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The range of a contractive projection in $L_p(H)$

Yves Raynaud

Abstract: We show that the range of a contractive projection on a Lebesgue-Bochner space of Hilbert valued functions $L_p(H)$ is isometric to a $\ell_p$-direct sum of Hilbert-valued $L_p$-spaces. We explicit the structure of contractive projections. As a consequence for every $1 < p < \infty$ the class $C_p$ of $\ell_p$-direct sums of Hilbert-valued $L_p$-spaces is axiomatizable (in the class of all Banach spaces).

Introduction

It was a remarkable achievement in the isometric theory of Banach spaces of the years 1960’s to characterize the contractive linear projections of Lebesgue $L_p$ spaces ($p \neq 2$). In the case of $L_p$ spaces of a probability space it was done by Douglas [D] in the case $p = 1$ and Andô [A] in the case $1 < p < \infty$, $p \neq 2$. They showed that the range of such a contractive projection is itself isometric to a $L_p$ space (for the same $p$, but a different measure space); if moreover the projection is positive then its range is a sublattice of the initial $L_p$ space and is lattice isomorphically isometric to a $L_p$ space. This was extended to the non-sigma-finite measure space setting by Tzafriri ([T]). In the case of a probability space, the structure of contractive projections is elucidated by Douglas-Andô works: a general contractive projection $P$ on $L_p(\Omega, \Sigma, \mu)$ has the form:

$$P = M_\varepsilon \hat{P} M_\varepsilon^{-1} + V$$

(0.1)

where $M_\varepsilon$ is the multiplication operator by a function $\varepsilon$ with $|\varepsilon| = 1$, $\hat{P}$ is a positive contractive projection, and $V = 0$ if $p > 1$, while if $p = 1$, then $V$ is a contraction from $L_1$ into the range $R(P)$ of $P$ which vanishes on the band generated by $R(P)$. Moreover $\hat{P}$ is a weighted conditional expectation, i.e. there exist a sub-sigma algebra $B$, an element $B \in B$ and a nonnegative function $w \in L^p$ such that $\mathbb{E}(w^p | B) = 1$ and

$$\hat{P}f = w\mathbb{E}(1_B f, w^{p-1} | B)$$

for every $f \in L_p$ (in particular if $P1 = 1$ then $P$ is a conditional expectation). This last formula can also be written

$$\hat{P}f = w\mathbb{E}_\nu(1_B f w^{-1} | B)$$

where $\mathbb{E}_\nu$ is the conditional expectation relative to the measure $\nu = w^p.\mu$. If we denote by $S$ the isometric isomorphism $f \mapsto w. f$ of $L_p(\Omega, \Sigma, \nu)$ onto $L_p(\Omega, \Sigma, \nu)$ and by $M_B$ the multiplication operator by the indicator function $1_B$, we have:

$$\hat{P} = SM_B \mathbb{E}_\nu( | B)S^{-1}$$

(0.2)

The structure of contractive projections in the non-sigma finite case was treated by Bernau and Lacey ([BL]); their main result can be rephrased in saying that if we assume (as we


Key words: Contractive projections, Vector-valued $L_p$-spaces.
may) that the measure space \((\Omega, \Sigma, \mu)\) is localizable ([F]) then formulas (0.1) and (0.2) are still valid; now \(w\) is some \(\Sigma\)-measurable positive function, \(\nu = w^p, \mu\) and \(B\) is some semi-finite sigma-subalgebra of \(\Sigma\).

The task of extending these results to various classical spaces was considered by numerous authors; see the recent survey paper [R2] and the references inside. Here we are more specifically interested in the case of vector-valued Lebesgue \(L_p\) spaces, in particular mixed norm spaces \(L_p(L_q)\). Since the survey paper [Dt] on this specific subject, several partial results appeared. In particular B. Randrianantoanina ([R1]) succeeded in solving thoroughly the complex sequential case \(\ell_p(\ell_q)\) using hermitian operator techniques introduced in the subject by Kalton and Wood. More recently the case of finite dimensional real Banach spaces with \(C^2\) norm was considered by the authors of [LG]; under some additional conditions on the dual norm (in particular it is assumed to be \(C^2\) on the complementary set of the coordinate hyperplanes associated to a distinguished basis) the contractively complemented subspaces are shown to be necessarily generated by a block-basis of the given basis. This can be applied in particular to the real spaces \(\ell_p(\ell_q^n)\), when \(2 < p, q < \infty\) (or by duality when \(1 < p, q < 2\), obtaining the same description of their contractively complemented subspaces as in the complex case [R3].

In the present paper we examine the case of Lebesgue spaces of Hilbert valued functions \(L_p(H)\); this is done in the most general case (without any assumption of sigma-finiteness of \(L_p\)-space or separability of the Hilbert space; in fact we have in mind some applications to the ultrapowers of such spaces, which are neither separable nor sigma-finite). It turns out that the range of a contractive projection is a \(\ell_p\)-direct sum of spaces of the type \(L_p(H)\).

More precisely:

**Theorem 0.1.** Let \(1 \leq p < \infty, p \neq 2; H\) be a Hilbert space and \(L_p = L_p(\Omega, \Sigma, \mu)\). The range of every contractive projection \(P : L_p(H) \to L_p(H)\) is isometric to a \(\ell_p\)-direct sum of Hilbert-valued \(L_p\)-spaces, i. e.

\[
R(P) \approx_1 \left( \bigoplus_{i \in I} L_p(\Omega_i, B_i, \mu_i; H_i) \right)_{\ell_p}
\]

where \((\Omega_i)_i\) is a family of pairwise almost disjoint members of \(\Sigma\), each \(B_i\) is a subsigma-algebra of the trace \(\Sigma_i\) of \(\Sigma\) on \(\Omega_i\); \(\mu_i\) is the trace on \(\Omega_i\) of the measure \(\mu\); and the Hilbert spaces \(H_i\) have Hilbertian dimension not greater than the Hilbertian dimension of \(H\).

Conversely a \(\ell_p\)-sum \(\left( \bigoplus_{i \in I} L_p(\Omega_i, B_i, \mu_i; H_i) \right)_{\ell_p}\) embeds isometrically into \(L_p(H)\), where

\[
L_p = \left( \bigoplus_{i \in I} L_p(\Omega_i, B_i, \mu_i) \right)_{\ell_p}
\]

and \(H = (\bigoplus_{i \in I} H_i)_{\ell_2}\). Hence a contractively complemented subspace of a \(\ell_p\)-direct sum of Hilbert-valued \(L_p\)-spaces is still a \(\ell_p\)-direct sum of Hilbert-valued \(L_p\)-spaces. In other words:

**Corollary 0.2.** The class \(C_p\) of \(\ell_p\)-direct sums of Hilbert-valued \(L_p\)-spaces is stable under contractive projections.

The structure of the contractive projection \(P\) can be easily explained in the case where the space \(H\) is separable (the non-separable case is analogous and will be described in section 5). Recall that given two Banach spaces \(X, Y\), a family of operators \(T_\omega : X \to Y\) is said to be strong-operator \(\Sigma\)-measurable if for every \(x \in X\), the map \(\omega \mapsto T_\omega x\) is \(\Sigma\)-measurable as a map \(\Omega \to Y\). If moreover \(\text{Esssup}_\omega \| T_\omega \| < \infty\), such a measurable family induces a bounded
linear map $T$ from $L_p(\Omega, \Sigma, \mu; X)$ into $L_p(\Omega, \Sigma, \mu; Y)$ by the equation:

$$(Tf)(\omega) = T_\omega(f_\omega)$$

**Theorem 0.3.** Under the conditions of Th. 0.1, if moreover $H$ is separable, then

$$P = \sum_{i \in I} S_i(\tilde{P}_i \otimes 1_{H_i}) S_i^* M_{\Omega_i} + V$$

where $\tilde{P}_i$ is a positive contractive projection in $L_p(\Omega_i, \Sigma_i, \mu_i); S_i$ is an isometric embedding of $L_p(\Omega_i, \Sigma_i, \mu_i; H_i)$ into $L_p(\Omega_i, \Sigma_i, \mu_i; H)$ associated with a (strong-operator)-measurable family $(S_{i,\omega})_{\omega \in \Omega_i}$ of isometric embeddings $H_i \to H$, while $S_i^*$ is associated with the adjoint family $(S_{i,\omega}^*)_{\omega \in \Omega_i}$ of projections $H \to H_i; M_{\Omega_i} : L_p(\Omega; H) \to L_p(\Omega_i; H_i)$ is the multiplication operator by the indicator function $1_{\Omega_i}$; and $V = 0$ if $p > 1$, while if $p = 1$ then $V$ is a contraction of $L_1(\Omega, \Sigma, \mu; H)$ vanishing on every $L_1(\Omega_i, \Sigma_i, \mu_i; H)$ and taking values in the range of $P$.

Let us present shortly an application of the Thm 0.1 which was in fact our main motivation for starting this study. If $X,Y$ are Banach spaces, we say that $X$ is an ultraroot of $Y$ if $Y$ is isometric to some ultrapower of $X$. Recall that a Banach space $X$ embeds canonically isometrically in every of its ultrapowers $X_\mathcal{U}$, and that if $X$ is reflexive, then this canonical image is contractively complemented in $X_\mathcal{U}$. As a consequence of Th. 0.1 we see that every ultraroot of a $L_p(H)$ space, $p > 1$ is a member of $\mathcal{C}_p$. By Cor. 0.2 the same is true for ultraroots of members of $\mathcal{C}_p$. On the other hand it was proved in [LR] that every ultraproduct of $L_p(H)$ spaces is isometric to a $\ell_p$-direct sum of Hilbert-valued $L_p$-spaces. More generally every ultraproduct of members of $\mathcal{C}_p$ is itself isometric to a member of $\mathcal{C}_p$. Hence we obtain:

**Corollary 0.4.** For every $1 < p < \infty$ the class $\mathcal{C}_p$ of $\ell_p$-direct sums of Hilbert-valued $L_p$-spaces is stable under ultraproducts and ultraroots.

In other words the class $\mathcal{C}_p$ is axiomatisable in the sense of Henson-Iovino [HI] in their language of normed spaces structures (see [HI], th. 13.8).

The paper is organized as follows: after a section devoted to definitions, notations and a general result on orthogonally complemented subspaces of $L_p(H)$, we have two sections of preliminary results distinguishing the case $p = 1$ (Section 2) from the case $p > 1$ (Section 3). In these sections it is proved that if $f$ belongs to the range of a contractive projection $P$, then the whole subspace $Z_f := L_\infty(\Omega).f$ is preserved by $P$ (i.e. $PZ_f \subset Z_f$) which suggests clearly a possible reduction to the scalar case. It is also proved that the “orthogonal projection” onto $Z_f$ preserves the range of $P$. This allows to find an “orthogonal system” in $R(P)$ which generates $Z_p := L_\infty(\Sigma).R(P)$ over $L_\infty(\Sigma)$ which will furnish the orthogonal bases of the Hilbert spaces $H_i$ of Thm. 0.1. Section 4 is devoted to the proof of Thm. 0.1; a key point consists in proving that the different subalgebras of $\Sigma$ given by the scalar theorem (applied to each $Z_f$) are induced by the same sigma-subalgebra $\mathcal{F}$ of $\Sigma$. Finally Thm 0.3 is proved in Section 5 (in a more general version not requiring separability).
1. General preliminaries

1.1 Definitions and notations

Let \( 1 \leq p < \infty \); \( H \) be an Hilbert space and \((\Omega, \Sigma, \mu)\) be a measure space. In the following we denote (when there is no ambiguity) by \( L_p(H) \) the Lebesgue-Bochner space \( L_p(\Omega, \Sigma, \mu; H) \) of classes of \( H \)-valued \( p \)-integrable functions (for \( \mu \)-a.e. equality). Similarly \( L_\infty(H) \) will be the space of classes of Bochner measurable, essentially bounded \( H \)-valued functions. These spaces can be defined directly from the Banach lattices \( L_p \) (resp. \( L_\infty \)) and the Hilbert space \( H \), but we adopt the functional point of view for the simplicity of the exposition. In the case where \((\Omega, \Sigma, \mu)\) is not sigma-finite, it is preferable to suppose that this measure space is localizable: the measure \( \mu \) contains a further one of positive and finite measure \( \mu \), but we adopt the functional point of view for the simplicity of the exposition.

The set \( \{ f \in \) \( H \) \( \text{ measurable } \mid \int_{\Omega} |f|^p d\mu < \infty \} \) is defined (up to a \( \mu \)-null set) by the conditions:

\[
A \supset A_i \text{ for every } i \in I
\]

If \( B \in \Sigma \) and \( B \supset A_i \) for every \( i \in I \) then \( B \supset A \)

where \( B \supset A \) means \( \mu(A \setminus B) = 0 \) (define similarly \( A \supset B \) and \( A = B \)). We say that \( B, C \) are almost disjoint if \( A \cap B = \emptyset \).

