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Nonsmooth dynamic frictional contact of a thermoviscoelastic body

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This paper studies a system of two hemivariational inequalities modeling a dynamic thermoviscoelastic contact problem with general nonmonotone and multivalued subdifferential boundary conditions. Thermal effects are included in the Kelvin–Voigt thermoviscoelastic constitutive law and in the boundary conditions, and so in frictional heat generation, which takes place on the boundary and enters the condition for the temperature. The existence of a weak solution to the problem is established using a recent surjectivity result for differential inclusions associated with pseudomonotone operators.

1. Introduction

There has been considerable progress in the Mathematical Theory of Contact Mechanics and recently a number of interesting results have been obtained. The majority of these results concerns the solvability of contact problems, uniqueness of the solution, continuous dependence on the data, asymptotic behavior of solutions, etc. Moreover, theoretical results in numerical analysis, numerical simulations and other computational issues have been obtained, as well. These mathematical studies are motivated by a variety of applications to concrete industrial processes and everyday practical situations. In contact problems, we naturally consider various deformable materials subjected to contact and friction conditions that lead to complex and nonstandard boundary value problems of static, quasistatic, and dynamic type.

This paper studies a model for the dynamic frictional contact between a thermoviscoelastic object or body and an obstacle or foundation. The main novelty of the model lies in the general nonmonotone multivalued subdifferential boundary condition used to describe the processes on the contact boundary. We employ three such boundary conditions between the normal components of the displacement and the stress, between the tangential components of the velocity and the stress, and between temperature and the heat flux vector. These conditions on the contact surface are natural generalizations leading to a nonmonotone normal compliance condition, a multivalued friction law and a heat exchange conduction, respectively.

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We note that the contact problems for elastic or viscoelastic bodies with thermal effects have been investigated in a number of papers. The first result on unique solvability of a contact problem with friction in elastodynamics were delivered by Duvaut and Lions in [1]. The normal compliance model of contact with friction for isothermal viscoelastic materials was studied by Martins and Oden [2]. Next, these results were extended to thermoelastic and thermoviscoelastic models by Figueiredo and Trabucho [3] and Shi and Shillor [4]. A thermoelastic contact problem was investigated in [5]. Dynamic frictional contact problems for elastic and viscoelastic materials with or without thermal effects have been studied in e.g. Adly et al. [6], Amassad et al. [7], Andrews et al. [8–10], Chau et al. [11], Han and Sofonea [12], Ionescu and Paumier [13], Jarušek [14], Kuttler and Shillor [15,16], Kuttler et al. [17,18], Migórski [19,20], Migórski et al. [21–23], Rochdi and Shillor [24], Szafraniec [25] and the references therein. In these contributions, various methods of proofs were exploited: the Galerkin method, compactness arguments, regularization techniques, fixed points, and surjectivity theorems. Results for hemivariational inequalities that model static, quasistatic, and dynamic contact problems can be found in Panagiotopoulos [26,27], Naniewicz and Panagiotopoulos [28], and Migórski et al. [22].

The main novelty of this work lies in the analysis of a system that contains strong couplings in the multivalued boundary conditions: both the normal compliance condition and the friction law depend on the boundary temperature (see (8) and (9)) and the relation between the boundary temperature and the heat flux depends on the tangential velocity, via frictional heat generation (10). To our best knowledge, dynamic contact problems with such a combination of a multivalued normal compliance condition and a multivalued normal damped response condition have not been studied in the literature. A simpler system that takes into account nonmonotone friction was studied by Denkowski and Migórski in [29], however, since that publication there has been considerable progress in the field and the results in this work are new, since the model here is considerably more sophisticated. The result that we establish here were announced in the conference [30].

Here, we establish the existence of a solution to a problem set in a weak formulation. The latter consists of a system of two coupled nonlinear hemivariational inequalities of hyperbolic and parabolic types for the displacements and the temperature, respectively. All subdifferentials in this paper are understood in the sense of Clarke and are considered for locally Lipschitz, and in general, nonconvex and nonsmooth superpotentials. The multivalued boundary conditions (8)–(10) cover several types of boundary conditions, see e.g. [12,22,31]. We note that when these superpotentials are convex, the hemivariational inequalities reduce to the usual variational inequalities. The uniqueness of a solution to the system is not expected because of the strong coupling in the constitutive laws and boundary conditions. Also, in contrast to [23], our approach does not need a fixed point argument, since it is fully based on a surjectivity result for a class of multivalued pseudomonotone operators, [32].

It is of interest to extend these results to more general settings, such as viscoplastic materials with hardening, and to include Joules heating by adding the equation for static electrical potential that describes the electric currents in the system, which cause changes in the temperature.

The rest of the paper is structured as follows. Preliminary material is recalled in Section 2. In Section 3, we present the physical setting and the classical formulation of the problem. Section 4 provides the variational formulation of the contact problem and states the main existence result in Theorem 4.1. The proof of the theorem is carried out in Section 5.

2. Preliminaries

We recall in this section the notations, definitions, and basic results used in the paper. More details can be found in [28,33–35].

Let X be a Banach space and X^* denote its dual space. The Clarke subdifferential of a locally Lipschitz function $\varphi: X \rightarrow \mathbb{R}$ at $x \in X$ is defined by

$$\partial\varphi(x) = \{x^* \in X^* \mid \langle x^*, v \rangle_{X^* \times X} \leq \varphi^0(x; v) \text{ for all } v \in X\},$$

where

$$\varphi^0(x; v) = \limsup_{y \rightarrow x, \lambda \downarrow 0} \frac{\varphi(y + \lambda v) - \varphi(y)}{\lambda}$$

is the generalized directional derivative of φ at x in the direction $v \in X$. A function φ is regular (in the sense of Clarke) at $x \in X$ if for all $v \in X$ the one-sided directional derivative $\varphi'(x; v)$ exists and is equal to $\varphi^0(x; v)$.

A multivalued operator $F: X \rightarrow 2^{X^*}$ is said to be bounded if it sends bounded sets of X into bounded ones in X^* . An operator F is said to be coercive if there exists a function $\beta: \mathbb{R}_+ \rightarrow \mathbb{R}$ with $\beta(r) \rightarrow \infty$, as $r \rightarrow \infty$ such that $\langle x^*, x \rangle_{X^* \times X} \geq \beta(\|x\|_X) \|x\|_X$ for all $x^* \in F(x)$. A multifunction F is called upper semicontinuous if for all closed sets $D \subset X^*$, the weak inverse image of D under F given by $F^-(D) = \{x \in X \mid F(x) \cap D \neq \emptyset\}$ is a closed subset in X .

Let X be a reflexive Banach space and $L: D(L) \subset X \rightarrow X^*$ be a linear densely defined maximal monotone operator. An operator $F: X \rightarrow 2^{X^*}$ is said to be pseudomonotone with respect to $D(L)$ (L -pseudomonotone for short) iff

- (a) for all $x \in X$, $F(x)$ is a nonempty, convex, and weakly compact set in X^* ,
- (b) F is upper semicontinuous from every finite dimensional subspace of X into X^* equipped with the weak topology (for short, w - X^*),
- (c) if $\{x_n\} \subset D(L)$ is such that $x_n \rightarrow x$ weakly in X , $Lx_n \rightarrow Lx$ weakly in X^* , $x_n^* \in F(x_n)$, $x_n^* \rightarrow x^*$ weakly in X^* and $\limsup \langle x_n^*, x_n \rangle_{X^* \times X} \leq \langle x^*, x \rangle_{X^* \times X}$, then $x^* \in F(x)$ and $\langle x_n^*, x_n \rangle_{X^* \times X} \rightarrow \langle x^*, x \rangle_{X^* \times X}$.

The following proposition is crucial in the proof of the main theorem of this paper.

Proposition 2.1 ([32], **Theorem 2.1**): *If X is a reflexive Banach space, $L: D(L) \subset X \rightarrow X^*$ is a linear maximal monotone operator and $F: X \rightarrow 2^{X^*} \setminus \{\emptyset\}$ is bounded coercive and L -pseudomonotone operator, then $L + F$ is surjective.*

Note that in Theorem 2.1 in [32], it is assumed that X is also strictly convex. However, we skip this assumption since, due to the Troyanski renorming theorem (see e.g. [34, Proposition 32.23]), every reflexive Banach space can be endowed with an equivalent norm such that X becomes strictly convex. Also, in Theorem 2.1 in [32], the result of Proposition 2.1 is stated under the hypothesis that L is densely defined. We skip this assumption too, since by Theorem 32.L in [34], it follows that every linear maximal monotone operator is densely defined.

Finally, we denote by $\mathcal{L}(X, Y)$ the space of all linear continuous mappings from a normed space X to a normed space Y .

3. Classical formulation

We now describe the physical setting of the contact problem and provide its classical formulation. We follow the exposition of the model described in [1,8,23,36–38] and employ the thermoviscoelastic version of the Kelvin–Voigt constitutive law. Thus, the thermomechanical behavior of the material is assumed to be linear while the nonlinear effects occur only in the contact boundary conditions.

The physical setting consists of a thermoviscoelastic structure, the ‘body’, represented in the reference configuration by an open bounded domain $\Omega \subset \mathbb{R}^d$ whose boundary $\partial\Omega = \Gamma$ is divided into three measurable and disjoint parts $\Gamma = \Gamma_D \cup \Gamma_N \cup \Gamma_C$. On the part Γ_D , of positive surface measure, the body is clamped, while known tractions of density f_N act on Γ_N . On the potential contact surface Γ_C the body may come in frictional contact with a reactive thermally active foundation. The reference configuration Ω is assumed to be stress-free and at a constant reference temperature, conveniently scaled to be zero. We assume that the temperature changes which accompany the deformations are small and so the material parameters are supposed to be temperature independent. However, for the sake of generality, the material is assumed to be anisotropic. We denote by f_0 the

density (per unit mass) of the applied body forces acting in Ω , such as gravity. The material density ρ assumed to be constant, scaled so that $\rho = 1$. Moreover, the body is subjected to a heat source term per unit volume g , such as Joule heating. Here, we study the evolution of the state of the system on a finite time interval $[0, T]$, with $T > 0$.

We denote by $u = (u_i) \in \mathbb{R}^d$ the *displacement vector*, by $\sigma = (\sigma_{ij}) \in \mathbb{S}^d$ the *stress tensor*, by $\varepsilon(u) = (\varepsilon_{ij}) \in \mathbb{S}^d$ the linearized (small) *strain tensor*, i.e. $\varepsilon_{ij}(u) = (\partial u_i / \partial x_j + \partial u_j / \partial x_i) / 2$, where $i, j = 1, \dots, d$, and by $\theta \in \mathbb{R}$ the *temperature*. The latter is measured with respect to a reference temperature Θ_{ref} that may be taken as the ambient temperature (in 0K). We also use the prime to denote the time derivative, so $v = u' = (u'_i)$ represents the *velocity vector*. In what follows, the summation convention over the repeated indices is used, and the notation $\partial_j = \partial / \partial x_j$ refers to the partial derivative. The functions $u: \Omega \times [0, T] \rightarrow \mathbb{R}^d$, $\sigma: \Omega \times [0, T] \rightarrow \mathbb{S}^d$ and $\theta: \Omega \times [0, T] \rightarrow \mathbb{R}$ are the unknowns of the problem. We also use the notation $Q = \Omega \times (0, T)$, $\Sigma_D = \Gamma_D \times (0, T)$, $\Sigma_N = \Gamma_N \times (0, T)$ and $\Sigma_C = \Gamma_C \times (0, T)$, and sometimes we suppress the explicit dependence on the variables x and t .

