilon \geqslant \epsilon_0$.

On the other hand

$$\begin{aligned}\|(\tilde{q}-q)\phi_k\|_{H_0^1(M)} &\leqslant \|\tilde{q}-q\|_{W^{1,\infty}(M)}\|\phi_k\|_{H_0^1(M)}\\ &\leqslant 2N\sqrt{\lambda_k}\\ &\leqslant ck^{1/n} \text{ by (5.14)}.\end{aligned}$$

This in (5.17) gives

$$C\|(\tilde{q}-q)\phi_k\|_{L^2(M)} \leqslant e^{\kappa\epsilon}e^{ck^{2/n}}\|\partial_\nu v\|_{H^1((0,\tau),L^2(\Gamma))} + \frac{k^{1/n}}{\epsilon}, \quad \epsilon \geqslant \epsilon_0.$$

But we have by Cauchy-Schwarz's inequality

$$(\tilde{q}-q,\phi_k)^2 \leqslant \mathrm{Vol}(M)\|(\tilde{q}-q)\phi_k\|_{L^2(M)}.$$

Whence

$$C(\tilde{q}-q,\phi_k)^2 \leqslant e^{\kappa\epsilon}e^{ck^{2/n}}\|\partial_\nu v\|_{H^1((0,\tau),L^2(\Gamma))} + \frac{k^{\frac{1}{n}}}{\epsilon}, \quad \epsilon \geqslant \epsilon_0.$$

Also

$$\begin{aligned}\|\tilde{q}-q\|_{L^2(M)}^2 &= \sum_{k\leqslant\ell}(\tilde{q}-q,\phi_k)^2 + \sum_{k>\ell}(\tilde{q}-q,\phi_k)^2\\ &\leqslant \sum_{k\leqslant\ell}(\tilde{q}-q,\phi_k)^2 + \frac{1}{\lambda_{\ell+1}}\sum_{k>\ell}\lambda_k(\tilde{q}-q,\phi_k)^2\\ &\leqslant \sum_{k\leqslant\ell}(\tilde{q}-q,\phi_k)^2 + \frac{N^2}{(\ell+1)^{2/n}}.\end{aligned}$$

Thus

$$C\|\tilde{q}-q\|_{L^2(M)}^2 \leqslant \ell e^{\kappa\epsilon}e^{c\ell^{2/n}}\|\partial_\nu v\|_{H^1((0,\tau),L^2(\Gamma))} + \frac{1}{(\ell+1)^{2/n}} + \frac{\ell^{1+1/n}}{\epsilon}. \tag{5.18}$$

Let $s \geqslant 1$ be a real number and let ℓ be the unique integer so that $\ell \leqslant s < \ell+1$. Then (5.18) with that ℓ yields

$$C\|\tilde{q}-q\|_{L^2(M)}^2 \leqslant se^{\kappa\epsilon}e^{cs^{2/n}}\|\partial_\nu v\|_{H^1((0,\tau),L^2(\Gamma))} + \frac{1}{s^{2/n}} + \frac{s^{1+1/n}}{\epsilon}. \tag{5.19}$$

We then get by taking $\epsilon = s^{3/n+1}$ in (5.19), where $s_0 = \max\left(1,\epsilon_0^{n/(n+3)}\right)$,

$$C\|\tilde{q}-q\|_{L^2(M)}^2 \leqslant \frac{1}{s^{2/n}} + e^{cs^{2/n+1}}e^{\kappa s^{3/N+1}}\|\partial_\nu v\|_{H^1((0,\tau),L^2(\Gamma))}, \quad s \geqslant s_0.$$

Therefore

$$C\|\tilde{q}-q\|_{L^2(M)}^2 \leqslant \frac{1}{s^{2/n}} + e^{cs^{3/n+1}}\|\partial_\nu v\|_{H^1((0,\tau),L^2(\Gamma))}, \quad s \geqslant s_0$$

or equivalently

$$C\|\tilde{q}-q\|_{L^2(M)} \leqslant \frac{1}{s^{1/n}} + e^{cs^{3/n+1}}\|\partial_\nu v\|_{H^1((0,\tau),L^2(\Gamma))}, \quad s \geqslant s_0.$$

We end up getting the expected inequality by minimizing with respect to s. □

5.4. **Inverse source problem for the heat equation.** Consider the following IBVP for the heat equation

$$\begin{cases} \partial_t u - \Delta u + q(x)u = g(t)f(x) & \text{in } M \times (0,\tau), \\ u = 0 & \text{on } \partial M \times (0,\tau), \\ u(\cdot,0) = 0 \end{cases} \tag{5.20}$$

and set $Q = M \times (0,\tau)$.

