



**HAL**  
open science

# What Can Abrupt Events Tell Us About Sustainability ?

Can Askan Mavi

► **To cite this version:**

| Can Askan Mavi. What Can Abrupt Events Tell Us About Sustainability ?. 2017. hal-01628682

**HAL Id: hal-01628682**

**<https://hal.science/hal-01628682>**

Preprint submitted on 3 Nov 2017

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# What Can Abrupt Events Tell Us About Sustainability ?

Can Askan Mavi<sup>a\*</sup>

November 3, 2017

<sup>a</sup> *Université Paris 1 Panthéon Sorbonne, Paris School of Economics*

## Abstract

This paper aims to analyze the overlooked link between abrupt events and sustainability, through limit cycle analysis. We use the well known [Calvo and Obstfeld \(1988\)](#) framework in order to distinguish individual's and social planner's discount rate and show that individual discount rate could lead to a Hopf bifurcation, only if the economy is exposed to abrupt event risk. This result is in contrast with the literature which shows that individual discount rate does not have any effect on the optimum trajectory of aggregate consumption. More importantly, the existence of limit cycles implies that consumption and natural resource stock are exposed to cycles at the long run, meaning that the path of utility does not conform with the prominent Sustainable Development criterion. Lastly, we analyze the economic reasons behind limit cycles and show that protecting the environment makes limit cycles less likely to occur.

## 1 Introduction

The fact that uncertain abrupt climate events could cause large scale damages is widely recognized ([Alley et al. \(2003\)](#), [Field et al. \(2012\)](#)). A very considerable number of studies focus on decision making concerning exploitation policy of natural resources under uncertainty ([Bretschger and Vinogradova \(2017\)](#), [Tsur and Zemel \(1998, 2007a, 2016a\)](#), [Clarke and Reed \(1994\)](#)). Some recent literature examines also the ways to deal with the uncertainty. For this purpose, adaptation and mitigation policies and their implications under uncertainty is one of the major point of interest ([Zemel \(2015\)](#), [Tsur and Zemel \(2015\)](#), [Mavi \(2017\)](#)).

Other than the studies on uncertainty and resource exploitation, another branch of the literature concentrates on the relationship between discounting and sustainability which is one of the lasting important debate in economics literature. Especially, the debate has been intensified in the context of climate change ([Stern \(2006\)](#), [Weitzman \(2007\)](#), [Heal \(2009\)](#)). Some of the studies within this debate concern the role of individual time preferences (see [Endress et al. \(2014\)](#), [Schneider et al. \(2012\)](#), [Marini and Scaramozzino \(1995, 2008\)](#), [Burton \(1993\)](#)). The presence of individual time preferences in an economic model is appealing because infinitely lived agent model (ILA hereafter) is criticized for not respecting consumer sovereignty<sup>1</sup>.

---

\*Corresponding Author : Can Askan Mavi, Université Paris 1 Panthéon-Sorbonne, Paris School of Economics, 48, Boulevard Jourdan 75014 Paris. e-mail : [camavi@univ-paris1.fr](mailto:camavi@univ-paris1.fr)

<sup>†</sup>I am grateful to Antoine d'Autume, Katheline Schubert, Hippolyte d'Albis, Katrin Millock, Mouez Fodha, Bertrand Wigniolle, Fabien Prieur, Lionel Ragot and Karine Constant for helpful comments and discussions. I also thank Regulation and Environment and DDEE seminar participants at PSE and University of Paris-Nanterre - EconomiX. All errors and omissions remain mine.

<sup>1</sup>When there is only a unique social discount rate, we can not distinguish individual impatience from social planner's impatience level. This is one of the ethical objections to ILA model.

The articles cited above which are incorporating individual discount rate are based on the famous framework proposed by [Calvo and Obstfeld \(1988\)](#). Authors are introducing individual time preferences (i.e individual discount rate) in an OLG model. Then, they find statically aggregate consumption level of all generations at a given time. Once aggregation is made, the model reduces to representative agent framework. This framework is well used in environmental economics to treat various problem as inter-generational equity<sup>2</sup> by the above cited papers. In these models, there is any effect of individual discount rate on aggregate long term dynamics, neither on transitional dynamics. One may easily say that model introduces a clear dichotomy between an OLG and ILA model since individual discount rate has nothing to tell about the aggregate dynamics of the economy.

However, on one hand, the studies treating long term impacts of uncertainty on resource exploitation policy do not take into account sustainability and intergenerational equity. On the other hand, the strand of the literature on sustainability and intergenerational equity does not take into account uncertain events. In this sense, to the best of our knowledge, we can argue that the link between sustainability and abrupt events is overlooked in environmental economics literature.

In this paper, our aim is to make a bridge between sustainability and uncertain abrupt events. To do this, we take a step further to answer the following central questions : What are the implications of abrupt events regarding the sustainability of an economy ? What can individual time preferences tell about sustainability in an economy subject to abrupt events ?

The contribution of this paper is threefold : First, by using the [Calvo and Obstfeld \(1988\)](#), we show that an economy exposed to an abrupt event could have limit cycles (Hopf bifurcation) at the long run. On the one hand, the economy accumulates physical capital and creates wastes. In this sense, the environment is used as a “sink”. This can be considered as an economic goal. On the other hand, since the inflicted damage after the abrupt event is proportional to remaining natural resource stock, the economy wishes to protect the natural resource. This is the environmental goal. When it becomes difficult to steer between two conflicting policies, it may be optimal to cycle around the optimal steady state<sup>3</sup>.

One may argue that this trade-off between the economic and environmental goal is usual in growth models with environment. For this purpose, we show rigorously that with waste stemming from physical capital accumulation and the abrupt event probability, the preferences of the economy over time are not independent in the sense of Koopmans (see [Koopmans \(1960\)](#)). Then, when the economy faces the above mentioned trade-off coupled with complementarity over time, it is likely that the economy is exposed to limit cycles (see [Dockner and Feichtinger \(1991\)](#), [Heal and Ryder \(1973\)](#)).

Indeed, limit cycles are examined extensively in environmental economics. [Wirl \(1999, 2004\)](#) and [Bosi and Desmarchelier \(2016a, 2017\)](#) study the existence of limit cycles in models with representative agent framework. Nonetheless, none of these studies link limit cycles to sustainability.

At this point, the question to be addressed is : what are the implications of limit cycles regarding sustainability ? The Sustainable Development Criterion requires that the utility of consumption should have a non-decreasing path (i.e  $\frac{du(c(t))}{dt} \geq 0$ ). If the economy is exposed to limit cycles due to the trade-off between the environmental and the economic goal, consumption increases at one date and decreases at another date.

---

<sup>2</sup>Contributions made in intergenerational equity discussions argue that ILA framework as a utilitarian social welfare function corresponds to a different generation at each point of time (see [Schneider et al. \(2012\)](#)).

<sup>3</sup>Note that cycles around steady state are optimal.

Then, the Sustainable Development Criterion is not respected since the utility has a cyclical behavior at the long run.

Secondly, contrary to the [Calvo and Obstfeld \(1988\)](#) framework and to the articles using this framework, we show that individual time preferences could change the stability properties of the model. This result disproves the conventional result which says that aggregate dynamics are solely governed by the social planner's discount rate (see [Endress et al. \(2014\)](#), [Schneider et al. \(2012\)](#), [Marini and Scaramozzino \(1995, 2008\)](#), [Burton \(1993\)](#)). Indeed, we show that the individual discount rate has an important role concerning the sustainability of an economy.

Since the first part of the model is an OLG model, there is an intra-generational allocation of consumption which is stable over time. We show that intra-generational equity comes as a cost since it is more likely to have a stable equilibrium when social planner allocates unequally the consumption over different cohorts of age. It is shown that when the consumption is allocated equally between generations, the economy faces limit cycles at the long run which compromises the sustainability of the economy.

The reason why individual discount rate can raise limit cycles is the following : when the social planner aggregates the consumption over all agents, she finds an aggregate utility function and post value function which are dependent on social and individual discount rate. Different levels of individual discount rate changes the weight of utility function and post value function in the objective function of the social planner. This is equivalent to say that the trade-off between the economic and the environmental goal changes with respect to different individual discount rate.

It is argued that sustainability and intergenerational equity are generally perceived as normative inquiries ([Solow \(2005, 2006\)](#)). Then, a social planner who pays attention to the sustainability and intergenerational equity should seek to avoid limit cycles. We show that the social planner could avoid limit cycles by protecting the environment. This is due to the fact that a higher natural resource stock implies a lower marginal utility of consumption. As a result, different levels of individual discount rate could not change so much the trade-off between the economic and the environmental goal. Consequently, there are less chances that an economy traps to limit cycles at the long run.

The remainder of the paper is organized as follows. The section 2 presents the benchmark model and explains all economic mechanisms behind limit cycles in detail. Section 3 explains the model with environmental protection and focuses from a normative point of view in order to avoid limit cycle and last section concludes the paper.

## 2 Model

Similar to [Calvo and Obstfeld \(1988\)](#), the model is constituted by two parts. In the first part, the social planner allocates consumption across different cohorts of age in an overlapping generations model (OLG). At each time  $t$ , the economy is composed by individuals ranged between age 0 and  $N$  and also no two individuals have the same age. In the second part, the model is solved first by aggregating the consumption over all agents. At this point, the model reduces to an infinitely-lived agent model and one can maximize intertemporal utility and find optimal aggregate trajectories of the economy.

The lifetime utility of an individual born at date  $T$  (or equivalently, the aggregate utility of all living generations at a given date  $t$ ) is written

$$U(c(t)) = \int_0^N u(c(t, \tau)) e^{-\beta\tau} d\tau \quad (1)$$

where  $\beta$  is the individual discount.  $c(t, \tau)$  is the consumption level of an agent with age  $\tau$  at time  $t$ . The utility of the society is a weighted sum of discounted utilities of individuals of the society augmented by the post-event value.

$$W_b^C = E_{\bar{b}} \left[ \int_{-N}^{\bar{b}} \left\{ \int_0^N u(c(t, \tau)) e^{-(\beta-\rho)\tau} d\tau \right\} e^{-\rho b} db + e^{-\rho\bar{b}} \varphi(S(\bar{b})) \right] \quad (2)$$

The social planner weights generations by her discount rate  $\rho$ . An abrupt event occurs at an uncertain date  $\bar{b}$  at which a generation born. This generation can be considered as an “unlucky” generation. The variable  $S$  represents the natural capital stock. The expression  $\varphi(S)$  stands for the post-value function that describes the value of the problem after a catastrophic event. The constant parameter  $\theta$  represents Poisson probability of catastrophic event. In our model, similar to [Bretschger and Vinogradova \(2017\)](#), the use of constant hazardous event probability implies that the environmental shock is to a flow variable rather than a stock variable.

