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Yves Raynaud

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The range of a contractive projection in $L_p(H)$

Yves Raynaud

Abstract: We show that the range of a contractive projection on a Lebesgue-Bochner space of Hilbert-valued functions $L_p(H)$ is isometric to a $\ell_p$-direct sum of Hilbert-valued $L_p$-spaces. We explicit the structure of contractive projections. As a consequence for every $1 < p < \infty$ the class $C_p$ of $\ell_p$-direct sums of Hilbert-valued $L_p$-spaces is axiomatizable (in the class of all Banach spaces).

Introduction

It was a remarkable achievement in the isometric theory of Banach spaces of the years 1960’s to characterize the contractive linear projections of Lebesgue $L_p$ spaces ($p \neq 2$). In the case of $L_p$ spaces of a probability space it was done by Douglas [D] in the case $p = 1$ and Andô [A] in the case $1 < p < \infty$, $p \neq 2$. They showed that the range of such a contractive projection is itself isometric to a $L_p$ space (for the same $p$, but a different measure space); if moreover the projection is positive then its range is a sublattice of the initial $L_p$ space and is lattice isomorphically isometric to a $L_p$ space. This was extended to the non-sigma-finite measure space setting by Tzafriri ([T]). In the case of a probability space, the structure of contractive projections is elucidated by Douglas-Andô works: a general contractive projection $P$ on $L_p(\Omega, \Sigma, \mu)$ has the form:

$$P = M_\varepsilon \hat{P} M_\varepsilon^{-1} + V$$

(0.1)

where $M_\varepsilon$ is the multiplication operator by a function $\varepsilon$ with $|\varepsilon| = 1$, $\hat{P}$ is a positive contractive projection, and $V = 0$ if $p > 1$, while if $p = 1$, then $V$ is a contraction from $L_1$ into the range $R(P)$ of $P$ which vanishes on the band generated by $R(P)$. Moreover $\hat{P}$ is a weighted conditional expectation, i. e. there exist a sub-sigma algebra $\mathcal{B}$, an element $B \in \mathcal{B}$ and a nonnegative function $w \in L^p$ such that $\mathbb{E}(w^p | \mathcal{B}) = 1$ and

$$\hat{P} f = w\mathbb{E}(1_B f, w^{p-1} | \mathcal{B})$$

for every $f \in L_p$ (in particular if $P1 = 1$ then $P$ is a conditional expectation). This last formula can also be written

$$\hat{P} f = w\mathbb{E}_\nu(1_B f w^{-1} | \mathcal{B})$$

where $\mathbb{E}_\nu$ is the conditional expectation relative to the measure $\nu = w^p \mu$. If we denote by $S$ the isometric isomorphism $f \mapsto w^p f$ of $L_p(\Omega, \Sigma, \nu)$ onto $L_p(\Omega, \Sigma, \nu)$ and by $M_B$ the multiplication operator by the indicator function $1_B$, we have:

$$\hat{P} = SM_B \mathbb{E}_\nu( | \mathcal{B})S^{-1}$$

(0.2)

The structure of contractive projections in the non-sigma finite case was treated by Bernau and Lacey ([BL]); their main result can be rephrased in saying that if we assume (as we
may) that the measure space \((\Omega, \Sigma, \mu)\) is localizable ([F]) then formulas (0.1) and (0.2) are still valid; now \(w\) is some \(\Sigma\)-measurable positive function, \(\nu = w^p \mu\) and \(B\) is some semi-finite sigma-subalgebra of \(\Sigma\).

The task of extending these results to various classical spaces was considered by numerous authors; see the recent survey paper [R2] and the references inside. Here we are more specifically interested in the case of vector-valued Lebesgue \(L_p\) spaces, in particular mixed norm spaces \(L_p(L_q)\). Since the survey paper [Dt] on this specific subject, several partial results appeared. In particular B. Randrianantoanina ([R1]) succeeded in solving thoroughly the complex sequential case \(\ell_p(\ell_q)\) using hermitian operator techniques introduced in the subject by Kalton and Wood. More recently the case of finite dimensional real Banach spaces with \(C^2\) norm was considered by the authors of [LG]; under some additional conditions on the dual norm (in particular it is assumed to be \(C^2\) on the complementary set of the coordinate hyperplanes associated to a distinguished basis) the contractively complemented subspaces are shown to be necessarily generated by a block-basis of the given basis. This can be applied in particular to the real spaces \(\ell_p(\ell_q^n)\), when \(2 < p, q < \infty\) (or by duality when \(1 < p, q < 2\), obtaining the same description of their contractively complemented subspaces as in the complex case [R3].

In the present paper we examine the case of Lebesgue spaces of Hilbert valued functions \(L_p(H)\); this is done in the most general case (without any assumption of sigma-finiteness of \(L_p\)-space or separability of the Hilbert space; in fact we have in mind some applications to the ultrapowers of such spaces, which are neither separable nor sigma-finite). It turns out that the range of a contractive projection is a \(\ell_p\)-direct sum of spaces of the type \(L_p(H)\). More precisely:

**Theorem 0.1.** Let \(1 \leq p < \infty, p \neq 2; H\) be a Hilbert space and \(L_p = L_p(\Omega, \Sigma, \mu)\). The range of every contractive projection \(P : L_p(H) \to L_p(H)\) is isometric to a \(\ell_p\)-direct sum of Hilbert-valued \(L_p\)-spaces, i.e.

\[
R(P) \approx_1 \left( \bigoplus_{i \in I} L_p(\Omega_i, B_i, \mu_i; H_i) \right)_{\ell_p}
\]

where \((\Omega_i)_i\) is a family of pairwise almost disjoint members of \(\Sigma\), each \(B_i\) is a subsigma-algebra of the trace \(\Sigma_i\) of \(\Sigma\) on \(\Omega_i\); \(\mu_i\) is the trace on \(\Omega_i\) of the measure \(\mu\); and the Hilbert spaces \(H_i\) have Hilbertian dimension not greater than the Hilbertian dimension of \(H\).

Conversely a \(\ell_p\)-sum \(\left( \bigoplus_{i \in I} L_p(\Omega_i, \Sigma_i, \mu_i; H_i) \right)_{\ell_p}\) embeds isometrically into \(L_p(H)\), where

\[
L_p = \left( \bigoplus_{i \in I} L_p(\Omega_i, B_i, \mu_i) \right)_{\ell_p} \quad \text{and} \quad H = (\bigoplus_{i \in I} H_i)_{\ell_2}.
\]

Hence a contractively complemented subspace of a \(\ell_p\)-direct sum of Hilbert-valued \(L_p\)-spaces is still a \(\ell_p\)-direct sum of Hilbert-valued \(L_p\)-spaces. In other words:

**Corollary 0.2.** The class \(C_p\) of \(\ell_p\)-direct sums of Hilbert-valued \(L_p\)-spaces is stable under contractive projections.

The structure of the contractive projection \(P\) can be easily explained in the case where the space \(H\) is separable (the non-separable case is analogous and will be described in section 5). Recall that given two Banach spaces \(X, Y\), a family of operators \(T_\omega : X \to Y\) is said to be strong-operator \(\Sigma\)-measurable if for every \(x \in X\), the map \(\omega \mapsto T_\omega(x)\) is \(\Sigma\)-measurable as a map \(\Omega \to Y\). If moreover \(\text{Esssup}_\omega \|T_\omega\| < \infty\), such a measurable family induces a bounded
linear map $T$ from $L_p(\Omega, \Sigma, \mu; X)$ into $L_p(\Omega, \Sigma, \mu; Y)$ by the equation:

$$(Tf)(\omega) = T_\omega(f_\omega)$$

**Theorem 0.3.** Under the conditions of Th. 0.1, if moreover $H$ is separable, then

$$P = \sum_{i \in I} S_i(\tilde{P}_i \otimes 1d_{H_i})S_i^* M_{\Omega_i} + V$$

where $\tilde{P}_i$ is a positive contractive projection in $L_p(\Omega_i, \Sigma_i, \mu_i)$; $S_i$ is an isometric embedding of $L_p(\Omega_i, \Sigma_i, \mu_i; H_i)$ into $L_p(\Omega_i, \Sigma_i, \mu_i; H)$ associated with a (strong-operator)-measurable family $(S_{i,\omega})_{\omega \in \Omega_i}$ of isometric embeddings $H_i \rightarrow H$, while $S_{i,\omega}^*$ is associated with the adjoint family $(S_{i,\omega}^*)_{\omega \in \Omega_i}$ of projections $H \rightarrow H_i$; $M_{\Omega_i} : L_p(\Omega; H) \rightarrow L_p(\Omega_i; H_i)$ is the multiplication operator by the indicator function $1_{\Omega_i}$; and $V = 0$ if $p > 1$, while if $p = 1$ then $V$ is a contraction of $L_1(\Omega, \Sigma, \mu; H)$ vanishing on every $L_1(\Omega_i, \Sigma_i, \mu_i; H)$ and taking values in the range of $P$.

Let us present shortly an application of the Thm 0.1 which was in fact our main motivation for starting this study. If $X, Y$ are Banach spaces, we say that $X$ is an ultraroot of $Y$ if $Y$ is isometric to some ultrapower of $X$. Recall that a Banach space $X$ embeds canonically isometrically in every of its ultrapowers $X_I$, and that if $X$ is reflexive, then this canonical image is contractively complemented in $X_I$. As a consequence of Th. 0.1 we see that every ultraroot of a $L_p(H)$ space, $p > 1$ is a member of $C_p$. By Cor. 0.2 the same is true for ultraroots of members of $C_p$. On the other hand it was proved in [LR] that every ultraproduct of $L_p(H)$ spaces is isometric to a $\ell_p$-direct sum of Hilbert-valued $L_p$-spaces. More generally every ultraproduct of members of $C_p$ is itself isometric to a member of $C_p$. Hence we obtain:

**Corollary 0.4.** For every $1 < p < \infty$ the class $C_p$ of $\ell_p$-direct sums of Hilbert-valued $L_p$-spaces is stable under ultraproducts and ultraroots.

In other words the class $C_p$ is axiomatisable in the sense of Henson-Iovino [HI] in their language of normed spaces structures (see [HI], th. 13.8).

The paper is organized as follows: after a section devoted to definitions, notations and a general result on orthogonally complemented subspaces of $L_p(H)$, we have two sections of preliminary results distinguishing the case $p = 1$ (Section 2) from the case $p > 1$ (Section 3). In these sections it is proved that if $f$ belongs to the range of a contractive projection $P$, then the whole subspace $Z_f := L_\infty(\Omega).f$ is preserved by $P$ (i.e. $PZ_f \subset Z_f$) which suggests clearly a possible reduction to the scalar case. It is also proved that the “orthogonal projection” onto $Z_f$ preserves the range of $P$. This allows to find an “orthogonal system” in $R(P)$ which generates $Z_p := L_\infty(\Sigma).R(P)$ over $L_\infty(\Sigma)$ which will furnish the orthogonal bases of the Hilbert spaces $H_i$ of Thm. 0.1. Section 4 is devoted to the proof of Thm. 0.1; a key point consists in proving that the different subalgebras of $\Sigma$ given by the scalar theorem (applied to each $Z_f$) are induced by the same sigma-subalgebra $F$ of $\Sigma$. Finally Thm 0.3 is proved in Section 5 (in a more general version not requiring separability).
1. General preliminaries

1.1 Definitions and notations

Let $1 \leq p < \infty$; $H$ be an Hilbert space and $(\Omega, \Sigma, \mu)$ be a measure space. In the following we denote (when there is no ambiguity) by $L_p(H)$ the Lebesgue-Bochner space $L_p(\Omega, \Sigma, \mu; H)$ of classes of $H$-valued $p$-integrable functions (for $\mu$-a.e. equality). Similarly $L_\infty(H)$ will be the space of classes of Bochner measurable, essentially bounded $H$-valued functions. These spaces can be defined directly from the Banach lattices $L_p$ (resp. $L_\infty$) and the Hilbert space $H$, but we adopt the functional point of view for the simplicity of the exposition. In the case where $(\Omega, \Sigma, \mu)$ is not sigma-finite, it is preferable to suppose that this measure space is localizable: the measure $\mu$ is semifinite (every set in $\Sigma$ of positive measure contains a further one of positive and finite measure) and $L_\infty(\Omega, \Sigma, \mu)$ is order complete. In particular every family $(A_i)_{i \in I}$ in $\Sigma$ has a supremum $A$, denoted by $\bigvee_{i \in I} A_i$. The set $A$ is defined (up to a $\mu$-null set) by the conditions:

$$A \supset A_i \text{ for every } i \in I$$

$$\text{If } B \in \Sigma \text{ and } B \supset A_i \text{ for every } i \in I \text{ then } B \supset A$$

where $B \supset A$ means $\mu(A \setminus B) = 0$ (define similarly $A \subset B$ and $A \equiv B$). We say that $B, C$ are almost disjoint if $A \cap B = \emptyset$.