We recall that the anisotropic Sobolev space $H^{2,1}(Q)$ is given as follows

$$H^{2,1}(Q) = L^2((0,\tau), H^2(M)) \cap H^1((0,\tau), L^2(M)).$$

From classical parabolic regularity theorems for any $f \in L^2(M)$, $g \in L^2(0,\tau)$ and $q \in L^\infty(M)$ the IBVP (5.20) has a unique solution

$$u = u(q,f,g) \in H^{2,1}(Q).$$

Furthermore if $\aleph > 0$ then there exists a constant $C > 0$ so that

$$\|u\|_{H^{2,1}(Q)} \leqslant C\|g\|_{L^2(0,\tau)}\|f\|_{L^2(M)} \tag{5.21}$$

for any $q \in L^\infty(M)$ satisfying $\|q\|_{L^\infty(M)} \leqslant \aleph$.

If in addition $g \in H^1(0,\tau)$ then it is not hard to check that $\partial_t u$ is the solution of the IBVP (5.20) with g substituted by g'. Hence $\partial_t u \in H^{2,1}(Q)$ and

$$\|\partial_t u\|_{H^{2,1}(Q)} \leqslant C\|g'\|_{L^2(0,\tau)}\|f\|_{L^2(M)} \tag{5.22}$$

for any $q \in L^\infty(M)$ satisfying $\|q\|_{L^\infty(M)} \leqslant \aleph$, where C is the constant in (5.21).

We derive that $\partial_\nu u$ is well defined as an element of $H^1((0,\tau), L^2(\Gamma))$. Therefore by (5.21), (5.22) and the continuity of the trace operator on Γ we have

$$\|\partial_\nu u\|_{H^1((0,\tau),L^2(\Gamma))} \leqslant C\|g\|_{H^1(0,\tau)}\|f\|_{L^2(M)},$$

where the constant C is as in (5.21).

The following result will be useful in the sequel.

Proposition 5.1. *Let $\aleph > 0$. There exist two constants $c > 0$ and $C > 0$ so that for any $q \in L^\infty(M)$ satisfying $\|q\|_{L^\infty(M)} \leqslant \aleph$, $f \in H_0^1(M)$ and $g \in H^1(0,\tau)$ with $g(0) \neq 0$ we have*

$$C\|f\|_{L^2(M)} \leqslant \frac{1}{\sqrt{\epsilon}}\|f\|_{H_0^1(M)} + \frac{1}{|g(0)|}e^{\tau\|g'\|^2_{L^2(0,\tau)}/|g(0)|^2}e^{c\epsilon}\|\partial_\nu u\|_{H^1((0,\tau),L^2(\Gamma))} \tag{5.23}$$

for any $\epsilon \geqslant 1$, where $u = u(q,f,g)$ is the solution of the IBVP (5.20).

Proof. Pick $q \in L^\infty(M)$ satisfying $\|q\|_{L^\infty(M)} \leqslant \aleph$, $f \in H_0^1(M)$ and $g \in H^1(0,\tau)$ with $g(0) \neq 0$. We may assume without loss of generality that $q \geqslant 0$. This is achieved by substituting u by $ue^{-\aleph t}$, which is the solution of the IBVP (5.20) when q is replaced by $q + \aleph$.