The system of equations of motion, assuming small displacements and the law of energy conservation, takes the form

$$\begin{aligned} u''_i - \partial_j \sigma_{ij} &= f_{0i} \quad \text{in } Q, \\ \theta' + \partial_i q_i &= -c_{ij} \Theta_{ref} \partial_j u'_i + g \quad \text{in } Q. \end{aligned}$$

The behavior of the body is governed by the thermoviscoelastic constitutive law

$$\sigma_{ij} = a_{ijkl} \varepsilon_{kl}(u') + b_{ijkl} \varepsilon_{kl}(u) - c_{ij} \theta \quad \text{in } Q,$$

and the heat flux vector $q = (q_i) \in \mathbb{R}^d$ is given by the anisotropic version of the Fourier law of heat conduction

$$q_i = -k_{ij} \partial_j \theta \quad \text{in } Q.$$

Here, $\mathcal{A}_e = (a_{ijkl})$ and $\mathcal{B}_e = (b_{ijkl})$ are the elasticity and viscosity tensors, respectively, $\mathcal{C}_e = (c_{ij})$ and $k = (k_{ij})$ are the tensors of thermal expansion and of thermal conductivity, respectively.

We turn to describe the boundary conditions, so we denote by $\nu = (\nu_i)$ the unit outward normal on Γ . Throughout the paper, we assume that the boundary Γ is Lipschitz continuous and, therefore, the unit outward normal on Γ exists a.e. on the boundary. Let $u_\nu = u \cdot \nu$ and $\sigma_\nu = \sigma \cdot \nu$ be the normal components of u and σ on Γ , and let $u_\tau = u - u_\nu \nu$ and $\sigma_\tau = \sigma \nu - \sigma_\nu \nu$ be their tangential components. Then, on the $\Gamma_D \cup \Gamma_N$ portion of the boundary, we impose the following conditions

$$\begin{aligned} u &= \mathbf{0} \quad \text{on } \Sigma_D, \\ \sigma \nu &= f_N \quad \text{on } \Sigma_N, \end{aligned}$$

and finally, for the sake of simplicity assuming that the ambient temperature Θ_{ref} is constant, we set

$$\theta = 0 \quad \text{on } \Sigma_D \cup \Sigma_N.$$

We now describe the thermomechanical boundary conditions on the potential contact surface Γ_C . We note that frictional heat generation can be substantial and, therefore, the properties of the contact surface could drastically change with the temperature, thus, it is important to allow the various coefficients to depend on the temperature.

First, we describe the normal contact conditions. The foundation is assumed to be reactive and the reaction force is a function of the normal penetration $(u_\nu - g_0)_+$, where $(\cdot)_+$ denotes the positive part function and g_0 is the gap function measuring the distance between the body and a foundation in the direction of the normal. We use it since when $u_\nu \leq g_0$ there is no contact and, therefore, no

reaction. We note that in the literature (see [2,38–40] and references therein) the power law normal compliance condition has been used, namely

$$-\sigma_\nu = k_c(u_\nu - g_0)_+^{n_c} \text{ on } \Sigma_C,$$

where k_c and n_c , both positive constants, are the normal compliance stiffness coefficient and exponent, respectively. Here, we model the foundation's reaction with a very general contact law that includes the power law normal compliance condition as a special case, i.e.

$$-\sigma_\nu = p(\theta, u_\nu - g_0) \text{ on } \Sigma_C,$$

where $p = p(x, t, \theta, r)$ with $(x, t) \in \Sigma_C$, and $\theta, r \in \mathbb{R}$, is a given nonnegative function that satisfies appropriate growth and smoothness conditions. This function contains the information on the normal contact process and depends on the surface temperature and the interpenetration of the surface asperities. To take into account the multivalued character and possible discontinuities of p , we assume that the normal stress σ_ν and the normal displacement u_ν satisfy a nonmonotone normal compliance condition of the form

$$-\sigma_\nu(t) \in p_\nu(\theta(t))\partial j_\nu(t, u_\nu(t)) \text{ on } \Sigma_C,$$

where p_ν is a prescribed function of the temperature θ , and ∂j_ν represents the Clarke subdifferential of the normal contact pseudo-potential j_ν , assumed to be locally Lipschitz, with respect to its last variable.

The corresponding Coulomb law of dry friction may be stated as follows:

$$-\sigma_\tau = \left. \begin{array}{l} \|\sigma_\tau\| \leq F_b(\theta, u_\nu - g_0), \\ F_b(\theta, u_\nu - g_0) \frac{u'_\tau}{\|u'_\tau\|}, \text{ if } u'_\tau \neq 0 \end{array} \right\} \text{ on } \Sigma_C,$$

where $F_b = F_b(x, t, \theta, r)$ for $(x, t) \in \Sigma_C$, $\theta, r \in \mathbb{R}$ is the so-called friction bound, a given function which depends on the temperature θ and the contact penetration $(u_\nu - g_0)_+$. In this work, we use quite a general version of the friction condition described by the subdifferential condition

$$-\sigma_\tau(t) \in p_\tau(\theta(t))\partial j_\tau(t, u'_\tau(t)) \text{ on } \Sigma_C,$$

which describes a multivalued relation between the tangential force σ_τ and the tangential velocity u'_τ and includes thermal effects. Here, p_τ is a prescribed function of the temperature θ , j_τ is the friction pseudo-potential, and ∂j_τ is its Clarke subdifferential, assuming that j_τ is a locally Lipschitz function j_τ with respect to its last variable.

Next, we describe the boundary condition for the temperature on Γ_C . We assume that there is heat exchange between the surface and the foundation, which is at temperature θ_R , with a heat exchange coefficient $k_R > 0$. Moreover, since the flux of heat generated by the frictional contact forces is proportional to the tangential velocity u'_τ , we assume a boundary condition of the following form

$$k_{ij} \partial_i \theta \nu_j = F_C(\theta) \|u'_\tau\| - k_R(\theta - \theta_R) \text{ on } \Sigma_C,$$

where $F_C = F_C(x, t, \theta)$ for $(x, t) \in \Sigma_C$, $\theta \in \mathbb{R}$ is a prescribed function. The inclusion of such a term is required if the effects of the frictional heat generation are to be taken into account, which is essential in many applications. For example, the sudden application of car brakes may lead to the dissipation of energy in the form of frictional heating at a rate of over 100 HP. Again, for the sake of generality, we extend the boundary condition to the following relation between the boundary temperature and the heat flux vector of the form $q_i \nu_i \in p(u'_\tau) \partial j(\theta)$ which we write as

$$-\frac{\partial\theta(t)}{\partial\nu_k} = -k_{ij}\partial_i\theta(t)\nu_j \in p(u'_\tau(t)) \partial j(t, \theta(t)) \text{ on } \Sigma_C.$$

Here, p is a prescribed function of the tangential velocity u'_τ , j is the surface thermal pseudo-potential, and ∂j represents its Clarke subdifferential with respect to its last variable, since we assume that j is a locally Lipschitz function.

Collecting the equations and conditions above, the classical formulation of the problem of thermoviscoelastic contact with a reactive foundation and the frictional heat generation can be stated as follows.

Problem 1: Find a displacement field $u: Q \rightarrow \mathbb{R}^d$, a stress field $\sigma: Q \rightarrow \mathbb{S}^d$, a temperature field $\theta: Q \rightarrow \mathbb{R}$, and a heat flux $q: Q \rightarrow \mathbb{R}^d$ such that

$$u''(t) - \text{Div } \sigma(t) = f_0(t) \quad \text{in } Q, \quad (1)$$

$$\sigma(t) = \mathcal{A}_e \boldsymbol{\varepsilon}(u'(t)) + \mathcal{B}_e \boldsymbol{\varepsilon}(u(t)) + \mathcal{C}_e \theta(t) \quad \text{in } Q, \quad (2)$$

$$\theta'(t) + \text{div } q(t) = g(t) - \Theta_{\text{ref}} \mathcal{C} \boldsymbol{\varepsilon}(u'(t)) \quad \text{in } Q, \quad (3)$$

$$q(t) = -k \nabla \theta(t) \quad \text{in } Q, \quad (4)$$

$$u(t) = 0 \quad \text{on } \Sigma_D, \quad (5)$$

$$\sigma(t) \boldsymbol{\nu} = f_N(t) \quad \text{on } \Sigma_N, \quad (6)$$

$$\theta(t) = 0 \quad \text{on } \Sigma_D \cup \Sigma_N, \quad (7)$$

$$-\sigma_\nu(t) \in p_\nu(\theta(t)) \partial j_\nu(t, u_\nu(t)) \quad \text{on } \Sigma_C, \quad (8)$$

$$-\sigma_\tau(t) \in p_\tau(\theta(t)) \partial j_\tau(t, u'_\tau(t)) \quad \text{on } \Sigma_C, \quad (9)$$

$$-\frac{\partial\theta(t)}{\partial\nu_k} \in p(u'_\tau(t)) \partial j(t, \theta(t)) \quad \text{on } \Sigma_C, \quad (10)$$

$$u(0) = u_0, \quad u'(0) = v_0, \quad \theta(0) = \theta_0 \quad \text{in } \Omega. \quad (11)$$

Here, u_0 , v_0 , and θ_0 are the prescribed initial displacement, velocity and temperature, respectively. To analyze Problem 1, we need to reformulate it in a weak form.

4. Weak formulation

We turn to a weak formulation of Problem 1 and the statement of our main theorem on its solvability.

First, we introduce the standard Lebesgue and Sobolev spaces. Let $\mathcal{H} = L^2(\Omega; \mathbb{S}^d)$. For the displacement and the temperature we use the spaces

$$E = \{v \in H^1(\Omega; \mathbb{R}^d) \mid v = 0 \text{ on } \Gamma_D\}, \quad V = \{\eta \in H^1(\Omega) \mid \eta = 0 \text{ on } \Gamma_D \cup \Gamma_N\}.$$

The spaces E and V are Hilbert when endowed with the norms $\|v\|_E = \|\boldsymbol{\varepsilon}(v)\|_{\mathcal{H}}$ for $v \in E$, and $\|\eta\|_V = \|\nabla \eta\|_{L^2(\Omega; \mathbb{R}^d)}$ for $\eta \in V$, respectively. It follows from the Korn and Poincaré inequalities (see, e.g. [12,34]), these norms are equivalent to the respective standard norms on E and V . The spaces $E \subset L^2(\Omega; \mathbb{R}^d) \subset E^*$ and $V \subset L^2(\Omega) \subset V^*$ form the Gelfand triples. We also need the following spaces of vector valued functions for displacement and temperature, respectively,

$$\begin{aligned} \mathcal{E} &= L^2(0, T; E) \text{ and } \mathbb{E} = \{v \in \mathcal{E} \mid v' \in \mathcal{E}^*\}, \\ \mathcal{V} &= L^2(0, T; V) \text{ and } \mathbb{V} = \{\eta \in \mathcal{V} \mid \eta' \in \mathcal{V}^*\}. \end{aligned}$$

Here, the time derivative is understood in the sense of vector valued distributions. It is well-known that the spaces \mathbb{E} and \mathbb{V} endowed with the standard graph norms are reflexive separable Banach spaces and the embeddings

$$\mathbb{E} \subset \mathcal{E} \subset L^2(0, T; L^2(\Omega; \mathbb{R}^d)) \subset \mathcal{E}^*, \quad \mathbb{V} \subset \mathcal{V} \subset L^2(0, T; L^2(\Omega)) \subset \mathcal{V}^*$$

are dense and continuous. It is also known (see e.g. [35]) that the embeddings $\mathbb{E} \subset C(0, T; L^2(\Omega; \mathbb{R}^d))$ and $\{v \in \mathcal{E} \mid v' \in \mathbb{E}\} \subset C(0, T; E)$ are continuous and, in addition, the embedding $\mathbb{E} \subset L^2(0, T; L^2(\Omega; \mathbb{R}^d))$ is compact. Analogously, the embeddings $\mathbb{V} \subset C(0, T; L^2(\Omega))$ and $\{\eta \in \mathcal{V} \mid \eta' \in \mathbb{V}\} \subset C(0, T; V)$ are continuous and the embedding $\mathbb{V} \subset L^2(0, T; L^2(\Omega))$ is compact. Next, we introduce the trace operators $\gamma: E \rightarrow L^2(\Gamma; \mathbb{R}^d)$ and $\gamma_s: V \rightarrow L^2(\Gamma)$ for vector and scalar valued functions.

