# How to use observability inequalities to solve some inverse problems for evolution equations ? An unified approach

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# HOW TO USE OBSERVABILITY INEQUALITIES TO SOLVE SOME INVERSE PROBLEMS FOR EVOLUTION EQUATIONS ? AN UNIFIED APPROACH

KAÏS AMMARI, MOURAD CHOULLI, AND FAOUZI TRIKI

Abstract. We survey some of our recent results on inverse problems for evolution equations. The goal is to provide an unified approach to solve various type of evolution equations. The inverse problems we consider consist in determining unknown coefficients from boundary measurements by varying initial conditions. Based on observability inequalities, and a special choice of initial conditions we provide uniqueness and stability estimates for the recovery of volume and boundary lower order coefficients in wave and heat equations. Some of the results presented here are slightly improved from their original versions.

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## 1. Introduction

Inverse coefficient problems in heat and wave equations using control results has been developed by a large community of people (see for instance [7, 26, 16, 17] and the references therein). In [2] the authors initiated a general method to deal with inverse source problems for evolution equations. Starting from the ideas in [2], we developed an approach based on observability inequalities and spectral decompositions to solve some inverse coefficients problems in evolution equations [3, 4, 5]. The measurements are made on a sub-boundary by varying initial conditions. The key idea in our analysis consists in reducing the inverse coefficients problems to inverse source problems. This is achieved by using spectral decompositions.

For clarity's sake we limit ourselves to initial boundary value problems for wave and heat equations. But our analysis can be extended to other type of evolution equation such as dynamical Schrödinger equation.

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The main ingredient in our approach is observability inequalities. We point out that the wave and the heat equations have different observability properties. We know that, under some appropriate conditions, the wave equation is exactly observable, while the heat equation is only final time observable. We refer to Section 2 for details.

## 2. Observability inequalities

In this section we collect various observability inequalities, necessary to the analysis of the inverse problems we want to tackle in this text. Since most of these results are well recorded in the literature, we limit ourselves to give precise statement of them and provide the references where the proofs can be found.

2.1. **Wave and heat equations for the Laplace-Beltrami operator.** Let $n \geqslant 2$ be an integer, and consider $M = (M, g)$ a compact $n$-dimensional Riemannian manifold with boundary. By a manifold with boundary, we mean a $C^\infty$ manifold and its boundary is $C^\infty$ manifold of dimension $n-1$. Throughout this text, we adopt the Einstein convention summation for repeated indices. If in any term the same index name appears twice, as both an upper and a lower index, that term is assumed to be summed from 1 to $n$.

In local coordinates system $x = (x^1, \dots, x^n)$,

$$g = g_{ij} dx^i \otimes dx^j.$$

Let $(\partial_1, \dots, \partial_n)$ be the dual basis to $(x^1, \dots, x^n)$. For two vector fields $X = X^i \partial_i$ and $Y = Y^j \partial_j$ over $M$,

$$\langle X, Y \rangle = g_{ij} X^i Y^j.$$

Set $|X| = \sqrt{\langle X, X \rangle}$.

As usual, the gradient of $u \in C^\infty(M)$ is the vector field given by

$$\nabla u = g^{ij} \partial_i u \partial_j$$

and the Laplace-Beltrami operator is the operator acting as follows

$$\Delta u = \frac{1}{\sqrt{\det g}} \partial_i \left( \sqrt{\det g}\, g^{ij} \partial_j u \right),$$

where $(g^{ij})$ denote the inverse of the metric $(g_{ij})$.

We are first concerned with observability inequalities for the wave equation.

Consider the following initial-boundary value problem, abbreviated to IBVP's in the sequel, for the wave equation:

$$\begin{cases} \partial_t^2 u - \Delta u + q(x) u + a(x) \partial_t u = 0 & \text{in } M \times (0, \tau), \\ u = 0 & \text{on } \partial M \times (0, \tau), \\ u(\cdot, 0) = u_0,\ \partial_t u(\cdot, 0) = u_1. \end{cases} \tag{2.1}$$

Recall that the usual energy space for the wave equation is

$$\mathcal{H}_0 = H_0^1(M) \oplus L^2(M).$$

According to [10, sections 5 and 6, Chapter XVIII] or [6, Chapter 2]), for any $q, a \in L^\infty(\Omega)$, $\tau > 0$ and $(u_0, u_1) \in \mathcal{H}_0$, the IBVP (2.1) has a unique solution

$$u = u(q, a, (u_0, u_1)) \in C([0, \tau], H_0^1(M))$$

so that $\partial_t u \in C([0, \tau], L^2(M))$. If, additionally

$$\|q\|_\infty + \|a\|_\infty \leqslant N,$$

for some constant $N > 0$, then, by the energy estimate,

$$\|u\|_{C([0,\tau], H_0^1(M))} + \|\partial_t u\|_{C([0,\tau], L^2(M))} \leqslant C \|(u_0, u_1)\|_{\mathcal{H}_0}, \tag{2.2}$$

holds with $C = C(N) > 0$ is a nondecreasing function.

Denote by $\nu$ the unit normal vector field pointing inward $M$ and set $\partial_\nu u = \langle \nabla u, \nu \rangle$. From [6, Lemma 2.4.1], $\partial_\nu u \in L^2(B)$ and

$$\|\partial_\nu u\|_{L^2(\partial M)} \leqslant c_M \left( \|(u_0, u_1)\|_{\mathcal{H}_0} + \|qu + a\partial_t u\|_{L^1((0,\tau),L^2(M))} \right), \tag{2.3}$$

where $c_M$ is a constant depending only on $M$.

In light of (2.2), (2.3) entails

$$\|\partial_\nu u\|_{L^2(\partial M)} \leqslant C\|(u_0, u_1)\|_{\mathcal{H}_0}, \tag{2.4}$$

with a constant $C$ of the same form as in (2.2).

Let $\Gamma$, a non empty open subset of $\partial M$, and $\tau > 0$ so that $(\Gamma, \tau)$ geometrically control $M$. This means that if every generalized geodesic traveling at speed one in $M$ meets $\Gamma$ in a non-diffractive point at a time $t \in (0, \tau)$. We refer to [20] for a precise definition of this assumption.

In light of [20, theorem page 169] (see also [18, Theorem 1.5]), and bearing in mind that exact controllability is equivalent to exact observability, we can state the following inequality

$$2\kappa_0\|(u_0, u_1)\|_{\mathcal{H}_0} \leqslant \|\partial_\nu u^0\|_{L^2(\Gamma\times(0,\tau))},$$

for some constant $\kappa_0 > 0$, where we set $u^0 = u(0, 0, (u_0, u_1))$ for $(u_0, u_1) \in \mathcal{H}_0$.

The perturbation argument in [25, Proposition 6.3.3, page 189] allows us red to show the existence of $\beta > 0$ so that, for any $(q, a) \in W^{1,\infty}(M) \oplus L^\infty(M)$ satisfying $\|(q, a)\|_{W^{1,\infty}(M)\oplus L^\infty(M)} \leqslant \beta$,

$$\kappa_0\|(u_0, u_1)\|_{\mathcal{H}_0} \leqslant \|\partial_\nu u\|_{L^2(\Gamma\times(0,\tau))}.$$

Here $\kappa_0 > 0$ is as in previous inequality and $u = u(q, a, (u_0, u_1))$.

Fix $N > 0$. Repeating the preceding argument $\left[\frac{N}{\beta}\right] + 1$ times in order to deduce that there exist $\kappa > 0$ so that $(q, a) \in W^{1,\infty}(M) \oplus L^\infty(M)$ satisfying

$$\|(q, a)\|_{W^{1,\infty}(M)\oplus L^\infty(M)} \leqslant N,$$

we have

$$\kappa\|(u_0, u_1)\|_{\mathcal{H}_0} \leqslant \|\partial_\nu u\|_{L^2(\Gamma\times(0,\tau))},$$

where $u = u(q, a, (u_0, u_1))$.

We point out that the authors in [8] have established an observability inequality in which the constant $\kappa$ depends only on the $L^\infty$ bound of the potential.

**Theorem 2.1.** *Let $N > 0$ and assume that $(\Gamma, \tau)$ geometrically control $M$. There exists $\kappa > 0$ so that, for any $(q, a) \in W^{1,\infty}(M) \oplus L^\infty(M)$ satisfying*

$$\|(q, a)\|_{W^{1,\infty}(M)\oplus L^\infty(M)} \leqslant N,$$

*we have*

$$\kappa\|(u_0, u_1)\|_{\mathcal{H}_0} \leqslant \|\partial_\nu u\|_{L^2(\Gamma\times(0,\tau))}, \tag{2.5}$$

*where $u = u(q, a, (u_0, u_1))$.*

Next, we examine the case where we do not assume that $(\Gamma, \tau)$ geometrically control $M$. To this end, define

$$\mathbf{d}(\Gamma) = \sup\{d(x, \Gamma);\ x \in M\}.$$

Introduce the notation

$$v = v(q, (u_0, v_0)) = u(q, 0, (u_0, u_1))$$

and set

$$\mathcal{H}_{-1} = L^2(M) \oplus H^{-1}(M).$$

In light of [18, Corollary 3.2] , we have

**Theorem 2.2.** *Let $N > 0$ and assume that $\mathbf{d}(\Gamma) < \infty$. Under the assumption $\tau > 2\mathbf{d}(\Gamma)$, there exist strictly positive constants $C$, $\kappa$ and $\epsilon_0$ so that for any $q \in L^\infty(M)$ with $\|q\|_{L^\infty(M)} \leqslant N$, we have*

$$C\|(u_0,u_1)\|_{\mathcal{H}_{-1}} \leqslant e^{\kappa\epsilon}\|\partial_\nu v\|_{L^2(\Gamma\times(0,\tau))} + \frac{1}{\epsilon}\|(u_0,u_1)\|_{\mathcal{H}_0}, \ \ (u_0,u_1) \in \mathcal{H}_0, \ \epsilon \geqslant \epsilon_0. \tag{2.6}$$

*Here $v = v(q,(u_0,v_0))$.*

Next, we a give an observability inequality for a parabolic equation. Consider then the IBVP

$$\begin{cases} \partial_t u - \Delta u + q(x)u = 0 & \text{in } M\times(0,\tau), \\ u = 0 & \text{on } \partial M\times(0,\tau), \\ u(\cdot,0) = u_0. \end{cases} \tag{2.7}$$

For $q \in L^\infty(M)$, let $A_q = \Delta - q$ with domain $D(A_q) = H_0^1(M) \cap H^2(M)$. As $A_0$ is an m-dissipative operator, we deduce form the well established theory of continuous semigroups that $A_q$ generates a strongly continuous semigroup $e^{tA_q}$. Therefore, for any $u_0 \in L^2(M)$ the IBVP has a unique solution

$$u = u(q,u_0) = e^{tA_q}u_0 \in C([0,\tau],L^2(M)) \cap C^1(]0,\tau],H^2(M)\cap H_0^1(M)).$$

The perturbation argument we used previously for the wave equation is stated in general abstract setting [25, Proposition 6.3.3, page 189]. So it is in particular [25, Proposition 6.3.3, page 189] applicable for the heat equation. This argument together with [19, Corollary 4] yield the following final time observability inequality

**Theorem 2.3.** *Let $\tau > 0$, $\Gamma$ a non empty open subset of $\partial M$ and $N > 0$. There exists a constant $\kappa > 0$ so that, for any $q \in L^\infty(M)$ satisfying $\|q\|_{L^2(M)} \leqslant N$, we have*

$$\|u(\cdot,\tau)\|_{L^2(M)} \leqslant C\|\partial_\nu u\|_{L^2(\Gamma\times(0,\tau))}, \tag{2.8}$$

*where $u = u(q,u_0)$ with $u_0 \in L^2(M)$.*

2.2. **Wave equation in a rectangular domain with boundary damping.** In this subsection $\Omega = (0,1)\times(0,1)$. Consider

$$\begin{cases} \partial_t^2 u - \Delta u = 0 & \text{in } \Omega\times(0,\tau), \\ u = 0 & \text{on } \Gamma_0\times(0,\tau), \\ \partial_\nu u + a\partial_t u = 0 & \text{on } \Gamma_1\times(0,\tau), \\ u(\cdot,0) = u_0, \ \partial_t u(\cdot,0) = u_1. \end{cases} \tag{2.9}$$

Here

$$\Gamma_0 = ((0,1)\times\{1\}) \cup (\{1\}\times(0,1)),$$
$$\Gamma_1 = ((0,1)\times\{0\}) \cup (\{0\}\times(0,1))$$

and $\partial_\nu = \nu\cdot\nabla$ is the derivative along $\nu$, the unit normal vector pointing outward of $\Omega$. Note that $\nu$ is everywhere defined except at the vertices of $\Omega$.