Let $v = v(q,f) \in H^{2,1}(Q)$ be the unique solution of the IBVP

$$\begin{cases} \partial_t v - \Delta v + q(x)v = 0 & \text{in } M \times (0,\tau), \\ v = 0 & \text{on } \partial M \times (0,\tau), \\ v(\cdot,0) = f. \end{cases}$$

Then $\partial_\nu v$ is well defined as an element of $L^2(\Gamma \times (0,\tau))$. As for the wave equation we have

$$\partial_\nu u|_\Gamma(\cdot,t) = \int_0^t g(t-s)\partial_\nu v|_\Gamma(\cdot,s)ds.$$

Therefore

$$\|\partial_\nu v\|_{L^2(\Gamma\times(0,\tau))} \leqslant \frac{\sqrt{2}}{|g(0)|} e^{\tau\|g'\|^2_{L^2((0,\tau))}/|g(0)|^2} \|\partial_\nu u\|_{H^1((0,\tau),L^2(\Gamma))}. \tag{5.24}$$

From the final time observability inequality in Theorem 2.3 we have

$$\|v(\cdot,\tau)\|_{L^2(M)} \leqslant K\|\partial_\nu v\|_{L^2(\Gamma\times(0,\tau))}, \tag{5.25}$$

for some constant $K > 0$ independent of q and f.

A combination of (5.24) and (5.25) yields

$$C\|v(\cdot,\tau)\|_{L^2(M)} \leqslant \frac{1}{|g(0)|} e^{\tau\|g'\|^2_{L^2((0,\tau))}/|g(0)|^2} \|\partial_\nu u\|_{H^1((0,\tau),L^2(\Gamma))}. \tag{5.26}$$

Denote by $0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_k \to \infty$ the sequence of eigenvalues of the $-\Delta + q$ with domain $H^1_0(M) \cap H^2(M)$. Let (ϕ_k) be a sequence of eigenfunctions, each ϕ_k is associated to λ_k, so that (ϕ_k) form an orthonormal basis of $L^2(M)$.

We have

$$v(\cdot,\tau) = \sum_{\ell\geqslant 1} e^{-\lambda_k\tau}(f,\phi_\ell)\phi_\ell,$$

where $(\cdot,\cdot)$ is the usual scalar product on $L^2(M)$. Hence

$$(f,\phi_\ell)^2 \leqslant e^{2\lambda_\ell\tau}\|v(\cdot,\tau)\|^2_{L^2(M)}, \quad \ell \geqslant 1.$$

Whence

$$\sum_{\ell=1}^{k}(f,\phi_\ell)^2 \leqslant ke^{2\lambda_k\tau}\|v(\cdot,\tau)\|^2_{L^2(M)}$$

for any integer $k \geqslant 1$.

This and the fact that $\left(\sum_{\ell\geqslant 1}\lambda_\ell(\cdot,\phi_\ell)^2_{L^2(\Omega)}\right)^{1/2}$ is an equivalent norm on $H^1_0(M)$ lead

$$\begin{aligned}
\|f\|^2_{L^2(M)} &= \sum_{\ell=1}^{k}(f,\phi_\ell)^2 + \sum_{\ell\geqslant k+1}(f,\phi_\ell)^2 \\
&\leqslant \sum_{\ell=1}^{k}(f,\phi_\ell)^2 + \frac{1}{\lambda_{k+1}}\sum_{\ell\geqslant k+1}\lambda_\ell(f,\phi_\ell)^2 \\
&\leqslant ke^{2\lambda_k\tau}\|v(\cdot,\tau)\|^2_{L^2(M)} + \frac{1}{\lambda_{k+1}}\|f\|^2_{H^1_0(M)}.
\end{aligned}$$

Until the end of this proof C and c denote generic constants independent of q, f and g.

We get from inequality (5.14)

$$C\|f\|^2_{L^2(M)} \leqslant ke^{ck^{2/n}}\|v(\cdot,\tau)\|^2_{L^2(M)} + \frac{1}{(k+1)^{2/n}}\|f\|^2_{H^1_0(M)}. \tag{5.27}$$

Let $\epsilon \geqslant 1$ and $k \geqslant 1$ be the unique integer so that $k \leqslant \epsilon^{n/2} < k+1$. We obtain in a straightforward manner from (5.27)

$$C\|f\|^2_{L^2(M)} \leqslant e^{c\epsilon}\|v(\cdot,\tau)\|^2_{L^2(M)} + \frac{1}{\epsilon}\|f\|^2_{H^1(M)}. \tag{5.28}$$