To every \( f \in L_p(H) \) we associate its “random norm” \( N(f) = \| f \|_H \), its vectorial function support \( \text{VS}(f) = \text{Supp}(N(f)) \) and its “random direction”, i.e. the element \( u_f \) of \( L_\infty(H) \) defined by \( u_f(\omega) = \frac{f(\omega)}{N(f)(\omega)} \) if \( \omega \in \text{VS}(f) \), \( = 0 \) if \( \omega \notin \text{VS}(f) \). If \( M \subset L_p(H) \) we set \( \text{VS}(M) = \{ \text{VS}(f) \mid f \in M \} \). If \( f \in L_p(H) \), \( g \in L_q(H) \) we define their random scalar product \( \langle f, g \rangle \in L_r \) (where \( \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \) by \( \langle f | g \rangle(\omega) = \langle f(\omega) | g(\omega) \rangle_H \), where \( \langle . | . \rangle_H \) denotes the scalar product in \( H \) (which we suppose left linear, right antilinear in the complex case). When \( p, q \) are conjugate \( (\frac{1}{p} + \frac{1}{q} = 1) \), we obtain a sesquilinear pairing

\[
\langle f, g \rangle = \int_{\Omega} \langle f | g \rangle d\mu
\]

which gives rise to a canonical antilinear identification of \( L_q(H) \) with \( L_p(H)^* \) (if \( 1 < p, q < \infty \); the case \( p = 1, q = \infty \) is more delicate); it is the usual duality pairing in the real spaces case. We have also:

\[
\forall f \in L_p(H), \langle f | u_f \rangle = N(f)
\]

We say that two elements \( f, g \in L_p(H) \) are orthogonal, and we write \( f \perp g \) if \( \langle f | g \rangle = 0 \). A related notation is the following. We set:

\[
\{ f \perp g \} = \{ \omega \in \Omega \mid \langle f | g \rangle(\omega) = 0 \}
\]

We have then \( f \perp g \iff \{ f \perp g \} = \Omega \).

Let \( H, K \) two Hilbert spaces. We say that a linear operator \( T : L_p(H) \rightarrow L_p(K) \) is \( \Sigma \)-modular iff \( T(\varphi.f) = \varphi.Tf \) for every \( f \in L_p(H) \) and \( \varphi \in L_\infty(\Sigma) \). It is modularly contractive, resp modularly isometric iff \( N(Tf) \leq N(f), \) resp. \( N(Tf) = N(f) \) for every \( f \in L_p(H) \): it is then automatically \( \Sigma \)-modular (and, of course, contractive, resp. isometric). If \( H \) is separable, then a modularly contractive, resp. modularly isometric operator \( T \) is associated with a measurable family of contractions, resp. isometries \( T_\omega : H \rightarrow K \).
Let $\mathcal{F}$ be a sub-sigma-algebra of $\Sigma$; a linear subspace $Z$ of $L_p(H)$ is a $L_\infty(\mathcal{F})$-submodule iff $\varphi.f \in Z$ for every $f \in Z$ and $\varphi \in L_\infty(\Omega,\mathcal{F},\mu)$.

To every $f \in L_p(H)$ we associate the bounded $\Sigma$-modular operator:

$$E_f : L_p(H) \to L_p(H), \quad g \mapsto \langle g|u_f \rangle u_f$$

We have $N(E_f g) = |\langle g|u_f \rangle| \leq N(g)$, hence $E_f$ is modularly contractive.

We have clearly $E_f(f) = N(f)u_f = f$. Consequently for every $\varphi \in L_\infty$, we have

$$E_f((\varphi N(f)).u_f) = E_f(\varphi f) = \varphi f = (\varphi N(f)).u_f$$

and by density we deduce that $E_f(\psi.u_f) = \psi.u_f$ for every $\psi \in L_p$. In particular $E_f(E_f g) = E_f g$, so $E_f$ is a projection (with range $R(E_f) = L_p(\Omega).u_f$). It is not hard to see that $R(E_f)$ is exactly the closed $L_\infty(\Sigma)$-submodule generated by $f$. Note also that if $f,g \in L_p(H)$,

$$f \perp g \iff E_f g = 0 \iff E_g f = 0$$

1.2 Orthogonal projections

We end this section by considering a special class of contractive projections, namely the orthogonal ones. A projection $Q$ in $L_p(H)$ is said to be orthogonal if $(f - Qf) \perp Qf$ for every $f \in L_p(H)$. Such a projection is trivially modularly contractive since

$$N(f)^2 = N(Qf)^2 + N((I - Q)f)^2 \geq N(Qf)^2$$

Note that by polarization we have for every $f,g \in L_p(H)$:

$$\langle f | g \rangle = \langle Qf | Qg \rangle + \langle (I - Q)f | (I - Q)g \rangle$$

Replacing $g$ by $Qg$, we have

$$\langle f | Qg \rangle = \langle Qf | Qg \rangle$$

that is $(I - Q)f \perp Qg$; hence $\ker Q = R(I - Q) \perp R(Q)$.

Conversely if $f \perp R(I - Q)$ then $f - Qf \perp R(I - Q)$ and in particular $f - Qf \perp f - Qf$, i.e. $f = Qf \in R(Q)$. Hence $R(Q) = \ker Q^\perp := \{ f \in L_p(H) | f \perp \ker Q \}$ and similarly (exchanging the roles of $Q$ and $I - Q$) we have: $\ker Q = R(Q)^\perp$.

If $A$ is a subset of $L_p(H)$ then $A^\perp$ is a closed $L_\infty(\Sigma)$-submodule of $L_p(H)$. In particular the range of any orthogonal projection in $L_p(H)$ is a closed $L_\infty(\Sigma)$-submodule. The converse is true:

**Lemma 1.2.** If $Z$ is a closed $L_\infty(\Sigma)$-submodule of $L_p(\Omega,\Sigma,\mu; H)$ there exists a unique orthogonal projection $Q_Z$ in $L_p(H)$ with range $Z$.

**Proof:** Let $(f_\alpha)_{\alpha \in A}$ be a maximal family of pairwise orthogonal non zero elements of $Z$. For every family $(\varphi_\alpha)_{\alpha}$ in $L_p(\Omega)$ and every finite subset $B$ of $A$ we have:

$$\| \sum_{\alpha \in B} \varphi_\alpha u_{f_\alpha} \|_{L_p(H)} = \| N(\sum_{\alpha \in B} \varphi_\alpha u_{f_\alpha}) \|_p = \| (\sum_{\alpha \in B} \mathbf{1}_{\Sigma}(f_\alpha) | \varphi_\alpha|^2)^{1/2} \|_p$$
Proof: For every \( \varepsilon \in L_1(\Omega) \) with \( 0 \leq \varepsilon \leq N(f) \) we have:

\[
\|f\| - \|\varepsilon \cdot u_f\| = \int N(f) \, d\mu - \int N(\varepsilon u_f) \, d\mu = \int (N(f) - \varepsilon) \, d\mu
\]

\[
= \|(N(f) - \varepsilon) \cdot u_f\| = \|f - \varepsilon \cdot u_f\|
\]

\[
\geq \|P(f - \varepsilon \cdot u_f)\| = \|f - P(\varepsilon \cdot u_f)\|
\]

\[
\geq \|f\| - \|P(\varepsilon \cdot u_f)\|
\]

\[
\geq \|f\| - \|\varepsilon \cdot u_f\|
\]

hence by Cauchy’s criterion, \( \sum_{\alpha \in A} \varphi_{\alpha} u_{f\alpha} \) converges in \( L_p(H) \) iff \( (\sum_{\alpha \in A} 1_{VS}(f_\alpha)|\varphi_{\alpha}|^2)^{1/2} \) exists in \( L_p \) and

\[
\| \sum_{\alpha \in A} \varphi_{\alpha} u_{f\alpha} \|_{L_p(H)} = \| (\sum_{\alpha \in A} 1_{VS}(f_\alpha)|\varphi_{\alpha}|^2)^{1/2} \|_p
\]

If now \( f \in L_p(H) \) and \( B \) is a finite subset of \( A \) we have:

\[
N(\sum_{\alpha \in B} \|\langle f \| u_{f\alpha} \| \rangle u_{f\alpha} \|)^2 = \sum_{\alpha \in B} \|\langle f \| u_{f\alpha} \| \rangle u_{f\alpha} \|^2 = \langle f, \sum_{\alpha \in B} \|\langle f \| u_{f\alpha} \| \rangle u_{f\alpha} \rangle \leq N(f)N(\sum_{\alpha \in B} \|\langle f \| u_{f\alpha} \| \rangle u_{f\alpha} \|)
\]

whence

\[
N(\sum_{\alpha \in B} \|\langle f \| u_{f\alpha} \| \rangle u_{f\alpha} \|) = (\sum_{\alpha \in B} \|\langle f \| u_{f\alpha} \| \rangle u_{f\alpha} \|)^2 \leq N(f)
\]

so

\[
(\sum_{\alpha \in A} \|\langle f \| u_{f\alpha} \| \rangle u_{f\alpha} \|)^2 \leq N(f)
\]

consequently \( Qf := \sum_{\alpha \in A} \|\langle f \| u_{f\alpha} \| \rangle u_{f\alpha} = \sum_{\alpha \in A} E_{f\alpha}f \) converges in \( L_p(H) \) (with \( \|Qf\| \leq \|f\| \)). Since \( R(E_{f\alpha}) \) is the closed \( L_\infty(\Sigma) \)-submodule generated by \( f_{\alpha} \), we have \( R(E_{f\alpha}) \subset Z \) for each \( \alpha \) and consequently \( Qf \in Z \) for every \( f \in L_p(H) \). The map \( Q \) is modular for the action of \( L_\infty(\Omega) \), and clearly \( QI_{\beta} = f_{\beta} \) for every \( \beta \in A \). It results easily that \( Qf = f \) for every \( f = \sum_{\alpha \in A} \varphi_{\alpha} u_{f\alpha} \) (when this series converges), i.e. \( Q \) is a contractive projection in \( L_p(H) \) with range

\[
R(Q) = \{ \sum_{\alpha \in A} \varphi_{\alpha} u_{f\alpha} \mid \left( \sum_{\alpha} |\varphi_{\alpha}|^2 \right)^{1/2} \in L_p(\Omega) \}
\]

\[
= \{ \sum_{\alpha \in B} \psi_{\alpha} f_{\alpha} \mid \left( \sum_{\alpha} |\psi_{\alpha}|^2 N(f_{\alpha})^2 \right)^{1/2} \in L_p(\Omega) \}
\]

Since clearly \( \langle Qf \| f_{\alpha} \rangle = \langle f \| f_{\alpha} \rangle \) for every \( \alpha \in A \) we have \( (f - Qf) \perp f_{\alpha} \) for every \( \alpha \in A \). By maximality of the system \( (f_{\alpha}) \) we deduce that

\[
f = Qf \text{ for every } f \in Z
\]

so \( R(Q) \) contains \( Z \), hence coincides with \( Z \). Note also that \( f - Qf \perp Z \) for all \( f \in L_p(H) \), and so \( Q \) is orthogonal.

The unicity of the orthogonal projection onto \( Z \) is a consequence of the fact that its image and kernel are uniquely determined \( (R(Q) = Z \text{ and ker } Q = Z^\perp) \). \( \square \)

2. Preliminary results: the case \( p = 1 \).

Lemma 2.1. Let \( P \) be a contractive projection in \( L_1(H) \). Then for every \( f \in R(P) \) we have:

\[
P E_f = E_f P E_f
\]

Proof: For every \( \varphi \in L_1(\Omega) \) with \( 0 \leq \varphi \leq N(f) \) we have:

\[
\|f\| - \|\varphi \cdot u_f\| = \int N(f) \, d\mu - \int N(\varphi u_f) \, d\mu = \int (N(f) - \varphi) \, d\mu
\]

\[
= \|(N(f) - \varphi) \cdot u_f\| = \|f - \varphi \cdot u_f\|
\]

\[
\geq \|P(f - \varphi \cdot u_f)\| = \|f - P(\varphi \cdot u_f)\|
\]

\[
\geq \|f\| - \|P(\varphi \cdot u_f)\|
\]

\[
\geq \|f\| - \|\varphi \cdot u_f\|
\]

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hence all the inequalities are equalities, and in particular:

\[ \| f - P(\varphi.u_f) \| = \| f \| - \| P(\varphi.u_f) \| \]

that is:

\[ \int N(f - P(\varphi.u_f)) \, d\mu = \int [N(f) - N(P(\varphi.u_f))] \, d\mu \]

which implies

\[ N(f - P(\varphi.u_f)) = N(f) - N(P(\varphi.u_f)) \]

(equality as elements of \( L_1(\Omega) \)). Since \( H \) is strictly convex this implies that

\[ P(\varphi.u_f) = \alpha.f \]

for some \( \alpha \in L_\infty^+(\Omega) \). Hence

\[ E_f P(\varphi.u_f) = E_f(\alpha.f) = \alpha.f = P(\varphi.u_f) \]  \hspace{1cm} (2.1)

This property has been proved for \( \varphi \in L_1(\Omega) \) with \( 0 \leq \varphi \leq N(f) \); it is extended by linearity and density to every \( \varphi \in L_1(\Omega) \). In particular if we take \( \varphi = \langle h|u_f \rangle \), we obtain:

\[ \forall h \in L_1(H), \quad E_f P E_f h = P E_f h \]

that is \( E_f P E_f = P E_f \). \( \square \)

**Lemma 2.2.** Let \( P \) be a contractive projection in \( L_1(H) \). Then for every \( f, g \in R(P) \) we have: \( E_g f \in R(P) \). In other words \( E_g P = P E_g P \).