In the following model, note that natural capital accumulation is negatively affected by physical capital accumulation, which is similar to [Wirl \(2004\)](#) and [Ayong Le Kama \(2001\)](#). In this sense, nature is considered as a “sink” for wastes coming from physical capital. Essentially, this feature creates a trade-off between capital accumulation and its negative effects to nature that we will analyze in depth within this study. The social planner seeks to choose the optimal level of consumption of agents  $c(t, \tau)$  for each time  $t$  subject to two physical constraints, i.e.,

$$\begin{cases} \max_{c(t, \tau)} W_b^C \\ \dot{K}(t) = f(K(t)) - \delta K(t) - \int_0^N c(t, \tau) d\tau \\ \dot{S}(t) = G(S(t)) - \gamma f(K(t)) \end{cases} \quad (3)$$

The maximization program can be solved with optimal control techniques. Since the state variables  $K$  and  $S$  evolves in terms of pure time  $t$ , we can solve the problem of the social planner by taking the expectations of the objective function and reformulating it in terms of time  $t$ , instead of born date  $b$  of generations (see Appendix for calculations). For this purpose, we switch from  $(b, \tau)$  to  $(t, \tau)$  by using  $b = t - \tau$ . We can reformulate the social planner’s program in the light of these information.

$$\max_{c(t)} W = \int_0^\infty \{U(c(t)) + \theta\varphi(S(t))\} e^{-(\rho+\theta)t} dt \quad (4)$$

subject to

$$\begin{cases} \dot{K}(t) = f(K(t)) - \delta K(t) - c(t) \\ \dot{S}(t) = G(S(t)) - \gamma f(K(t)) \end{cases} \quad (5)$$

where the objective function consists of the allocation of consumption among different groups of age by the social planner.

$$U(c(t)) + \theta\varphi(S(t)) = \max \int_0^N [u(c(t, \tau)) + \theta\varphi(S(\bar{b}))] e^{-(\beta-(\rho+\theta))\tau} d\tau$$

The utility function of different cohorts is maximized statically with respect to constraint

$$\int_0^N c(t, \tau) d\tau \leq c(t) \quad (6)$$

Once the social planner allocates the consumption between different cohorts of agent, she aggregates the consumption over all agents at a given date  $t$ .

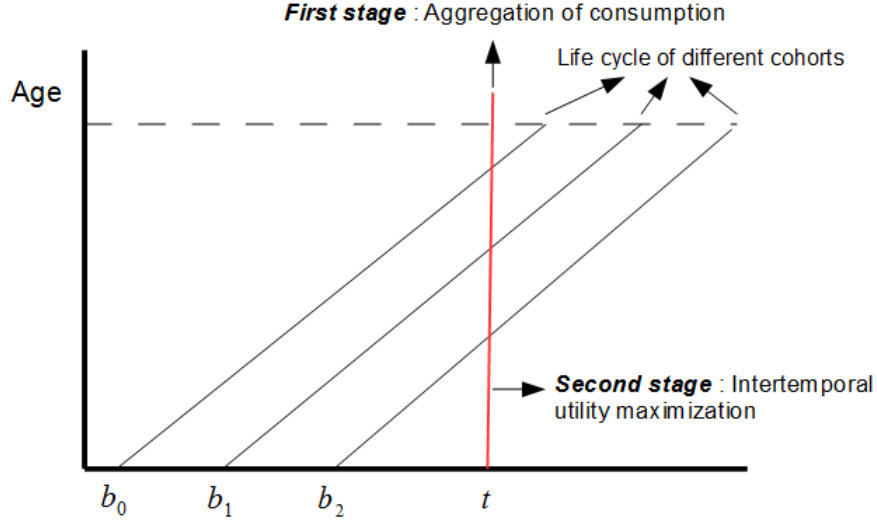


Figure 1: Allocation and aggregation of consumption

The use of exogenous event probability can be also justified by the first stage optimization. In the OLG model, one should ensure that the allocation of consumption across individuals is constant over time. Otherwise, with the use of an endogenous hazardous event, since the utility becomes non-stationary over time (see [Koopmans \(1960\)](#)), the social planner's decisions will be time inconsistent.

### First stage optimization : A static problem

In the first stage, the social planner finds a relationship between  $c(t, 0)$  and  $c(t, \tau)$ . Indeed, the social planner finds the optimal consumption allocation between different agents. This stage of the problem consists on a static optimization in an OLG model where  $c(t)$  is the aggregate consumption level at time  $t$ . We write the static optimization problem

$$\mathcal{L} = \int_0^N [u(c(t, \tau)) e^{-(\beta-(\rho+\theta))\tau} + \theta\varphi(S(b))] d\tau + \lambda(t) \left[ c(t) - \int_0^N c(t, \tau) d\tau \right]$$

For the sake of simplicity, after the abrupt event, the social planner does not maximize the utility since the consumption level is assumed to reduce to a subsistence level  $c_{min}$  (see [Bommier et al. \(2015\)](#), [Tsur and Zemel \(2016a\)](#)). Then the post-event value does not depend on  $c(t, \tau)$ . From this problem, we find easily

$$u_c(c(t, \tau)) = u_c(c(t, 0)) e^{(\beta-(\rho+\theta))\tau} \quad (7)$$

By using a CRRA utility function,

$$u(c(t, \tau)) = \frac{c(t, \tau)^{1-\sigma} - c_{min}^{1-\sigma}}{1-\sigma} \quad (8)$$

The maximization of the static program yields

$$c(t, \tau) = c(t, 0) e^{-\frac{(\beta - (\rho + \sigma))\tau}{\sigma}}$$

The social planner gives more weight to younger generations when the individual discount rate is higher than her discount rate. This makes sense when one thinks about the sum of discounted utility of young and old agents. Since the utility of young agents are discounted over a longer time horizon, the social planner gives much more weight to a young agent than an old one. In addition, the social planner allocates equally consumption across agents when both individual and social discount rate coincide. Note that social planner tends to treat generations more equally when individual discount rate is closer to her own discount rate. Graphically, it is easy to see how a social planner could allocate the consumption between different cohorts at some time  $t$ .

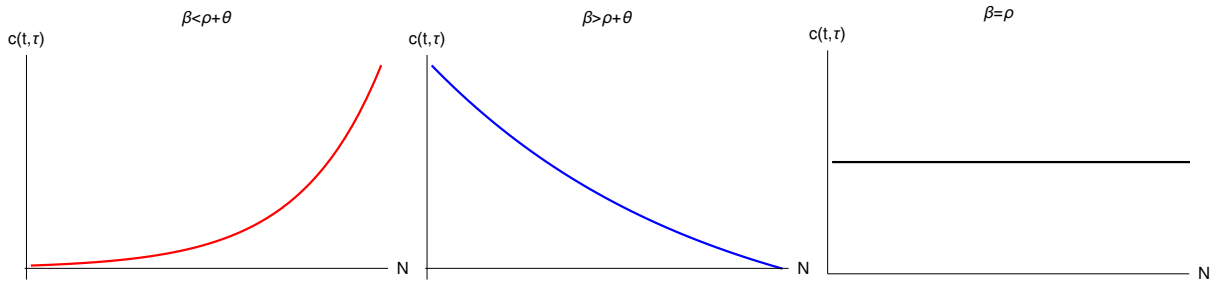


Figure 2: Allocation of consumption across generations

The analysis of the effect of abrupt event probability on intragenerational allocation is also appealing. Regarding the explanation above, one can observe that a higher abrupt event probability shifts the consumption allocation from young to older cohorts.

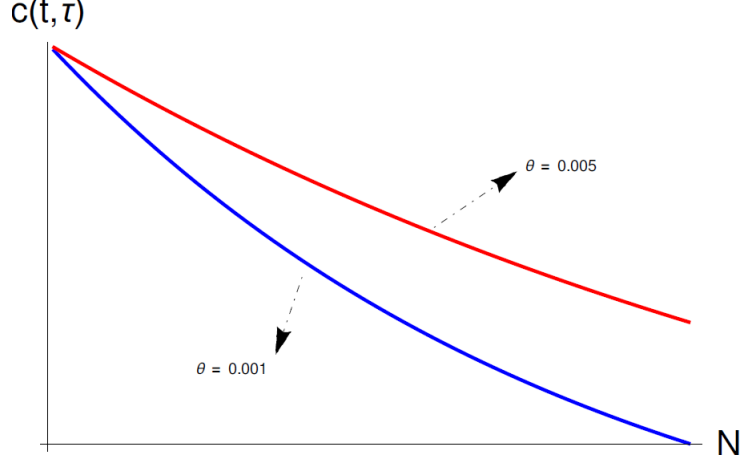


Figure 3: Effect of risk on intragenerational allocation

This is due to the fact that a higher event probability, adjusted discount rate of social planner  $\rho + \theta$  increases. Then, older agent's utility starts to take more weight with a higher abrupt event probability, which shifts the consumption from younger agents to older cohorts. We find aggregate utility function (see Appendix for calculations).

Once the social planner allocates consumption across agents, we need to find the utility of aggregate level of consumption and post-event value. With the CRRA form utility function, we have (see Appendix for details.)

$$U(c(t)) = \frac{c(t)^{1-\sigma} - c_{min}^{1-\sigma}}{1-\sigma} (Z_1)^\sigma \quad (9)$$

where  $Z_1$  is the aggregation term which includes individual discount rate. The post value function  $\varphi(S) = u(c_{min}) - \psi(S)$  can be written in a similar way to [Tsur and Zemel \(1998, 2016a\)](#). After the aggregation of post-event value, we have

$$\varphi(S) = Z_2 (u(c_{min}) - \psi(S)) \quad (10)$$

where  $Z_2$  is the aggregation term of the post-value function. We take the simplest possible form of post-value function in line with [Tsur and Zemel \(2016a\)](#). (see Appendix for the derivation concerning the aggregation of different cohorts of age.) The single occurrence event implies that consumption is reduced to a minimum level<sup>4</sup> which does not provide utility. The penalty rate due to abrupt event is proportional to the level of natural capital stock after catastrophe.

$$\psi(S) = \bar{\psi}(\omega_1 - \omega_2 \log S) \quad (11)$$

where  $\omega_1$  is the unrecoverable part of penalty. We put the condition that  $\omega_1 > \omega_2 \log S$  to ensure that some part of the damage to the environment is permanent. The use of the penalty function implies that more the natural capital stock is protected, the less the economy suffers from penalty.