Then (5.26) in (5.28) gives the expected inequality. □

Minimizing the right hand side of (5.23) with respect to ϵ we obtain the following result in which $\Phi(\rho) = |\ln\rho|^{-1/2} + \rho$ for $\rho > 0$ and $\Phi(0) = 0$

Corollary 5.1. *Let $\aleph > 0$, $q \in L^\infty(M)$ and $g \in H^1(0,\tau)$ satisfying $g(0) \neq 0$. There exists a constant $C > 0$, depending of $\aleph$, q and g, so that for any $f \in H_0^1(\Omega)$ with $\|f\|_{H_0^1(M)} \leqslant \aleph$ we have*

$$C\|f\|_{L^2(M)} \leqslant \Phi\left(\|\partial_\nu u\|_{H^1((0,\tau),L^2(\Gamma))}\right),$$

where $u = u(q,f,g)$ is the solution of the IBVP (5.20).

5.5. **Determining the zeroth order coefficient in a heat equation.** Consider the IBVP

$$\begin{cases} \partial_t u - \Delta u + q(x)u = 0 & \text{in } M \times (0,\tau), \\ u = 0 & \text{on } \partial M \times (0,\tau), \\ u(\cdot,0) = u_0. \end{cases} \tag{5.29}$$

Again, with reference to classical regularity theorems we have that, for $q \in L^\infty(M)$ and $u_0 \in H_0^1(M)$, the IBVP(5.29) has unique solution $u = u(q,u_0) \in H^{2,1}(Q)$. Furthermore for any $\aleph > 0$ there exists a constant $C > 0$ so that

$$\|u(q,u_0)\|_{H^{2,1}(M\times(0,\tau))} \leqslant C\|u_0\|_{H_0^1(M)} \tag{5.30}$$

for any $q \in L^\infty(M)$ satisfying $\|q\|_{L^\infty(M)} \leqslant \aleph$.

Define

$$\mathcal{H}_0(M) = \{w \in H_0^1(M);\ \Delta w \in H_0^1(M)\}$$

that we equip with its natural norm

$$\|w\|_{\mathcal{H}_0(M)} = \|w\|_{H_0^1(M)} + \|\Delta w\|_{H_0^1(M)}.$$

If $q \in W^{1,\infty}(M)$ and $u_0 \in \mathcal{H}_0(M)$ then it is straightforward to check that

$$\partial_t u(q,u_0) = u(q,\Delta u_0 - qu_0).$$

We get by applying (5.30) with u_0 substituted by $\Delta u_0 - qu_0$

$$\|\partial_t u\|_{H^{2,1}(M\times(0,\tau))} \leqslant C\|u_0\|_{\mathcal{H}_0(M)} \tag{5.31}$$

for any $q \in W^{1,\infty}(M)$ satisfying $\|q\|_{W^{1,\infty}(M)} \leqslant \aleph$, where the constant C is independent of q.

Bearing in mind that the trace operator

$$w \in H^{2,1}(Q) \mapsto \partial_\nu w \in L^2(\Gamma \times (0,\tau))$$

is bounded we obtain that $\partial_\nu u \in H^1((0,\tau),L^2(\Gamma))$ provided that $u_0 \in \mathcal{H}_0(M)$ and $q \in W^{1,\infty}(M)$. Further we get from (5.30) and (5.31)

$$\|\partial_\nu u\|_{H^1((0,\tau),L^2(\Gamma))} \leqslant C\|u_0\|_{\mathcal{H}_0(M)}$$

for any $q \in W^{1,\infty}(M)$ satisfying $\|q\|_{W^{1,\infty}(M)} \leqslant \aleph$, where the constant C is independent of q.

That is we proved that the operator

$$\mathcal{N}(q) : u_0 \in \mathcal{H}_0(M) \mapsto \partial_\nu u \in H^1((0,\tau),L^2(\Gamma))$$

is bounded and

$$\|\mathcal{N}(q)\|_{\mathscr{B}(\mathcal{H}_0(M),H^1((0,\tau),L^2(\Gamma)))} \leqslant C$$

for any $q \in W^{1,\infty}(M)$ satisfying $\|q\|_{W^{1,\infty}(M)} \leqslant N$, where the constant C is independent of q.