In the sequel we identify $a_{|(0,1)\times\{0\}}$ by $a_1 = a_1(x)$, $x \in (0,1)$ and $a_{|\{0\}\times(0,1)}$ by $a_2 = a_2(y)$, $y \in (0,1)$. In that case it is natural to identify $a$, defined on $\Gamma_1$, by the pair $(a_1,a_2)$.

Fix $\frac{1}{2} < \alpha \leqslant 1$ and let

$$\mathscr{A} = \{b = (b_1,b_2) \in C^\alpha([0,1]) \oplus C^\alpha([0,1]), \ b_1(0) = b_2(0), \ b_j \geqslant 0\}.$$

Let $V = \{u \in H^1(\Omega); \ u = 0 \text{ on } \Gamma_0\}$ and consider on $V \oplus L^2(\Omega)$ the operator $A_a$, $a \in \mathscr{A}$, given by

$$A_a = (w, \Delta v), \quad D(A_a) = \{(v,w) \in V\oplus V; \ \Delta v \in L^2(\Omega) \text{ and } \partial_\nu v = -aw \text{ on } \Gamma_1\}.$$

From [4], $A_a$ generates a strongly continuous semigroup $e^{tA_a}$. Whence, for any $(u_0,u_1) \in D(A_a)$, the IBVP (2.9) has a solution $u = u(a,(u_0,u_1))$ so that

$$(u,\partial_t u) \in C([0,\tau],D(A_a)) \cap C^1([0,\tau],V\oplus L^2(\Omega)).$$

We proved in [4, Corollary 2.2] the following observability inequality

**Theorem 2.4.** *Fix $0 < \delta_0 < \delta_1$. Then there exist $\tau_0 > 0$ and $\kappa > 0$, depending only on $\delta_0$ and $\delta_1$ so that for any $\tau \geqslant \tau_0$ and $a \in \mathscr{A}$ satisfying $\delta_0 \leqslant a \leqslant \delta_1$ on $\Gamma_1$,*

$$\kappa \|(u_0, u_1)\|_{V \oplus L^2(\Omega)} \leqslant \|\partial_\nu u\|_{L^2(\Gamma_1 \times (0,\tau))},$$

*where $u = u(a, (u_0, u_1))$, with $(u_0, u_1) \in D(A_a)$.*

We notice here that this result is a direct consequence of the fact that the observation is made on $\Gamma_1$ which fulfill the geometric control condition for $\tau \geqslant \tau_0$. If the observation is made only on one side of $\Gamma_1$ we obtain a Hölder type estimate (see [24]).

## 3. Weighted interpolation inequalities

We aim in the present section to establish two weighted interpolation inequalities. These inequalities are useful in the proof of Hölder stability estimates for certain inverse problems we will discuss in the coming sections.

As in the preceding section, $M = (M, g)$ is a compact $n$-dimensional Riemannian manifold with boundary.

Introduce the assumption:

($\mathscr{H}$) $M$ is embedded in a $n$-dimensional complete manifold without boundary $M_0 = (M_0, g)$ and the following Hardy's inequality is fulfilled

$$\int_M |\nabla f(x)|^2 dV \geqslant c \int_M \frac{|f(x)|^2}{d(x, \partial M)^2} dV, \quad f \in H_0^1(M), \tag{3.1}$$

for some constant $c > 0$, where $dV$ is the volume form on $M$, $d$ is the geodesic distance introduced previously, and $d(\cdot, \partial M)$ is the distance to $\partial M$.

Recall that $C_x \subset T_x M$ is $\alpha$-angled cone if, for some $X \in T_x M$, $C_x$ is of the form

$$C_x = \{Y \in T_x M;\ \angle\langle X, Y\rangle < \alpha\}.$$

Define $r_x(v) = \inf\{|t|;\ \gamma_{x,v}(t) \notin M\}$, where $\gamma_{x,v}$ is the geodesic satisfying the initial condition $\gamma_{x,v}(0) = x$ and $\dot\gamma_{x,v}(0) = v$. Following [23], Hardy's inequality (3.1) holds for $M$ whenever $M$ has the following uniform interior cone property: there are an angle $\alpha > 0$ and a constant $c_0 > 0$ so that, for any $x \in M$, there exists an $\alpha$-angled cone $C_x \subset T_x M$ with the property that $r_x(v) \leqslant c_0 d(x, \partial M)$, for all $v \in C_x$. The proof in [23] follows the method by Davies [11, page 25] for the flat case. Note that Hardy's inequality holds for any bounded Lipschitz domain of $\mathbb{R}^n$, and $c \leqslant \frac{1}{4}$ with equality when $\Omega$ is convex.

The following Hopf's maximum principle is a key ingredient for establishing our first weighted interpolation inequality.

**Lemma 3.1.** *Let $q \in C(M)$ and $u \in C^2(M) \cap H_0^1(M)$ satisfying $q \leqslant 0$ and $\Delta u + qu \leqslant 0$. If $u$ is non identically equal to zero, then $u > 0$ in $\mathring{M}$ and $\partial_\nu u(y) = \langle \nabla u(y), \nu(y)\rangle > 0$ for any $y \in \partial M$.*

*Proof.* Similar to that of [13, Lemma 3.4, page 34 and Theorem 3.5, page 35]. The tangent ball in the classical Hopf's lemma is substitute by a tangent geodesic ball (see the construction in [22, Proof of Theorem 9.2, page 51]). □

**Proposition 3.1.** *Let $q \in C(M)$ and $u \in C^2(M) \cap H_0^1(M)$ satisfying $q \leqslant 0$ and $\Delta u + qu \leqslant 0$. If $u$ is non identically equal to zero, then*

$$u(x) \geqslant c_u d(x, \partial M), \quad x \in M,$$

*the constant $c_u$ only depends on $u$ and $M$.*

*Proof.* Let $0 < \epsilon$ to be specified later. Let $x \in M$ so that $d(x, \partial M) \leqslant \epsilon$ and $y \in \partial M$ satisfying $d(x, \partial M) = d(x, y)$. Since $M$ is complete, there exist a unit speed minimizing geodesic $\gamma : [0, r] \to M$ such that $\gamma(0) = y$, $\gamma(r) = x$ and $\dot\gamma(0) = \nu(y)$, where we set $r = d(x, \partial M)$ (see for instance [21, page 150]).

Define $\phi(t) = u(\gamma(t))$. Then

$$\phi'(t) = du(\gamma(t))(\dot\gamma(t))$$
$$\phi''(t) = d^2u(\gamma(t))(\dot\gamma(t),\dot\gamma(t)) + du(\gamma(t))(\ddot\gamma(t)).$$

Here $\dot\gamma(t) = \dot\gamma^i(t)\partial_i \in T_{\gamma(t)}$. Observe that by the geodesic equation

$$\ddot\gamma^k(t) = -\dot\gamma^i(t)\dot\gamma^j(t)\Gamma^k_{ij}(\gamma(t)),$$

where $\Gamma^k_{ij}$ are the Christoffel symbols associated to the metric $g$.

Taking into account that $\phi'(0) = du(y)(\nu(y)) = \langle\nabla u(y),\nu(y)\rangle = \partial_\nu u(y)$, we get

$$\phi(r) = r\partial_\nu u(y) + \frac{r^2}{2}\phi''(st),$$

for some $0 < s < 1$. Hence, there exist $c > 0$ depending on $u$ and $M$ so that

$$\phi(r) \geqslant 2r\eta - cr^2 \geqslant r\eta + r(\eta - c\epsilon)$$

with $2\eta = \min_{y\in\Gamma}\partial_\nu u(y) > 0$ (by the compactness of $\partial M$ and Lemma 3.1). Thus,

$$\phi(r) \geqslant r\eta$$

provided that $\epsilon \leqslant \eta/c$. In other words, we proved

$$u(x) = \phi(r) \geqslant r\eta = \eta d(x,\partial M). \tag{3.2}$$

On the other hand, an elementary compactness argument yields, where $M^\epsilon = \{x \in M;\ d(x,\partial M) \geqslant \epsilon\}$,

$$u(x) \geqslant \min_{z\in M^\epsilon} u(z) \geqslant \frac{\min_{z\in M^\epsilon} u(z)}{\max_{z\in M^\epsilon} d(z,\partial M)} d(x,\partial M),\quad x \in M^\epsilon. \tag{3.3}$$

In light of (3.2) and (3.3), we end up getting

$$u(x) \geqslant c_u d(x,\partial M),\quad x \in M.$$

□

A consequence of Proposition 3.1 is

**Corollary 3.1.** *Assume that ($\mathscr{H}$) is satisfied. Let $q \in C(M)$, $q \leqslant 0$, and $u \in C^2(M)\cap H^1_0(M)$ non identically equal to zero satisfying $\Delta u + qu \leqslant 0$. There exists a constant $C_u$, only depending on $u$ and $M$, so that, for any $f \in H^2(M)$, we have*

$$\|f\|_{L^2(M)} \leqslant C_u \|fu\|^{\frac12}_{L^2(M)}\|f\|^{\frac12}_{H^2(M)}.$$

*Proof.* By Proposition 3.1, $u(x) \geqslant c_u d(x,\partial M)$. Therefore

$$\int_M f(x)^2 dV(x) \leqslant c_u^{-1}\int_M \frac{f(x)^2u(x)^2}{d(x,\partial M)^2} dV(x).$$

Combined with Hardy's inequality (3.1), this estimate gives

$$\int_M f(x)^2 dV(x) \leqslant c_u^{-1} c\int_M |\nabla(fu)(x)|^2 dV(x). \tag{3.4}$$

But, from usual interpolation inequalities,

$$\|fu\|_{H^1(M)} \leqslant C\|fu\|^{\frac12}_{L^2(M)}\|fu\|^{\frac12}_{H^2(M)},$$

where the constant $C$ only depends on $M$.

Whence, (3.4) implies

$$\|f\|_{L^2(M)} \leqslant C_u \|fu\|^{\frac12}_{L^2(M)}\|f\|^{\frac12}_{H^2(M)},$$

which is the expected inequality □

Let $0 \leqslant q \in C^1(M)$ be fixed and consider the operator $A = -\Delta + q$, with domain $D(A) = H^2(M) \cap H_0^1(M)$. An extension of [13, Theorem 8.38, page 214] to a compact Riemannian manifold with boundary shows that the first eigenvalue of $A$, denoted by $\lambda_1$ is simple and has a positive eigenfunction. Let then $\phi_1 \in C^2(M)$ (by elliptic regularity) be the unique first eigenfunction satisfying $\phi_1 > 0$ and normalized by $\|\phi_1\|_{L^2(M)} = 1$. Since $\Delta\phi_1 - q\phi_1 = -\lambda_1\phi_1$, the Hopf's maximum principle is applicable to $\phi_1$. Therefore, a particular weight in the preceding corollary is obtained by taking $u = \phi_1$.

**Corollary 3.2.** *Under assumption $(\mathscr{H})$, there exists a constant $c > 0$, depending on $\phi_1$, so that, for any $f \in H^2(M)$, we have*

$$\|f\|_{L^2(\mathring{M})} \leqslant c\|f\phi_1\|_{L^2(M)}^{\frac{1}{2}}\|f\|_{H^2(M)}^{\frac{1}{2}}. \tag{3.5}$$

The second weighted interpolation inequality relies on the following proposition.

**Proposition 3.2.** *Let $p \in L^\infty(M)$ and $u \in W^{2,n}(M)$ satisfying $(\Delta + p)\varphi = 0$ in $M$ and $\varphi^2 \in W^{2,n}(M)$. Then there exists $\delta = \delta(\varphi) > 0$ so that $|\varphi|^{-\delta} \in L^1(M)$.*

It is worth mentioning that $\delta < 1$ as soon as $\varphi$ vanishes at some point $x_0 \in M$. Indeed, in the flat case $\psi(x) \sim |x - x_0|^k$ near $x_0$ if $x_0$ is a zero of order $k$. Hence $|\varphi|^{-\delta}$ is locally integrable in a neighborhood of $x_0$ if and only if $\delta k < 1$.

*Sketch of the proof. First step.* Denote by $B$ the unit ball of $\mathbb{R}^n$ and let $B_+ = B \cap \mathbb{R}^n_+$, with $\mathbb{R}^n_+ = \{x = (x', x_n) \in \mathbb{R}^n;\ x_n > 0\}$. Let $L$ be a second order differential operator acting as follows

$$Lu = \partial_j(a_{ij}\partial_i u) + C \cdot \nabla u + du.$$

Assume that $(a_{ij})$ is a symmetric matrix with entries in $C^1(2\overline{B_+})$, $C \in L^\infty(2B_+)^n$ is real valued and $d \in L^\infty(2B_+)$ is complex valued. Assume moreover that

$$a_{ij}(x)\xi_j \cdot \xi_j \geqslant \kappa_0|\xi|^2,\ \ x \in 2B_+,\ \xi \in \mathbb{R}^n,$$

for some $\kappa_0 > 0$.