<sup>4</sup>The minimum level of consumption can be thought as a subsistence consumption. The reduction of consumption can be imposed by the social planner as a political response to appease the public outrage. (See [Tsur and Zemel \(2016a\)](#) for a detailed discussion.)



## Second stage optimization : A dynamic problem

The social planner maximizes the program (4) subject to physical and natural capital accumulation constraints (5). The current-value Hamiltonian for maximizing  $W$

$$\mathcal{H} = U(c) + \theta\varphi(S) + \lambda(f(K) - \delta K - c) + \mu(G(S) - \gamma f(K)) \quad (12)$$

The first order conditions and dynamics of the economy are as follows

$$\begin{cases} U_c = Z_1^{-\sigma} c^{-\sigma} = \lambda \\ \dot{K} = f(K) - \delta K - Z_1 \lambda^{-\frac{1}{\sigma}} \\ \dot{S} = G(S) - \gamma f(K) \\ \dot{\lambda} = (\rho + \theta)\lambda - (f_K - \delta)\lambda + \mu\gamma f_K \\ \dot{\mu} = (\rho + \theta)\mu - \mu G_S - \theta\varphi_S \end{cases} \quad (13)$$

By using the functional forms above, the steady-state of the economy can be written as a function of renewable resource stock  $S$  :

$$\begin{cases} K^*(S) = \frac{(1-S^*)S^*}{\gamma A} \\ \lambda^*(S) = \frac{\gamma A \mu^*(S)}{((A-\delta) - (\rho+\theta))} \\ \mu^*(S) = \frac{Z_2 \psi \omega_2 \theta}{S^*((\rho+\theta) - (1-2S^*))} \end{cases} \quad (14)$$

**Proposition 1.** *In an economy with catastrophic event probability, a Hopf bifurcation is possible for different critical parameters of individual discount rate  $\beta$ .*

It is important to figure out the economic reasons behind the occurrence of limit cycles. On one hand, more natural capital stock is high, less the penalty rate would be. (see Tsur and Zemel, 1998) This can be described as an *environmental goal*. On the other hand, the resource stock in this model is used as a sink for waste coming from physical capital accumulation<sup>5</sup>. This represents the *economic goal*. Trying to decide between these two opposite strategies could lead to limit cycles around steady state when it becomes torn to steer between two goals. (see Wirl (2004))

One may argue that this trade-off is usual in growth models with environment and does not cause instability. Indeed, the limit cycles and unstable behavior in this economic model is due to conflicting economic and environmental goals coupled with *complementarity of preferences over time* (see Dockner and Feichtinger (1991), Heal and Ryder (1973)). An incremental increase of consumption at time  $t_1$  shifts the preferences from  $t_3$  to  $t_2$  (i.e. adjacent complementarity and distant complementarity vice versa.) if there is complementarity of preferences over time. If not, an incremental change of consumption at time  $t_1$  does not have any effect on the preferences of other dates. This implies that preferences are intertemporally independent.

The model is shown to have complementarity of preferences over time (see Appendix). To understand the implications of complementarity, consider an increase of consumption near date  $t_1$  which shifts a part of consumption of date  $t_3$  to  $t_2$ . At time  $t_1$ , physical capital accumulation decreases. It follows that the economy accumulates less waste and protects the natural stock. On the contrary, since at date  $t_3$ , consumption

<sup>5</sup>See Ayong Le Kama (2001) and Wirl (2004) for a similar model.

decreases and the economy accumulates more physical capital. Then, there is more waste accumulation which harms the environment and so on. From this mechanism, we can understand the intuition behind optimal limit cycles. Due to the complementarity over time, natural resource stock increases and decreases at consecutive dates, from which a cyclical behavior occurs.

Note that this mechanism does not hold if there is not waste stemming from physical capital accumulation. We also show that when waste rate  $\gamma$  approaches zero, the complementarity over time vanishes. (see Appendix.) We can also remark that the dynamic system (13) reduces to a block recursive system of  $(K, \lambda)$  and  $(S, \mu)$  which admits a saddle path equilibrium when there is not waste accumulation in the economy.

The legitimate question to be addressed is why and how individual preferences have a role in explaining limit cycles and unstable behavior in this economy. It is evident that the individual discount rate  $\beta$  is crucial since it changes the difficulty of steering between economic and environmental goal. This can be remarked through the utility of aggregate consumption which depends on the individual discount factor. More importantly, not every level of individual discount rate causes limit cycles or unstable spirals but a special set of parameters causes.

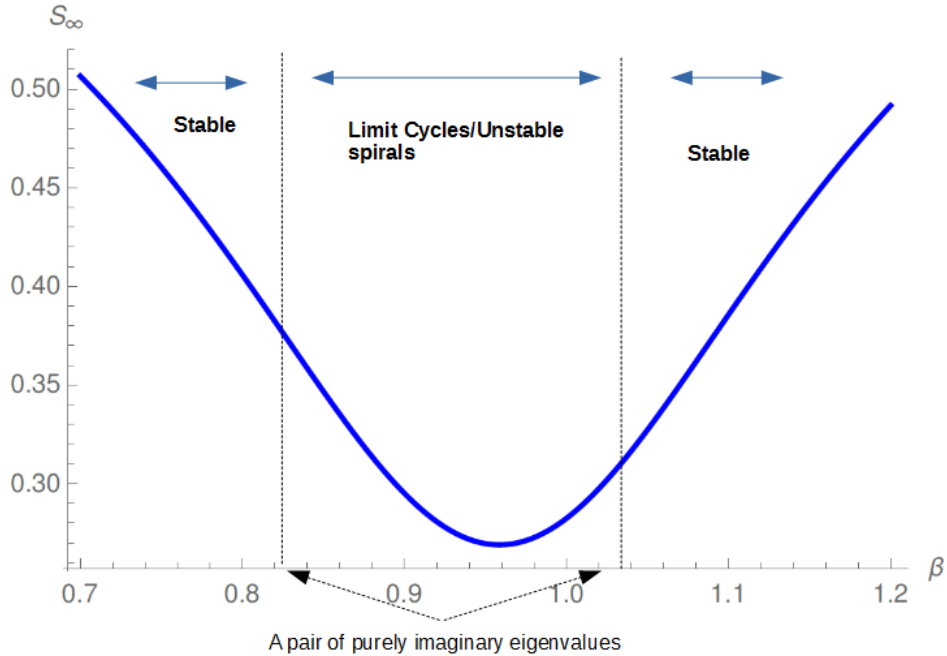


Figure 4: Steady state level of natural resource stock for variations in individual discount rate  $\beta$

Despite the fact that we try simple functional forms, it is always a hard task to find analytically the critical value for bifurcation parameter. (see Wirl (1999, 2004)) As in Wirl (2004), we refer to a numerical analysis to study in which cases bifurcation is a possible outcome. Figure (4) shows the steady state values of natural resource stock with respect to individual discount rate  $\beta$ .

**Proposition 1.** *In an economy with abrupt event probability, a Hopf bifurcation is possible for different critical parameters  $\beta$ .*<sup>6</sup>

<sup>6</sup>When we choose  $\beta$  as a bifurcation parameter, we use following set of parameters ;  $\rho = 0.85$ ,  $N = 60$ ,  $A = 1$ ,  $\omega_2 = 0.1$ ,

The steady state level is stable for low and high levels of individual discount rate  $\beta^7$ . At  $\beta = 0.925644$  and  $\beta = 0.974355$ , we have a pair of purely imaginary eigenvalues where Hopf bifurcation occurs. When we have  $\beta \in [0.925644.., 0.974355..]$ , the economy is exposed to unstable spirals. As mentioned above, it is possible to understand the intuition behind why limit cycles and unstable spirals take place but we are unable to offer an economic explanation why Hopf bifurcation takes place at given two different critical values <sup>8</sup>.

It is clear that the economy becomes more aggressive to exploit natural resources when the individual discount rate is higher until the threshold after which individual discount rate exceeds the augmented discount rate of social planner. (see Figure (4) ) After this threshold, since the economy starts to be more precautionary since the social planner is more patient than individuals.

In the figure (4), we observe that instability and limit cycles occur in a region where individual discount rate decreases and increases the optimum steady state level of the natural resource stock. This is equivalent to say that a higher individual discount rate does not imply necessarily a lower stock of natural resources.

The main reason why the optimum steady state has a convex form with respect to individual discount rate relies also on the trade-off between environmental and economic goal. In order to understand why optimum steady state level of natural resource stock  $S$  with respect to different individual discount level has a convex form, we can reformulate the steady state level of consumption in the following way

$$c^*(S) = \frac{Z [S^* ((\rho + \theta) - (1 - 2S^*)) ((A - \delta) - (\rho + \theta))]}{(\gamma A \bar{\psi} \omega_2 \theta) (A - \delta)} \quad (15)$$

where  $Z = \frac{Z_1}{(Z_2)^\sigma}$ . With a higher individual discount rate  $\beta$ , the aggregation term for utility and post value function  $Z_1$  and  $Z_2$  are shown to decrease with respect to individual discount rate  $\beta$  (see Appendix). The first effect is that in the case where  $Z$  decreases, consumption decreases, physical capital stock tends to increase and waste increases also. Then, the natural resource stock decreases. We have the reverse mechanism when the term  $Z$  increases.

According to the figure ((4)), the instability occurs in an area where it is hard to decide for the management of natural resources. When the individual discount rate is just  $\epsilon$  below the social planner's adjusted discount rate, a higher individual discount rate makes decrease the natural resource stock. Contrary to this, when individual discount rate is  $\epsilon$  above the social planner's discount rate, the natural resource stock starts to be conserved. One may understand from this explanation that it is difficult to steer between environmental and economic goal when the marginal utility of consumption is very sensible to small variations of individual discount rate  $\beta$ . We can also say that agents are in a sort of dilemma to consume more or less when their discount rate is closer to social planner's adjusted discount rate. Consequently, in this case, it becomes optimal to cycle around the steady state.