Henceforward for convenience $\|\mathcal{N}(\tilde{q}) - \mathcal{N}(q)\|_{\mathscr{B}(\mathcal{H}_0(M),H^1((0,\tau),L^2(\Gamma)))}$ is simply denoted by $\|\mathcal{N}(\tilde{q}) - \mathcal{N}(q)\|$.

Theorem 5.6. *Let $\aleph > 0$. There exists a constant $C > 0$ so that for any q, $\tilde{q} \in W^{1,\infty}(M)$ satisfying*

$$\|q\|_{W^{1,\infty}(M)} \leqslant \aleph, \quad \|\tilde{q}\|_{W^{1,\infty}(M)} \leqslant \aleph,$$

we have

$$C\|\tilde{q} - q\|_{L^2(M)} \leqslant \Theta\left(\|\mathcal{N}(\tilde{q}) - \mathcal{N}(q)\|\right).$$

Here $\Theta(\rho) = |\ln \rho|^{-1/(1+4n)} + \rho$ for $\rho > 0$ and $\Theta(0) = 0$.

Proof. Let q, $\tilde{q} \in W^{1,\infty}(M)$ satisfying

$$\|q\|_{W^{1,\infty}(M)} \leqslant \aleph, \quad \|\tilde{q}\|_{W^{1,\infty}(M)} \leqslant \aleph.$$

As in the preceding subsection we may assume without loss of generality that $q \geqslant 0$.

Denote by $0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_k \to \infty$ the sequence of eigenvalues of the operator $-\Delta + q$ with domain $H_0^1(M) \cap H^2(M)$. Let (ϕ_k) a sequence of the corresponding eigenfunctions so that (ϕ_k) form an orthonormal basis of $L^2(M)$.

Taking into account that $u(q, \phi_k) = e^{-\lambda_k t}\phi_k$ we obtain that

$$v = u(\tilde{q}, \phi_k) - u(q, \phi_k)$$

is the solution of the IBVP

$$\begin{cases} \partial_t v - \Delta v + q(x)v = (\tilde{q} - q)\phi_k e^{-\lambda_k t} & \text{in } M \times (0, \tau), \\ uv = 0 & \text{on } \partial M \times (0, \tau), \\ v(\cdot, 0) = 0. \end{cases}$$

Therefore

$$\mathcal{N}(\tilde{q})(\phi_k) - \mathcal{N}(q)(\phi_k) = \partial_\nu v$$

from which we deduce

$$\|\partial_\nu v\|_{H^1((0,\tau),L^2(\Gamma))} \leqslant C\lambda_k \|\mathcal{N}(\tilde{q}) - \mathcal{N}(q)\|.$$

Here and henceforth C and c denote generic constants independent of q and $\tilde{q}$.

As in the preceding subsection we get from (5.23)

$$C|(\tilde{q} - q, \phi_k)| \leqslant \frac{\sqrt{\lambda_k}}{\sqrt{\epsilon}} + e^{\tau\lambda_k^2} e^{c\epsilon} \lambda_k^2 \|\mathcal{N}(\tilde{q}) - \mathcal{N}(q)\| \tag{5.32}$$

for any $\epsilon \geqslant 1$, where we used the inequality $\|(\tilde{q} - q)\phi_k\|_{H_0^1(M)} \leqslant C\sqrt{\lambda_k}$.

Inequality (5.32) then gives

$$C \sum_{k=1}^{\ell} |(\tilde{q} - q, \phi_k)^2_{L^2(M)} \leqslant \frac{\ell\lambda_\ell}{\epsilon} + \ell e^{c\lambda_\ell^2} e^{c\epsilon} \|\mathcal{N}(\tilde{q}) - \mathcal{N}(q)\|^2 \tag{5.33}$$

for any arbitrary integer $\ell \geqslant 1$.