To every $f \in L_p(H)$ we associate its "random norm" $N(f) \in L_p^\infty$ defined by $N(f)(\omega) = \|f(\omega)\|_H$, its vectorial function support $\mathbf{VS}(f) = \text{Supp}(N(f))$ and its "random direction", i.e. the element $u_f$ of $L_\infty(H)$ defined by $u_f(\omega) = \frac{f(\omega)}{N(f)(\omega)}$ if $\omega \in \mathbf{VS}(f)$, $=0$ if $\omega \notin \mathbf{VS}(f)$. If $M \subset L_p(H)$ we set $\mathbf{VS}(M) = \bigvee\{\mathbf{VS}(f) \mid f \in M\}$. If $f \in L_p(H), g \in L_q(H)$ we define their random scalar product $\langle f, g \rangle \in L_r$ (where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$) by $\langle f | g \rangle(\omega) = \langle f(\omega) | g(\omega) \rangle_H$, where $\langle \cdot | \cdot \rangle_H$ denotes the scalar product in $H$ (which we suppose left linear, right antilinear in the complex case). When $p, q$ are conjugate ($\frac{1}{p} + \frac{1}{q} = 1$), we obtain a sesquilinear pairing

$$\langle f, g \rangle = \int_\Omega \langle f | g \rangle d\mu$$

(1.1)

which gives rise to a canonical antilinear identification of $L_q(H)$ with $L_p(H)^*$ (if $1 < p, q < \infty$; the case $p = 1, q = \infty$ is more delicate); it is the usual duality pairing in the real spaces case. We have also:

$$\forall f \in L_p(H), \langle f | u_f \rangle = N(f)$$

We say that two elements $f, g \in L_p(H)$ are orthogonal, and we write $f \perp g$ if $\langle f | g \rangle = 0$. A related notation is the following. We set:

$$\{f \perp g\} = \{\omega \in \Omega \mid \langle f | g \rangle(\omega) = 0\}$$

We have then $f \perp g \iff \{f \perp g\} = \Omega$.

Let $H, K$ two Hilbert spaces. We say that a linear operator $T : L_p(H) \to L_p(K)$ is $\Sigma$-modular iff $T(\varphi.f) = \varphi.T f$ for every $f \in L_p(H)$ and $\varphi \in L_\infty(\Omega, \Sigma, \mu)$. It is modularly contractive, resp modularly isometric iff $N(T f) \leq N(f)$, resp. $N(T f) = N(f)$ for every $f \in L_p(H)$: it is then automatically $\Sigma$-modular (and, of course, contractive, resp. isometric). If $H$ is separable, then a modularly contractive, resp. modularly isometric operator $T$ is associated with a measurable family of contractions, resp. isometries $T_\omega : H \to K$.

4
Let $\mathcal{F}$ be a sub-sigma-algebra of $\Sigma$; a linear subspace $Z$ of $L_p(H)$ is a $L_\infty(\mathcal{F})$-submodule iff $\varphi.f \in Z$ for every $f \in Z$ and $\varphi \in L_\infty(\Omega, \mathcal{F}, \mu)$.

To every $f \in L_p(H)$ we associate the bounded $\Sigma$-modular operator:

$$E_f : L_p(H) \to L_p(H), \quad g \mapsto \langle \langle g|u_f \rangle \rangle u_f$$

We have $N(Efg) = |\langle \langle g|u_f \rangle \rangle| v_{\Sigma}(f) \leq N(g)$, hence $E_f$ is modularly contractive.

We have clearly $E_f(f) = N(f)u_f = f$. Consequently for every $\varphi \in L_\infty$, we have

$$E_f((\varphi N(f)).u_f) = E_f(\varphi f) = \varphi f = (\varphi N(f)).u_f$$

and by density we deduce that $E_f(\psi.u_f) = \psi.u_f$ for every $\psi \in L_p$. In particular $E_f(Efg) = E_f g$, so $E_f$ is a projection (with range $R(E_f) = L_p(\Omega).u_f$). It is not hard to see that $R(E_f)$ is exactly the closed $L_\infty(\Sigma)$-submodule generated by $f$. Note also that if $f, g \in L_p(H)$,

$$f \perp g \iff E_f g = 0 \iff E_g f = 0$$

1.2 Orthogonal projections

We end this section by considering a special class of contractive projections, namely the orthogonal ones. A projection $Q$ in $L_p(H)$ is said to be 
m\textit{orthogonal} if $(f - Qf) \perp Qf$ for every $f \in L_p(H).$ Such a projection is trivially modularly contractive since

$$N(f)^2 = N(Qf)^2 + N((I - Q)f)^2 \geq N(Qf)^2$$

Note that by polarization we have for every $f, g \in L_p(H)$:

$$\langle \langle f | g \rangle \rangle = \langle \langle Qf | Qg \rangle \rangle + \langle \langle (I - Q)f | (I - Q)g \rangle \rangle$$

Replacing $g$ by $Qg$, we have

$$\langle \langle f | Qg \rangle \rangle = \langle \langle Qf | Qg \rangle \rangle$$

that is $(I - Q)f \perp Qg$; hence $\ker Q = R(I - Q) \perp R(Q)$.

Conversely if $f \perp R(I - Q)$ then $f - Qf \perp R(I - Q)$ and in particular $f - Qf \perp f - Qf$, i.e. $f = Qf \in R(Q)$. Hence $R(Q) = \ker Q^\perp := \{ f \in L_p(H) \mid f \perp \ker Q \}$ and similarly (exchanging the roles of $Q$ and $I - Q$) we have: ker $Q = R(Q^\perp)$.

If $A$ is a subset of $L_p(H)$ then $A^\perp$ is a closed $L_\infty(\Sigma)$-submodule of $L_p(H)$. In particular the range of any orthogonal projection in $L_p(H)$ is a closed $L_\infty(\Sigma)$-submodule. The converse is true:

\textbf{Lemma 1.2.} If $Z$ is a closed $L_\infty(\Sigma)$-submodule of $L_p(\Omega, \Sigma, \mu; H)$ there exists a unique orthogonal projection $Q_Z$ in $L_p(H)$ with range $Z$.

\textbf{Proof:} Let $(f_\alpha)_{\alpha \in A}$ be a maximal family of pairwise orthogonal non zero elements of $Z$.

For every family $(\varphi_\alpha)_{\alpha \in A}$ in $L_p(\Omega)$ and every finite subset $B$ of $A$ we have:

$$\| \sum_{\alpha \in B} \varphi_\alpha u_{f_\alpha} \|_{L_p(H)} = \| N(\sum_{\alpha \in B} \varphi_\alpha u_{f_\alpha}) \|_p = \| (\sum_{\alpha \in B} v_{\Sigma}(f_\alpha)|\varphi_\alpha|^2)^{1/2} \|_p$$
Proof: For every \( \sum_{\alpha \in A} \varphi_{\alpha} u_{\alpha} \) converges in \( L_p(H) \) iff \( (\sum_{\alpha \in A} 1_{\text{VS}}(f_{\alpha})|\varphi_{\alpha}|^2)^{1/2} \) exists in \( L_p \) and
\[
\| \sum_{\alpha \in A} \varphi_{\alpha} u_{\alpha} \|_{L_p(H)} = \|(\sum_{\alpha \in A} 1_{\text{VS}}(f_{\alpha})|\varphi_{\alpha}|^2)^{1/2}\|_p
\]

If now \( f \in L_p(H) \) and \( B \) is a finite subset of \( A \) we have:
\[
N(\sum_{\alpha \in B} \langle f | u_{\alpha} \rangle u_{\alpha})^2 = \sum_{\alpha \in B} |\langle f | u_{\alpha} \rangle|^2 = \langle f, \sum_{\alpha \in B} \langle f | u_{\alpha} \rangle u_{\alpha} \rangle \leq N(f) N(\sum_{\alpha \in B} \langle f | u_{\alpha} \rangle u_{\alpha})
\]
whence
\[
N(\sum_{\alpha \in B} \langle f | u_{\alpha} \rangle u_{\alpha}) = (\sum_{\alpha \in B} |\langle f | u_{\alpha} \rangle|^2)^{1/2} \leq N(f)
\]
so
\[
(\sum_{\alpha \in A} |\langle f | u_{\alpha} \rangle|^2)^{1/2} \leq N(f)
\]
consequently \( Qf := \sum_{\alpha \in A} \langle f | u_{\alpha} \rangle u_{\alpha} = \sum_{\alpha \in A} E_{f_{\alpha}} f \) converges in \( L_p(H) \) (with \( \|Qf\| \leq \|f\| \)). Since \( R(E_{f_{\alpha}}) \) is the closed \( L_\infty(\Sigma) \)-submodule generated by \( f_{\alpha} \), we have \( R(E_{f_{\alpha}}) \subset Z \) for each \( \alpha \) and consequently \( Qf \in Z \) for every \( f \in L_p(H) \). The map \( Q \) is modular for the action of \( L_\infty(\Omega) \), and clearly \( Qf_{\beta} = f_{\beta} \) for every \( \beta \in A \). It results easily that \( Qf = f \) for every \( f = \sum_{\alpha \in A} \varphi_{\alpha} u_{\alpha} \) (when this series converges), i.e. \( Q \) is a contractive projection in \( L_p(H) \) with range
\[
R(Q) = \{ \sum_{\alpha \in A} \varphi_{\alpha} u_{\alpha} \mid (\sum_{\alpha} |\varphi_{\alpha}|^2)^{1/2} \in L_p(\Omega) \} = \{ \sum_{\alpha \in A} \psi_{\alpha} f_{\alpha} \mid (\sum_{\alpha} |\psi_{\alpha}|^2 N(f_{\alpha})^2)^{1/2} \in L_p(\Omega) \}
\]
Since clearly \( \langle Qf | f_{\alpha} \rangle = \langle f | f_{\alpha} \rangle \) for every \( \alpha \in A \) we have \( (f - Qf) \perp f_{\alpha} \) for every \( \alpha \in A \). By maximality of the system \( (f_{\alpha}) \) we deduce that
\[
f = Qf \text{ for every } f \in Z
\]
so \( R(Q) \) contains \( Z \), hence coincides with \( Z \). Note also that \( f - Qf \perp Z \) for all \( f \in L_p(H) \), and so \( Q \) is orthogonal.

The unicity of the orthogonal projection onto \( Z \) is a consequence of the fact that its image and kernel are uniquely determined (\( R(Q) = Z \) and \( \ker Q = Z^\perp \)). \[\]

2. Preliminary results: the case \( p = 1 \).

**Lemma 2.1.** Let \( P \) be a contractive projection in \( L_1(H) \). Then for every \( f \in R(P) \) we have:
\[
P(E_f) = E_f P(E_f)
\]

**Proof:** For every \( \varphi \in L_1(\Omega) \) with \( 0 \leq \varphi \leq N(f) \) we have:
\[
\|f\| - \|\varphi.u_f\| = \int N(f) \, d\mu - \int N(\varphi.u_f) \, d\mu = \int (N(f) - \varphi) \, d\mu
\]
\[
= \|N(f) - \varphi).u_f\| = \|f - \varphi.u_f\|
\]
\[
\geq \|P(f - \varphi.u_f)\| = \|f - P(\varphi.u_f)\|
\]
\[
\geq \|f\| - \|P(\varphi.u_f)\|
\]
\[
\geq \|f\| - \|\varphi.u_f\|
\]
hence all the inequalities are equalities, and in particular:
\[ \| f - P(\varphi, u_f) \| = \| f \| - \| P(\varphi, u_f) \| \]
that is:
\[ \int N(f - P(\varphi, u_f)) \, d\mu = \int [N(f) - N(P(\varphi, u_f))] \, d\mu \]
which implies
\[ N(f - P(\varphi, u_f)) = N(f) - N(P(\varphi, u_f)) \]
(equality as elements of \( L_1(\Omega) \)). Since \( H \) is strictly convex this implies that
\[ P(\varphi, u_f) = \alpha f \]
for some \( \alpha \in L_+^\infty(\Omega) \). Hence
\[ E f P(\varphi, u_f) = E f (\alpha f) = \alpha f = P(\varphi, u_f) \]
(2.1)
This property has been proved for \( \varphi \in L_1(\Omega) \) with \( 0 \leq \varphi \leq N(f) \); it is extended by linearity and density to every \( \varphi \in L_1(\Omega) \). In particular if we take \( \varphi = \langle h|u_f \rangle \), we obtain:
\[ \forall h \in L_1(H), \quad E f P E f h = P E f h \]
that is \( E f P E f = P E f \). \( \square \)

**Lemma 2.2.** Let \( P \) be a contractive projection in \( L_1(H) \). Then for every \( f, g \in R(P) \) we have: \( E g f \in R(P) \). In other words \( E g P = P E g P \).