**Proof:** We have \( (f - E_g f) \perp g \), while (by lemma 2.1) \( E_g f - P E_g f = E_g (f - P E_g f) \in L_1(\Omega).u_g \). Hence \( (f - E_g f) \perp (E_g f - P E_g f) \). It results that:

\[ N(f - P E_g f) = [N(f - E_g f)^2 + N((E_g f - P E_g f)^2)]^{1/2} \geq N(f - E_g f) \]  \hspace{1cm} (2.2)

Hence:

\[ \| f - P E_g f \| \geq \| f - E_g f \| \]

\[ \geq \| P (f - E_g f) \| \]

\[ = \| f - P E_g f \| \]

Hence the inequalities are equalities. In view of (2.2), the equality \( \| f - P E_g f \| = \| f - E_g f \| \) implies

\[ N(f - P E_g f) = [N(f - E_g f)^2 + N((E_g f - P E_g f)^2)]^{1/2} = N(f - E_g f) \]

which implies in turn that \( N(E_g f - P E_g f) = 0 \), that is \( E_g f = P E_g f \). So \( E_g f \in R(P) \). \( \square \)

3. Preliminary results: the case \( p > 1 \).

**Notations.** Let \( p_* \) be the conjugate exponent of \( p \). If \( T : L_p(H) \to L_p(H) \) is a bounded operator, we define its adjoint \( T^* : L_{p_*}(H) \to L_{p_*}(H) \) by

\[ \forall f \in L_{p_*}(H), \forall g \in L_p(H) \quad \langle T^* f, g \rangle = \langle f, T g \rangle \]

where \( \langle ., . \rangle \) denotes the sesquilinear pairing given by eq.(1.1).

If \( f \in L_p(H), \ f \neq 0 \) let \( J f \in L_{p_*}(H) \) be the unique norm-one element such that \( \langle f, J f \rangle = \| f \|^2 \). In fact it will be easier to consider the \( (p - 1) \)-homogeneous functional \( J_p(h) = \| h \|^{p-1} J(h) \). We have \( J_p(h) = N(h)^{p-1} u_h = N(h)\sqrt{p-2} h \), hence \( J_p \) is random direction preserving. Note that \( p J_p \) is the derivative of the \( p^{th} \) power of the norm.
**Lemma 3.1.** Let $1 < p < \infty$, $p \neq 2$ and $P$ be a contractive projection in $L_p(H)$. Then for every $f, g \in R(P)$ the function $F(f, g) := \text{sgn} \langle g \mid f \rangle f + \gamma_p \mathbf{1}_{\{f \perp g\}} N(f)u_g$ belongs to $R(P)$, where $\gamma_p$ is a positive constant depending only on $p$.

**Proof:** a) Case $2 < p < \infty$.

Recall that since $L_p(H)$ is smooth the duality map $J$ maps $R(P)$ into $R(P^*)$ (see e. g. [DHP], Lemma 4.8); hence $J_p(f + tg) \in R(P^*)$ for every $t \geq 0$. The derivative $\frac{\partial}{\partial t} J_p(f + tg)$ exists at $t = 0$ (since the norm to the power $p$ is twice differentiable) and it belongs to $R(P^*)$ too. We have:

$$
\frac{\partial}{\partial t} J_p(f + tg) = N(f + tg)^{p-2}g + \left(\frac{p-2}{2} \frac{\partial}{\partial t} N(f + tg)^2\right) N(f + tg)^{p-4}(f + tg)
$$

hence:

$$
A(f, g) := \frac{\partial}{\partial t} J_p(f + tg) \mid_{t=0} = N(f)^{p-2}g + (p-2) \text{Re} (\langle f \mid g \rangle) N(f)^{p-4}f
$$

$$
= N(f)^{p-2}[g + (p-2) \text{Re} (\langle f \mid g \rangle) u_f] \in R(P^*) \quad (3.1)
$$

In the complex case, replacing $f$ by $if$ we obtain

$$
B(f, g) := N(f)^{p-2}[g - i(p-2)\Im (\langle f, g \rangle) u_f] \in R(P^*) \quad (3.1\text{bis})
$$

adding:

$$
N(f)^{p-2}[2g + (p-2)\langle g, u_f \rangle u_f] \in R(P^*)
$$

With $E_f g = \langle g, u_f \rangle u_f$ we obtain

$$
N(f)^{p-2}[2(g - E_f g) + pE_f g] \in R(P^*) \quad (3.2)
$$

In the case of a real space (3.1) is valid without the symbol $\text{Re}$ and we obtain

$$
N(f)^{p-2}[(g - E_f g) + (p-1)E_f g] \in R(P^*) \quad (3.3)
$$

If $h \in R(P^*)$ then $J_{p, h} = N(h)^{p-1}u_h \in R(P)$, hence if we set $Tg = \alpha_p (g - E_f g) + E_f g$, with $\alpha_p = \frac{2}{p}$ in the complex case, $\alpha_p = \frac{1}{p-1}$ in the real case, we obtain:

$$
\Phi(g) := N(f)^{p-2}(p-1)N(Tg)^{p-1}u_Tg \in R(P)
$$

Since $T$ is $\Sigma$-modular we have $u_T(\varphi, u_h) = \mathbf{1}_{\text{Supp} \varphi, u_T h}$ for every $h \in L_p(H)$ and $\varphi \in L_p$, and more generally $u_T(\varphi, u_{k h}) = \mathbf{1}_{\text{Supp} \varphi, u_{T^k h}}$ for every $k \geq 1$. It is easily deduced that $u_T(\varphi, u_g) = \mathbf{1}_{\text{VS}(f), u_{T^k+1} g}$ for every $k \geq 0$. Then
\[ u_{\Phi^n(g)} = u_{\Phi(\Phi^{n-1}(g))} = \textbf{1vs}(f).u_{T\Phi^{n-1}(g)} = \textbf{1vs}(f).u_{T\Phi^{n-2}(g)} = \textbf{1vs}(f).u_{T\Phi^{n-3}(g)} \ldots = \textbf{1vs}(f).u_{T^n.g} \]

for every \( n \geq 1 \). If \( E_fg(\omega) \neq 0 \) we have:

\[ u_{T^n.g}(\omega) = \frac{\alpha_n^p(g - E_fg)(\omega) + E_fg(\omega)}{N(\alpha_n^p(g - E_fg) + E_fg)(\omega)} \rightarrow \frac{E_fg(\omega)}{N(E_fg)(\omega)} = u_{E_fg}(\omega) \]  

(3.4)

(norm convergence in \( H \)) while if \( E_fg(\omega) = 0 \):

\[ u_{T^n.g}(\omega) = \frac{(g - E_fg)(\omega)}{N((g - E_fg)(\omega))} = u_{g-E_fg}(\omega) = u_g(\omega) \]  

(3.5')

Since \( g - E_fg \perp E_fg \) we have \( N(Tg) \leq N(g) \). Hence:

\[ N(\Phi(g)) = N(f)^{2-p}N(Tg)^{p-1} \leq N(f)^{2-p}N(g)^{p-1} \]  

(3.6)

In particular

\[ N(\Phi(g)) \leq \max(N(f), N(g)) \]  

(3.7)

Reiterating (3.6) we obtain for every \( n \geq 1 \)

\[ N(\Phi^n(g)) \leq N(f)^{(2-p)\sum_{k=0}^{n-1}(p-1)^k}N(g)^{(p-1)^n} = N(f)^{1-(p-1)^n}N(g)^{(p-1)^n} \]

Since \( 0 < p - 1 < 1 \) we obtain

\[ \lim_{n \to \infty} N(\Phi^n(g)) \leq \textbf{1vs}(g)N(f) \]  

(3.8)

We try now to be more precise. If \( E_fg(\omega) = 0 \) we have \( N(Tg)(\omega) = \alpha_pN(g)(\omega) \). Hence

\[ N(\Phi(g))(\omega) = N(f)(\omega)^{2-p}\alpha_pN(g)(\omega)^{p-1} \]  

(3.9)

Moreover since in this case \( u_{\Phi^n(g)}(\omega) = u_g(\omega) \) we have \( E_fg(\Phi^n(g))(\omega) = 0 \) for every \( n \), and we can reiterate. We obtain:

\[ N(\Phi^n(g))(\omega) = (\alpha_p^{p-1}N(f)(\omega)^{(2-p)\sum_{k=0}^{n-1}(p-1)^k}N(g)(\omega)^{(p-1)^n} \]

hence

\[ \lim_{n \to \infty} N(\Phi^n(g))(\omega) = \alpha_p^{(p-1)/(2-p)}\textbf{1vs}(g)(\omega)N(f)(\omega) = \alpha_p^{1/(p-2)}\textbf{1vs}(g)(\omega)N(f)(\omega) \]  

(3.10)

If now \( E_f(g)(\omega) \neq 0 \), we have also \( E_f(\Phi^n(g))(\omega) \neq 0 \) for every \( n \geq 0 \). Set

\[ \beta_n(\omega) = \frac{N(E_f\Phi^n(g))(\omega)}{N(\Phi^n(g))(\omega)} \]  

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We have then:

$$N(T\Phi^n(g))(\omega) \geq \beta_n(\omega)N(\Phi^n(g))(\omega)$$

and consequently:

$$N(\Phi^{n+1}(g))(\omega) \geq N(f)^{2-p} \beta_n(\omega)N(\Phi^n(g))(\omega)^{p-1}$$

(3.11)

On the other hand

$$\beta_n(\omega) = \frac{|\langle u_{\Phi^n}(g), u_f \rangle(\omega)|}{|\langle u_{T^n}(g), u_f \rangle(\omega)|} = \frac{N(E_f T^n(g))(\omega)}{N(T^n(g))(\omega)}$$

and since $N(T^n g) = (\alpha_p^2 N(g - E_f g)^2 + N(E_f g)^2)^{1/2} \gamma N(E_f g)$ pointwise (as $\alpha_p < 1$) we have $\beta_n(\omega) \not\geq 1$ on the set \{\(\omega \mid E_f g(\omega) \neq 0\)\}. Reiterating (3.11) from the step $n = n_0$ we obtain then

$$\liminf_{n \to \infty} N(\Phi^n(g))(\omega) \geq (\beta_{n_0}(\omega))^{1/(p-2)} 1_{\text{VS}(\Phi_{n_0}(g))}(\omega)N(f)(\omega)$$

and letting $n_0 \to \infty$, we have since $\text{VS}(\Phi_n(g)) = \text{VS}(g) \cap \text{VS}(f)$ for every $n$:

$$\liminf_{n \to \infty} N(\Phi^n(g))(\omega) \geq 1_{\text{VS}(g)}(\omega)N(f)(\omega)$$

(3.12)

From (3.4),(3.5),(3.5'), and (3.10),(3.8),(3.12) we deduce that:

$$\Phi^n(g) \to N(f)[u_{E_f,g} + \alpha_p^{1/(p-2)} 1_{\{f \perp g\}} u_g]$$

(3.13)

almost everywhere in $H$-norm, hence in $L_p(H)$-norm by (3.7) and Lebesgue's Theorem. Hence the right-hand member of (3.13) belongs to $R(P)$. Since $u_{E_f,g} = \text{sgn}(\langle g \mid f \rangle) u_f$ the right member of (3.13) can be written:

$$\text{sgn}(\langle g \mid f \rangle) f + \gamma_p 1_{\{f \perp g\}} N(f) u_g = F_p(f,g)$$

(3.14)

where we have set $\gamma_p = \alpha_p^{1/(p-2)}$.

b) Case $1 < p < 2$.