Similarly to the proof of Theorem 5.5 inequality (5.33) yields

$$C\|\tilde{q} - q\|^2_{L^2(M)} \leqslant \frac{s^{1+2/n}}{\epsilon} + \frac{1}{s^{2/n}} + e^{cs^{1+4/n}} e^{c\epsilon} \|\mathcal{N}(\tilde{q}) - \mathcal{N}(q)\|^2, \quad s \geqslant 1.$$

The proof is then completed in the same manner like that of Theorem 5.5. □

6. Determining a boundary coefficient in a wave equation

6.1. **Inverse source problem for the wave equation with boundary damping.** In this subsection $\Omega = (0,1)\times(0,1)$ and

$$\Gamma_0 = ((0,1)\times\{1\}) \cup (\{1\}\times(0,1)),$$
$$\Gamma_1 = ((0,1)\times\{0\}) \cup (\{0\}\times(0,1)).$$

Consider the IBVP

$$\begin{cases} \partial_t^2 u - \Delta u = \lambda(t) w & \text{in } \Omega\times(0,\tau), \\ u = 0 & \text{on } \Gamma_0\times(0,\tau), \\ \partial_\nu u + a\partial_t u = 0 & \text{on } \Gamma_1\times(0,\tau), \\ u(\cdot,0) = 0, \quad \partial_t u(\cdot,0) = 0. \end{cases} \tag{6.1}$$

Fix $\frac{1}{2} < \alpha \leqslant 1$ and let

$$\mathscr{A} = \{b = (b_1,b_2) \in C^\alpha([0,1]) \oplus C^\alpha([0,1]),\ b_1(0) = b_2(0),\ b_j \geqslant 0\}.$$

Let $V = \{u \in H^1(\Omega);\ u = 0 \text{ on } \Gamma_0\}$ and define on $V \oplus L^2(\Omega)$ the operator A_a, $a \in \mathscr{A}$, by

$$A_a = (w, \Delta v),$$
$$D(A_a) = \{(v,w) \in V\oplus V;\ \Delta v \in L^2(\Omega) \text{ and } \partial_\nu v = -aw \text{ on } \Gamma_1\}.$$

We are going to apply Theorem 4.2 with $H = V\oplus L^2(\Omega)$, $H_1 = D(A_a)$ equipped with its graph norm and $Y = L^2(\Gamma_1)$.

Denote by H_{-1} the dual of H_1 with respect to the pivot space H.

If $(0,w) \in H_{-1}$ and $\lambda \in H^1(0,\tau)$ then the IBVP (6.1) has a unique solution $u(w)$ so that $(u(w), \partial_t u(w)) \in C([0,\tau]; V\oplus L^2(\Omega))$ and $\partial_\nu u(w)|_{\Gamma_1\times(0,\tau)} \in L^2(\Gamma_1\times(0,\tau))$.

Taking into account that $\{0\}\times V' \subset H_{-1}$, where V' is the dual space of V, we obtain the following consequence of Theorem 4.2.

Proposition 6.1. *There exists a constant $C > 0$ so that for any $\lambda \in H^1(0,\tau)$ and $w \in V'$ we have*

$$\|w\|_{V'} \leqslant C|\lambda(0)| e^{\tau\|\lambda'\|^2_{L^2(0,\tau)}/|\lambda(0)|^2} \|\partial_\nu u_w\|_{L^2(\Gamma_1\times(0,\tau))}. \tag{6.2}$$

6.2. **Determining the boundary damping coefficient in a wave equation.** Let Ω and Γ_i, $i = 1,2$ as in the preceding subsection and consider the IBVP

$$\begin{cases} \partial_t^2 u - \Delta u = 0 & \text{in } \Omega\times(0,\tau), \\ u = 0 & \text{on } \Gamma_0\times(0,\tau), \\ \partial_\nu u + a\partial_t u = 0 & \text{on } \Gamma_1\times(0,\tau), \\ u(\cdot,0) = u_0, \quad \partial_t u(\cdot,0) = u_1. \end{cases} \tag{6.3}$$

