Poincaré and Sobolev inequalities for differential forms in Heisenberg groups

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Abstract

Poincaré and Sobolev inequalities for differential forms on Heisenberg balls, involving Rumin's differentials, are given. Furthermore, a global homotopy of Rumin's complex which improves differentiability of Rumin forms is provided on any bounded geometry contact manifold.

1. Introduction

1.1. **Sobolev and Poincaré inequalities for differential forms.** Sobolev inequality in $\mathbb{R}^n$ deals with compactly supported 0-forms, i.e. functions u on $\mathbb{R}^n$, and 1-forms, their differentials du. It states that

$$\|u\|_q \le C_{p,q,n}\|du\|_p$$

whenever

$$1 \le p, q < +\infty, \quad \frac{1}{p} - \frac{1}{q} = \frac{1}{n}.$$

A local version, for functions supported in the unit ball, holds under the weaker assumption

$$1 \le p, q < +\infty, \quad \frac{1}{p} - \frac{1}{q} \le \frac{1}{n}.$$

Poincaré's inequality is a variant for functions u defined on but not necessarily compactly supported in the unit ball B. It states that there exists a real number c_u such that

$$\|u - c_u\|_q \le C_{p,q,n}\|du\|_p.$$

Alternatively, given a closed 1-form ω on B, there exists a function u on B such that $du = \omega$ on B, and such that

$$\|u\|_q \le C_{p,q,n}\|\omega\|_p.$$

This suggests the following generalization for higher degree differential forms.

Let M be a Riemannian manifold. We say that a *strong Poincaré inequality* (p,q)-Poincaré(k) holds on M, if there exists a positive constant $C = C(M,p,q)$ such that for every closed k-form ω on M, belonging to L^p, there exists a $k-1$-form ϕ such that $d\phi = \omega$ and

$$\|\phi\|_q \le C\,\|\omega\|_p.$$

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A *strong Sobolev inequality* (p,q)-Sobolev(k) holds on M, if for every closed compactly supported k-form ω on M, belonging to L^p, there exists a compactly supported $k-1$-form ϕ such that $d\phi=\omega$ and

$$\|\phi\|_q \leq C\,\|\omega\|_p.$$

Both statements should be thought of as quantitative versions of the statement that every closed k-form is exact.

For Euclidean domains, the validity of Poincaré inequality is sensitive to irregularity of boundaries. One way to eliminate such a dependance is to allow a loss on domain. Say an *interior Poincaré inequality* (p,q)-Poincaré(k) holds on M if for every small enough $r>0$ and large enough $\lambda\geq 1$, there exists a constant $C=C(M,p,q,r,\lambda)$ such that for every $x\in M$ and every closed k-form ω on $B(x,\lambda r)$, belonging to L^p, there exists a $(k-1)$-form ϕ on $B(x,r)$ such that $d\phi=\omega$ on $B(x,r)$ and

$$\|\phi\|_{L^q(B(x,r))} \leq C\,\|\omega\|_{L^p(B(x,\lambda r))}.$$

For *interior Sobolev inequalities*, merely add the word compactly supported. Both properties should be thought of as quantitative versions of the statement that, locally, every closed k-form is exact.

It turns out that in several situations, the loss on domain is harmless. This is the case for $L^{q,p}$-cohomological applications, see [17].

1.2. **Contact manifolds.** A contact structure on a manifold M is a smooth distribution of hyperplanes H which is maximally nonintegrable in the following sense: if θ is a locally defined smooth 1-form such that $H=\ker(\theta)$, then $d\theta$ restricts to a non-degenerate 2-form on H. A contact manifold is the data of a smooth manifold M and a contact structure H on M. M must be odd-dimensional. Contactomorphisms are contact structure preserving diffeomorphisms between contact manifolds. The prototype of a contact manifold is the Heisenberg group $\mathbb{H}^n$, the simply connected Lie group whose Lie algebra is the central extension $\mathfrak{h}=\mathfrak{h}_1\oplus\mathfrak{h}_2$, $\mathfrak{h}_2=\mathbb{R}=Z(\mathfrak{h})$, with bracket $\mathfrak{h}_1\otimes\mathfrak{h}_1\to\mathfrak{h}_2=\mathbb{R}$ being a non-degenerate skew-symmetric 2-form. The contact structure is obtained by left-translating $\mathfrak{h}_1$. According to Darboux, every contact manifold is locally contactomorphic to $\mathbb{H}^n$. The Heisenberg Lie algebra admits a one parameter group of automorphisms δ_t,

$$\delta_t = t \text{ on } \mathfrak{h}_1, \quad \delta_t = t^2 \text{ on } \mathfrak{h}_2,$$

which are analogues of Euclidean homotheties. However, differential forms on $\mathfrak{h}$ split into 2 eigenspaces under δ_t, therefore de Rham complex lacks scale invariance under these anisotropic dilations.

A substitute for de Rham's complex, that recovers scale invariance under δ_t has been defined by M. Rumin, [18]. It makes sense for arbitrary contact manifolds (M,H). Let $\Omega^\bullet$ denote the space of smooth differential forms on M, let $\mathcal{I}^\bullet$ denote the differential ideal generated by 1-forms that vanish on H, let $\mathcal{J}^\bullet$ denote its annihilator. Exterior differential $d:\Omega^\bullet\to\Omega^\bullet$ descends to first order differential operators $d_c:\Omega^\bullet/\mathcal{I}^\bullet\to\Omega^\bullet/\mathcal{I}^\bullet$ and $d_c:\mathcal{J}^\bullet\to\mathcal{J}^\bullet$. It turns out that $\Omega^h/\mathcal{I}^h=0$ for $h\geq n+1$ and $\mathcal{J}^h=0$ for $h\leq n$. If $\omega\in\Omega^n/\mathcal{I}^n$, there is a unique lift $\tilde\omega\in\Omega^n$ such that $d\tilde\omega\in\mathcal{J}^{n+1}$. Set $d_c\omega=d\tilde\omega$. This defines a linear second order differential operator $\Omega^n/\mathcal{I}^{n+1}\to\mathcal{J}^{n+1}$ which completes Rumin's complex, which is homotopic to de Rham's complex. The homotopy is a first order differential operator.

Elements of $\Omega^\bullet/\mathcal{I}^\bullet$ and $\mathcal{J}^\bullet$ can be viewed as smooth sections of sub-bundles $\mathcal{E}_0^\bullet$ of $\Lambda^\bullet H^*$ and $\Lambda^\bullet H^*\otimes(TM/H)$ respectively. A Euclidean norm on H determines Euclidean norms on $\Lambda^\bullet H^*$. Locally, a 1-form θ vanishing on H such that $|d\theta_{|H}|=1$ is uniquely determined up to sign, hence a norm on TM/H. The measure on M defined by the locally defined top degree form $\theta\wedge(d\theta)^n$ only depends on the norm on H as well. Whence L^p-norms on spaces of sections of bundles $\mathcal{E}_0^\bullet$.

The data of (M,H) equipped with a Euclidean norm defined on sub-bundle H only is called a *sub-Riemannian* contact manifold. Poincaré and Sobolev inequalities for differential forms make sense on contact sub-Riemannian manifolds: merely replace d with d_c. All left-invariant sub-Riemannian metrics on Heisenberg group are bi-Lipschitz equivalent, hence we may refer to sub-Riemannian Heisenberg group without referring to a specific left-invariant metric. On the other hand, in absence of symmetry assumptions, large scale behaviours of sub-Riemannian contact manifolds are diverse.

1.3. **Results on Poincaré and Sobolev inequalities.** In this paper, we prove strong contact Poincaré and Sobolev inequalities and interior contact Poincaré and Sobolev inequalities in Heisenberg groups, where the word "contact" is meant to stress that the exterior differential is replaced by Rumin's d_c. The range of parameters differs slightly from the Euclidean case, due to the fact that d_c has order 2 in middle dimension. Let $h\in\{0,\dots,2n+1\}$. Say that assumption $E(h,p,q,n)$ holds if $1<p\le q<\infty$ satisfy

$$\frac{1}{p}-\frac{1}{q}=\begin{cases}\frac{1}{2n+2} & \text{if } h\neq n+1,\\ \frac{2}{2n+2} & \text{if } h=n+1.\end{cases}$$

Say that assumption $I(h,p,q,n)$ holds if $1<p\le q<\infty$ satisfy

$$\frac{1}{p}-\frac{1}{q}\le\begin{cases}\frac{1}{2n+2} & \text{if } h\neq n+1,\\ \frac{2}{2n+2} & \text{if } h=n+1.\end{cases}$$

Theorem 1.1. *Under assumption $E(h,p,q,n)$, strong (p,q)-Poincaré and (p,q)-Sobolev inequalities hold for h-forms on $\mathbb{H}^n$.*

Theorem 1.2. *Under assumption $I(h,p,q,n)$, interior (p,q)-Poincaré and (p,q)-Sobolev inequalities hold for h-forms on $\mathbb{H}^n$.*

Precise formulations of interior Poincaré and Sobolev inequalities are given in section 6.

Here is a sample consequence of these results. Combining both theorems with results from [17], we get

Corollary 1.3. *Under assumption $E(h,p,q,n)$, the $\ell^{q,p}$-cohomology in degree h of $\mathbb{H}^n$ vanishes.*

1.4. **Bounded geometry and smoothing.** Along the way, we construct local smoothing operators for differential forms. They can be combined to yield a global smoothing operator on sub-Riemannian contact manifolds, which has independent interest (see Theorem 1.5 below). This operator is bounded on L^p provided the sub-Riemannian metric has bounded geometry in the following sense.

Definition 1.4. *Let $k\ge 2$. Let $B(e,1)$ denote the unit sub-Riemannian ball in $\mathbb{H}^n$. We say that a sub-Riemannian contact manifold (M,H,g) has* bounded C^k-geometry *is there exist constants $r>0$, C such that, for every $x\in M$, if we* *denote by $B(x,r)$ the sub-Riemannian ball for (M,H,g) centered at x and of radius r, there exists a contactomorphism (i.e. a diffeomorphism preserving the contact forms) $\phi_x : B(e,1) \to M$*

(1) $B(x,r) \subset \phi_x(B(e,1))$.
(2) ϕ_x *is C-bi-Lipschitz.*
(3) *Coordinate changes $\phi_x \circ \phi_y^{-1}$ and their first k derivatives with respect to unit left-invariant horizontal vectorfields are bounded by C.*

On sub-Riemannian Heisenberg balls, Sobolev spaces can be defined as follows. Fix an orthonormal basis of left-invariant vector fields W_i. Express forms in this frame, and differentiate along these vector-fields only. Let $\ell = 0, \dots, k$. Say that a differential form on unit ball B belongs to $W^{\ell,p}$ if all derivatives up to order k of its components belong to $L^p(B)$. Using C^k-bounded charts, this local notion extends to C^k-bounded geometry sub-Riemannian contact manifolds M, and the global $W^{k,p}$ norm on globally defined differential forms is defined by

$$\left(\sum_j \|\omega_{|B(x_j,r)}\|_{W^{k,p}(B(x_j,r))} \right)^{1/p},$$

where x_i is an r-dense uniformly discrete subset of M (it will be shown in section 5 that this norm does not depend on choices, up to multiplicative constants). By duality, Sobolev spaces with negative $\ell = -k+1, \dots, -1$ can be defined.

Theorem 1.5. *Let (M,H,g) be a sub-Riemannian contact manifold of bounded C^k-geometry. Under assumption $I(h,p,q,n)$, there exist operators S and T on h-forms on M which are bounded from $W^{j-1,p}$ to $W^{j,q}$ for all $0 \le j \le k$, and such that $1 = S + d_cT + Td_c$.*

Iterating S yields an operator which is bounded from L^p to $W^{k,q}$, and still acts trivially on cohomology. For instance, this allows to replace a closed form, up to adding a controlled exact form, with a much more regular differential form.

1.5. **Questions.** Keeping in mind the analogous inequalities in the scalar case, the following questions naturally arise.

1. Do balls is Heisenberg group satisfy strong (p,q)-Poincaré and (p,q)-Sobolev inequalities? In other words, do Poincaré and Sobolev inequalities hold without lack on domain?
2. Do interior (p,q)-Poincaré and (p,q)-Sobolev inequalities hold for limiting values, i.e. for $p = 1$ or $q = \infty$?
3. How much of these results does extend to more general Carnot groups?