This case is treated by duality. Set $\gamma_p = \gamma_p^{p_*-1}$ and define $F_p(f,g)$ by the formula (3.14). If $g = J_p g'$, $f = J_p h'$ with $f', g' \in L_{p_*}(H)$ we have

$$\text{sgn}(\langle g \mid f \rangle) f = J_{p_*}(\text{sgn}(\langle g' \mid f' \rangle) f')$$

hence

$$\text{sgn}(\langle g \mid f \rangle) f = J_{p_*}(\text{sgn}(\langle g' \mid f' \rangle) f')$$

and similarly

$$N(f) = N(J_{p_*} f') = N(f')^{p_*-1}$$

hence

$$N(f) u_g = N(f')^{p_*-1} u_g = J_{p_*}(N(f') u_g')$$

finally since $\{f \perp g\} = \{f' \perp g'\}$ and $J_{p_*}$ is additive on elements with disjoint functional supports, and positively homogeneous of degree $p_* - 1$,

$$F_p(f,g) = J_{p_*}(F_p(f',g'))$$

Then since $f' = J_p f$, $g' = J_p g$ belong to $R(P^*)$, the function $F_p(f',g')$ belongs to $R(P^*)$ too by the case (a), and $F_p(f,g)$ belongs to $R(P)$. \[\square\]
Corollary 3.2. Let $p$ and $P$ be as in Lemma 3.1. Then for every $f, g \in R(P)$ the three elements $\text{sgn} \langle g \mid f \rangle f, 1_{\{f \perp g\}} f$ and $1_{\{f \perp g\}} N(f) u_g$ belong to $R(P)$.

Proof: The set $\Lambda$ of scalars $\lambda$ such that the set $\{ \omega \in \text{VS}(f) \mid \langle \langle g \mid f \rangle \rangle(\omega) = -\lambda \}$ has positive measure is at most countable. This set is also the set of $\lambda$'s such that $\{(g + \lambda f) \perp f\} \cap \text{VS}(f)$ has positive measure. Choose a sequence $(\varepsilon_n)$ of positive numbers not in $\Lambda \cup (-\Lambda)$ with converges to 0. Then by Lemma 3.1

$$\text{sgn} \langle g \pm \varepsilon_n f \mid f \rangle f \in R(P)$$

for every $n \geq 1$. Since

$$\text{sgn} \langle g \pm \varepsilon_n f \mid f \rangle(\omega) \to \begin{cases} \text{sgn} \langle g \mid f \rangle(\omega) & \text{if } \langle g \mid f \rangle(\omega) \neq 0 \\ \pm 1 & \text{if } \langle g \mid f \rangle(\omega) = 0 \text{ and } f(\omega) \neq 0 \end{cases}$$

we have

$$\text{sgn} \langle g \mid f \rangle f \pm 1_{\{f \perp g\}} f = \lim_n \text{sgn} \langle g \pm \varepsilon_n f \mid f \rangle f \in R(P)$$

and consequently $\text{sgn} \langle g \mid f \rangle f$ and $1_{\{f \perp g\}} f$ belong to $R(P)$. Then $F_p(f, g) - \text{sgn} \langle g \mid f \rangle f = \gamma_p 1_{\{f \perp g\}} N(f) u_g$ belongs to $R(P)$ too. []

Corollary 3.3. Let $p$ and $P$ be as in Lemma 3.1. Then for every $f, g \in R(P)$ we have $1_{\text{vs}(g)} f \in R(P)$.

Proof: By Cor. 3.2, $h := G(f, g) := 1_{\{f \perp g\}} N(f) u_g$ belongs to $R(P)$. Then $G(h, f) = 1_{\{f \perp g\}} 1_{\{u_g \neq 0\}} N(f) u_f = 1_{\text{vs}(g) \cap \{f \perp g\}} f$ belongs to $R(P)$ too. By Cor. 3.2, $f - 1_{\{f \perp g\}} f = 1_{\{f \perp g\}} f \in R(P)$, thus $1_{\text{vs}(g)} f = 1_{\{f \perp g\}} f + 1_{\text{vs}(g) \cap \{f \perp g\}} f \in R(P)$. []

Remark 3.4. In the complex case, for every $f, g \in R(P)$ the elements $\text{sgn}(\text{Re} \langle g \mid f \rangle) f$ and $1_{\{\text{sgn}(\text{Re} \langle g \mid f \rangle) = 0\}} f$ belong to $R(P)$ too. For, $L_p(H)$ is a real Hilbert-valued $L_p(K)$ space, where $K$ is the real vector space $H$ equipped with the scalar product $(x, y)_K = \text{Re}(x \mid y)_H$. As a consequence, the element $1_{\{\text{Re} \langle g \mid f \rangle > 0\}} f = \frac{1}{2}(\text{sgn} \text{Re} \langle g \mid f \rangle + 1_{\{\text{sgn}(\text{Re} \langle g \mid f \rangle) \neq 0\}}) f$ belongs to $R(P)$.

Lemma 3.5. Let $p$ and $P$ be as in Lemma 3.1. For every $f, g \in R(P)$ denote by $\Sigma_{f, \varphi}$ the $\sigma$-field generated by the element $\langle g \mid f \rangle$. Then for every $\Sigma_{f, \varphi}$-measurable function $\varphi$ such that $\varphi \cdot N(f) \in L_p(\Omega, \Sigma, \mu)$, the element $\varphi f$ belongs to $R(P)$.

Proof: Since $R(P)$ is a closed linear subspace, it is sufficient to prove this for indicator functions of $\Sigma_{f, \varphi}$-measurable sets. The sigma-algebra $\Sigma_{f, \varphi}$ is generated by the sets $\{ \text{Re} \langle g \mid f \rangle \lambda \}, \{- \text{Re} \langle g \mid f \rangle \lambda \}, \{ 3 \text{Re} \langle g \mid f \rangle \lambda \}, \{ -3 \text{Re} \langle g \mid f \rangle \lambda \}, \lambda \in \mathbb{R}_+$. If $A_{f, g, \lambda} = \{ \text{Re} \langle g \mid f \rangle \lambda \}$ we have $A_{f, g, \lambda} \ominus (\text{Re} \langle g - \lambda f \mid f \rangle > 0)$, hence $1_{A_{f, g, \lambda}} f \in R(P)$ by Rem. 4. The conclusion is the same for the three others kinds of sets (replacing $g$ by $-g$ or $\pm ig$). Now if $1_B f \in R(P)$ then $A_{f, g, \lambda} \cap B = A'_{f, g, \lambda}$ with $f' = 1_B f$, hence $1_{A_{f, g, \lambda} \cap B} f = f'$. It results that the class $C$ of the sets $A \in \Sigma$ such that $1_A f \in R(P)$ contains finite intersections of sets of the four preceding types. Since $C$ is closed by complementation and monotone limits, it contains the sigma-algebra $\Sigma_{f, \varphi}$. []
Corollary 3.6. Let $p$ and $P$ be as in Lemma 3.1. For every $f \in R(P)$ we have $E_fP = PE_f$.

**Proof:** Let $g \in R(P)$. Applying Lemma 3.5 to the function $\varphi = \frac{\langle g(f) \rangle}{\|f\|}$ we obtain that $E_fg \in R(P)$. Hence for every $h \in L_p(H)$, we have $E_fPh \in R(P)$, i.e. $E_fPh = PE_fPh$; thus $E_fP = PE_fP$. Similarly, reasoning with the contractive projection $P^*$ in $L_p(H)$, and the element $J_{p,f}$ of $R(P^*)$, we have $E_{J_{p,f}}P^* = P^*E_{J_{p,f}}P^*$. Dualizing we obtain $PE_{J_{p,f}}^* = PE_{J_{p,f}}^*P$. We claim that $E_{J_{p,f}}^* = E_{J_{p,f}}$. This will show that $PE_f = PE_fP = E_fP$. Let us show this claim. Since $u_{J_{p,f}} = u_f$, we have for every $g \in L_p(H)$ and $h' \in L_p(H)$:

$$
\langle E_fh, h' \rangle = \int \langle E_fh,h' \rangle \, d\mu = \int \langle \langle g, u_f \rangle u_f, h' \rangle \, d\mu \\
= \int \langle g, u_f \rangle \langle u_f, h' \rangle \, d\mu \\
= \int \langle g, \langle h', u_f \rangle u_f \rangle \, d\mu \\
= \int \langle g, E_{J_{p,f}}h' \rangle \, d\mu = \langle g, E_{J_{p,f}}h' \rangle \quad \Box
$$

**Remark:** The preceding proof of Cor. 3.6 is essentially a real one. In the complex case it can be replaced by a shorter one, of more algebraic nature, due to Arazy and Friedman in the context of spaces $C_p$ (see [AF]). It seemed interesting to us to reproduce this proof in the Annex (see §6), after simplifying it considerably by eliminating the unnecessary non-commutative apparatus.

4. The range of a contractive projection

This section is devoted to the proof of Thm. 0.1, which consists in four lemmas.

**Lemma 4.1.** The closed $L_\infty(\Sigma)$-module $Z$ generated by $R(P)$ in $L_p(H)$ is generated (as $L_\infty$-module) by a family $(f_\alpha)_{\alpha \in A}$ of pairwise orthogonal elements of $R(P)$. We have in fact a Schauder (orthogonal) decomposition

$$
Z = \bigoplus_{\alpha \in A} L_p(\Omega).u_{f_\alpha}
$$

**Proof:** Let $(f_\alpha)_{\alpha \in A}$ be a maximal family of pairwise orthogonal non zero elements of $R(P)$ and $Z_0$ be the closed $L_\infty(\Sigma)$-submodule generated by the family $(f_\alpha)_{\alpha \in A}$. Let $QZ_0$ be the orthogonal projection onto $Z_0$. By the proof of Lemma 1.2 we know that $QZ_0 = \sum_{\alpha \in A} E_{f_\alpha}$ (convergence in strong operator topology) hence by Lemma 2.2 if $p = 1$, resp. Cor. 3.6 if $p > 1$, $QZ_0f \in R(P)$ for every $f \in R(P)$. Since $QZ_0$ is orthogonal and $f_\alpha \in R(QZ_0)$ we have $(f - QZ_0f) \perp f_\alpha$ for every $\alpha \in A$. By maximality of the system $(f_\alpha)$ we deduce that

$$
f = QZ_0f \text{ for every } f \in R(P)
$$

i.e. $QZ_0P = P$. Then $Z_0 = R(QZ_0)$ is a closed $L_\infty$-module containing $R(P)$ and generated by a subset of $R(P)$; hence it coincides with the closed $L_\infty$-module generated by $R(P)$.  

**Lemma 4.2.** There exists a sub-$\sigma$-algebra $F$ of $\Sigma$ containing the vectorial function supports of all elements of $R(P)$ such that for every $f \in R(P)$ and $\varphi \in L_p(\Omega, \Sigma, \mu)$, the product $\varphi.u_f$
belongs to $R(P)$ iff $1_{VS}(f)N(f)^{-1}\varphi$ is $\mathcal{F}$-measurable. In particular $R(P)$ is a $L_\infty(\Omega, \mathcal{F}, \mu)$-submodule.

**Proof:** Since $PE_f = E_f PE_f$ by Lemma 2.1 (if $p = 1$) or by Cor. 3.6 (if $p > 1$), we have $P(\varphi.u_f) \in L_p(\Omega).u_f$ for every $f \in R(P)$ and $\varphi \in L_p(\Omega, \Sigma, \mu)$. We may write $P(\varphi.u_f) = (P_f \varphi).u_f$, with $\text{Supp}(P_f \varphi) \subset VS(f)$. Clearly $P_f$ is linear, $\tilde{P} f_2 = \tilde{P} f$ and

$$\|\tilde{P}f\varphi\|_p = \|P(\varphi.u_f)\| \leq \|\varphi.u_f\| \leq \|\varphi\|_p$$

hence $\tilde{P} f$ is a contractive projection in $L_p(\Omega, \Sigma, \mu)$. Moreover $\tilde{P} f(N(f)) = N(f)$ and $\tilde{P} f \psi = 0$ for every $\psi \in L_p(\Omega, \Sigma, \mu)$ disjoint from $N(f)$.

It results from Douglas’ theorem (in case $p = 1$) or Andô’s theorem (in case $p > 1$) that $\tilde{P} f$ is positive and

$$\tilde{P}f(\varphi) = N(f)E_{\tilde{P} f}(\frac{1_{\text{Supp}(N(f))\varphi}}{N(f)})$$

where $E_{\tilde{P} f}$ is the conditional expectation with respect to some subalgebra $\mathcal{F}_f$ of $\Sigma$ containing $VS(f)$ and to the measure $\nu_f = N(f)^p \ d\mu$. (We may assume that $\Omega \setminus \text{VS}(f)$ is an atom of $\mathcal{F}_f$). In particular $L_p(\Omega, \Sigma, \mu).u_f \cap R(P) = L_p(\Omega, \mathcal{F}_f, \nu_f).f$ is a $L_\infty(\Omega, \mathcal{F}_f, \mu)$-module.