**Proof:** We have \((f - E g f) \perp g\), while (by Lemma 2.1) \( E g f - P E g f = E g(f - P E g f) \in L_1(\Omega), u_g \). Hence \((f - E g f) \perp (E g f - P E g f)\). It results that:
\[ N(f - P E g f) = [N(f - E g f)^2 + N((E g f - P E g f)^2)]^{1/2} \geq N(f - E g f) \]
(2.2)
Hence:
\[ \| f - P E g f \| \geq \| f - E g f \|
\[ \quad \geq \| P(f - E g f) \|
\[ \quad = \| f - P E g f \| \]
Hence the inequalities are equalities. In view of (2.2), the equality \( \| f - P E g f \| = \| f - E g f \| \) implies
\[ N(f - P E g f) = [N(f - E g f)^2 + N((E g f - P E g f)^2)]^{1/2} = N(f - E g f) \]
which implies in turn that \( N(E g f - P E g f) = 0 \), that is \( E g f = P E g f \). So \( E g f \in R(P) \). \( \square \)

3. Preliminary results: the case \( p > 1 \).

**Notations.** Let \( p_* \) be the conjugate exponent of \( p \). If \( T : L_p(H) \to L_p(H) \) is a bounded operator, we define its adjoint \( T^* : L_{p_*}(H) \to L_{p_*}(H) \) by
\[ \forall f \in L_{p_*}(H), \forall g \in L_p(H) \quad \langle T^* f, g \rangle = \langle f, T g \rangle \]
where \( \langle \cdot, \cdot \rangle \) denotes the sesquilinear pairing given by eq.(1.1).

If \( f \in L_p(H) \), \( f \neq 0 \) let \( J f \in L_{p_*}(H) \) be the unique norm-one element such that \( \langle f, J f \rangle = \| f \| \). In fact it will be easier to consider the \( (p - 1) \)-homogeneous functional \( J_p(h) = \| h \|^{p-1} J(h) \). We have \( J_p(h) = N(h)^{p-1} u_h = N(h)^{p-2} h \), hence \( J_p \) is random direction preserving. Note that \( p J_p \) is the derivative of the \( p^\text{th} \) power of the norm.
Lemma 3.1. Let $1 < p < \infty$, $p \neq 2$ and $P$ be a contractive projection in $L_p(H)$. Then for every $f, g \in R(P)$ the function $F(f, g) := \text{sgn}(\langle g \mid f \rangle) f + \gamma_p 1_{\{f \perp g\}} N(f) u_g$ belongs to $R(P)$, where $\gamma_p$ is a positive constant depending only on $p$.

Proof: a) Case $2 < p < \infty$.

Recall that since $L_p(H)$ is smooth the duality map $J$ maps $R(P)$ into $R(P^*)$ (see e.g. [DHP], Lemma 4.8); hence $J_p(f + tg) \in R(P^*)$ for every $t \geq 0$. The derivative $\frac{\partial}{\partial t} J_p(f + tg)$ exists at $t = 0$ (since the norm to the power $p$ is twice differentiable) and it belongs to $R(P^*)$ too. We have:

$$\frac{\partial}{\partial t} J_p(f + tg) = N(f + tg)^{p-2} g + \left(\frac{p-2}{2} \frac{\partial}{\partial t} N(f + tg)^2\right) N(f + tg)^{p-4}(f + tg)$$

hence:

$$A(f, g) := \frac{\partial}{\partial t} J_p(f + tg) \big|_{t=0} = N(f)^{p-2} g + (p-2) \text{Re} \left(\langle f \mid g \rangle\right) N(f)^{p-4} f$$

$$= N(f)^{p-2} \left[g + (p-2) \text{Re} \left(\langle u_f, g \rangle\right) u_f\right] \in R(P^*) \quad (3.1)$$

In the complex case, replacing $f$ by $i f$ we obtain

$$B(f, g) := N(f)^{p-2} \left[g - i(p-2) \Im \left(\langle u_f, g \rangle\right) u_f\right] \in R(P^*) \quad (3.1\text{bis})$$

adding:

$$N(f)^{p-2} \left[2g + (p-2) \langle g, u_f \rangle u_f\right] \in R(P^*)$$

With $E_{fg} = \langle g, u_f \rangle u_f$ we obtain

$$N(f)^{p-2} \left[2(g - E_{fg}) + pE_{fg}\right] \in R(P^*) \quad (3.2)$$

In the case of a real space (3.1) is valid without the symbol $\text{Re}$ and we obtain

$$N(f)^{p-2} \left[(g - E_{fg}) + (p-1)E_{fg}\right] \in R(P^*) \quad (3.3)$$

If $h \in R(P^*)$ then $J_p, h = N(h)^{p-1} u_h \in R(P)$, hence if we set $T g = \alpha p(g - E_{fg}) + E_{fg}$, with $\alpha_p = \frac{2}{p}$ in the complex case, $\alpha_p = \frac{1}{p-1}$ in the real case, we obtain:

$$\Phi(g) := N(f)^{(p-2)(p-1)} N(T g)^{(p-1)} u_{T g} \in R(P)$$

Since $T$ is $\Sigma$-modular we have $u_{T(\varphi, h)} = 1\text{supp} \varphi . u_{T h}$ for every $h \in L_p(H)$ and $\varphi \in L_p$, and more generally $u_{T^k(\varphi, h)} = 1\text{supp} \varphi . u_{T^k h}$ for every $k \geq 1$. It is easily deduced that $u_{T^k \Phi(g)} = 1_{\text{Vs}}(f) . u_{T^k+1 g}$ for every $k \geq 0$. Then
\[ u_{\Phi^n(g)} = u_{\Phi(\Phi^{n-1}(g))} = 1 \mathbf{v}_f(u, u_{T\Phi}^{n-1}(g)) \]
\[ = 1 \mathbf{v}_f(u, u_{T\Phi}^{n-2}(g)) = 1 \mathbf{v}_f(u, u_{T\Phi}^{n-2}(g)) \ldots \]
\[ = 1 \mathbf{v}_f(u, u_{T^n g}) \quad (3.4) \]

for every \( n \geq 1 \). If \( E_f g(\omega) \neq 0 \) we have:

\[ u_{T^n g}(\omega) = \frac{\alpha_p^n (g - E_f g)(\omega) + E_f g(\omega)}{N(\alpha_p^n (g - E_f g) + E_f g)(\omega)} \rightarrow \frac{E_f g(\omega)}{N(E_f g(\omega))} = u_{E_f g}(\omega) \quad (3.5) \]

(norm convergence in \( H \)) while if \( E_f g(\omega) = 0 \):

\[ u_{T^n g}(\omega) = \frac{(g - E_f g)(\omega)}{N((g - E_f g)(\omega))} = u_{(g-E_f g)}(\omega) = u_g(\omega) \quad (3.5') \]

Since \( g - E_f g \perp E_f g \) we have \( N(Tg) \leq N(g) \). Hence:

\[ N(\Phi(g)) = N(f)^{2-p_+} N(Tg)^{p_+ - 1} \leq N(f)^{2-p_+} N(g)^{p_+ - 1} \quad (3.6) \]

In particular

\[ N(\Phi(g)) \leq \max(N(f), N(g)) \quad (3.7) \]

Reiterating (3.6) we obtain for every \( n \geq 1 \)

\[ N(\Phi^n(g)) \leq N(f)^{(2-p_+) \sum_{k=0}^{n-1} (p_+ - 1)^k} N(g)^{(p_+ - 1)^n} = N(f)^{1-(p_+ - 1)^n} N(g)^{(p_+ - 1)^n} \]

Since \( 0 < p_+ - 1 < 1 \) we obtain

\[ \lim_{n \to \infty} N(\Phi^n(g)) \leq 1 \mathbf{v}_f(g) N(f) \quad (3.8) \]

We try now to be more precise. If \( E_f g(\omega) = 0 \) we have \( N(Tg)(\omega) = \alpha_p N(g)(\omega) \). Hence

\[ N(\Phi(g))(\omega) = N(f)(\omega)^{2-p_+} (\alpha_p N(g)(\omega))^{p_+ - 1} \quad (3.9) \]

Moreover since in this case \( u_{\Phi^n(g)}(\omega) = u_g(\omega) \) we have \( E_f \Phi^n(g)(\omega) = 0 \) for every \( n \), and we can reiterate. We obtain:

\[ N(\Phi^n(g))(\omega) = (\alpha_p^{p_+ - 1} N(f)(\omega)^{(2-p_+)} \sum_{k=0}^{n-1} (p_+ - 1)^k) N(g)(\omega)^{(p_+ - 1)^n} \]

hence

\[ \lim_{n \to \infty} N(\Phi^n(g))(\omega) = \alpha_p^{(p_+ - 1)/(2-p_+)} \mathbf{v}_f(g)(\omega) N(f)(\omega) = \alpha_p^{1/(p-2)} \mathbf{v}_f(g)(\omega) N(f)(\omega) \quad (3.10) \]

If now \( E_f(g)(\omega) \neq 0 \), we have also \( E_f(\Phi^n(g))(\omega) \neq 0 \) for every \( n \geq 0 \). Set

\[ \beta_n(\omega) = \frac{N(E_f \Phi^n(g))(\omega)}{N(\Phi^n(g))(\omega)} \]

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We have then:
\[ N(T\Phi^n(g))(\omega) \geq \beta_n(\omega)N(\Phi^n(g))(\omega) \]
and consequently:
\[ N(\Phi^{n+1}(g))(\omega) \geq N(f)^2 - \rho_n(\omega)N(\Phi^n(g))(\omega)^{p-1} \]  
(3.11)
On the other hand
\[ \beta_n(\omega) = |\langle u_{\Phi^n(g)}, u_f \rangle(\omega)| = \frac{N(E_f T^n(g))(\omega)}{N(T^n(g))(\omega)} = \frac{N(E_f g)(\omega)}{N(T^n(g))(\omega)} \]
and since \( N(T^n g) = (\alpha_{p_1}^2 + N(g - E_f g)^2 + N(E_f g)^2)^{1/2} \gamma_n N(E_f g) \) pointwise (as \( \alpha_p < 1 \)) we have \( \beta_n(\omega) \not\nearrow 1 \) on the set \( \{ \omega \mid E_f g(\omega) \neq 0 \} \). Reiterating (3.11) from the step \( n = n_0 \) we obtain then
\[ \liminf_{n \to \infty} N(\Phi^n(g))(\omega) \geq (\beta_{n_0}(\omega))^{1/(p-2)} \mathbf{1}_{\Phi_{n_0}(g)}(\omega)N(f)(\omega) \]
and letting \( n_0 \to \infty \), we have since \( \mathbf{V} \mathbf{S}(\Phi_n(g)) = \mathbf{V} \mathbf{S}(g) \cap \mathbf{V} \mathbf{S}(f) \) for every \( n \):
\[ \liminf_{n \to \infty} N(\Phi^n(g))(\omega) \geq \mathbf{1}_{\mathbf{V} \mathbf{S}(g)}(\omega)N(f)(\omega) \]  
(3.12)
From (3.4),(3.5),(3.5)', and (3.10),(3.8),(3.12) we deduce that:
\[ \Phi^n(g) \to N(f)u_{E_f(g)} + \alpha_p^{1/(p-2)} \mathbf{1}_{\{ f_{\perp g} \}} u_g \]  
(3.13)
almost everywhere in \( H \)-norm, hence in \( L_p(H) \)-norm by (3.7) and Lebesgue’s Theorem. Hence the right-hand member of (3.13) belongs to \( R(P) \). Since \( u_{E_f g} = \text{sgn} \langle g \mid f \rangle u_f \) the right member of (3.13) can be written:
\[ \text{sgn} \langle g \mid f \rangle f + \gamma_p \mathbf{1}_{\{ f_{\perp g} \}} N(f) u_g = F_p(f, g) \]  
(3.14)
where we have set \( \gamma_p = \alpha_p^{1/(p-2)} \).

b) Case \( 1 < p < 2 \).

