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Smoothing cones over K3 surfaces

Stephen Coughlan Taro Sano

Abstract

We prove that the affine cone over a general primitively polarised K3 surface of genus g is smoothable if and only if $g \leq 10$ or $g = 12$. We discuss connections with the classification of Fano threefolds, and several examples of singularities with special behaviour.

1 Introduction

1.1 In 1974, Pinkham [23] showed that the cone over a normal elliptic curve is smoothable if and only if the curve has degree ≤ 9 . Schlessinger's 1973 criterion [27] can be used to show that the cone over an abelian variety of dimension ≥ 2 is never smoothable, and Mumford also used the same criterion to show that the cone over a curve of genus ≥ 2 embedded in sufficiently high degree is a non-smoothable singularity [22].

The cone over a K3 surface is a natural 3-dimensional generalisation of the cone over an elliptic curve, and our main result is the analogue of Pinkham's theorem.

Theorem 1.2. *Let S be a general K3 surface with primitive polarisation of genus g . Then the affine cone over S is smoothable if and only if $g \leq 10$ or $g = 12$.*

We make more precise statements about what "general" means in the article. This is connected with Brill–Noether loci as well as the rank of the Picard lattice.

1.3 A polarised K3 surface (S, L) of genus g is a K3 surface S with an ample line bundle L such that $c_1(L)^2 = 2g - 2 > 0$. We define

$$R(S) = \bigoplus_{n=0}^{\infty} H^0(S, L^{\otimes n}),$$

so that the affine cone over (S, L) is $X = C_a(S, L) = \text{Spec } R(S)$. Now X is normal ([31, (3.1)]) and Gorenstein [11, §5], and if S is nonsingular, then X has an isolated singularity at the vertex.

1.4 The singularity at the vertex P is resolved by a single blow up $f: \tilde{X} \rightarrow X$ at P , with exceptional divisor $E \subset \tilde{X}$ that is isomorphic to S with normal bundle L^\vee , the dual of L . By the adjunction formula, $K_{\tilde{X}} = f^*K_X - E$, so P is a log canonical 3-fold singularity. Moreover, P is an elliptic singularity; that is, $R^2f_*\mathcal{O}_{\tilde{X}} = \mathbb{C}_P$ and $R^1f_*\mathcal{O}_{\tilde{X}} = 0$. Such singularities are important in the study of 3-folds of general type and the boundary of their moduli spaces (see [8]).

1.5 By [27, 23], the \mathbb{C}^* -action on X induces a grading on T_X^1 , the space of isomorphism classes of infinitesimal deformations of X :

$$T_X^1 = \bigoplus_{k \in \mathbb{Z}} T_X^1(k).$$

Example 1.6. Suppose S is a quartic hypersurface in \mathbb{P}^3 defined by $\sum_{i=0}^3 x_i^4 = 0$, and let $X \subset \mathbb{A}^4$ be the affine cone over $(S, \mathcal{O}_S(1))$ for $\mathcal{O}_S(1) := \mathcal{O}_{\mathbb{P}^3}(1)|_S$. Then

$$T_X^1 = \mathbb{C}[x_0, x_1, x_2, x_3]/(x_0^3, x_1^3, x_2^3, x_3^3),$$

where the grading is shifted by -4 . Thus $T_X^1(k)$ is nonzero in degrees $-4 \leq k \leq 4$ and the graded pieces have dimensions $1, 4, 10, 16, 19, 16, 10, 4, 1$. Since X is a hypersurface singularity, it is clearly smoothable.

1.7 The “only if” part of Theorem 1.2 proceeds by showing that for $g = 11$ or $g \geq 13$, T_X^1 is concentrated in degree zero; that is, $T_X^1(k)$ vanishes for $k \neq 0$. It then follows from work of Schlessinger, that the only deformations of X are cones. In fact, we show that $T_X^1(k)$ vanishes for $|k| \geq 2$ by Green’s conjecture for curves on K3 surfaces. For $|k| \geq 1$, we interpret vanishing of $T_X^1(k)$ in terms of ramification of the map of moduli stacks $\varphi_{g,k}: \mathcal{P}_{g,k} \rightarrow \overline{\mathcal{M}}_{g_k}$, where $\varphi_{g,k}$ maps a pair $S, C \in |L^{\otimes k}|$ to the stable curve C (see Section 4 for more precise statements). This extends work of Mori and Mukai on the uniruledness of the moduli space of curves of genus 11 [19].

1.8 The “if” part is proved by *sweeping out the cone*. By [3], the general K3 surface S of genus $g \leq 10$ or $g = 12$ is the anticanonical section of a Fano 3-fold. Thus the affine cone X over S is realised as a hyperplane section through the vertex of the cone Y over the Fano. If we vary the hyperplane section so that it misses the vertex, then we obtain a smoothing of the vertex of X .

1.9 Another viewpoint, is that sweeping out the cone gives a smoothing of the projective cone $C_p(V, L)$ over a variety V polarised by L . That is, $C_p(V, L) = \text{Proj } R(V)[x]$, where x is the adjoined cone variable. A naive guess would be that all smoothings of the affine cone $C_a(V, L)$ over a variety are induced by smoothings of the projective cone $C_p(V, L)$. This is not true: Pinkham [24] gave an example of a variety V whose affine cone $C_a(V, L)$ is a smoothable singularity, even though the projective cone $C_p(V, L)$ is not. At first sight, Pinkham’s example seems to be quite special; V is 0-dimensional and $C_a(V, L)$ is Cohen–Macaulay but not Gorenstein or normal. We exhibit a 3-dimensional, normal and Gorenstein singularity $C_a(V, L)$, which is smoothable even though $C_p(V, L)$ is not. Our example is the affine cone over a particular surface of general type in its canonical model. Smoothing affine cones is more general than smoothing projective cones. On the other hand, if V is a nonsingular Calabi–Yau variety, in light of Pinkham’s theorem on elliptic curves and Theorem 1.2 above, we may reasonably ask:

Are all smoothings of $C_a(V, L)$ induced by deformations of $C_p(V, L)$?

Smoothings of projective cones over canonical curves and K3 surfaces are studied in [5, 6], where the authors show (among other things) that the projective cone over a general K3 surface of Picard rank 1 is smoothable iff $g \leq 10$ or $g = 12$.

1.10 We also give examples of K3 surfaces of genus 11 and ≥ 13 whose affine cone is smoothable. These are hyperplane sections of the anticanonical model of a Fano 3-fold with $b_2 \geq 2$, from the classification of Mori–Mukai [18]. Moreover, we exhibit K3 surfaces of genus 7 whose affine cone has at least two topologically distinct smoothings, analogous to the affine cone over a del Pezzo surface of degree 6.

1.11 We describe the contents of this paper. In section 2 we review certain criteria related to the vanishing of graded pieces of T_X^1 , a formula for computing graded pieces of T_X^1 , and our example (2.19) of a 3-dimensional singularity whose projective cone is not smoothable, from 1.9. We also define certain deformation functors and morphisms between them, which relax projective normality conditions (2.8, 2.10, 2.14). We then establish generalised criteria for formal smoothness of those morphisms of functors (2.11, 2.14). As applications we give a short proof of Pinkham’s theorem about smoothability of the affine cone over an elliptic curve (2.13) and prove that the affine cone over any polarised abelian variety of dimension ≥ 2 is not smoothable (2.17). In Section 3 we study vanishing of $T_X^1(k)$ using Wahl’s criterion and Green’s conjecture for curves on a K3 surface. This also has

an interesting interpretation via the classification of Fano 3-folds of index > 1 in terms of Clifford index. The main technical part of the proof is in Section 4, where we prove vanishing of $T_X^1(1)$ for a general K3 surface by extending work of Mori and Mukai on the uniruledness of the moduli space of curves of genus 11. The last section contains the proof of Theorem 1.2 and the examples mentioned in 1.10. We also mention further questions regarding hyperelliptic and trigonal K3 surfaces, and quasismooth K3 surfaces.

We work over the complex numbers unless otherwise stated.

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2 Review and properties of graded T_X^1

2.1 Criterion for negative gradedness Let $\text{Art}_{\mathbb{C}}$ be the category of Artin local \mathbb{C} -algebras with residue field \mathbb{C} and (Sets) be the category of sets. For an algebraic scheme X , let $\text{Def}_X : \text{Art}_{\mathbb{C}} \rightarrow (\text{Sets})$ be the usual deformation functor (cf. [28, 2.4.1]). For a projective scheme $X \hookrightarrow \mathbb{P}^N$, let $\text{Hilb}_X := \text{Hilb}_{X \hookrightarrow \mathbb{P}^N} : \text{Art}_{\mathbb{C}} \rightarrow (\text{Sets})$ be the Hilbert functor parametrizing embedded deformations of $X \hookrightarrow \mathbb{P}^N$ (cf. [28, 3.2.1]).

Let (X, L) be a *polarised manifold*, that is, X is a smooth projective variety such that $\dim X \geq 1$ and L is an ample line bundle. Let

$$C_a(X, L) := \text{Spec} \bigoplus_{k \geq 0} H^0(X, L^{\otimes k})$$

be the *affine cone* over (X, L) . By [14, 8.8.6], $C_a(X, L)$ is normal. We also have the following property.