Let us denote $E^f \psi = E^f(1_{VS(f)} \psi)$, we have then $P(\psi.f) = E^f(\psi).f$ for every $\psi \in L_\infty(\Omega, \Sigma, \mu)$. Let now $f, g \in R(P)$. If $g = h.u_f$ with $h \in L^p(\Omega)$ then $h = h_\mathcal{F}_f$ is $\mathcal{F}_f$-measurable and for every $\varphi \in L_\infty(\Omega, \Sigma, \mu)$ we have:

$$E^g(\varphi).g = P(\varphi.h.u_f) = N(f)E^f(\frac{\varphi.h}{N(f)}).u_f = hE^f(\varphi).u_f = E^f(\varphi).g$$

hence

$$E^g(\varphi) = 1_{VS}(g).E^f(\varphi) = 1_{\text{Supp} h}E^f(\varphi)$$

(4.0)

For every $\varphi \in L_\infty(\Omega)$ the equation

$$P(\varphi.(f + g)) = P(\varphi.f) + P(\varphi.g)$$

is equivalent to

$$E^{f+g}(\varphi).(f + g) = E^f(\varphi).f + E^g(\varphi).g$$

(4.1)

Let $g = h.u_f + g'$ be the orthogonal decomposition, i. e. $h = \langle g \mid u_f \rangle$ and $g' \perp f$. Note that $h.u_f = E_f g \in R(P)$. Set $A = VS(f), B = VS(g)$ and $B' = VS(g')$. Taking the images of both sides of (4.1) by the orthogonal projection $I - E_f$ we obtain:

$$E^{f+g}(\varphi).g' = E^g(\varphi).g'$$

hence $1_{B'}E^{f+g}(\varphi) = 1_{B'}E^g(\varphi)$. Then by (4.1) again, $1_{B'}E^{f+g}(\varphi)f = 1_{B'}E^f(\varphi)f$ and finally

$$1_{A \cap B'}E^{f+g}(\varphi) = 1_{A \cap B'}E^f(\varphi) = 1_{A \cap B'}E^g(\varphi)$$

(4.2)

On the other hand similarly to (4.1) we have:

$$E^{h.u_f - g}(\varphi).(h.u_f - g) = E^{h.u_f}(\varphi).h.u_f - E^g(\varphi).g$$

(4.3)
since \( h.u_f - g = -g' \) we deduce that

\[
1_{\Omega \setminus B'} \mathbb{E}^{h u_f} (\varphi). h u_f = 1_{\Omega \setminus B'} \mathbb{E}^g (\varphi). g
\]

hence:

\[
1_{B \setminus B'} \mathbb{E}^{h u_f} (\varphi) = 1_{B \setminus B'} \mathbb{E}^g (\varphi) \tag{4.4}
\]

We have \( \mathbb{E}^{h u_f} (\varphi) = 1_{\text{Supp} h} \mathbb{E}^f (\varphi) \) by eq. (4.0). Hence since \( B \setminus B' \subseteq \text{Supp} h \), eq. (4.4) gives

\[
1_{B \setminus B'} \mathbb{E}^f (\varphi) = 1_{B \setminus B'} \mathbb{E}^g (\varphi) \tag{4.5}
\]

which together with eq. (4.2) gives:

\[
1_{A \cap B} \mathbb{E}^f (\varphi) = 1_{A \cap B} \mathbb{E}^g (\varphi) \tag{4.6}
\]

for every \( \varphi \in L_\infty (\Omega, \Sigma, \mu) \). In particular

\[
\text{1 vs} (f) \cap \text{vs} (g) = \text{1 vs} (f) \cap \text{vs} (g) \mathbb{E}^g (1_{\text{vs} (g)})
\]

\[
= 1_{\text{vs} (f) \cap \text{vs} (g)} \mathbb{E}^f (1_{\text{vs} (g)})
\]

\[
= 1_{\text{vs} (f) \cap \text{vs} (g)} \mathbb{E}^f (1_{\text{vs} (f) \cap \text{vs} (g)})
\]

hence

\[
\mathbb{E}^f (1_{\text{vs} (f) \cap \text{vs} (g)}) \geq 1_{\text{vs} (f) \cap \text{vs} (g)}
\]

and since \( \mathbb{E}^f \) is a contraction in \( L_p (\Omega, \Sigma, N(f)^p, \mu) \) we have in fact

\[
\mathbb{E}^f (1_{\text{vs} (f) \cap \text{vs} (g)}) = 1_{\text{vs} (f) \cap \text{vs} (g)}
\]

that is \( \text{VS} (f) \cap \text{VS} (g) \in \mathcal{F}_f \). In particular \( 1_{\text{vs} (g)} \cdot f = 1_{\text{vs} (g)} \cap \text{vs} (g) \cdot f \in R_P \). More generally for every \( A \in \mathcal{F}_g \), its trace \( \text{VS} (f) \cap A \) belongs to \( \mathcal{F}_f \) (as is easily seen by treating separately the cases \( A \subseteq \text{VS} (g) \) and \( A = \Omega \setminus \text{VS} (g) \)). Let \( \mathcal{F} \) be the \( \sigma \)-algebra consisting of sets \( A \in \Sigma \) such that \( \forall N \text{VS} (f) \) belongs to \( \mathcal{F}_f \) for every \( f \in R_P \). Then for every \( f \in R_P \) and \( \varphi \in L_0 (\Omega, \Sigma, \mu) \) the function \( 1_{\text{vs} (f)} \varphi \) is \( \mathcal{F} \)-measurable iff it is \( \mathcal{F}_f \)-measurable, and the Lemma follows.

**Lemma 4.3.** There is a weight \( w \in L_0 (\Omega, \Sigma, \mu) \) with support \( \text{VS} (R_P) \) such that for every \( f \in R_P \), \( w^{-1} N(f) \) is \( \mathcal{F} \)-measurable.

**Proof:**

a) First we claim that for every \( f, g \in R_P \) then \( 1_{\text{vs} (f)} N(g) \) is \( \mathcal{F} \)-measurable. Since \( E_{\varphi} g = \langle g | u_f \rangle u_f \in R_P \) by Lemma 2.2, it results from Lemma 4.2 that \( N(f)^{-1} \langle g | u_f \rangle = N(f)^{-2} \langle g | f \rangle \) is \( \mathcal{F} \)-measurable; hence its absolute value \( N(f)^{-2} |\langle g | f \rangle| \) is \( \mathcal{F} \)-measurable, and similarly \( N(g)^{-2} |\langle f | g \rangle| \) is \( \mathcal{F} \)-measurable too. Then the ratio of these functions, that is \( 1_{\text{Supp} \langle g | f \rangle} N(g)^2 N(f)^{-2} \) is \( \mathcal{F} \)-measurable, and so is its square root \( 1_{\text{Supp} \langle g | f \rangle} N(g) N(f)^{-1} \). Replacing \( g \) by \( g_\varepsilon = g + \varepsilon f, \varepsilon > 0 \) we obtain that \( 1_{\text{Supp} \langle g_\varepsilon | f \rangle} N(g_\varepsilon) N(f)^{-1} \) is \( \mathcal{F} \)-measurable. When \( \varepsilon \to 0 \) we have \( g_\varepsilon \to g, N(g_\varepsilon) \to N(g) \) (in \( L_p \)-norm) and \( \text{Supp} \langle g_\varepsilon | f \rangle \to \text{Supp} N(f) = \text{VS} (f) \) (in probability). At the limit \( 1_{\text{vs} (f)} N(g) \) is \( \mathcal{F} \)-measurable.

b) Let \( (f_i)_{i \in I} \) be a maximal family in \( R_P \) with pairwise almost disjoint functional supports \( \text{VS} (f_i) \). Then \( \text{VS} (R_P) = \bigvee_{i \in I} \text{VS} (f_i) \); if \( f \in R_P \) then since \( S = \bigvee_{i \in I} \text{VS} (f_i) \) belongs to \( \mathcal{F} \), so does its complementary set \( S^c \), and thus \( 1_{S^c} f \in R_P \); then by maximality
of the family \((f_i)\) we have \(1_{S^c}.f = 0\), that is \(f = 1_S.f\). We set \(w = \sum_{i \in I} N(f_i)\) (which converges in \(L_0(\Omega, \Sigma, \mu)\)): this is a \(\Sigma\)-measurable weight with support \(\mathbf{VS}(R(P))\). For every \(f \in R(P)\) and every \(i \in I\), \(1_{\mathbf{VS}(f_i)}w^{-1}N(f) = 1_{\mathbf{VS}(f_i)}N(f_i)^{-1}N(f)\) is \(\mathcal{F}\)-measurable; hence \(w^{-1}N(f) = \sum_{i \in I} 1_{\mathbf{VS}(f_i)}w^{-1}N(f)\) is \(\mathcal{F}\)-measurable. \(\square\)

We can now give the proof of the Th 0.1. For, consider the new measure \(\nu = w^{\mu}\), which has support \(\Omega_\nu = \mathbf{VS}(R(P))\) and set \(T : L_p(\Omega_\nu, \Sigma, \mu) \to L_p(\Omega_\nu, \Sigma, \nu)\), defined by \(Tf = w^{-1}f\) (we denote by \(\Sigma_\nu\) the trace of \(\Sigma\) on \(\Omega_\nu\)). Then \(T\) is an isometry; \(Y := (T \otimes \text{Id}_H)(R(P))\) is a \(L_\infty(\mathcal{F}_\nu)\)-module isometric to \(R(P)\) and for every \(f \in Y\) its new random norm \(\tilde{N}(f) = w^{-1}N(f)\) belongs to \(L_p(\Omega_\nu, \mathcal{F}_\nu, \nu)\). It results from an argument in [LR] that \(Y\) is isometric to \(\bigoplus_{i \in I} L_p(\Omega_i, \mathcal{F}_{[\Omega_i]} , \nu_{[\Omega_i]}; H_i)\), for some families \((\Omega_i)\) of pairwise almost disjoint sets in \(\mathcal{F}\) and \((H_i)\) of Hilbert spaces. Set then \(\tilde{w}_i = (\mathbb{E}(1_{\Omega_i}w^{\mu} | \mathcal{F}))^{1/p}\), and define an isometry \(S_i : L_p(\Omega_i, \mathcal{F}_{[\Omega_i]}, \nu_{[\Omega_i]} ) \to L_p(\Omega_i, \mathcal{F}_{[\Omega_i]}, \mu_{[\Omega_i]} )\) by \(S_if = \tilde{w}_if\). Then each \(S_i \otimes \text{Id}_{H_i}\) is an onto isometry \(L_p(\Omega_i, \mathcal{F}_{[\Omega_i]}, \nu_{[\Omega_i]}; H_i) \to L_p(\Omega_i, \mathcal{F}_{[\Omega_i]}, \mu_{[\Omega_i]}; H_i)\); the collection of these isometries induces an isometry of the corresponding \(\ell_p\)-direct sums. The proof of Th. 0.1 is complete.

Let us finally adapt to the present situation the argument of [LR] for the commodity of the reader (and for further reference in section 5).

**Lemma 4.4.** Let \((\Omega, \Sigma, \nu)\) be a localizable measure space, \(\mathcal{F}\) be a sub-sigma algebra such that \((\Omega, \mathcal{F}, \nu)\) is still localizable and \(H\) be a Hilbert space. Let \(Y\) be a closed \(L_\infty(\mathcal{F})\)-submodule of \(L_p(\Omega, \Sigma, \nu; H)\) such that for every \(f \in Y\) its random norm \(N(f)\) is \(\mathcal{F}\)-measurable. Then there exists a family \((\Omega_i)_{i \in I}\) of pairwise almost disjoint members of \(\mathcal{F}\), a family \((H_i)\) of Hilbert spaces (of lower Hilbertian dimension than \(H\)) and a random norm preserving isometry from \(Y\) onto \(\bigoplus_{i \in I} L_p(\Omega_i, \mathcal{F}_{[\Omega_i]}, \nu_{[\Omega_i]}; H_i)\).

**Proof:** Note that by an elementary polarization argument all the scalar products \(\langle f | g \rangle\), \(f, g \in Y\) are \(\mathcal{F}\)-measurable. Hence for every \(f \in Y\), the projection \(E_f\) restricts to a projection from \(Y\) onto \(L_p(\Omega, \mathcal{F}); uf\). It results that for every closed \(L_\infty(\mathcal{F})\)-submodule \(Z\) of \(Y\) there is an orthogonal projection from \(Y\) onto \(Z\) (which is the restriction of the orthogonal projection from \(L_p(\Omega, \Sigma; H)\) onto the closed \(L_\infty(\Sigma)\)-submodule generated by \(Z\)). In particular \(Y = Z \oplus (Z^\perp \cap Y)\).

Remark that if \(A \subset \mathcal{F}\) is \(\nu\)-sigma-finite and \(M \subset Y\) is a closed \(L_\infty(\mathcal{F})\)-submodule such that \(\mathbf{VS}(M) \supset A\) then there exists \(g \in M\) such that \(\mathbf{VS}(g) = A\): take a maximal family \((g_n)\) in \(M\) of norm-one elements with almost disjoint functional supports included in \(A\); this family is necessarily at most countable and \(\bigvee_n \mathbf{VS}(g_n) = A\); then set \(g = \sum_n 2^{-n}g_n\).