This case is treated by duality. Set \( \gamma_p = \gamma_p^{p_1 - 1} \) and define \( F_p(f, g) \) by the formula (3.14). If \( g = J_p g' \), \( f = J_p h' \) with \( f', g' \in L_{p_1}(H) \) we have
\[ \text{sgn} \langle g \mid f \rangle f = J_{p_1} (\text{sgn} \langle g' \mid f' \rangle f') \]
hence
\[ \text{sgn} \langle g \mid f \rangle f = J_{p_1} (\text{sgn} \langle g' \mid f' \rangle f') \]
and similarly
\[ N(f) = N(J_{p_1} f') = N(f')^{p_1 - 1} \]
hence
\[ N(f) u_g = N(f')^{p_1 - 1} u_g = J_{p_1} (N(f') u_g) \]
finally since \( \{ f \perp g \} = \{ f' \perp g' \} \) and \( J_{p_1} \) is additive on elements with disjoint functional supports, and positively homogeneous of degree \( p_1 - 1 \),
\[ F_p(f, g) = J_{p_1} (F_{p_1}(f', g')) \]
Then since \( f' = J_p f, g' = J_p g \) belong to \( R(P^*) \), the function \( F_p(f', g') \) belongs to \( R(P^*) \) too by the case (a), and \( F_p(f, g) \) belongs to \( R(P) \). \( \square \)
Corollary 3.2. Let $p$ and $P$ be as in Lemma 3.1. Then for every $f, g \in R(P)$ the three elements $\text{sgn}(\langle g \mid f \rangle, f, 1_{\{f \perp g\}}f$ and $1_{\{f \perp g\}}N(f)u_g$ belong to $R(P)$.

Proof: The set $\Lambda$ of scalars $\lambda$ such that the set $\{\omega \in \text{VS}(f) \mid \frac{\langle g \mid f \rangle(\omega)}{\langle f \mid f \rangle(\omega)} = -\lambda\}$ has positive measure is at most countable. This set is also the set of $\lambda$ such that $\{f + \lambda f \perp f\} \cap \text{VS}(f)$ has positive measure. Choose a sequence $(\varepsilon_n)$ of positive numbers not in $\Lambda \cup (-\Lambda)$ which converges to 0. Then by Lemma 3.1

$$\text{sgn}(\langle g \pm \varepsilon_n f \mid f \rangle)f \in R(P)$$

for every $n \geq 1$. Since

$$\text{sgn}(\langle g \pm \varepsilon_n f \mid f \rangle)(\omega) \rightarrow \begin{cases} \text{sgn}(\langle g \mid f \rangle)(\omega) & \text{if } \langle g \mid f \rangle(\omega) \neq 0 \\ \pm 1 & \text{if } \langle g \mid f \rangle(\omega) = 0 \text{ and } f(\omega) \neq 0 \end{cases}$$

we have

$$\text{sgn}(\langle g \mid f \rangle)f \pm 1_{\{f \perp g\}}f = \lim_{n} \text{sgn}(\langle g \pm \varepsilon_n f \mid f \rangle)f \in R(P)$$

and consequently $\text{sgn}(\langle g \mid f \rangle)f$ and $1_{\{f \perp g\}}f$ belong to $R(P)$. Then $F_p(f, g) - \text{sgn}(\langle g \mid f \rangle)f = \gamma_p 1_{\{f \perp g\}}N(f)u_g$ belongs to $R(P)$ too. \[\]

Corollary 3.3. Let $p$ and $P$ be as in Lemma 3.1. Then for every $f, g \in R(P)$ we have $1_{\text{VS}(g)f \in R(P)}$.

Proof: By Cor. 3.2, $h := G(f, g) := 1_{\{f \perp g\}}N(f)u_g$ belongs to $R(P)$. Then $G(h, f) = 1_{\{f \perp g\}}1_{\{u_g \neq 0\}}N(f)u_f = 1_{\text{VS}(g) \cap \{f \perp g\}}f$ belongs to $R(P)$ too. By Cor. 3.2, $f - 1_{\{f \perp g\}}f = 1_{\{f \perp g\}}f \in R(P)$, thus $1_{\text{VS}(g)f} = 1_{\{f \perp g\}}f + 1_{\text{VS}(g) \cap \{f \perp g\}}f \in R(P)$. \[\]

Remark 3.4. In the complex case, for every $f, g \in R(P)$ the elements $\text{sgn}(\text{Re} \langle g \mid f \rangle)f$ and $1_{\{\text{sgn}(\text{Re} \langle g \mid f \rangle) = 0\}}f$ belong to $R(P)$ too. For, $L_p(H)$ is a real Hilbert-valued $L_p(K)$ space, where $K$ is the real vector space $H$ equipped with the scalar product $(x, y)_K = \text{Re}(x \cdot y)_H$. As a consequence, the element $1_{\{\text{Re} \langle g \mid f \rangle > 0\}}f = \frac{1}{2}(\text{sgn} \text{Re} \langle g \mid f \rangle + 1_{\{\text{sgn}(\text{Re} \langle g \mid f \rangle) = 0\}})f$ belongs to $R(P)$.

Lemma 3.5. Let $p$ and $P$ be as in Lemma 3.1. For every $f, g \in R(P)$ denote by $\Sigma_{f,g}$ the $\sigma$-field generated by the element $\langle g \mid f \rangle$. Then for every $\Sigma_{f,g}$-measurable function $\varphi$ such that $\varphi. N(f) \in L_p(\Omega, \Sigma, \mu)$, the element $\varphi.f$ belongs to $R(P)$.

Proof: Since $R(P)$ is a closed linear subspace, it is sufficient to prove this for indicator functions of $\Sigma_{f,g}$-measurable sets. The sigma-algebra $\Sigma_{f,g}$ is generated by the sets $\{\text{Re} \langle g \mid f \rangle > \lambda\}$, $\{-\text{Re} \langle g \mid f \rangle > \lambda\}$, $\{3\langle g \mid f \rangle > \lambda\}$, and $\{-3\langle g \mid f \rangle > \lambda\}$, $\lambda \in \mathbb{R}_+$. If $A_{f,g,\lambda} = \{\text{Re} \langle g \mid f \rangle > \lambda\}$ we have $A_{f,g,\lambda} = \{\text{Re} \langle g - \lambda f \mid f \rangle > 0\}$, hence $1_{A_{f,g,\lambda}}f \in R(P)$ by Rem. 4. The conclusion is the same for the three other kinds of sets (replacing $g$ by $-g$ or $\pm ig$). Now if $1_{B}f \in R(P)$ then $A_{f,g,\lambda} \cap B = A_{f,g,\lambda} \cap f' = 1_{B}f$, hence $1_{A_{f,g,\lambda} \cap B}f = 1_{A_{f,g,\lambda} \cap B}f' \in R(P)$. It results that the class $\mathcal{C}$ of the sets $A \in \Sigma$ such that $1_{A}f \in R(P)$ contains finite intersections of sets of the four preceding types. Since $\mathcal{C}$ is closed by complementation and monotone limits, it contains the sigma-algebra $\Sigma_{f,g}$. \[\]
Corollary 3.6. Let \( p \) and \( P \) be as in Lemma 3.1. For every \( f \in R(P) \) we have \( E_f P = PE_f \).

Proof: Let \( g \in R(P) \). Applying Lemma 3.5 to the function \( \varphi = \frac{\langle g, f \rangle}{\langle f, f \rangle} \) we obtain that \( E_f g \in R(P) \). Hence for every \( h \in L_p(H) \), we have \( E_f Ph \in R(P) \), i.e. \( E_f Ph = PE_f Ph \); thus \( E_f P = PE_f P \). Similarly, reasoning with the contractive projection \( P^* \) in \( L_p(H) \), and the element \( J_p f \) of \( R(P^*) \), we have \( E_{J_p f} P^* = P^* E_{J_p f} P^* \). Dualizing we obtain \( PE_{J_p f} = PE_{J_p f}^* P \). We claim that \( E_{J_p f}^* = E_{J_p f} \). This will show that \( PE_f = PE_{J_p f} P = E_f P \). Let us show this claim. Since \( u_{J_p f} = u_f \), we have for every \( g \in L_p(H) \) and \( h' \in L_p(H) \):

\[
\langle E_f g, h' \rangle = \int \langle E_f g, h' \rangle \, d\mu = \int \langle g, u_f \rangle u_f, h' \rangle \, d\mu = \int \langle g, \langle h', u_f \rangle u_f \rangle \, d\mu = \int \langle g, E_{J_p f} h' \rangle \, d\mu = \langle g, E_{J_p f} h' \rangle \]

Remark: The preceding proof of Cor. 3.6 is essentially a real one. In the complex case it can be replaced by a shorter one, of more algebraic nature, due to Arazy and Friedman in the context of spaces \( C_p \) (see [AF]). It seemed interesting to us to reproduce this proof in the Annex (see §6), after simplifying it considerably by eliminating the unnecessary non-commutative apparatus.

4. The range of a contractive projection

This section is devoted to the proof of Thm. 0.1, which consists in four lemmas.

Lemma 4.1. The closed \( L_\infty(\Sigma) \)-module \( Z \) generated by \( R(P) \) in \( L_p(H) \) is generated (as \( L_\infty \)-module) by a family \( (f_\alpha)_{\alpha \in A} \) of pairwise orthogonal elements of \( R(P) \). We have in fact a Schauder (orthogonal) decomposition

\[ Z = \bigoplus_{\alpha \in A} L_p(\Omega).u_{f_\alpha} \]

Proof: Let \( (f_\alpha)_{\alpha \in A} \) be a maximal family of pairwise orthogonal non zero elements of \( R(P) \) and \( Z_0 \) be the closed \( L_\infty(\Sigma) \)-submodule generated by the family \( (f_\alpha)_{\alpha \in A} \). Let \( QZ_0 \) be the orthogonal projection onto \( Z_0 \). By the proof of Lemma 1.2 we know that \( QZ_0 = \sum_{\alpha \in A} E_{f_\alpha} \) (convergence in strong operator topology) hence by Lemma 2.2 if \( p = 1 \), resp. Cor. 3.6 if \( p > 1 \), \( QZ_0 f \in R(P) \) for every \( f \in R(P) \). Since \( QZ_0 \) is orthogonal and \( f_\alpha \in R(QZ_0) \) we have \( (f - QZ_0 f) \perp f_\alpha \) for every \( \alpha \in A \). By maximality of the system \( (f_\alpha) \) we deduce that

\[ f = QZ_0 f \quad \text{for every } f \in R(P) \]

i.e. \( QZ_0 P = P \). Then \( Z_0 = R(QZ_0) \) is a closed \( L_\infty \)-module containing \( R(P) \) and generated by a subset of \( R(P) \); hence it coincides with the closed \( L_\infty \)-module generated by \( R(P) \). \[ \square \]

Lemma 4.2. There exists a sub-\( \sigma \)-algebra \( \mathcal{F} \) of \( \Sigma \) containing the vectorial function supports of all elements of \( R(P) \) such that for every \( f \in R(P) \) and \( \varphi \in L_p(\Omega, \Sigma, \mu) \), the product \( \varphi.u_f \)
belongs to $R(P)$ iff $1_{\mathbf{VS}(f)}N(f)^{-1}\varphi$ is $\mathcal{F}$-measurable. In particular $R(P)$ is a $L_\infty(\Omega,\mathcal{F},\mu)$-submodule.

**Proof:** Since $PE_f = E_fPE_f$ by Lemma 2.1 (if $p = 1$) or by Cor. 3.6 (if $p > 1$), we have $P(\varphi,u_f) \in L_p(\Omega).u_f$ for every $f \in R(P)$ and $\varphi \in L_p(\Omega,\Sigma,\mu)$. We may write $P(\varphi.u_f) = (P_f\varphi).u_f$, with $\text{Supp}(\tilde{P}_f) \subset \mathbf{VS}(f)$. Clearly $\tilde{P}_f$ is linear, $\tilde{P}_f^2 = \tilde{P}_f$ and

$$\|\tilde{P}_f\varphi\|_p = \|P(\varphi,u_f)\| \leq \|\varphi.u_f\| \leq \|\varphi\|_p$$

hence $\tilde{P}_f$ is a contraction map in $L_p(\Omega,\Sigma,\mu)$. Moreover $\tilde{P}_f(N(f)) = N(f)$ and $\tilde{P}_f\psi = 0$ for every $\psi \in L_p(\Omega,\Sigma,\mu)$ disjoint from $N(f)$.