2. Scheme of proof

2.1. **Global homotopy operators.** The most efficient way to prove a Poincaré inequality is to find a homotopy between identity and 0 on the complex of differential forms, i.e. a linear operator K that raises the degree by 1 and satisfies

$$1 = dK + Kd.$$

More generally, we shall deal with homotopies between identity and other operators P, i.e. of the form

$$1 - P = dK + Kd.$$

In Euclidean space, the Laplacian provides us with such a homotopy. Write $\Delta = d\delta + \delta d$. Denote by Δ^{-1} the operator of convolution with the fundamental solution of the Laplacian. Then Δ^{-1} commutes with d and its adjoint δ, hence $K_e = \delta\Delta^{-1}$ satisfies $1 = dK_e + K_e d$ on globally defined L^p differential forms. Furthermore, K_e is bounded $L^p \to W^{1,q}$ provided $\frac{1}{p} - \frac{1}{q} = \frac{1}{n}$. This proves the strong (p,q)-Poincaré inequality for Euclidean space. Rumin defines a Laplacian Δ_c by $\Delta_c = d_c\delta_c + \delta_c d_c$ when both d_c's are first order, and by $\Delta_c = (d_c\delta_c)^2 + \delta_c d_c$ or $\Delta_c = d_c\delta_c + (\delta_c d_c)^2$ near middle dimension, when one of them has order 2. This leads to a homotopy of the form $K_0 = \delta_c\Delta_c^{-1}$ or $K_0 = \delta_c d_c \delta_c \Delta_c^{-1}$ depending on degree. Again, K_0 is bounded $L^p \to W^{1,q}$ under assumption $E(h,p,q,n)$. This proves the strong contact (p,q)-Poincaré(h) inequality for Heisenberg group, Theorem 1.1.

2.2. **Local homotopy operators.** We pass to local results. In Euclidean space, Poincaré's Lemma asserts that every closed form on a ball is exact. We need a quantitative version of this statement. The standard proof of Poincaré's Lemma relies on a homotopy operator which depends on the choice of an origin. Averaging over origins yields a bounded operator $K : L^p \to L^q$, as was observed by Iwaniec and Lutoborski, [13]. This proves the strong Euclidean (p,q)-Poincaré(h) inequality for convex Euclidean domains. A support preserving variant $J : L^p \to L^q$ appears in Mitrea-Mitrea-Monniaux, [16] and this proves the strong Euclidean (p,q)-Sobolev inequality for bounded convex Euclidean domains. Incidentally, since, for balls, constants do not depend on the radius of the ball, this reproves the strong Euclidean (p,q)-Sobolev inequality for Euclidean space.

In this paper a sub-Riemannian counterpart is obtained using the homotopy of de Rham's and Rumin's complexes. Since this homotopy is a differential operator, a preliminary smoothing operation is needed. This is obtained by localizing (multiplying the kernel with cut-offs) the global homotopy K_0 provided by the inverse of Rumin's (modified) Laplacian.

Hence the proof goes as follows (see Section 6):

(1) Show that the inverse K_0 of Rumin's modified Laplacian on all of $\mathbb{H}^n$ is given by a homogeneous kernel k_0. Deduce bounds $L^p \to W^{1,q}$. Conclude that K_0 is an exact homotopy for globally defined L^p forms.
(2) Split $k_0 = k_1 + k_2$ where k_1 has small support and k_2 is smooth. Hence $T = K_1$ is a homotopy on balls (with a loss on domain) of identity to $S = d_c K_2 + K_2 d_c$ which is smoothing. This provides the required local smoothing operation.
(3) Compose Iwaniec and Lutoborski's averaged Poincaré homotopy for the de Rham complex and Rumin's homotopy, and apply the result to smoothed forms. This proves an interior Poincaré inequality in Heisenberg group. Replacing Iwaniec and Lutoborski's homotopy with Mitrea-Mitrea-Monniaux's homotopy leads to an interior Sobolev inequality.

2.3. **Global smoothing.** Let (M,H,g) be a bounded C^k-geometry sub-Riemannian contact manifold. Pick a uniform covering by equal radius balls. Let χ_j be a partition of unity subordinate to this covering. Let ϕ_j be the corresponding charts from the unit Heisenberg ball. Let S_j and T_j denote the smoothing and homotopy operators transported by ϕ_j. Set

$$T = \sum_j T_j\chi_j, \quad S = \sum_j S_j\chi_j + T_j[\chi_j, d_c].$$

When d_c is first order, the commutator $[\chi_j, d_c]$ is an order 0 differential operator, hence $T_j[\chi_j, d_c]$ gains 1 derivative. When d_c is second order, $[\chi_j, d_c]$ is a first order differential operator. It turns out that precisely in this case, T_j gains 2 derivatives, hence $T_j[\chi_j, d_c]$ gains 1 derivative in this case as well.

This is detailed in section 7.

3. Heisenberg groups and Rumin's complex $(E_0^\bullet, d_c)$

3.1. **Differential forms on Heisenberg group.** We denote by $\mathbb{H}^n$ the n-dimensional Heisenberg group, identified with $\mathbb{R}^{2n+1}$ through exponential coordinates. A point $p \in \mathbb{H}^n$ is denoted by $p = (x, y, t)$, with both $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}$. If p and $p' \in \mathbb{H}^n$, the group operation is defined by

$$p \cdot p' = (x + x', y + y', t + t' + \frac{1}{2} \sum_{j=1}^{n} (x_j y'_j - y_j x'_j)).$$

The unit element of $\mathbb{H}^n$ is the origin, that will be denote by e.

For a general review on Heisenberg groups and their properties, we refer to [19], [11] and to [20]. We limit ourselves to fix some notations, following [10].

The Heisenberg group $\mathbb{H}^n$ can be endowed with the homogeneous norm (Korányi norm)

$$\varrho(p) = \big(|p'|^4 + p_{2n+1}^2\big)^{1/4}, \tag{1}$$

and we define the gauge distance (a true distance, see [19], p. 638, that is equivalent to Carnot–Carathéodory distance) as

$$d(p, q) := \varrho(p^{-1} \cdot q). \tag{2}$$

Finally, set $B_\rho(p, r) = \{q \in \mathbb{H}^n;\ d(p, q) < r\}$.

A straightforward computation shows that there exists $c_0 > 1$ such that

$$c_0^{-2}|p| \le \rho(p) \le |p|^{1/2}, \tag{3}$$

provided p is close to e. In particular, for $r > 0$ small, if we denote by $B_{\text{Euc}}(e, r)$ the Euclidean ball centred ad e of radius r,

$$B_{\text{Euc}}(e, r^2) \subset B_\rho(e, r) \subset B_{\text{Euc}}(e, c_0^2 r). \tag{4}$$

It is well known that the topological dimension of $\mathbb{H}^n$ is $2n+1$, since as a smooth manifold it coincides with $\mathbb{R}^{2n+1}$, whereas the Hausdorff dimension of $(\mathbb{H}^n, d)$ is $Q := 2n + 2$.

We denote by $\mathfrak{h}$ the Lie algebra of the left invariant vector fields of $\mathbb{H}^n$. The standard basis of $\mathfrak{h}$ is given, for $i = 1, \dots, n$, by

$$X_i := \partial_{x_i} - \frac{1}{2} y_i \partial_t, \quad Y_i := \partial_{y_i} + \frac{1}{2} x_i \partial_t, \quad T := \partial_t.$$

The only non-trivial commutation relations are $[X_j, Y_j] = T$, for $j = 1, \dots, n$. The *horizontal subspace* $\mathfrak{h}_1$ is the subspace of $\mathfrak{h}$ spanned by $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$. Coherently, from now on, we refer to $X_1, \dots, X_n, Y_1, \dots, Y_n$ (identified with first order differential operators) as to the *horizontal derivatives*. Denoting by $\mathfrak{h}_2$ the linear span of T, the 2-step stratification of $\mathfrak{h}$ is expressed by

$$\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2.$$

The stratification of the Lie algebra $\mathfrak{h}$ induces a family of non-isotropic dilations δ_λ, $\lambda > 0$ in $\mathbb{H}^n$. The homogeneous dimension of $\mathbb{H}^n$ with respect to δ_λ, $\lambda > 0$ equals Q.

The vector space $\mathfrak{h}$ can be endowed with an inner product, indicated by $\langle\cdot,\cdot\rangle$, making $X_1,\dots,X_n$, $Y_1,\dots,Y_n$ and T orthonormal.

Throughout this paper, we write also

$$W_i := X_i, \quad W_{i+n} := Y_i, \quad W_{2n+1} := T, \quad \text{for } i = 1,\cdots,n. \tag{5}$$

The dual space of $\mathfrak{h}$ is denoted by $\bigwedge^1\mathfrak{h}$. The basis of $\bigwedge^1\mathfrak{h}$, dual to the basis $\{X_1,\dots,Y_n,T\}$, is the family of covectors $\{dx_1,\dots,dx_n,dy_1,\dots,dy_n,\theta\}$ where

$$\theta := dt - \frac{1}{2}\sum_{j=1}^{n}(x_j dy_j - y_j dx_j)$$

is called the *contact form* in $\mathbb{H}^n$.

We indicate as $\langle\cdot,\cdot\rangle$ also the inner product in $\bigwedge^1\mathfrak{h}$ that makes $(dx_1,\dots,dy_n,\theta)$ an orthonormal basis.

Coherently with the previous notation (5), we set

$$\omega_i := dx_i, \quad \omega_{i+n} := dy_i, \quad \omega_{2n+1} := \theta, \quad \text{for } i = 1,\cdots,n.$$

We put $\bigwedge_0\mathfrak{h} := \bigwedge^0\mathfrak{h} = \mathbb{R}$ and, for $1 \le k \le 2n+1$,

$$\bigwedge^k\mathfrak{h} := \mathrm{span}\{\omega_{i_1}\wedge\cdots\wedge\omega_{i_k} : 1 \le i_1 < \cdots < i_k \le 2n+1\}.$$

The volume $(2n+1)$-form $\theta_1\wedge\cdots\wedge\theta_{2n+1}$ will be also written as dV.

The same construction can be performed starting from the vector subspace $\mathfrak{h}_1 \subset \mathfrak{h}$, obtaining the *horizontal k-covectors*

$$\bigwedge^k\mathfrak{h}_1 := \mathrm{span}\{\omega_{i_1}\wedge\cdots\wedge\omega_{i_k} : 1 \le i_1 < \cdots < i_k \le 2n\}.$$

Definition 3.1. *If $\eta \neq 0$, $\eta \in \bigwedge^1\mathfrak{h}_1$, we say that η has* weight 1*, and we write $w(\eta) = 1$. If $\eta = \theta$, we say $w(\eta) = 2$. More generally, if $\eta \in \bigwedge^h\mathfrak{h}$, we say that η has* pure weight k *if η is a linear combination of covectors $\omega_{i_1}\wedge\cdots\wedge\omega_{i_h}$ with $w(\omega_{i_1}) + \cdots + w(\omega_{i_h}) = k$.*

Notice that, if $\eta,\zeta \in \bigwedge^h\mathfrak{h}$ and $w(\eta) \neq w(\zeta)$, then $\langle\eta,\zeta\rangle = 0$.

3.2. **Rumin's complex on Heisenberg groups.** The exterior differential d does not preserve weights. It splits into

$$d = d_0 + d_1 + d_2$$

where d_0 preserves weight, d_1 increases weight by 1 unit and d_2 increases weight by 2 units. d_0 is a differential operator of order 0; in degree k, it vanishes on forms of weight k and if β is a $k-1$-form of weight $k-1$, $d_0(\theta\wedge\beta) = d\theta\wedge\beta$. A first attempt in trying to invert d is to invert d_0. For this, let us pick a complement $\mathcal{W}$ to $\ker(d_0)$ in $\bigwedge^\bullet\mathfrak{h}$ and a complement $\mathcal{V}$ to $\mathrm{Im}(d_0)$ in $\bigwedge^\bullet\mathfrak{h}$ containing $\mathcal{W}$. This allows to define d_0^{-1} to be 0 on $\mathcal{V}$ and the inverse of $d_0 : \mathcal{W} \to \mathrm{Im}(d_0)$. This defines a left-invariant order 0 operator on smooth forms on $\mathbb{H}^n$. Denote by V (resp. W) the space of smooth sections of $\mathcal{V}$ (resp. $\mathcal{W}$).

Rumin shows that

$$r = 1 - d_0^{-1}d - dd_0^{-1}$$

is the projector onto the subspace

$$E = V \cap d^{-1}V$$

along the subspace

$$F = W + dW.$$

Hence, in the sequel, it will be denoted by Π_E. The weight-preserving part of r,

$$r_0 = 1 - d_0^{-1}d_0 - d_0 d_0^{-1},$$

has order 0, it is the projector onto $\mathcal{E}_0 := \mathcal{V} \cap \ker(d_0)$ along $\mathcal{W} \oplus \mathrm{Im}(d_0)$. Hence, in the sequel, it will be denoted by Π_{E_0}, where E_0 is the space of smooth sections of $\mathcal{E}_0$. $\Pi_{E_0 \,|E}$ and $\Pi_{E|E_0}$ are inverses of each other. We use them to conjugate $d_{|E}$ to an operator

$$d_c = \Pi_{E_0} d \Pi_E \Pi_{E_0}$$

on E_0. By construction, the complex (E_0, d_c) is isomorphic to (E, d), which is homotopic to the full de Rham complex.

3.3. **Contact manifolds.** We now sketch Rumin's construction of the intrinsic complex for general contact manifolds (M, H). Locally, H is the kernel of a smooth contact 1-form θ. Let $L : \bigwedge^\bullet H^* \to \bigwedge^\bullet H^*$ denote multiplication by $d\theta_{|H}$.