Now we claim that for every \(A \subset \mathcal{F}\), \(A \subset \mathbf{VS}(Y)\) with positive measure, there exists a \(\mathcal{F}\)-measurable subset \(B\) of \(A\) of positive measure and a family of pairwise orthogonal elements \((f_\gamma)_{\alpha \in \Gamma_\alpha}\) such that \(\mathbf{VS}(f_\gamma) = B\) for every \(\gamma \in \Gamma_\alpha\), which generates \(1_B.Y\) as closed \(L_\infty(\mathcal{F})\)-submodule. For, let \(A' \subset A\) be a sigma-finite \(\mathcal{F}\)-measurable subset with positive measure, and \((g_\gamma)_{\gamma \in \Gamma}\) be a maximal family of pairwise orthogonal elements of \(Y\) with \(\mathbf{VS}(g_\gamma) = A'\). If this family generates \(1_{A'}Y\) as closed \(L_\infty(\mathcal{F})\)-submodule we can take \(B = A'\). If not, consider the set \(M = \{f \in Y \mid f \perp g_\gamma, \forall \gamma \in \Gamma\}\). Then \(M\) is a closed \(L_\infty(\mathcal{F})\)-submodule of \(Y\), and \(\mathbf{VS}(M) \not\supset A'\) by the maximality of \((g_\gamma)_{\gamma \in \Gamma}\) (and the preceding remark). Let \(B = A' \setminus \mathbf{VS}(M)\), then \((1_B g_\gamma)_{\gamma \in \Gamma}\) is a maximal family in \(1_B.Y\) of nonzero, pairwise orthogonal elements of \(1_B.Y\). Consequently it generates \(1_B.Y\) as \(L_\infty(\mathcal{F})\)-submodule, and moreover \(\mathbf{VS}(1_B g_\gamma) = B\) for every \(\gamma \in \Gamma\).

Let now \((\Omega_i)_{i \in I}\) be a maximal family of \(\mathcal{F}\)-measurable almost disjoint subsets of \(\mathbf{VS}(Y)\),
of positive measure, such that there exists for each \(i \in I\) a family \((f_\gamma^i)_{\gamma \in \Gamma_i}\), of pairwise orthogonal elements with \(\mathbf{VS}(f_\gamma^i) = \Omega_i\) for every \(\gamma \in \Gamma_i\), which generates \(1_{\Omega_i}Y\) as closed \(L_\infty(\mathcal{F})\)-submodule. By the claim, we have \(\bigvee_{i \in I} \Omega_i = \mathbf{VS}(Y)\). Every \(f \in 1_{\Omega_i}Y\) can be written \(f = \sum_{\gamma \in \Gamma_i} \varphi_\gamma f_\gamma^i\) with \(\varphi_\gamma \in L_0((\Omega_i, \mathcal{F}_{|\Omega_i}, \nu)\); then \(N(f) = \sum_{\gamma \in \Gamma_i} |\varphi_\gamma|^2 N(f_\gamma^i)^2)^{1/2} \in L_\sigma(\Omega_i, \mathcal{F}_{|\Omega_i}, \nu)\).

Note that, by refining if necessary the “partition” \((\Omega_i)\) we may suppose that each \(\Omega_i\) has finite \(\nu\)-measure. Then, replacing each \(f_\gamma^i\) by \(u f_\gamma^i = N(f_\gamma^i)^{-1} f_\gamma^i\), we may assume that \(N(f_\gamma^i) = 1_{\Omega_i}\). We have then \(N(f) = \sum_{\gamma \in \Gamma_i} |\varphi_\gamma|^2)^{1/2}\) for each \(f = \sum_{\gamma \in \Gamma_i} \varphi_\gamma f_\gamma^i\) in \(1_{\Omega_i}Y\). Let \(\mathcal{H}_i = \ell^2(\Gamma_i)\). Then \(T_i: L_p(\Omega_i, \mathcal{F}_{|\Omega_i}, \nu; \mathcal{H}_i) \to 1_{\Omega_i}Y, \sum_{\gamma \in \Gamma_i} \varphi_\gamma e_\gamma \mapsto \sum_{\gamma \in \Gamma_i} \varphi_\gamma u f_\gamma^i\) is an (onto) isometry preserving the random norm, and finally \(Y\) is isometric to \((\bigoplus_{i \in I} L_p(\Omega_i; \mathcal{H}_i))_{\ell_p}\).

For proving the assertion about Hilbertian dimension of \(\mathcal{H}_i\), suppose that for some \(i \in I\), the hilbertian dimension \(d_H\) is strictly smaller than that, \(d_{H_i}\), of \(H_i\). We distinguish two cases:

- if \(H\) is finite dimensional: select a finite subset \(\Gamma_i^f\) of \(\Gamma_i\) with cardinality \(d_H + 1\); since \(\langle f_\gamma^i | f_\delta^i \rangle = 0\) for every \(\gamma \neq \delta \in \Gamma_i^f\), there exists \(\omega \in \Omega\) such that \(\langle f_\gamma^i | f_\delta^i \rangle(\omega) = 0\), i.e. the vectors \(f_\gamma^i(\omega), \gamma \in \Gamma_i^f\) of \(H\) are pairwise orthogonal: a contradiction.

- if \(H\) is infinite dimensional: for every \(x \in H\) the set \(\{ \gamma \in \Gamma_i \mid \langle f_\gamma^i | 1_{\Omega_i}x \rangle \neq 0 \}\) is at most countable (since \(\sum_{\gamma} |\langle f_\gamma^i | 1_{\Omega_i}x \rangle|^2 \leq N(1_{\Omega_i}) \|x\|^2\)), hence if \(D\) is a dense set in \(H\) of cardinality \(d_H\), the set \(\{ \gamma \in \Gamma_i \mid \exists x \in D, \langle f_\gamma^i | 1_{\Omega_i}x \rangle \neq 0 \}\) has cardinality \(d_H < d_{H_i} = \# \Gamma_i\). Hence there exists some \(\gamma \in \Gamma_i\) such that \(f_\gamma^i \perp 1_{\Omega_i}x\) for every \(x \in D\), and thus for every \(x \in H\), which means \(f_\gamma^i = 0\), a contradiction. \(\square\)

**Remark 4.5:** The final argument in the proof of Lemma 4.4 shows indeed that \(L_p(\Omega, \Sigma, \nu; \mathcal{H})\) embeds in \(L_p(\Omega, \Sigma, \nu; H)\) by a modularly isometric map then \(\dim \mathcal{H} \leq \dim H\).

**Remark 4.6:** In a forthcoming paper ([HR]) it will be proved that contractively complemented sublattices of \(L_p(L_q)\) are isometric to “abstract \(L_p(L_q)\) spaces”, i.e. bands in \(L_p(L_q)\) spaces. Let us show how this permits to deduce shortly the essence of Th. 0.1 from Lemma 4.1.

As in the proof of Lemma 4.1 let \((f_\alpha)_{\alpha \in A}\) be a maximal family of non zero, pairwise orthogonal elements of \(R(P)\) and \(Z = \bigoplus_{\alpha} L_p(\Omega, \Sigma, \mu).u_{f_\alpha}\) be the closed \(L_\infty(\Sigma)\)-submodule generated by \(R(P)\). There is clearly a \(\Sigma\)-modular isometry \(U\) from the closed submodule \(Z\) onto a band \(Y\) of the Banach lattice \(L_p(\Omega, \Sigma, \mu; \mathcal{H})\) where \(\mathcal{H}\) is the discrete Banach lattice \(\ell_2(A)\), such that \(Ue_\alpha = N(f_\alpha)e_\alpha\), where \((e_\alpha)_{\alpha \in A}\) is a Hilbertian basis of \(\mathcal{H}\). Then \(P | Z\) is similar by \(U\) to a contractive projection \(\hat{P}\) of \(Y\) which preserves the spaces \(Y_\alpha = L_p(A_\alpha).e_\alpha\) (where \(A_\alpha = \text{Supp } N(f_\alpha)\)) by Lemma 2.1 if \(p = 1\) and Cor. 3.6 if \(p > 1\), as well as the elements \(N(f_\alpha).e_\alpha\). By the classical (scalar) theorem of Douglas if \(p = 1\), Andô if \(p > 1\), \(\hat{P} | Y_\alpha\) is positive and its image is a sublattice of \(Y_\alpha\). Since \(Y = \bigoplus_{\alpha} Y_\alpha\) is a decomposition in disjoint subbands, \(\hat{P}\) is itself positive and its range is a sublattice of \(Y\), hence of \(L_p(\mathcal{H})\). By the analysis of contractive projections on sublattices in \(L_p(L_q)\)-spaces developed in [HR], the range \(R(\hat{P})\) is an abstract \(L_p(L_2)\)-space, hence by [LR] it is Banach-isometric to a \(\ell_p\)-direct sum \(\bigoplus_{i \in I} L_p(\Omega_i, H_i)\), where the \(H_i\) are Hilbert spaces. \(\square\)
5. Structure of the contractive projections

**Theorem 5.1.** Let $1 \leq p < \infty$, $p \neq 2$. For every contractive projection $P$ of $L_p(H)$ there exist a family $(u_\gamma)_{\gamma \in \Gamma}$ of pairwise orthogonal elements of $L_\infty(H)$, a positive contractive projection $\tilde{P}$ of $L_p(\Omega)$ and, if $p = 1$, a contractive linear operator $V : L_1(H) \to L_1(H)$ verifying $\ker V \supset 1_A L_1(H)$ where $A = \bigvee_{\gamma \in \Gamma} \mathbf{VS}(u_\gamma)$, and $R(V) \subset \sum_{\gamma} R(\tilde{P}).u_\gamma$, such that:

$$Pf = \begin{cases} \sum_{\gamma} \tilde{P}(\langle f \mid u_\gamma \rangle)u_\gamma & \text{if } p \neq 1 \\ \sum_{\gamma} \tilde{P}(\langle f \mid u_\gamma \rangle)u_\gamma + V(f) & \text{if } p = 1 \end{cases} \quad (5.1)$$

for every $f \in L_p(H)$.

Conversely for every family $(u_\gamma)_{\gamma \in \Gamma}$ of pairwise orthogonal elements of $L_\infty(H)$, every positive contractive projection $\tilde{P}$ of $L_p(\Omega)$ [and every linear contraction $V$ of $L_1(H)$ satisfying the previous conditions of kernel and range in the case $p = 1$], the formula (5.1) defines a contractive projection $P$ of $L_p(H)$.

Moreover if $p \neq 1$ the inequality $N(Pf) \leq \tilde{P}(N(f))$ holds for every $f \in L_p(H)$ [this happens also for a contractive projection of $L_1(H)$ for which the operator $V$ of formula (5.1) is zero]. ($\tilde{P}$ is a “majorizing $L_p$-contraction” for $P$ in the terminology of [GR]).

The proof of Th. 5.1 will require the two following Lemmas, the first of which is specific to the $p = 1$ case:

**Lemma 5.2.** Let $P$ be a contractive projection in $L_1(H)$. Then $Pf = 0$ for every $f \in L_1(H)$ with $\mathbf{VS}(f) \subset \mathbf{VS}(R(P))$ and $f \perp R(P)$.

**Proof:** Assume that $f \perp R(P)$ and $\mathbf{VS}(f) \subset \mathbf{VS}(h)$ for some $h \in R(P)$. Then $g := Pf + 1_{\mathbf{VS}(Pf)^\circ}.h$ belongs to $R(P)$ and $\mathbf{VS}(g) \supset \mathbf{VS}(f) \cup \mathbf{VS}(Pf)$. We have for every $t > 0$:

$$\int (N(g)^2 + t^2 N(f)^2)^{1/2} d\mu = \|g + tf\|
\geq \|P(g + tf)\| = \|g + tPf\|
= (1 + t)\|Pf\| + \|1_{\mathbf{VS}(Pf)^\circ}.h\|
= \|g\| + t\|Pf\|$$

Hence:

$$\|Pf\| \leq \lim_{t \to 0} \left( \frac{\|g + tf\| - \|g\|}{t} \right) = \lim_{t \to 0} \int \frac{N(g)^2 + t^2 N(f)^2)^{1/2} - N(g)}{t} d\mu = 0 \quad \square$$

**Lemma 5.3.** Let $P$ be a contractive projection in $L_p(H)$. There exists a positive contractive projection $\tilde{P}$ on $L_p(\Omega, \Sigma, \mu)$ such that $P(\varphi.u_f) = (\tilde{P}\varphi).u_f$ for every $f \in R(P)$ and $\varphi \in L_p(\Omega, \Sigma, \mu)$.

**Proof:** Let $\mathcal{F}$ be the $\sigma$-algebra of Lemma 4.2 and $w$ be the weight of Lemma 4.3. Define $\tilde{P}_f(\varphi)$ as in the proof of Lemma 4.2. Recall that for every $f \in R(P)$ the function $w^{-1}N(f)$
The formula (5.1) in Thm. 5.1 is now clear if we set

\[ \tilde{P}_f \varphi = 1_{\mathcal{V}(f)} \tilde{P} \varphi \]

if we set \( \tilde{P}_f = w \tilde{P}_{w_{\mu}}(1_{\mathcal{V}(R(P))} w^{-1} \varphi) \) for every \( \varphi \in L_p(\Omega, \Sigma, \mu) \). Then \( P \) is a positive contractive projection in \( L_p(\Omega, \Sigma, \mu) \) and \( P(\varphi, u_f) = \tilde{P}(\varphi).u_f \) for every \( f \in R(P) \) and \( \varphi \in L_p(\Omega, \Sigma, \mu) \). 