Let $u \in W^{2,n}(2B_+) \cap C^0(2\overline{B_+})$ be a weak solution of $Lu = 0$ satisfying $u = 0$ on $\partial(2B_+) \cap \overline{\mathbb{R}^n_+}$ and $|u|^2 \in W^{2,n}(2B_+) \cap C^0(2\overline{B_+})$. From [1, Theorem 1.1, page 942], there exists a constant $C$, that can depend on $u$, so that the following doubling inequality at the boundary

$$\int_{B_{2r}\cap B_+} |u|^2 dx \leqslant C \int_{B_r \cap B_+} |u|^2 dx,$$

holds for any ball $B_{2r}$, of radius $2r$, contained in $2B$.

On the other hand simple calculations yield, where $v = \Re u$ and $w = \Im u$,

$$\partial_j(a_{ij}\partial_i|u|^2) + 2C \cdot \nabla|u|^2 + 4(|\Re d| + |\Im d|)|u|^2 \geqslant 2a_{ij}\partial_i v\partial_j v + 2a_{ij}\partial_i w\partial_j w \geqslant 0 \ \text{ in } 2B_+$$

and $|u|^2 = 0$ on $\partial(2B_+) \cap \overline{\mathbb{R}^n_+}$.

Harnak's inequality at the boundary (see [13, Theorem 9.26, page 250]) entails

$$\sup_{B_r \cap B_+} |u|^2 \leqslant \frac{C}{|B_{2r}|} \int_{B_{2r}\cap B_+} |u|^2 dx,$$

for any ball $B_{2r}$, of radius $2r$, contained in $2B$.

Define $\widetilde{u}$ by

$$\widetilde{u}(x', x_n) = u(x', x_n) \ \text{ if } (x', x_n) \in 2B_+, \quad \widetilde{u}(x', x_n) = u(x', -x_n) \text{ if } (x', -x_n) \in 2B_+.$$

Therefore $\widetilde{u}$ belongs to $H^1(2B) \cap L^\infty(2B)$ and satisfies

$$\int_{B_{2r}} |\widetilde{u}|^2 dx \leqslant C \int_{B_r} |\widetilde{u}|^2 dx, \tag{3.6}$$

$$\sup_{B_r} |\widetilde{u}|^2 \leqslant \frac{C}{|B_{2r}|} \int_{B_{2r}} |\widetilde{u}|^2 dx, \tag{3.7}$$

for any ball $B_{2r}$, of radius $2r$, contained in $2B$.

Inequalities (3.6) and (3.7) at hand, we mimic the proof of [9, Theorem 4.2, page 1784] in order to obtain that $|\widetilde{u}|^{-\delta} \in L^1(B)$ for some $\delta > 0$ depending on $u$. Whence $|u|^{-\delta} \in L^1(B_+)$.

*Second step.* As $\partial M$ is compact, there exists a finite cover $(U_\alpha)$ of $\partial M$ and $C^\infty$-diffeomorphisms $f_\alpha : U_\alpha \to 2B$ so that $f_\alpha(U_\alpha \cap \overset{\circ}{M}) = 2B_+$, $f_\alpha(U_\alpha \cap \partial M) = 2B \cap \overline{\mathbb{R}^n_+}$ and, for any $x \in \partial M$, $x \in V_\alpha = f_\alpha^{-1}(B)$, for some $\alpha$. Then $u_\alpha = \varphi \circ f_\alpha^{-1}$ satisfies $L_\alpha u_\alpha = 0$ in $2B$ and $u = 0$ on $\partial(2B_+) \cap \overline{\mathbb{R}^n_+}$ for some $L = L_\alpha$ obeying to the conditions of the first step. Hence $|u_\alpha|^{-\delta_\alpha} \in L^1(B_+)$ and then $|\varphi|^{-\delta_\alpha} \in L^1(V_\alpha)$. Let $V$ the union of $V_\alpha$'s. Since $u \in L^\infty(V)$, $|u|^{-\delta_0} \in L^1(V)$ with $\delta_0 = \min \delta_\alpha$. Next, let $\epsilon$ sufficiently small in such a way that $M \backslash M_\epsilon \subset V$, where $M_\epsilon = \{x \in M;\ \mathrm{dist}(x, \partial M) > \epsilon\}$. Proceeding as previously it is not hard to get that there exists $\delta_1$ so that $|\varphi|^{-\delta_1} \in L^1(M_{\epsilon/2})$. Finally, as it is expected, we derive that $|\varphi|^{-\delta} \in L^1(M)$ with $\delta = \min(\delta_0, \delta_1)$.

□

**Lemma 3.2.** *Let $\delta > 0$ be as in Proposition 3.2. There exists a constant $C > 0$, depending on $\varphi$, so that, for any $f \in L^\infty(M)$, we have*

$$\|f\|_{L^2(M)} \leqslant C \|f\|_{L^\infty(M)}^{\frac{2}{2+\delta}} \|f\varphi\|_{L^2(M)}^{\frac{\delta}{2+\delta}}.$$

*Proof.* Since $\varphi$ in Proposition 3.2 belongs to $L^\infty(M)$, substituting $\delta$ by $\min(1, \delta)$ if necessary, we assume that $\delta < 2$. We get by applying Cauchy-Schwarz's inequality

$$\int_M |f|^{\delta/2} dV \leqslant \||f\varphi|^\delta\|_{L^1(M)}^{1/2} \||\varphi|^{-\delta}\|_{L^1(M)}^{1/2}.$$

But, by Hölder's inequality,

$$\||f\varphi|^\delta\|_{L^1(M)}^{1/2} \leqslant \mathrm{Vol}(M)^{(2-\delta)/4} \|f\varphi\|_{L^2(M)}^{\delta/2}.$$

Whence

$$\||f|^{\delta/2}\|_{L^1(M)} \leqslant \mathrm{Vol}(M)^{(2-\delta)/4} \|f\varphi\|_{L^2(M)}^{\delta/2} \||\varphi|^{-\delta}\|_{L^1(M)}^{1/2}. \tag{3.8}$$

On the other hand

$$\|f\|_{L^2(M)} \leqslant \|f\|_{L^\infty(M)}^{1-\delta/4} \||f|^{\delta/2}\|_{L^1(M)}^{1/2}. \tag{3.9}$$

A combination of (3.8) and (3.9) yields

$$\|f\|_{L^2(M)} \leqslant C \|f\|_{L^\infty(M)}^{1-\delta/4} \|f\varphi\|_{L^2(M)}^{\delta/4},$$

which is the expected inequality.

□

## 4. Inverse source problem: abstract framework

Let $H$ be a Hilbert space and $A : D(A) \subset H \to H$ be the generator of continuous semigroup $T(t)$. An operator $C \in \mathscr{B}(D(A), Y)$, $Y$ is another Hilbert space which is identified with its dual space, is called an admissible observation for $T(t)$ if for some (and hence for all) $\tau > 0$, the operator $\Psi \in \mathscr{B}(D(A), L^2((0,\tau), Y))$, given by

$$(\Psi x)(t) = CT(t)x, \ \ t \in [0, \tau], \ \ x \in D(A),$$

has a bounded extension to $H$.

We introduce the notion of exact observability for the system

$$z'(t) = Az(t), \ \ z(0) = x, \tag{4.1}$$

$$y(t) = Cz(t), \tag{4.2}$$

where $C$ is an admissible observation for $T(t)$. Following the usual definition, the pair $(A, C)$ is said exactly observable at time $\tau > 0$ if there is a constant $\kappa$ such that the solution $(z, y)$ of (4.1) and (4.2) satisfies

$$\int_0^\tau \|y(t)\|_Y^2 dt \geqslant \kappa^2 \|x\|_H^2, \ \ x \in D(A).$$

Or equivalently

$$\int_0^\tau \|(\Psi x)(t)\|_Y^2 dt \geqslant \kappa^2 \|x\|_H^2, \quad x \in D(A). \tag{4.3}$$

Consider the Cauchy problem

$$z'(t) = Az(t) + \lambda(t)x, \quad z(0) = 0 \tag{4.4}$$

and set

$$y(t) = Cz(t), \quad t \in [0, \tau]. \tag{4.5}$$

By Duhamel's formula, we have

$$y(t) = \int_0^t \lambda(t-s)CT(s)xds = \int_0^t \lambda(t-s)(\Psi x)(s)ds. \tag{4.6}$$

Let

$$H_\ell^1((0,\tau),Y) = \left\{u \in H^1((0,\tau),Y);\ u(0) = 0\right\}.$$

We define the operator $S : L^2((0,\tau),Y) \longrightarrow H_\ell^1((0,\tau),Y)$ by

$$(Sh)(t) = \int_0^t \lambda(t-s)h(s)ds. \tag{4.7}$$

If $E = S\Psi$, then (4.6) takes the form

$$y(t) = (Ex)(t).$$

**Theorem 4.1.** *We assume that $(A, C)$ is exactly observable for $\tau \geqslant \tau_0$, for some $\tau_0 > 0$. Let $\lambda \in H^1(0,\tau)$ satisfies $\lambda(0) \neq 0$. Then $E$ is one-to-one from $H$ onto $H_\ell^1((0,\tau),Y)$ and*

$$\frac{\kappa|\lambda(0)|}{\sqrt{2}} e^{-\tau \frac{\|\lambda'\|^2_{L^2(0,\tau)}}{|\lambda(0)|^2}} \|x\|_H \leqslant \|Ex\|_{H_\ell^1((0,\tau),Y)}, \quad x \in H. \tag{4.8}$$

*Proof.* First, taking the derivative with respect to $t$ of both sides of the integral equation

$$\int_0^t \lambda(t-s)\varphi(s)ds = \psi(t),$$

we get a Volterra integral equation of second kind

$$\lambda(0)\varphi(t) + \int_0^t \lambda'(t-s)\varphi(s)ds = \psi'(t).$$

Mimicking the proof of [15, Theorem 2, page 33], we obtain that this integral equation has a unique solution $\varphi \in L^2((0,\tau),Y)$ and

$$\begin{aligned} \|\varphi\|_{L^2((0,\tau),Y)} &\leqslant C\|\psi'\|_{L^2((0,\tau),Y)} \\ &\leqslant C\|\psi\|_{H_\ell^1((0,\tau),Y)}. \end{aligned}$$

Here $C = C(\lambda)$ is a constant.

To estimate the constant $C$ above, we first use the elementary convexity inequality $(a+b)^2 \leqslant 2(a^2+b^2)$ in order to get

$$\||\lambda(0)|\varphi(t)\|_Y^2 \leqslant 2\left(\int_0^t \frac{|\lambda'(t-s)}{|\lambda(0)|}\left[|\lambda(0)|\|\varphi(s)\|_Y\right]ds\right)^2 + 2\|\psi'(t)\|_Y^2.$$

Thus,

$$|\lambda(0)|^2\|\varphi(t)\|_Y^2 \leqslant 2\frac{\|\lambda'\|^2_{L^2((0,\tau))}}{|\lambda(0)|^2}\int_0^t |\varphi(0)|^2\|\varphi(s)\|_Y^2 ds + 2\|\psi'(t)\|_Y^2$$

by the Cauchy-Schwarz's inequality. Therefore, using Gronwall's lemma, we obtain in a straightforward manner

$$\|\varphi\|_{L^2((0,\tau),Y)} \leqslant \frac{\sqrt{2}}{|\lambda(0)|} e^{\tau \frac{\|\lambda'\|^2_{L^2((0,\tau))}}{|\lambda(0)|^2}} \|\psi'\|_{L^2((0,\tau),Y)}$$

and then

$$\|\varphi\|_{L^2((0,\tau),Y)} \leqslant \frac{\sqrt{2}}{|\lambda(0)|} e^{\tau \frac{\|\lambda'\|^2_{L^2((0,\tau))}}{|\lambda(0)|^2}} \|S\varphi\|_{H^1_\ell((0,\tau),Y)}.$$

In light of (4.3), we end up getting

$$\|Ex\|_{H^1_\ell((0,\tau),Y)} \geqslant \frac{\kappa|\lambda(0)|}{\sqrt{2}} e^{-\tau \frac{\|\lambda'\|^2_{L^2((0,\tau))}}{|\lambda(0)|^2}} \|x\|_H.$$

□

We shall need a variant of Theorem 4.1. If $(A, C)$ is as in Theorem 4.1, then, as in the preceding section, by the perturbation argument in [25, Proposition 6.3.3, page 189], for any $N > 0$, there is $\kappa > 0$ such that for any $P \in \mathscr{B}(H)$ satisfying $\|P\| \leqslant N$, $(A + P, C)$ is exactly observable with $\kappa(P + A) \geqslant \kappa$.

We define $E^P$ similarly to $E$ by substituting $A$ by $A + P$.