It is well known that, for every $h \le n-1$, $L^{n-h} : \bigwedge^h H^* \to \bigwedge^{2n-h} H^*$ is an isomorphism. It follows that $\ker(L^{n-h+1})$ is a complement of $\mathrm{Im}(L)$ in $\bigwedge^h H^*$, if $h \le n$, and that $\mathrm{Im}(L) = \bigwedge^h H^*$ if $h \ge n+1$. Therefore we set

$$\mathcal{V}^h = \begin{cases} \{\alpha \in T^*M\,;\, L^{n-h+1}(\alpha_{|H}) = 0\} & \text{if } h \le n, \\ \{\alpha \in T^*M\,;\, \alpha_{|H} = 0\} & \text{otherwise.} \end{cases}$$

Similarly, $\mathrm{Im}(L^{h-n+1})$ is a complement of $\ker(L)$ in $\bigwedge^h H^*$ if $h \ge n$, and $\ker(L) = \{0\}$ in $\bigwedge^h H^*$ if $h \le n-1$. Therefore we set

$$\mathcal{W}^h = \begin{cases} \{\alpha \in T^*M\,;\, \alpha_{|H} = 0\} & \text{if } h \le n-1, \\ \{\alpha \in T^*M\,;\, \alpha \in \theta \wedge \mathrm{Im}(L^{h-n+1})\} & \text{otherwise.} \end{cases}$$

Changing θ to an other smooth 1-form $\theta' = f\theta$ with kernel H does not change $\mathcal{V}$ and $\mathcal{W}$. With these choices, spaces of smooth sections V and W depend only on the plane field H. We can define subspaces of smooth differential forms $E = V \cap d^{-1}V$ and $F = W + dW$ and the projector Π_E. Since no extra choices are involved, E, F and Π_E are invariant under contactomorphisms.

In degrees $h \ge n+1$, $\mathcal{E}_0 = \theta \wedge (\bigwedge^h H^* \cap \ker(L))$ is a contact invariant. Since

$$(\Pi_{E_0})_{|E} = ((\Pi_E)_{|E_0})^{-1},$$

the operator $d_c = ((\Pi_E)_{|E_0})^{-1} \circ d \circ (\Pi_E)_{|E_0}$ is a contact invariant.

In degrees $h \le n$, the restriction of differential forms to H is an isomorphism of $\mathcal{E}_0$ to $\mathcal{E}_0' := \bigwedge^h H^* \cap \ker(L^{n-h+1})$. We note that for a differential form ω such that $\omega_{|H} \in \mathcal{E}_0'$, $\Pi_E(\omega)$ only depends on $\omega_{|H}$. Indeed, $d_0^{-1}\omega = 0$. Furthermore, if $\omega = \theta \wedge \beta$, $d_0^{-1} d\omega = d_0^{-1}(d\theta \wedge \beta) = \omega$, hence $\Pi_E(\omega) = \omega - dd_0^{-1}\omega = 0$. It follows that $(\Pi_E)_{|E_0}$ can be viewed as defined on the space E_0' of sections of $\mathcal{E}_0'$, which is a contact invariant. Since

$$(\Pi_{E_0})_{|E} = ((\Pi_E)_{|E_0})^{-1}, \quad \text{it follows that } (\Pi_{E_0'})_{|E} = ((\Pi_E)_{|E_0'})^{-1}$$

and d_c viewed as an operator on E'_0,

$$((\Pi_E)_{|E'_0})^{-1} \circ d \circ (\Pi_E)_{|E'_0}$$

is a contact invariant. In the sequel, we shall ignore the distinction between E_0 and E'_0. The connection with the description provided in the introduction is easy.

Alternate contact invariant descriptions of Rumin's complex can be found in [4] and [5].

By construction,

i) $d_c^2 = 0$;
ii) the complex $\mathcal{E}_0 := (E_0^\bullet, d_c)$ is homotopically equivalent to the de Rham complex $\Omega := (\Omega^\bullet, d)$. Thus, if $D \subset \mathbb{H}^n$ is an open set, unambiguously we write $H^h(D)$ for the h-th cohomology group;
iii) $d_c : E_0^h \to E_0^{h+1}$ is a homogeneous differential operator in the horizontal derivatives of order 1 if $h \neq n$, whereas $d_c : E_0^n \to E_0^{n+1}$ is an homogeneous differential operator in the horizontal derivatives of order 2.

Since the exterior differential d_c on E_0^h can be written in coordinates as a left-invariant homogeneous differential operator in the horizontal variables of order 1 if $h \neq n$ and of order 2 if $h = n$, the proof of the following Leibniz' formula is easy.

Lemma 3.2. *If ζ is a smooth real function, then*

- *if $h \neq n$, then on E_0^h we have:*

$$[d_c, \zeta] = P_0^h(W\zeta),$$

where $P_0^h(W\zeta) : E_0^h \to E_0^{h+1}$ is a homogeneous differential operator of degree zero with coefficients depending only on the horizontal derivatives of ζ;

- *if $h = n$, then on E_0^n we have*

$$[d_c, \zeta] = P_1^n(W\zeta) + P_0^n(W^2\zeta),$$

where $P_1^n(W\zeta) : E_0^n \to E_0^{n+1}$ is a homogeneous differential operator of degree 1 with coefficients depending only on the horizontal derivatives of ζ, and where $P_0^h(W^2\zeta) : E_0^n \to E_0^{n+1}$ is a homogeneous differential operator in the horizontal derivatives of degree 0 with coefficients depending only on second order horizontal derivatives of ζ.

4. Kernels

If f is a real function defined in $\mathbb{H}^n$, we denote by ${}^{\mathrm{v}}f$ the function defined by ${}^{\mathrm{v}}f(p) := f(p^{-1})$, and, if $T \in \mathcal{D}'(\mathbb{H}^n)$, then ${}^{\mathrm{v}}T$ is the distribution defined by $\langle {}^{\mathrm{v}}T|\phi\rangle := \langle T|{}^{\mathrm{v}}\phi\rangle$ for any test function ϕ.

Following e.g. [8], we can define a group convolution in $\mathbb{H}^n$: if, for instance, $f \in \mathcal{D}(\mathbb{H}^n)$ and $g \in L^1_{\mathrm{loc}}(\mathbb{H}^n)$, we set

$$f * g(p) := \int f(q)g(q^{-1} \cdot p)\, dq \quad \text{for } q \in \mathbb{H}^n. \tag{6}$$

We remind that, if (say) g is a smooth function and P is a left invariant differential operator, then

$$P(f * g) = f * Pg.$$

We remind also that the convolution is again well defined when $f, g \in \mathcal{D}'(\mathbb{H}^n)$, provided at least one of them has compact support. In this case the following identities hold

$$\langle f * g | \phi \rangle = \langle g | {}^{\mathrm{v}}f * \phi \rangle \quad \text{and} \quad \langle f * g | \phi \rangle = \langle f | \phi * {}^{\mathrm{v}}g \rangle \tag{7}$$

for any test function ϕ.

As in [8], we also adopt the following multi-index notation for higher-order derivatives. If $I = (i_1, \dots, i_{2n+1})$ is a multi–index, we set $W^I = W_1^{i_1} \cdots W_{2n}^{i_{2n}}\, T^{i_{2n+1}}$. By the Poincaré–Birkhoff–Witt theorem, the differential operators W^I form a basis for the algebra of left invariant differential operators in $\mathbb{H}^n$. Furthermore, we set $|I| := i_1 + \cdots + i_{2n} + i_{2n+1}$ the order of the differential operator W^I, and $d(I) := i_1 + \cdots + i_{2n} + 2i_{2n+1}$ its degree of homogeneity with respect to group dilations.

Suppose now $f \in \mathcal{E}'(\mathbb{H}^n)$ and $g \in \mathcal{D}'(\mathbb{H}^n)$. Then, if $\psi \in \mathcal{D}(\mathbb{H}^n)$, we have

$$\begin{aligned} \langle (W^I f) * g | \psi \rangle &= \langle W^I f | \psi * {}^{\mathrm{v}}g \rangle = (-1)^{|I|} \langle f | \psi * (W^I\, {}^{\mathrm{v}}g) \rangle \\ &= (-1)^{|I|} \langle f * {}^{\mathrm{v}}W^I\, {}^{\mathrm{v}}g | \psi \rangle. \end{aligned} \tag{8}$$

Following [7], we remind now the notion of *kernel of type* μ.

Definition 4.1. *A kernel of type μ is a homogeneous distribution of degree $\mu - Q$ (with respect to group dilations δ_r), that is smooth outside of the origin.*

The convolution operator with a kernel of type μ is still called an operator of type μ.

Proposition 4.2. *Let $K \in \mathcal{D}'(\mathbb{H}^n)$ be a kernel of type μ.*

i) *${}^{\mathrm{v}}K$ is again a kernel of type μ;*
ii) *WK and KW are associated with kernels of type $\mu - 1$ for any horizontal derivative W;*
iii) *If $\mu > 0$, then $K \in L^1_{\mathrm{loc}}(\mathbb{H}^n)$.*

Theorem 4.3. *Suppose $0 < \alpha < Q$, and let K be a kernel of type α. Then*

i) *if $1 < p < Q/\alpha$, and $1/q := 1/p - \alpha/Q$, then*
$$\|u * K\|_{L^q(\mathbb{H}^n)} \le C \|u\|_{L^p(\mathbb{H}^n)}$$
for all $u \in L^p(\mathbb{H}^n)$.
ii) *If $p \ge Q/\alpha$ and $B, B' \subset \mathbb{H}^n$ are fixed balls, then for any $q \ge p$*
$$\|u * K\|_{L^q(B')} \le C \|u\|_{L^p(\mathbb{H}^n)}$$
for all $u \in L^p(\mathbb{H}^n)$ with $\operatorname{supp} u \subset B$.
iii) *If K is a kernel of type 0 and $1 < p < \infty$, then*
$$\|u * K\|_{L^p(\mathbb{H}^n)} \le C \|u\|_{L^p(\mathbb{H}^n)}.$$

Proof. For statements i) and iii), we refer to [7], Propositions 1.11 and 1.9. As for ii), if $p \ge Q/\alpha$, we choose $1 < \tilde{p} < Q/\alpha$ such that $1/\tilde{p} \le 1/q + \alpha/Q$. If we set $1/\tilde{q} := 1/\tilde{p} - \alpha/Q < 1/q$, then

$$\begin{aligned} \|u*K\|_{L^q(B')} &\le C_{B'} \|u * K\|_{L^{\tilde{q}}(B')} \le C_{B'} \|u * K\|_{L^{\tilde{q}}(\mathbb{H}^n)} \\ &\le C'(B') \|u\|_{L^{\tilde{p}}(\mathbb{H}^n)} \le C'(B, B') \|u\|_{L^p(B)}. \end{aligned}$$

$\square$

Lemma 4.4. *Suppose $0 < \alpha < Q$. If K is a kernel of type α and $\psi \in \mathcal{D}(\mathbb{H}^n)$, $\psi \equiv 1$ in a neighborhood of the origin, then the statements i) and ii) of Proposition 4.3 still hold if we replace K by $(1-\psi)K$.*

Analogously, if K is a kernel of type 0 and $\psi \in \mathcal{D}(\mathbb{H}^n)$, then statement iii) of Proposition 4.3 still hold if we replace K by $(\psi - 1)K$.

Proof. As in [7], Proposition 1.11, we have only notice that $|(1-\psi)K(x)| \le C_\psi |x|^{\alpha - Q}$, so that $(1-\psi)K \in L^{Q/(Q-\alpha),\infty}(\mathbb{H}^n)$, and thereforet i) and ii) hold true.

Suppose now $\alpha = 0$. Notice that $(\psi - 1)K \in L^{1,\infty}(\mathbb{H}^n)$, and therefore also $u \to ((\psi-1)K) * u$ is $L^p - L^p$ continuous by Hausdorff-Young Theorem. This proves that iii) holds true. □

Remark 4.5. *By Theorem 4.3, Lemma 4.4 still holds if we replace $(1-\psi)K$ by ψK.*

The following (well known) estimate will be useful in the sequel.

Lemma 4.6. *Let g be a a kernel of type $\mu > 0$. Then, if $f \in \mathcal{D}(\mathbb{H}^n)$ and R is an homogeneous polynomial of degree $\ell \ge 0$ in the horizontal derivatives, we have*

$$R(f * g)(p) = O(|p|^{\mu - Q - \ell}) \quad \text{as } p \to \infty.$$

On the other hand, if g is a smooth function in $\mathbb{H}^n \setminus \{0\}$ that satisfies the logarithmic estimate $|g(p)| \le C(1 + |\ln|p||)$ and in addition its horizontal derivatives are homogeneous of degree -1 with respect to group dilations, then, if $f \in \mathcal{D}(\mathbb{H}^n)$ and R is an homogeneous polynomial of degree $\ell \ge 0$ in the horizontal derivatives, we have

$$\begin{aligned} R(f*g)(p) &= O(|p|^{-\ell}) \quad \text{as } p \to \infty \quad \text{if } \ell > 0; \\ R(f*g)(p) &= O(\ln|p|) \quad \text{as } p \to \infty \quad \text{if } \ell = 0. \end{aligned}$$

Since we have fixed a left-invariant moving frame for $E_0^\bullet$, a $(N_h \times N_k)$-matrix whose entries are kernels of type α defines in a natural way an operator from E_0^h to E_0^k. We still refer to this operator as to an operator associated with a (matrix-valued) kernel of type α.