Proposition 2.2. *Let (X, L) be a polarised manifold. Assume that $H^i(X, L^{\otimes k}) = 0$ for all $0 < i < \dim X$ and $k \in \mathbb{Z}$. Then we have the following:*

- (i) *The cone $C_a(X, L)$ is Cohen–Macaulay;*
- (ii) *If $\omega_X \simeq L^{\otimes m}$ for some $m \in \mathbb{Z}$, then $C_a(X, L)$ is Gorenstein.*

Proof. For (i), it is enough to check the conditions (a) and (b) in [11, 5.1.6(ii)]. We can check (a) by the construction of $C_a(X, L)$. The condition (b) is nothing but our assumption. Part (ii) follows from [11, 5.1.9]. \square

Write $C_a := C_a(X, L)$ for the affine cone over X . For $k \in \mathbb{Z}_{\geq 0}$, let $A_k := \mathbb{C}[t]/(t^{k+1})$ and $T_{C_a}^1 := \text{Def}_{C_a}(A_1)$ be the tangent space to Def_{C_a} . Then by [23, Proposition 2.2] or [31, (3.2)], the \mathbb{C}^* -action on C_a induces a grading $T_{C_a}^1 = \bigoplus_{k \in \mathbb{Z}} T_{C_a}^1(k)$.

Let $\tilde{C}_a \rightarrow C_a$ be the blowup at the vertex $P \in C_a$. Then $\tilde{C}_a \simeq \text{Tot}(L^\vee) := \text{Spec}_{\mathcal{O}_X} \bigoplus_{k \geq 0} L^{\otimes k}$ is the total space of the dual line bundle L^\vee of L . The punctured cone $C'_a := C_a \setminus P$ is isomorphic to $\tilde{C}_a \setminus X$ if we view X as the zero section in \tilde{C}_a . Note that $C'_a \simeq \text{Spec}_{\mathcal{O}_X} \bigoplus_{k \in \mathbb{Z}} L^{\otimes k}$. We have the inclusion map $\iota: C'_a \rightarrow C_a$, the bundle $\pi: \tilde{C}_a \rightarrow X$, and the \mathbb{C}^* -bundle $C'_a \rightarrow X$, which we also denote by π .

Restricting the short exact sequence $0 \rightarrow \pi^*L \rightarrow \mathcal{T}_{\tilde{C}_a} \rightarrow \pi^*\mathcal{T}_X \rightarrow 0$ of [27, §4] to C'_a , we have

$$0 \rightarrow \mathcal{O}_{C'_a} \rightarrow \mathcal{T}_{C'_a} \rightarrow \pi^*\mathcal{T}_X \rightarrow 0, \quad (1)$$

because $\pi^*L|_{C'_a}$ is the trivial line bundle. More precisely, by [31, Proposition 3.3], we have isomorphisms

$$\mathcal{T}_{C'_a} \simeq \pi^*\mathcal{E}_L \simeq \mathcal{T}_{\tilde{C}_a}(-\log X) \otimes \mathcal{O}_{C'_a} \quad (2)$$

where \mathcal{E}_L is the extension

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{E}_L \rightarrow \mathcal{T}_X \rightarrow 0 \quad (3)$$

corresponding to $c_1(L) \in H^1(X, \Omega_X^1) \simeq \text{Ext}^1(\mathcal{T}_X, \mathcal{O}_X)$.

We use the following criterion about the vanishing of the graded pieces $T_{C_a}^1(k)$ when L induces a projectively normal embedding in \mathbb{P}^n .

Proposition 2.3. ([2, Theorem 12.1]) *Let (X, L) be a polarised manifold. Assume that L induces a projectively normal embedding $\Phi_{|L|}: X \hookrightarrow \mathbb{P}^N$.*

- (i) (Pinkham [23, Theorem 5.1]) *Suppose that $T_{C_a}^1(k) = 0$ for all $k > 0$. Write $C_p(X, L) \subset \mathbb{P}^{N+1}$ for the projective cone over $\Phi_{|L|}: X \hookrightarrow \mathbb{P}^N$ as defined in 1.9. Then the restriction map*

$$\text{Hilb}_{C_p(X, L)} \rightarrow \text{Def}_{C_a(X, L)}$$

is formally smooth.

- (ii) (Schlessinger [27, §4.3]) *Suppose that $T_{C_a}^1(k) = 0$ for all $k \neq 0$. Then we have a canonical morphism of functors*

$$\text{Hilb}_X \rightarrow \text{Def}_{C_a(X, L)}$$

and it is formally smooth. Every deformation of $C_a(X, L)$ is a cone.

We describe the eigenspace $T_{C_a}^1(k)$ for an arithmetically Gorenstein embedding $X \subset \mathbb{P}^n$.

Proposition 2.4. *Let $X \subset \mathbb{P}^n$ be a smooth, arithmetically Gorenstein variety of dimension ≥ 2 , and let $C_a = C_a(X, \mathcal{O}_X(1))$ be the affine cone over X . If $\dim X \geq 3$, then*

$$T_{C_a}^1(k) = H^1(X, \mathcal{T}_X(k)).$$

If X is a surface with $\omega_X = \mathcal{O}(c)$, then

$$T_{C_a}^1(c - k) \simeq T_{C_a}^1(c + k) \simeq H^1(X, \mathcal{T}_X(c + k)) \text{ for } k \neq 0,$$

and $T_{C_a}^1(c) \subseteq H^1(X, \mathcal{T}_X(c))$.

Proof. We start from $T_{C_a}^1 = \text{Ext}^1(\Omega_{C_a}^1, \mathcal{O}_{C_a})$. Since $\text{codim}_P C_a \geq 3$ and C_a is Cohen–Macaulay, we have

$$T_{C_a}^1 = \text{Ext}^1(\Omega_{C_a}^1, \mathcal{O}_{C_a}) = H^1(C'_a, \mathcal{T}_{C'_a}).$$

By the projection formula and the Leray spectral sequence, we know that $H^1(\mathcal{O}_{C'_a}) = H^1(\bigoplus_{k \in \mathbb{Z}} \mathcal{O}_X(k))$ and $H^1(\pi^* \mathcal{T}_X) = H^1(\bigoplus_{k \in \mathbb{Z}} \mathcal{T}_X(k))$.

If X is not a surface, then $H^1(\mathcal{O}_X(k)) = H^2(\mathcal{O}_X(k)) = 0$ for all k because X is arithmetically Gorenstein. Thus the long exact sequence associated to (1) above gives $T_{C_a}^1(k) = H^1(\mathcal{T}_X(k))$.

Now suppose that X is an arithmetically Gorenstein surface with $\omega_X = \mathcal{O}(c)$. Then $H^1(\mathcal{O}_X(c + k))$ vanishes for all k , and this gives an inclusion of $T_{C_a}^1(c + k)$ in $H^1(\mathcal{T}_X(c + k))$ for all k . For $k > 0$, we have $H^2(\mathcal{O}_X(c + k)) = 0$ by Kodaira vanishing, and thus $T_{C_a}^1(c + k) = H^1(\mathcal{T}_X(c + k))$ for all $k > 0$. Now $T_{C_a}^1(c + k) \cong T_{C_a}^1(c - k)$ by a theorem of Wahl [32, §2.3], and this completes the proof. \square

2.5 Generalised criteria In the following, we generalise Proposition 2.3 (i) and (ii) for a general ample line bundle L . That is, we do not assume that X is projectively normal. For that purpose, we prepare several notions on projective cones and two deformation functors in Definitions 2.6 and 2.8. We do not use these directly in the proof of Theorem 1.2, but we give some other applications, and the functors may be useful in studying some of the questions raised in Section 5.

Definition 2.6. (cf. Appendices to [17] and [10]) Let (X, L) be a polarised manifold. Let

$$R(X, L)[t] := \bigoplus_{i \geq 0} \bigoplus_{k \geq 0} H^0(X, L^{\otimes k}) \cdot t^i$$

be a graded \mathbb{C} -algebra such that $\deg(\sigma_k \cdot t^i) = k+i$, where $\sigma_k \in H^0(X, L^{\otimes k})$. We define $C_p(X, L) := \text{Proj } R(X, L)[t]$ to be the *projective cone* over (X, L) and $H_t \subset C_p(X, L)$ to be the divisor defined by the variable t . There is a natural \mathbb{C}^* -action on $R(X, L)[t]$ which is trivial on t , and this induces a \mathbb{C}^* -action on $C_p(X, L)$.

Remark 2.7. Let $\mathcal{O}_{C_p}(1)$ be the tautological sheaf corresponding to the graded module $R(X, L)[t](1)$ shifted by degree 1. Then we can check that this is invertible and $H_t \in |\mathcal{O}_{C_p}(1)|$. Note that $\mathcal{O}_{C_p}(1)$ is not necessarily very ample. We can also check that $C_p \setminus H_t \simeq C_a$. Let $\tilde{C}_p \rightarrow C_p$ be the blowup at the vertex P . Then $\tilde{C}_p \simeq \mathbb{P}(\mathcal{O}_X \oplus L^\vee)$ is the projectivization of $\mathcal{O}_X \oplus L^\vee$. We also have the punctured cone $C'_p := C_p \setminus P$ and $C'_p \simeq \tilde{C}_p \setminus H_0 \simeq \text{Tot}(L) \simeq \text{Spec}_{\mathcal{O}_X} \bigoplus_{k \geq 0} (L^\vee)^{\otimes k}$, where $H_0 \subset \tilde{C}_p$ is the negative section.