We next present the hypotheses on the problem data of Problem 1 that allow for the use of the appropriate mathematical tools.

$H(\mathcal{A}_e)$: $\mathcal{A}_e: \Omega \times \mathbb{S}^d \rightarrow \mathbb{S}^d$ is such that

- (a) $\mathcal{A}_e(x, \boldsymbol{\varepsilon}) = a(x)\boldsymbol{\varepsilon}$, $a(x) = (a_{ijkl}(x))$, $a_{ijkl} = a_{jikl} = a_{lkij} \in L^\infty(\Omega)$,
- (b) $a_{ijkl}(x)\varepsilon_{ij}\varepsilon_{kl} \geq \alpha_{\mathcal{A}} \varepsilon_{ij}\varepsilon_{ij}$ for all $\boldsymbol{\varepsilon} = (\varepsilon_{ij}) \in \mathbb{S}^d$, a.e. $x \in \Omega$ with $\alpha_{\mathcal{A}} > 0$.

$H(\mathcal{B}_e)$: $\mathcal{B}_e: \Omega \times \mathbb{S}^d \rightarrow \mathbb{S}^d$ is such that

- (a) $\mathcal{B}_e(x, \boldsymbol{\varepsilon}) = b(x)\boldsymbol{\varepsilon}$, $b(x) = (b_{ijkl}(x))$, $b_{ijkl} = b_{jikl} = b_{lkij} \in L^\infty(\Omega)$,
- (b) $b_{ijkl}(x)\varepsilon_{ij}\varepsilon_{kl} \geq 0$ for all $\boldsymbol{\varepsilon} = (\varepsilon_{ij}) \in \mathbb{S}^d$, a.e. $x \in \Omega$.

$H(\mathcal{C}_e, \mathcal{C})$: $\mathcal{C}_e: \Omega \times \mathbb{R} \rightarrow \mathbb{S}^d$ and $\mathcal{C}: \Omega \times \mathbb{S}^d \rightarrow \mathbb{R}$ are such that

- $\mathcal{C}_e(x, r) = c(x)r$ for all $r \in \mathbb{R}$, $\mathcal{C}(x, \boldsymbol{\varepsilon}) = \Theta_{ref} c_{ij}(x)\varepsilon_{ij}$ for all $\boldsymbol{\varepsilon} = (\varepsilon_{ij}) \in \mathbb{S}^d$,
- a.e. $x \in \Omega$ with $c(x) = (c_{ij}(x))$, $c_{ij} = c_{ji} \in L^\infty(\Omega)$.

$H(k)$: $k: \Omega \rightarrow \mathbb{S}^d$ is such that

- (a) $k(x) = (k_{ij}(x))$, $k_{ij} = k_{ji} \in L^\infty(\Omega)$,
- (b) $k_{ij}(x)\xi_i\xi_j \geq \alpha_k \xi_i\xi_i$ for all $\boldsymbol{\xi} = (\xi_i) \in \mathbb{R}^d$, a.e. $x \in \Omega$ with $\alpha_k > 0$.

$H(p_\nu, p_\tau, p)$: $p_\nu, p_\tau: \Gamma_C \times \mathbb{R} \rightarrow \mathbb{R}$ and $p: \Gamma_C \times \mathbb{R}^d \rightarrow \mathbb{R}$ are such that

- (a) $p_\nu(\cdot, r), p_\tau(\cdot, r)$ are measurable for all $r \in \mathbb{R}$,
- (b) $p_\nu(x, \cdot), p_\tau(x, \cdot)$ are continuous for a.e. $x \in \Gamma_C$,
- (c) $0 \leq p_\nu(x, r) \leq \bar{p}_\nu, 0 \leq p_\tau(x, r) \leq \bar{p}_\tau$ for all $r \in \mathbb{R}$, a.e. $x \in \Gamma_C$ with $\bar{p}_\nu, \bar{p}_\tau > 0$,
- (d) $p(\cdot, \boldsymbol{\xi})$ is measurable for all $\boldsymbol{\xi} \in \mathbb{R}^d$,
- (e) $p(x, \cdot)$ is continuous for a.e. $x \in \Gamma_C$,
- (f) $0 \leq p(x, \boldsymbol{\xi}) \leq \bar{p}$ for all $\boldsymbol{\xi} \in \mathbb{R}^d$, a.e. $x \in \Gamma_C$ with $\bar{p} > 0$.

$H(j_\nu)$: $j_\nu: \Sigma_C \times \mathbb{R} \rightarrow \mathbb{R}$ is such that

- (a) $j_\nu(\cdot, \cdot, r)$ is measurable for all $r \in \mathbb{R}$, $j_\nu(\cdot, \cdot, e_0(\cdot)) \in L^1(\Sigma_C)$ with $e_0 \in L^2(\Gamma_C)$,
- (b) $j_\nu(x, t, \cdot)$ is locally Lipschitz for a.e. $(x, t) \in \Sigma_C$,
- (c) $|\partial j_\nu(x, t, r)| \leq c_{0\nu}(x, t)$ for all $r \in \mathbb{R}$, a.e. $(x, t) \in \Sigma_C$ with $c_{0\nu} \in L^2(0, T; L^\infty(\Gamma_C))$, $c_{0\nu} \geq 0$,
- (d) $j_\nu(x, t, \cdot)$ or $-j_\nu(x, t, \cdot)$ is regular on \mathbb{R} for a.e. $(x, t) \in \Sigma_C$.

$H(j_\tau)$: $j_\tau: \Sigma_C \times \mathbb{R}^d \rightarrow \mathbb{R}$ is such that

- (a) $j_\tau(\cdot, \cdot, \boldsymbol{\xi})$ is measurable for all $\boldsymbol{\xi} \in \mathbb{R}^d$, $j_\tau(\cdot, \cdot, e_1(\cdot)) \in L^1(\Sigma_C)$ with $e_1 \in L^2(\Gamma_C; \mathbb{R}^d)$,
- (b) $j_\tau(x, t, \cdot)$ is locally Lipschitz for a.e. $(x, t) \in \Sigma_C$,
- (c) $\|\partial j_\tau(x, t, \boldsymbol{\xi})\| \leq c_{0\tau}(x, t) + c_{1\tau}\|\boldsymbol{\xi}\|$ for all $\boldsymbol{\xi} \in \mathbb{R}^d$, a.e. $(x, t) \in \Sigma_C$ with $c_{0\tau} \in L^2(0, T; L^\infty(\Gamma_C))$, $c_{0\tau}, c_{1\tau} \geq 0$,

- (d) $j_\tau^0(x, t, \xi; -\xi) \leq d_\tau(1 + \|\xi\|)$ for all $\xi \in \mathbb{R}^d$, a.e. $(x, t) \in \Sigma_C$ with $d_\tau \geq 0$,
(e) $j_\tau(x, t, \cdot)$ or $-j_\tau(x, t, \cdot)$ is regular on \mathbb{R}^d for a.e. $(x, t) \in \Sigma_C$.

$H(j)$: $j: \Sigma_C \times \mathbb{R} \rightarrow \mathbb{R}$ is such that

- (a) $j(\cdot, \cdot, r)$ is measurable for all $r \in \mathbb{R}$, $j(\cdot, \cdot, e_2(\cdot)) \in L^1(\Sigma_C)$ with $e_2 \in L^2(\Gamma_C)$,
(b) $j(x, t, \cdot)$ is locally Lipschitz for a.e. $(x, t) \in \Sigma_C$,
(c) $|\partial j(x, t, r)| \leq c_0(x, t) + c_1|r|$ for all $r \in \mathbb{R}$, a.e. $(x, t) \in \Sigma_C$ with $c_0 \in L^2(0, T; L^\infty(\Gamma_C))$, $c_0, c_1 \geq 0$,
(d) $j^0(x, t, r; -r) \leq d(1 + |r|)$ for all $r \in \mathbb{R}$, a.e. $(x, t) \in \Sigma_C$ with $d \geq 0$,
(e) $j(x, t, \cdot)$ or $-j(x, t, \cdot)$ is regular on \mathbb{R}^d for a.e. $(x, t) \in \Sigma_C$.

(H_0) : $f_0 \in \mathcal{E}^*$, $f_N \in L^2(\Sigma_N; \mathbb{R}^d)$, $g \in \mathcal{V}^*$, $u_0, v_0 \in E$ and $\theta_0 \in V$.

Examples of potentials that satisfy conditions $H(j_\nu)$, $H(j_\tau)$ and $H(j)$ can be found in [20,22,28,31] and in the references therein.