**Theorem 4.2.** *Let $N > 0$ and assume that $(A, C)$ is exactly observable for $\tau \geqslant \tau_0$, for some $\tau_0 > 0$. Let $\lambda \in H^1(0, \tau)$ satisfies $\lambda(0) \neq 0$. For any $P \in \mathscr{B}(H)$, $E^P$ is one-to-one from $H$ onto $H^1_\ell((0, \tau), Y)$ and*

$$\frac{\kappa|\lambda(0)|}{\sqrt{2}} e^{-\tau \frac{\|\lambda'\|^2_{L^2(0,\tau)}}{|\lambda(0)|^2}} \|x\|_H \leqslant \|E^P x\|_{H^1_\ell((0,\tau),Y)}, \quad x \in H. \tag{4.9}$$

*Here $\kappa$ is the observability inequality of $A + P$, depending on $N$ but not in $P$.*

We will consider inverse source problems with singular sources. So we need to extend Theorem 4.1. Fix then $\beta$ in the resolvent set of $A$. Let $H_1$ be the space $D(A)$ equipped with the norm $\|x\|_1 = \|(\beta - A)x\|$ and denote by $H_{-1}$ the completion of $H$ with respect to the norm $\|x\|_{-1} = \|(\beta - A)^{-1}x\|$. As it is observed in [25, Proposition 4.2, page 1644] and its proof, when $x \in H_{-1}$ (which is the dual space of $H_1$ with respect to the pivot space $H$) and $\lambda \in H^1(0, \tau)$, then, according to the classical extrapolation theory of semigroups, the Cauchy problem (4.1) has a unique solution $z \in C([0, \tau], H)$. Additionally, $y$ given in (4.2) belongs to $L^2((0, \tau), Y)$.

When $x \in H$, we have by Duhamel's formula

$$y(t) = \int_0^t \lambda(t - s)CT(s)x ds = \int_0^t \lambda(t - s)(\Psi x)(s) ds. \tag{4.10}$$

Let

$$H^1_\ell((0, \tau), Y) = \left\{u \in H^1((0, \tau), Y);\ u(0) = 0\right\}.$$

We define the operator $S : L^2((0, \tau), Y) \longrightarrow H^1_\ell((0, \tau), Y)$ by

$$(Sh)(t) = \int_0^t \lambda(t - s)h(s) ds. \tag{4.11}$$

If $E = S\Psi$, then (4.10) takes the form

$$y(t) = (Ex)(t).$$

Let $\mathcal{Z} = (\beta - A^*)^{-1}(X + C^*Y)$.

**Theorem 4.3.** *Assume that $(A, C)$ is exactly observable at time $\tau$. Then*
*(i) $E$ is one-to-one from $H$ onto $H^1_\ell((0, \tau), Y)$.*
*(ii) $E$ is extended to an isomorphism, denoted by $\widetilde{E}$, from $\mathcal{Z}'$ onto $L^2((0, \tau), Y)$.*
*(iii) There exists a constant $\widetilde{\kappa}$, independent on $\lambda$, so that*

$$\|x\|_{\mathcal{Z}'} \leqslant \widetilde{\kappa}|\lambda(0)| e^{\frac{\|\lambda'\|^2_{L^2((0,\tau))}}{|\lambda(0)|^2}\tau} \|\widetilde{E}x\|_{L^2((0,\tau),Y)}. \tag{4.12}$$

*Proof.* We give the proof of (ii) and (iii). (i) is contained in Theorem 4.1. To do this, we start by observing that $S^*$, the adjoint of $S$, maps $L^2((0,\tau),Y)$ into $H^1_r((0,\tau),Y)$, where

$$H^1_r((0,\tau),Y)=\left\{u\in H^1((0,\tau),Y);\ u(\tau)=0\right\}.$$

Moreover

$$S^*h(t)=\int_t^\tau \lambda(s-t)h(s)ds,\quad h\in L^2((0,\tau),Y).$$

Fix $h\in L^2((0,\tau),Y)$ and set $k=S^*h$. Then

$$k'(t)=\lambda(0)h(t)-\int_t^\tau \lambda'(s-t)h(s)ds.$$

Hence

$$\begin{aligned}|\lambda(0)\|h(t)\|^2&\leqslant\left(\int_t^\tau\frac{|\lambda'(s-t)|}{|\lambda(0)|}[|\lambda(0)|\|h(s)\|]ds+\|k'(t)\|\right)^2\\&\leqslant 2\left(\int_t^\tau\frac{|\lambda'(s-t)|}{|\lambda(0)|}[|\lambda(0)|\|h(s)\|]ds\right)^2+2\|k'(t)\|^2\\&\leqslant 2\frac{\|\lambda'\|^2_{L^2((0,\tau))}}{|\lambda(0)|^2}\int_0^t[|\lambda(0)|\|h(s)\|]^2ds+2\|k'(t)\|^2.\end{aligned}$$

The last estimate is obtained by applying Cauchy-Schwarz's inequality.

A simple application of Gronwall's lemma entails

$$[|\lambda(0)|\|h(t)\|]^2\leqslant 2e^{2\frac{\|\lambda'\|^2_{L^2((0,\tau))}}{|\lambda(0)|^2}\tau}\|k'(t)\|^2.$$

Therefore,

$$\|h\|_{L^2((0,\tau),Y)}\leqslant\frac{\sqrt{2}}{|\lambda(0)|}e^{\frac{\|\lambda'\|^2_{L^2((0,\tau))}}{|\lambda(0)|^2}\tau}\|k'\|_{L^2((0,\tau),Y)}.$$

This inequality yields

$$\|h\|_{L^2((0,\tau),Y)}\leqslant\frac{\sqrt{2}}{|\lambda(0)|}e^{\frac{\|\lambda'\|^2_{L^2((0,\tau))}}{|\lambda(0)|^2}\tau}\|S^*h\|_{H^1_r((0,\tau),Y)}.\tag{4.13}$$

The adjoint operator of $S^*$, acting as a bounded operator from $[H^1_r((0,\tau);Y)]'$ into $L^2((0,\tau);Y)$, gives an extension of $S$. We denote by $\widetilde{S}$ this operator. By [25, Proposition 4.1, page 1644] $\widetilde{S}$ defines an isomorphism from $[H_r((0,1);Y)]'$ onto $L^2((0,\tau);Y)$. In light of the identity

$$\|\widetilde{S}\|_{\mathscr{B}([H^1_r((0,\tau);Y)]';L^2((0,\tau),Y))}=\|S^*\|_{\mathscr{B}(L^2((0,\tau);Y);H^1_r((0,\tau),Y))},$$

(4.13) implies

$$\frac{|\lambda(0)|}{\sqrt{2}}e^{-\frac{\|\lambda'\|^2_{L^2((0,\tau))}}{|\lambda(0)|^2}\tau}\leqslant\|\widetilde{S}\|_{\mathscr{B}([H^1_r((0,\tau);Y)]';L^2((0,\tau);Y))}.\tag{4.14}$$

On the other hand, according to [25, Proposition 2.13, page 1641], $\Psi$ possesses a unique bounded extension, denoted by $\widetilde{\Psi}$ from $\mathcal{Z}'$ into $[H^1_r((0,\tau);Y)]'$ and there exists a constant $c>0$ so that

$$\|\widetilde{\Psi}\|_{\mathcal{B}(\mathcal{Z}';[H^1_r((0,\tau);Y)]')}\geqslant c.\tag{4.15}$$

$\widetilde{E}=\widetilde{S}\widetilde{\Psi}$ gives the unique extension of $E$ to an isomorphism from $\mathcal{Z}'$ onto $L^2((0,\tau),Y)$.

We end up the proof by noting that (4.12) is a consequence of (4.14) and (4.15). □

## 5. Inverse problems for evolution equations associated to Laplace-Beltrami operator

In this section $M=(M,g)$ is a compact $n$-dimensional Riemannian manifold with boundary, $\tau>0$ and $\Gamma$ is a nonempty subset of $\partial M$.

5.1. **Inverse source problem for the wave equation.** Consider the IBVP for the wave equation

$$\begin{cases} \partial_t^2 u - \Delta u + q(x)u + a(x)\partial_t u = g(t)f(x) & \text{in } M\times(0,\tau),\\ u = 0 & \text{on } \partial M\times(0,\tau),\\ u(\cdot,0) = 0,\ \partial_t u(\cdot,0) = 0. \end{cases} \tag{5.1}$$

Let $\Gamma$ be a nonempty subset of $\partial M$ and assume that $(\Gamma,\tau)$ geometrically control $M$. Fix $N>0$ and denote by $\kappa$ the observability constant corresponding to arbitrary $(q,a)\in W^{1,\infty}(M)\oplus L^\infty(M)$ satisfying

$$\|(q,a)\|_{W^{1,\infty}(M)\oplus L^\infty(M)}\leqslant N.$$

For $f\in L^2(M)$ and $g\in H^1(0,\tau)$ with $g(0)\neq 0$. According to Theorem 4.1, we have

$$\|f\|_{L^2(M)}\leqslant \frac{\sqrt{2}}{\kappa|g(0)|}e^{\tau\frac{\|g'\|^2_{L^2(0,\tau)}}{|g(0)|^2}}\|\partial_\nu u\|_{H^1((0,\tau),L^2(\Gamma))}, \tag{5.2}$$

where $u=u(q,a,f,g)$ denotes the solution of the IBVP (5.1).

As an immediate consequence of this inequality, we have

**Theorem 5.1.** *Assume that $(\Gamma,\tau)$ geometrically control $M$. Let $N>0$ and $g\in H^1(0,\tau)$ satisfying $g(0)\neq 0$. Then there exists a constant $C$, depending on $\kappa$, $\Gamma$, $\tau$ and $g$ so that, for any $(q,a)\in W^{1,\infty}(M)\oplus L^\infty(M)$ with*

$$\|(q,a)\|_{W^{1,\infty}(M)\oplus L^\infty(M)}\leqslant N,$$

*we have*

$$\|f\|_{L^2(M)}\leqslant C\|\partial_\nu u\|_{H^1((0,\tau),L^2(\Gamma))},$$

*Here $u=u(q,a,f,g)$ denotes the solution of the IBVP (5.1).*

For simplicity's sake, set $v=v(q,f,g)=u(q,0,f,g)$. That is $v$ is the solution of the IBVP

$$\begin{cases} \partial_t^2 u - \Delta u + q(x)u = g(t)f(x) & \text{in } M\times(0,\tau),\\ u = 0 & \text{on } \partial M\times(0,\tau),\\ u(\cdot,0) = 0,\ \partial_t u(\cdot,0) = 0. \end{cases} \tag{5.3}$$

Using Duhamel's formula, it is not hard to check that

$$v(x,t)=\int_0^t g(t-s)w(x,s)ds,$$

where $w=w(f)$ is the solution of the IBVP

$$\begin{cases} \partial_t^2 w - \Delta w + q(x)w = 0 & \text{in } M\times(0,\tau),\\ w = 0 & \text{on } \partial M\times(0,\tau),\\ w(\cdot,0) = f,\ \partial_t w(\cdot,0) = 0. \end{cases} \tag{5.4}$$

Let

$$H^1_\ell((0,\tau),L^2(\Gamma))=\big\{u\in H^1((0,\tau),L^2(\Gamma));\ u(0)=0\big\}$$

and define the operator $S: L^2(\Gamma\times(0,\tau))\longrightarrow H^1_\ell((0,\tau),L^2(\Gamma))$ by

$$(Sh)(t)=\int_0^t g(t-s)h(s)ds.$$

We have seen in the proof of Theorem 4.1 that $S$ is an isomorphism and

$$\|h\|_{L^2(\Gamma\times(0,\tau))}\leqslant \frac{\sqrt{2}}{\kappa|g(0)|}e^{\tau\frac{\|g'\|^2_{L^2((0,\tau))}}{|g(0)|^2}}\|Sh\|_{H^1((0,\tau),L^2(\Gamma))}.$$

Whence

$$\|\partial_\nu w\|_{L^2(\Gamma\times(0,\tau))}\leqslant \frac{\sqrt{2}}{\kappa|g(0)|}e^{\tau\frac{\|g'\|^2_{L^2((0,\tau))}}{|g(0)|^2}}\|\partial_\nu v\|_{H^1((0,\tau),L^2(\Gamma))}. \tag{5.5}$$

Let $N > 0$, assume that $\mathbf{d}(\Gamma) < \infty$ and $\tau > 2\mathbf{d}(\Gamma)$. From Theorem 2.2, there exist three constants $C$, $\kappa$ and $\epsilon_0$ so that for any $q \in L^\infty(M)$ with $\|q\|_{L^\infty(M)} \leqslant N$, we have

$$C\|f\|_{L^2(M)} \leqslant e^{\kappa\epsilon}\|\partial_\nu w\|_{L^2(\Gamma\times(0,\tau))} + \frac{1}{\epsilon}\|f\|_{H_0^1(M)}, \quad \epsilon \geqslant \epsilon_0. \tag{5.6}$$

Now (5.5) in (5.6) yields

$$C\|f\|_{L^2(M)} \leqslant e^{\kappa\epsilon}\frac{\sqrt{2}}{\kappa|g(0)|}e^{\tau\frac{\|g'\|^2_{L^2((0,\tau))}}{|g(0)|^2}}\|\partial_\nu v\|_{H^1((0,\tau),L^2(\Gamma))} + \frac{1}{\epsilon}\|f\|_{H_0^1(M)}, \quad \epsilon \geqslant \epsilon_0. \tag{5.7}$$

Let

$$\Psi(\rho) = |\ln\rho|^{-1} + \rho, \quad \rho > 0,$$

extended by continuity at $\rho = 0$ by setting $\Psi(0) = 0$.