Definition 4.7. *In $\mathbb{H}^n$, following [18], we define the operator $\Delta_{\mathbb{H},h}$ on E_0^h by setting*

$$\Delta_{\mathbb{H},h} = \begin{cases} d_c\delta_c + \delta_c d_c & \text{if} \quad h \ne n, n+1; \\ (d_c\delta_c)^2 + \delta_c d_c & \text{if} \quad h = n; \\ d_c\delta_c + (\delta_c d_c)^2 & \text{if} \quad h = n+1. \end{cases}$$

Notice that $-\Delta_{\mathbb{H},0} = \sum_{j=1}^{2n}(W_j^2)$ is the usual sub-Laplacian of $\mathbb{H}^n$.

For sake of simplicity, since a basis of E_0^h is fixed, the operator $\Delta_{\mathbb{H},h}$ can be identified with a matrix-valued map, still denoted by $\Delta_{\mathbb{H},h}$

$$\Delta_{\mathbb{H},h} = (\Delta_{\mathbb{H},h}^{ij})_{i,j=1,\dots,N_h} : \mathcal{D}'(\mathbb{H}^n, \mathbb{R}^{N_h}) \to \mathcal{D}'(\mathbb{H}^n, \mathbb{R}^{N_h}), \tag{9}$$

where $\mathcal{D}'(\mathbb{H}^n, \mathbb{R}^{N_h})$ is the space of vector-valued distributions on $\mathbb{H}^n$.

This identification makes possible to avoid the notion of currents: we refer to [2] for a more elegant presentation.

It is proved in [18] that $\Delta_{\mathbb{H},h}$ is hypoelliptic and maximal hypoelliptic in the sense of [12]. In general, if $\mathcal{L}$ is a differential operator on $\mathcal{D}'(\mathbb{H}^n, \mathbb{R}^{N_h})$, then $\mathcal{L}$ is said hypoelliptic if for any open set $\mathcal{V} \subset \mathbb{H}^n$ where $\mathcal{L}\alpha$ is smooth, then α is smooth in $\mathcal{V}$. In addition, if $\mathcal{L}$ is homogeneous of degree $a \in \mathbb{N}$, we say that $\mathcal{L}$ is maximal hypoelliptic if for any $\delta > 0$ there exists $C = C(\delta) > 0$ such that for any homogeneous polynomial P in $W_1, \dots, W_{2n}$ of degree a we have

$$\|P\alpha\|_{L^2(\mathbb{H}^n,\mathbb{R}^{N_h})} \le C\Big(\|\mathcal{L}\alpha\|_{L^2(\mathbb{H}^n,\mathbb{R}^{N_h})} + \|\alpha\|_{L^2(\mathbb{H}^n,\mathbb{R}^{N_h})}\Big).$$

for any $\alpha \in \mathcal{D}(B_\rho(0,\delta), \mathbb{R}^{N_h})$.

Combining [18], Section 3, and [3], Theorems 3.1 and 4.1, we obtain the following result.

Theorem 4.8 (see [1], Theorem 4.6)**.** *If $0 \le h \le 2n+1$, then the differential operator $\Delta_{\mathbb{H},h}$ is hypoelliptic of order a, where $a = 2$ if $h \ne n, n+1$ and $a = 4$ if $h = n, n+1$ with respect to group dilations. Then*

i) *for $j = 1, \dots, N_h$ there exists*

$$K_j = \big(K_{1j}, \dots, K_{N_h j}\big), \quad j = 1, \dots N_h \tag{10}$$

with $K_{ij} \in \mathcal{D}'(\mathbb{H}^n) \cap \mathcal{E}(\mathbb{H}^n \setminus \{0\})$, $i, j = 1, \dots, N$;

ii) *if $a < Q$, then the K_{ij}'s are kernels of type a for $i, j = 1, \dots, N_h$*

If $a = Q$, then the K_{ij}'s satisfy the logarithmic estimate $|K_{ij}(p)| \le C(1 + |\ln \rho(p)|)$ and hence belong to $L^1_{\mathrm{loc}}(\mathbb{H}^n)$. Moreover, their horizontal derivatives $W_\ell K_{ij}$, $\ell = 1, \dots, 2n$, are kernels of type $Q-1$;

iii) *when $\alpha \in \mathcal{D}(\mathbb{H}^n, \mathbb{R}^{N_h})$, if we set*

$$\mathcal{K}\alpha := \big(\sum_j \alpha_j * K_{1j}, \dots, \sum_j \alpha_j * K_{N_h j}\big), \tag{11}$$

then $\Delta_{\mathbb{H},h}\mathcal{K}\alpha = \alpha$. Moreover, if $a < Q$, also $\mathcal{K}\Delta_{\mathbb{H},h}\alpha = \alpha$.

iv) *if $a = Q$, then for any $\alpha \in \mathcal{D}(\mathbb{H}^n, \mathbb{R}^{N_h})$ there exists $\beta_\alpha := (\beta_1, \dots, \beta_{N_h}) \in \mathbb{R}^{N_h}$, such that*

$$\mathcal{K}\Delta_{\mathbb{H},h}\alpha - \alpha = \beta_\alpha.$$

Remark 4.9. *Coherently with formula* (9)*, the operator $\mathcal{K}$ can be identified with an operator (still denoted by $\mathcal{K}$) acting on smooth compactly supported differential forms in $\mathcal{D}(\mathbb{H}^n, E_0^h)$. Moreover, when the notation will not be misleading, we shall denote by $\alpha \to \Delta^{-1}_{\mathbb{H},h}\alpha$ the convolution with $\mathcal{K}$ acting on forms of degree h.*

Lemma 4.10. *If $\alpha \in \mathcal{D}(\mathbb{H}^n, E_0^h)$*

i) $d_c\Delta^{-1}_{\mathbb{H},h}\alpha = \Delta^{-1}_{\mathbb{H},h+1}d_c\alpha, \qquad h = 0, 1, \dots, 2n, \qquad h \ne n-1, n+1.$
ii) $d_c\Delta^{-1}_{\mathbb{H},n-1}\alpha = d_c\delta_c\Delta^{-1}_{\mathbb{H},n}d_c\alpha \qquad (h = n-1).$
iii) $d_c\delta_c d_c\Delta^{-1}_{\mathbb{H},n+1}\alpha = \Delta^{-1}_{\mathbb{H},n+2}d_c\alpha, \qquad (h = n+1).$
iv) $\delta_c\Delta^{-1}_{\mathbb{H},h}\alpha = \Delta^{-1}_{\mathbb{H},h-1}\delta_c\alpha \qquad h = 1, \dots, 2n+1, \qquad h \ne n, n+2.$
v) $\delta_c\Delta^{-1}_{\mathbb{H},n+2}\alpha = \delta_c d_c\Delta^{-1}_{\mathbb{H},n+1}\delta_c\alpha \qquad (h = n+2).$
vi) $\delta_c d_c\delta_c\Delta^{-1}_{\mathbb{H},n}\alpha = \Delta^{-1}_{\mathbb{H},n-1}\delta_c\alpha, \qquad (h = n).$

Proof. Let us prove i), ii), iii). The remaining assertions will follow by Hodge duality. Put

$$\begin{aligned}
\omega_h &:= d_c\Delta^{-1}_{\mathbb{H},h}\alpha - \Delta^{-1}_{\mathbb{H},h+1}d_c\alpha \qquad \text{if } h \ne n-1, n+1,\\
\omega_{n-1} &:= d_c\Delta^{-1}_{\mathbb{H},n-1}\alpha - d_c\delta_c\Delta^{-1}_{\mathbb{H},n}d_c\alpha\\
\omega_{n+1} &:= d_c\delta_c d_c\Delta^{-1}_{\mathbb{H},n+1}\alpha - \Delta^{-1}_{\mathbb{H},n+2}d_c\alpha.
\end{aligned}$$

We notice first that, by Theorem 4.8 and Proposition 4.2, for all $h = 1, \dots, 2n$, $\omega_h = M_h * \alpha$, where M_h is a kernel of type 1. Thus, by Lemma 4.6

$$\omega_h(x) = O(|x|^{1-Q}) \qquad \text{as } x \to \infty. \tag{12}$$

We want to show now that

$$\Delta^{-1}_{\mathbb{H},h+1}\omega_h = 0 \qquad \text{for } h = 1, \dots, 2n. \tag{13}$$

Suppose first $h \neq n-1, n, n+1$. By Theorem 4.8, we have:

$$\begin{aligned}\Delta_{\mathbb{H},h+1}\omega_h &= d_c\delta_c d_c \Delta^{-1}_{\mathbb{H},h}\alpha - d_c\alpha \\ &= d_c\Delta_{\mathbb{H},h}\Delta^{-1}_{\mathbb{H},h}\alpha - d_c\alpha = 0.\end{aligned}$$

If $h = n-1$, then

$$\begin{aligned}\Delta_{\mathbb{H},n}\omega_{n-1} &= d_c\delta_c d_c\delta_c\Big(d_c\Delta^{-1}_{\mathbb{H},n-1}\alpha - d_c\delta_c\Delta^{-1}_{\mathbb{H},n}d_c\alpha\Big) \\ &= d_c\delta_c d_c\Delta_{\mathbb{H},n-1}\Delta^{-1}_{\mathbb{H},n-1}\alpha - d_c\delta_c\Delta^{-1}_{\mathbb{H},n}\Delta^{-1}_{\mathbb{H},n}d_c\alpha = 0.\end{aligned}$$

If $h = n$, then (keeping in mind that $d_c\Delta^{-1}_{\mathbb{H},n}\alpha$ is a form of degree $n+1$ and $\Delta^{-1}_{\mathbb{H},n}\alpha$ is a form of degree n)

$$\begin{aligned}\Delta_{\mathbb{H},n+1}\omega_n &= ((\delta_c d_c)^2 + d_c\delta_c)d_c\Delta^{-1}_{\mathbb{H},n}\alpha - d_c\alpha \\ &= d_c(\delta_c d_c + (d_c\delta_c)^2)\Delta^{-1}_{\mathbb{H},n}\alpha - d_c\alpha \\ &= d_c\Delta_{\mathbb{H},n}\Delta^{-1}_{\mathbb{H},n}\alpha - d_c\alpha = 0.\end{aligned}$$

Finally, if $h = n+1$, then

$$\begin{aligned}\Delta_{\mathbb{H},n+2}\omega_{n+1} &= d_c\delta_c d_c\delta_c d_c\Delta^{-1}_{\mathbb{H},n+1}\alpha - d_c\alpha \\ &= d_c\Delta_{\mathbb{H},n+1}\Delta^{-1}_{\mathbb{H},n+1}\alpha - d_c\alpha = 0.\end{aligned}$$

This proves (13).

Thus, by [3], Proposition 3.2, ω is a polynomial coefficient form. Then, by (12) necessarily $\omega \equiv 0$.

This proves i), ii), iii).

□

5. Function spaces

5.1. **Sobolev spaces.** Since here we are dealing only with integer order Folland-Stein function spaces, we can give this simpler definition (for a general presentation, see e.g. [7]).

Definition 5.1. *If $U \subset \mathbb{H}^n$ is an open set, $1 \le p \le \infty$ and $m \in \mathbb{N}$, then the space $W^{m,p}(U)$ is the space of all $u \in L^p(U)$ such that*

$$W^I u \in L^p(U) \quad \textit{for all multi-indices } I \textit{ with } d(I) = m,$$

endowed with the natural norm.

Theorem 5.2. *If $U \subset \mathbb{H}^n$, $1 \le p < \infty$, and $k \in \mathbb{N}$, then*

i) *$W^{k,p}(U)$ is a Banach space;*
ii) *$W^{k,p}(U) \cap C^\infty(U)$ is dense in $W^{k,p}(U)$;*
iii) *if $U = \mathbb{H}^n$, then $\mathcal{D}(\mathbb{H}^n)$ is dense in $W^{k,p}(U)$.*

Definition 5.3. *If $1 \le p < \infty$, we denote by $\overset{\circ}{W}{}^{k,p}(U)$ the completion of $\mathcal{D}(U)$ in $W^{k,p}(U)$. If U is bounded, then by (iterated) Poincaré inequality (see e.g. [14]), it follows that the norms*

$$\|u\|_{W^{k,p}(U)} \quad \text{and} \quad \sum_{d(I)=k} \|W^I u\|_{L^p(U)}$$

are equivalent on $\overset{\circ}{W}{}^{k,p}(U)$ when $1 \le p < \infty$.

Finally, $W^{k,p}_{\mathrm{Euc}}(U)$ denotes the usual Sobolev space.

5.2. **Negative spaces.**

Definition 5.4. *If $U \subset \mathbb{H}^n$ is an open set and $1 < p < \infty$, $W^{-k,p}(U)$ is the dual space of $\overset{\circ}{W}{}^{k,p'}(U)$, where $1/p + 1/p' = 1$. It is well known that*

$$W^{-k,p}(U) = \{f_0 + \sum_{d(I)=k} W^I f_I,\ f_0, f_I \in L^p(U) \text{ for any } I \text{ such that } d(I) = k\},$$

and

$$\|u\|_{W^{-k,p}(U)} \approx \inf\{\|f_0\|_{L^p(U)} + \sum_I \|f_I\|_{L^p(U)}\,;\, d(I) = k, f_0 + \sum_{d(I)=k} W^I f_I = u\}.$$

If U is bounded, then we can take $f_0 = 0$.