We now construct a weak formulation of Problem 1. To this end, we assume that (u, σ, θ, q) are sufficiently smooth functions that solve Problem 1. Let $v \in E$ and $t \in (0, T)$. We multiply the equation of motion (1) by v and use Green's formula (see [22, Theorem 2.25]) to find that

$$\langle u''(t), v \rangle_{E^* \times E} + \langle \sigma(t), \boldsymbol{\varepsilon}(v) \rangle_{\mathcal{H}} = \langle f_0(t), v \rangle_{E^* \times E} + \int_{\Gamma} \sigma(t) \boldsymbol{\nu} \cdot \gamma v \, d\Gamma. \quad (12)$$

Taking into account the boundary condition (6) and the fact that $v = 0$ on Γ_D , yields

$$\int_{\Gamma} \sigma(t) \boldsymbol{\nu} \cdot \gamma v \, d\Gamma = \int_{\Gamma_N} f_N(t) \cdot \gamma v \, d\Gamma + \int_{\Gamma_C} (\sigma_\nu(t) v_\nu + \sigma_\tau(t) \cdot v_\tau) \, d\Gamma. \quad (13)$$

On the other hand, by the definition of the Clarke subdifferential, it follows from (8) and (9), that on Σ_C

$$\begin{aligned} -\sigma_\nu(t) v_\nu &\leq p_\nu(\theta(t)) j_\nu^0(t, u_\nu(t); v_\nu), \\ -\sigma_\tau(t) \cdot v_\tau &\leq p_\tau(\theta(t)) j_\tau^0(t, u'_\tau(t); v_\tau). \end{aligned} \quad (14)$$

Here and in what follows, we simplify the notation of $\gamma_s \theta(t)$ on the boundary Γ_C and simply write $\theta(t)$. Similarly, we skip γ when writing the normal and tangential components of γu , $\gamma u'$ and $\gamma(\sigma \boldsymbol{\nu})$. Inserting (13) and (14) in (12), we find

$$\begin{aligned} &\langle u''(t), v \rangle_{E^* \times E} + \langle \sigma(t), \boldsymbol{\varepsilon}(v) \rangle_{\mathcal{H}} \\ &+ \int_{\Gamma_C} \left(p_\nu(\theta(t)) j_\nu^0(t, u_\nu(t); v_\nu) + p_\tau(\theta(t)) j_\tau^0(t, u'_\tau(t); v_\tau) \right) d\Gamma \geq \langle f(t), v \rangle_{E^* \times E}, \end{aligned} \quad (15)$$

where

$$\langle f(t), v \rangle_{E^* \times E} = \langle f_0(t), v \rangle_{E^* \times E} + \langle f_N(t), \gamma v \rangle_{L^2(\Gamma_N; \mathbb{R}^d)}$$

for all $v \in E$, a.e. $t \in (0, T)$. We note that hypothesis (H_0) implies that $f \in \mathcal{E}^*$.

Next, let $\eta \in V$ and $t \in (0, T)$. We multiply (3) by η and then apply Green's formula again and obtain

$$\langle \theta'(t), \eta \rangle_{V^* \times V} - \langle q(t), \nabla \eta \rangle_{L^2(\Omega; \mathbb{R}^d)} = \langle g(t) - \mathcal{C} \boldsymbol{\varepsilon}(u'(t)), \eta \rangle_{V^* \times V} + \int_{\Gamma} q(t) \cdot \boldsymbol{\nu} \gamma_s \eta \, d\Gamma.$$

Since $\eta = 0$ on $\Gamma_D \cup \Gamma_N$, it follows that

$$\int_{\Gamma} q(t) \cdot \boldsymbol{\nu} \gamma_s \eta \, d\Gamma = \int_{\Gamma_C} q(t) \cdot \boldsymbol{\nu} \gamma_s \eta \, d\Gamma,$$

and then (10) implies that

$$-q(t) \cdot \mathbf{v} \gamma_s \eta \leq p(u'_\tau(t)) j^0(t, \gamma_s \theta(t); \gamma_s \eta) \text{ on } \Sigma_C.$$

Hence, we find

$$\begin{aligned} \langle \theta'(t), \eta \rangle_{V^* \times V} - \langle q(t), \nabla \eta \rangle_{L^2(\Omega; \mathbb{R}^d)} + \int_{\Gamma_C} p(u'_\tau(t)) j^0(t, \gamma_s \theta(t); \gamma_s \eta) \, d\Gamma \\ \geq \langle g(t), \eta \rangle_{V^* \times V} - \langle \mathcal{C} \boldsymbol{\varepsilon}(u'(t)), \eta \rangle_{V^* \times V}. \end{aligned} \quad (16)$$

Finally, using the constitutive law (2), the heat conduction condition (4), the inequalities (15) and (16), and the initial conditions (11), we obtain the following weak formulation of Problem 1.

Problem 2: Find $(u, \theta) \in \mathcal{E} \times \mathcal{V}$ such that $u' \in \mathbb{E}$, $\theta' \in \mathcal{V}^*$ and

$$\begin{aligned} \langle u''(t), v \rangle_{E^* \times E} + \langle \mathcal{A}_e \boldsymbol{\varepsilon}(u'(t)) + \mathcal{B}_e \boldsymbol{\varepsilon}(u(t)) + \mathcal{C}_e \theta(t), \boldsymbol{\varepsilon}(v) \rangle_{\mathcal{H}} \\ + \int_{\Gamma_C} \left(p_\nu(\theta(t)) j_\nu^0(t, u_\nu(t); \nu_\nu) + p_\tau(\theta(t)) j_\tau^0(t, u'_\tau(t); \nu_\tau) \right) \, d\Gamma \geq \langle f(t), v \rangle_{E^* \times E}, \end{aligned}$$

for all $v \in E$, a.e. $t \in (0, T)$,

$$\begin{aligned} \langle \theta'(t), \eta \rangle_{V^* \times V} + \langle k \nabla \theta(t), \nabla \eta \rangle_{L^2(\Omega; \mathbb{R}^d)} + \langle \mathcal{C} \boldsymbol{\varepsilon}(u'(t)), \eta \rangle_{V^* \times V} \\ + \int_{\Gamma_C} p(u'_\tau(t)) j^0(t, \gamma_s \theta(t); \gamma_s \eta) \, d\Gamma \geq \langle g(t), \eta \rangle_{V^* \times V} \end{aligned}$$

for all $\eta \in V$, a.e. $t \in (0, T)$,

$$u(0) = u_0, \quad u'(0) = v_0, \quad \theta(0) = \theta_0.$$

It is seen that Problem 2 consists of a system of two strongly coupled hemivariational inequalities. Our main existence result in the study of Problem 2 and in this work, which we prove in the next section, is as follows.

Theorem 4.1: Assume that $H(\mathcal{A}_e)$, $H(\mathcal{B}_e)$, $H(\mathcal{C}_e, \mathcal{C})$, $H(k)$, $H(p_\nu, p_\tau, p)$, $H(j_\nu)$, $H(j_\tau)$, $H(j)$ and (H_0) hold true. Then, Problem 2 has a solution.

We conclude that Problem 1 has a weak solution.

5. Proof of Theorem 4.1

We establish Theorem 4.1 in this section. The main idea of the proof is to solve a related system of evolutionary inclusions and to show that a solution of this system is also a solution to Problem 2. The proof is carried out in the following steps.

Step 1. We start by introducing a system of coupled evolutionary inclusions related to Problem 2. To that end we let $A, B: E \rightarrow E^*$, $C_1: V \rightarrow E^*$, $C_2: V \rightarrow V^*$ and $C_3: E \rightarrow V^*$ be the operators given by

$$\langle Au, v \rangle_{E^* \times E} = \langle \mathcal{A}_e \boldsymbol{\varepsilon}(u), \boldsymbol{\varepsilon}(v) \rangle_{\mathcal{H}}, \quad (17)$$

$$\langle Bu, v \rangle_{E^* \times E} = \langle \mathcal{B}_e \boldsymbol{\varepsilon}(u), \boldsymbol{\varepsilon}(v) \rangle_{\mathcal{H}}, \quad (18)$$

$$\langle C_1 \theta, v \rangle_{E^* \times E} = \langle \mathcal{C}_e \theta, \boldsymbol{\varepsilon}(v) \rangle_{\mathcal{H}}, \quad (19)$$

$$\langle C_2 \theta, \eta \rangle_{V^* \times V} = \langle k \nabla \theta, \nabla \eta \rangle_{L^2(\Omega; \mathbb{R}^d)}, \quad (20)$$

$$\langle C_3 u, \eta \rangle_{V^* \times V} = \langle \mathcal{C} \boldsymbol{\varepsilon}(u), \eta \rangle_{L^2(\Omega)} \quad (21)$$

for all $u, v \in E$ and $\theta, \eta \in V$. Next, we define functionals $J_1, J_2: (0, T) \times L^2(\Gamma_C) \times L^2(\Gamma_C; \mathbb{R}^d) \rightarrow \mathbb{R}$ and $J_3: (0, T) \times L^2(\Gamma_C; \mathbb{R}^d) \times L^2(\Gamma_C) \rightarrow \mathbb{R}$ by

$$J_1(t, \theta, v) = \int_{\Gamma_C} p_v(\theta) j_v(t, v_v) \, d\Gamma, \quad (22)$$

$$J_2(t, \theta, v) = \int_{\Gamma_C} p_\tau(\theta) j_\tau(t, v_\tau) \, d\Gamma, \quad (23)$$

$$J_3(t, v, \theta) = \int_{\Gamma_C} p(v_\tau) j(t, \theta) \, d\Gamma, \quad (24)$$

for all $\theta \in L^2(\Gamma_C)$ and $v \in L^2(\Gamma_C; \mathbb{R}^d)$, a.e. $t \in (0, T)$.

We now state the properties of the operators and functionals defined above in the following lemmas. The proof of the first lemma is presented in [29, Lemma 3], and the proofs of the other three lemmas are analogous to each other and can be found in [20, Lemma 5].

Lemma 5.1: *If $H(\mathcal{A}_e)$, $H(\mathcal{B}_e)$, $H(\mathcal{C}_e, \mathcal{C})$, $H(k)$ hold, then the operators A, B, C_1, C_2 , and C_3 , given by (17)–(21), respectively, are such that*

- (i) $A \in \mathcal{L}(E, E^*)$, A is symmetric, $\langle Av, v \rangle_{E^* \times E} \geq \alpha_A \|v\|_E^2$ for $v \in E$,
- (ii) $B \in \mathcal{L}(E, E^*)$, B is symmetric, $\langle Bv, v \rangle_{E^* \times E} \geq 0$ for $v \in E$,
- (iii) $C_1 \in \mathcal{L}(V, E^*)$, $|\langle C_1\theta, v \rangle_{E^* \times E}| \leq c \|\theta\|_{L^2(\Omega)} \|v\|_E$ for $\theta \in V, v \in E$ with $c > 0$,
- (iv) $C_2 \in \mathcal{L}(V, V^*)$, C_2 is symmetric, $\langle C_2\theta, \theta \rangle_{V^* \times V} \geq \alpha_k \|\theta\|_V^2$ for $\theta \in V$,
- (v) $C_3 \in \mathcal{L}(E, V^*)$, $|\langle C_3v, \theta \rangle_{V^* \times V}| \leq c \|\theta\|_{L^2(\Omega)} \|v\|_E$ for $\theta \in V, v \in E$ with $c > 0$,
- (vi) $\langle C_1\theta, v \rangle_{E^* \times E} + \langle C_3v, \theta \rangle_{V^* \times V} = 0$ for all $v \in E, \theta \in V$.