Next we define two deformation functors associated to an algebraic scheme with a closed subscheme and a line bundle.

Definition 2.8. (cf. [28, 3.3.3, 3.4.4]) Let X be an algebraic scheme, $D \subset X$ be its closed subscheme, and L be a line bundle on X .

- (i) For $A \in \text{Art}_{\mathbb{C}}$, a *deformation of a pair* (X, D) over A is a pair (ξ, D_A) which consists of $\xi = (X \hookrightarrow X_A \rightarrow \text{Spec } A) \in \text{Def}_X(A)$, and a closed subscheme $D_A \subset X_A$ which is flat over A and has an isomorphism $D_A \otimes_A \mathbb{C} \simeq D$. Let $\text{Def}_{(X,D)}(A)$ be the isomorphism classes of deformations of (X, D) over $A \in \text{Art}_{\mathbb{C}}$. Then we obtain a deformation functor $\text{Def}_{(X,D)}: \text{Art}_{\mathbb{C}} \rightarrow (\text{Sets})$ of the pair (X, D) .
- (ii) For $A \in \text{Art}_{\mathbb{C}}$, a *deformation of a pair* (X, L) over A is a pair (ξ, L_A) which consists of $\xi = (X \hookrightarrow X_A \rightarrow \text{Spec } A) \in \text{Def}_X(A)$, and $L_A \in \text{Pic } X_A$ with an isomorphism $L_A|_X \simeq L$. Let $\text{Def}_{(X,L)}(A)$ be the set of isomorphism classes of deformations of (X, L) over A . Then we obtain a deformation functor $\text{Def}_{(X,L)}: \text{Art}_{\mathbb{C}} \rightarrow (\text{Sets})$ of the pair (X, L) .

Remark 2.9. Let (X, L) be a polarised manifold. Let $T_{(X,L)}^1 := \text{Def}_{(X,L)}(A_1)$ be the tangent space for $\text{Def}_{(X,L)}$. It is known that $T_{(X,L)}^1 \simeq H^1(X, \mathcal{E}_L)$ and we can take $H^2(X, \mathcal{E}_L)$ as an obstruction space, where \mathcal{E}_L is the sheaf as in (3) (cf. [28, Theorem 3.3.11]).

Definition 2.10. Let $\text{Def}_{(C_p(X,L), H_t)}: \text{Art}_{\mathbb{C}} \rightarrow (\text{Sets})$ be the deformation functor of a pair $(C_p(X, L), H_t)$ (cf. [28, 3.4.4]). Then we have a restriction morphism

$$\Phi: \text{Def}_{(C_p(X,L), H_t)} \rightarrow \text{Def}_{C_a(X,L)}$$

which is induced by the open immersion $C_a(X, L) \hookrightarrow C_p(X, L)$.

Using this morphism of functors, we can generalise Proposition 2.3(i) as follows.

Proposition 2.11. *Let (X, L) be a polarised manifold and let $C_p := C_p(X, L)$, $C_a := C_a(X, L)$ be the projective (respectively affine) cone over (X, L) . Assume that $T_{C_a}^1(k)$ vanishes for all $k > 0$. Then the restriction morphism $\Phi: \text{Def}_{(C_p, H_t)} \rightarrow \text{Def}_{C_a}$ is formally smooth.*

Proof. We follow the approach of [2, Theorem 12.1], that is, we shall prove the following:

- (a) The tangent map $d\Phi: T_{(C_p, H_t)}^1 \rightarrow T_{C_a}^1$ is surjective;
- (b) Given $\xi_A := (C_{p,A}, H_{t,A}) \in \text{Def}_{(C_p, H_t)}(A)$. Let $\bar{\xi}_A := \Phi(\xi_A) \in \text{Def}_{C_a}(A)$ be its image and assume that $\bar{\xi}_A$ can be lifted over a small extension $A' \in \text{Art}_{\mathbb{C}}$ of A . Then there exists a lift $\xi_{A'} \in \text{Def}_{(C_p, H_t)}(A')$ of ξ_A over A' .

First we shall check (a). Let $\Phi': \text{Def}_{(C'_p, H_t)} \rightarrow \text{Def}_{C'_a}$ be the restriction by an open immersion $\iota': C'_a \hookrightarrow C'_p$. We have a commutative diagram

$$\begin{array}{ccc} T_{(C_p, H_t)}^1 & \xrightarrow{d\Phi} & T_{C_a}^1 \\ \downarrow \iota_p^* & & \downarrow \iota_a^* \\ T_{(C'_p, H_t)}^1 & \xrightarrow{d\Phi'} & T_{C'_a}^1. \end{array}$$

The vertical homomorphisms are induced by open immersions $\iota_p: C'_p \hookrightarrow C_p$ and $\iota_a: C'_a \hookrightarrow C_a$ and they are injective since C_a and C_p are normal. We will show that $T_{(C_p, H_t)}^1 = (d\Phi')^{-1}(T_{C_a}^1)$. This is sufficient since we can construct a deformation of (C_p, H_t) from $\eta \in (d\Phi')^{-1}(T_{C_a}^1) \subset T_{(C'_p, H_t)}^1$ by taking the push-forward of the structure sheaf by the open immersion $\iota_p: C'_p \hookrightarrow C_p$.

Let $T_{(C'_p, H_t)}^1 = \bigoplus_{k \in \mathbb{Z}} T_{(C'_p, H_t)}^1(k)$ and $T_{C'_a}^1 = \bigoplus_{k \in \mathbb{Z}} T_{C'_a}^1(k)$ be the eigendecompositions with respect to the \mathbb{C}^* -actions on C'_p and C'_a . Since all the homomorphisms in the above diagram are \mathbb{C}^* -equivariant, we have a homomorphism $d\Phi'(k): T_{(C'_p, H_t)}^1(k) \rightarrow T_{C'_a}^1(k)$ for all $k \in \mathbb{Z}$. By the assumption on $T_{C_a}^1(k)$, it is enough to check that $d\Phi'(k): T_{(C'_p, H_t)}^1(k) \rightarrow T_{C'_a}^1(k)$ is surjective for all $k \leq 0$.

Let \mathcal{E}_{L^\vee} be the locally free sheaf on X defined by the extension associated to $c_1(L^\vee) \in H^1(X, \Omega_X^1)$ as in short exact sequence (3). In fact, since $c_1(L) = -c_1(L^\vee)$, we have $\mathcal{E}_L \simeq \mathcal{E}_{L^\vee}$ by [28, 3.3.10]. Then by [31, Proposition 3.3], we have $\mathcal{T}_{C'_p}(-\log H_t) \simeq \pi_p^* \mathcal{E}_{L^\vee}$ where $\pi_p: C'_p \simeq \text{Tot}(L) \rightarrow X$ is the projection.

Since π_p is an affine morphism, we obtain an isomorphism

$$T_{(C'_p, H_t)}^1 \simeq H^1(C'_p, \mathcal{T}_{C'_p}(-\log H_t)) \simeq \bigoplus_{k \leq 0} H^1(X, \mathcal{E}_{L^\vee} \otimes L^{\otimes k})$$

by the projection formula and $(\pi_p)_* \mathcal{O}_{C'_p} \simeq \bigoplus_{k \geq 0} \mathcal{O}_X(L^{\otimes -k})$. Hence, for $k \leq 0$, we obtain

$$T_{(C'_p, H_t)}^1(k) \simeq H^1(X, \mathcal{E}_{L^\vee} \otimes L^{\otimes k}).$$

Let $\pi_a: C'_a \rightarrow X$ be the \mathbb{C}^* -bundle projection. Then we have $\mathcal{T}_{C'_a} \simeq \pi_a^* \mathcal{E}_{L^\vee}$ by (2) and

$$T_{C'_a}^1 \simeq H^1(C'_a, \mathcal{T}_{C'_a}) \simeq \bigoplus_{k \in \mathbb{Z}} H^1(X, \mathcal{E}_{L^\vee} \otimes L^{\otimes k}) \quad (4)$$

by the projection formula for the affine morphism π_a and $(\pi_a)_* \mathcal{O}_{C'_a} \simeq \bigoplus_{k \in \mathbb{Z}} L^{\otimes k}$. Hence, for $k \in \mathbb{Z}$, we obtain

$$T_{C'_a}^1(k) \simeq H^1(X, \mathcal{E}_{L^\vee} \otimes L^{\otimes k}).$$

We see that $d\Phi'(k)$ is an isomorphism for all $k \leq 0$ by the description of $T_{(C'_p, H_t)}^1(k)$ and $T_{C'_a}^1(k)$, and the following commutative diagram for $C'_p \simeq \text{Spec}_{\mathcal{O}_X} \bigoplus_{k \leq 0} L^{\otimes k}$ and $C'_a \simeq \text{Spec}_{\mathcal{O}_X} \bigoplus_{k \in \mathbb{Z}} L^{\otimes k}$:

$$\begin{array}{ccc} (\pi_p)_* \mathcal{O}_{C'_p} & \longrightarrow & (\pi_a)_* \mathcal{O}_{C'_a} \\ \downarrow \simeq & & \downarrow \simeq \\ \bigoplus_{k \leq 0} L^{\otimes k} & \longrightarrow & \bigoplus_{k \in \mathbb{Z}} L^{\otimes k}. \end{array}$$