**Definition 1.** *A path of utility respects the Sustainable Development criterion if  $dU(t)/dt \geq 0$ .*

The Sustainable Development criterion states that utility of consumption should follow a non-decreasing or at least a constant path in order to ensure the sustainability of the economy. This means that limit cycles for consumption violate sustainability criterion since consumption decreases at some moments of time  $t$ .

An important point is that the model reduces to a representative agent model at the second stage. Consequently, we can not talk about the intergenerational equity as in [Schumacher and Zou \(2008\)](#) for

---

$\delta = 0.001, \gamma = 0.0001, \bar{\psi} = 1, \theta = 0.1$

<sup>7</sup>The Lyapunov number for both critical points are negative. This means that model shows a sub-critical Hopf bifurcation at two different critical individual discount rate  $\beta$ . For  $\beta = 0.925644$  and  $\beta = 0.974355$ , the Lyapunov numbers are -0.011645276 and -0.0116452956 respectively.

<sup>8</sup>See Wirl (1991,1992) for a detailed discussion regarding the difficulty of explaining economically bifurcation points.

aggregate dynamics because we don't know the length of life cycle of a generation along time  $t$ . For example, each generation can “equally” face a decrease or an increase of the consumption. Then, the equity between generations is respected. For this reason, we are focusing on the link between limit cycles and sustainability notion in our analysis.

From a normative point of view, the social planner should think about the way of avoiding cycles. We will address this issue in the following section.

**Proposition 2.** *When social planner treats all generations equally (i.e  $\beta = \rho + \theta$ ), sustainability criterion is not respected<sup>9</sup>.*

Another interesting result is that the way how a social planner allocates the consumption over different generations has an impact on long term dynamics of the economy (i.e sustainability). Treating generations equally comes at a cost since limit cycles and instability occur when individual discount rate is closer to social planner's adjusted discount rate ( $\rho + \theta$ ). When social discount rate is close to individual discount rate, the social planner tends to treat generations equally but she is also likely to trap the economy to limit cycles which compromises sustainability. . As a result, a new trade-off between intragenerational equity and sustainability arises in this framework when the economy faces a difficulty of steering between economic and environmental goal.

The economic intuition behind limit cycles can be understood from a mathematical point of view. (see Appendix for proof.) Note that limit cycles occur mostly when there exists complementarity over time. (Dockner and Feichtinger (1991)) This is equivalent to say that preferences of economic agents are interdependent, different than the growth models where the preferences of agents over time are independent in the sense of Koopmans (see Koopmans (1960)). Wirl (1992) explains that limit cycles take place when synergistic between control and state and/or between states is strong. In the model, it is evident that waste accumulation creates a strong link between two state variables. We can also verify the claim of Wirl (1992) by looking at complementarity over time. It is shown that complementarity over time vanishes when there does not exist any waste coming from physical capital accumulation. (see Appendix.)

**Proposition 3.** (a) *In an economy without abrupt event, individual discount rate has no effect on long term dynamics.*

(b) *The economy admits a saddle path equilibrium when there is any abrupt event probability.*

*Proof. See Appendix.*

This result is quite plausible when one thinks about the absence of the trade-off between decreasing effect of natural stock on penalty rate (environmental goal) and the use of it as a sink for waste (economic goal). Then, the abrupt event probability plays a crucial role since it changes the marginal utility which changes also the above mentioned trade-off between capital accumulation and the environment.

Intuitively, one may say that a very high catastrophic event may push the society to be more precautionary about environment since its importance is high in order to have a lower penalty rate. Equivalently, this means there would be less physical capital accumulation in this scenario.

On the contrary, when there is relatively very low catastrophe event probability, society becomes indifferent to environmental concerns and use the environment as a sink without caring much about it. In a nutshell, when social planner faces a catastrophic event probability between “too high” and “too low” probabilities,

---

<sup>9</sup>Note that we can not put exactly  $\beta = \rho + \theta$  since the aggregation term  $Z_1$  and  $Z_2$  turn out to be indeterminate. In the numerical analysis, we also skip the point where individual discount is exactly equal to the adjusted discount rate.

she can not decide firmly between these two strategies. Consequently, it becomes optimal to cycle around the steady state. We also show in appendix that when the economy faces an abrupt event probability very high or closer to zero, the complementarity of preferences over time vanishes.

Another remark could be that individual discount rate  $\beta$  has no effect on stability of the equilibrium if there is not a catastrophic event probability. As to which extent this trade-off is important, mentioned above, the steady state is stable for relatively high and low levels of catastrophe probability. Figure (5) shows limit cycles for given parameters in a phase diagram with plane  $(K, S)$ <sup>10</sup>

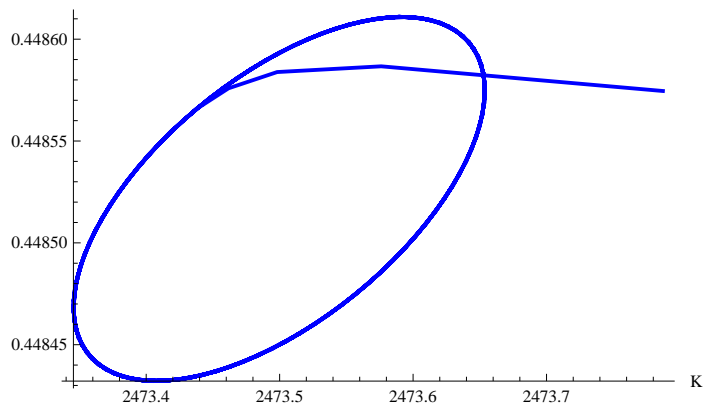


Figure 5: Limit cycles on a phase plane  $(K, S)$  with bifurcation parameter  $\beta$

A closer intuitive explanation to the mechanism explained for limit cycles in this study is proposed by Heal (1982) and Bosi and Desmarchelier (2016a) which are based on *compensation effect*. Assume that economy is at steady state at a given date  $t$  and assume that natural capital stock  $S$ <sup>11</sup> decreases exogenously. The degradation of natural capital pushes agents to increase their consumption since they would like to compensate the disutility due to the decrease of natural capital stock. It follows that capital accumulation decreases as well, which increases the natural capital as there is less waste and so on : deterministic cycles arises. Since in our specification, the objective function is separable in consumption and natural resource stock, we are unable to justify the limit cycles by a compensation effect. However, we show that even with a non-additive objective function, we show that limit cycles exist for a set of parameters.

**Proposition 4.** *Sustainability Development criterion is violated for a combination of parameters  $\beta$ .*

<sup>10</sup>The phase diagram is plotted for the first critical bifurcation point where  $\beta = 0.925644$ .

<sup>11</sup>Note that Bosi and Desmarchelier (2016a) use pollution stock instead of natural capital stock in their model.

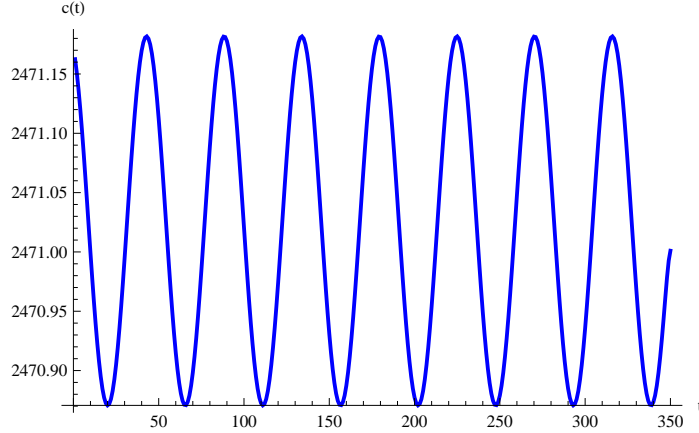


Figure 6: Limit cycles across time for utility function

What model stipulates is that not only the level of natural resource stock or an abrupt event as a fact would violate Sustainable Development criterion but even a probability of an extreme event and individual preferences could compromise sustainability. Figure (6) shows that at the first bifurcation point, the utility is exposed to cycles and does not converge to a stable point. From a normative point of view, a social planner that pays attention to sustainability should avoid any path that leads to bifurcations. The next section analyzes the necessary policy that avoids limit cycles.

### 3 Model with abatement activities

In this section, our focus will be on the implications of abatement activities on limit cycles. The social planner's program augmented by abatement activities is

$$W = \int_0^{\infty} \{U(c(t)) + \theta\varphi(S(t))\} e^{-(\rho+\theta)t} dt \quad (16)$$

The dynamics of capital accumulation contains the cost of mitigation  $M$ .

$$\begin{cases} \dot{K}(t) = f(K(t)) - \delta K(t) - c(t) - M(t) \\ \dot{S}(t) = G(S(t)) + \Gamma(M(t)) - \gamma f(K(t)) \end{cases} \quad (17)$$

where

$$\Gamma(M) = M^\alpha, \alpha > 0$$

holds for abatement activities such as reforestation, desalination of water stock, enhancing carbon sinks etc. The specification of abatement is in the same line with [Chimeli and Braden \(2005\)](#). Alternatively, function  $\Gamma(M)$  can be thought as “environmental protection function”. The expenditures for environmental protection may be directed not only toward pollution mitigation but also toward protection of forests, recovery of degraded areas. Equivalently, abatement activity in this model can be seen as a means of improving the environmental quality.