For $(u_0,u_1) \in H_1$ the IBVP (6.3) admits a unique solution $u = u(a,(u_0,u_1))$ so that

$$(u_a, \partial_t u_a) \in C([0,\infty), H_1) \cap C^1([0,\infty), H).$$

Fix $0 < \underline{a} \leqslant \aleph$ and set

$$\underline{\mathscr{A}} = \{b = (b_1,b_2) \in \mathscr{A} \cap H^1(0,1)\oplus H^1(0,1);\ \underline{a} \leqslant b_1,\ b_2,\ \|b\|^2_{H^1(0,1)\oplus H^1(0,1)} \leqslant \aleph\}.$$

Let $\mathcal{U}_0$ given by

$$\mathcal{U}_0 = \{v \in V;\ \Delta v \in L^2(\Omega) \text{ and } \partial_\nu v = 0 \text{ on } \Gamma_1\}$$

and observe that $\mathcal{U}_0 \times \{0\} \subset H_1$ for any $a \in \mathscr{A}$. We endow $\mathcal{U}_0$ with the norm

$$\|u_0\|_{\mathcal{U}_0} = \left(\|u_0\|_V^2 + \|\Delta u_0\|_{L^2(\Omega)}^2\right)^{1/2}.$$

Define the initial-to-boundary operator

$$\Lambda(a) : u_0 \in \mathcal{U}_0 \mapsto \partial_\nu u \in L^2(\Gamma_1 \times (0,\tau)),$$

where $u = u(a,(u_0,u_1)$ is the solution of the IBVP (6.3). Then

$$\Lambda(a) \in \mathscr{B}(\mathcal{U}_0, L^2(\Gamma_1 \times (0,\tau))).$$

Henceforward for convenience the norm of $\Lambda(a) - \Lambda(0)$ in $\mathscr{B}(\mathcal{U}_0, L^2(\Gamma_1 \times (0,\tau)))$ will simply denoted by $\|\Lambda(a) - \Lambda(0)\|$.

The following Hölder stability inequality improve the result in [4].

Theorem 6.1. *Let $\delta \in (0,1)$. There exists a constant C only depending of δ, $\underline{a}$ and $\aleph$ so that*

$$\|a - 0\|_{L^2(0,1)\oplus L^2(0,1)} \leqslant C\|\Lambda(a) - \Lambda(0)\|^{\delta/[2(2+\delta)]} \tag{6.4}$$

for each $a \in \underline{\mathscr{A}}$.

Proof. We first observe that $u(a)$ is also the unique solution of

$$\begin{cases} \displaystyle\int_\Omega u''(t)v dx = \int_\Omega \nabla u(t)\cdot\nabla v dx - \int_{\Gamma_1} a u'(t) v, \quad v \in V. \\ \\ u(0) = u_0, \quad u'(0) = u_1. \end{cases}$$

Therefore $u = u(a) - \underline{u}$, where $\underline{u} = u(0,(u_0,u_1))$, is the solution of the following problem

$$\begin{cases} \displaystyle\int_\Omega u''(t)v dx = \int_\Omega \nabla u(t)\cdot\nabla v dx - \int_{\Gamma_1} a u'(t) v - \int_{\Gamma_1} a u'(0)(t) v, \quad v \in V. \\ \\ u(0) = 0, \quad u'(0) = 0. \end{cases} \tag{6.5}$$

For k, $\ell \in \mathbb{Z}$ set

$$\lambda_{k\ell} = \left[\left(k+\frac{1}{2}\right)^2 + \left(\ell+\frac{1}{2}\right)^2\right]\pi^2$$

$$\phi_{k\ell}(x,y) = 2\cos\left(\left(k+\frac{1}{2}\right)\pi x\right)\cos\left(\left(\ell+\frac{1}{2}\right)\pi y\right).$$

and observe that $\underline{u} = \cos(\sqrt{\lambda_{k\ell}}\,t)\phi_{k\ell}$ when $(u_0,u_1) = (\phi_{k\ell},0)$.