Next consider the situation in (b). We shall construct a lift $(C_{p,A'}, H_{t,A'}) \in \text{Def}_{(C_p, H_t)}(A')$ of $(C_{p,A}, H_{t,A})$. The transformation $\Phi': \text{Def}_{(C'_p, H_t)} \rightarrow \text{Def}_{C'_a}$ induces a homomorphism

$$o(\Phi'): H^2(C'_p, \mathcal{T}_{C'_p}(-\log H_t)) \rightarrow H^2(C'_a, \mathcal{T}_{C'_a})$$

between the obstruction spaces of the functors. This is injective since we have a commutative diagram

$$\begin{array}{ccc} H^2(C'_p, \mathcal{T}_{C'_p}(-\log H_t)) & \xrightarrow{\simeq} & \bigoplus_{k \leq 0} H^2(X, \mathcal{E}_{L^\vee} \otimes L^{\otimes k}) \\ \downarrow & & \downarrow \\ H^2(C'_a, \mathcal{T}_{C'_a}) & \xrightarrow{\simeq} & \bigoplus_{k \in \mathbb{Z}} H^2(X, \mathcal{E}_{L^\vee} \otimes L^{\otimes k}). \end{array}$$

Thus we can lift $\iota_p^*(\xi_A)$ in $\text{Def}_{(C'_p, H_t)}(A)$ to A' . Let $(C'_{p, A'}, H_{t, A'})$ in $\text{Def}_{(C'_p, H_t)}(A')$ be the corresponding deformation. We see that $(\iota_p)_* \mathcal{O}_{C'_{p, A'}}$ is flat over A' since $\bar{\xi}_A$ can be lifted over A' . Thus we obtain a lift of $(C_{p, A}, H_{t, A})$ over A' as required. \square

We define smoothability as follows.

Definition 2.12. Let X be an algebraic scheme. We say that X is smoothable if there exists a deformation $\phi: \mathcal{X} \rightarrow S$ of X over some quasi-projective curve S whose fibre $\mathcal{X}_s := \phi^{-1}(s)$ is smooth for general $s \in S$.

As a corollary of Proposition 2.11, we obtain a quick proof of Pinkham's theorem on cones over elliptic curves.

Corollary 2.13. ([23, §9]) *Let E be an elliptic curve and L be an ample line bundle on E of degree d . Then $C_a(E, L)$ has a smoothing if and only if $d \leq 9$.*

Proof. Note that $T_{C'_a}^1(k) \simeq H^1(E, \mathcal{E}_{L^\vee} \otimes L^{\otimes k})$ as in (4). This vanishes for all $k > 0$ by the exact sequence (3) and $H^1(E, L^{\otimes k}) = 0$ for all $k > 0$. Since C_a is normal, we have a \mathbb{C}^* -equivariant injection $T_{C_a}^1 \hookrightarrow T_{C'_a}^1$ and this induces an injection $T_{C_a}^1(k) \hookrightarrow T_{C'_a}^1(k)$. Thus we obtain $T_{C_a}^1(k) = 0$ for all $k > 0$ and, by Proposition 2.11, the restriction morphism $\Phi: \text{Def}_{(C_p(E, L), H_t)} \rightarrow \text{Def}_{C_a(E, L)}$ is formally smooth.

Assume that $C_a(E, L)$ is smoothable. By the formal smoothness of Φ , the projective cone $C_p(E, L)$ also has a smoothing $\mathcal{C}_p \rightarrow S$ over some quasi-projective curve S . Since $-K_{C_p(E, L)}$ is ample, the general fibre $\mathcal{C}_{p, s}$ is a smooth del Pezzo surface of degree d . Hence we obtain $d \leq 9$ by the classification of del Pezzo surfaces.

Conversely assume that $d \leq 9$. For a given (E, L) , we can explicitly construct a smooth del Pezzo surface S such that $E \in |-K_S|$ and $L \simeq \mathcal{O}_S(-K_S)|_C$. For example, first choose an embedding $\iota_E: E \hookrightarrow \mathbb{P}^2$ as a cubic curve and take $p_1, \dots, p_{9-d} \in E$ such that $L \simeq \mathcal{O}_E(3) \otimes \mathcal{O}_E(-p_1 - \dots - p_{9-d})$. Let $S \rightarrow \mathbb{P}^2$ be the blow-up at p_1, \dots, p_{9-d} . Note that we can choose the embedding ι_E such that p_1, \dots, p_{9-d} are in general position and S is del Pezzo. Then we see that $L \simeq \mathcal{O}_S(-K_S)|_C$ by the construction. For this S , we see that $C_a(E, L)$ is a hypersurface in $C_a(S, -K_S)$ since $H^0(S, -mK_S) \rightarrow H^0(E, mL)$ is surjective by $H^1(S, -(m-1)K_S) = 0$ for all $m \geq 1$. Thus we can construct a smoothing by sweeping out the cone. \square

Using the functor $\text{Def}_{(X, L)}$ as in Definition 2.8, we have the following analogue of Proposition 2.3 (ii) for a general polarization.

Proposition 2.14. *Let X be a smooth projective variety and L an ample line bundle on X . Assume that $\dim X \geq 1$ and $H^1(X, L^{\otimes k}) = 0$ for all $k > 0$.*

(i) We can define a canonical morphism of functors

$$\Gamma: \text{Def}_{(X,L)} \rightarrow \text{Def}_{C_a(X,L)}.$$

(ii) Suppose that $T_{C_a}^1(k) = 0$ for $k \neq 0$. Then the morphism Γ is formally smooth. Thus $C_a(X, L)$ has only conical deformations.

Remark 2.15. The morphism Γ of Proposition 2.14 can also be formulated as a morphism of functors

$$\Gamma_w: \text{Hilb}_X^w \rightarrow \text{Def}_{C_a(X,L)},$$

where Hilb_X^w denotes the Hilbert functor of embedded deformations of X in weighted projective space $w\mathbb{P}^n$ via the Proj-construction $X = \text{Proj} \bigoplus_{k \geq 0} H^0(X, L^{\otimes k})$.

Remark 2.16. Christophersen–Kleppe treated a similar result (cf. [4, Corollary 5.2]). However they assume $H^2(X, \mathcal{O}_X) = 0$ which implies that the deformations preserve polarizations (cf. [15, Example 21.2.5]). A general projective variety X has non-algebraic deformations. By considering deformations of (X, L) , we can avoid this problem.

Proof. (i) For $A \in \text{Art}_{\mathbb{C}}$ and $(X_A, L_A) \in \text{Def}_{(X,L)}(A)$, let

$$C_a(X_A, L_A) := \text{Spec} \bigoplus_{k \geq 0} H^0(X_A, L_A^{\otimes k}).$$

We see that $H^0(X_A, L_A^{\otimes k})$ is flat over A by [31, Corollary 0.4.4] and $H^1(X, L^{\otimes k}) = 0$ for all $k > 0$. Hence $C_a(X_A, L_A)$ is a deformation of $C_a(X, L)$ and we can define Γ by letting $\Gamma_A((X_A, L_A)) := C_a(X_A, L_A)$.

(ii) We follow the approach of [2, Theorem 12.1] as in Proposition 2.11. That is, we shall prove the following:

- (a) The tangent map $d\Gamma: T_{(X,L)}^1 \rightarrow T_{C_a}^1$ is surjective;
- (b) Given $\xi_A := (X_A, L_A) \in \text{Def}_{(X,L)}(A)$. Let $\bar{\xi}_A := \Gamma(\xi_A) \in \text{Def}_{C_a}(A)$ be its image and assume that $\bar{\xi}_A$ can be lifted over a small extension $A' \in \text{Art}_{\mathbb{C}}$ of A . Then there exists a lift $\xi_{A'} \in \text{Def}_{(X,L)}(A')$ of ξ_A over A' .

Let $\iota: C'_a \hookrightarrow C_a$ be the open immersion of the punctured neighborhood. Then $\iota^*: \text{Def}_{C_a} \rightarrow \text{Def}_{C'_a}$ is the restriction by ι , and we define $\Gamma' := \iota^* \circ \Gamma: \text{Def}_{(X,L)} \rightarrow \text{Def}_{C'_a}$ to be the composition. Then the tangent map $d\Gamma'$ is decomposed as

$$d\Gamma': T_{(X,L)}^1 \xrightarrow{d\Gamma} T_{C_a}^1 \xrightarrow{\iota^*} T_{C'_a}^1$$

and ι^* is injective since the cone C_a is normal.

We shall prove $d\Gamma$ is surjective. We can describe $d\Gamma'$ as the natural homomorphism

$$d\Gamma': H^1(X, \mathcal{E}_L) \rightarrow H^1(C'_a, \mathcal{T}_{C'_a}) \simeq H^1(C'_a, \pi_a^* \mathcal{E}_L),$$

where $\pi_a: C'_a \rightarrow X$ is the \mathbb{C}^* -bundle projection. Note that we have $H^1(C'_a, \pi_a^* \mathcal{E}_L) \simeq \bigoplus_{k \in \mathbb{Z}} H^1(X, \mathcal{E}_L \otimes L^{\otimes k})$ and $d\Gamma'$ is induced by the adjunction morphism $\mathcal{E}_L \rightarrow (\pi_a)_* \pi_a^* \mathcal{E}_L$. Thus we see that $d\Gamma'$ is an isomorphism onto the degree 0 part of the target space. Hence $d\Gamma$ is an isomorphism onto $T_{C_a}^1(0)$. By this and the assumption $T_{C_a}^1(k) = 0$ for $k \neq 0$, we see that $d\Gamma$ is surjective. This proves (a).