Finally, we stress that

$$\{f_0 + \sum_{d(I)=k} W^I f_I,\ f_0, f_I \in \mathcal{D}(U) \text{ for any } I \text{ such that } d(I) = k\}$$

is dense in $W^{-k,p}(U)$.

Definition 5.5. *If $U \subset \mathbb{H}^n$ is an open set, $0 \le h \le 2n+1$, $1 \le p \le \infty$ and $m \ge 0$, we denote by $W^{m,p}(U, \bigwedge^h \mathfrak{h})$ (by $\overset{\circ}{W}{}^{m,p}(U, \bigwedge^h \mathfrak{h})$) the space of all sections of $\bigwedge^h \mathfrak{h}$ such that their components with respect to a given left-invariant frame belong to $W^{m,p}(U)$ (to $\overset{\circ}{W}{}^{m,p}(U)$, respectively), endowed with its natural norm. Clearly, this definition is independent of the choice of the frame itself.*

The spaces $W^{m,p}(U, E_0^h)$ and $\overset{\circ}{W}{}^{m,p}(U, E_0^h)$ are defined in the same way.

On the other hand, the spaces

$$W^{-m,p}(U, E_0^h) := \left(\overset{\circ}{W}{}^{m,p'}(U, E_0^h)\right)^*$$

can be viewed as spaces of currents on $(E_0^\bullet, d_c)$ as in [2], Proposition 3.14. Again as in [2], Proposition 3.14, an element of $W^{-m,p}(U, E_0^h)$ can be identified (with respect to our basis) with a N_h-ple

$$(T_1, \dots, T_{N_h}) \in \left(W^{-m,p}(U, E_0^h)\right)^{N_h}$$

(this is nothing but the intuitive notion of "currents as differential form with distributional coefficients"). The action of $u \in W^{-m,p}(U, E_0^h)$ associated with $(T_1, \dots, T_{N_h})$ on the form $\sum_j \alpha_j \xi_j^h \in \overset{\circ}{W}{}^{m,p'}(U, E_0^h)$ is given by

$$\langle u | \alpha \rangle := \sum_j \langle T_j | \alpha_j \rangle.$$

On the other hand, suppose for sake of simplicity that U is bounded, then by Definition 5.4 there exist $f_I^j \in L^p(U)$, $j = 1, \dots, N_h$, $i = 1, \dots, 2n+1$ such that

$$\langle u|\alpha\rangle = \sum_j \sum_{d(I)=m} \int_U f_I^j(x) W^I \alpha_j(x)\, dx. \tag{14}$$

Alternatively, one can express duality in spaces of differential forms using the pairing between h-forms and $2n+1-h$-forms defined by

$$\alpha, \beta \mapsto \int_U \alpha \wedge \beta.$$

Note that this makes sense for Rumin forms and is a nondegenerate pairing. In this manner, the dual of $L^p(U, E_0^h)$ is $L^{p'}(U, E_0^{2n+1-h})$. Hence $W^{-m,p}(U, E_0^h)$ consists of differential forms of degree $2n+1-h$ whose coefficients are distributions belonging to $W^{-m,p}(U)$.

5.3. **Contact invariance.**

Lemma 5.6. *Let U, V be open subsets of $\mathbb{H}^n$. Let $\phi : U \to V$ be a C^k-bounded contact diffeomorphism. Let $\ell = -k+1, \dots, k-1$. Then the pull-back operator $\phi^\sharp$ from $W^{\ell,p}$ forms on V to $W^{\ell,p}$ forms on U is bounded, and its norm depends only on the C^k norms of ϕ and ϕ^{-1}.*

When $\ell \geq 0$, this follows from the chain rule and the change of variables formula. According to the change of variables formula

$$\int_U \phi^\sharp \alpha \wedge \phi^\sharp \beta = \int_V \alpha \wedge \beta,$$

the adjoint of $\phi^\sharp$ with respect to the above pairing is $(\phi^{-1})^\sharp$. Hence $\phi^\sharp$ is bounded on negative Sobolev spaces of differential forms as well.

5.4. **Sobolev spaces on contact sub-Riemannian manifolds.** We define Sobolev spaces (involving a positive or negative number of derivatives) on bounded geometry contact sub-Riemannian manifolds.

Let (M, H, g) be a bounded C^k-geometry sub-Riemannian contact manifold. Pick a uniform covering $\mathcal{U}$ by equal radius balls (uniform means that distances between centers are bounded below). Let $\phi_j : B \to U_j$ be C^k-bounded contact charts from the unit Heisenberg ball. Given a differential form ω on M, let $\omega_j = \phi_j^\sharp \omega$. Let $-k+1 \leq \ell \leq k-1$ be an integer. Define

$$\|\omega\|_{\mathcal{U},\ell,p} = \left(\sum_j \|\omega_j\|^p_{W^{\ell,p}(B)} \right)^{1/p}.$$

Let us show that an other uniform covering $\mathcal{U}'$ and other choices of controlled charts lead to an equivalent norm. Every piece U of $\mathcal{U}$ is covered with boundedly many pieces U_i' of $\mathcal{U}'$. Thus

$$\|\omega_j\|^p_{W^{\ell,p}} \leq \sum \|\omega_{j|\phi_j{}^{-1}(U_i')}\|^p_{W^{\ell,p}(\phi_j{}^{-1}(U_i'))}.$$

Since $\omega_{j|\phi_j{}^{-1}(U_i')}$ is the pull-back by the contactomorphism $\phi = \phi_j \circ \phi_i'^{-1}$ of $\omega_{i|\phi_i'^{-1}(U_j)}$, Lemma 5.6 implies that

$$\|\omega_{j|\phi_j{}^{-1}(U_i')}\|_{W^{\ell,p}(\phi_j{}^{-1}(U_i'))} \leq C\, \|\omega_{i|\phi_i'^{-1}(U_j)}\|_{W^{\ell,p}(\phi_i'^{-1}(U_j))},$$

where the constant only depends on the uniform bound on horizontal derivatives of order $\leq k$ of ϕ. Thus

$$\|\omega_j\|^p_{W^{\ell,p}} \leq \sum \|\omega'_i\|^p_{W^{\ell,p}(B)}.$$

When summing over j, each term $\|\omega'_i\|$ on the right hand side occurs only a bounded number N of times. This yields

$$\|\omega\|_{\mathcal{U},\ell,p} \leq CN^{1/p}\|\omega\|_{\mathcal{U}',\ell,p}.$$

6. Homotopy formulae and Poincaré and Sobolev inequalities

In this paper we are mainly interested to obtain functional inequalities for differential forms that are the counterparts of the classical (p,q)-Sobolev and Poincaré inequalities on a ball $B \subset \mathbb{R}^n$ with sharp exponents of the form

$$\|u - u_B\|_{L^q}(B) \leq C(r)\|\nabla u\|_{L^p}(B)$$

(as well as of its counterpart for compactly supported functions). In this case, we can choose $q = pn/(n-p)$, provided $p < n$.

Definition 6.1. *Take $\lambda > 1$ and set $B = B(e,1)$ and $B' = B(e,\lambda)$, where the $B(x,r)$'s are the Korányi balls in $\mathbb{H}^n$ (in particular the balls centered at $x = e$, and then all balls, are convex). If $1 \leq k \leq 2n+1$ and $q \geq p \geq 1$, we say that the interior (p,q)-Poincaré inequality holds in E_0^k if there exists a constant C such that, for every d_c-closed differential k-form ω in $L^p(B'; E_0^k)$ there exists a differential $k-1$-form ϕ in $L^q(B, E_0^{k-1})$ such that $d_c\phi = \omega$ and*

$$\|\phi\|_{L^q(B,E_0^{k-1})} \leq C\,\|\omega\|_{L^p(B',E_0^k)} \qquad \text{interior}\,\mathbb{H}\text{-Poincaré}_{p,q}(k)).$$

Remark 6.2. *If $k = 1$ and $Q > p \geq 1$, then* ($\mathbb{H}$-Poincaré$_{p,q}(1)$) *is nothing but the usual Poincaré inequality with $\dfrac{1}{p} - \dfrac{1}{q} = \dfrac{1}{Q}$ (see e.g.* [9], [6], [15]*).*

Remark 6.3. *If we replace Rumin's complex $(E_0^\bullet, d_c)$ by the usual de Rham's complex $(\Omega^\bullet, d)$ in $\mathbb{R}^{2n+1}$, then the (p,q)-Poincaré inequality holds on Euclidean balls for $k = 1$ and $n > p \geq 1$. If $k > 1$, then the (p,q)-Poincaré inequality for $2n+1 > p > 1$ and $\dfrac{1}{p} - \dfrac{1}{q} = \dfrac{1}{2n+1}$ is proved by Iwaniec & Lutoborski (see* [13]*, Corollary 4.2).*

The $\mathbb{H}$-Poincaré$_{p,q}(k)$ inequality (as well as its Euclidean counterpart) can be formulated by duality as follows.

Definition 6.4. *Take $\lambda > 1$ and set $B = B(e,1)$ and $B' = B(e,\lambda)$. If $1 \leq k \leq 2n$, $1 \leq p \leq q < \infty$ and $q \geq p$, we say that the (local)* $\mathbb{H}$-Sobolev$_{p,q}(k)$ *inequality holds if there exists a constant C such that for every compactly supported smooth d_c-closed differential k-form ω in $L^p(B; E_0^k)$ there exists a smooth compactly supported differential $(k-1)$-form ϕ in $L^q(B', E_0^{k-1})$ such that $d_c\phi = \omega$ in B' and*

$$\|\phi\|_{L^q(B',E_0^{k-1})} \leq C\,\|\omega\|_{L^p(B,E_0^k)}. \tag{15}$$

Notice that, in this case, we do not distinguish interior inequalities (in other words, we can always assume $B = B'$), basically since, when dealing with compactly supported forms, the structure of the boundary does not affect the estimates.

Remark 6.5. *If $k = 1$ and $Q > p \geq 1$, then* ($\mathbb{H}$-Sobolev$_{p,q}(1)$) *is nothing but the usual Sobolev inequality with $\dfrac{1}{p} - \dfrac{1}{q} = \dfrac{1}{Q}$.*

In [13], starting from Cartan's homotopy formula, the authors proved that, if $D \subset \mathbb{R}^N$ is a convex set, $1 < p < \infty$, $1 < k < N$, then there exists a linear bounded map:

$$K_{\mathrm{Euc},k} : L^p(D, \textstyle\bigwedge^k) \to W^{1,p}(D, \textstyle\bigwedge^{k-1}) \tag{16}$$

that is a homotopy operator, i.e.

$$\omega = dK_{\mathrm{Euc},k}\omega + K_{\mathrm{Euc},k+1}d\omega \qquad \text{for all } \omega \in C^\infty(D, \textstyle\bigwedge^k) \tag{17}$$

(see Proposition 4.1 and Lemma 4.2 in [13]). More precisely, K_{Euc} has the form

$$K_{\mathrm{Euc},k}\omega(x) = \int_D \psi(y) K_y\omega(x)\, dy, \tag{18}$$

where $\psi \in \mathcal{D}(D)$, $\int_D \psi(y)\, dy = 1$, and

$$\langle K_y\omega(x) | \xi_1 \wedge \cdots \wedge \xi_{k-1}) \rangle := \int_0^1 t^{k-1} \langle \omega(tx + (1-t)y) | (x-y) \wedge \xi_1 \wedge \cdots \wedge \xi_{k-1}) \rangle. \tag{19}$$

Starting from [13], in [16], Section 4, the authors define a compact homotopy operator $J_{\mathrm{Euc},k}$ in Lipschitz star-shaped domains in the Euclidean space $\mathbb{R}^N$, providing an explicit representation formulas for $J_{\mathrm{Euc},k}$, together with continuity properties among Sobolev spaces. More precisely, if $D \subset \mathbb{R}^N$ is a star-shaped Lipschitz domain and $1 < k < N$, then there exists

$$J_{\mathrm{Euc},k} : L^p(D, \textstyle\bigwedge^k) \to W_0^{1,p}(D, \textstyle\bigwedge^{k-1})$$

such that

$$\omega = dJ_{\mathrm{Euc},k}\omega + J_{\mathrm{Euc},k+1}d\omega \qquad \text{for all } \omega \in \mathcal{D}(D, \textstyle\bigwedge^k).$$

Take now $D = B(e, 1) =: B$ and $N = 2n + 1$. If $\omega \in C^\infty(B, E_0^k)$, then we set

$$K = \Pi_{E_0} \circ \Pi_E \circ K_{\mathrm{Euc}} \circ \Pi_E \tag{20}$$

(for sake of simplicity, from now on we drop the index k - the degree of the form - writing, e.g., K_{Euc} instead of $K_{\mathrm{Euc},k}$.

Analogously, we can define

$$J = \Pi_{E_0} \circ \Pi_E \circ J_{\mathrm{Euc}} \circ \Pi_E. \tag{21}$$

Then K and J invert Rumin's differential d_c on closed forms of the same degree. More precisely, we have:

Lemma 6.6. *If ω is d_c-closed, then*

$$\omega = d_c K\omega \quad \text{if } 1 \le k \le 2n+1 \qquad \text{and} \qquad \omega = d_c J\omega \quad \text{if } 1 \le k \le 2n. \tag{22}$$

In addition, if ω is compactly supported in B, then $J\omega$ is still compactly supported in B.