The first order conditions and dynamics of the economy augmented by abatement activity are as follows

$$\begin{cases} U_c = Z_1 \lambda^{-\frac{1}{\sigma}} = \lambda \\ \dot{K} = f(K) - \delta K - Z_1 \lambda^{-\frac{1}{\sigma}} - M \\ \dot{S} = G(S) + \Gamma(M) - \gamma f(K) \\ \dot{\lambda} = (\rho + \theta) \lambda - (f_K - \delta) \lambda + \mu \gamma f_K \\ \dot{\mu} = (\rho + \theta) \mu - \mu G_S - \theta \varphi_S \end{cases} \quad (18)$$

The steady-state of the economy as a function of natural resource stock is

$$\begin{cases} K^*(S) = \frac{(1-S^*)S^* + \left(\alpha \frac{\mu^*(S^*)}{\lambda^*(S^*)}\right)^{\frac{\alpha}{1-\alpha}}}{\frac{\gamma A}{1-\alpha}} \\ M^*(S) = \left(\alpha \frac{\mu^*(S^*)}{\lambda^*(S^*)}\right)^{\frac{1}{1-\alpha}} \\ \lambda^*(S) = \frac{\gamma A \mu^*(S^*)}{((A-\delta) - (\rho+\theta))} \\ \mu^*(S) = \frac{\psi \omega_2 \theta}{S^* ((\rho+\theta) - (1-2S^*))} \end{cases} \quad (19)$$

**Proposition 5.** *Abatement activity makes limit cycles less likely to occur by increasing the determinant and decreasing the sum of sub-matrices of Jacobian, in which case the economy admits a saddle-path stable equilibrium.*

*Proof. See Appendix.*

We also show numerically that model augmented by abatement activities admits always a saddle path equilibrium.

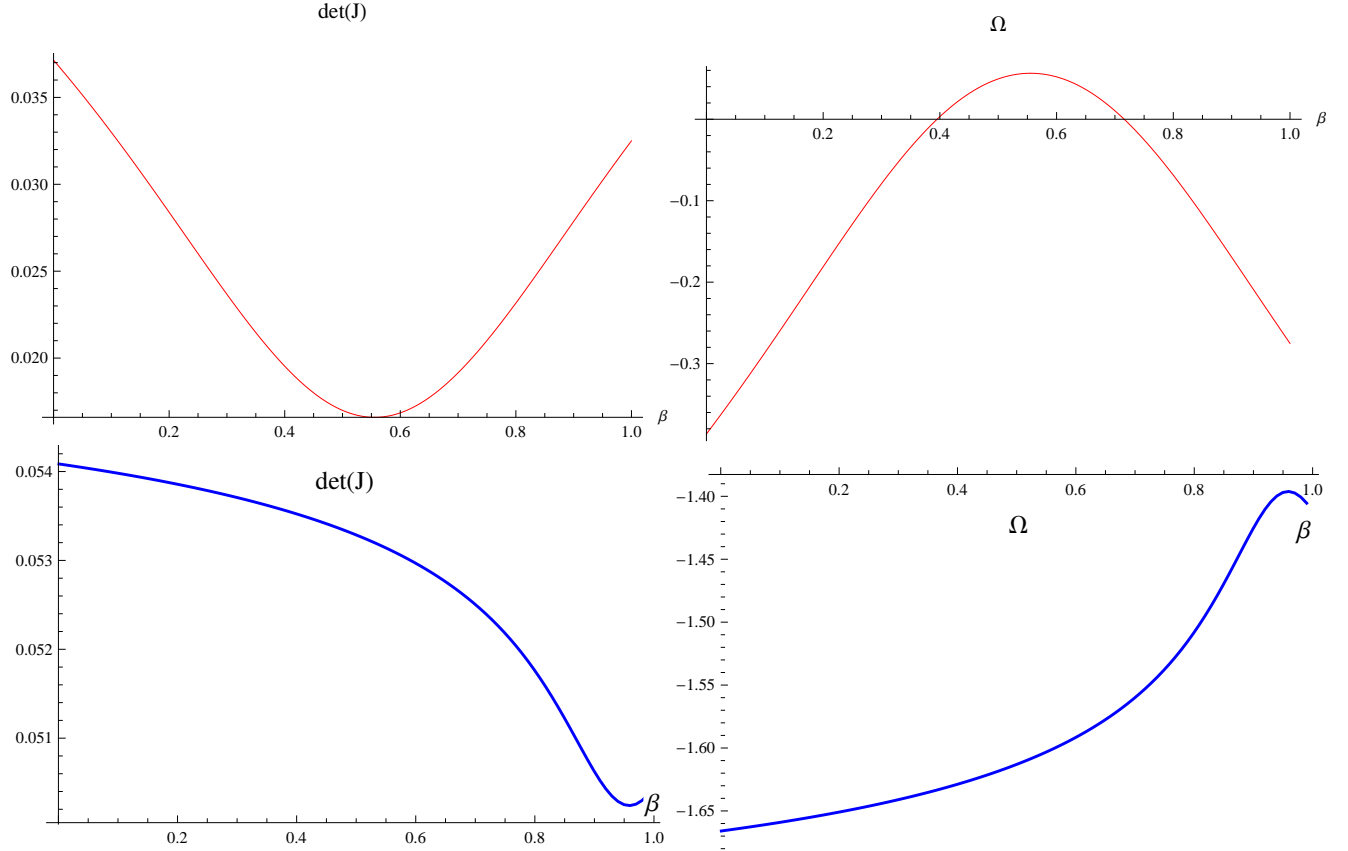


Figure 7:  $\det(J)$  and  $\Omega$  in benchmark (blue) and augmented model (red)

It is obvious that the determinant and the sum of sub matrices of the model with abatement activity increases and decreases respectively in the model with abatement activity. Numerical exercise confirms that whatever the level of individual discount rate is, the economy always admits a saddle path equilibrium. The economic explanation is the following : when the steady state level of natural resource stock is higher due to environmental protection activity, the marginal utility of consumption (see (19)) is lower. In this case, the variations of consumption due to different levels of individual discount rate  $\beta$  would have a lower impact on marginal utility. Indeed, when the economy is trapped to limit cycles due to a tight trade-off between economic and environmental goal, the abatement activity relaxes this tight trade-off by lowering the marginal utility. Then, the economy escapes the dilemma between environmental and economic goal by giving more weight to environmental goal.

## 4 Conclusion

In this paper, we showed the existence of limit cycles in an economy exposed to abrupt event probability. The limit cycles are caused by conflicting economic and environmental goals coupled with complementarity of preferences over time. An interesting finding of the paper is that individual time preferences of agents other than the social planner's discount rate is crucial not only for intra-generational equity but also for the sustainability of an economy. It is shown that when individual discount rate is close to social planner's discount rate, intra-generational equity is ensured. However, the equality between living generations comes



at a cost since it is more likely that the economy is trapped at limit cycles at the long run. This result also disproves a wide spread result in the literature which says that aggregate dynamics are solely governed by the social planner's discount rate. Since we adopt a normative point of view, the limit cycles are considered an undesirable result since it compromises the sustainability according to the prominent Sustainable Development Criterion. Then, we seek to avoid these cycles by protecting the environment. The result in the model augmented by environmental protection activities shows that it is less likely to have limit cycles at the long run with an environmental policy aiming at improving/protecting the environment.

## 5 Appendix

### 5.1 Aggregate Economy facing a catastrophic event

In order to understand clearly the problem, firstly we present the economy without a catastrophic event from the starting point as in [Calvo and Obstfeld \(1988\)](#);

$$W^{NC} = \int_{-N}^{\infty} \left\{ \int_0^N u(c(t, \tau)) e^{-\beta\tau} d\tau \right\} e^{-\rho b} db \quad (20)$$

where  $b$  is the born date of a cohort and script  $NC$  indicates the welfare in a no-catastrophe situation. A social planner, once she makes intra-generational allocation, decides for an optimal allocation of consumption between generations. In this form of the social welfare function  $W$ , time does not enter directly in the utility function of an agent. Since born date  $b$  is equal to  $t - \tau$  where  $\tau$  stands for age, we can reformulate the social planner's program in a way that reflects directly time  $t$  by switching the space of the integral from born date  $b$  to calendar time  $t$ .

$$W^{NC} = \int_0^{\infty} \left\{ \int_0^N u(c(t, \tau)) e^{-(\beta-\rho)\tau} d\tau \right\} e^{-\rho t} dt \quad (21)$$

Differently from [Calvo and Obstfeld \(1988\)](#), what happens if there is an abrupt event probability in this economy ? The basic way to reformulate this problem is the following

$$W^C = E_{\bar{b}} \left[ \int_{-N}^{\bar{b}} \left\{ \int_0^N u(c(t, \tau)) e^{-(\beta-\rho)\tau} d\tau \right\} e^{-\rho b} db + e^{-\rho \bar{b}} \varphi(S(\bar{b})) \right] \quad (22)$$

where  $\bar{b}$  is the uncertain date at which a catastrophic event occurs. This is equivalent to say that there is an “unlucky” generation which born with an abrupt event. In case where an abrupt event does not happen, we have  $\bar{b} \rightarrow \infty$ . Then, the maximization program reduces to  $W^{NC}$ . In this problem, we seek to maximize expected future utility stream. With integration by parts of the equation (22), we find the deterministic equivalent of the expected value of the uncertainty problem in (22),

$$\begin{aligned} W^C &= \int_0^{\infty} f(\bar{b}) \left[ \int_{-N}^{\bar{b}} \left\{ \int_0^N u(c(t, \tau)) e^{-\beta\tau} d\tau \right\} e^{-\rho t} dt + e^{-\rho \bar{b}} \varphi(S(\bar{b})) \right] d\bar{b} \\ &= \underbrace{\int_0^{\infty} f(\bar{b}) \left[ \int_{-N}^{\bar{b}} \left\{ \int_0^N u(c(t, \tau)) e^{-\beta\tau} d\tau \right\} e^{-\rho t} dt \right] d\bar{b}}_A + \underbrace{\int_0^{\infty} f(\bar{b}) \left[ e^{-\rho \bar{b}} \varphi(S(\bar{b})) \right] d\bar{b}}_B \end{aligned}$$

Note that probability distribution and density function are

$$f(b) = \theta e^{-\theta b} \text{ and } F(b) = 1 - e^{-\theta b}$$

Integrating by parts  $A$  and keeping in mind the definition (1) and fundamental theorem of calculus yield

$$dX = f(\bar{b}) \implies X = \int_{-N}^{\bar{b}} f(s) ds$$

$$Y = \int_{-N}^{\bar{b}} \left\{ \int_0^N u(c(t, t-b)) e^{-\beta\tau} d\tau \right\} e^{-\rho b} db \implies dY = \left\{ \int_0^N u(c(t, t-\bar{b})) e^{-\beta\tau} d\tau \right\} e^{-\rho\bar{b}}$$