Next we shall prove (b). The obstruction class to lifting ξ_A to A' is $o(\xi_A) \in H^2(X, \mathcal{E}_L)$. We show that $o(\xi_A) = 0$. The morphism $\Gamma': \text{Def}_{(X,L)} \rightarrow \text{Def}_{C'_a}$ induces a linear map

$$o_{\Gamma'}: H^2(X, \mathcal{E}_L) \rightarrow H^2(C'_a, \mathcal{T}_{C'_a})$$

between the obstruction spaces. By the construction of the obstruction map and $\pi_a^* \mathcal{E}_L \simeq \mathcal{T}_{C'_a}$, we have a commutative diagram

$$\begin{array}{ccc} H^2(X, \mathcal{E}_L) & \xrightarrow{o_{\Gamma'}} & H^2(C'_a, \mathcal{T}_{C'_a}) \\ & \searrow^{o'} & \downarrow \simeq \\ & & H^2(C'_a, \pi_a^* \mathcal{E}_L) \\ & \searrow^{o''} & \downarrow \simeq \\ & & H^2(X, (\pi_a)_* \pi_a^* \mathcal{E}_L). \end{array}$$

Note that o'' is induced by the adjunction homomorphism $\mathcal{E}_L \rightarrow (\pi_a)_* \pi_a^* \mathcal{E}_L$ which is a split injection. Hence o'' is injective and $o_{\Gamma'}$ is also injective.

We see that $o_{\Gamma'}(o(\xi_A)) = 0$ since $\bar{\xi}_A \in \text{Def}_{C_a}(A)$ and its image $\bar{\xi}'_A \in \text{Def}_{C'_a}(A)$ can be extended over A' . Hence we have $o(\xi_A) = 0$ and obtain (b). This concludes the proof of Proposition 2.14. \square

We think that the following result is known to experts, but we could not find it in the literature. We prove it as an application of Proposition 2.14.

Corollary 2.17. *Let X be an abelian variety of dimension $n \geq 2$ and L an ample line bundle on X . Then the affine cone $C_a = C_a(X, L)$ has only conical deformations.*

Proof. We see that $\iota^*: T_{C_a}^1 \rightarrow T_{C'_a}^1$ is injective since C_a is S_2 and $\text{codim}_{C_a} P \geq 2$. We have an isomorphism $T_{C'_a}^1 \simeq \bigoplus_{k \in \mathbb{Z}} H^1(X, \mathcal{E}_L \otimes L^{\otimes k})$. Note that $\mathcal{T}_X \simeq \mathcal{O}_X^{\oplus n}$. Thus $H^1(X, \mathcal{T}_X \otimes L^{\otimes k}) = 0$ and $H^1(X, L^{\otimes k}) = 0$ for any $k \neq 0$ by Serre duality, Kodaira vanishing and $n \geq 2$. Hence we obtain $H^1(X, \mathcal{E}_L \otimes L^{\otimes k}) = 0$ and thus $T_{C_a}^1(k) = 0$ for $k \neq 0$. To conclude the proof, we apply Proposition 2.14 (ii). \square

2.18 A smoothable affine cone with a non-smoothable projectivization

In [24, Ex. 2.11], Pinkham exhibited a Cohen–Macaulay curve singularity as the affine cone $C_a(X)$ over a certain projective 0-dimensional scheme X such that $C_a(X)$ is smoothable, but there is no smoothing induced by a deformation of the projective cone $C_p(X)$. Below, we construct a Gorenstein normal 3-fold singularity, the cone over a certain surface X of general type, such that $C_a(X)$ is smoothable but $C_p(X)$ is not.

Example 2.19. Let $X \subset \mathbb{A}^6$ be the codimension 3 variety defined by the 4×4 Pfaffians of the skew matrix M with homogeneous entries of degrees $\begin{pmatrix} -1 & 1 & 1 & 1 \\ & 1 & 1 & 1 \\ & & 3 & 3 \\ & & & 3 \end{pmatrix}$. The general such M is

$$\begin{pmatrix} 0 & x_1 & x_2 & x_3 \\ & x_4 & x_5 & x_6 \\ & & f_1 & f_2 \\ & & & f_3 \end{pmatrix},$$

and $X = C_a(S, K_S)$ is the cone over a divisor S of bidegree $(3, 4)$ in $\mathbb{P}^1 \times \mathbb{P}^2$ under the Segre embedding in \mathbb{P}^5 (S is the canonical model of a surface of general type with $p_g = 6$ and $K^2 = 11$). Indeed, the 4×4 Pfaffians of M are

$$x_1x_5 - x_2x_4, \quad x_1x_6 - x_3x_4, \quad x_2x_6 - x_3x_5, \quad x_1f_3 - x_2f_2 + x_3f_1, \quad x_4f_3 - x_5f_2 + x_6f_1,$$

the first three define the Segre embedding, and the last two cut out the divisor S .

All deformations of X are obtained by varying the entries of M [16, 33]. Thus after coordinate changes, the general fibre X' of any deformation of X is defined by the Pfaffians of

$$M' = \begin{pmatrix} \lambda & x_1 & x_2 & x_3 \\ & x_4 & x_5 & x_6 \\ & & f'_1 & f'_2 \\ & & & f'_3 \end{pmatrix},$$

where $f'_i = f_i + h_i$ for some polynomials h_i .

We first show that X is smoothable. Let λ be a nonzero constant, and choose h_i sufficiently general with some terms of degree ≤ 1 . Since λ is constant, Pfaffians 1 and 2 are redundant, and X' is a nonsingular complete intersection for suitably chosen h_i .

Now suppose that we restrict ourselves to deformations X' that are induced by a deformation of the projective cone $C_p(S, K_S)$ over S . Then $\lambda \equiv 0$ for degree reasons, and h_i must have degree ≤ 3 — in particular, we see that the above smoothing is not induced by $C_p(S, K_S)$. Since $\lambda = 0$, X' passes through the origin, and a computation

of the partial derivatives of Pfaffians 3, 4 and 5 shows that the Jacobian matrix of X' must have rank ≤ 2 there. Thus X' must be singular at the origin.

Remark 2.20. The above example is quite flexible. For example, we get 3-fold singularities with similar properties by taking a divisor S_k in $\mathbb{P}^1 \times \mathbb{P}^2$ of bidegree $(k, k+1)$ for any $k \geq 3$.

3 Vanishing of $T_X^1(k)$ for $|k| \geq 2$

Theorem 3.1. *Let S be a K3 surface with primitive polarisation L of Clifford index > 2 . Let X be the affine cone over (S, L) , then $T_X^1(k)$ vanishes for $|k| \geq 2$.*

3.2 The Clifford index of a smooth curve C is

$$\text{Cliff } C = \min\{d - 2r \mid r \geq 1, d \leq g - 1\},$$

computed over all special linear systems g_d^r on C . Clifford index is a refinement of gonality. The general curve has maximal Clifford index $\lfloor \frac{g-1}{2} \rfloor$, and using this terminology, Clifford's theorem states that $\text{Cliff } C \geq 0$ with equality if and only if C has a g_2^1 . It follows from work of Green–Lazarsfeld [13] (see also Reid [25]), that the Clifford index is constant for all curves in a linear system $|C|$ on a K3 surface. Thus we define the Clifford index of a K3 surface (S, L) to be $\text{Cliff } C$ for any C in $|L|$.

The generic polarised K3 surface of genus g has maximal Clifford index $\lfloor \frac{g-1}{2} \rfloor$, and so the hypothesis of Theorem 3.1 holds for general K3 surfaces of genus $g \geq 7$. Curves of Clifford index 0 are hyperelliptic, index 1 means trigonal or a plane quintic, and index 2 means tetragonal or a plane sextic [9, §0].

Example 3.3. The K3 surface of genus 6 is a complete intersection $H_1 \cap H_2 \cap H_3 \cap Q$ inside the Plücker embedding of $\text{Gr}(2, 5)$ in \mathbb{P}^9 , where H_i are hyperplanes and Q is a hyperquadric. We compute T_X^1 has nonzero graded pieces in degrees $-2, -1, 0, 1, 2$ with dimensions 1, 10, 19, 10, 1. The generic curve of genus 6 has Clifford index 2, while the generic curve of genus 7 has Clifford index 3. Thus the theorem is sharp.

3.4 Wahl's criterion Theorem 3.1 is proved by using Koszul cohomology and Green's conjecture for curves on a K3 surface, to show that S satisfies Wahl's criterion for vanishing of $T^1(k)$ for $k \leq -2$.