A standard minimization argument, with respect to $\epsilon$ in (5.7) enables us to establish

**Theorem 5.2.** *Let $N > 0$, $R > 0$, assume that $\mathbf{d}(\Gamma) < \infty$ and $\tau > 2\mathbf{d}(\Gamma)$. Let $g \in H^1(0,\tau)$ satisfying $g(0) \neq 0$. Then there exists a constant $C > 0$, depending on $N$, $R$, $\Gamma$, $\tau$ and $g$ so that, for any $q \in L^\infty(M)$ with $\|q\|_{L^\infty(M)} \leqslant N$ and any $f \in H_0^1(M)$ satisfying $\|f\|_{H_0^1(M)} \leqslant R$, we have*

$$\|f\|_{L^2(M)} \leqslant C\Psi\left(\|\partial_\nu v\|_{H^1((0,\tau),L^2(\Gamma))}\right),$$

*where $v = v(q,f,g)$ is the solution of the IBVP* (5.3)

5.2. **Determining the potential and the damping coefficient in a wave equation.** Introduce the IBVP for the wave equation

$$\begin{cases} \partial_t^2 u - \Delta u + q(x)u + a(x)\partial_t u = 0 & \text{in } M\times(0,\tau), \\ u = 0 & \text{on } \partial M\times(0,\tau), \\ u(\cdot,0) = u_0,\ \partial_t u(\cdot,0) = u_1. \end{cases} \tag{5.8}$$

Recall that

$$\mathcal{H}_0 = H_0^1(M)\oplus L^2(M)$$

and let $N > 0$.

We have seen in Section 1 that, for any $(q,a) \in L^\infty(M)\oplus L^\infty(M)$, $\tau > 0$ and $(u_0,u_1) \in \mathcal{H}_0$, the IBVP (5.8) has a unique solution

$$u = u(q,a,(u_0,u_1)) \in C([0,\tau],H_0^1(M))$$

so that $\partial_t u \in C([0,\tau],L^2(M))$ and $\partial_\nu u \in L^2(\partial M\times(0,\tau))$. Moreover, under the assumption

$$\|(q,a)\|_{L^\infty(M)\oplus L^\infty(M)} \leqslant N,$$

we have

$$\|u\|_{C([0,\tau],H_0^1(M))} + \|\partial_t u\|_{C([0,\tau],L^2(M))} \leqslant C\|(u_0,u_1)\|_{\mathcal{H}_0} \tag{5.9}$$

and

$$\|\partial_\nu u\|_{L^2(\partial M\times(0,\tau))} \leqslant C\|(u_0,u_1)\|_{\mathcal{H}_0}. \tag{5.10}$$

Here $C = C(N)$ is a nondecreasing function.

Define the initial-to-boundary operator $\Lambda(q,a)$ as follows

$$\Lambda(q,a) : (u_0,u_1) \in \mathcal{H}_0 \mapsto \partial_\nu u(q,a,(u_0,u_1)) \in L^2(\Gamma\times(0,\tau)).$$

Let

$$\mathcal{H}_1 = (H_0^1(M)\cap H^2(M))\oplus H_0^1(M).$$

Observing that

$$\partial_t u(q,a,(u_0,u_1)) = u(q,a,(u_1,\Delta u_0 - qu_0 - au_1)),$$

we easily obtain that $\Lambda(q,a) \in \mathscr{B}(\mathcal{H}_1,H^1((0,\tau),L^2(\Gamma)))$. Additionally, as a consequence of (5.10), we have

$$\|\Lambda(q,a)\|_{\mathscr{B}(\mathcal{H}_1,H^1((0,\tau),L^2(\Gamma)))} \leqslant C,$$

where the constant $C$ is similar to that in (5.10).

**Theorem 5.3.** *Assume that $(\Gamma,\tau)$ geometrically control $M$. Let $N>0$ and $0\leqslant q\in H^2(M)$. There exists a constant $C>0$, depending on $N$ and $q$, so that, for any $(\widetilde{q},\widetilde{a})\in H^2(M)\oplus H^2(M)$ satisfying*

$$\|(\widetilde{q},\widetilde{a})\|_{H^2(M)\oplus H^2(M)}\leqslant N,$$

*we have*

$$\|\widetilde{q}-q\|_{L^2(M)}+\|\widetilde{a}-0\|_{L^2(M)}\leqslant C\|\Lambda(\widetilde{q},\widetilde{a})-\Lambda(q,0)\|^{\frac{1}{2}}_{\mathscr{B}(\mathcal{H}_1,H^1((0,\tau),L^2(\Gamma)))}.$$

*Proof.* Let $0\leqslant\phi_1$ be the first eigenfunction of the operator $-\Delta+q$ with domain $H^2(M)\cap H^1_0(M)$. This eigenfunction is normalized by $\|\phi_1\|_{L^2(M)}=1$. If

$$u=u(q,0,(\phi_1,i\sqrt{\lambda_1}\phi_1))=e^{i\sqrt{\lambda_1}\,t}\phi_1\ \text{ and }\ \widetilde{u}=u(\widetilde{q},\widetilde{a},(\phi_1,i\sqrt{\lambda_1}\,\phi_1)),$$

then $v=\widetilde{u}-u$ is the solution of the following IBVP

$$\left\{\begin{array}{ll}\partial_t^2v-\Delta v+\widetilde{q}v+\widetilde{a}(x)\partial_tv=-[(\widetilde{q}-q)+i\sqrt{\lambda_1}\widetilde{a}]e^{i\sqrt{\lambda_1}t}\phi_1 & \text{in } M\times(0,\tau),\\ v=0 & \text{on } \partial M\times(0,\tau),\\ v(\cdot,0)=0,\ \partial_tv(\cdot,0)=0.\end{array}\right.\tag{5.11}$$

Bearing in mind that $(\Gamma,\tau)$ geometrically control $M$, we get from Theorems 5.1

$$\|\phi_1(\widetilde{q}-q)\|_{L^2(M)}+\|\phi_1\widetilde{a}\|_{L^2(M)}\leqslant C\|\partial_\nu v\|_{H^1((0,\tau),L^2(\partial M))}.$$

This inequality, combined with Corollary 3.1, yields

$$\|\widetilde{q}-q\|_{L^2(M)}+\|\widetilde{a}-0\|_{L^2(M)}\leqslant C\|\partial_\nu v\|^{\frac{1}{2}}_{H^1((0,\tau),L^2(\Gamma))}.$$

This inequality entails in a straightforward manner the expected one. □

Denote the sequence of eigenvalues, counted according to their multiplicity, of $A=-\Delta$ with domain $H^2(M)\cap H^1_0(M)$, by $0<\lambda_1<\lambda_2\leqslant\ldots\leqslant\lambda_k\to\infty$.

Consider, on $\mathcal{H}_0$, the operators

$$\mathcal{A}=\begin{pmatrix}0 & I\\ -A & 0\end{pmatrix},\ \ D(\mathcal{A})=\mathcal{H}_1$$

and $\mathcal{A}(q,a)=\mathcal{A}+\mathcal{B}(q,a)$ with $D(\mathcal{A}(q,a))=D(\mathcal{A})$, where

$$\mathcal{B}(q,a)=\begin{pmatrix}0 & 0\\ -q & -a\end{pmatrix}\in\mathscr{B}(\mathcal{H}_0).$$

From [25, Proposition 3.7.6, page 100], $\mathcal{A}$ is skew-adjoint operator with $0\in\rho(\mathcal{A})$ and

$$\mathcal{A}_0^{-1}=\begin{pmatrix}0 & -A^{-1}\\ I & 0\end{pmatrix}.$$

We note that, since $\mathcal{A}^{-1}:\mathcal{H}\to\mathcal{H}_1$ is bounded and the embedding $\mathcal{H}_1\hookrightarrow\mathcal{H}$ is compact, $\mathcal{A}^{-1}:\mathcal{H}\to\mathcal{H}$ is compact.

Also, from [25, Proposition 3.7.6, page 100], $\mathcal{A}$ is diagonalizable and its spectrum consists in the sequence $(i\sqrt{\lambda_k})$.

Introduce the bounded operator $\mathcal{C}(q,a)=(i\mathcal{A}_0^{-1})(-i\mathcal{B}(q,a))(i\mathcal{A}^{-1})$. Let $s_k(\mathcal{C}(q,a))$ be the singular values of $\mathcal{C}(q,a)$, that is the eigenvalues of $[\mathcal{C}(q,a)^*\mathcal{C}(q,a)]^{\frac{1}{2}}$. In light of [14, formulas (2.2) and (2.3), page 27], we have

$$s_k(\mathcal{C}(q,a))\leqslant\|\mathcal{B}(q,a)\|s_k(i\mathcal{A}^{-1})^2=\|\mathcal{B}(q,a)\|\lambda_k^{-1},$$

where $\|\mathcal{B}(q,a)\|$ denote the norm of $\mathcal{B}(q,a)$ in $\mathscr{B}(\mathcal{H}_0)$.

On the other hand, referring to Weyl's asymptotic formula, we have $\lambda_k=O(k^{\frac{2}{n}})$. Hence, $\mathcal{C}_{q,a}$ belongs to the Shatten class $\mathcal{S}_p$ for any $p>\frac{n}{2}$, that is

$$\sum_{k\geqslant1}[s_k(\mathcal{C}(q,a))]^p<\infty.$$

We apply [14, Theorem 10.1, page 276] in order to get that the spectrum of $\mathcal{A}_{q,a}$ consists in a sequence of eigenvalues $(\mu_k(q,a))$, counted according to their multiplicity, and the corresponding eigenfunctions $(\phi_k(q,a))$ form a Riesz basis of $\mathcal{H}_0$.

Fix $(q,a)$ and $k$. Set $\mu = \mu_k(q,a)$ and $\phi = \phi_k(q,a) = (\varphi,\psi) \in \mathcal{H}_1$ be an eigenfunction associated to $\mu$. Then it is straightforward to check that $\psi = \mu\varphi$ and $(-\Delta + q + a\mu + \mu^2)\varphi = 0$ in $M$. Since $-\Delta\varphi = f$ in $M$ with $f = (q + a\mu + \mu^2)\varphi$, we can use iteratively [13, Corollary 7.11, page 158] (Sobelev embedding theorem) together with [13, Theorem 9.15, page 241] in order to obtain that $\varphi \in W^{2,p}(M)$ for any $1 < p < \infty$. In particular $\varphi, |\varphi|^2 \in W^{2,n}(M) \cap C^0(M)$. In other words, $\varphi$ satisfies the assumption of Proposition 3.2.