It results from Douglas’ theorem (in case $p = 1$) or Andô’s theorem (in case $p > 1$) that $\tilde{P}_f$ is positive and

$$\tilde{P}_f(\varphi) = N(f)\mathbf{E}^{f}_\varphi\left(\frac{1_{\text{Supp}(N(f))}\varphi}{N(f)}\right)$$

where $\mathbf{E}^{f}_\varphi$ is the conditional expectation with respect to some subalgebra $\mathcal{F}_f$ of $\Omega$ containing $\mathbf{VS}(f)$ and to the measure $\nu_f = N(f)^p d\mu$. (We may assume that $\Omega \setminus \mathbf{VS}(f)$ is an atom of $\mathcal{F}_f$). In particular $L_p(\Omega,\Sigma,\mu).u_f \cap R(P) = L_p(\Omega,\mathcal{F}_f,u_f).f$ is a $L_\infty(\Omega,\mathcal{F}_f,\mu)$-module.

Let us denote $\mathbf{E}^{f}_\varphi = \mathbf{E}^{f}_\varphi(1_{\mathbf{VS}(f)}\varphi)$, we have then $P(\psi.f) = \mathbf{E}^{f}_\psi(\psi.f)$ for every $\psi \in L_\infty(\Omega,\Sigma,\mu)$. Let now $f,g \in R(P)$. If $g = h.u_f$ with $h \in L^p(\Omega)$ then $\frac{h}{\mathbf{N}(f)}$ is $\mathcal{F}_f$-measurable and for every $\varphi \in L_\infty(\Omega,\Sigma,\mu)$ we have:

$$\mathbf{E}^{g}_\varphi.f = P(\varphi.h.u_f) = N(f)\mathbf{E}^{f}_\varphi\left(\frac{\varphi.h}{N(f)}\right).u_f = h\mathbf{E}^{f}_\varphi(\varphi.u_f) = \mathbf{E}^{f}_\varphi(\varphi).g$$

hence

$$\mathbf{E}^{g}_\varphi = 1_{\mathbf{VS}(g)}\mathbf{E}^{f}_\varphi = 1_{\text{Supp}\mathbf{E}^{f}_\varphi}$$

(4.0)

For every $\varphi \in L_\infty(\Omega)$ the equation

$$P(\varphi.(f + g)) = P(\varphi.f) + P(\varphi.g)$$

is equivalent to

$$\mathbf{E}^{f + g}_\varphi.(f + g) = \mathbf{E}^{f}_\varphi(\varphi.f) + \mathbf{E}^{g}_\varphi$$

(4.1)

Let $g = h.u_f + g'$ be the orthogonal decomposition, i.e. $h = \langle g \mid u_f \rangle$ and $g' \perp f$. Note that $h.u_f = E_f g \in R(P)$. Set $A = \mathbf{VS}(f)$, $B = \mathbf{VS}(g)$ and $B' = \mathbf{VS}(g')$. Taking the images of both sides of (4.1) by the orthogonal projection $I - E_f$ we obtain:

$$\mathbf{E}^{f + g}_\varphi.g' = \mathbf{E}^{g}_\varphi.g'$$

hence $1_B\mathbf{E}^{f + g}_\varphi = 1_B\mathbf{E}^{g}_\varphi$. Then by (4.1) again, $1_B\mathbf{E}^{f + g}_\varphi f = 1_B\mathbf{E}^{f}_\varphi f$ and finally

$$1_{A \cap B'}\mathbf{E}^{f + g}_\varphi = 1_{A \cap B'}\mathbf{E}^{f}_\varphi = 1_{A \cap B'}\mathbf{E}^{g}_\varphi$$

(4.2)

On the other hand similarly to (4.1) we have:

$$\mathbf{E}^{h.u_f - g}_\varphi.(h.u_f - g) = \mathbf{E}^{h.u_f}_\varphi.h.u_f - \mathbf{E}^{g}_\varphi.g$$

(4.3)
Replacing

When

a) First we claim that for every

Proof:

\[ E_{\text{Supp}}N \]

every

\[ f \in \mathcal{F} \]

belongs to

There is a weight

Lemma 4.3.

since

\[ N \]

supports

1

\[ E \]

and since

\[ \phi \]

Lemma follows.

that is

\[ \langle \langle \text{Supp} f \rangle \rangle \]

\[ u \]

\[ \phi \]

\[ A \]

\[ \epsilon \]

\[ \text{VS} \]

is a contraction in

\[ f \]

\[ \Omega \]

\[ \Sigma \]

\[ \mu \]

\[ M \]

\[ \chi \]

\[ \mathcal{F}_f \]

as is easily seen by treating separately the cases

\[ A \subset \text{VS} (g) \]

and

\[ A = \Omega \setminus \text{VS} (g) . \]

Let \( \mathcal{F} \) be the \( \sigma \)-algebra consisting of

sets \( A \in \Sigma \)

such that

\[ A \cap \text{VS} (f) \]

belongs to \( \mathcal{F}_f \) for every

\[ f \in \mathcal{R}(P) . \]

Then for every

\[ f \in \mathcal{R}(P) \]

and

\[ \varphi \in L_0(\Omega, \Sigma, \mu) . \]

the function

\[ \text{1}_{\text{VS} (f) \cap \text{VS} (g)} \varphi \]

is \( \mathcal{F} \) measurable iff it is \( \mathcal{F}_f \)-measurable, and the

Lemma follows.

Lemma 4.3. There is a weight

\[ w \in L_0(\Omega, \Sigma, \mu) \]

with support \( \text{VS} (\mathcal{R}(P)) \)

such that for
every

\[ f \in \mathcal{R}(P), w^{-1}N(f) \]

is \( \mathcal{F} \)-measurable.

Proof: a) First we claim that for every

\[ f, g \in \mathcal{R}(P) \]

then

\[ \text{1}_{\text{VS} (f) \cap \text{VS} (g)} \]

is \( \mathcal{F} \)-measurable. Since

\[ E_{fg} = \langle \langle g \mid u_f \rangle \rangle u_f \in \mathcal{R}(P) \]

by Lemma 2.2, it results from Lemma 4.2 that

\[ N(f)^{-1} \}

\[ \langle \langle g \mid u_f \rangle \rangle \]

\[ N(f)^{-2} \}

\[ \langle \langle g \mid f \rangle \rangle \]

\[ N(f)^{-1} \}

is \( \mathcal{F} \)-measurable; hence its absolute value

\[ N(f)^{-2} \}

\[ \langle \langle g \mid f \rangle \rangle \]

\[ \mathcal{F} \]

measurable, and similarly

\[ N(g)^{-2} \}

\[ \langle \langle f \mid g \rangle \rangle \]

\[ \mathcal{F} \]

measurable too. Then the ratio of these functions, that is

\[ \text{1}_{\text{Supp}} \}

\[ \langle \langle g \mid f \rangle \rangle \}

\[ N(f)^{-2} \}

\[ \mathcal{F} \]

measurable, and so is its square root

\[ \text{1}_{\text{Supp}} \}

\[ \langle \langle g \mid f \rangle \rangle \]

\[ N(g)N(f)^{-1} \}

Replacing \( g \) by

\[ g_{\epsilon} = g + \varepsilon f, \]

\[ \varepsilon > 0 \]

we obtain that

\[ \text{1}_{\text{Supp}} \}

\[ \langle \langle g_{\epsilon} \mid f \rangle \rangle \]

\[ N(g_{\epsilon})N(f)^{-1} \}

is \( \mathcal{F} \)

measurable. When

\[ \varepsilon \to 0 \]

we have

\[ g_{\epsilon} \to g, \]

\[ N(g_{\epsilon}) \to N(g) \]

(in \( L_p \)-norm) and

\[ \text{Supp} \}

\[ \langle \langle g_{\epsilon} \mid f \rangle \rangle \to \text{Supp} \}

\[ N(f) \]

\[ \text{VS} (f) \]

(in probability). At the limit

\[ \text{1}_{\text{VS} (f) \cap \text{VS} (g)} \]

is \( \mathcal{F} \)-measurable.

b) Let \( (f_i)_{i \in I} \)

be a maximal family in \( \mathcal{R}(P) \)

with pairwise almost disjoint functional supports \( \text{VS} (f_i) \).

Then

\[ \text{VS} (\mathcal{R}(P)) = \bigvee_{i \in I} \text{VS} (f_i) ; \]

if \( f \in \mathcal{R}(P) \)

then since

\[ S = \bigvee_{i \in I} \text{VS} (f_i) \]

belongs to \( \mathcal{F} \), so does its complementary set

\[ S^c \]

and thus

\[ \text{1}_{S^c} f \in \mathcal{R}(P) ; \]

then by maximality
of the family \((f_i)\) we have \(1_{S^r}f = 0\), that is \(f = 1_S f\). We set \(w = \sum_{i \in I} N(f_i)\) (which converges in \(L_0(\Omega, \Sigma, \mu)\)): this is a \(\Sigma\)-measurable weight with support \(\mathbf{VS}(R(P))\). For every \(f \in R(P)\) and every \(i \in I\), \(1_{\mathbf{VS}(f_i)}w^{-1}N(f) = 1_{\mathbf{VS}(f_i)}N(f_i)^{-1}N(f)\) is \(F\)-measurable; hence \(w^{-1}N(f) = \sum_{i \in I} 1_{\mathbf{VS}(f_i)}w^{-1}N(f)\) is \(F\)-measurable.

We can now give the proof of the Th 0.1. For, consider the new measure \(\nu = w^\mu\), which has support \(\Omega_p = \mathbf{VS}(R(P))\) and set \(T : L_p(\Omega_p, \Sigma_p, \mu) \to L_p(\Omega_p, \Sigma_p, \nu)\), defined by \(Tf = w^{-1}f\) (we denote by \(\Sigma_p\) the trace of \(\Sigma\) on \(\Omega_p\)). Then \(T\) is an isometry; \(Y := (T \otimes \mathbb{I})_!(R(P))\) is a \(L_\infty(F_p)\)-module isometric to \(R(P)\) and for every \(f \in Y\) its new random norm \(\tilde{N}(f) = w^{-1}N(f)\) belongs to \(L_p(\Omega_p, F_p, \nu)\). It results from an argument in [LR] that \(Y\) is isometric to \(\bigoplus_{i \in I} L_p(\Omega_i, F\{\Omega_i, \nu|\Omega_i, H_i\})\) \(\ell_p\), for some families \((\Omega_i)\) of pairwise almost disjoint sets in \(F\) and \((H_i)\) of Hilbert spaces. Set then \(\tilde{w}_i = (\mathbb{E}(1_{\Omega_i}, w^\mu | F))^{1/p}\), and define an isometry \(S_i : L_p(\Omega_i, F\{\Omega_i, \nu|\Omega_i, H_i\}) \to L_p(\Omega_i, F\{\Omega_i, \mu|\Omega_i\})\) by \(S_i f = \tilde{w}_i f\). Then each \(S_i \otimes \mathbb{I}_{H_i}\) is an onto isometry \(L_p(\Omega_i, F\{\Omega_i, \nu|\Omega_i, H_i\}) \to L_p(\Omega_i, F\{\Omega_i, \mu|\Omega_i, H_i\})\); the collection of these isometries induces an isometry of the corresponding \(\ell_p\)-direct sums. The proof of Th. 0.1 is complete.

Let us finally adapt to the present situation the argument of [LR] for the commodity of the reader (and for further reference in section 5).

**Lemma 4.4.** Let \((\Omega, \Sigma, \nu)\) be a localizable measure space, \(F\) be a sub-sigma algebra such that \((\Omega, F, \nu)\) is still localizable and \(H\) be a Hilbert space. Let \(Y\) be a closed \(L_\infty(F)\)-submodule of \(L_p(\Omega, \Sigma, \nu; H)\) such that for every \(f \in Y\) its random norm \(N(f)\) is \(F\)-measurable. Then there exist a family \((\Omega_i)_{i \in I}\) of pairwise almost disjoint members of \(F\), a family \((H_i)\) of Hilbert spaces (of lower Hilbertian dimension than \(H\)) and a random norm preserving isometry from \(Y\) onto \(\bigoplus_{i \in I} L_p(\Omega_i, F\{\Omega_i, \nu|\Omega_i, H_i\})\) \(\ell_p\).