Theorem 3.5. *(Wahl [32, Corollary 2.8]) Suppose the free resolution of \mathcal{O}_S begins with*

$$\mathcal{O}_S \leftarrow \mathcal{O}_{\mathbb{P}} \leftarrow \mathcal{O}_{\mathbb{P}}(-2)^a \leftarrow \mathcal{O}_{\mathbb{P}}(-3)^b \leftarrow \dots \quad (5)$$

Then $T_X^1(k) = 0$ for $k \leq -2$.

Let (S, L) be a polarised K3 surface. By [26], we can choose $C \in |L|$ a nonsingular irreducible curve. Since C is a hyperplane section of $S \subset \mathbb{P}^g$ and the coordinate ring of S is Gorenstein, the Betti numbers of \mathcal{O}_C are the same as those of \mathcal{O}_S . Moreover, by adjunction $C \subset \mathbb{P}^{g-1}$ is a canonical curve, so we are reduced to studying the equations and syzygies of canonical curves.

3.6 Green's conjecture We refer to [12] for details on Koszul cohomology and Green's conjecture. For simplicity, we formulate everything in terms of Betti numbers. For a nonhyperelliptic canonical curve $C \subset \mathbb{P}^{g-1}$ of genus g , the free resolution of \mathcal{O}_C as an $\mathcal{O}_{\mathbb{P}^{g-1}}$ -module is

$$\mathcal{O}_C \leftarrow \mathcal{F}_0 \leftarrow \mathcal{F}_1 \leftarrow \cdots \leftarrow \mathcal{F}_{g-2} \leftarrow 0$$

where $\mathcal{F}_0 = \mathcal{O}_{\mathbb{P}}^{\beta_{0,0}}$, $\mathcal{F}_i = \bigoplus_{j=1,2} \mathcal{O}_{\mathbb{P}}(-i-j)^{\beta_{i,j}}$ for $i = 1, \dots, g-3$ and $\mathcal{F}_{g-2} = \mathcal{O}_{\mathbb{P}}(-g-2)^{\beta_{g-2,3}}$. This data is represented in a Betti table as follows:

$$\begin{array}{ccccccc} 0 & 1 & 2 & \dots & g-4 & g-3 & g-2 \\ \hline \beta_{0,0} & & & & & & \\ & \beta_{1,1} & \beta_{2,1} & \dots & \beta_{g-4,1} & \beta_{g-3,1} & \\ & \beta_{1,2} & \beta_{2,2} & \dots & \beta_{g-4,2} & \beta_{g-3,2} & \\ & & & & & & \beta_{g-2,3} \end{array}$$

Thus for Wahl's criterion (5) to be verified, we need $\beta_{1,2} = \beta_{2,2} = 0$. This is equivalent to $\beta_{g-3,1} = \beta_{g-4,1} = 0$ by Koszul duality. Now, Green's conjecture relates non-vanishing of certain Betti numbers with existence of special linear systems on C :

Conjecture 3.7 (Green [12]). *Let C be a canonical curve in \mathbb{P}^{g-1} . Then*

$$\beta_{p,1}(C, K_C) \neq 0 \iff C \text{ has a } g_d^r \text{ with } d \leq g-1, r \geq 1 \text{ and } d-2r \leq g-2-p.$$

Proof of Theorem 3.1. Green's conjecture holds for curves on any K3 surface by Voisin [29, 30] and Aprodu–Farkas [1]. Thus S satisfies Wahl's criterion if and only if C does not have a g_d^r with $d-2r \leq g-2-(g-4) = 2$, which means that the Clifford index of S must be > 2 . \square

3.8 Higher index Fano 3-folds, imprimitive embeddings and smoothings

In this subsection we recover the classification of higher index Fano 3-folds from smoothings of affine cones over K3 surfaces embedded imprimitively. Ciro Ciliberto informed us that this is similar to the line of thought in [6]. Let S be a general K3 surface of genus ≤ 6 , and write $X = C_a(S, \mathcal{O}_S(1))$. Now S has Clifford index ≤ 2 , and in fact, $T_X^1(k)$ does not vanish for some $|k| \geq 2$. Here, we study the connection between this nonvanishing, imprimitive embeddings of K3 surfaces and Fano 3-folds of higher Fano index.

Fix $I > 1$ and take the affine cone $Y = C_a(S, \mathcal{O}_S(I))$ over the nonprimitive embedding of S by $\mathcal{O}_S(I)$. By Proposition 2.4, we have

$$T_Y^1(k) \cong T_X^1(kI).$$

In the table below, we list all nonsingular Fano 3-folds W, A of Fano index $I > 1$ with $\text{Pic } W \simeq \mathbb{Z}$ and ample generator A satisfying $-K_W = IA$. Each entry of the table has two interpretations in terms of smoothings of general K3 surfaces of genus ≤ 6 . Firstly, as a special subspace $\bigoplus_{k \leq 0} T_X^1(kI)$ of T_X^1 corresponding to a deformation of $C_a(S, \mathcal{O}_S(1))$ with total space $C_a(\bar{W}, A)$. Secondly, as a deformation of $C_a(S, \mathcal{O}_S(I))$ with total space $C_a(W, -K_W)$. We work this out in a series of examples below.

g	(W, A)	Index	Fano description
2	$W_6 \subset \mathbb{P}(1, 1, 1, 2, 3)$	2	del Pezzo 3-fold of degree 1
3	$W_4 \subset \mathbb{P}(1^4, 4)$	4	\mathbb{P}^3
	$W_4 \subset \mathbb{P}(1^4, 2)$	2	del Pezzo 3-fold of degree 2
4	$W_{2,3} \subset \mathbb{P}(1^5, 2)$	2	cubic 3-fold
	$W_{2,3} \subset \mathbb{P}(1^5, 3)$	3	quadric 3-fold
5	$W_{2,2,2} \subset \mathbb{P}(1^6, 2)$	2	intersection of two quadrics
6	$W = H_1 \cap H_2 \cap H_3 \cap \text{Gr}(2, 5) \subset \mathbb{P}^6$	2	del Pezzo 3-fold of degree 5

Remark 3.9. The only higher index Fano 3-folds that are missing from the table are $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2 \times \mathbb{P}^2$. These have Picard rank $\rho > 1$, and in these cases, the K3 surfaces in question have genus 7 but do not have maximal Clifford index. These 3-folds make an appearance in Section 5.7.

Example 3.10. Consider the cone $X = C_a(S, \mathcal{O}_S(1))$ over the quartic K3 surface from Example 1.6. Now, T_X^1 contains an 11-dimensional subspace $T_X^1(-2) \oplus T_X^1(-4)$, corresponding to the deformation $\mathcal{X} \rightarrow \Delta$ defined by

$$\mathcal{X}: (x_0^4 + x_1^4 + x_2^4 + x_3^4 + t_{00}x_0^2 + t_{01}x_0x_1 + \cdots + t_{33}x_3^2 + u = 0) \subset \mathbb{A}^4 \times \Delta,$$

where $\Delta = \mathbb{C}^{11}(t_{ij}, u)$. Let $\lambda, \mu: \mathbb{C} \rightarrow \Delta$ be maps

$$\lambda: x \mapsto (x^2, \dots, x^2, x^4) \text{ and } \mu: y \mapsto (y, \dots, y, y^2),$$

where for simplicity, we assume all coefficients are 1. Performing base change with respect to λ or μ induces one parameter smoothings of X , which we denote by \mathcal{X}_λ and \mathcal{X}_μ . The total space of \mathcal{X}_λ is the affine cone $C_a(V, \mathcal{O}_V(1))$ over a quartic Fano 3-fold $V_4 \subset \mathbb{P}^4$, and λ sweeps out the hyperplane section in $C_a(V, \mathcal{O}_V(1))$. On the other hand, the total space of \mathcal{X}_μ is the affine cone $C_a(W, \mathcal{O}_W(1))$ over $W_4 \subset \mathbb{P}(1, 1, 1, 1, 2)$, and μ sweeps out the weighted hyperplane section of weight 2 inside $C_a(W, \mathcal{O}_W(1))$.

The subspace $T_X^1(-2) \oplus T_X^1(-4)$ was chosen so that \mathcal{X}_μ admits a weighted \mathbb{C}^* -action. The subspaces $T_X^1(-3)$ and $T_X^1(-4)$ have similar properties, giving rise to smoothings of X that sweep out weighted hyperplanes in the cone $C_a(W, \mathcal{O}_W(1))$, where W is the 3-fold $W_4 \subset \mathbb{P}^4(1, 1, 1, 1, k)$ for $k = 3, 4$. When $k = 3$, W has a $\frac{1}{3}(1, 1, 1)$ quotient singularity, while $k = 4$ gives $W \simeq \mathbb{P}^3$.

Example 3.11. Continuing with the quartic K3 surface S , we now take $I = 4$ and consider the affine cone $Y = C_a(S, \mathcal{O}_S(4))$. We see that Y is smoothable, because it is a hyperplane section of $C_a(\mathbb{P}^3, -K_{\mathbb{P}^3})$. The smoothing given by sweeping out this hyperplane corresponds to the 1-dimensional vector space $T_X^1(-4) \cong T_Y^1(-1)$.