Proof. Consider for instance $d_c K\omega$. If $d_c\omega = 0$, then $d(\Pi_E\omega) = 0$, and hence

$$\Pi_E\omega = dK_{\mathrm{Euc}}(\Pi_E\omega),$$

by (17). By (20) (and recalling that $d\Pi_E = \Pi_E d$ and $\Pi_E\Pi_{E_0}\Pi_E = \Pi_E$),

$$\begin{aligned} d_c K\omega &= \Pi_{E_0} d\Pi_E \Pi_{E_0} \Pi_E K_{\mathrm{Euc}} \Pi_E \omega = \Pi_{E_0} d\Pi_E K_{\mathrm{Euc}} \Pi_E \omega \\ &= \Pi_{E_0} \Pi_E dK_{\mathrm{Euc}} \Pi_E \omega = \Pi_{E_0} \Pi_E \Pi_E \omega = \Pi_{E_0} \Pi_E \Pi_{E_0} \omega = \omega\,. \end{aligned}$$

Finally, if $\operatorname{supp}\omega \subset B$, then $\operatorname{supp} J\omega \subset B$ since both Π_E and Π_{E_0} preserve the support. $\square$

Lemma 6.7. *Put $B = B(e,1)$. Then:*

i) *if $1 < p < \infty$ and $k = 1, \dots, 2n+1$, then $K : W^{1,p}(B, E_0^k) \to L^p(B, E_0^{k-1})$ is bounded;*
ii) *if $1 \le p \le \infty$ and $n+1 < k \le 2n+1$, then $K : L^p(B, E_0^k) \to L^p(B, E_0^{k-1})$ is compact;*
iii) *if $1 < p < \infty$ and $k = n+1$, then $K : L^p(B, E_0^{n+1}) \to L^p(B, E_0^n)$ is bounded.*

Analogous assertions hold for $1 \le k \le 2n$ when we replace K by J. In addition, $\operatorname{supp} J\omega \subset B$.

Proof. By its very definition, $\Pi_E : W^{1,p}(B, E_0^k) \to L^p(B, E_0^k)$ is bounded. By (16), K_{Euc} is continuous from $L^p(B, E_0^k)$ to $W^{1,p}(B, E_0^{k-1})$ and hence, in particular, from $L^p(B, E_0^k)$ to $W^{1,p}(B, E_0^{k-1})$. Then we can conclude the proof of i), keeping again into account that Π_E is a differential operator of order ≤ 1 in the horizontal derivatives.

To prove ii) it is enough to remind that $K = \Pi_{E_0} K_{\mathrm{Euc}}$ of forms of degree $h > n$, together with Remark 4.1 in [13].

As for iii), the statement can be proved similarly to i), noticing that $K = \Pi_{E_0} \Pi_E K_{\mathrm{Euc}}$ on forms of degree $n+1$.

Finally, $\operatorname{supp} J\omega \subset B$ since both Π_E and Π_{E_0} preserve the support.

□

The operators K and J provide a local homotopy in Rumin's complex, but fail to yield the Sobolev and Poincaré inequalities we are looking for, since, because of the presence of the projection operator Π_E (that on forms of low degree is a first order differential operator) they loose regularity as is stated in Lemma 6.7, ii) above. In order to build "good" local homotopy operators with the desired gain of regularity, we have to combine them with homotopy operators which, though not local, in fact provide the "good" gain of regularity.

Proposition 6.8. *If $\alpha \in \mathcal{D}(\mathbb{H}^n, E_0^h)$ for $p > 1$ and $h = 1, \dots, 2n$, then the following homotopy formulas hold:*

- *if $h \neq n, n+1$, then $\alpha = d_c K_1 \alpha + \tilde K_1 d_c \alpha$, where K_1 and $\tilde K_1$ are associated with kernels $k_1, \tilde k_1$ of type 1;*
- *if $h = n$, then $\alpha = d_c K_1 \alpha + \tilde K_2 d_c \alpha$, where K_1 and $\tilde K_2$ are associated with kernels $k_1, \tilde k_2$ of type 1 and 2, respectively;*
- *if $h = n+1$, then $\alpha = d_c K_2 \alpha + \tilde K_1 d_c \alpha$, where K_2 and $\tilde K_1$ are associated with kernels $k_2, \tilde k_1$ of type 2 and 1, respectively.*

Proof. Suppose $h \neq n-1, n, n+1$. By Lemma 4.10, we have:

$$\begin{aligned}\alpha = \Delta_{\mathbb{H},h} \Delta_{\mathbb{H},h}^{-1} \alpha &= d_c(\delta_c \Delta_{\mathbb{H},h}^{-1})\alpha + \delta_c(d_c \Delta_{\mathbb{H},h}^{-1})\alpha \\ &= d_c(\delta_c \Delta_{\mathbb{H},h}^{-1})\alpha + (\delta_c \Delta_{\mathbb{H},h+1}^{-1}) d_c \alpha.\end{aligned}$$

where $\delta_c \Delta_{\mathbb{H},h}^{-1}$ and $\delta_c \Delta_{\mathbb{H},h+1}^{-1}$ are associated with a kernel of type 1 (by Proposition 4.2 and Theorem 4.8).

Analogously, if $h = n-1$

$$\begin{aligned}\alpha = \Delta_{\mathbb{H},n-1} \Delta_{\mathbb{H},n-1}^{-1} \alpha &= d_c(\delta_c \Delta_{\mathbb{H},n-1}^{-1})\alpha + \delta_c(d_c \Delta_{\mathbb{H},n-1}^{-1})\alpha \\ &= d_c(\delta_c \Delta_{\mathbb{H},n-1}^{-1})\alpha + (\delta_c d_c \delta_c \Delta_{\mathbb{H},n}^{-1}) d_c \alpha.\end{aligned}$$

Again $\delta_c\Delta_{\mathbb{H},n-1}^{-1}$ and $\delta_c d_c\delta_c\Delta_{\mathbb{H},n}^{-1}$ are associated with kernels of type 1.

Take now $h=n$. Then

$$\begin{aligned}\alpha &= \Delta_{\mathbb{H},n}\Delta_{\mathbb{H},n}^{-1}\alpha = (d_c\delta_c)^2\Delta_{\mathbb{H},n}^{-1}\alpha + \delta_c(d_c\Delta_{\mathbb{H},n}^{-1})\alpha\\ &= d_c(\delta_c d_c\delta_c\Delta_{\mathbb{H},n}^{-1})\alpha + \delta_c\Delta_{\mathbb{H},n+1}^{-1}d_c\alpha\end{aligned}$$

where $\delta_c d_c\delta_c\Delta_{\mathbb{H},n}^{-1}$ and $\delta_c\Delta_{\mathbb{H},n+1}^{-1}$ are associated with a kernel of type 1 and 2, respectively).

Finally, take $h=n+1$. Then

$$\begin{aligned}\alpha &= \Delta_{\mathbb{H},n+1}\Delta_{\mathbb{H},n+1}^{-1}\alpha = d_c\delta_c\Delta_{\mathbb{H},n+1}^{-1}\alpha + (\delta_c d_c)^2\Delta_{\mathbb{H},n+1}^{-1}\alpha\\ &= d_c\delta_c\Delta_{\mathbb{H},n+1}^{-1}\alpha + \delta_c\Delta_{\mathbb{H},n+2}^{-1}d_c\alpha\end{aligned}$$

where $\delta_c\Delta_{\mathbb{H},n+1}^{-1}$ and $\delta_c\Delta_{\mathbb{H},n+2}^{-1}$ associated with kernels of type 2 and 1, respectively. $\square$

The $L^p - L^q$ continuity properties of convolution operators associated with Folland's kernels yields the following strong $\mathbb{H}$-Poincaré$_{p,q}(h)$ inequality in $\mathbb{H}^n$ (the strong $\mathbb{H}$-Sobolev$_{p,q}(h)$ is obtained in Corollary 6.17).

Corollary 6.9. *Take $1\le h\le 2n+1$. Suppose $1<p<Q$ if $h\neq n+1$ and $1<p<Q/2$ if $h=n+1$. Let $q\ge p$ defined by*

$$\frac{1}{p}-\frac{1}{q} := \begin{cases}\frac{1}{Q} & \text{if } h\neq n+1,\\ \frac{2}{Q} & \text{if } h=n+1.\end{cases} \tag{23}$$

Then for any d_c-closed form $\alpha\in\mathcal{D}(\mathbb{H}^n,E_0^h)$ there exists $\phi\in L^q(\mathbb{H}^{\cdot}E_0^{h-1})$ such that $d_c\phi=\alpha$ and

$$\|\phi\|_{L^q(\mathbb{H}^n,E_0^{h-1})}\le C\|\alpha\|_{L^p(\mathbb{H}^n,E_0^{h-1})}$$

(i.e., the strong $\mathbb{H}$-Poincaré$_{p,q}(h)$ inequality holds for $1\le h\le 2n+1$).

Theorem 6.10. *Let $B=B(e,1)$ and $B'=B(e,\lambda)$, $\lambda>1$, be concentric balls of $\mathbb{H}^n$. If $1\le h\le 2n+1$, there exist operators T and $\tilde T$ from $C^\infty(B',E_0^\bullet)$ to $C^\infty(B,E_0^{\bullet-1})$ and S from $C^\infty(B',E_0^\bullet)$ to $C^\infty(B,E_0^\bullet)$ satisfying*

$$d_cT+\tilde T d_c+S=I \qquad \text{on } B. \tag{24}$$

In addition

i) *$\tilde T: W^{-1,p}(B',E_0^{h+1})\to L^p(B,E_0^h)$ if $h\neq n$, and $\tilde T: W^{-2,p}(B,E_0^{n+1})\to L^p(B,E_0^n)$;*
ii) *$T: L^p(B',E_0^h)\to W^{1,p}(B,E_0^{h-1})$, $h\neq n+1$, $T:T:L^p(B',E_0^{n+1})\to W^{2,p}(B,E_0^n)$ if $h=n+1$,*
iii) *$S: L^p(B',E_0^h)\to W^{s,p}(B,E_0^h)$,*

so that (24) still holds in $L^p(B,E_0^\bullet)$. In addition, for every (h,p,q) satisfying inequalities

$$1<p\le q<\infty,\quad \frac{1}{p}-\frac{1}{q}\le\begin{cases}\frac{1}{Q} & \text{if } h\neq n+1,\\ \frac{2}{Q} & \text{if } h=n+1,\end{cases} \tag{25}$$

we have:

iv) *$T: L^p(B',E_0^h)\to L^q(B,E_0^{h-1})$;*
v) *$S: L^p(B',E_0^h)\to W^{s,q}(B,E_0^h)$;*
vi) *$W^{1,p}(B',E_0^h)\to W^{s,q}(B,E_0^{h-1})$ for any $s>0$.*

Proof. Suppose first $h \neq n, n+1$. We consider a cut-off function ψ_R supported in a R-neighborhood of the origin, such that $\psi_R \equiv 1$ near the origin. With the notations of Proposition 6.8, we can write $k_1 = k_1\psi_R + (1-\psi_R)k_1$ and $\tilde{k}_1 = \tilde{k}_1\psi_R + (1-\psi_R)\tilde{k}_1$. Let us denote by $K_{1,R}$, $\tilde{K}_{1,R}$ the convolution operators associated with $\psi_R k_1$, $\psi_R \tilde{k}_1$, respectively. Le us fix two balls B_0, B_1 with

$$B \Subset B_0 \Subset B_1 \Subset B', \tag{26}$$

and a cut-off function $\chi \in \mathcal{D}(B_1)$, $\chi \equiv 1$ on B_0. If $\alpha \in C^\infty(B', E_0^\bullet)$, we set $\alpha_0 = \chi\alpha$, continued by zero outside B_1.

Keeping in mind (8) and Proposition 4.2, we have

$$\alpha_0 = d_c K_{1,R}\alpha_0 + \tilde{K}_{1,R} d_c \alpha_0 + S_0\alpha_0, \tag{27}$$

where S_0 is

$$S_0\alpha_0 := d_c((1-\psi_R)k_1 * \alpha_0) + (1-\psi_R)\tilde{k}_1 * d_c\alpha_0.$$

We set

$$T\alpha := K_{1,R}\alpha_0, \qquad \tilde{T}\alpha := \tilde{K}_{1,R} d_c\alpha_0, \qquad S\alpha := S_0\alpha_0.$$

We notice that, provided $R > 0$ is small enough, the definition of T and $\tilde{T}$ does not depend on the continuation of α outside B_0. By (27) we have

$$\alpha = d_c T\alpha + \tilde{T} d_c \alpha + S\alpha \qquad \text{in } B.$$

If $h = n$ we can carry out the same construction, replacing $\tilde{k}_1$ by $\tilde{k}_2$ (keep in mind that $\tilde{k}_2$ is a kernel of type 2). Analogously, if $h = n+1$ we can carry out the same construction, replacing k_1 by k_2 (again a kernel of type 2).