Lemma 5.2: *If $H(j_v)$ and $H(p_v, p_\tau, p)$ hold, then the functional $J_1: (0, T) \times L^2(\Gamma_C) \times L^2(\Gamma_C; \mathbb{R}^d) \rightarrow \mathbb{R}$ given by (22) is such that*

- (i) $J_1(\cdot, \theta, v)$ is measurable for all $\theta \in L^2(\Gamma_C), v \in L^2(\Gamma_C; \mathbb{R}^d)$,
- (ii) $J_1(t, \theta, \cdot)$ is well defined and locally Lipschitz for $\theta \in L^2(\Gamma_C)$, a.e. t ,
- (iii) $J_1^0(t, \theta, v; w) \leq \int_{\Gamma_C} p_v(\theta) j_v^0(t, v_v; w_v) \, d\Gamma$ for $\theta \in L^2(\Gamma_C), v, w \in L^2(\Gamma_C; \mathbb{R}^d)$, a.e. $t \in (0, T)$,
- (iv) $\|\partial J_1(t, \theta, v)\|_{L^2(\Gamma_C; \mathbb{R}^d)} \leq \bar{c}_{0v}(t)$ with $\bar{c}_{0v} \in L^2(0, T)$, $\bar{c}_{0v}(t) = \sqrt{2|\Gamma_C|} \bar{p}_v c_{0v}(t)$,
- (v) $J_1^0(t, \theta, v; -v) \leq \bar{p}_v \|c_{0v}\|_{L^\infty(\Sigma_C)} \sqrt{|\Gamma_C|} \|v\|_{L^2(\Gamma_C; \mathbb{R}^d)}$ for $v \in L^2(\Gamma_C; \mathbb{R}^d)$,
- (vi) $\partial J_1(t, \cdot, \cdot)$ has a closed graph in the $L^2(\Gamma_C) \times L^2(\Gamma_C; \mathbb{R}^d) \times (w\text{-}L^2(\Gamma_C; \mathbb{R}^d))$ topology for a.e. $t \in (0, T)$.

Lemma 5.3: *If $H(j_\tau)$ and $H(p_v, p_\tau, p)$ hold, then the functional $J_2: (0, T) \times L^2(\Gamma_C) \times L^2(\Gamma_C; \mathbb{R}^d) \rightarrow \mathbb{R}$ given by (23) is such that*

- (i) $J_2(\cdot, \theta, v)$ is measurable for all $\theta \in L^2(\Gamma_C), v \in L^2(\Gamma_C; \mathbb{R}^d)$,
- (ii) $J_2(t, \theta, \cdot)$ is well defined and locally Lipschitz for $\theta \in L^2(\Gamma_C)$, a.e. t ,
- (iii) $J_2^0(t, \theta, v; w) \leq \int_{\Gamma_C} p_\tau(\theta) j_\tau^0(t, v_\tau; w_\tau) \, d\Gamma$ for $\theta \in L^2(\Gamma_C), v, w \in L^2(\Gamma_C; \mathbb{R}^d)$, a.e. $t \in (0, T)$,
- (iv) $\|\partial J_2(t, \theta, v)\|_{L^2(\Gamma_C; \mathbb{R}^d)} \leq \bar{c}_{0\tau}(t) + \bar{c}_{1\tau} \|v\|_{L^2(\Gamma_C; \mathbb{R}^d)}$ with $\bar{c}_{0\tau} \in L^2(0, T)$, $\bar{c}_{0\tau}(t) = \sqrt{2|\Gamma_C|} \bar{p}_\tau c_{0\tau}(t)$, $\bar{c}_{1\tau} = \sqrt{2} \bar{p}_\tau c_{1\tau}$,
- (v) $J_2^0(t, \theta, v; -v) \leq \bar{d}_\tau (1 + \|v\|_{L^2(\Gamma_C; \mathbb{R}^d)})$ for $v \in L^2(\Gamma_C; \mathbb{R}^d)$ with $\bar{d}_\tau = \bar{p}_\tau d_\tau$,
- (vi) $\partial J_2(t, \cdot, \cdot)$ has a closed graph in $L^2(\Gamma_C) \times L^2(\Gamma_C; \mathbb{R}^d) \times (w\text{-}L^2(\Gamma_C; \mathbb{R}^d))$ topology, for a.e. $t \in (0, T)$.

Lemma 5.4: *If $H(j)$ and $H(p_v, p_\tau, p)$ hold, then the functional $J_3: (0, T) \times L^2(\Gamma_C; \mathbb{R}^d) \times L^2(\Gamma_C) \rightarrow \mathbb{R}$ given by (24) is such that*

- (i) $J_3(\cdot, v, \theta)$ is measurable for all $v \in L^2(\Gamma_C; \mathbb{R}^d), \theta \in L^2(\Gamma_C)$,
- (ii) $J_3(t, v, \cdot)$ is well defined and locally Lipschitz for $v \in L^2(\Gamma_C; \mathbb{R}^d)$, a.e. t ,
- (iii) $J_3^0(t, v, \theta; \eta) \leq \int_{\Gamma_C} p(v_\tau) j^0(t, \theta; \eta) d\Gamma$ for $v \in L^2(\Gamma_C; \mathbb{R}^d), \theta, \eta \in L^2(\Gamma_C)$,
a.e. $t \in (0, T)$,
- (iv) $\|\partial J_3(t, v, \theta)\|_{L^2(\Gamma_C)} \leq \bar{c}_0(t) + \bar{c}_1 \|\theta\|_{L^2(\Gamma_C)}$ with $\bar{c}_0(t) = \sqrt{2|\Gamma_C|} \bar{p} c_0(t)$,
 $\bar{c}_0 \in L^2(0, T), \bar{c}_1 = \sqrt{2} \bar{p} c_1$,
- (v) $J_3^0(t, v, \theta; -\theta) \leq \bar{d} (1 + \|\theta\|_{L^2(\Gamma_C)})$ for $v \in L^2(\Gamma_C; \mathbb{R}^d), \theta \in L^2(\Gamma_C)$
with $\bar{d} = d \bar{p}$,
- (vi) $\partial J_3(t, \cdot, \cdot)$ has a closed graph in the $L^2(\Gamma_C; \mathbb{R}^d) \times L^2(\Gamma_C) \times (w\text{-}L^2(\Gamma_C))$ topology for a.e. $t \in (0, T)$.

Consider now the following system of evolutionary inclusions.

Problem 3: *Find $u \in \mathcal{E}$ with $u' \in \mathbb{E}$ and $\theta \in \mathbb{V}$ such that*

$$\begin{cases} u''(t) + Au'(t) + Bu(t) + C_1\theta(t) + \gamma^* \partial J_1(t, \gamma_s \theta(t), \gamma u(t)) \\ \quad + \gamma^* \partial J_2(t, \gamma_s \theta(t), \gamma u'(t)) \ni f(t) \text{ a.e. } t \in (0, T) \\ \theta'(t) + C_2\theta(t) + C_3u'(t) + \gamma_s^* \partial J_3(t, \gamma u'(t), \gamma_s \theta(t)) \ni g(t) \text{ a.e. } t \in (0, T) \\ u(0) = u_0, u'(0) = v_0, \theta(0) = \theta_0. \end{cases}$$

The following is the link between the two problems.

Proposition 5.5: *Assume $H(j_v), H(j_\tau), H(j)$ and $H(p_v, p_\tau, p)$. Then, every solution to Problem 3 is a solution to Problem 2.*

Proof: If (u, θ) is a solution to Problem 3, then

$$u''(t) + Au'(t) + Bu(t) + C_1\theta(t) + \gamma^* \xi^*(t) + \gamma^* \zeta^*(t) = f(t), \quad (25)$$

$$\theta'(t) + C_2\theta(t) + C_3u'(t) + \gamma_s^* \eta^*(t) = g(t) \quad (26)$$

for a.e. $t \in (0, T)$, where $\xi^*(t) \in \partial J_1(t, \gamma_s \theta(t), \gamma u(t))$, $\zeta^*(t) \in \partial J_2(t, \gamma_s \theta(t), \gamma u'(t))$ and $\eta^*(t) \in \partial J_3(t, \gamma u'(t), \gamma_s \theta(t))$ for a.e. $t \in (0, T)$. Let $v \in E$ and $\eta \in V$. Using the definition of the generalized subdifferential and Lemmas 5.2(iii), 5.3(iii) and 5.4(iii), we obtain

$$\begin{aligned} \langle \gamma^* \xi^*(t), v \rangle_{E^* \times E} &= \langle \xi^*(t), \gamma v \rangle_{L^2(\Gamma_C; \mathbb{R}^d)} \leq J_1^0(t, \gamma_s \theta(t), \gamma u(t); \gamma v) \\ &\leq \int_{\Gamma_C} p_v(\gamma_s \theta(t)) j_v^0(t, u_v(t); v_v) d\Gamma, \\ \langle \gamma^* \zeta^*(t), v \rangle_{E^* \times E} &= \langle \zeta^*(t), \gamma v \rangle_{L^2(\Gamma_C; \mathbb{R}^d)} \leq J_2^0(t, \gamma_s \theta(t), \gamma u'(t); \gamma v) \\ &\leq \int_{\Gamma_C} p_\tau(\gamma_s \theta(t)) j_\tau^0(t, u'_\tau(t); v_\tau) d\Gamma, \\ \langle \gamma_s^* \eta^*(t), \eta \rangle_{V^* \times V} &= \langle \eta^*(t), \gamma_s \eta \rangle_{L^2(\Gamma_C)} \leq J_3^0(t, \gamma u'(t), \gamma_s \theta(t); \gamma_s \eta) \\ &\leq \int_{\Gamma_C} p(u'_\tau(t)) j^0(t, \gamma_s \theta(t); \gamma_s \eta) d\Gamma. \end{aligned}$$

Next, we test equation (25) with $v \in E$, and equation (26) with $\eta \in V$. Using the definition of operators A, B, C_1, C_2, C_3 and the above inequalities, we see that (u, θ) is a solution to Problem 2. \square

Step 2. We reformulate Problem 3 in this step as an abstract multivalued first order evolutionary inclusion. We let the operator $K: \mathcal{E} \rightarrow C(0, T; E)$ be defined by

$$(Kv)(t) = \int_0^t v(s) ds + u_0 \text{ for } v \in \mathcal{E}.$$

We observe that operator K has the following properties:

- (i) $\|Kv\|_{C(0,T;E)} \leq \sqrt{T} \|v\|_{\mathcal{E}} + \|u_0\|_E$, for all $v \in \mathcal{E}$,
- (ii) $\|Kv_1 - Kv_2\|_{C(0,T;E)} \leq \sqrt{T} \|v_1 - v_2\|_{\mathcal{E}}$, for all $v_1, v_2 \in \mathcal{E}$,
- (iii) $K: \mathcal{E} \rightarrow \mathcal{E}$ is weakly continuous, i.e. $v_n \rightarrow v$ weakly in \mathcal{E} implies $Kv_n \rightarrow Kv$ weakly in \mathcal{E} .

The first two properties follow immediately from the definition. The proof of (iii) is as follows. The operator $M: \mathcal{E} \rightarrow C(0, T; E)$, defined by $(Mv)(t) = \int_0^t v(s) ds$, for $v \in \mathcal{E}$, is linear and continuous, $M \in \mathcal{L}(\mathcal{E}, \mathcal{E})$, thus $M \in \mathcal{L}(w\text{-}\mathcal{E}, w\text{-}\mathcal{E})$. Therefore, if $v_n \rightarrow v$ weakly in \mathcal{E} , then $Mv_n \rightarrow Mv$ weakly in \mathcal{E} , which implies that $Kv_n \rightarrow Kv$ weakly in \mathcal{E} .

Using the operator K , Problem 3 can be equivalently formulated as follows.