**Proof:** Note that by an elementary polarization argument all the scalar products \(\langle f, g \rangle\), \(f, g \in Y\) are \(F\)-measurable. Hence for every \(f \in Y\), the projection \(E_f\) restricts to a projection from \(Y\) onto \(L_p(\Omega, F)\)\(u_f\). It results that for every closed \(L_\infty(F)\)-submodule \(Z\) of \(Y\) there is an orthogonal projection from \(Y\) onto \(Z\) (which is the restriction of the orthogonal projection from \(L_p(\Omega, \Sigma; H)\) onto the closed \(L_\infty(\Sigma)\)-submodule generated by \(Z\)). In particular \(Y = Z \oplus (Z^\perp \cap Y)\).

Remark that if \(A \in F\) is \(\nu\)-sigma-finite and \(M \subseteq Y\) is a closed \(L_\infty(F)\)-submodule such that \(\mathbf{VS}(M) \supset A\) then there exists \(g \in M\) such that \(\mathbf{VS}(g) = A\): take a maximal family \((g_n)\) in \(M\) of norm-one elements with almost disjoint functional supports included in \(A\); this family is necessarily at most countable and \(\bigvee_n \mathbf{VS}(g_n) = A\); then set \(g = \sum_n 2^{-n}g_n\).

Now we claim that for every \(A \in F\), \(A \subseteq \mathbf{VS}(Y)\) with positive measure, there exists a \(F\)-measurable subset \(B\) of \(A\) of positive measure and a family of pairwise orthogonal element \((f_\gamma)_{\alpha \in \Gamma}\), such that \(\mathbf{VS}(f_\gamma) = B\) for every \(\gamma \in \Gamma\), which generates \(1_B Y\) as closed \(L_\infty(F)\)-submodule. For, let \(A' \subseteq A\) be a sigma-finite \(F\)-measurable subset with positive measure, and \((g_\gamma)_{\gamma \in \Gamma}\) be a maximal family of pairwise orthogonal elements of \(Y\) with \(\mathbf{VS}(g_\gamma) = A'\). If this family generates \(1_{A'} Y\) as closed \(L_\infty(F)\)-submodule we can take \(B = A'\). If not, consider the set \(M = \{ f \in Y \mid f \perp g_\gamma, \forall \gamma \in \Gamma\}\). Then \(M\) is a closed \(L_\infty(F)\)-submodule of \(Y\), and \(\mathbf{VS}(M) \not\supset A'\) by the maximality of \((g_\gamma)_{\gamma \in \Gamma}\) (and the preceding remark). Let \(B = A' \setminus \mathbf{VS}(M)\), then \((1_B g_\gamma)_{\gamma \in \Gamma}\) is a maximal family in \(1_B Y\) of nonzero, pairwise orthogonal elements of \(1_B Y\). Consequently it generates \(1_B Y\) as \(L_\infty(F)\)-submodule, and moreover \(\mathbf{VS}(1_B g_\gamma) = B\) for every \(\gamma \in \Gamma\).

Let now \((\Omega_i)_{i \in I}\) be a maximal family of \(F\)-measurable almost disjoint subsets of \(\mathbf{VS}(Y)\)
of positive measure, such that there exists for each \( i \in I \) a family \((f^i_\gamma)_{\gamma \in \Gamma_i}\) of pairwise orthogonal elements with \(\text{VS}(f^i_\gamma) = \Omega_i\) for every \( \gamma \in \Gamma_i \), which generates \(1_{\Omega_i}Y\) as closed \(L_\infty(\mathcal{F})\)-submodule. By the claim, we have \(\bigvee_{i \in I} \Omega_i = \text{VS}(Y)\). Every \( f \in 1_{\Omega_i}Y \) can be written \( f = \sum_{\gamma \in \Gamma_i} \varphi_\gamma f^i_\gamma \) with \( \varphi_\gamma \in L_0((\Omega_i, \mathcal{F}_{\Omega_i}, \nu) \); then \( N(f) = (\sum_{\gamma \in \Gamma_i} |\varphi_\gamma|^2 N(f^i_\gamma)^2)^{1/2} \in L_p(\Omega_i, \mathcal{F}_{\Omega_i}, \nu)\).

Note that, by refining if necessary the “partition” \((\Omega_i)\) we may suppose that each \( \Omega_i \) has finite \( \nu \)-measure. Then, replacing each \( f^i_\gamma \) by \( u f^i_\gamma = N(f^i_\gamma)^{-1} f^i_\gamma \), we may assume that \( N(f^i_\gamma) = 1_{\Omega_i} \). We have then \( N(f) = (\sum_{\gamma \in \Gamma_i} |\varphi_\gamma|^2)^{1/2} \) for each \( f = \sum_{\gamma \in \Gamma_i} \varphi_\gamma f^i_\gamma \) in \( 1_{\Omega_i}Y \). Let \( \mathcal{H}_i = \ell^2(\Gamma_i) \). Then \( T_i : L_p(\Omega_i, \mathcal{F}_{\Omega_i}, \nu; \mathcal{H}_i) \to 1_{\Omega_i}Y \), \( \sum_{\gamma \in \Gamma_i} \varphi_\gamma e_\gamma \mapsto \sum_{\gamma \in \Gamma_i} \varphi_\gamma u f^i_\gamma \) is an (onto) isometry (preserving the random norm), and finally \( Y \) is isometric to \((\bigoplus_{i \in I} L_p(\Omega_i; \mathcal{H}_i))_\ell_p\).

For proving the assertion about Hilbertian dimension of \( \mathcal{H}_i \), suppose that for some \( i \in I \), the hilbertian dimension \( d_H \) is strictly smaller than that, \( d_{H_i} \), of \( H_i \). We distinguish two cases:

- if \( H \) is finite dimensional: select a finite subset \( \Gamma'_i \) of \( \Gamma_i \) with cardinality \( d_H + 1 \); since \( \langle f^i_\gamma | f^i_\delta \rangle = 0 \) for every \( \gamma \neq \delta \in \Gamma_i \), there exists \( \omega \in \Omega \) such that \( \langle f^i_\gamma | f^i_\delta \rangle(\omega) = 0 \), i.e. the vectors \( f^i_\gamma(\omega), \gamma \in \Gamma'_i \) of \( H \) are pairwise orthogonal: a contradiction.

- if \( H \) is infinite dimensional: for every \( x \in H \) the set \( \{ \gamma \in \Gamma_i \mid \langle f^i_\gamma | 1_{\Omega_i}x \rangle \neq 0 \} \) is at most countable (since \( \sum_{\gamma} |\langle f^i_\gamma | 1_{\Omega_i}x \rangle|^2 \leq N(1_{\Omega_i}, x)^2 = 1_{\Omega_i} \|x\|^2 \)\), hence if \( D \) is a dense set in \( H \) of cardinality \( d_H \), the set \( \{ \gamma \in \Gamma_i \mid \exists x \in D, \langle f^i_\gamma | 1_{\Omega_i}x \rangle \neq 0 \} \) has cardinality \( d_H < d_{H_i} = \#\Gamma_i \). Hence there exists some \( \gamma \in \Gamma_i \) such that \( f^i_\gamma \perp 1_{\Omega_i}x \) for every \( x \in D \), and thus for every \( x \in H \), which means \( f^i_\gamma = 0 \), a contradiction. \( \square \)

**Remark 4.5:** The final argument in the proof of Lemma 4.4 shows indeed that \( L_p(\Omega, \Sigma, \nu; \mathcal{H}) \) embeds in \( L_p(\Omega, \Sigma, \nu; H) \) by a modularly isometric map then \( \dim \mathcal{H} \leq \dim H \).

**Remark 4.6:** In a forthcoming paper ([HR]) it will be proved that contractively complemented sublattices of \( L_p(L_q) \) are isometric to “abstract \( L_p(L_q) \) spaces”, i.e. bands in \( L_p(L_q) \) spaces. Let us show how this permits to deduce shortly the essence of Th. 0.1 from Lemma 4.1.

As in the proof of Lemma 4.1 let \((f_\alpha)_{\alpha \in \mathcal{A}}\) be a maximal family of non zero, pairwise orthogonal elements of \( R(P) \) and \( Z = \bigoplus_\alpha L_p(\Omega, \Sigma, \mu).u f_\alpha \) be the closed \( L_\infty(\Sigma)\)-submodule generated by \( R(P) \). There is clearly a \( \Sigma \)-modular isometry \( U \) from the closed submodule \( Z \) onto a band \( Y \) of the Banach lattice \( L_p(\Omega, \Sigma, \mu; \mathcal{H}) \) where \( \mathcal{H} \) is the discrete Banach lattice \( \ell_2(\mathcal{A}) \), such that \( U e_\alpha = N(f_\alpha)e_\alpha \), where \( (e_\alpha)_{\alpha \in \mathcal{A}} \) is a Hilbertian basis of \( \mathcal{H} \). Then \( P | Z \) is similar by \( U \) to a contractive projection \( \hat{P} \) of \( Y \) which preserves the spaces \( Y_\alpha = L_p(A_\alpha).e_\alpha \) (where \( A_\alpha = \text{Supp} N(f_\alpha) \)) by Lemma 2.1 if \( p = 1 \) and Cor. 3.6 if \( p > 1 \), as well as the elements \( N(f_\alpha).e_\alpha \). By the classical (scalar) theorem of Douglas if \( p = 1 \), Andô if \( p > 1 \), \( \hat{P} \mid Y_\alpha \) is positive and its image is a sublattice of \( Y_\alpha \). Since \( Y = \bigoplus_\alpha Y_\alpha \) is a decomposition in disjoint subbands, \( \hat{P} \) is itself positive and its range is a sublattice of \( Y \), hence of \( L_p(\mathcal{H}) \). By the analysis of contractive projections on sublattices in \( L_p(L_q) \)-spaces developed in [HR], the range \( R(\hat{P}) \) is an abstract \( L_p(L_2) \)-space, hence by [LR] it is Banach-isometric to a \( \ell_p \)-direct sum \( \bigoplus_{i \in I} L_p(\Omega_i, H_i) \), where the \( H_i \) are Hilbert spaces. \( \square \)
5. Structure of the contractive projections

**Theorem 5.1.** Let $1 \leq p < \infty$, $p \neq 2$. For every contractive projection $P$ of $L_p(H)$ there exist a family $(u_\gamma)_{\gamma \in \Gamma}$ of pairwise orthogonal elements of $L_\infty(H)$, a positive contractive projection $\tilde{P}$ of $L_p(\Omega)$ and, if $p=1$, a contractive linear operator $V: L_1(H) \rightarrow L_1(H)$ verifying 

$$\ker V \supset 1_A L_1(H)$$

where $A = \bigvee_{\gamma \in \Gamma} \text{VS}(u_\gamma)$, and $R(V) \subset \sum_{\gamma} R(\tilde{P}).u_\gamma$, such that:

$$Pf = \begin{cases} 
\sum_{\gamma} \tilde{P}(\langle f, u_\gamma \rangle)u_\gamma & \text{if } p \neq 1 \\
\sum_{\gamma} \tilde{P}(\langle f, u_\gamma \rangle)u_\gamma + V(f) & \text{if } p = 1
\end{cases} \quad (5.1)$$

for every $f \in L_p(H)$.

Conversely for every family $(u_\gamma)_{\gamma \in \Gamma}$ of pairwise orthogonal elements of $L_\infty(H)$, every positive contractive projection $\tilde{P}$ of $L_p(\Omega)$ [and every linear contraction $V$ of $L_1(H)$ satisfying the previous conditions of kernel and range in the case $p = 1$], the formula $(5.1)$ defines a contractive projection $P$ of $L_p(H)$.

Moreover if $p \neq 1$ the inequality $N(Pf) \leq \tilde{P}(N(f))$ holds for every $f \in L_p(H)$ [this happens also for a contractive projection of $L_1(H)$ for which the operator $V$ of formula $(5.1)$ is zero]. ($\tilde{P}$ is a “majorizing $L_p$-contraction” for $P$ in the terminology of [GR]).

The proof of Th. 5.1 will require the two following Lemmas, the first of which is specific to the $p = 1$ case:

**Lemma 5.2.** Let $P$ be a contractive projection in $L_1(H)$. Then $Pf = 0$ for every $f \in L_1(H)$ with $\text{VS}(f) \subset \text{VS}(R(P))$ and $f \perp R(P)$.