Let us prove i). Suppose $h \neq n$, and take $\beta \in W^{-1,p}(B', E_0^h)$. The operator $\tilde{K}_{1,R}$ is associated with a matrix-valued kernel $\psi_R(\tilde{k}_1)_{\ell,\lambda}$ and β is identified with a vector-valued distribution $(\beta_1, \dots, \beta_{N_h})$, with $\beta_j = \sum_i W_i f_i^j$ as in Definition 5.5 with

$$\sum_j \sum_i \|f_i^j\|_{L^p(B')} \leq C\|\beta\|_{W^{-1,p}(B',E_0^h)}.$$

Thus $(\beta_0)_j$, the j-th component of $\beta_0 = \chi\beta$ has the form

$$(\beta_0)_j = \sum_i W_i(\chi f_i^j) - \sum_i (W_i\chi) f_i^j =: \sum_i W_i (f_i^j)_0 - \sum_i (W_i\chi) f_i^j.$$

In order to estimate the norm of $\tilde{T}\beta$ in $L^p(B, E_0^h)$, we take

$$\phi = \sum_j \phi_j \xi_j^h \in \mathcal{D}(B, E_0^h), \qquad \text{with} \qquad \textstyle\sum_j \|\phi_j\|_{L^{p'}(B')} \leq 1,$$

and we estimate $\langle T\beta|\phi\rangle$, that, by (14), is a sum of terms of the form

$$\int_B (\psi_R\kappa * f_0)(x) W_i\phi(x)\, dx = \langle \psi_R\kappa * W_i f_0|\phi\rangle \tag{28}$$

or of the form

$$\int_B (\psi_R\kappa * (W_i\chi)f)(x)\phi(x)\, dx, \tag{29}$$

where κ denotes one of the kernels $(\tilde{k}_1)_{\ell,\lambda}$ of type 1 associated with $\tilde{k}_1$, f is one of the f_i^j's and ϕ one of the ϕ_j's, As for (28), by (8),

$$\begin{aligned}\langle \psi_R\kappa * W_i f_0|\phi\rangle &= \langle {}^{\mathrm{v}}W^{I\,\mathrm{v}}[\psi_R\kappa] * f_0|\phi\rangle \\ &= \langle \psi_R{}^{\mathrm{v}}W^{I\,\mathrm{v}}\kappa * f_0|\phi\rangle - \langle ({}^{\mathrm{v}}W^{I\,\mathrm{v}}\psi_R)\kappa * f_0|\phi\rangle\end{aligned}$$

We notice now that ${}^{\mathrm{v}}W^{I\,\mathrm{v}}\kappa$ is a kernel of type 0. Therefore, by Lemma 4.4

$$\begin{aligned}\langle \psi_R{}^{\mathrm{v}}W^{I\,\mathrm{v}}\kappa * f_0|\phi\rangle &\le \|\psi_R{}^{\mathrm{v}}W^{I\,\mathrm{v}}\kappa * f_0\|_{L^p(B)}\|\phi\|_{L^{p'}(B)} \\ &\le \|\psi_R{}^{\mathrm{v}}W^{I\,\mathrm{v}}\kappa * f_0\|_{L^p(B)} \le C\|f_0\|_{L^p(B')} \\ &\le C\|\beta\|_{W^{-1,p}(B',E_0^h)}.\end{aligned}$$

The term in (29) can be handled in the same way, keeping into account Remark (4.5). Eventually, combining (28) and (29) we obtain that

$$\|\tilde{T}\beta\|_{L^p(B)} \le C\|\beta\|_{W^{-1,p}(B',E_0^h)}.$$

The assertion for $h = n$ can be proved in the same way, taking into account that $\tilde{T}$ is built from a kernel of type 2, and that the space $W^{-2,p}(B, E_0^{n+1})$ is characterized by “second order divergences”.

Let us prove now ii). Suppose $h \neq n+1$ and take $\alpha = \sum_j \alpha_j \xi_j^h \in \mathcal{D}(B', E_0^h)$. Arguing as above, in order to estimate $\|T\alpha\|_{W^{1,p}(B,E_0^{h-1})}$ we have to consider terms of the form

$$W_\ell(\psi_R\kappa * (\chi\alpha_j)) = \psi_R\kappa * (W_\ell(\chi\alpha_j)) \tag{30}$$

(when we want to estimate the the L^p-norm of the horizontal derivatives of $T\alpha$), or of the form

$$\psi_R\kappa * (\chi\alpha_j) \tag{31}$$

(when we want to estimate the L^p-norm of $T\alpha$). Both (30) and (31) can be handled as in the case i) (no need here of the duality argument).

We point out that (31) yields a $L^p - L^q$ estimates (since, unlike (30), involves only kernels of type 1) and then assertion iv) follows.

Let us prove v). Then also iii) will follow straightforwardly.

It is easy to check that S_0 can be written as a convolution operator with matrix-valued kernel s_0. In turn, each entry of s_0 (that we still denote by s_0) is a sum of terms of the form

$$(1-\psi_R)W_\ell\kappa - (W_\ell\psi_R)\kappa.$$

Thus, the kernels are smooth and then regularizing from $\mathcal{E}'(B')$ to C^∞ of a neighborhood of B. Thus

$$\|W^I s_0 * \alpha_j\|_{L^q(B)} \le C\|\alpha_j\|_{L^p(B)},$$

for all p, q.

□

Remark 6.11. *Apparently, in previous theorem, two different homotopy operators T and $\tilde{T}$ appear. In fact, they coincide when acting on form of the same degree.*

More precisely, in Proposition 6.8 the homotopy formulas involve four operators $K_1, \tilde{K}_1, K_2, \tilde{K}_2$, where the notation is meant to distinguish operators acting on $d_c\alpha$ (the operators with tilde) from those on which the differential acts (the operators without tilde), whereas the lower index 1 or 2 denotes the type of the associated *kernels. Alternatively, a different notation could be used: if $\alpha \in \mathcal{D}(\mathbb{H}^n, E_0^h)$ we can write*

$$\alpha = d_c K_h + \tilde{K}_{h+1} d_c \alpha,$$

where the tilde has the same previous meaning, whereas the lower index refers now to the degree of the forms on which the operator acts.

It is important to notice that

$$K_{h+1} = \tilde{K}_{h+1}, \qquad h = 1, \dots, 2n.$$

Indeed, take $h < n-1$. Then $\tilde{K}_{h+1} = \delta_c \Delta^{-1}_{\mathbb{H},h+1}$ (as it appears in the homotopy formula at the degree h), that equals K_{h+1} (as it appears in the homotopy formula at the degree $h+1 \le n-1$). Take now $h = n-1$. Then $\tilde{K}_n = \delta_c d_c \delta_c \Delta^{-1}_{\mathbb{H},n}$ (as it appears in the homotopy formula at the degree n), that equals K_n (as it appears in the homotopy formula at the degree n). If $h = n$, then $\tilde{K}_{n+1} = \delta_c \Delta^{-1}_{\mathbb{H},n+1}$ (as it appears in the homotopy formula at the degree n), that equals K_{n+1} (as it appears in the homotopy formula at the degree $n+1$). Finally, if $h > n$, then $\tilde{K}_{h+1} = \delta_c \Delta^{-1}_{\mathbb{H},h+1}$ (as it appears in the homotopy formula at the degree h), that equals K_{h+1} (as it appears in the homotopy formula at the degree $h+1$).

Once this point is established, from now on we shall write

$$K := K_h = \tilde{K}_h$$

without ambiguity.

Therefore $T = \tilde{T}$ and the homotopy formula (24) reads as

$$d_c T + T d_c + S = I \qquad \text{on } B. \tag{32}$$

Remark 6.12. *By the arguments used in the proof of Theorem 6.10, i) the proof of the $L^p - W^{s,q}$ continuity of S can be adapted to prove that S is a smoothing operator, i.e for any $m, s \in \mathbb{N} \cup \{0\}$, S is bounded from $W^{-m,p}(B', E_0^\bullet)$ to $W^{s,q}(B, E_0^\bullet)$ when (25) holds. In particular, if $\alpha \in W^{-m,p}(B', E_0^\bullet)$ then $S\alpha \in C^\infty(B, E_0^\bullet)$.*

Remark 6.13. *It is worth pointing out the following fact: take $\alpha, \beta \in L^p(B', E_0^\bullet)$, $\alpha \equiv \beta$ on B_1 (B_1 has been introduced in (26)). Then $\alpha_0 \equiv \beta_0$ in B_0, so that $K_{1,R}\alpha_0 \equiv K_{1,R}\beta_0$ and $\tilde{K}_{1,R} d_c \alpha_0 \equiv \tilde{K}_{1,R} d_c \tilde{\beta}_0$ in B. In other words, $(d_c T + T d_c)\alpha = (d_c T + T d_c)\beta$ in B. Thus, by (32), $S\alpha = S\beta$ in B.*

The following commutation lemma will be helpful in the sequel.

Lemma 6.14. *We have:*

$$[S, d_c] = 0 \qquad \text{in } L^p(\mathbb{H}^n, E_0^\bullet).$$

Proof. Take first $\alpha \in C^\infty(B', E_0^h)$, $1 \le h \le 2n+1$. By (32), $Sd_c = d_c S$ on $\mathcal{D}(B', E_0^h)$.

Take now $\alpha \in L^p(B', E_0^h)$, and let χ_1 be a cut-off function supported in B', $\chi_1 \equiv 1$ on B_1 (B_1 has been defined in (26)). By convolution with usual Friedrichs' mollifiers, we can find a sequence $(\alpha_k)_{k \in \mathbb{N}}$ in $\mathcal{D}(B', E_0^h)$ converging to $\chi_1 \alpha$ in $L^p(B', E_0^h)$. By Theorem 6.10, $S\alpha_k \to S(\chi_1 \alpha)$ in $W^{2,p}(B, E_0^{h+1})$, and hence $d_c S\alpha_k \to d_c S(\chi_\alpha)$ in $L^p(B, E_0^h)$ as $k \to \infty$ (obviously, if $h \neq n-1$, it would have been enough to have $S\alpha_k \to S(\chi_1\alpha)$ in $W^{1,p}(B, E_0^{h+1})$). On the other hand, $\chi_1 \alpha \equiv \alpha$ in B_1, and then by Remark 6.13 $S(\chi_1 \alpha) = S\alpha$ in B, so that $d_c S\alpha_k \to d_c S\alpha$ in $L^p(B, E_0^h)$ as

$k \to \infty$. Moreover $d_c\alpha_k \to d_c(\chi_1\alpha)$ in $W^{-1,p}(B', E_0^h)$ (in $W^{-2,p}(B', E_0^h)$ if $h = n$) and hence, again by Theorem 6.10, $Sd_c\alpha_k \to Sd_c(\chi_1\alpha)$ in B as $n \to \infty$. Again $d_c(\chi_1\alpha) \equiv d_c\alpha$ in B_1 and then, by Remark 6.13, $Sd_c\alpha_k \to Sd_c\alpha$ in B as $k \to \infty$.

Finally, since $d_cS\alpha_k = Sd_c\alpha_k$ for all $k \in \mathbb{N}$, we can take the limits as $k \to \infty$ and the assertion follows. □

Theorem 6.15. *Let p, q, h s in* (25). *With the notations of Definitions 6.1, if $1 < p < \infty$, then both an interior $\mathbb{H}$-Poincaré$_{p,q}(h)$ and an interior $\mathbb{H}$-Sobolev$_{p,q}(h)$ inequalities hold for $1 \le h \le 2n$.*

Proof. $\mathbb{H}$-Poincaré$_{p,q}(h)$ inequality: let $\omega \in L^p(B', E_0^h)$ be such that $d_c\omega = 0$.