Fix k and ℓ and set $\lambda(t) = \cos(\sqrt{\lambda_{k\ell}}\,t)$. Define $w(a) \in V'$ by

$$w(a)(v) = -\sqrt{\lambda_{k\ell}}\int_{\Gamma_1} a\phi_{k\ell} v.$$

Whence, (6.5) becomes

$$\begin{cases} \displaystyle\int_\Omega u''(t)v dx = \int_\Omega \nabla u(t)\cdot\nabla v dx - \int_{\Gamma_1} a u'(t) v + \lambda(t) w(a)(v), \quad v \in V. \\ \\ u(0) = 0, \;\; u'(0) = 0. \end{cases}$$

In other words u is the solution of (6.1) with $w = w(a)$. We find by applying Proposition 6.1

$$\|w(a)\|_{V'} \leqslant Ce^{\lambda_{k\ell}\tau^2}\|\partial_\nu u\|_{L^2(\Gamma_1\times(0,\tau))}. \tag{6.6}$$

By noting that $(a_1 \otimes a_2)\phi_{k\ell} \in V$ even if $a_1 \otimes a_2 \notin V$ we obtain

$$\begin{aligned} a_1(0)\left|\int_{\Gamma_1}(a\phi_{k\ell})^2 d\sigma\right| &= \frac{1}{\sqrt{\lambda_{k\ell}}}\,|w(a)((a_1\otimes a_2)\phi_{k\ell})| \\ &\leqslant \frac{1}{\sqrt{\lambda_{k\ell}}}\|w(a)\|_{V'}\|(a_1\otimes a_2)\phi_{k\ell}\|_V, \end{aligned} \tag{6.7}$$

where we used that $a_1(0) = a_2(0)$ and

$$\|(a_1 \otimes a_2)\phi_{k\ell}\|_V \leqslant C_0\sqrt{\lambda_{kl}}\|a_1 \otimes a_2\|_{H^1(\Omega)}. \tag{6.8}$$

Here and henceforth C_0 is a generic constant independent of a and $\phi_{k\ell}$.

Now a combination of (6.6), (6.7) and (6.8) yields

$$\begin{aligned} a_1(0)\Big(\|a_1\phi_k\|^2_{L^2(0,1)}+\|a_2\phi_\ell\|^2_{L^2(0,1)}\Big) \\ \leqslant C\|a_1\|_{H^1(0,1)}\|a_2\|_{H^1(0,1)}e^{\lambda_{k\ell}\frac{\tau^2}{2}}\|\partial_\nu u\|_{L^2(\Gamma_1\times(0,\tau))}, \end{aligned}$$

where $\phi_k(s) = \sqrt{2}\cos\left((k+1/2)\,\pi s\right)$. This, $\underline{a} \leqslant a_j(0)$ and $\|a_j\|_{H^1(0,1)} \leqslant \aleph$ imply

$$\|a_1\phi_k\|^2_{L^2(0,1)} + \|a_2\phi_\ell\|^2_{L^2(0,1)} \leqslant C_0e^{\lambda_{k\ell}\frac{\tau^2}{2}}\|\partial_\nu u\|_{L^2(\Gamma_1\times(0,\tau))}.$$

Hence, where $j = 1$ or 2,

$$\|a_j\phi_k\|^2_{L^2(0,1)} \leqslant C_0e^{k^2\tau^2\pi^2}\|\partial_\nu u\|_{L^2(\Gamma_1\times(0,\tau))}. \tag{6.9}$$

Let $\delta \in (0,1)$ be fixed. Observing that $\phi_0(s) \sim \pi(s-1)/2$ as $s \to 1$ we deduce that $|\phi_0|^{-\delta} \in L^1(0,1)$. Then we obtain by following the proof of Lemma 3.2

$$\begin{aligned} \|a_j\|_{L^2(0,1)} &\leqslant \||\phi_0|^{-\delta}\|^{1/(2+\delta)}_{L^1(0,1)}\|a_j\|^{2/(2+\delta)}_{L^\infty(0,1)}\|a_j\phi_0\|^{\delta/(2+\delta)}_{L^2(0,1)} \\ &\leqslant C\|a_j\phi_0\|^{\delta/(2+\delta)}_{L^2(0,1)}. \end{aligned} \tag{6.10}$$

A combination of inequalities (6.10) and (6.9) with $k = 0$ yields

$$\|a_j\|_{L^2(0,1)} \leqslant C\|\partial_\nu u\|^{\delta/[2(2+\delta)]}_{L^2(\Gamma_1\times(0,\tau))}.$$

This achieves the proof of the expected inequality. □

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