Using  $\int Y dX = XY - \int X dY$  yields and inflicted penalty

$$A = \left[ \left( \int_{-N}^{\bar{b}} f(s) ds \right) \left( \int_{-N}^b \left\{ \int_0^N u(c(t, t-b)) e^{-\beta\tau} d\tau \right\} e^{-\rho b} db \right) \right]_{-N}^{\infty} - \int_{-N}^{\infty} F(\bar{b}) \left\{ \int_0^N u(c(t, t-\bar{b})) e^{-\beta\tau} d\tau \right\} e^{-\rho\bar{b}} d\bar{b}$$

Recall that  $\int_{-N}^{\infty} f(s) ds = 1$ . The part  $A$  leads to

$$\int_{-N}^{\infty} (1 - F(b)) \left\{ \int_0^N u(c(t, t-b)) e^{-\beta\tau} d\tau \right\} e^{-\rho b} db$$

We can reformulate part  $B$  in the following manner

$$B = \int_{-N}^{\infty} f(\bar{b}) e^{-\rho\bar{b}} \left[ \int_0^N \{ (u(c_{min}) - \psi(S(t, t-\bar{b}))) e^{-\beta\tau} d\tau \right] d\bar{b}$$

Since the natural resource stock is independent of age  $\tau$ <sup>12</sup>, one can simplify the term in integral,

$$B = \int_{-N}^{\infty} f(\bar{b}) \left[ \int_0^N \{ (u(c_{min}) - \psi(S(t)) e^{-\beta\tau}) d\tau \} d\bar{b} \right]$$

Then, the social planner's program becomes ;

$$A+B = W = \int_{-N}^{\infty} \left[ (1 - F(b)) \left\{ \int_0^N u(c(t, t-b)) e^{-\beta\tau} d\tau \right\} + f(b) \left[ \int_0^N \{ (u(c_{min}) - \psi(S(t)) e^{-\beta\tau}) d\tau \} \right] \right] e^{-\rho b} db$$

By changing the space of the integral by born date  $b$  to calendar time  $t$ , using probability distribution, density function, and taking the overall sum  $A+B$ , we have objective function to be maximized by the social planner,

$$\int_0^{\infty} [U(c(t)) + \theta\varphi(S(t))] e^{-(\rho+\theta)t} dt$$

where the post-value function can be written after integration of the integral

$$\varphi(S(t)) = Z_2 (u(c_{min}) - \psi(S(t)))$$

$$\text{where } Z_2 = \left( \frac{1 - e^{-(\beta - (\rho + \theta))N}}{(\beta - (\rho + \theta))} \right)$$

---

<sup>12</sup>Note that  $S(t, t-\bar{b}) = S(t)$

## 5.2 Aggregate Utility from OLG model

From equation (7) and using (8)

$$c(t, \tau) = c(t, 0) e^{-\frac{(\beta - (\rho + \theta))\tau}{\sigma}} \quad (23)$$

Aggregating the consumption of all generations gives

$$c(t) = \int_0^N c(t, \tau) d\tau = \int_0^N c(t, 0) e^{(\beta - (\rho + \theta))\tau} d\tau \quad (24)$$

This yields

$$c(t) = c(t, 0) \sigma \left( \frac{1 - e^{-\frac{(\beta - (\rho + \theta))N}{\sigma}}}{(\beta - (\rho + \theta))} \right) \quad (25)$$

In order to ease the presentation, we denote  $Z_1 = \sigma \left( \frac{1 - e^{-\frac{(\beta - (\rho + \theta))N}{\sigma}}}{(\beta - (\rho + \theta))} \right)$ . Plugging equation (23) in aggregate utility 26 gives

$$U(c(t)) = \int_0^N \frac{c(t, \tau)^{1-\sigma} - c_{min}^{1-\sigma}}{1-\sigma} e^{-(\beta - (\rho + \theta))\tau} d\tau \quad (26)$$

Plugging equation (23) in aggregate utility 26 gives

$$U(c(t)) = \int_0^N \frac{\left[ c(t, 0) \sigma \left( \frac{1 - e^{-\frac{(\beta - (\rho + \theta))N}{\sigma}}}{(\beta - (\rho + \theta))} \right) \right]^{1-\sigma}}{1-\sigma} e^{-(\beta - (\rho + \theta))\tau} d\tau \quad (27)$$

Calculating the integral and using equation (25) gives

$$U(c(t)) = \frac{c(t)^{1-\sigma} - c_{min}^{1-\sigma}}{1-\sigma} (Z_1)^\sigma$$

## 5.3 Proof of Proposition 1

We analyze an economy without catastrophic event to show how catastrophic event probability could have different implications on local stability of the economy. To see this, we equalize catastrophe probability to zero. The maximization program is indeed the reduced form of the social welfare function.

$$W^{NC} = \int_0^\infty U(c(t)) e^{-\rho t} dt \quad (28)$$

subject to

$$\begin{cases} \dot{K}(t) = f(K(t)) - \delta K(t) - c(t) \\ \dot{S} = G(S) - \gamma f(K(t)) \end{cases} \quad (29)$$

The differential system describing dynamics of the economy is

$$\begin{cases} \dot{K} = f(K) - \delta K - Z_1 \lambda^{-\frac{1}{\sigma}} \\ \dot{S} = G(S) - \gamma f(K) \\ \dot{\lambda} = \rho \lambda - (f_K - \delta) \lambda + \mu \gamma f_K \\ \dot{\mu} = \rho \mu - \mu G_S \end{cases} \quad (30)$$

In this version of the model, an important aspect is that social planner does not face a trade-off mentioned in the text. She takes into account only the negative effect of capital accumulation to the environment. Then, environment is just served as a sink and does not represent any amenity value.

In an economy without catastrophic events, some straightforward calculations allow us to find analytically the steady state equilibrium.

$$\begin{cases} S^* = \frac{1-\rho}{2} \\ K^*(S) = \frac{(1-S^*)S^*}{\gamma A} \\ \lambda^*(S) = \frac{1}{M_1} \left( N - \frac{(A-\delta)(1-S^*)S^*}{\gamma A} \right) \\ \mu^*(S) = \frac{\lambda^*(S)[(A-\delta)-\rho]}{\gamma A} \end{cases} \quad (31)$$

Then, just recall the  $\det(J)$  becomes the following one when there is not an abrupt event

$$\det(J) = 2(A\gamma)^2 \left( \frac{(A-\delta)}{(A-\delta)-\rho} \left( \frac{1-\rho}{2} \right) \left( 1 - \frac{1-\rho}{2} \right) \right) > 0$$

$$K = (A-\delta)(\rho - (A-\delta)) < 0$$

Then, it is easy to remark that the aggregation term  $Z_1$  cancels out when there is not an abrupt event probability. It is also obvious that  $K$  does not depend on  $Z_1$  neither.

## 5.4 Proof of Proposition 2

The Jacobian matrix of the differential system with using given functional forms is

$$J = \begin{bmatrix} A-\delta & 0 & \frac{Z_1}{\sigma} \lambda^{-\frac{1}{\sigma}-1} & 0 \\ -A\gamma & (1-2S) & 0 & 0 \\ 0 & 0 & (\rho+\theta) - (A-\delta) & A\gamma \\ 0 & 2\mu + \frac{\theta \bar{\psi} \omega_2}{S^2} & 0 & (\rho+\theta) - (1-2S) \end{bmatrix} \quad (32)$$

Following [Dockner and Feichtinger \(1991\)](#), the characteristic polynomial associated with Jacobian is

$$v^4 - \text{tr}Jv^3 + b_2v^2 - b_3v + \det(J) = 0 \quad (33)$$

where  $b_2$  and  $b_3$  are the sum of second and third order minors of Jacobian respectively. We have

$$\text{tr}J = 2(\rho+\theta) \quad \text{and} \quad -b_3 + (\rho+\theta)b_2 - (\rho+\theta)^3 = 0 \quad (34)$$

The eigenvalues of the Jacobian matrix are calculated from first order conditions.

$$v_i = \frac{(\rho + \theta)}{2} \pm \sqrt{\left(\frac{\rho + \theta}{2}\right)^2 - \frac{\Omega}{2} \pm \sqrt{\Omega^2 - \det(J)}} \quad (35)$$

The sum of the determinants of sub-matrices of Jacobian can be specified

$$\Omega = \begin{bmatrix} A - \delta & \frac{Z_1}{\sigma} \lambda^{-\frac{1}{\sigma} - 1} \\ 0 & (\rho + \theta) - (A - \delta) \end{bmatrix} + \begin{bmatrix} (1 - 2S) & 0 \\ 2\mu + \frac{\theta \bar{\psi} \omega_2}{S^2} & (\rho + \theta) - (1 - 2S) \end{bmatrix} + 2 \begin{bmatrix} 0 & 0 \\ 0 & A\gamma \end{bmatrix} \quad (36)$$

Then we have

$$\Omega = (A - \delta) [(\rho + \theta) - (A - \delta)] + (1 - 2S) [(\rho + \theta) - (1 - 2S)] \quad (37)$$

and

$$\det(J) = [(\rho + \theta) - (1 - 2S)] [(A - \delta) (1 - 2S) [(\rho + \theta) - (A - \delta)]] + (A\gamma)^2 \frac{Z_1 \lambda^{-\frac{1}{\sigma} - 1}}{\sigma} \left( 2\mu + \frac{\theta \bar{\psi} \omega_2}{S^2} \right) \quad (38)$$

For the possibility of a Hopf bifurcation,  $\Omega$  should be positive. In this framework, this one is possible when following condition is ensured

$$\rho + \theta > G_S > 0 \quad (39)$$

In this case, we can observe that an economy in which all levels of the natural resource stock exceeding maximum sustainable yield is stable. A Hopf bifurcation in a 4x4 dimension system occurs when two of eigenvalues have only imaginary parts. This means that real part of these two eigenvalues crosses zero for some parameters. More precisely, the derivative of the real part of eigenvalues with respect to chosen bifurcation parameter is non-zero. The necessary condition to have Hopf bifurcation

$$\det(J) - \left(\frac{\Omega}{2}\right)^2 > 0 \quad (40)$$

$$\det(J) - \left(\frac{\Omega}{2}\right)^2 - (\rho + \theta)^2 \frac{\Omega}{2} = 0 \quad (41)$$

are necessary and sufficient in order to have four complex eigenvalues and two having only imaginary parts. A saddle-path equilibrium is a possible outcome if and only if

$$\det(J) > 0 \text{ and } \Omega < 0 \quad (42)$$

Besides, the inequality  $0 < \det(J) < \frac{1}{2}\Omega^2$  is sufficient in order to have all eigenvalues with real parts, which implies local monotonicity.