By (32) we can write $\omega = d_cT\omega + S\omega$ in B. By Remark 6.12 and Lemma 6.14, $S\omega \in \mathcal{E}(B, E_0^h)$, and $d_cS\omega = 0$. Thus we can apply (22) to $S\omega$ and we get $S\omega = d_cKS\omega$, where K is defined in (20). In B, put now

$$\phi := (KS + T)\omega.$$

Trivially $d_c\phi = d_cKS\omega + d_cT\omega = S\omega + \omega - S\omega = \omega$. By Theorem 6.10,

$$\begin{aligned} \|\phi\|_{L^q(B,E_0^{h-1})} &\le \|KS\omega\|_{L^q(B,E_0^{h-1})} + \|T\omega\|_{L^q(B,E_0^{h-1})} \\ &\le \|KS\omega\|_{L^q(B,E_0^{h-1})} + C\|\omega\|_{L^p(B',E_0^{h-1})} \\ &\le C\{\|S\omega\|_{W^{1,q}(B,E_0^{h-1})} + \|\omega\|_{L^p(B',E_0^{h-1})}\} \qquad \text{(by Lemma 6.7)} \\ &\le C\|\omega\|_{L^p(B',E_0^{h-1})}. \end{aligned} \tag{33}$$

$\mathbb{H}$-Sobolev$_{p,q}(h)$ inequality: let $\omega \in L^p(B, E_0^h)$ be a compactly supported form such that $d_c\omega = 0$. Since ω vanishes in a neighborhood of ∂B, without loss of generality we can assume that it is continued by zero on B'. In addition, $\omega = \chi\omega$. By (32) we have $\omega = d_cT\omega + S\omega$. On the other hand, $T\omega$ is supported in B_0 (since R is small), so that also $S\omega$ is supported in B_0. Again as above $S\omega \in C^\infty(B, E_0^h)$, and $d_cS\omega = 0$. Thus we can apply (22) to $S\omega$ and we get $S\omega = d_cJS\omega$, where J is defined in (21). By Lemma 6.6, $JS\omega$ is supported in $B_0 \subset B'$. Thus, if we set $\phi := (JS + T)\omega$, then ϕ is supported in B'. Moreover $d_c\phi = d_cKS\omega + d_cT\omega = S\omega + \omega - S\omega = \omega$. At this point, we can repeat the estimates (33) and we get eventually

$$\|\phi\|_{L^q(B',E_0^{h-1})} \le C\|\omega\|_{L^p(B,E_0^{h-1})}.$$

This completes the proof of the theorem. □

Let $B(p, r)$ a Korányi ball of center $p \in \mathbb{H}^n$ and radius $r > 0$. The map $x \to f(x) := \tau_p\delta_r(x)$ provides a contact diffeomorphism from $B(e, \rho)$ to $B(p, r\rho)$ for $\rho > 0$. Therefore the pull-back $f^\# : E_0^\bullet \to E_0^\bullet$. In addition, if $\alpha \in E_0^h$, then

$$f^\#\alpha = r^h\alpha \circ f \qquad \text{if } h \le n \qquad \text{and} \qquad f^\#\alpha = r^{h+1}\alpha \circ f \qquad \text{if } h > n.$$

Theorem 6.16. *Take $1 \le h \le 2n + 1$. Suppose $1 < p < Q$ if $h \ne n + 1$ and $1 < p < Q/2$ if $h = n + 1$. Let $q \ge p$ such that*

$$\frac{1}{p} - \frac{1}{q} \le \begin{cases} \frac{1}{Q} & \text{if } h \ne n+1, \\ \frac{2}{Q} & \text{if } h = n+1. \end{cases} \tag{34}$$

Then there exists a constant C such that, for every d_c-closed differential h-form ω in $L^p(B(p, \lambda r); E_0^h)$ there exists a $h-1$-form ϕ in $L^q(B(p, r), E_0^{h-1})$ such that $d_c\phi = \omega$ and

$$\|\phi\|_{L^q(B(p,r),E_0^{h-1})} \le C\, r^{Q/q-Q/p+1}\, \|\omega\|_{L^p(B(p,\lambda r),E_0^h)} \qquad \text{if } h \ne n+1$$

and

$$\|\phi\|_{L^q(B(p,r),E_0^n)} \le C\, r^{Q/q-Q/p+2}\, \|\omega\|_{L^p(B(p,\lambda r),E_0^{n+1})}.$$

Analogously there exists a constant C such that, for every compactly supported d_c-closed h-form ω in $L^p(B(p,r);E_0^h)$ there exists a compactly supported $(h-1)$-form ϕ in $L^q(B(p,\lambda r),E_0^{h-1})$ such that $d_c\phi=\omega$ in $B(p,\lambda r)$ and

$$\|\phi\|_{L^q(B(p,\lambda r),E_0^{k-1})} \le C\, \|\omega\|_{L^p(B(p,r),E_0^k)} \tag{35}$$

Proof. We have just to take the pull-back $f^\#\omega$ and then apply Theorem 6.15. □

If the choice of q is sharp (i.e. in (34) the equality holds), then the constant on the right hand side of (35) is independent of the radius of the ball, so that a global $\mathbb{H}$-Sobolev$_{p,q}(h)$ inequality holds.

Corollary 6.17. *Take $1\le h\le 2n+1$. Suppose $1<p<Q$ if $h\neq n+1$ and $1<p<Q/2$ if $h=n+1$. Let $q\ge p$ defined by*

$$\frac{1}{p}-\frac{1}{q} := \begin{cases} \frac{1}{Q} & \text{if } h\neq n+1,\\ \frac{2}{Q} & \text{if } h=n+1. \end{cases} \tag{36}$$

Then $\mathbb{H}$-Sobolev$_{p,q}(h)$ inequality holds for $1\le h\le 2n+1$.

7. Contact manifolds and global smoothing

Throughout this section, (M,H,g) will be a sub-Riemannian contact manifold of bounded C^k-geometry as in Definition 1.4. We shall denote by $(E_0^\bullet, d_c)$ both the Rumin's complex in (M,H,g) and in the Heisenberg group.

Proposition 7.1. *If ϕ is a contactomorphism from an open set $\mathcal{U}\subset\mathbb{H}^n$ to M, and we set $\mathcal{V}:=\phi(\mathcal{U})$, we have*

i) $\phi^\# E_0^\bullet(\mathcal{V}) = E_0^\bullet(\mathcal{U})$;
ii) $d_c\phi^\# = \phi^\# d_c$;
iii) *if ζ is a smooth function in M, then the differential operator in $\mathcal{U}\subset\mathbb{H}^n$ defined by $v\to\phi^\#[d_c,\zeta](\phi^{-1})^\# v$ is a differential operator of order zero if $v\in E_0^h(\mathcal{U})$, $h\neq n$ and a differential operator of order 1 if $v\in E_0^n(\mathcal{U})$.*

Proof. Assertions i) and ii) follow straightforwardly since ϕ is a contact map. Assertion iii) follows from Lemma 3.2, since, by definition,

$$\phi^\#[d_c,\zeta](\phi^{-1})^\# v = [d_c,\zeta\circ\phi]v.$$

□

Remark 7.2. *Let $\{\phi_{x_j}(B(e,1))\}$ a countable locally finite subcovering of $\{\phi_x(B(e,1)), x\in M\}$. From now on, for sake of simplicity, we shall write $\phi_j:=\phi_{x_j}$. Without loss of generality, we can replace $B(e,1)$ by $B(e,\lambda)$, where $\lambda>1$ is fixed (just to be congruent in the sequel with the notations of previous sections).*

Let $\{\chi_j\}$ be a partition of the unity subordinated to the covering $\{\phi_j(B(e,\lambda))\}$ of M. As above, without loss of generality, we can assume $\phi_j^{-1}(\operatorname{supp}\chi_j)\subset B(e,1)$.

If $u\in L^p(M,E_0^\bullet)$, we write

$$u=\sum_j \chi_j u$$

We can write

$$\chi_j u = (\phi_j^{-1})^{\#} \phi_j^{\#}(\chi_j u) =: (\phi_j^{-1})^{\#} v_j.$$

We use now the homotopy formula in $\mathbb{H}^n$ (see Theorem 6.10):

$$v_j = d_c T v_j + T d_c v_j + S v_j \qquad \text{in } B(e,1).$$

Without loss of generality, we can assume that $R > 0$ in the definition of the kernel of T has been chosen in such a way that the R-neighborood of $\phi_j^{-1}(\text{supp } \chi_j) \subset B(e,1)$. In particular $v_j - d_c T v_j - T d_c v_j$ is supported in $B(0,1)$ and therefore also $S v_j$ is supported in $B(0,1)$.

In particular, $(\phi_j^{-1})^{\#}\big(d_c T v_j + T d_c v_j + S v_j\big)$ is supported in $\phi_j(B(e,1))$ so that it can be continued by zero on M.

Thus

$$\begin{aligned} u &= \sum_j (\phi_j^{-1})^{\#}\big(d_c T v_j + T d_c v_j + S v_j\big) \\ &= d_c \sum_j \, (\phi_j^{-1})^{\#} T \phi_j^{\#}(\chi_j u) \\ &+ \sum_j ((\phi_j^{-1})^{\#} T \phi_j^{\#} \chi_j) d_c u - \sum_j (\phi_j^{-1})^{\#} T \phi_j^{\#}([\chi_j, d_c] u) \\ &+ \sum_j ((\phi_j^{-1})^{\#} (S \phi_j^{\#} \chi_j) u. \end{aligned}$$

We set

$$Tu := \sum_j (\phi_j^{-1})^{\#} T \phi_j^{\#}(\chi_j u) \tag{37}$$

and

$$Su := \sum_j (\phi_j^{-1})^{\#} S \phi_j^{\#}(\chi_j u) - \sum_j (\phi_j^{-1})^{\#} T \phi_j^{\#}([\chi_j, d_c] u). \tag{38}$$

The core of this section consists in the following approximate homotopy formula, where the "error term" S_M has the maximal regularising property compatible with the regularity of M.

Theorem 7.3. *Let (M, H, g) be a bounded C^k-geometry sub-Riemannian contact manifold, $k \geq 2$. Then*

$$I = d_c T_M + T_M d_c + S_M, \tag{39}$$

where

$$T_M := \big(\sum_{i=0}^{k-1} S^i\big) T, \qquad S_M := S^k,$$

and T and S are defined in (37) *and* (38)*.*

By definition

$$d_c S u = S d_c u. \tag{40}$$

In addition, the following maps are continuous:

i) $T_M : W^{-1,p}(M, E_0^{h+1}) \to L^p(M, E_0^h)$ *if* $h \neq n$*, and* $T_M : W^{-2,p}(M, E_0^{n+1}) \to L^p(M, E_0^n)$*;*

ii) $T_M : L^p(M, E_0^h) \to W^{1,p}(M, E_0^{h-1})$*,* $h \neq n+1$*,* $T_M : L^p(M, E_0^{n+1}) \to W^{2,p}(M, E_0^n)$ *if* $h = n+1$*,*

iii) $S_M : L^p(M, E_0^h) \to W^{k,p}(M, E_0^h)$.

In order to prove Theorem 7.3, let us prove the following preliminary result:

Lemma 7.4. *Let (M, H, g) be a bounded C^k-geometry sub-Riemannian contact manifold. If $2 \le \ell \le k-1$ and T and S are defined in (37) and (38), then*

$$I = d_c T + T d_c + S. \tag{41}$$

In addition, the following maps are continuous:

i) $T : W^{-1,p}(M, E_0^{h+1}) \to L^p(M, E_0^h)$ *if* $h \neq n$, *and* $T : W^{-2,p}(M, E_0^{n+1}) \to L^p(M, E_0^n)$;

ii) $T : L^p(M, E_0^h) \to W^{1,p}(M, E_0^{h-1})$, $h \neq n+1$, $T : L^p(M, E_0^{n+1}) \to W^{2,p}(M, E_0^n)$ *if* $h = n+1$,

iii) *if* $1 \le \ell \le k$, *then* $S : W^{\ell-1,p}(M, E_0^h) \longrightarrow W^{\ell,p}(M, E_0^h)$.

Proof. First of all, we notice that, if α is supported in $\phi_j(B(e,\lambda))$, then, by Definition 1.4 the norms

$$\|\alpha\|_{W^{m,p}(M,E_0^\bullet)} \qquad \text{and} \qquad \|\phi_j^\# \alpha\|_{W^{m,p}(\mathbb{H}^n,E_0^\bullet)}$$

are equivalent for $-k \le m \le k$, with equivalence constants independent of j. Thus, assertions i) and ii) follow straightforwardly from Theorem 6.10.

To get iii) we only need to note that the operators $(\phi_j^{-1})^\# T \phi_j^\# [\chi_j, d_c]$ are bounded from $W^{\ell-1,p}(M, E_0^\bullet) \to W^{\ell,p}(M, E_0^\bullet)$ in every degree. Indeed, by Lemma 3.2 above, the differential operator in $\mathbb{H}^n$ $\phi_j^\# [\chi_j, d_c](\phi_j^{-1})^\#$ has order 1 if $h = n$, and order 0 if $h \neq n$. Since the kernel of T can be estimated by kernel of type 2 if acts on forms of degree $h = n$, and of type 1 if acts on forms of degree $h \neq n$, the assertion follows straightforwardly

Summing up in j and keeping into account that the sum is locally finite, we obtain:

$$\begin{aligned}
\| \sum_j \phi_j^\# T_j (\phi_j^{-1})^\# [\chi_j, d_c] \|_{W^{\ell,p}(M)} &\le \sum_j \|\phi_j^\# T_j (\phi_j^{-1})^\# [\chi_j, d_c]\|_{W^{\ell,p}(\phi(\mathcal{U}_j))} \\
&\le C \sum_j \|T_j \phi_j^\# [\chi_j, d_c]\|_{W^{\ell,p}(\mathcal{U}_j)} \le C \sum_j \|\phi_j^\# u\|_{W^{\ell-1,p}(U_j)} \\
&\le C \|u\|_{W^{\ell-1,p}(M)}.
\end{aligned}$$

□

Proof of Theorem 7.3. By (40)

$$\begin{aligned}
&d_c T_M + T_M d_c + S_M \\
&\qquad = d_c \big(\sum_{i=0}^{k-1} S^i \big) T + \big(\sum_{i=0}^{k-1} S^i \big) \tilde{T} d_c + S^k \\
&\qquad = \sum_{i=0}^{k-1} S^i \big(d_c T + T d_c \big) + S^k \\
&\qquad = \sum_{i=0}^{k-1} S^i \big(I - S \big) + S^k = I.
\end{aligned}$$

Then statements i), ii) and iii) follow straightforwardly from i), ii) and iii) of Lemma 7.4.

□

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