Find $w \in \mathbb{E}$ and $\theta \in \mathbb{V}$ such that

$$\begin{cases} w'(t) + Aw(t) + B(Kw)(t) + C_1\theta(t) + \gamma^* \partial J_1(t, \gamma_s \theta(t), \gamma(Kw)(t)) \\ \quad + \gamma^* \partial J_2(t, \gamma_s \theta(t), \gamma w(t)) \ni f(t) \text{ a.e. } t \in (0, T), \\ \theta'(t) + C_2\theta(t) + C_3w(t) + \gamma_s^* \partial J_3(t, \gamma w(t), \gamma_s \theta(t)) \ni g(t) \text{ a.e. } t \in (0, T), \\ w(0) = v_0, \theta(0) = \theta_0. \end{cases} \quad (27)$$

To deal with the initial conditions, we perform the translation $\tilde{w}(t) = w(t) - v_0$ and $\tilde{\theta}(t) = \theta(t) - \theta_0$. Then, we can rewrite (27) in the equivalent form.

Find $\tilde{w} \in \mathbb{E}$ and $\tilde{\theta} \in \mathbb{V}$ such that

$$\begin{cases} \tilde{w}'(t) + A(\tilde{w}(t) + v_0) + B(K(\tilde{w}(\cdot) + v_0))(t) + C_1(\tilde{\theta}(t) + \theta_0) \\ \quad + \gamma^* \partial J_1(t, \gamma_s(\tilde{\theta}(t) + \theta_0), \gamma(K(\tilde{w}(\cdot) + v_0))(t)) \\ \quad + \gamma^* \partial J_2(t, \gamma_s(\tilde{\theta}(t) + \theta_0), \gamma(\tilde{w}(t) + v_0)) \ni f(t) \text{ a.e. } t \in (0, T), \\ \tilde{\theta}'(t) + C_2(\tilde{\theta}(t) + \theta_0) + C_3(\tilde{w}(t) + v_0) \\ \quad + \gamma_s^* \partial J_3(t, \gamma(\tilde{w}(t) + v_0), \gamma_s(\tilde{\theta}(t) + \theta_0)) \ni g(t) \text{ a.e. } t \in (0, T), \\ \tilde{w}(0) = 0, \tilde{\theta}(0) = 0. \end{cases} \quad (28)$$

We introduce the following Nemitsky operators defined by

$$\mathcal{A}: \mathcal{E} \rightarrow \mathcal{E}^*, (\mathcal{A}v)(t) = A(v(t) + v_0), \quad (29)$$

$$\mathcal{B}: \mathcal{E} \rightarrow \mathcal{E}^*, (\mathcal{B}v)(t) = B(K(v(\cdot) + v_0))(t), \quad (30)$$

$$\mathcal{C}_1: \mathcal{V} \rightarrow \mathcal{E}^*, (\mathcal{C}_1\theta)(t) = C_1(\theta(t) + \theta_0), \quad (31)$$

$$\mathcal{C}_2: \mathcal{V} \rightarrow \mathcal{V}^*, (\mathcal{C}_2\theta)(t) = C_2(\theta(t) + \theta_0), \quad (32)$$

$$\mathcal{C}_3: \mathcal{E} \rightarrow \mathcal{V}^*, (\mathcal{C}_3v)(t) = C_3(v(t) + v_0) \quad (33)$$

and the multivalued Nemitsky operators $\mathcal{N}_1, \mathcal{N}_2: \mathcal{E} \times \mathcal{V} \rightarrow 2^{\mathcal{E}^*}$ and $\mathcal{N}_3: \mathcal{E} \times \mathcal{V} \rightarrow 2^{\mathcal{V}^*}$

$$\mathcal{N}_1(v, \theta) = \{v^* \in \mathcal{E}^* \mid v^*(t) \in \gamma^* \partial J_1(t, \gamma_s(\theta(t) + \theta_0), \gamma(K(v + v_0))(t)) \text{ a.e. } t\}, \quad (34)$$

$$\mathcal{N}_2(v, \theta) = \{v^* \in \mathcal{E}^* \mid v^*(t) \in \gamma^* \partial J_2(t, \gamma_s(\theta(t) + \theta_0), \gamma(v(t) + v_0)) \text{ a.e. } t\}, \quad (35)$$

$$\mathcal{N}_3(v, \theta) = \{\theta^* \in \mathcal{V}^* \mid \theta^*(t) \in \gamma_s^* \partial J_3(t, \gamma(v(t) + v_0), \gamma_s(\theta(t) + \theta_0)) \text{ a.e. } t\} \quad (36)$$

for $(v, \theta) \in \mathcal{E} \times \mathcal{V}$. Next, we define the operator

$$\mathcal{G}: \mathcal{E} \times \mathcal{V} \rightarrow 2^{(\mathcal{E} \times \mathcal{V})^*} \text{ by } \mathcal{G}(v, \theta) = \begin{pmatrix} \mathcal{A}v + \mathcal{B}v + \mathcal{C}_1\theta + \mathcal{N}_1(v, \theta) + \mathcal{N}_2(v, \theta) \\ \mathcal{C}_2\theta + \mathcal{C}_3v + \mathcal{N}_3(v, \theta) \end{pmatrix},$$

for $(v, \theta) \in \mathcal{E} \times \mathcal{V}$ and the functional

$$F \in \mathcal{E}^* \times \mathcal{V}^* \text{ by } F = \begin{pmatrix} f \\ g \end{pmatrix}.$$

Furthermore, we introduce the operators

$$\begin{aligned} L_1: D(L_1) \subset \mathcal{E} &\rightarrow \mathcal{E}^*, L_1v = v' \text{ with } D(L_1) = \{v \in \mathbb{E} \mid v(0) = 0\}, \\ L_2: D(L_2) \subset \mathcal{V} &\rightarrow \mathcal{V}^*, L_2\eta = \eta' \text{ with } D(L_2) = \{\eta \in \mathbb{V} \mid \eta(0) = 0\}, \end{aligned}$$

and $L: D(L) = D(L_1) \times D(L_2) \subset \mathcal{E} \times \mathcal{V} \rightarrow (\mathcal{E} \times \mathcal{V})^*$ defined by

$$L(v, \eta) = \begin{pmatrix} L_1v \\ L_2\eta \end{pmatrix} \text{ for } (v, \eta) \in D(L).$$

Using these notations, we can formulate Problem (28), and hence also Problem 3, in an abstract form as follows.

Problem 4: Find $(w, \theta) \in D(L)$ such that $F \in (L + \mathcal{G})(w, \theta)$.

Step 3. This step takes into account Lemma 5.5 and the fact that Problem 3 is equivalent to Problem 4, and proves an existence result for Problem 4.

Lemma 5.6: Assume $H(\mathcal{A}_e)$, $H(\mathcal{B}_e)$, $H(\mathcal{C}_e, \mathcal{C})$, $H(k)$, $H(p_v, p_\tau, p)$, $H(j_v)$, $H(j_\tau)$, $H(j)$ and (H_0) . Then, Problem 4 has a solution.

Proof: It is well-known that L is a linear and maximal monotone operator (cf. [34, Proposition 32.10]). In order to show that Problem 4 has a solution $(w, \theta) \in D(L)$, we prove that

- (i) \mathcal{G} is bounded,
- (ii) \mathcal{G} is coercive,
- (iii) \mathcal{G} is L -pseudomonotone.

Then, we apply the surjectivity result given in Proposition 2.1.

The auxiliary results that are used to prove these properties of the operator \mathcal{G} , which rely on the properties of the operators \mathcal{A} , \mathcal{B} , \mathcal{C}_1 , \mathcal{C}_2 , \mathcal{C}_3 , \mathcal{N}_1 , \mathcal{N}_2 , and \mathcal{N}_3 , are presented in Lemmas A.1–A.7 in Appendix.

Proof of (i). We conclude from Lemmas A.1(i), A.2(i), A.3(i), A.4(i)–(ii), A.5(ii), A.6(ii) and A.7(ii) that

$$\|\mathcal{G}(v, \theta)\|_{(\mathcal{E} \times \mathcal{V})^*} \leq c(1 + \|v\|_{\mathcal{E}} + \|\theta\|_{\mathcal{V}}) \leq c(1 + \|(v, \theta)\|_{\mathcal{E} \times \mathcal{V}})$$

for all $(v, \theta) \in \mathcal{E} \times \mathcal{V}$ with $c > 0$, hence that \mathcal{G} is a bounded operator.

Proof of (ii). The coercivity of \mathcal{G} follows from Lemmas A.1(ii), A.2(iii), A.3(ii), A.4(iii), A.5(iii), A.6(iii) and A.7(iii). Indeed, for $(v, \theta) \in \mathcal{E} \times \mathcal{V}$ and $(v^*, \theta^*) \in \mathcal{G}(v, \theta)$, we obtain

$$\begin{aligned} \langle (v^*, \theta^*), (v, \theta) \rangle_{(\mathcal{E} \times \mathcal{V})^* \times (\mathcal{E} \times \mathcal{V})} &= \langle \mathcal{A}v, v \rangle_{\mathcal{E}^* \times \mathcal{E}} + \langle \mathcal{B}v, v \rangle_{\mathcal{E}^* \times \mathcal{E}} + \langle \mathcal{C}_1\theta, v \rangle_{\mathcal{E}^* \times \mathcal{E}} \\ &\quad + \langle \mathcal{C}_2\theta, \theta \rangle_{\mathcal{V}^* \times \mathcal{V}} + \langle \mathcal{C}_3v, \theta \rangle_{\mathcal{V}^* \times \mathcal{V}} + \langle \xi^*, v \rangle_{\mathcal{E}^* \times \mathcal{E}} \\ &\quad + \langle \xi^*, v \rangle_{\mathcal{E}^* \times \mathcal{E}} + \langle \eta^*, \theta \rangle_{\mathcal{V}^* \times \mathcal{V}} \\ &\geq \alpha_{\mathcal{A}} \|v\|_{\mathcal{E}}^2 + \alpha_k \|\theta\|_{\mathcal{V}}^2 - c_1 \|(v, \theta)\|_{\mathcal{E} \times \mathcal{V}} - c_2, \end{aligned}$$

where $\xi^* \in \mathcal{N}_1(v, \theta)$, $\zeta^* \in \mathcal{N}_2(v, \theta)$ and $\eta^* \in \mathcal{N}_3(v, \theta)$ with $c_1, c_2 \in \mathbb{R}$.

Proof of (iii). Next, we show that \mathcal{G} is an L -pseudomonotone operator. It follows from Lemmas A.5(i), A.6(i), and A.7(i), that \mathcal{G} has nonempty, convex, and weakly compact values in $(\mathcal{E} \times \mathcal{V})^*$. We prove that the operator \mathcal{G} is upper semicontinuous in $(\mathcal{E} \times \mathcal{V}) \times (w\text{-}(\mathcal{E} \times \mathcal{V})^*)$ topology. To that end, it is enough to show ([35]) that if $D \subset (\mathcal{E} \times \mathcal{V})^*$ is weakly closed, then $\mathcal{G}^-(D) = \{(v, \theta) \in \mathcal{E} \times \mathcal{V} \mid \mathcal{G}(v, \theta) \cap D \neq \emptyset\}$ is closed in $\mathcal{E} \times \mathcal{V}$.