**Proof of Thm. 5.1:** Let \( Q \) be the orthogonal projection from \( L_p(H) \) onto the closed submodule generated by \( R(P) \). It results from Lemma 4.1 that if \( (f_\gamma)_{\gamma \in \Gamma} \) is a maximal family of pairwise orthogonal elements of \( R(P) \) then \( Q = \sum_{\gamma \in \Gamma} E_{f_\gamma} \) (convergence for s.o.t.), hence \( PQ = \sum_{\gamma \in \Gamma} PE_{f_\gamma} \). If \( p > 1 \) we know by Cor. 3.6 that \( E_{f_\gamma} P = PE_{f_\gamma} \) for every \( \gamma \), hence \( P = QP = PQ \). If \( p = 1 \) let \( \Pi : L_1(H) \to L_1(H) \) the projection defined by \( \Pi f = 1_{\mathcal{V}(R(P))} f \), then \( \Pi \) and \( I - \Pi \) are contractive. We have \( Q \Pi = \Pi Q = Q \) and it results from the preceding Lemma 5.2 that \( P(I - Q) \Pi = 0 \). Hence \( P = PQ + V \), where \( V = P(I - \Pi) \).

Let us express now \( PE_f \) when \( f \in R(P) \). If \( \tilde{P} \) is the positive projection in \( L_p(\Omega, \Sigma, \mu) \) defined in Lemma 5.3 we have for every \( g \in L_p(\Omega, \Sigma, \mu; H) \)

\[ PE_f g = P(\langle g \mid u_f \rangle).u_f = \tilde{P}(\langle g \mid u_f \rangle).u_f \]

The formula (5.1) in Thm. 5.1 is now clear if we set \( u_\gamma = u_{f_\gamma} \).

Conversely given \( (u_\gamma) \), \( \tilde{P} \) and \( V \), let us prove first that \( P \) is a contraction. We have for every finite subset \( G \) of \( \Gamma \) (using the positivity of \( \tilde{P} \)):

\[
N(\sum_{\gamma \in G} \tilde{P}(\langle f \mid u_\gamma \rangle)u_\gamma) = (\sum_{\gamma \in G} |\tilde{P}(\langle f \mid u_\gamma \rangle)|^2)^{1/2} \\
= \sqrt{\left\{ \sum_{\gamma \in G} a_\gamma \tilde{P}(\langle f \mid u_\gamma \rangle) \right\}^2} | a_\gamma |^2 \leq 1 \\
\leq \tilde{P} \left( \sqrt{\left\{ \sum_{\gamma \in G} a_\gamma \langle f \mid u_\gamma \rangle \right\}^2} | a_\gamma |^2 \leq 1 \right) \\
= \tilde{P} \left( \sum_{\gamma \in G} |\langle f \mid u_\gamma \rangle|^2 \right)^{1/2}
\]
Hence \( \| \sum_{\gamma \in \Gamma} \tilde{P}(\langle f | u_\gamma \rangle) u_\gamma \|_p^p \leq \int (\sum_{\gamma \in \Gamma} |\langle f | u_\gamma \rangle|^2)^{p/2} \, d\mu \) and the sum \( P_0 f := \sum_{\gamma \in \Gamma} \tilde{P}(\langle f | u_\gamma \rangle) u_\gamma \) converges in \( L_p(H) \). Moreover

\[
N(P_0 f) \leq \tilde{P} \left( \langle \sum_{\gamma \in \Gamma} |\langle f | u_\gamma \rangle|^2 \rangle^{1/2} \right) \leq \tilde{P}(N(1_A \cdot f))
\]

and

\[
\| P_0 f \| \leq \| 1_A \cdot f \|
\]

where \( A = \sqrt{\mathbf{V} \mathbf{S}}(u_\gamma) \). That \( P_0 \) is a projection follows immediately from the fact that \( \tilde{P} \) is. If \( p = 1 \) we have to care with the contraction \( V \). Since \( \| V f \| \leq \| 1_{A^c} \cdot f \| \) we obtain \( \| P f \| \leq \| 1_A \cdot f \| + \| 1_{A^c} \cdot f \| = \| f \| \). Then since \( V P_0 = 0, P_0 V = V \), it follows clearly that \( P = P_0 + V \) is a projection. \( \qed \)

We can now give the structure theorem for contractive projections:

**Theorem 5.4.** For every contractive projection \( P \) of \( L_p(\Omega, \Sigma, \mu; H) \) \((1 \leq p < \infty, p \neq 2)\) there exist:

- a modularly isometric automorphism \( W \) of \( L_p(H) \);
- a family \((\Omega_i)_{i \in I}\) of pairwise almost disjoint \( \Sigma \)-measurable subsets of \( \Omega \) of positive measure;
- a family \((\mathcal{H}_i)_{i \in I}\) of Hilbert spaces;
- for every \( i \in I \) a (strong operator) measurable family \((U_{i,\omega})_{\omega \in \Omega_i}\) of isometric embeddings of \( \mathcal{H}_i \) into \( H \);
- a positive contractive projection \( \tilde{P} \) of \( L_p(\Omega, \Sigma, \mu) \) commuting with the band projections associated with the sets \( \Omega_i \);
- and if \( p = 1 \) a contraction \( V \) from \( L_1(S, \Sigma_{|S}, \mu_{|S}; H) \) into \( R(P) \), where \( S = \Omega \setminus \bigcup_i \Omega_i \) such that (setting \( V = 0 \) if \( p > 1)\):

\[
P = WP(U_i \cdot \omega, \Id_{\mathcal{H}_i}) U_i^* W^{-1} + V
\]

where \( M_{\Omega_i} \) denotes the multiplication operator by the characteristic function \( 1_{\Omega_i} \); \( U \) is the modularly isometric embedding of \( \bigoplus L_p(\Omega_i, \Sigma_{|\Omega_i}, \mu_{|\Omega_i}; \mathcal{H}_i) \) into \( L_p(H) \) naturally associated with the family \((U_{i,\omega})_{i \in I, \omega \in \Omega_i}\) by mean of the formula:

\[
(U f)(\omega) = U_{i,\omega}(f(\omega)) \quad \text{when} \quad \omega \in \Omega_i
\]

and similarly \( U_i^* : L_p(H) \rightarrow \bigoplus L_p(\Omega_i, \Sigma_{|\Omega_i}, \mu_{|\Omega_i}; \mathcal{H}_i) \) is the modularly contractive map associated with the family \((U_{i,\omega}^*)_{i \in I, \omega \in \Omega_i}\).

**Remark 5.5.** In fact the families \((U_{i,\omega})_{\omega \in \Omega_i}\) may be chosen locally constant, i.e., there is a partition of \( \Omega_i \) in \( \Sigma \)-measurable subsets of positive measure on which \( U_{i,\omega} \) is constant.

**Remark 5.6.** In the case where \( H \) is separable, it is a standard (and easy) fact that every modularly isometric automorphism \( W \) of \( L_p(H) \) is associated with a measurable family \((W_{\omega})_{\omega \in \Omega} \) of unitary operators on \( H \); so we recover the Theorem 0.3 of the Introduction.
Proof: By the proof of Thm 0.1 in section 4, there are a sub-$\sigma$-algebra $F$ of $\Sigma$, a family $(\Omega_i)_{i \in I}$ of pairwise almost disjoint elements of $F$, a positive weight $w$ on $\Omega$ with support $\bigvee_{i \in I} \Omega_i$, a family $(\mathcal{H}_i)_{i \in I}$ of Hilbert spaces and for every $i \in I$ an isometry $T_i$ from $L_p(\Omega_i, F_{\Omega_i}, \nu_{\Omega_i}; \mathcal{H}_i)$ into $L_p(\Omega, F_{\Omega}, \nu_{\Omega}; H)$ such that $R(P) = \bigoplus_{i \in I} w_i R(T_i)$ and moreover $N(T_if) = N(f)$ for all $f \in L_p(\Omega_i, F_{\Omega_i}, \nu_{\Omega_i}; H)$ (recall that $\nu = w^p \mu$). Moreover $P$ commutes with the action of $L_\infty(F)$, in particular with the multiplication operators $M_{\Omega_i}$.

Each $T_i$ extends unica to a modularly isometric map $\tilde{T}_i$ from $L_p(\Omega_i, \Sigma_{\Omega_i}, \mu_{\Omega_i}; H_i)$ onto the closed $L_\infty(\Sigma)$-submodule generated by $R(T_i)$ in $L_p(\Omega, \Sigma, \nu; H)$; set simply $\tilde{T}_i(\sum_k \varphi_k f_k) = \sum_k \varphi_k T_i(f_k)$ when $\varphi_1, \ldots, \varphi_n \in L_\infty(\Sigma, \Sigma_\Omega)$ and $f_1, \ldots f_k \in L_p(\Omega_i, F_{\Omega_i}, \nu_{\Omega_i}; H_i)$ and verify that $N(\sum_k \varphi_k T_i(f_k)) = N(\sum_k \varphi_k f_k)$ (since $T_i$ preserves the random scalar products).

Now define $S_i : L_p(\Omega_i, \Sigma_{\Omega_i}, \mu_{\Omega_i}; H_i) \to L_p(\Omega, \Sigma, \mu; H)$ by $S_i f = w \tilde{T}_i(w^{-1} f)$: the range $R(S_i) = w R(\tilde{T}_i)$ is exactly $1_{\Omega_i} Z_i$, where $Z_i$ is the closed $L_\infty(\Sigma)$-submodule generated by $R(P)$. We can glue up the maps $S_i$ and obtain a modularly isometric embedding $S$ from $\bigoplus_{i \in I} L_p(\Omega_i, \Sigma_{\Omega_i}, \mu_{\Omega_i}; H_i)$ into $L_p(\Omega, \Sigma, H)$, with range $R(S) = Z$.

By Lemma 5.3 there exists a positive projection $\tilde{P}$ on $L_p(\Omega, \Sigma, \mu)$ such that $P(\varphi_{u_f}) = (\tilde{P} \varphi).u_f$ for every $f \in R(P)$ and $\varphi \in L_p(\Omega, \Sigma, \mu)$. Note that $\tilde{P}$ is $\mathcal{F}$-modular, in particular it commutes with every multiplication operator $M_{\Omega_i}, i \in I$.

If $A \subseteq F$ is a $\nu$-integrable subset of $\Omega_i$ and $e \in H_i$ we have $S_i(1_A w \otimes e) = w T_i(1_A \otimes e) \in R(P)$, and $N(S_i(1_A w \otimes e)) = N(1_A w \otimes e) = 1_A w$, and consequently for $f = S_i(1_A w \otimes e)$ we have $f = w u_f$. Thus for every $\psi \in L_\infty(\Omega, \Sigma, \mu) \cap L_p(\Omega, \Sigma, \mu)$ we have

$$
PS_i(\psi 1_A w \otimes e) = P(\psi. w u_f) = \tilde{P}(\psi. w 1_{\Omega_i}).u_f = \tilde{P}(\psi. w 1_{\Omega_i}).w^{-1} S_i(1_A w \otimes e)
$$

$$
= S_i(\tilde{P}(\psi. w 1_{\Omega_i})).w^{-1} 1_A w \otimes e = S_i(\tilde{P}(\psi. w 1_{\Omega_i})1_A \otimes e)
$$

hence by linearity and density we have for every $\varphi \in L_p(\Omega_i, \Sigma_{\Omega_i}, \mu_{\Omega_i})$ and $e \in H_i$:

$$
PS_i(\varphi \otimes e) = S_i(\tilde{P}(\varphi) \otimes e)
$$

that is, the restriction of $P$ to $1_{\Omega_i} Z_i$ is similar by $S_i$ to the projection $\tilde{P} \otimes \text{id}_{H_i}$; consequently the restriction of $P$ to $Z$ is similar by $S_i$ to the projection $\sum_{i \in I} \tilde{P} M_{\Omega_i} \otimes \text{id}_{H_i}$.

In the case where $Z = L_p(\Omega, \Sigma, \mu; H)$ we have necessarily $\dim H = \dim H_i$ for every $i \in I$ since $S_i$ is a modularly isometric map from $L_p(\Omega_i; H_i)$ onto $1_{\Omega_i} Z = L_p(\Omega; H)$ (see Remark 4.5). Thus we may assume that $H_i = H$ and the conclusion of Thm. 5.4 is obtained with $W = S$ and $U = \text{Id}$.