If

$$u = u(q,a,\phi) \text{ and } \widetilde{u} = u(\widetilde{q},\widetilde{a},\phi),$$

and $v = \widetilde{u} - u$, then, similarly to Theorem 5.3, we prove

$$\|\varphi(\widetilde{q}-q)\|_{L^2(M)} + \|\varphi(\widetilde{a}-a)\|_{L^2(M)} \leqslant C\|\partial_\nu v\|_{H^1((0,\tau),L^2(\partial M))}.$$

This and Lemma 3.1 yield

**Theorem 5.4.** *Assume that $(\Gamma,\tau)$ geometrically control $M$. Let $N > 0$ and $(q,a) \in W^{1,\infty}(M) \oplus L^\infty(M)$. There exists two constants $C > 0$ and $\alpha > 0$, depending on $N$ and $(q,a)$ so that, for any $(\widetilde{q},\widetilde{a}) \in W^{1,\infty}(M) \oplus L^\infty(M)$ satisfying*

$$\|(\widetilde{q},\widetilde{a})\|_{W^{1,\infty}(M)\oplus L^\infty(M)} \leqslant N,$$

*we have*

$$\|\widetilde{q}-q\|_{L^2(M)} + \|\widetilde{a}-a\|_{L^2(M)} \leqslant C\|\Lambda(\widetilde{q},\widetilde{a}) - \Lambda(q,a)\|^\alpha_{\mathscr{B}(\mathcal{H}_1,H^1((0,\tau),L^2(\Gamma)))}.$$

5.3. **Determining the potential in a wave equation without geometric control assumption.** Consider the IBVP

$$\begin{cases} \partial_t^2 u - \Delta u + q(x)u = 0 & \text{in } M\times(0,\tau), \\ u = 0 & \text{on } \partial M\times(0,\tau), \\ u(\cdot,0) = u_0,\ \partial_t u(\cdot,0) = 0. \end{cases} \tag{5.12}$$

From the preceding subsection, the initial-to-boundary mapping

$$\Lambda(q) : u_0 \in H_0^1(M)\cap H^2(M) \mapsto \partial_\nu u \in H^1((0,\tau),L^2(\Gamma)),$$

where, $u = u(q,u_0)$ is the solution on the IBVP, defines a bounded operator. Moreover, for any $N > 0$, there exists a constant $C > 0$, depending on $N$, so that for any $q \in L^\infty(M)$ satisfying $\|q\|_{L^\infty(M)} \leqslant N$, we have

$$\|\Lambda(q)\|_{\mathscr{B}(H_0^1(M)\cap H^2(M),H^1((0,\tau),L^2(\Gamma)))} \leqslant C.$$

**Theorem 5.5.** *Let $N > 0$, assume that $\mathbf{d}(\Gamma) < \infty$ and $\tau > 2\mathbf{d}(\Gamma)$. There exists a constant $C > 0$ so that, for any $0 \leqslant q \in L^\infty(M)$, $\widetilde{q} \in L^\infty(M)$ satisfying $q - \widetilde{q} \in W^{1,\infty}(M)$ and*

$$\|q\|_{L^\infty(M)},\ \|\widetilde{q}\|_{L^\infty(M)},\ \|q-\widetilde{q}\|_{W^{1,\infty}(M)} \leqslant N,$$

*we have*

$$\|q-\widetilde{q}\|_{L^2(M)} \leqslant C\Phi\left(\|\Lambda(q) - \Lambda(\widetilde{q})\|_{\mathscr{B}(H_0^1(M)\cap H^2(M),H^1((0,\tau),L^2(\Gamma)))}\right),$$

*with $\Phi(\rho) = |\ln\rho|^{-\frac{1}{n+3}} + \rho$, $\rho > 0$, extended by continuity at $\rho = 0$ by setting $\Phi(0) = 0$.*

*Proof.* Let $0 \leqslant q \in L^\infty(M)$ satisfying $\|q\|_{L^\infty(M)} \leqslant N$. Denote by $0 < \lambda_1 \leqslant \lambda_2 \ldots \leqslant \lambda_k \ldots$ the eigenvalues of the operator $-\Delta + q$ with domain $H_0^1(M)\cap H^2(M)$. Let $(\phi_k)$ an orthonormal basis of $L^2(M)$ consisting in eigenfunctions, each $\phi_k$ is an eigenvalue for $\lambda_k$. Note that according to the usual elliptic regularity, $\phi_k \in C^\infty(M)$ for each $k$.

By the Weyl's asymptotic formula and the min-max principle, there exists a constant $\beta > 1$, depending on $N$ but not in $q$, so that

$$\beta^{-1}k^{\frac{2}{n}} \leqslant \lambda_k \leqslant \beta k^{\frac{2}{n}}. \tag{5.13}$$

For $\widetilde{q}\in L^\infty(M)$ with $\|\widetilde{q}\|_{L^\infty(M)}\leqslant N$, set

$$u=u(q,\phi_k)=\cos(\lambda_k t)\phi_k \ \text{ and } \ \widetilde{u}=u(\widetilde{q},\phi_k).$$

Then $v=\widetilde{u}-u$ is the solution of the IBVP, where $g_k(t)=\cos(\sqrt{\lambda_k}t)$,

$$\left\{\begin{array}{ll}\partial_t^2 u-\Delta u+\widetilde{q}u=(\widetilde{q}-q)\phi_k g_k(t) & \text{in } M\times(0,\tau),\\ u=0 & \text{on } \partial M\times(0,\tau),\\ u(\cdot,0)=0,\ \partial_t u(\cdot,0)=0.\end{array}\right. \tag{5.14}$$

We have $\|g_k'\|^2_{L^2(0,\tau)}\leqslant\lambda_k\tau$. Hence

$$\|g_k'\|^2_{L^2(0,\tau)}\leqslant\beta\tau k^{\frac{2}{n}}, \tag{5.15}$$

by (5.13)

In the rest of this proof, $C$ and $c$ denote generic constant, depending only on $M$, $N$, $\Gamma$ and $\tau$. From (5.7), we have

$$C\|(\widetilde{q}-q)\phi_k\|_{L^2(M)}\leqslant e^{\kappa\epsilon}e^{ck^{\frac{2}{n}}}\|\partial_\nu v\|_{H^1((0,\tau),L^2(\Gamma))}+\frac{1}{\epsilon}\|(\widetilde{q}-q)\phi_k\|_{H^1_0(M)},\ \ \epsilon\geqslant\epsilon_0. \tag{5.16}$$

On the other hand,

$$\begin{aligned}\|(\widetilde{q}-q)\phi_k\|_{H^1_0(M)}&\leqslant\|\widetilde{q}-q\|_{W^{1,\infty}(M)}\|\phi_k\|_{H^1_0(M)}\\ &\leqslant 2N\sqrt{\lambda_k}\\ &\leqslant ck^{\frac{1}{n}}\ \text{ by (5.13).}\end{aligned}$$

This in (5.16) gives

$$C\|(\widetilde{q}-q)\phi_k\|_{L^2(M)}\leqslant e^{\kappa\epsilon}e^{ck^{\frac{2}{n}}}\|\partial_\nu v\|_{H^1((0,\tau),L^2(\Gamma))}+\frac{k^{\frac{1}{n}}}{\epsilon},\ \ \epsilon\geqslant\epsilon_0.$$

On the other hand, by Cauchy-Schwarz inequality

$$(\widetilde{q}-q,\phi_k)^2\leqslant\mathrm{Vol}(M)\|(\widetilde{q}-q)\phi_k\|_{L^2(M)}.$$

Whence

$$C(\widetilde{q}-q,\phi_k)^2\leqslant e^{\kappa\epsilon}e^{ck^{\frac{2}{n}}}\|\partial_\nu v\|_{H^1((0,\tau),L^2(\Gamma))}+\frac{k^{\frac{1}{n}}}{\epsilon},\ \ \epsilon\geqslant\epsilon_0.$$

But

$$\begin{aligned}\|\widetilde{q}-q\|^2_{L^2(M)}&=\sum_{k\leqslant\ell}(\widetilde{q}-q,\phi_k)^2+\sum_{k>\ell}(\widetilde{q}-q,\phi_k)^2\\ &\leqslant\sum_{k\leqslant\ell}(\widetilde{q}-q,\phi_k)^2+\frac{1}{\lambda_{\ell+1}}\sum_{k>\ell}\lambda_k(\widetilde{q}-q,\phi_k)^2\\ &\leqslant\sum_{k\leqslant\ell}(\widetilde{q}-q,\phi_k)^2+\frac{N^2}{(\ell+1)^{\frac{2}{n}}}.\end{aligned}$$

Thus

$$C\|\widetilde{q}-q\|^2_{L^2(M)}\leqslant\ell e^{\kappa\epsilon}e^{c\ell^{\frac{2}{n}}}\|\partial_\nu v\|_{H^1((0,\tau),L^2(\Gamma))}+\frac{1}{(\ell+1)^{\frac{2}{n}}}+\frac{\ell^{1+\frac{1}{n}}}{\epsilon}. \tag{5.17}$$

Let $s\geqslant 1$ be a real number and let $\ell$ be the unique integer so that $\ell\leqslant s<\ell+1$. Then (5.17) with that $\ell$ entails

$$C\|\widetilde{q}-q\|^2_{L^2(M)}\leqslant se^{\kappa\epsilon}e^{cs^{\frac{2}{n}}}\|\partial_\nu v\|_{H^1((0,\tau),L^2(\Gamma))}+\frac{1}{s^{\frac{2}{n}}}+\frac{s^{1+\frac{1}{n}}}{\epsilon}. \tag{5.18}$$

Take $\epsilon = s^{\frac{3}{n}+1}$ in (5.18) in order to get, where $s_0 = \max\left(1, \epsilon_0^{\frac{n}{n+3}}\right)$,

$$C\|\widetilde{q} - q\|^2_{L^2(M)} \leqslant \frac{1}{s^{\frac{2}{n}}} + e^{cs^{\frac{2}{n}+1}} e^{\kappa s^{\frac{3}{n}+1}} \|\partial_\nu v\|_{H^1((0,\tau),L^2(\Gamma))}, \quad s \geqslant s_0.$$

Therefore

$$C\|\widetilde{q} - q\|^2_{L^2(M)} \leqslant \frac{1}{s^{\frac{2}{n}}} + e^{cs^{\frac{3}{n}+1}} \|\partial_\nu v\|_{H^1((0,\tau),L^2(\Gamma))}, \quad s \geqslant s_0,$$

or equivalently,

$$C\|\widetilde{q} - q\|_{L^2(M)} \leqslant \frac{1}{s^{\frac{1}{n}}} + e^{cs^{\frac{3}{n}+1}} \|\partial_\nu v\|_{H^1((0,\tau),L^2(\Gamma))}, \quad s \geqslant s_0.$$

We end up getting the expected inequality by minimizing with respect to $s$. □

5.4. **Inverse source problem for the heat equation.** Consider the IBVP for the heat equation

$$\begin{cases} \partial_t u - \Delta u + q(x)u = g(t)f(x) & \text{in } Q := M \times (0,\tau), \\ u = 0 & \text{on } \partial M \times (0,\tau), \\ u(\cdot,0) = 0. \end{cases} \tag{5.19}$$

From classical parabolic regularity theorems in anisotropic Sobolev space, where

$$H^{2,1}(Q) = L^2((0,\tau), H^2(M)) \cap H^1((0,\tau), L^2(M)),$$

for any $f \in L^2(M)$, $g \in L^2(0,\tau)$ and $q \in L^\infty(M)$, the IBVP (5.19) has a unique solution

$$u = u(q,f,g) \in H^{2,1}(Q).$$

Moreover, for any $N > 0$, there exists a constant $C > 0$ so that, for any $q \in L^\infty(M)$ satisfying $\|q\|_{L^\infty(M)} \leqslant N$,

$$\|u\|_{H^{2,1}(Q)} \leqslant C\|g\|_{L^2(0,\tau)}\|f\|_{L^2(M)}. \tag{5.20}$$

If in addition $g \in H^1(0,\tau)$, then it is not hard to check that $\partial_t u$ is the solution of the IBVP (5.19) with $g$ substituted by $g'$. Hence $\partial_t u \in H^{2,1}(Q)$ and

$$\|\partial_t u\|_{H^{2,1}(Q)} \leqslant C\|g'\|_{L^2(0,\tau)}\|f\|_{L^2(M)}, \tag{5.21}$$

for any $q \in L^\infty(M)$ satisfying $\|q\|_{L^\infty(M)} \leqslant N$, where $C$ is the same constant as in (5.20).

We derive that $\partial_\nu u$ is well defined as an element of $H^1((0,\tau), L^2(\Gamma))$ and, by (5.20), (5.21) and the continuity of the trace on $\Gamma$,

$$\|\partial_\nu u\|_{H^1((0,\tau),L^2(\Gamma))} \leqslant C\|g\|_{H^1(0,\tau)}\|f\|_{L^2(M)},$$

the constant $C$ is as in (5.20).

The following result will be useful in the sequel.

**Proposition 5.1.** *Let $N > 0$. There exist two constants $c > 0$ and $C > 0$ so that, for any $q \in L^\infty(M)$ satisfying $\|q\|_{L^\infty(M)} \leqslant N$, $f \in H^1_0(M)$ and $g \in H^1(0,\tau)$ with $g(0) \neq 0$,*

$$C\|f\|_{L^2(M)} \leqslant \frac{1}{\sqrt{\epsilon}}\|f\|_{H^1_0(M)} + \frac{1}{|g(0)|} e^{\tau \frac{\|g'\|^2_{L^2(0,\tau)}}{|g(0)|^2}} e^{c\epsilon}\|\partial_\nu u\|_{H^1((0,\tau),L^2(\Gamma))}, \quad \epsilon \geqslant 1, \tag{5.22}$$

*where $u = u(q,f,g)$ is the solution of the BVP (5.19).*

*Proof.* Pick $q \in L^\infty(M)$ satisfying $\|q\|_{L^\infty(M)} \leqslant N$, $f \in H^1_0(M)$ and $g \in H^1(0,\tau)$ with $g(0) \neq 0$. Without loss of generality, we may assume that $q \geqslant 0$. Indeed, we have only to substitute $u$ by $ue^{-Nt}$, which is the solution of the IBVP (5.19) when $q$ is replaced by $q + N$.