Let $(v_n, \theta_n) \in \mathcal{G}^-(D)$ and $(v_n, \theta_n) \rightarrow (v, \theta)$ in $\mathcal{E} \times \mathcal{V}$. Then, we can find $(v_n^*, \theta_n^*) \in D$ such $(v_n^*, \theta_n^*) \in \mathcal{G}(v_n, \theta_n)$. Since \mathcal{G} is a bounded operator, the sequence $\{(v_n^*, \theta_n^*)\}$ is bounded in $(\mathcal{E} \times \mathcal{V})^*$. Hence, by passing to a subsequence, if necessary, we may assume that

$$(v_n^*, \theta_n^*) \rightarrow (v^*, \theta^*) \text{ weakly in } (\mathcal{E} \times \mathcal{V})^*.$$

Moreover, we have

$$v_n^* = \mathcal{A}v_n + \mathcal{B}v_n + \mathcal{C}_1\theta_n + \xi_n^* + \zeta_n^* \text{ in } \mathcal{E}^*, \quad (37)$$

$$\theta_n^* = \mathcal{C}_2\theta_n + \mathcal{C}_3v_n + \eta_n^* \text{ in } \mathcal{V}^*, \quad (38)$$

with $\xi_n^* \in \mathcal{N}_1(v_n, \theta_n)$, $\zeta_n^* \in \mathcal{N}_2(v_n, \theta_n)$ and $\eta_n^* \in \mathcal{N}_3(v_n, \theta_n)$. Since \mathcal{N}_i , $i = 1, 2, 3$ are bounded operators (cf. Lemmas A.5(ii), A.6(ii), A.7(ii)), we have, possibly for subsequences,

$$\xi_n^* \rightarrow \xi^*, \quad \zeta_n^* \rightarrow \zeta^* \text{ weakly in } \mathcal{E}^*, \quad \eta_n^* \rightarrow \eta^* \text{ weakly in } \mathcal{V}^*. \quad (39)$$

Hence, from the closedness of graph of $\partial J_i(t, \cdot, \cdot)$, $i = 1, 2, 3$, in suitable topologies (cf. Lemmas 5.2(vi), 5.3(vi), 5.4(vi)), we obtain

$$\xi^* \in \mathcal{N}_1(v, \theta), \quad \zeta^* \in \mathcal{N}_2(v, \theta), \quad \eta^* \in \mathcal{N}_3(v, \theta). \quad (40)$$

We now pass to the weak limit in (37) and (38). We use (39) and the continuity of \mathcal{A} , \mathcal{B} , \mathcal{C}_1 , \mathcal{C}_2 , and \mathcal{C}_3 (cf. Lemmas A.1(iv), A.2(ii), A.3(iv), and A.4(iv)) to deduce that

$$\begin{aligned} v^* &= \mathcal{A}v + \mathcal{B}v + \mathcal{C}_1\theta + \xi^* + \zeta^* \text{ in } \mathcal{E}^*, \\ \theta^* &= \mathcal{C}_2\theta + \mathcal{C}_3v + \eta^* \text{ in } \mathcal{V}^*, \end{aligned}$$

which together with (40) yield $(v^*, \theta^*) \in \mathcal{G}(v, \theta)$. Next, since D is weakly closed in $(\mathcal{E} \times \mathcal{V})^*$ and $(v_n^*, \theta_n^*) \rightarrow (v^*, \theta^*)$ weakly in $(\mathcal{E} \times \mathcal{V})^*$, we have also $(v^*, \theta^*) \in D$. Therefore, $(v, \theta) \in \mathcal{G}^-(D)$ which shows that \mathcal{G} is upper semicontinuous in a suitable topology.

To complete the proof of L -pseudomonotonicity of \mathcal{G} , we need to show that for every sequence $\{(v_n, \theta_n)\} \subset D(L)$ such that $(v_n, \theta_n) \rightarrow (v, \theta)$ weakly in $\mathcal{E} \times \mathcal{V}$, $(v_n', \theta_n') \rightarrow (v', \theta')$ weakly in $(\mathcal{E} \times \mathcal{V})^*$, $(v_n^*, \theta_n^*) \in \mathcal{G}(v_n, \theta_n)$, $(v_n^*, \theta_n^*) \rightarrow (v^*, \theta^*)$ weakly in $(\mathcal{E} \times \mathcal{V})^*$ and $\limsup \langle (v_n^*, \theta_n^*), (v_n, \theta_n) - (v, \theta) \rangle_{(\mathcal{E} \times \mathcal{V})^* \times (\mathcal{E} \times \mathcal{V})} \leq 0$, we obtain

$$(v^*, \theta^*) \in \mathcal{G}(v, \theta), \quad (41)$$

$$\langle (v_n^*, \theta_n^*), (v_n, \theta_n) - (v, \theta) \rangle_{(\mathcal{E} \times \mathcal{V})^* \times (\mathcal{E} \times \mathcal{V})} \rightarrow 0. \quad (42)$$

First, we note that (41) is a consequence of the condition $(v_n^*, \theta_n^*) \in \mathcal{G}(v_n, \theta_n)$. This can be proved analogously as in the above proof of the upper semicontinuity of \mathcal{G} . Second, to prove (42), we need to show that

$$\langle \xi_n^* + \zeta_n^*, v_n - v \rangle_{\mathcal{E}^* \times \mathcal{E}} \rightarrow 0, \quad \langle \eta_n^*, \theta_n - \theta \rangle_{\mathcal{V}^* \times \mathcal{V}} \rightarrow 0, \quad (43)$$

$$\langle \mathcal{C}_1 \theta_n, v_n - v \rangle_{\mathcal{E}^* \times \mathcal{E}} + \langle \mathcal{C}_3 v_n, \theta_n - \theta \rangle_{\mathcal{V}^* \times \mathcal{V}} \rightarrow 0, \quad (44)$$

$$\langle \mathcal{A} v_n, v_n - v \rangle_{\mathcal{E}^* \times \mathcal{E}} \rightarrow 0, \quad \langle \mathcal{C}_2 \theta_n, \theta_n - \theta \rangle_{\mathcal{V}^* \times \mathcal{V}} \rightarrow 0, \quad (45)$$

$$\langle \mathcal{B} v_n, v_n - v \rangle_{\mathcal{E}^* \times \mathcal{E}} \rightarrow 0. \quad (46)$$

The convergences (43) follow directly from Lemmas A.5(iv), A.6(iv), and A.7(iv). Next, Lemma 5.1(vi) and the weak continuity of \mathcal{C}_1 and \mathcal{C}_3 (cf. Lemma A.4(iv)) lead to

$$\begin{aligned} & \langle \mathcal{C}_1 \theta_n, v_n - v \rangle_{\mathcal{E}^* \times \mathcal{E}} + \langle \mathcal{C}_3 v_n, \theta_n - \theta \rangle_{\mathcal{V}^* \times \mathcal{V}} \\ &= \int_0^T \left(\langle \mathcal{C}_1 \theta_n(t), v_n(t) \rangle_{\mathcal{E}^* \times \mathcal{E}} + \langle \mathcal{C}_3 v_n(t), \theta_n(t) \rangle_{\mathcal{V}^* \times \mathcal{V}} \right) dt \\ & \quad + \langle \mathcal{C}_1 \theta_0, v_n \rangle_{\mathcal{E}^* \times \mathcal{E}} + \langle \mathcal{C}_3 v_0, \theta_n \rangle_{\mathcal{V}^* \times \mathcal{V}} - \langle \mathcal{C}_1 \theta_n, v \rangle_{\mathcal{E}^* \times \mathcal{E}} - \langle \mathcal{C}_3 v_n, \theta \rangle_{\mathcal{V}^* \times \mathcal{V}} \\ & \rightarrow \langle \mathcal{C}_1 \theta_0, v \rangle_{\mathcal{E}^* \times \mathcal{E}} + \langle \mathcal{C}_3 v_0, \theta \rangle_{\mathcal{V}^* \times \mathcal{V}} - \langle \mathcal{C}_1 \theta, v \rangle_{\mathcal{E}^* \times \mathcal{E}} - \langle \mathcal{C}_3 v, \theta \rangle_{\mathcal{V}^* \times \mathcal{V}} \\ &= - \int_0^T \left(\langle \mathcal{C}_1 \theta(t), v(t) \rangle_{\mathcal{E}^* \times \mathcal{E}} + \langle \mathcal{C}_3 v(t), \theta(t) \rangle_{\mathcal{V}^* \times \mathcal{V}} \right) dt = 0, \end{aligned}$$

which establishes (44).

Next, we prove (45). From (43) and (44), we obtain

$$\begin{aligned} 0 &\geq \limsup \langle (v_n^*, \theta_n^*), (v_n, \theta_n) - (v, \theta) \rangle_{(\mathcal{E} \times \mathcal{V})^* \times (\mathcal{E} \times \mathcal{V})} \\ &\geq \limsup \left(\langle \mathcal{A} v_n, v_n - v \rangle_{\mathcal{E}^* \times \mathcal{E}} + \langle \mathcal{C}_2 \theta_n, \theta_n - \theta \rangle_{\mathcal{V}^* \times \mathcal{V}} \right) \\ &\quad + \liminf \langle \mathcal{B} v_n, v_n - v \rangle_{\mathcal{E}^* \times \mathcal{E}}. \end{aligned}$$

The monotonicity of the operator \mathcal{B} (cf. Lemma A.2(iv)) implies

$$\begin{aligned} \liminf \langle \mathcal{B} v_n, v_n - v \rangle_{\mathcal{E}^* \times \mathcal{E}} &= \liminf \langle \mathcal{B} v_n - \mathcal{B} v, v_n - v \rangle_{\mathcal{E}^* \times \mathcal{E}} \\ &\quad + \lim \langle \mathcal{B} v, v_n - v \rangle_{\mathcal{E}^* \times \mathcal{E}} \geq 0. \end{aligned} \quad (47)$$

It follows that

$$\limsup \left(\langle \mathcal{A} v_n, v_n - v \rangle_{\mathcal{E}^* \times \mathcal{E}} + \langle \mathcal{C}_2 \theta_n, \theta_n - \theta \rangle_{\mathcal{V}^* \times \mathcal{V}} \right) \leq 0. \quad (48)$$

We claim that

$$\limsup \langle \mathcal{A} v_n, v_n - v \rangle_{\mathcal{E}^* \times \mathcal{E}} \leq 0 \quad \text{and} \quad \limsup \langle \mathcal{C}_2 \theta_n, \theta_n - \theta \rangle_{\mathcal{V}^* \times \mathcal{V}} \leq 0. \quad (49)$$

Indeed, assume that $\limsup \langle \mathcal{A} v_n, v_n - v \rangle_{\mathcal{E}^* \times \mathcal{E}} > 0$. Therefore, there exist $m > 0$ and a subsequence such that $\lim \langle \mathcal{A} v_n, v_n - v \rangle_{\mathcal{E}^* \times \mathcal{E}} = m > 0$. Then, from (48), we obtain $\limsup \langle \mathcal{C}_2 \theta_n, \theta_n - \theta \rangle_{\mathcal{V}^* \times \mathcal{V}} \leq -m < 0$ and subsequently the L_2 -pseudomonotonicity of \mathcal{C}_2 (cf. Lemma A.3(iii)) implies $\lim \langle \mathcal{C}_2 \theta_n, \theta_n - \theta \rangle_{\mathcal{V}^* \times \mathcal{V}} = 0$. It follows that $m \leq 0$, which contradicts the choice $m > 0$. Hence, the first inequality in (49) holds true. Similarly, we can show the second inequality in (49).