**Proof:** Assume that $f \perp R(P)$ and $\text{VS}(f) \subset \text{VS}(h)$ for some $h \in R(P)$. Then $g := Pf + 1_{\text{VS}(Pf)}h$ belongs to $R(P)$ and $\text{VS}(g) \supset \text{VS}(f) \cup \text{VS}(Pf)$. We have for every $t > 0$:

$$\int (N(g)^2 + t^2 N(f)^2)^{1/2} d\mu = \|g + tf\|
\geq \|P(g + tf)\| = \|g + tPf\|
= (1 + t)\|Pf\| + \|1_{\text{VS}(Pf)}h\|
= \|g\| + t\|Pf\|$$

Hence:

$$\|Pf\| \leq \lim_{t \to 0} \left( \frac{\|g + tf\| - \|g\|}{t} \right) = \lim_{t \to 0} \int \frac{N(g)^2 + t^2 N(f)^2)^{1/2} - N(g)}{t} d\mu = 0$$

**Lemma 5.3.** Let $P$ be a contractive projection in $L_p(H)$. There exists a positive contractive projection $\tilde{P}$ on $L_p(\Omega, \Sigma, \mu)$ such that $P(\varphi.u_f) = (\tilde{P}\varphi).u_f$ for every $f \in R(P)$ and $\varphi \in L_p(\Omega, \Sigma, \mu)$.

**Proof:** Let $F$ be the $\sigma$-algebra of Lemma 4.2 and $w$ be the weight of Lemma 4.3. Define $\tilde{P}_f(\varphi)$ as in the proof of Lemma 4.2. Recall that for every $f \in R(P)$ the function $w^{-1}N(f)$
Let $\tilde{P}$ be the orthogonal projection from $L_p(H)$ onto the closed submodule generated by $R(P)$. It results from Lemma 5.2 that $P = \Pi$ and $P(\varphi, u_f) = P(\varphi, u_f)$ for every $f \in R(P)$ and $\varphi \in L_p(\Omega, \Sigma, \mu)$. Hence $P$ is a positive contractive projection in $L_p(\Omega, \Sigma, \mu)$ and $P(\varphi, u_f) = P(\varphi, u_f) = P(\varphi, u_f)$. Then $P$ is a positive contractive projection in $L_p(\Omega, \Sigma, \mu)$ and $P(\varphi, u_f) = P(\varphi, u_f)$. Hence $P = \Pi$. 

Let us express now $PE_f$ when $f \in R(P)$. If $\tilde{P}$ is the positive projection in $L_p(\Omega, \Sigma, \mu)$ defined in Lemma 5.3 we have for every $g \in L_p(\Omega, \Sigma, \mu; H)$

$$PE_f g = P(\langle g | u_f \rangle).u_f = P(\langle g | u_f \rangle).u_f$$

The formula (5.1) in Thm. 5.1 is now clear if we set $u_\gamma = u_{f_\gamma}$.

Conversely given $(u_\gamma)$, $\tilde{P}$ and $V$, let us prove first that $P$ is a contraction. We have for every finite subset $G$ of $\Gamma$ (using the positivity of $\tilde{P}$):

$$N \left( \sum_{\gamma \in G} \tilde{P}(\langle f | u_\gamma \rangle) u_\gamma \right) = \left( \sum_{\gamma \in G} |\tilde{P}(\langle f | u_\gamma \rangle)|^2 \right)^{1/2}$$

$$\leq \tilde{P} \left( \sum_{\gamma \in G} |a_\gamma|^2 \right)^{1/2}$$

$$\leq \tilde{P} \left( \sum_{\gamma \in G} |\langle f | u_\gamma \rangle|^2 \right)^{1/2}$$

$$= \tilde{P} \left( \sum_{\gamma \in G} |\langle f | u_\gamma \rangle|^2 \right)^{1/2}$$
For every contractive projection $P$ of $L_p(\Omega, \Sigma, \mu; H)$ $(1 \leq p < \infty, p \neq 2)$ there exist:

- a modularly isometric automorphism $W$ of $L_p(H)$;
- a family $(\Omega_i)_{i \in I}$ of pairwise almost disjoint $\Sigma$-measurable subsets of $\Omega$ of positive measure;
- a family $(\mathcal{H}_i)_{i \in I}$ of Hilbert spaces;
- for every $i \in I$ a (strong operator) measurable family $(U_{i, \omega})_{\omega \in \Omega}$ of isometric embeddings of $\mathcal{H}_i$ into $H$;
- a positive contractive projection $\tilde{P}$ of $L_p(\Omega, \Sigma, \mu)$ commuting with the band projections associated with the sets $\Omega_i$;
- and if $p = 1$ a contraction $V$ from $L_1(S, \Sigma|_S, \mu|_S; H)$ into $R(P)$, where $S = \Omega \setminus \bigcup_i \Omega_i$ such that (setting $V = 0$ if $p > 1$):

$$P = WU(\sum_i \tilde{P}M_{\Omega_i} \otimes \text{Id}_{\mathcal{H}_i})U^*W^{-1} + V$$

where $M_{\Omega}$ denotes the multiplication operator by the characteristic function $1_{\Omega_i}$; $U$ is the modularly isometric embedding of $\bigoplus L_p(\Omega_i, \Sigma|_{\Omega_i}, \mu|_{\Omega_i}; \mathcal{H}_i)$ into $L_p(H)$ naturally associated with the family $(U_{i, \omega})_{i \in I, \omega \in \Omega}$ by mean of the formula:

$$(Uf)(\omega) = U_{i, \omega}(f(\omega)) \quad \text{when } \omega \in \Omega_i$$

and similarly $U^*: L_p(H) \to \bigoplus L_p(\Omega_i, \Sigma|_{\Omega_i}, \mu|_{\Omega_i}; \mathcal{H}_i)$ is the modularly contractive map associated with the family $(U_{i, \omega}^*)_{i \in I, \omega \in \Omega}$.

**Remark 5.5.** In fact the families $(U_{i, \omega})_{\omega \in \Omega}$ may be chosen locally constant, i.e. there is a partition of $\Omega_i$ in $\Sigma$-measurable subsets of positive measure on which $U_{i, \omega}$ is constant.

**Remark 5.6.** In the case where $H$ is separable, it is a standard (and easy) fact that every modularly isometric automorphism $W$ of $L_p(H)$ is associated with a measurable family $(W_\omega)_{\omega \in \Omega}$ of unitary operators on $H$; so we recover the Theorem 0.3 of the Introduction.
Remark 4.5). Thus we may assume that $H \in \text{support}^\prime$ of $f$ and verify that $\sum_{k=1}^n \varphi_k T_i(f_k)$ is a $T_i$-integral subset of $\Omega$. We may also refine the partition $(\Omega_i', \Sigma_i, \mu|_{\Omega_i}; H_i)$ and verify that $N(\sum_{k=1}^n \varphi_k T_i(f_k)) = N(\sum_{k=1}^n \varphi_k f_k)$ (since $T_i$ preserves the random scalar products).

Now define $S_i : L_p(\Omega_i, \Sigma_i, \mu|_{\Omega_i}; H_i) \rightarrow L_p(\Omega, \Sigma, \mu; H)$ by $S_i f = wT_i(w^{-1}f)$: the range $R(S_i) = wR(T_i)$ is exactly $1_{\Omega_i}.Z$, where $Z$ is the closed $L_\infty(\Sigma)$-submodule generated by $R(P)$. We can glue up the maps $S_i$ and obtain a modularly isometric embedding $S$ from $\bigoplus_{i \in I} L_p(\Omega_i, \Sigma_i, \mu|_{\Omega_i}; H_i)$ into $L_p(\Omega, \Sigma, H)$, with range $R(S) = Z$.

By Lemma 5.3 there exists a positive projection $\tilde{P}$ on $L_p(\Omega, \Sigma, \mu)$ such that $P(\varphi.u_f) = (\tilde{P}\varphi).u_f$ for every $f \in R(P)$ and $\varphi \in L_p(\Omega, \Sigma, \mu)$. Note that $\tilde{P}$ is $\mathcal{F}$-modular, in particular it commutes with every multiplication operator $M_{\Omega_i}, i \in I$.

If $A \in \mathcal{F}$ is a $\nu$-integrable subset of $\Omega_i$ and $e \in H_i$ we have $S_i(1_A w \otimes e) = wT_i(1_A \otimes e) \in R(P)$, and $N(S_i(1_A w \otimes e)) = N(1_A w \otimes e) = 1_A w$, and consequently for $f = S_i(1_A w \otimes e)$ we have $f = w.u_f$. Thus for every $\psi \in L_\infty(\Omega, \Sigma, \mu) \cap L_p(\Omega, \Sigma, \mu)$ we have

$$PS_i(\psi 1_A w \otimes e) = P(\psi.wu_f) = \tilde{P}(\psi.w1_{\Omega_i}).u_f = \tilde{P}(\psi.w1_{\Omega_i}).w^{-1}S_i(1_A w \otimes e) = S_i(\tilde{P}(\psi.w1_{\Omega_i})1_A \otimes e)$$

hence by linearity and density we have for every $\varphi \in L_p(\Omega_i, \Sigma_i, \mu|_{\Omega_i})$ and $e \in H_i$:

$$PS_i(\varphi \otimes e) = S_i(\tilde{P}(\varphi) \otimes e)$$

that is, the restriction of $P$ to $1_{\Omega_i}.Z$ is similar by $S_i$ to the projection $\tilde{P} \otimes \text{id}_{H_i}$; consequently the restriction of $P$ to $Z$ is similar by $S_i$ to the projection $\sum_{i \in I} \tilde{P}M_{\Omega_i} \otimes \text{id}_{H_i}$.

In the case where $Z = L_p(\Omega, \Sigma, \mu; H)$ we have necessarily $\text{dim} H = \text{dim} H_i$ for every $i \in I$ since $S_i$ is a modularly isometric map from $L_p(\Omega_i; H_i)$ onto $1_{\Omega_i}.Z = L_p(\Omega_i; H)$ (see Remark 4.5). Thus we may assume that $H_i = H$ and the conclusion of Thm. 5.4 is obtained with $W = S$ and $U = \text{Id}$.

In the general case we apply Lemma 4.4 to the $L_\infty(\Sigma)$ submodule $Z^\perp$. We find a family $(\Omega_j'_{i,j} \in J$ of pairwise almost disjoint members of $\Sigma$, a family $(K_{i,j})_{i \in I, j \in J}$ of Hilbert spaces and a modularly isometric map $S'$ from $(\bigoplus_{j \in J} L_p(\Omega'_{j,i}, \Sigma|_{\Omega'_{j,i}}, \mu|_{\Omega'_{j,i}}; K_{j}))_{i \in I}$ onto $Z^\perp$. Note that now the sets $\Omega'_{j,i}$ have no reason to belong to the smaller $\sigma$-algebra $\mathcal{F}$. We have $\bigvee_{j} \Omega'_j = V(S)(Z^\perp)$. For the commodity of the notation we may assume $\bigvee_{j} \Omega'_j = \Omega$, adding if necessary one extra set $\Omega'_0 = \Omega \setminus \bigvee_{j} \Omega'_j$ for which we set $K_0 = \{0\}$, the 0-dimensional Hilbert space. Similarly, up to the cost of adding one extra set $\Omega'_0 = \Omega \setminus \Omega_P$ and setting $H_0 = \{0\}$, we may assume that $\bigvee_{j} \Omega_i = \Omega$. We may also refine the partition $(\Omega'_j)$ by setting $\Omega'_j = \Omega_i \cap \Omega'_j$ and removing the $\Omega'_{ij}$ which are almost void. This operation gives a doubly indexed family $(\Omega'_{ij})_{i \in I, j \in J}$.