## 5.5 Proof of Proposition 4

The Jacobian matrix of the differential system with abatement activity at steady state

$$J = \begin{bmatrix} A - \delta & 0 & \frac{Z_1}{\sigma} \lambda^{-\frac{1}{\sigma}-1} + \frac{1}{1-\alpha} \left(\alpha \frac{\mu}{\lambda}\right)^{\frac{1}{1-\alpha}} \lambda^{-1} & -\frac{1}{1-\alpha} \left(\alpha \frac{\mu}{\lambda}\right)^{\frac{1}{1-\alpha}} \mu^{-1} \\ -A\gamma & (1-2S) & 0 & 0 \\ 0 & 0 & -\frac{\alpha}{1-\alpha} \left(\alpha \frac{\mu}{\lambda}\right)^{\frac{\alpha}{1-\alpha}} \lambda^{-1} & \frac{\alpha}{1-\alpha} \left(\alpha \frac{\mu}{\lambda}\right)^{\frac{\alpha}{1-\alpha}} \mu^{-1} \\ 0 & 2\mu + \frac{\theta \bar{\psi} \omega_2}{S^2} & 0 & (\rho + \theta) - (1-2S) \end{bmatrix} \quad (43)$$

The sum of the determinants of sub-matrices and determinant of Jacobian are

$$\Omega = (A - \delta) [(\rho + \theta) - (A - \delta)] + (1 - 2S) [(\rho + \theta) - (1 - 2S)] - \left(2\mu + \frac{\theta \bar{\psi} \omega_2}{S^2}\right) \frac{\alpha M^\alpha}{(1 - \alpha) \mu} \quad (44)$$

and

$$\det(J) = [(\rho + \theta) - (1 - 2S)] [(A - \delta) (1 - 2S) [(\rho + \theta) - (A - \delta)]] + (A\gamma)^2 \frac{Z_1 \lambda^{-\frac{1}{\sigma}-1}}{\sigma} \left(2\mu + \frac{\theta \bar{\psi} \omega_2}{S^2}\right) \quad (45)$$

The proof is easy to follow. Let  $S^A > S^B$  where  $S^A$  is the steady state level of natural stock with abatement activity and  $S^B$  stands for the steady state level of natural stock for the benchmark model without abatement activity. By replacing steady state value of  $\mu$  in (44), we can reformulate the sum of sub-matrices of Jacobian

$$\Omega = (A - \delta) [(\rho + \theta) - (A - \delta)] + (1 - 2S^A) [(\rho + \theta) - (1 - 2S^A)] - \underbrace{\frac{\alpha M^\alpha}{(1 - \alpha)} \left(\frac{1 - (\rho + \theta)}{S^A}\right)}_{>0}$$

Notice that  $M$  depends only on constant parameters in the long run. The presence of abatement activity decreases the sum of sub-matrices of Jacobian  $\Omega$ , which makes less likely that unstable spirals and cycles occur. To see the effect of a higher natural stock level on the determinant of the Jacobian matrix, we look at the first derivative with respect to  $S$

$$\begin{aligned} \frac{\partial(\det(J))}{\partial S} &= \underbrace{((\rho + \theta) - (A - \delta))2(A - \delta) [2(1 - 2S^A) - (\rho + \theta)]}_{<0} - \underbrace{\frac{(A\gamma)^2}{\sigma} Z_1 \left(2\frac{\mu}{\lambda} + \frac{\theta \bar{\psi} \omega_2}{S^2 \lambda}\right) \frac{\partial \lambda(S^A)}{\partial S}}_{<0} \\ &\quad + \underbrace{\frac{((A - \delta) - (\rho + \theta))(1 - (\rho + \theta))}{\gamma A S^2}}_{>0} > 0 \end{aligned}$$

The two last terms are unambiguously negative and positive respectively. The determinant increases unambiguously with respect to  $S$  if  $2(1 - 2S^A) - (\rho + \theta) < 0$ .

## 5.6 Proof for complementarity effect between different time periods

This proof aims to show the existence of the complementarity of preferences over time within this framework. The objective of the proof is to point out the fact that complementarity over time vanishes when waste rate  $\gamma$  tends to zero.

We write down the objective function in form

$$J[c(\cdot)] = \int_0^{\infty} u(c(t), S(t)) e^{-(\rho+\theta)t} dt \quad (46)$$

In order to ease the calculations, we don't take the quadratic form for regeneration function  $G(S)$  but simply take a linear regeneration function  $G(S) = mS$ .<sup>13</sup> At this point, [Wirl \(1992, 1994\)](#) show that the occurrence of limit cycles does not have any link with the form of regeneration function. The author also shows that a linear regeneration function can also generate limit cycles. Our proof also supports this idea by showing the existence of complementarity of preferences over time. Then, the use of a linear regeneration function does not enter in a conflict with the aim of the proof. We can express physical and natural capital from constraints

$$S(t) = e^{mt} \gamma A \int_t^{\infty} K(s) e^{-ms} ds \quad (47)$$

$$K(t) = e^{(A-\delta)t} \int_t^{\infty} c(s) e^{-(A-\delta)s} ds \quad (48)$$

To understand the complementarity effects between state variables and control and state variables, we refer to Volterra derivatives ([Heal and Ryder \(1973\)](#), [Dockner and Feichtinger \(1991\)](#)). This requires to look at the marginal rate of substitution between different time periods  $t_1, t_2, t_3$  etc. For example, the marginal utility at time  $t_1$  is  $J' [c(\cdot), t_1]$  which is a Volterra derivative. To sum up, a small incremental increase of consumption at the neighborhood of time  $t_1$  can be calculated by using Volterra derivatives. The concept of Volterra derivative is useful to show how a change in consumption at a given date shifts the allocation of consumption between other dates. The marginal rate of substitution between consumption at dates  $t_1$  and  $t_2$

$$R[c(\cdot), t_1, t_2] = \frac{J' [c(\cdot), t_1]}{J' [c(\cdot), t_2]} \quad (49)$$

In order to see the effect of an incremental change of consumption near date  $t_3$ , we take the Volterra derivative of  $R[c(\cdot), t_1, t_2]$ .

$$R' [c(\cdot), t_1, t_2; t_3] = \frac{J' [c(\cdot), t_2] J'' [c(\cdot), t_1, t_3] - J' [c(\cdot), t_1] J'' [c(\cdot), t_2, t_3]}{(J' [c(\cdot), t_2])^2} \quad (50)$$

If  $R' [c(\cdot), t_1, t_2; t_3] > 0$ , an incremental increase of consumption at date  $t_3$  shifts consumption from  $t_2$  to  $t_1$ , in which case there is complementarity between  $t_1$  and  $t_3$ . This represents a *distant* complementarity. If  $R' [c(\cdot), t_1, t_2; t_3] < 0$ , the preferences shift from  $t_1$  to  $t_2$  where two neighboring dates hold for *adjacent* complementarity. Taking the derivatives of (46), (47) and (48),

$$J' [c(\cdot), t_1] = e^{-(\rho+\theta)t_1} u_c(c(t_1), S(t_1)) + f_c(c(t_1)) e^{-mt_1} \int_{t_1}^{\infty} e^{-(\rho+\theta-m)t} u_s(c(t), s(t)) dt \quad (51)$$

$$J'' [c(\cdot), t_1, t_2] = f_c(c(t_1)) f_c(c(t_2)) e^{-m(t_1+t_2)} \int_{t_2}^{\infty} e^{-(\rho+\theta-2m)t} u_{ss}(c(t), s(t)) dt \quad (52)$$

where  $f_c$  is the derivative of  $S$  with respect to  $c$ . Regarding the equation (47), we know that  $S$  is a function of  $c$  since  $K$  depends on  $c$ . For example, when consumption changes marginally, capital accumulation changes

<sup>13</sup>Otherwise, in case of the use of logistic growth function for natural regeneration, one should deal with Riccati differential equation which yields very tedious calculations.



and consequently the trajectory of natural capital accumulation. In this sense, we can simply say  $S = f(c)$  but we can not know the form of this function analytically out of steady state (see [Dockner and Feichtinger \(1991\)](#)). Note also that since we have an additive objective function,  $u_{ss}$  does not depend directly on  $c$  in our model.

For the sake of simplicity regarding the analysis, we can restrict our attention to a constant investment path (i.e steady state) similar to [Heal and Ryder \(1973\)](#) and [Dockner and Feichtinger \(1991\)](#). Using equations (47) and (48), we can write  $mS^* = \gamma K^* = \frac{\gamma c}{A}$ . It is obvious that with this simplification, we can find the form of  $f_c$  which is equal to  $\frac{\gamma}{mA} > 0$ . We write simplified form of equations (51) and (52),

$$J' [c(\cdot), t_1] = e^{-(\rho+\theta)t_1} \left[ u_c(c(t_1), S(t_1)) + \frac{f_c(c(t_1)) u_s(c(t_1), S(t_1))}{(\rho + \theta - m)} \right]$$

$$J' [c(\cdot), t_1, t_2] = f_c(c(t_1)) f_c(c(t_2)) e^{-m(t_1+t_2)-(\rho+\theta-2m)t_2} \frac{u_{ss}}{\rho + \theta - 2m}$$

With all these elements, it is easy to express the effect of marginal increase of consumption at date  $t_3$  on the marginal rate of substitution between  $t_1$  and  $t_2$ ,

$$R' [c(\cdot), t_1, t_2; t_3] = \frac{\frac{\gamma^2 c(t_1)c(t_2)}{A^2} \frac{u_{ss}(c(t_1), S(t_1))}{(\rho+\theta-2m)}}{u_c(c(t_1), S(t_1)) + \frac{\gamma c(t_1)}{A} \frac{u_s(c(t_1), S(t_1))}{(\rho+\theta-m)}} [\alpha(t_3 - t_1) - \alpha(t_3 - t_2)]$$

where  $0 < t_1 < t_2$ . and  $u_{ss} < 0$ . We know that the date  $t_2$  and  $t_3$  is placed after  $t_1$  but we don't know the order of dates for  $t_2$  and  $t_3$ . In order to understand the effect of a small increase near date  $t_3$ , we claim that date  $t_3$  is situated before  $t_2$ . Otherwise, it is evident on the expression above that there is no effect of a variation of consumption at date  $t_3$  on the marginal rate of substitution of consumption between dates  $t_1$  and  $t_2$ . This makes sense because when we are near date  $t_3$ , all decisions at  $t_1$  and  $t_2$  are already made. Similar to [Heal and Ryder \(1973\)](#), we write

$$\alpha(t) = e^{-(\rho+\theta-m)t} \text{ for } t > 0$$

$$\alpha(t) = e^{mt} \text{ for } t < 0$$

We observe that  $R' < 0$  which means that there exists an adjacent complementarity between dates  $t_2$  and  $t_3$ .

$$t_3 < \frac{(\rho + \theta - m)t_1 - mt_2}{(\rho + \theta - 2m)} \quad (53)$$

Note that right hand side of the inequality increases when  $\theta$  increases. Limit cycles can appear in the model with both adjacent and distant complementarity, which is in line with the results of [Dockner and Feichtinger \(1991\)](#).