Since \mathcal{A} is L_1 -pseudomonotone (cf. Lemma A.1(iii)) and \mathcal{C}_2 is L_2 -pseudomonotone (cf. Lemma A.3(iii)), we conclude that (45) holds true.

Finally, to justify (46), we use the hypotheses and the convergences (43)–(45), and the fact that

$$0 \geq \limsup \langle (v_n^*, \theta_n^*), (v_n, \theta_n) - (v, \theta) \rangle_{(\mathcal{E} \times \mathcal{V})^* \times (\mathcal{E} \times \mathcal{V})} \geq \limsup \langle \mathcal{B} v_n, v_n - v \rangle_{\mathcal{E}^* \times \mathcal{E}}.$$

This, together with (47) implies that $\lim \langle \mathcal{B} v_n, v_n - v \rangle_{\mathcal{E}^* \times \mathcal{E}} = 0$, which completes the proof of (46), and consequently also of (42). This finishes the proof of (iii).

We have all necessary ingredients to apply Proposition 2.1 and conclude that Problem 4 has a solution $(w, \theta) \in D(L)$. \square

Finally, we use Lemma 5.6 to infer that Problem 4 has a solution $(w, \theta) \in D(L)$. Hence $(u = K(w + v_0), \theta + \theta_0)$ is a solution to Problem 3. Then, Proposition 5.5 implies that $(u, \theta) \in \mathcal{E} \times \mathcal{V}$ is a solution to Problem 2. The proof of Theorem 4.1 is now complete.

Remark 1: We note that we may weaken the assumptions on the initial data. Indeed, Theorem 4.1 still holds if the initial conditions are less regular, i.e. $v_0 \in L^2(\Omega; \mathbb{R}^d)$ and $\theta_0 \in L^2(\Omega)$. In this case we approximate these elements by sequences of more regular functions, i.e. $\{v_0^n\} \subset E$, $\{\theta_0^n\} \subset V$ such that $v_0^n \rightarrow v_0$ in $L^2(\Omega; \mathbb{R}^d)$ and $\theta_0^n \rightarrow \theta_0$ in $L^2(\Omega)$. Then, we may consider a sequence of versions of Problem 2, called Problems 2_n , with suitable initial conditions (i.e. $u_n'(0) = v_0^n$ and $\theta_n(0) = \theta_0^n$) and apply Theorem 4.1. Then, a subsequence of solutions of Problems 2_n converges to a solution of Problem 2.

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Appendix 1. Auxiliary results

We provide a collection of results that are used in the proofs in the paper, especially in establishing the properties of the operator \mathcal{G} in Section 5. The Lemmas below are generalizations of similar results in [29].

Lemma A.1 ([29, Lemma 8]): *If $H(\mathcal{A}_e)$ holds, then the operator $\mathcal{A} : \mathcal{E} \rightarrow \mathcal{E}^*$ defined by (29) satisfies:*

- (i) $\|\mathcal{A}v\|_{\mathcal{E}^*} \leq c(1 + \|v\|_{\mathcal{E}})$ for all $v \in \mathcal{E}$ with $c > 0$,
- (ii) $\langle \mathcal{A}v, v \rangle_{\mathcal{E}^* \times \mathcal{E}} \geq \alpha_{\mathcal{A}} \|v\|_{\mathcal{E}}^2 - c_1 \|v\|_{\mathcal{E}} - c_2$ for all $v \in \mathcal{E}$ with $c_1, c_2 \in \mathbb{R}$,
- (iii) \mathcal{A} is L_1 -pseudomonotone,
- (iv) \mathcal{A} is continuous and weakly continuous.

Lemma A.2 ([29, Lemma 9]): *If $H(\mathcal{B}_e)$ holds, then the operator $\mathcal{B} : \mathcal{E} \rightarrow \mathcal{E}^*$ defined by (30) satisfies:*

- (i) $\|\mathcal{B}v\|_{\mathcal{E}^*} \leq c(1 + \|v\|_{\mathcal{E}})$ for all $v \in \mathcal{E}$ with $c > 0$,
- (ii) $\|\mathcal{B}v_1 - \mathcal{B}v_2\|_{\mathcal{E}^*} \leq c\|v_1 - v_2\|_{\mathcal{E}}$ for all $v_1, v_2 \in \mathcal{E}$ with $c > 0$,
- (iii) $\langle \mathcal{B}v, v \rangle_{\mathcal{E}^* \times \mathcal{E}} \geq -c_1 \|v\|_{\mathcal{E}} - c_2$ for all $v \in \mathcal{E}$ with $c_1 \geq 0$ and $c_2 \in \mathbb{R}$,
- (iv) \mathcal{B} is monotone,
- (v) \mathcal{B} is weakly continuous.

Lemma A.3 ([29, Lemma 10]): *If $H(k)$ holds, then the operator $\mathcal{C}_2 : \mathcal{V} \rightarrow \mathcal{V}^*$ defined by (32) satisfies:*

- (i) $\|\mathcal{C}_2\theta\|_{\mathcal{V}^*} \leq c(1 + \|\theta\|_{\mathcal{V}})$ for all $\theta \in \mathcal{V}$ with $c > 0$,
- (ii) $\langle \mathcal{C}_2\theta, \theta \rangle_{\mathcal{V}^* \times \mathcal{V}} \geq \alpha_k \|\theta\|_{\mathcal{V}}^2 - c_1 \|\theta\|_{\mathcal{V}} - c_2$ for all $\theta \in \mathcal{V}$ with $c_1, c_2 \in \mathbb{R}$,

- (iii) C_2 is L_2 -pseudomonotone,
- (iv) C_2 is continuous and weakly continuous.

Lemma A.4 ([29, Lemma 13]): *If $H(C_e, C)$ holds, then the operators $C_1: \mathcal{V} \rightarrow \mathcal{E}^*$ and $C_3: \mathcal{E} \rightarrow \mathcal{V}^*$ defined by (31) and (33), respectively, satisfy:*

- (i) $\|C_1\theta\|_{\mathcal{E}^*} \leq c(1 + \|\theta\|_{\mathcal{V}})$ for $\theta \in \mathcal{V}$ with $c > 0$,
- (ii) $\|C_3v\|_{\mathcal{V}^*} \leq c(1 + \|v\|_{\mathcal{E}})$ for $v \in \mathcal{E}$ with $c > 0$,
- (iii) $\langle C_1\theta, v \rangle_{\mathcal{E}^* \times \mathcal{E}} + \langle C_3v, \theta \rangle_{\mathcal{V}^* \times \mathcal{V}} \geq -c\|(v, \theta)\|_{\mathcal{E} \times \mathcal{V}}$ for $(v, \theta) \in \mathcal{E} \times \mathcal{V}$ with $c > 0$,
- (iv) C_1 and C_3 are continuous and weakly continuous.

The proofs of the next Lemmas are analogous to each other and they are similar to the results in [29, Lemmas 11 and 12] or [20, Lemma 13].

Lemma A.5: *If $H(j_\nu)$ and $H(p_\nu, p_\tau, p)$ hold, then the operator $\mathcal{N}_1: \mathcal{E} \times \mathcal{V} \rightarrow 2^{\mathcal{E}^*}$ defined by (34) satisfies:*

- (i) \mathcal{N}_1 has nonempty, convex, and weakly compact values in \mathcal{E}^* ,
- (ii) $\|\mathcal{N}_1(v, \theta)\|_{\mathcal{E}^*} \leq c\|v\|_{\mathcal{E}}$ for all $v \in \mathcal{E}, \theta \in \mathcal{V}$ with $c > 0$,
- (iii) $\langle v^*, v \rangle_{\mathcal{E}^* \times \mathcal{E}} \geq -c\|v\|_{\mathcal{E}}$ for all $v^* \in \mathcal{N}_1(v, \theta), v \in \mathcal{E}, \theta \in \mathcal{V}$ with $c > 0$,
- (iv) If $v_n^* \in \mathcal{N}_1(v_n, \theta_n)$ with $(v_n, \theta_n) \rightarrow (v, \theta)$ weakly in $\mathbb{E} \times \mathbb{V}$ and $v_n^* \rightarrow v^*$ weakly in \mathcal{E}^* , then $v^* \in \mathcal{N}_1(v, \theta)$ and $\langle v_n^*, v_n - v \rangle_{\mathcal{E}^* \times \mathcal{E}} \rightarrow 0$.

Lemma A.6: *If $H(j_\tau)$ and $H(p_\nu, p_\tau, p)$ hold, then the operator $\mathcal{N}_2: \mathcal{E} \times \mathcal{V} \rightarrow 2^{\mathcal{E}^*}$ defined by (35) satisfies:*

- (i) \mathcal{N}_2 has nonempty, convex, and weakly compact values in \mathcal{E}^* ,
- (ii) $\|\mathcal{N}_2(v, \theta)\|_{\mathcal{E}^*} \leq c(1 + \|v\|_{\mathcal{E}})$ for $(v, \theta) \in \mathcal{E} \times \mathcal{V}$ with $c > 0$,
- (iii) $\langle v^*, v \rangle_{\mathcal{E}^* \times \mathcal{E}} \geq -c_1\|v\|_{\mathcal{E}} - c_2$ for $v^* \in \mathcal{N}_2(v, \theta), (v, \theta) \in \mathcal{E} \times \mathcal{V}$ with $c_1, c_2 > 0$,
- (iv) If $v_n^* \in \mathcal{N}_2(v_n, \theta_n)$ with $(v_n, \theta_n) \rightarrow (v, \theta)$ weakly in $\mathbb{E} \times \mathbb{V}$ and $v_n^* \rightarrow v^*$ weakly in \mathcal{E}^* , then $v^* \in \mathcal{N}_2(v, \theta)$ and $\langle v_n^*, v_n - v \rangle_{\mathcal{E}^* \times \mathcal{E}} \rightarrow 0$.

Lemma A.7: *If $H(j)$ and $H(p_\nu, p_\tau, p)$ hold, then the operator $\mathcal{N}_3: \mathcal{E} \times \mathcal{V} \rightarrow 2^{\mathcal{V}^*}$ given by (36) satisfies:*

- (i) \mathcal{N}_3 has nonempty, convex, and weakly compact values in \mathcal{V}^* ,
- (ii) $\|\mathcal{N}_3(v, \theta)\|_{\mathcal{V}^*} \leq c(1 + \|\theta\|_{\mathcal{V}})$ for $(v, \theta) \in \mathcal{E} \times \mathcal{V}$ with $c > 0$,
- (iii) $\langle \theta^*, \theta \rangle_{\mathcal{V}^* \times \mathcal{V}} \geq -c_1\|\theta\|_{\mathcal{V}} - c_2$ for $\theta^* \in \mathcal{N}_3(v, \theta), (v, \theta) \in \mathcal{E} \times \mathcal{V}$ with $c_1, c_2 > 0$,
- (iv) If $\theta_n^* \in \mathcal{N}_3(v_n, \theta_n)$ with $(v_n, \theta_n) \rightarrow (v, \theta)$ weakly in $\mathbb{E} \times \mathbb{V}$ and $\theta_n^* \rightarrow \theta^*$ weakly in \mathcal{V}^* , then $\theta^* \in \mathcal{N}_3(v, \theta)$ and $\langle \theta_n^*, \theta_n - \theta \rangle_{\mathcal{V}^* \times \mathcal{V}} \rightarrow 0$.