Example 3.12. Consider the affine cone $X = C_a(S, \mathcal{O}_S(1))$ over the K3 surface S of genus 6 from Example 3.3. The subspace $T_X^1(-2)$ in T_X^1 corresponds to a one parameter deformation of X , whose total space is the affine cone $C_a(W, \mathcal{O}_W(1))$ over the del Pezzo 3-fold of index 2: $W = H_1 \cap H_2 \cap H_3 \cap \text{Gr}(2, 5)$ in \mathbb{P}^6 . The deformation is realised by varying the hyperquadric section $Q = 0$ cutting out X in $C_a(W, \mathcal{O}_W(1))$, to $Q = t$, where t is the deformation parameter.

4 Vanishing of $T^1(k)$ for $|k| = 1$

The following is a key result of this section.

Theorem 4.1 (cf. Beauville [3, §5.2], Mukai [20, §4]). *Let S be a general K3 surface with primitive polarisation L of genus $g = 11$ or $g \geq 13$. Then*

$$H^1(S, \Omega_S^1 \otimes L^{\otimes k}) = 0$$

for any $k \geq 1$.

Remark 4.2. After the first version of this article appeared, Angelo Lopez and Shigeru Mukai informed us that φ_g is actually birational onto its image for $g = 11$ and $g \geq 13$,

cf. [5], [21]. Moreover, Ciro Ciliberto informed us that [6, Lemma 2.3] shows the same vanishing as Theorem 4.1, using rather different methods, under the assumption that S is general of Picard rank 1. It is quite likely that Theorem 4.1 and [6, Lemma 2.3] give different (intersecting) dense open sets in the moduli space of K3 surfaces. In particular, our result shows vanishing for some K3 surfaces of higher Picard rank.

4.3 Some moduli spaces (stacks) of K3 surfaces Before proving Theorem 4.1, we introduce the necessary material on moduli spaces of K3 surfaces. Good references for this are [3] and [7, §6]. The main point is that the spaces we consider are smooth as stacks, and so we can use infinitesimal methods to study morphisms between them.

Let \mathcal{F}_g be the moduli stack of polarised K3 surfaces (S, L) such that L is primitive and $c_1(L)^2 = 2g - 2$. We define $\mathcal{P}_{g,k}$ to be the moduli stack of pairs (S, C) , where C is a stable curve in $|L^{\otimes k}|$ for some line bundle L such that $(S, L) \in \mathcal{F}_g$. Finally, we write $\overline{\mathcal{M}}_g$ for the moduli stack of stable curves of genus g . Then we have a forgetful map

$$\varphi_{g,k}: \mathcal{P}_{g,k} \rightarrow \overline{\mathcal{M}}_{g_k} \quad (6)$$

sending (S, C) to C , where g_k is the genus of $C \in |L^{\otimes k}|$. Following the construction of [3, §3], we can show that $\mathcal{P}_{g,k}$ is a smooth irreducible Deligne-Mumford stack.

When $k = 1$, Beauville [3, (5.1)] shows that the vanishing in Theorem 4.1 is equivalent to generic finiteness of the morphism of stacks $\varphi_g: \mathcal{P}_g \rightarrow \overline{\mathcal{M}}_g$, and Mukai [20, Theorem 7] proves that φ_g is generically finite when $g = 11$ and $g \geq 13$ (see below for definitions and notation). Beauville's proof actually shows that for each $k > 0$, the vanishing of $H^1(S, \Omega_S^1 \otimes L^{\otimes k})$ is equivalent to generic finiteness of the forgetful map $\varphi_{g,k}$ defined in (6) above.

4.4 When $k \geq 2$, the following lemma is a bit stronger than the generic vanishing proved using Mori–Mukai. Thus it is enough to show the vanishing for $k = 1, 2$ in Theorem 4.1.

Lemma 4.5. *Let (S, L) be a polarised K3 surface as in Theorem 4.1. Suppose $k \geq 2$ and $H^1(S, \Omega_S^1 \otimes L^{\otimes k}) = 0$. Then $H^1(S, \Omega_S^1 \otimes L^{\otimes k+1}) = 0$.*

Proof. Let $C \in |L|$ be a smooth member. We compute $H^1(S, T_S \otimes L^{\otimes k})$. First use the long exact sequence associated to

$$0 \rightarrow L^{\otimes k}|_C \rightarrow (\Omega_S^1 \otimes L^{\otimes k+1})|_C \rightarrow \Omega_C^1 \otimes L^{\otimes k+1}|_C \rightarrow 0$$

to show that $h^1(C, (\Omega_S^1 \otimes L^{\otimes k+1})|_C) = 0$ for $k \geq 2$. Then for $k \geq 2$, the lemma follows from the long exact sequence associated to

$$0 \rightarrow \Omega_S^1 \otimes L^{\otimes k} \rightarrow \Omega_S^1 \otimes L^{\otimes k+1} \rightarrow \Omega_S^1 \otimes L^{\otimes k+1}|_C \rightarrow 0.$$

□

Remark 4.6. By the Riemann–Roch formula, we can estimate the dimension of $T_X^1(1)$ using $h^1(T_S \otimes L) \approx -\chi(T_S \otimes L) = 20 - (2g - 2)$. This shows that Fano 3-folds of genus $g > 10$ are superabundant: they are “not expected” to exist.

Proof of Theorem 4.1. Let (S, L) be a general K3 surface polarised by L with $c_1(L)^2 = 2g - 2$, and take a smooth curve $C \in |L^{\otimes k}|$. By duality, it is enough to show that $H^1(S, \Omega_S^1(-C)) = 0$.

Twisting the standard short exact sequence $0 \rightarrow \Omega_S^1 \rightarrow \Omega_S^1(\log C) \rightarrow \mathcal{O}_C \rightarrow 0$ by $\mathcal{O}(-C)$, we get the following exact sequence of cohomology

$$0 \rightarrow H^1(S, \Omega_S^1(-C)) \xrightarrow{\kappa} H^1(S, \Omega_S^1(\log C)(-C)) \xrightarrow{\tau} H^1(C, \mathcal{O}_C(-C)) \quad (7)$$

where κ is injective because $h^0(\mathcal{O}_C(-C)) = h^1(2K_C) = 0$ by Serre duality. Now by [3, (5.1)], τ is the tangent map at (S, C) to the forgetful map $\varphi_{g,k}$ from (6). Hence $\varphi_{g,k}$ is unramified at (S, C) if and only if τ is injective. Thus by Proposition 4.9 below, $H^1(S, \Omega_S^1(-C)) = 0$ for general (S, C) . □

4.7 Mukai’s construction In order to prove that $\varphi_{g,k}$ is generically finite, we use the following theorem of Mori–Mukai [19]:

Assumption (*) Let $S \subset \mathbb{P}^m$ be a smooth K3 surface with $m \geq 5$ such that S is set-theoretically an intersection of quadrics, and the map $H^0(\mathcal{O}_{\mathbb{P}^m}(1)) \rightarrow H^0(\mathcal{O}_S(1))$ is an isomorphism. Suppose that S contains an irreducible smooth curve C such that $H^0(\mathcal{O}_S(1)) \rightarrow H^0(\mathcal{O}_C(1))$ is an isomorphism and $\deg C \geq m + 1$, so that $p_a(C) > 0$. Let $H_{\mathbb{P}} \subset \mathbb{P}^m$ be a general hyperplane such that $H := S \cap H_{\mathbb{P}}$ is a smooth transverse hyperplane section of S and $C \cap H_{\mathbb{P}}$ is in general position in $H_{\mathbb{P}}$.

Theorem 4.8 (Mori–Mukai [19, Theorem 1.2]). *Let $S, \Gamma := C + H$ be a pair such that S, C, H satisfy assumption (*). Then for every embedding $i: \Gamma \rightarrow S'$ into a K3 surface S' , there exists an isomorphism $I: S \rightarrow S'$ whose restriction to Γ coincides with i .*

Proposition 4.9. *The map $\varphi_{g,k}$ is generically finite for $g = 11$ or $g \geq 13$.*

Proof. It is enough to construct a pair $(S, \Gamma_{g,k}) \in \mathcal{P}_{g,k}$ such that

$$\varphi_{g,k}^{-1}(\Gamma_{g,k}) = \{(S, \Gamma_{g,k})\}.$$

In [20, §4], Mukai constructed such pairs for $g = 11$, $g \geq 13$ when $k = 1$. We summarise the construction here.

Let $E \subset \mathbb{P}^5$ be a sextic normal elliptic curve. Let $S := Q_1 \cap Q_2 \cap Q_3$ be a smooth complete intersection of three quadrics Q_i which contain E . Since S contains an elliptic curve, there is an elliptic fibration $S \rightarrow \mathbb{P}^1$. Let $H \in |\mathcal{O}_S(1)|$ be a general hyperplane section. We can assume that S contains the following singular fibres:

$$(I_3) \ E_1 \cup E_2 \cup E_3 \text{ with } E_i \cdot E_j = 1 \text{ for all } i \neq j$$

$$(I_2) \ E'_2 \cup E_4 \text{ with } E'_2 \cdot E_4 = 2,$$

where $E_i \simeq \mathbb{P}^1$ and $E_i \cdot H = i$ for all $i = 1, \dots, 4$. Let $\Gamma := E \cup H$. Then Γ is a stable curve of genus 11. We can check that (S, Γ) satisfies Assumption (*).