It is important to point out the presence of waste in the model. It is obvious to see

$$\lim_{\gamma \rightarrow 0} R' = 0 \quad (54)$$

$$\lim_{\theta \rightarrow 0} R' = 0 \quad (55)$$

$$\lim_{\theta \rightarrow \infty} R' = 0 \quad (56)$$

The limit of the condition (54) means that when the rate of waste coming from physical capital accumulation is lower, the complementarity over time becomes lower as well. Also, since stated in the text, when the abrupt event probability is too low or too high, the complementarity over time vanishes as well. This gives a mathematical support to see how waste from physical capital is a source of limit cycles in the economy. The limit cycles occur at the presence of waste but for which parameter sets, does it occur ?

## 5.7 Variation of Aggregation Term w.r.t $\beta$

We differentiate the aggregation term with respect to  $x = \beta - (\rho + \theta)$

$$\frac{\partial Z_1}{\partial x} = \frac{\sigma \left[ \left( e^{-\frac{xN}{\sigma}} \left( 1 + \frac{N}{\sigma} \right) \right) - 1 \right]}{x^2} < 0$$

The variation is unambiguously negative when individual discount rate is higher than social planner's discount rate.  $\left( e^{-\frac{xN}{\sigma}} \left( 1 + \frac{N}{\sigma} \right) \right)$  attains 1 when  $x = 0$  after it decreases monotonically.

## 6 References

- R.B. Alley, J. Marotzke, and Overpeck J.T. Peteet D.M. Pielke R.A. Pierrehumbert R.T. Rhines P.B. Stocker T.F.-Talley L.D. Wallace J.M. Nordhaus, W.D. Abrupt climate change. *Science*, 299 (5615) 2005 - 2010., 2003.
- Alain Ayong Le Kama. Sustainable growth, renewable resources and pollution. *Journal of Economic Dynamics & Control*, 2001.
- Antoine Bommier, Bruno Lanz, and Stéphane Zuber. Models-as-usual for unusual risks? On the value of catastrophic climate change. *Journal of Environmental Economics and Management*, 74:1–22, 2015. ISSN 00950696. doi: 10.1016/j.jeem.2015.07.003. URL <http://linkinghub.elsevier.com/retrieve/pii/S0095069615000625>.
- Stefano Bosi and David Desmarchelier. Limit Cycles Under a Negative Effect of Pollution on Consumption Demand: The Role of an Environmental Kuznets Curve. *Environmental and Resource Economics*, 2016a. ISSN 0924-6460. doi: 10.1007/s10640-016-0082-7. URL <http://link.springer.com/10.1007/s10640-016-0082-7>.
- Stefano Bosi and David Desmarchelier. Natural cycles and pollution Stefano Bosi - David Desmarchelier Natural cycles and pollution. *FAERE Working Papers*, 2017.
- Lucas Bretschger and Alexandra Vinogradova. Best policy response to environmental shocks : Applying a stochastic framework. *Journal of Environmental Economics and Management*, 2017. ISSN 0095-0696. doi: 10.1016/j.jeem.2017.07.003. URL <http://dx.doi.org/10.1016/j.jeem.2017.07.003>.
- P. S. Burton. Intertemporal Preferences and Intergenerational Equity Considerations in Optimal Resource Harvesting. *Journal of Environmental Economics and Management*, 24:119–132, 1993.

- G. Calvo and M. Obstfeld. Optimal Time-Consistent Fiscal Policy with Finite Lifetimes. *Econometrica*, 1988.
- Ariaster B. Chimeli and John B. Braden. Total factor productivity and the environmental Kuznets curve. *Journal of Environmental Economics and Management*, 49(2):366–380, 2005. ISSN 00950696. doi: 10.1016/j.jeem.2004.06.003.
- HR Clarke and WJ Reed. Consumption/pollution tradeoffs in an environment vulnerable to pollution-related catastrophic collapse. *Journal of Economic Dynamics and Control*, 1994. URL <http://www.sciencedirect.com/science/article/pii/0165188994900426>.
- EJ Dockner and G Feichtinger. On the optimality of limit cycles in dynamic economic systems. *Journal of Economics*, 1991. URL <http://link.springer.com/article/10.1007/BF01227014>.
- Lee H. Endress, Sittidaj Pongkijvorasin, James Roumasset, and Christopher Wada. Intergenerational equity with individual impatience in a model of optimal and sustainable growth. *Resource and Energy Economics*, 2014.
- C.B. Field, V. Barros, and Qin D. Dokken D.J. Ebi K.L. Mastrandrea M.D. Mach K.J. Plattner G.-K. Allen S.K. Tignor M. Midgley P.M. Stocker, T.F. Managing the risks of extreme events and disasters to advance climate change adaptation. *A Special Report of Working Groups I and II of the Intergovernmental Panel on Climate Change*. Cambridge University Press, Cambridge, UK., 2012.
- Geoffrey Heal. The use of common property resources. In: *Smith VK, Krutilla JV (eds) Explorations in natural resource economics*. The Johns Hopkins University Press for Resources for the Future, Baltimore, 1982.
- Geoffrey M Heal and H. Ryder. Optimal Growth with Intertemporally Dependent Preferences. *Review of Economic Studies*, 40(1):1–31, 1973.
- G.M. Heal. Climate economics: a meta-review and some suggestions for future research. *Review of Environmental Economics and Policy*, 3 (1):4 – 21, 2009.
- By Tjalling C Koopmans. Stationary Ordinal Utility and Impatience Author ( s ): Tjalling C . Koopmans. *Econometrica*, 28(2):287–309, 1960.
- Giancarlo Marini and Pasquale Scaramozzino. Overlapping Generations and Environmental Control. *Journal of Environmental Economics and Management*, 1995.
- Giancarlo Marini and Pasquale Scaramozzino. Social time preference: A rejoinder. *Journal of Population Economics*, 21(3):623–625, 2008. ISSN 09331433. doi: 10.1007/s00148-006-0089-4.
- C. A Mavi. Can a hazardous event be another source of poverty traps? *FAERE Working Paper*, 2017.
- M. Schneider, C. Traeger, and R. Winkler. Trading off generations: Equity, discounting, and climate change. *European Economic Review*, 2012.
- Ingmar Schumacher and Benteng Zou. Pollution perception: A challenge for intergenerational equity. *Journal of Environmental Economics and Management*, 55(3):296–309, may 2008. ISSN 00950696. doi: 10.1016/j.jeem.2007.12.001. URL <http://linkinghub.elsevier.com/retrieve/pii/S0095069607001210>.

- R. Solow. Sustainability, an economist perspective. In: *Stavins, R. (Ed.), Economics of the Environment*, pages pp. 503 – 513., 2005.
- R. Solow. An almost practical step toward sustainability. In: *Oates, W. (Ed.), The RFF Reader in Environmental and Resource Policy*. RFF Press, Washington., pages pp. 253 – 264., 2006.
- N. Stern. Report on the economics of climate change. avail via the national archives. 2006.
- Y Tsur and A Zemel. Pollution control in an uncertain environment. *Journal of Economic Dynamics and Control*, 1998. URL <http://www.sciencedirect.com/science/article/pii/S0165188997000870>.
- Yacov Tsur and Amos Zemel. Regulating environmental threats. *Environmental and Resource Economics*, 39(3):297–310, 2007a. ISSN 0924-6460. doi: 10.1007/s10640-007-9127-2. URL <http://link.springer.com/10.1007/s10640-007-9127-2>.
- Yacov Tsur and Amos Zemel. Policy tradeoffs under risk of abrupt climate change. *Journal of Economic Behavior and Organization*, 2015. ISSN 01672681. doi: 10.1016/j.jebo.2016.05.020. URL <http://dx.doi.org/10.1016/j.jebo.2016.05.020>.
- Yacov Tsur and Amos Zemel. The management of fragile resources: A long term perspective. *Environmental and Resource Economics*, 65(3):639–655, Nov 2016a. ISSN 1573-1502. doi: 10.1007/s10640-016-0005-7. URL <https://doi.org/10.1007/s10640-016-0005-7>.
- Martin L. Weitzman. A Review of The Stern Review on the Economics of Climate Change. *Journal of Economic Literature*, 19(September):703–724, 2007.
- Franz Wirl. CYCLICAL STRATEGIES IN TWO-DIMENSIONAL OPTIMAL CONTROL MODELS: NECESSARY CONDITIONS AND EXISTENCE. *Annals of Operations Research*, 37:345–356, 1992.
- Franz Wirl. A new route to cyclical strategies optimal control models. *Recherche Economique*, 48:165–173, 1994.
- Franz Wirl. Complex, dynamic environmental policies. *Resource and Energy Economics*, 21(1):19–41, 1999. ISSN 09287655. doi: 10.1016/S0928-7655(98)00031-1.
- Franz Wirl. Sustainable growth, renewable resources and pollution: Thresholds and cycles, 2004. ISSN 01651889.
- Amos Zemel. Adaptation, mitigation and risk: An analytic approach. *Journal of Economic Dynamics & Control*, 51:133–147, 2015.