Let $v = v(q,f) \in H^{2,1}(Q)$ be the unique solution of the IBVP

$$\begin{cases} \partial_t v - \Delta v + q(x)v = 0 & \text{in } M \times (0,\tau), \\ v = 0 & \text{on } \partial M \times (0,\tau) \\ v(\cdot,0) = f. \end{cases}$$

Then $\partial_\nu v$ is well defined as an element of $L^2(\Gamma\times(0,\tau))$. As for the wave equation

$$\partial_\nu u_{|\Gamma}(\cdot,t)=\int_0^t g(t-s)\partial_\nu v_{|\Gamma}(\cdot,s)ds.$$

Therefore

$$\|\partial_\nu v\|_{L^2(\Gamma\times(0,\tau))}\leqslant\frac{\sqrt{2}}{|g(0)|}e^{\tau\frac{\|g'\|^2_{L^2((0,\tau))}}{|g(0)|^2}}\|\partial_\nu u\|_{H^1((0,\tau),L^2(\Gamma))}. \tag{5.23}$$

From the final time observability inequality in Theorem 2.3, we have

$$\|v(\cdot,\tau)\|_{L^2(M)}\leqslant K\|\partial_\nu v\|_{L^2(\Gamma\times(0,\tau))}, \tag{5.24}$$

for some constant $K>0$, independent on $q$ and $f$.

A combination of (5.23) and (5.24) yields

$$C\|v(\cdot,\tau)\|_{L^2(M)}\leqslant\frac{1}{|g(0)|}e^{\tau\frac{\|g'\|^2_{L^2((0,\tau))}}{|g(0)|^2}}\|\partial_\nu u\|_{H^1((0,\tau),L^2(\Gamma))}. \tag{5.25}$$

Denote by $0<\lambda_1\leqslant\lambda_2\leqslant\ldots\leqslant\lambda_k\to\infty$ the sequence of eigenvalues of the $-\Delta+q$ with domain $H^1_0(M)\cap H^2(M)$. Let $(\phi_k)$ be a sequence of eigenfunctions, each $\phi_k$ corresponds to $\lambda_k$, so that $(\phi_k)$ form an orthonormal basis of $L^2(M)$.

By usual spectral decomposition, we have

$$v(\cdot,\tau)=\sum_{\ell\geqslant1}e^{-\lambda_k\tau}(f,\phi_\ell)\phi_\ell.$$

Here $(\cdot,\cdot)$ is the usual scalar product on $L^2(M)$. In particular,

$$(f,\phi_\ell)^2\leqslant e^{2\lambda_\ell\tau}\|v(\cdot,\tau)\|^2_{L^2(M)},\ \ \ell\geqslant1.$$

Whence, for any integer $k\geqslant1$,

$$\sum_{\ell=1}^k(f,\phi_\ell)^2\leqslant ke^{2\lambda_k\tau}\|v(\cdot,\tau)\|^2_{L^2(M)}.$$

This and the fact that $\left(\sum_{\ell\geqslant1}\lambda_\ell(\cdot,\phi_\ell)^2_{L^2(\Omega)}\right)^{\frac12}$ is an equivalent norm on $H^1_0(M)$ lead

$$\begin{aligned}\|f\|^2_{L^2(M)}&=\sum_{\ell=1}^k(f,\phi_\ell)^2+\sum_{\ell\geqslant k+1}(f,\phi_\ell)^2\\&\leqslant\sum_{\ell=1}^k(f,\phi_\ell)^2+\frac{1}{\lambda_{k+1}}\sum_{\ell\geqslant k+1}\lambda_\ell(f,\phi_\ell)^2\\&\leqslant ke^{2\lambda_k\tau}\|v(\cdot,\tau)\|^2_{L^2(M)}+\frac{1}{\lambda_{k+1}}\|f\|^2_{H^1_0(M)}.\end{aligned}$$

in the rest of this proof $C$ and $c$ are generic constants, independent on $q$, $f$ and $g$.

Applying inequality (5.13), we get

$$C\|f\|^2_{L^2(M)}\leqslant ke^{ck^{\frac{2}{n}}}\|v(\cdot,\tau)\|^2_{L^2(M)}+\frac{1}{(k+1)^{\frac{2}{n}}}\|f\|^2_{H^1_0(M)}. \tag{5.26}$$

Let $\epsilon\geqslant1$ and $k\geqslant1$ be the unique integer so that $k\leqslant\epsilon^{n/2}<k+1$. We obtain in a straightforward manner from (5.26)

$$C\|f\|^2_{L^2(M)}\leqslant e^{c\epsilon}\|v(\cdot,\tau)\|^2_{L^2(M)}+\frac{1}{\epsilon}\|f\|^2_{H^1(M)}. \tag{5.27}$$

(5.25) in (5.27) gives the expected inequality. □

If $q \in L^\infty(M)$ and $g \in H^1(0,\tau)$ satisfying $g(0) \neq 0$ are fixed, we obtain, by minimizing with respect to $\epsilon$, the following corollary

**Corollary 5.1.** *Let $N > 0$, $q \in L^\infty(M)$ and $g \in H^1(0,\tau)$ satisfying $g(0) \neq 0$. There exists a constant $C > 0$, depending on $N$, $q$ and $g$ so that, for any $f \in H_0^1(\Omega)$ with $\|f\|_{H_0^1(M)} \leqslant N$, we have*

$$C\|f\|_{L^2(M)} \leqslant \Phi\left(\|\partial_\nu u\|_{H^1((0,\tau),L^2(\Gamma))}\right),$$

*where $u = u(q,f,g)$ is the solution of the IBVP (5.19) and $\Phi(\rho) = |\ln \rho|^{-\frac{1}{2}} + \rho$, $\rho > 0$, extended at $\rho = 0$ by setting $\Phi(0) = 0$.*

5.5. **Determining the zero order coefficient in a heat equation.** Consider the IBVP

$$\begin{cases} \partial_t u - \Delta u + q(x)u = 0 & \text{in } M \times (0,\tau), \\ u = 0 & \text{on } \partial M \times (0,\tau), \\ u(\cdot,0) = u_0. \end{cases} \tag{5.28}$$

One more time, referring to classical regularity theorems in anisotropic Sobolev spaces, we derive that, for $q \in L^\infty(M)$ and $u_0 \in H_0^1(M)$, the IBVP(5.28) has unique solution $u = u(q,u_0) \in H^{2,1}(M \times (0,\tau))$. Moreover, for any $N > 0$, there exists a constant $C > 0$ so that

$$\|u(q,u_0)\|_{H^{2,1}(M\times(0,\tau))} \leqslant C\|u_0\|_{H_0^1(M)}, \tag{5.29}$$

for any $q \in L^\infty(M)$ satisfying $\|q\|_{L^\infty(M)} \leqslant N$.

Define

$$\mathcal{H}_0(M) = \{w \in H_0^1(M);\ \Delta w \in H_0^1(M)\}$$

that we equip with its natural norm

$$\|u\|_{\mathcal{H}_0(M)} = \|u\|_{H_0^1(M)} + \|\Delta u\|_{H_0^1(M)}.$$

If $q \in W^{1,\infty}(M)$ and $u_0 \in \mathcal{H}_0(M)$, then it is straightforward to check that

$$\partial_t u(q,u_0) = u(q, \Delta u_0 - q u_0).$$

So applying (5.29), with $u_0$ substituted by $\Delta u_0 - q u_0$, we get

$$\|\partial_t u\|_{H^{2,1}(M\times(0,\tau))} \leqslant C\|u_0\|_{\mathcal{H}_0(M)}, \tag{5.30}$$

for any $q \in W^{1,\infty}(M)$ satisfying $\|q\|_{W^{1,\infty}(M)} \leqslant N$, where the constant $C$ is independent on $q$.

Bearing in mind that the trace operator $w \in H^{2,1}(M \times (0,\tau)) \mapsto \partial_\nu w \in L^2(\Gamma \times (0,\tau))$ is bounded, we obtain that $\partial_\nu u \in H^1((0,\tau),L^2(\Gamma))$ provided that $u_0 \in \mathcal{H}_0(M)$ and $q \in W^{1,\infty}(M)$. Additionally, from (5.29) and (5.30), we get

$$\|\partial_\nu u\|_{H^1((0,\tau),L^2(\Gamma))} \leqslant C\|u_0\|_{\mathcal{H}_0(M)},$$

for any $q \in W^{1,\infty}(M)$ satisying $\|q\|_{W^{1,\infty}(M)}$, where the constant $C$ is independent on $q$.

That is we proved that the operator

$$\mathcal{N}(q) : u_0 \in \mathcal{H}_0(M) \mapsto \partial_\nu u \in H^1((0,\tau),L^2(\Gamma))$$

is bounded and

$$\|\mathcal{N}(q)\|_{\mathscr{B}(\mathcal{H}_0(M),H^1((0,\tau),L^2(\Gamma)))} \leqslant C,$$

for any $q \in W^{1,\infty}(M)$ satisfying $\|q\|_{W^{1,\infty}(M)} \leqslant N$, where the constant $C$ is independent on $q$.

In the sequel, for simplicity's sake, $\|\mathcal{N}(\widetilde{q}) - \mathcal{N}(q)\|_{\mathscr{B}(\mathcal{H}_0(M),H^1((0,\tau),L^2(\Gamma)))}$ is denoted by $\|\mathcal{N}(\widetilde{q}) - \mathcal{N}(q)\|$.

**Theorem 5.6.** *Let $N > 0$. There exists a constant $C > 0$ so that, for any $q$, $\widetilde{q} \in W^{1,\infty}(M)$ satisfying*

$$\|q\|_{W^{1,\infty}(M)},\ \|\widetilde{q}\|_{W^{1,\infty}(M)} \leqslant N,$$

*we have*

$$C\|\widetilde{q} - q\|_{L^2(M)} \leqslant \Theta\left(\|\mathcal{N}(\widetilde{q}) - \mathcal{N}(q)\|\right).$$

*Here $\Theta(\rho) = |\ln \rho|^{-\frac{1}{1+4n}} + \rho$, $\rho > 0$, extended by continuity at $\rho = 0$ by setting $\Theta(0) = 0$.*

*Proof.* Let $q,\ \widetilde{q} \in W^{1,\infty}(M)$ satisfying

$$\|q\|_{W^{1,\infty}(M)},\ \|\widetilde{q}\|_{W^{1,\infty}(M)} \leqslant N.$$

As in the preceding subsection, without loss of generality, we assume that $q \geqslant 0$.

Denote by $0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_k \to \infty$ the sequence of eigenvalues of the operator $-\Delta + q$ with domain $H_0^1(M) \cap H^2(M)$. Let $(\phi_k)$ a sequence of the corresponding eigenfunctions so that $(\phi_k)$ form an orthonormal basis of $L^2(M)$.

Taking into account that $u(q, \phi_k) = e^{-\lambda_k t}\phi_k$, we obtain that

$$v = u(\widetilde{q}, \phi_k) - u(q, \phi_k)$$

is the solution of the IBVP

$$\begin{cases} \partial_t v - \Delta v + q(x)v = (\widetilde{q} - q)\phi_k e^{-\lambda_k t} & \text{in } M \times (0,\tau), \\ uv = 0 & \text{on } \partial M \times (0,\tau), \\ v(\cdot, 0) = 0. \end{cases}$$

Therefore

$$\mathcal{N}(\widetilde{q})(\phi_k) - \mathcal{N}(q)(\phi_k) = \partial_\nu v$$

from which we deduce

$$\|\partial_\nu v\|_{H^1((0,\tau),L^2(\Gamma))} \leqslant C\lambda_k \|\mathcal{N}(\widetilde{q}) - \mathcal{N}(q)\|.$$

Here and henceforth $C$ and $c$ denote generic constants, independent on $q$ and $\widetilde{q}$.

As in the preceding subsection, we get from (5.22), for any $\epsilon \geqslant 1$,

$$C|(\widetilde{q} - q, \phi_k)| \leqslant \frac{\sqrt{\lambda_k}}{\sqrt{\epsilon}} + e^{\tau\lambda_k^2} e^{c\epsilon} \lambda_k^2 \|\mathcal{N}(\widetilde{q}) - \mathcal{N}(q)\|, \tag{5.31}$$

where we used the estimate $\|(\widetilde{q} - q)\phi_k\|_{H_0^1(M)} \leqslant C\sqrt{\lambda_k}$.

A straightforward consequence of estimate (5.31) is

$$C \sum_{k=1}^{\ell} |(\widetilde{q} - q, \phi_k)^2_{L^2(M)} \leqslant \frac{\ell\lambda_\ell}{\epsilon} + \ell e^{c\lambda_\ell^2} e^{c\epsilon} \|\mathcal{N}(\widetilde{q}) - \mathcal{N}(q)\|^2, \tag{5.32}$$

for any arbitrary integer $\ell \geqslant 1$.