Let

$$\begin{aligned} \Gamma_{13} &:= \Gamma \cup E_3, & \Gamma_{16} &:= \Gamma \cup E_3 \cup E_4, \\ \Gamma_{14} &:= \Gamma \cup E_4, & \Gamma_{17} &:= \Gamma \cup E_1 \cup E_3 \cup E_4, \\ \Gamma_{15} &:= \Gamma \cup E_2 \cup E_3, & \Gamma_{18} &:= \Gamma \cup E_2 \cup E_3 \cup E_4. \end{aligned}$$

We construct Γ_g for $g \geq 19$ by adding smooth fibres to Γ_i for $13 \leq i \leq 18$. Then since $(S, \Gamma_g) \in \mathcal{P}_{g,1}$, we see by Theorem 4.8 that $\varphi_{g,1}^{-1}(\Gamma_g) = \{(S, \Gamma_g)\}$. This is the construction due to Mukai.

Next we consider the case $k > 1$. The linear system $|\Gamma_g|$ is free. Indeed, if there is a fixed curve $C \subset \text{Bs } |\Gamma_g|$, then we see that $C = E_i$ for some i and $(\Gamma_g - C)^2 = 0$ by Saint-Donat's classification [26]. This does not happen since $\Gamma_g - C$ is ample by construction. Take a smooth member $C_{g,k-1} \in |(k-1)\Gamma_g|$ and define $\Gamma_{g,k} := \Gamma_g \cup C_{g,k-1}$. Thus $(S, \Gamma_{g,k})$ is in $\mathcal{P}_{g,k}$ and $\varphi_{g,k}^{-1}(\Gamma_{g,k}) = \{(S, \Gamma_{g,k})\}$. \square

Note that for $g \leq 10$, φ_g is not generically finite for dimension reasons, and φ_{12} is also not generically finite (essentially due to the existence of Fano 3-folds of genus 12, cf. [3]). Thus combining Theorem 4.1 and Proposition 2.4 we get

Corollary 4.10. *Let S be a general K3 surface with primitive polarisation L of genus g and write $X = C_a(S, L)$. Then T_X^1 is concentrated in degree 0 if and only if $g = 11$ or $g \geq 13$.*

See the next section for various comments about the generality assumption.

5 Smoothings and Fano threefolds

We prove the main theorem, after which we present several remarks and examples.

Theorem 5.1. *Let S be a general K3 surface with primitive polarisation L of genus g . The affine cone $C_a(S, L)$ over (S, L) is smoothable if and only if $g \leq 10$ or $g = 12$.*

Proof. Let $X = C_a(S, L)$ denote the affine cone over S . If $g = 11$ or $g \geq 13$, then X has only conical deformations by Corollary 4.10 and Proposition 2.3. If $g \leq 10$ or $g = 12$ then by [3, Corollary 4.1], S is an anticanonical member of a smooth Fano 3-fold W , that is, $S \in |-K_W|$. Let $C_a(W, -K_W)$ be the affine cone over the pair $(W, -K_W)$ and $\sigma \in H^0(W, -K_W)$ be the defining section of S . Then we can regard $C_a(S, L)$ as a divisor in $C_a(W, -K_W)$ defined by the section σ since $H^0(W, -mK_W) \rightarrow H^0(S, L^{\otimes k})$ is surjective for all $m \geq 1$ by $H^1(W, -(m-1)K_W) = 0$. Let $\mathcal{X} \subset C_a(W, -K_W) \times \mathbb{A}^1$ be a divisor defined by a function $\sigma + \lambda$, where λ is the parameter of the affine line \mathbb{A}^1 . Then we have a deformation $\mathcal{X} \rightarrow \mathbb{A}^1$ of X . The fibre over 0 is X , and the general fibre \mathcal{X}_t is nonsingular, because the general fibres avoid the vertex. This is a smoothing of X by sweeping out the anticanonical member of W . \square

5.2 The cone over a K3 surface with $g = 11$ or $g \geq 13$ can nevertheless be smoothable If a K3 surface S is an anticanonical section of a Fano 3-fold with $b_2 \geq 2$ from the Mori–Mukai classification [18], then $C_a(S, \mathcal{O}_S(1))$ is smoothable. Thus there are K3 surfaces of genus 11 and ≥ 13 whose affine cone is smoothable. For such K3 surfaces, Theorem 4.1 does not apply, and $H^1(\Omega_S^1(L))$ does not vanish.

Example 5.3. Let S be a hypersurface of bidegree $(2, 3)$ in $\mathbb{P}^1 \times \mathbb{P}^2$. Since S is a section of $|-K_{\mathbb{P}^1 \times \mathbb{P}^2}|$, we see that $(S, -K_{\mathbb{P}^1 \times \mathbb{P}^2}|_S)$ is a K3 surface of genus 28. Nevertheless, we obtain a smoothing of the affine cone $C_a(S, \mathcal{O}_S(1))$, simply by sweeping out the cone inside $C_a(\mathbb{P}^1 \times \mathbb{P}^2, -K_{\mathbb{P}^1 \times \mathbb{P}^2})$.

Example 5.4. Suppose S is a K3 of genus 13. By [18], there are five distinct deformation families of Fano 3-folds with $g = 13$. Two each with $b_2 = 2$ and $b_2 = 3$, and one with $b_2 = 4$. The cone $C_a(S, \mathcal{O}_S(1))$ over a general S is not smoothable, but if we specialise $C_a(S, \mathcal{O}_S(1))$ to $C_a(S', \mathcal{O}_{S'}(1))$ where S' is the hyperplane section of one of the above Fano 3-folds, then $C_a(S', \mathcal{O}_{S'}(1))$ is smoothable. Thus we see that there are at least five strata in the moduli space of genus 13 K3 surfaces, for which the cone over a K3 surface in such a stratum is smoothable.

Example 5.5. Similarly, if S is a K3 of genus 11, then by [18], there are four families of Fano 3-folds with $g = 11$. Three with $b_2 = 2$ and one with $b_2 = 3$. The general K3 of genus 11 is not a hyperplane section of any Fano 3-fold.

5.6 K3 surfaces of genus > 32 Suppose S is a K3 surface of genus ≥ 13 . The only smoothings of $C_a(S, \mathcal{O}_S(1))$ that we know of, are induced by Fano 3-folds appearing in the classification of Mori–Mukai [18], in the same way as the above examples. If S has genus > 32 , exceeding the maximum appearing in [18], then any smoothing of $C_a(S, \mathcal{O}_S(1))$ does not lift to the projective cone $C_p(S, \mathcal{O}_S(1))$. In spite of Example 2.19, we expect that $C_a(S, \mathcal{O}_S(1))$ is not smoothable for *any* S of sufficiently large genus. An equivalent question (cf. [3, §5.4]) is the following:

Is $\varphi_{g,k}$ actually finite and unramified for $g > 32$?

5.7 K3 surfaces whose affine cone has at least two distinct smoothings

We give an example of a K3 surface S of genus 7 which is a hyperplane section of two topologically distinct anticanonical Fano 3-folds. It follows that the affine cone $C_a(S, \mathcal{O}_S(1))$ has two topologically distinct smoothings, obtained by sweeping out the cone over the two different Fano 3-folds. First recall the following famous example:

Example 5.8. The degree 6 del Pezzo surface Y is a hyperplane section of $V_1 = V: (1, 1) \subset \mathbb{P}^2 \times \mathbb{P}^2$ and $V_2 = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Thus $C_a(Y, -K_Y)$ has two distinct smoothings.

Inspired by this, we found the following example:

Example 5.9. Let Y be the degree 6 del Pezzo surface, and take $\pi: S \rightarrow Y$ a double cover branched in $B \in |-2K_Y|$. Then S is a K3 surface of degree 12 in \mathbb{P}^7 . By Example 5.8, $Y = V_i \cap H_i$ for some $H_i \in |-\frac{1}{2}K_{V_i}|$. Take $\pi_i: W_i \rightarrow V_i$ a double cover branched in $X_i \in |-K_{V_i}|$, where X_i are chosen so that $X_i \cap H_i = B$ since $H^0(V_i, -K_{V_i}) \rightarrow H^0(Y, -2K_Y)$ is surjective. The W_i are Fano 3-folds with distinct topology. Indeed, W_1 (respectively W_2) is number 2.6b (resp. 3.1) of the classification [18]. Moreover, both W_1 and W_2 contain S as a section of $|-K_{W_i}|$, because $W_i \cap \pi_i^* H_i = S$. Thus the affine cone $C_a(S, \mathcal{O}_S(1))$ is a hyperplane section of $C_a(W_i, -K_{W_i}) \subset \mathbb{A}^9$ for each i , and so $C_a(S, \mathcal{O}_S(1))$ has two topologically distinct smoothings.

5.10 Hyperelliptic and trigonal K3 surfaces In view of Theorem 3.1, it would be interesting to systematically study cones over hyperelliptic and trigonal K3 surfaces, and other K3 surfaces with Clifford index ≤ 2 . These K3 surfaces are not general in the sense of Theorem 1.2. For example, we would expect that the genus bound on smoothable cones over hyperelliptic K3 surfaces is given by the genus bound on hyperelliptic Fano 3-folds.

5.11 Quasismooth K3 surfaces It would be very interesting to generalise Theorem 1.2 to the case of affine cones over quasismooth K3 surfaces embedded in weighted projective space. Some applications of this are worked out in [8]. This motivates future work.

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