For every $i \in I, j \in J$, set $L_{ij} = H_i \oplus K_j$ (direct Hilbertian sum). Then $L_p(\Omega'_j; H_i)$ and $L_p(\Omega'_{ij}; K_j)$ identify naturally to a pair of mutually orthogonal $L_\infty(\Sigma)$-submodules of
\[ w \tau \hbar_{ij} \text{ if } u_{ij}^0 \text{ and } u_{ij}^0 \text{ are the inclusion maps of } \mathcal{H}_i, \text{ resp. } \mathcal{K}_j \text{ into } L_{ij} \text{ then } U_{ij}^0 = \text{id} \otimes u_{ij}^0 \text{ and } U_{ij}^0 = \text{id} \otimes u_{ij}^0 \text{ are the corresponding embeddings of } L_p(\Omega_{ij}; \mathcal{H}_i) \text{ and } L_p(\Omega_{ij}; \mathcal{K}_j) \text{ into } L_p(\Omega_{ij}; L_{ij}). \text{ Since } u_{ij}^0 \text{ and } u_{ij}^0 \text{ are the orthogonal projections } L_{ij} \to \mathcal{H}_i, \text{ resp. } L_{ij} \to \mathcal{K}_j, \text{ we see that } U_{ij}^0 \text{ and } U_{ij}^0 \text{ are the orthogonal projections (in the sense given in section 1.2) onto } L_p(\Omega_{ij}; \mathcal{H}_i), \text{ resp. } L_p(\Omega_{ij}; \mathcal{K}_j). \]

Now define \( W_{ij}^0 : L_p(\Omega_{ij}; L_{ij}) \to L_p(\Omega_{ij}; H) \) by \( W_{ij}^0 f = S_i(U_{ij}^0 f) + S_j(U_{ij}^0 f) : \) we have
\[
N(W_{ij}^0 f)^2 = N(S_i(U_{ij}^0 f))^2 + N(S_j(U_{ij}^0 f))^2 = N(U_{ij}^0 f)^2 + N(U_{ij}^0 f)^2 = N(f)^2
\]
since \( S_i \) and \( S_j \) are modularly isometric and have values in orthogonal subspaces \( Z \), resp. \( Z^\perp \). Hence \( W_{ij}^0 \) is modularly isometric and \( R(W_{ij}^0) = 1_{\Omega_{ij}^0} \oplus 1_{\Omega_{ij}^0} \oplus Z = L_p(\Omega_{ij}; H) \).

We know by the proof of Thm. 5.1 that \( P = PQ + V \), where \( Q \) is the orthogonal projection onto \( Z \). Since \( V \) satisfies the requirements of the theorem, we look only for a representation of \( P_0 = PQ \). From the first part of the proof we now that \( P_0 S_i = S_i(\tilde{P} \otimes \text{id}_{\mathcal{H}_i}) \); on the other hand \( P_0 S_j = 0 \) since \( R(S_j^0) \subset Z^\perp = \ker Q \). Hence for every \( f \in L_p(\Omega_{ij}; L_{ij}) \):
\[
P_0 W_{ij}^0 f = P_0 S_i U_{ij}^0 f + P_0 S_j U_{ij}^0 f = S_i(\tilde{P} \otimes \text{id}_{\mathcal{H}_i}) U_{ij}^0 f = W_{ij}^0 U_{ij}^0 (\tilde{P} \otimes \text{id}_{\mathcal{H}_i}) U_{ij}^0 f
\]
i.e. \( P_0 \) is similar to \( U_{ij}^0 \) to \( U_{ij}^0 (\tilde{P} M_{\Omega_{ij}} \otimes \text{id}_{\mathcal{H}_i}) U_{ij}^0 \).

Since \( L_p(\Omega_{ij}; L_{ij}) \) is modularly isometric to \( L_p(\Omega_{ij}; H) \) (by \( W_{ij}^0 \)), we have \( \dim L_{ij} = \dim H \) by Rem. 4.5, so we may identify \( L_{ij} \) with \( H \) by an isomorphism \( \theta_{ij} \). This isomorphism induces in turn a modular isometry \( \Theta_{ij} : \text{id} \otimes \theta_{ij} \) from \( L_p(\Omega_{ij}; L_{ij}) \) onto \( L_p(\Omega_{ij}; H) \). Set \( W_{ij} = W_{ij}^0 \Theta_{ij}^{-1} \); then \( W_{ij} \) is a modular automorphism of \( L_p(\Omega_{ij}; H) \). Let also \( u_{ij} = \theta_{ij} \otimes u_{ij}^0 \) be the embedding of \( \mathcal{H}_i \) into \( H \) resulting from this identification and \( U_{ij} = \text{id}_{L_p(\Omega_{ij})} \otimes u_{ij} = \theta_{ij} U_{ij}^0 \) be the associated embedding of \( L_p(\Omega_{ij}; \mathcal{H}_i) \) into \( L_p(\Omega_{ij}; H) \). Since \( \Theta_{ij}^{-1} = \Theta_{ij}^* \) we see that \( P_0 \) is similar by \( W_{ij} \) to \( U_{ij} (\tilde{P} M_{\Omega_{ij}} \otimes \text{id}_{\mathcal{H}_i}) U_{ij}^* \).

Finally we glue up the automorphisms \( W_{ij} \) to an automorphism \( W \) of \( L_p(\Omega; \mu; H) \) by setting
\[
W f = \sum_{i \in I} \sum_{j \in J_i} W_{ij} M_{\Omega_{ij}} f
\]
and similarly we glue up the embeddings \( U_{ij} \) to an embedding \( U \) of \( \bigoplus_{i \in I} L_p(\Omega_i; \mathcal{H}_i) \) into \( L_p(\Omega; H) \). The maps \( W \) and \( U \) are still modularly isometric and \( P_0 \) is similar by \( W \) to \( U(\sum_{i \in I} \tilde{P} M_{\Omega_i} \otimes \text{id}_{\mathcal{H}_i}) U^* \).

### 6. Annex: a proof of Corollary 3.6 specific to the complex case.

The following proof is an adaptation of that of Thm. 4.1 in [AF]. We assume that \( 2 < p < \infty \) (the case \( 1 < p < 2 \) follows by duality).

If \( f \in R(P) \) we introduce besides the projection \( E_f \) (defined in §1) the operators \( F_f \) and \( G_f \) defined by:
\[
F_f g = 1_{\mathbf{VS}(f)} \cdot g; \quad G_f g = 1_{\mathbf{VS}(f)} g - E_f g
\]
Then \( E_f, G_f \) and \( F_f \) are commuting modularly contractive projections in \( L_p(H) \) with \( E_f + F_f + G_f = I \).
Let \( f, g \in R(P) \), then the elements \( A(f, g) \) and \( B(f, g) \) defined in §3 (eqs 3.1 and 3.1bis) belong to the range of \( P^* \); so do the sum and difference: \( M_f(g) := A(f, g) + B(f, g) \) and \( \Gamma_f(g) := \frac{p}{p-2}[A(f, g) - B(f, g)] \) belong to \( R(P^*) \). Set:

\[
Q_fg = \langle u_f, g \rangle u_f
\]

We have then:

\[
M_f(g) = N(f)^{p-2}[2g + (p - 2)E_f g] \\
\Gamma_f(g) = pN(f)^{p-2}Q_f g
\]

Then \( M_f \), resp. \( \Gamma_f \) are bounded linear, resp. antilinear operators from \( L_p(H) \) into \( L_p(H) \), and \( Q_f \) is a contractive antilinear endomorphism of \( L_p(H) \) such that \( Q_f^2 = E_f \); moreover:

\[
M_f P = P^* M_f P; \quad \Gamma_f P = P^* \Gamma_f P \tag{6.1}
\]

Consider the positive symmetric bounded bilinear form defined on \( L_p(H) \) by:

\[
(g, h)_f := \langle M_f(g), h \rangle = \int N(f)^{p-2} \langle (2I + (p - 2)E_f)g \mid h \rangle \, d\mu
\]

Note that \( \Gamma_f = M_f Q_f = Q_{J_p} M_f \) and \( Q_f^* = Q_{J_p}; \) then \( Q_f \) is hermitian for \((.,.)_f\) since:

\[
(Q_f g, h)_f = \langle M_f Q_f g, h \rangle = \langle Q_f g, M_f h \rangle = \langle Q_{J_p} M_f h, g \rangle = \langle M_f Q_f h, g \rangle = \langle Q_f h, g \rangle
\]

On the other hand \( P \) is hermitian for \((.,.)_f\) since (using (6.1))

\[
(Pg, h)_f = \langle M_f Pg, h \rangle = \langle P^* M_f Pg, h \rangle = \langle M_f Pg, Ph \rangle = \langle Pg, Ph \rangle
\]

Let \( N_f \) be the kernel of the form \((.,.)_f\): we have \( g \in N_f \) iff \( (g, g)_f = 0 \) iff \( (g, h)_f = 0 \) for all \( h \in L_p(H) \) (by Cauchy-Schwartz inequality). Then \( PN_f \subset N_f \) since

\[
(Pg, Pg)_f = (g, Pg)_f = 0 \quad \text{if} \quad g \in N_f
\]

On the other hand the operator \( 2 \cdot 1\mathbf{VS}(f) + (p - 2)E_f \) maps \( L_p(H) \) onto \( 1\mathbf{VS}(f)L_p(H) \); hence \( g \in N_f \) iff \( \langle N(f)^{p-2}g, h \rangle \) = 0 for every \( h \in 1\mathbf{VS}(f)L_p(H) \) iff \( N(f)^{p-2}g = 0 \) iff \( 1\mathbf{VS}(f)g = 0 \).

We have thus: \( R(F_f) = N_f \) and consequently

\[
PF_f = F_f PF_f
\]

Since \( L_p(H) \) is a strictly convex Banach space as well as its dual, we have by the auxiliary Lemma 6.1 below:

\[
PF_f = F_f P
\]

Let us show that \( Q_f P \) is hermitian for \((.,.)_f\), using eq. (6.1) again:

\[
(Q_f Pg, h)_f = \langle M_f Q_f Pg, h \rangle = \langle \Gamma_f Pg, h \rangle \\
= \langle P^* \Gamma_f Pg, h \rangle = \langle \Gamma_f Pg, Ph \rangle \\
= \langle Q_f Pg, Ph \rangle = \langle Q_f Ph, Pg \rangle \\
= \langle Q_f Ph, g \rangle
\]
Since $Q_f$ and $P$ are separately hermitian for $(.,.)_f$ we have:

$$(Q_fP_g,h)_f = (PQ_fh,g)_f$$

hence $(PQ_f - Q_fP)h \in N_f$, i.e. $(I - F_f)PQ_f = (I - F_f)Q_fP$. Composing on the left by $G_f$ and on the right by $Q_f$, or conversely, we obtain:

$$G_fPE_f = 0 = E_fPG_f$$

Since on the other hand:

$$F_fPE_f = PF_fE_f = 0 = E_fF_fP = E_fPF_f$$

we obtain:

$$PE_f = E_fPE_f = E_fP$$

We state now and give a proof of the announced auxiliary Lemma.

**Lemma 6.1.** Let $X$ be a strictly convex Banach space with strictly convex dual, and $P,Q$ two contractive projections on $X$. The following conditions are equivalent:

i) $PQ$ is a projection

ii) $PQ = QPQ$

iii) $PQ = PQP$

If moreover the complementary projection $Q^\perp$ is contractive too then $PQ = QP$.

**Proof:** If (ii) is verified then $(PQ)^2 = PQPQ = PPQP = PQ$; while if (iii) is verified then $(PQ)^2 = PQPQ = PQQP = PQ$. Hence both (ii) and (iii) imply (i) (without any contractiveness assumption). Conversely if (i) is verified then for every $x \in R(PQ)$ we have $x = Qx = PQx$ (by [AF], Prop. 1.1 (iii); only the strict convexity of $X$ is needed) so $x \in R(P) \cap R(Q)$. Since the converse is trivial, we see that $R(PQ) = R(P) \cap R(Q)$; in particular $QPQ = PQ$ and (ii) is verified. Dualizing we have that $P^*$, $Q^*$ and $Q^*P^*$ are contractive projections in $X^*$; hence $Q^*P^* = P^*Q^*P^*$, so $PQ = PQP$ and (iii) is verified.

Now (iii) implies $PQ^\perp = PQP^\perp$, and if $Q^\perp$ is contractive this implies $PQ^\perp = QPQ^\perp$ by the preceding. Then

$$Q = PQ + PQ^\perp = PQP + QPQ^\perp$$

which in turn implies $QP = PQ = PQP$. □

**Remark:** The final assertion $PQ = QP$ of Lemma 6.1 is stated in [AF] (for $X = C_p$) as Cor. 1.7 without the assumption that the complementary projection $Q^\perp$ is contractive.

This statement is not correct: if $p \neq 2$ it is easy to construct rank 1 contractive projections $P,Q$ in $X = \ell_p$ or $C_p$ such that $PQ = 0 \neq QP$: choose non zero elements $a,b \in X$ such that their norming functionals $Ja, Jb$ verify $(Ja,b) = 0$ and $(Jb,a) \neq 0$ and set $P = a \otimes Ja$, $Q = b \otimes Jb$.

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References


Institut de Mathématiques de Jussieu (CNRS),
Projet Analyse Fonctionnelle (Case 186),
Université Paris-VI
4, place Jussieu, 75252 Paris Cedex 05, France.
E-mail: yr@ccr.jussieu.fr

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