Similarly to the proof of Theorem 5.5, inequality (5.32) yields, for any $s \geqslant 1$,

$$C\|\widetilde{q} - q\|^2_{L^2(M)} \leqslant \frac{s^{1+\frac{2}{n}}}{\epsilon} + \frac{1}{s^{\frac{2}{n}}} + e^{cs^{1+\frac{4}{n}}} e^{c\epsilon} \|\mathcal{N}(\widetilde{q}) - \mathcal{N}(q)\|^2.$$

The proof is then completed in the same manner to that of Theorem 5.5. □

## 6. Determining a boundary coefficient in a wave equation

6.1. **Inverse source problem for the wave equation with boundary damping.** In this subsection $\Omega = (0,1) \times (0,1)$ and

$$\Gamma_0 = ((0,1) \times \{1\}) \cup (\{1\} \times (0,1)),$$
$$\Gamma_1 = ((0,1) \times \{0\}) \cup (\{0\} \times (0,1)).$$

Consider the IBVP

$$\begin{cases} \partial_t^2 u - \Delta u = \lambda(t) w & \text{in } \Omega \times (0,\tau), \\ u = 0 & \text{on } \Gamma_0 \times (0,\tau), \\ \partial_\nu u + a\partial_t u = 0 & \text{on } \Gamma_1 \times (0,\tau), \\ u(\cdot, 0) = 0,\ \partial_t u(\cdot, 0) = 0. \end{cases} \tag{6.1}$$

Fix $\frac{1}{2} < \alpha \leqslant 1$ and let

$$\mathscr{A} = \{b = (b_1, b_2) \in C^\alpha([0,1]) \oplus C^\alpha([0,1]),\ b_1(0) = b_2(0),\ b_j \geqslant 0\}.$$

If $V = \{u \in H^1(\Omega);\ u = 0 \text{ on } \Gamma_0\}$, consider on $V \oplus L^2(\Omega)$ the operator $A_a$, $a \in \mathscr{A}$, given by

$$A_a = (w, \Delta v), \quad D(A_a) = \{(v, w) \in V \oplus V;\ \Delta v \in L^2(\Omega) \text{ and } \partial_\nu v = -aw \text{ on } \Gamma_1\}.$$

We are going to apply Theorem 4.2 with $H = V \oplus L^2(\Omega)$, $H_1 = D(A_a)$ equipped with its graph norm and $Y = L^2(\Gamma_1)$.

Denote by $H_{-1}$ the dual of $H_1$ with respect to the pivot space $H$.

If $(0, w) \in H_{-1}$ and $\lambda \in H^1(0, \tau)$, the IBVP (6.1) has a unique solution $u(w)$ so that $(u(w), \partial_t u(w)) \in C([0, \tau]; V \oplus L^2(\Omega))$ and $\partial_\nu u(w)_{|\Gamma_1} \in L^2(\Gamma_1 \times (0, \tau))$.

Taking into account that $\{0\} \times V' \subset H_{-1}$, where $V'$ is the dual space of $V$, we obtain as a consequence of Theorem 4.2

**Proposition 6.1.** *There exists a constant $C > 0$ so that for any $\lambda \in H^1(0, \tau)$ and $w \in V'$,*

$$\|w\|_{V'} \leqslant C|\lambda(0)| e^{\frac{\|\lambda'\|^2_{L^2(0,\tau)}}{|\lambda(0)|^2}\tau} \|\partial_\nu u_w\|_{L^2(\Gamma_1 \times (0,\tau))}. \tag{6.2}$$

6.2. **Determining the boundary damping coefficient in a wave equation.** Let $\Omega$ and $\Gamma_i$, $i = 1, 2$ as in the preceding subsection. Consider then the IBVP

$$\begin{cases} \partial_t^2 u - \Delta u = 0 & \text{in } \Omega \times (0, \tau), \\ u = 0 & \text{on } \Gamma_0 \times (0, \tau), \\ \partial_\nu u + a\partial_t u = 0 & \text{on } \Gamma_1 \times (0, \tau), \\ u(\cdot, 0) = u_0,\ \partial_t u(\cdot, 0) = u_1. \end{cases} \tag{6.3}$$

For $(u_0, u_1) \in H_1$, the IBVP (6.3) possesses a unique solution $u = u(a, (u_0, u_1))$ so that

$$(u_a, \partial_t u_a) \in C([0, \infty), H_1) \cap C^1([0, \infty), H).$$

Fix $0 < \underline{a} \leqslant N$ and set

$$\underline{\mathscr{A}} = \{b = (b_1, b_2) \in \mathscr{A} \cap H^1(0, 1) \oplus H^1(0, 1);\ \underline{a} \leqslant b_1,\ b_2,\ \|b\|^2_{H^1(0,1)\oplus H^1(0,1)} \leqslant N\}.$$

Let $\mathcal{U}_0$ given by

$$\mathcal{U}_0 = \{v \in V;\ \Delta v \in L^2(\Omega) \text{ and } \partial_\nu v = 0 \text{ on } \Gamma_1\}$$

and observe that $\mathcal{U}_0 \times \{0\} \subset H_1$, for any $a \in \mathscr{A}$. We endow $\mathcal{U}_0$ with the norm

$$\|u_0\|_{\mathcal{U}_0} = \left(\|u_0\|_V^2 + \|\Delta u_0\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}.$$

Define the initial-to-boundary operator

$$\Lambda(a) : u_0 \in \mathcal{U}_0 \mapsto \partial_\nu u \in L^2(\Sigma_1),$$

where $u = u(a, (u_0, u_1)$ is the solution of the IBVP (6.3). Then $\Lambda(a) \in \mathscr{B}(\mathcal{U}_0, L^2(\Gamma_1 \times (0, \tau)))$.

The norm of $\Lambda(a) - \Lambda(0)$ in $\mathscr{B}(\mathcal{U}_0, L^2(\Gamma_1 \times (0, \tau)))$ will simply denoted by $\|\Lambda(a) - \Lambda(0)\|$.

The following Hölder stability estimate is an improved version of the one derived in [4].

**Theorem 6.1.** *Let $\delta \in (0, 1)$ be fixed. There exists $\tau_0 > 0$ so that for any $\tau \geqslant \tau_0$, we find a constant $c > 0$ only depending on $\tau$ so that*

$$\|a - 0\|_{L^2(0,1)\oplus L^2(0,1)} \leqslant c(1 - \delta)^{\frac{-1}{2+\delta}} N^{\frac{1+\delta}{2+\delta}} \left(\underline{a}^{-1} \|\Lambda(a) - \Lambda(0)\|\right)^{\frac{\delta}{2(2+\delta)}}, \tag{6.4}$$

*for each $a \in \underline{\mathscr{A}}$.*

*Proof.* We first observe that $u(a)$ is also the unique solution of

$$\begin{cases} \displaystyle\int_\Omega u''(t) v dx = \int_\Omega \nabla u(t) \cdot \nabla v dx - \int_{\Gamma_1} a u'(t) v, \quad v \in V. \\ u(0) = u_0, \quad u'(0) = u_1. \end{cases}$$

Therefore, $u = u(a) - \underline{u}$, where $\underline{u} = u(0,(u_0,u_1))$, is the solution of the following problem

$$\begin{cases} \displaystyle\int_\Omega u''(t)v dx = \int_\Omega \nabla u(t)\cdot\nabla v dx - \int_{\Gamma_1} a u'(t)v - \int_{\Gamma_1} a u'(0)(t)v, \quad v\in V. \\ \\ u(0)=0, \quad u'(0)=0. \end{cases} \tag{6.5}$$

For $k$, $\ell\in\mathbb{Z}$, set

$$\lambda_{k\ell} = \left[\left(k+\frac{1}{2}\right)^2 + \left(\ell+\frac{1}{2}\right)^2\right]\pi^2$$
$$\phi_{k\ell}(x,y) = 2\cos\left(\left(k+\frac{1}{2}\right)\pi x\right)\cos\left(\left(\ell+\frac{1}{2}\right)\pi y\right).$$

Note that $\underline{u} = \cos(\sqrt{\lambda_{k\ell}}\,t)\phi_{k\ell}$ when $(u_0,u_1) = (\phi_{k\ell},0)$.

Fix $k$ and $\ell$ for the moment and set $\lambda(t) = \cos(\sqrt{\lambda_{k\ell}}\,t)$. Define $w(a)\in V'$ by

$$w(a)(v) = -\sqrt{\lambda_{k\ell}}\int_{\Gamma_1} a\phi_{k\ell}v.$$

Whence, (6.5) becomes

$$\begin{cases} \displaystyle\int_\Omega u''(t)v dx = \int_\Omega \nabla u(t)\cdot\nabla v dx - \int_{\Gamma_1} a u'(t)v + \lambda(t)w(a)(v), \quad v\in V. \\ \\ u(0)=0, \quad u'(0)=0. \end{cases}$$

In other words, $u$ is the solution of (6.1) with $w = w(a)$. Applying Proposition 6.1, we find

$$\|w(a)\|_{V'} \leqslant Ce^{\lambda_{k\ell}\tau^2}\|\partial_\nu u\|_{L^2(\Gamma_1\times(0,\tau))}. \tag{6.6}$$

But, noting that $(a_1\otimes a_2)\phi_{k\ell}\in V$ even if $a_1\otimes a_2\notin V$, we have

$$\begin{aligned} a_1(0)\Big|\int_{\Gamma_1}(a\phi_{k\ell})^2 d\sigma\Big| &= \frac{1}{\sqrt{\lambda_{k\ell}}}\big|w(a)((a_1\otimes a_2)\phi_{k\ell})\big| \\ &\leqslant \frac{1}{\sqrt{\lambda_{k\ell}}}\|w(a)\|_{V'}\|(a_1\otimes a_2)\phi_{k\ell}\|_V, \end{aligned} \tag{6.7}$$

where we used $a_1(0) = a_2(0)$, and

$$\|(a_1\otimes a_2)\phi_{k\ell}\|_V \leqslant C\sqrt{\lambda_{kl}}\|a_1\otimes a_2\|_{H^1(\Omega)}, \tag{6.8}$$

Here and henceforth, $C$ is a generic constant independent on $a$ and $\phi_{k\ell}$.

Now a combination of (6.6), (6.7) and (6.8) yields

$$\begin{aligned} a_1(0)\Big(\|a_1\phi_k\|^2_{L^2(0,1)} &+ \|a_2\phi_\ell\|^2_{L^2(0,1)}\Big) \\ &\leqslant C\|a_1\|_{H^1(0,1)}\|a_2\|_{H^1(0,1)}e^{\lambda_{k\ell}\frac{\tau^2}{2}}\|\partial_\nu u\|_{L^2(\Gamma_1\times(0,\tau))}, \end{aligned}$$

where $\phi_k(s) = \sqrt{2}\cos\left(\left(k+\frac{1}{2}\right)\pi s\right)$. This, $\underline{a}\leqslant a_j(0)$ and $\|a_j\|_{H^1(0,1)}\leqslant N$ imply

$$\|a_1\phi_k\|^2_{L^2(0,1)} + \|a_2\phi_\ell\|^2_{L^2(0,1)} \leqslant C\frac{N^2}{\underline{a}}e^{\lambda_{k\ell}\frac{\tau^2}{2}}\|\partial_\nu u\|_{L^2(\Gamma_1\times(0,\tau))},$$

Hence, where $j = 1$ or 2,

$$\|a_j\phi_k\|^2_{L^2(0,1)} \leqslant C\frac{N^2}{\underline{a}}e^{k^2\tau^2\pi^2}\|\partial_\nu u\|_{L^2(\Gamma_1\times(0,\tau))}. \tag{6.9}$$

Let $\delta \in (0,1)$ be fixed. A forward calculation shows that $|\phi_0|^{-\delta} \in L^1(0,1)$. There exists an universal constant $C>0$ such that

$$\begin{aligned}\int_0^1 |\phi_0(x)|^{-\delta} dx &= 2^{-\delta/2} \int_0^{\frac{\pi}{2}} \frac{1}{|\sin(x)|^{\delta}} dx \\ &\leqslant \frac{C}{1-\delta}.\end{aligned}$$

Following the proof of Lemma 3.2, we then obtain

$$\begin{aligned}\|a_j\|_{L^2(0,1)} &\leqslant \||\phi_0|^{-\delta}\|_{L^1(0,1)}^{\frac{1}{2+\delta}} \|a_j\|_{L^\infty(0,1)}^{\frac{2}{2+\delta}} \|a_j\phi_0\|_{L^2(0,1)}^{\frac{\delta}{2+\delta}} \\ &\leqslant C(1-\delta)^{\frac{-1}{2+\delta}} N^{\frac{1}{2+\delta}} \|a_j\phi_0\|_{L^2(0,1)}^{\frac{\delta}{2+\delta}}. \end{aligned} \tag{6.10}$$

A combination of inequalities (6.10) and (6.9), with $k=0$, yields

$$\|a_j\|_{L^2(0,1)} \leqslant C(1-\delta)^{\frac{-1}{2+\delta}} \underline{a}^{\frac{-\delta}{2(2+\delta)}} N^{\frac{1+\delta}{2+\delta}} \|\partial_\nu u\|_{L^2(\Gamma_1\times(0,\tau))}^{\frac{\delta}{2(2+\delta)}},$$

which achieves the proof of the expected result. □

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