In the general case we apply Lemma 4.4 to the $L_\infty(\Sigma)$ submodule $Z^\perp$. We find a family $(\Omega'_{ij})_{j \in J}$ of pairwise almost disjoint members of $\Sigma$, a family $(K_{ij})_{j \in J}$ of Hilbert spaces and a modularly isometric map $S'$ from $\bigoplus_{j \in J} L_p(\Omega'_{ij}, \Sigma_{\Omega'_{ij}}, \mu_{\Omega'_{ij}}; H')$ onto $Z^\perp$. Note that now the sets $\Omega'_{ij}$ have no reason to belong to the smaller $\sigma$-algebra $F$. We have $\bigvee_j \Omega'_{ij} = V S (Z^\perp)$. For the commodity of the notation we may assume $\bigvee_j \Omega'_{ij} = \Omega$, adding if necessary one extra set $\Omega_0 = \Omega \setminus \bigvee_j \Omega'_{ij}$ for which we set $K_0 = \{0\}$, the 0-dimensional Hilbert space. Similarly, up to the cost of adding one extra set $\Omega_0 = \Omega \setminus \Omega_P$ and setting $H_0 = \{0\}$, we may assume that $\bigvee_j \Omega_i = \Omega$. We may also refine the partition $(\Omega'_{ij})$ by setting $\Omega'_{ij} = \Omega_i \cap \Omega'_{ij}$ and removing the $\Omega'_{ij}$ which are almost void. This operation gives a doubly indexed family $(\Omega'_{ij})_{i \in I, j \in J}$.

For every $i \in I, j \in J$, set $L_{ij} = H_i \oplus K_j$ (direct Hilbertian sum). Then $L_p(\Omega'_{ij}; H_i)$ and $L_p(\Omega'_{ij}; K_j)$ identify naturally to a pair of mutually orthogonal $L_\infty(\Sigma)$-submodules of
Now define $W_{ij}^0 : L_p(\Omega_{ij}; L_{ij}) \to L_p(\Omega_{ij}; H)$ by $W_{ij}^0 f = S_i(U_{ij}^{0g} f) + S_j(U_{ij}^{0g} f)$: we have

$$N(W_{ij}^0 f)^2 = N(S_i(U_{ij}^{0g} f))^2 + N(S_j(U_{ij}^{0g} f))^2 = N(U_{ij}^{0g} f)^2 + N(U_{ij}^{0g} f)^2 = N(f)^2$$

since $S_i$ and $S_j'$ are modularly isometric and have values in orthogonal subspaces $Z$, resp. $Z^\perp$. Hence $W_{ij}^0$ is modularly isometric and $R(W_{ij}^0) = 1_{\Omega_{ij}} Z + 1_{\Omega_{ij}} Z^\perp = L_p(\Omega_{ij}; H)$.

We know by the proof of Thm. 5.1 that $P = PQ + V$, where $Q$ is the orthogonal projection onto $Z$. Since $V$ satisfies the requirements of the theorem, we look only for a representation of $P_0 = PQ$. From the first part of the proof we now that $P_0S_i = S_i(\tilde P \otimes id_{\mathcal H_i})$; on the other hand $P_0S_j' = 0$ since $R(S_j') \subset Z^\perp = ker Q$. Hence for every $f \in L_p(\Omega_{ij}; L_{ij})$:

$$P_0W_{ij}^0 f = P_0S_iU_{ij}^{0g} f + P_0S_j'U_{ij}^{0g} f = S_i(\tilde P \otimes id_{\mathcal H_i})U_{ij}^{0g} f = W_{ij}^0(U_{ij}^{0g}(\tilde P \otimes id_{\mathcal H_i})U_{ij}^{0g} f$$

i. e. $P_0$ is similar by $W_{ij}^0$ to $U_{ij}^0(\tilde PM_{\Omega_{ij}} \otimes id_{\mathcal H_i})U_{ij}^{0g}$.

Since $L_p(\Omega_{ij}; L_{ij})$ is modularly isometric to $L_p(\Omega_{ij}; H)$ (by $W_{ij}^0$), we have $\dim L_{ij} = \dim H$ by Rem. 4.5, so we may identify $L_{ij}$ with $H$ by an isomorphism $\theta_{ij}$. This isomorphism induces in turn a modular isometry $\Theta_{ij} = Id \otimes \theta_{ij}$ from $L_p(\Omega_{ij}; L_{ij})$ onto $L_p(\Omega_{ij}; H)$. Set $W_{ij} = W_{ij}^0 \Theta_{ij}^{-1}$: then $W_{ij}$ is a modular automorphism of $L_p(\Omega_{ij}; H)$. Let also $u_{ij} = \theta_{ij} \circ u_{ij}^0$ be the embedding of $\mathcal H_i$ into $H$ resulting from this identification and $U_{ij} = id_{L_p(\Omega_{ij})} \otimes u_{ij} = \Theta_{ij}U_{ij}^{0g}$ be the associated embedding of $L_p(\Omega_{ij}; \mathcal H_i)$ into $L_p(\Omega_{ij}; \mathcal H_i)$. Since $\Theta_{ij}^{-1} = \Theta_{ij}'$ we see that $P_0$ is similar by $W_{ij}$ to $U_{ij}(\tilde PM_{\Omega_{ij}} \otimes id_{\mathcal H_i})U_{ij}^\sharp$.

Finally we glue up the automorphisms $W_{ij}$ to an automorphism $W$ of $L_p(\Omega; \mu; H)$ by setting

$$Wf = \sum_{i \in I} \sum_{j \in J_i} W_{ij} M_{\Omega_{ij}} f$$

and similarly we glue up the embeddings $U_{ij}$ to an embedding $U$ of $\bigoplus_{i \in I} L_p(\Omega_i; \mathcal H_i)$ into $L_p(\Omega; H)$. The maps $W$ and $U$ are still modularly isometric and $P_0$ is similar by $W$ to $U(\sum_{i \in I} \tilde PM_{\Omega_i} \otimes id_{\mathcal H_i})U^\sharp$. \( \square \)


The following proof is an adaptation of that of Thm. 4.1 in [AF]. We assume that $2 < p < \infty$ (the case $1 < p < 2$ follows by duality).

If $f \in R(P)$ we introduce besides the projection $E_f$ (defined in §1) the operators $F_f$ and $G_f$ defined by:

$$F_fg = 1_{VS(f)}g ; \ G_fg = 1_{VS(f)}g - E_fg$$

Then $E_f, G_f$ and $F_f$ are commuting modularly contractive projections in $L_p(H)$ with $E_f + F_f + G_f = I$. 21
Let \( f, g \in R(P) \), then the elements \( A(f, g) \) and \( B(f, g) \) defined in §3 (eqs 3.1 and 3.1bis) belong to the range of \( P^* \); so do the sum and difference: \( M_f(g) := A(f, g) + B(f, g) \) and 
\[
\Gamma_f(g) := \frac{p}{p-2} [A(f, g) - B(f, g)]
\]
belong to \( R(P^*) \). Set:
\[
Q_f g = \langle u_f, g \rangle u_f
\]
We have then:
\[
M_f(g) = N(f)^{p-2} [2g + (p - 2) E_f g]
\]
\[
\Gamma_f(g) = p N(f)^{p-2} Q_f g
\]
Then \( M_f \), resp. \( \Gamma_f \) are bounded linear, resp. antilinear operators from \( L_p(H) \) into \( L_p(H) \), and \( Q_f \) is a contractive antilinear endomorphism of \( L_p(H) \) such that \( Q_f^2 = E_f \); moreover:
\[
M_f P = P^* M_f P \quad ; \quad \Gamma_f P = P^* \Gamma_f P
\]  
(6.1)
Consider the positive symmetric bounded bilinear form defined on \( L_p(H) \) by:
\[
(g, h)_f := \langle M_f(g), h \rangle = \int N(f)^{p-2} \langle (2I + (p - 2) E_f) g \mid h \rangle \, d\mu
\]
Note that \( \Gamma_f = M_f Q_f = Q_{J_P} M_f \) and \( Q_f^* = Q_{J_P} \); then \( Q_f \) is hermitian for \( (.,.)_f \) since:
\[
(Q_f g, h)_f = \langle M_f Q_f g, h \rangle = \langle Q_f g, M_f h \rangle = \langle Q_{J_P} M_f h, g \rangle = \langle M_f Q_f h, g \rangle = (Q_f h, g)_f
\]
On the other hand \( P \) is hermitian for \( (.,.)_f \) since (using (6.1))
\[
(P g, h)_f = \langle M_f P g, h \rangle = \langle P^* M_f P g, h \rangle = \langle M_f P g, Ph \rangle = (P g, Ph)_f
\]
\[
= \overline{(P h, P g)_f} = (P h, g)_f = (g, Ph)_f
\]
Let \( N_f \) be the kernel of the form \( (.,.)_f \): we have \( g \in N_f \) iff \( (g, g)_f = 0 \) iff \( (g, h)_f = 0 \) for all \( h \in L_p(H) \) (by Cauchy-Schwartz inequality). Then \( P N_f \subset N_f \) since
\[
(P g, P g)_f = (g, P g)_f = 0 \quad \text{if} \quad g \in N_f
\]
On the other hand the operator \( 2 \cdot 1_{VS}(f) + (p - 2) E_f \) maps \( L_p(H) \) onto \( 1_{VS}(f) L_p(H) \); hence \( g \in N_f \) iff \( \langle N(f)^{p-2} g, h \rangle = 0 \) for every \( h \in 1_{VS}(f) L_p(H) \) iff \( N(f)^{p-2} g = 0 \) iff \( 1_{VS}(f) g = 0 \).
We have thus: \( R(F_f) = N_f \) and consequently
\[
PF_f = F_f P F_f
\]
Since \( L_p(H) \) is a strictly convex Banach space as well as its dual, we have by the auxiliary Lemma 6.1 below:
\[
PF_f = F_f P
\]
Let us show that \( Q_f P \) is hermitian for \( (.,.)_f \), using eq. (6.1) again:
\[
(Q_f P g, h)_f = \langle M_f Q_f P g, h \rangle = \langle \Gamma_f P g, h \rangle
\]
\[
= \langle P^* \Gamma_f P g, h \rangle = \langle \Gamma_f P g, Ph \rangle
\]
\[
= \langle Q_f P g, Ph \rangle = (Q_f Ph, g)_f
\]
Since $Q_f$ and $P$ are separately hermitian for $(.,.)_f$ we have:

$$(Q_fP_g,h)_f = (PQ_fh,g)_f$$

hence $(PQ_f - Q_fP)h \in N_f$, i.e. $(I - F_f)PQ_f = (I - F_f)Q_fP$. Composing on the left by $G_f$ and on the right by $Q_f$, or conversely, we obtain:

$$G_fPE_f = 0 = E_fPG_f$$

Since on the other hand:

$$F_fPE_f = PF_fE_f = 0 = E_fF_fP = E_fPF_f$$

we obtain:

$$PE_f = E_fPE_f = E_fP$$

We state now and give a proof of the announced auxiliary Lemma.

**Lemma 6.1.** Let $X$ be a strictly convex Banach space with strictly convex dual, and $P, Q$ two contractive projections on $X$. The following conditions are equivalent:

i) $PQ$ is a projection

ii) $PQ = QPQ$

iii) $PQ = PQP$

If moreover the complementary projection $Q^\perp$ is contractive too then $PQ = QP$.

**Proof:** If (ii) is verified then $(PQ)^2 = PQPQ = PP = PQ$; while if (iii) is verified then $(PQ)^2 = PQPQ = PQP = PQ$. Hence both (ii) and (iii) imply (i) (without any contractiveness assumption). Conversely if (i) is verified then for every $x \in R(PQ)$ we have $x = Qx = PQx$ (by [AF], Prop. 1.1 (iii); only the strict convexity of $X$ is needed) so $x \in R(P) \cap R(Q)$. Since the converse is trivial, we see that $R(PQ) = R(P) \cap R(Q)$; in particular $Q P Q = PQ$ and (ii) is verified. Dualizing we have that $P^*, Q^*$ and $Q^*P^*$ are contractive projections in $X^*$; hence $Q^*P^* = P^*Q^*P^*$, so $PQ = PQP$ and (iii) is verified.

Now (iii) implies $PQ^\perp = PQ^\perp P$, and if $Q^\perp$ is contractive this implies $PQ^\perp = Q^\perp PQ^\perp$ by the preceding. Then

$$Q = PQ + P Q^\perp = PQQ + Q^\perp PQ^\perp$$

which in turn implies $QP = PQ = PQP$.

**Remark:** The final assertion $PQ = QP$ of Lemma 6.1 is stated in [AF] (for $X = C_p$) as Cor. 1.7 without the assumption that the complementary projection $Q^\perp$ is contractive. This statement is not correct: if $p \neq 2$ it is easy to construct rank 1 contractive projections $P, Q$ in $X = \ell_p$ or $C_p$ such that $PQ = 0 \neq QP$: choose non zero elements $a, b \in X$ such that their norming functionals $Ja, Jb$ verify $\langle Ja, b \rangle = 0$ and $\langle Jb, a \rangle \neq 0$ and set $P = a \otimes Ja$, $Q = b \otimes Jb$.

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References


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