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A MATHEMATICAL BRIDGE BETWEEN DISCRETIZED GAUGE THEORIES IN QUANTUM PHYSICS AND APPROXIMATE REASONNING IN PAIRWISE COMPARISONS

JEAN-PIERRE MAGNOT

ABSTRACT. We describe a mathematical link between aspects of information theory, called pairwise comparisons, and discretized gauge theories. The link is made by the notion of holonomy along the edges of a simplex. This correspondance leads to open questions in both field.

Keywords: Pairwise comparisons, discretized gauge theories, holonomy.
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INTRODUCTION

We present here an overview, addressed to physicists, of a possible bridge between gauge theories (and also some aspects of quantum gravity) with pairwise comparisons matrices and their applications in information theory and approximate reasoning. This is the reason why we fastly summarize features in physics (assuming that they are known by the reader), and give more details for selected features on Pairwise Comparisons (PC) matrices (assuming that this field is less known). This paper is a companion work to [13, 14, 15] where, after a tentative in [11], the extension of the notion of classical PC matrices to matrices with coefficients in a group is considered, partially motivated (in my case) by the striking similarities with mathematical constructions in discretized gauge theories. A not complete list of reference about PC matrices is [5, 8, 10], oriented in our perspective, and a very partial list of references about gauge theories and their discretized forms is [1, 2, 3, 4, 6, 7, 12, 16, 17, 18, 19, 20].

We begin with an oriented survey of selected problems in discretization of G -gauge theories, where G is a Lie group, and a selection of features in evaluation of inconsistency in pairwise comparisons with coefficients in \mathbb{R}_+^* . Then we describe, following [13, 14], a straightforward extension of PC matrices with coefficients in \mathbb{R}_+^* to a general Lie group G . The link with gauge theories is performed via holonomy, which appears in discretizations described in [15, 17]. We finish with the possible interpretations in both sides of this correspondance, first from quantities on PC matrices to gauge theories, and secondly from second quantization to approximate reasoning.

1. A SHORT AND NOT COMPLETE SURVEY OF EACH FIELD OF KNOWLEDGE

We present here the two fields under consideration, discretized gauge theories and pairwise comparisons in approximate reasoning, in a way to highlight the correspondances.

1.1. Gauge theories discretized. The phase space of a (continuum) gauge theory is a space of connections on a (finite dimensional) principal bundle P with structure

group G and with base M . We note by $C(P)$ the space of connections considered. If M is not compact and Riemannian, one often use the space of connections which are smooth and square-integrable. A gauge theory is defined by an action functional $S : C(P) \rightarrow \mathbb{R}$ which has to be minimized.

A discretized gauge theory is defined on a triangulation, a cubification or any other way to discretize the manifold M , and the principal bundle P can be often trivial. Let us highlight two kind of discretizations:

- Whitney's discretization [21], where a connection $\theta \in C(P)$ is (Riemann-)integrated on the 1-vertices of the chosen triangulation. This mimiks a finite element method of approximation for scalar functions, and the discretized connection θ_W generates a H^1 -approximation $\tilde{\theta}$ of θ , which H^1 -converges to θ when refining the triangulation. The action functional S is then evaluated on the finite dimensional space of cnnnections $\tilde{\theta}$ instead of the infinite dimensional space $C(P)$. This is, to our knowledge the most widely developped approach, but this approach seems to fail partially for non-abelian theories, partly because the trivialisaton is not gauge-covariant. This leads to gauge-fixing strategies.
- Holonomy discretization, mostly inspired by the ideas of quantum gravity [17], where connections along the edges are discretized through their holonomy. This approach requires mathematical precisions by fixing a preliminary gauge on the 1-vertices of the discretized manifold, but in a final analysis, only theories depending of secondary characteristic classes (e.g. Chern-Simons theory) can give rise to pathologies in gauge covariance, where as gauge theories arising from primary characteristic classes (e.g. Yang-Mills theories) are fully gauge covariant when discretized [15].

1.2. Pairwise comparisons, consistency and inconsistency. It is easy to explain the inconsistency in PC matrices when we consider cycles of three comparisons, called triads and represented here as (x, y, z) , which do not have the "morphism of groupoid" property such as

$$x.z \neq y$$

The use of "inconsistency" has a meaning of a measure of inconsistency in this study; not the concept itself. One approach to inconsistency (originated in [8] and generalized in [5]) can be reduced to a simple observation:

- search all triads (which generate all 3 by 3 PC submatrices) and locate the worst triad with an inconsistency indicator (ii),
- ii of the worst triad becomes ii of the entire PC matrix.

Expressing it a bit more formally in terms of triads (the upper triangle of a PC submatrix 3×3), we have:

$$ii_3(x, y, z) = 1 - \min \left\{ \frac{y}{xz}, \frac{xz}{y} \right\} = 1 - e^{-|\ln(\frac{y}{xz})|}$$

The expression $|\ln(\frac{y}{xz})|$ stands for the distance of the triad T to the "nearest" consistent PC matrix. When this distance increases, the $ii(x, y, z)$ also increases. It is important to notice here that this definition allows us to localize the inconsistency in the PC matrix, which is of a considerable importance for most applications. For high rank matrices, ii_3 is evaluated on each 3×3 matrices, taking the supremum of the obtained values.

Another possible definition of the inconsistency indicator can also be defined (following [10]) as:

$$\text{ii}_n(A) = 1 - \min_{1 \leq i < j \leq n} \min \left(\frac{a_{ij}}{a_{i,i+1} a_{i+1,i+2} \cdots a_{j-1,j}}, \frac{a_{i,i+1} a_{i+1,i+2} \cdots a_{j-1,j}}{a_{ij}} \right)$$

since the matrix A is consistent if and only if for any $1 \leq i < j \leq n$ the following equation holds:

$$a_{ij} = a_{i,i+1} a_{i+1,i+2} \cdots a_{j-1,j}.$$

This is equivalent to:

$$\text{ii}_n(A) = 1 - \max_{1 \leq i < j \leq n} \left(1 - e^{-\left| \ln \left(\frac{a_{ij}}{a_{i,i+1} a_{i+1,i+2} \cdots a_{j-1,j}} \right) \right|} \right)$$

One of the main features in applications is to minimize the inconsistency indicator ii . This is the reason why, instead of using the inconsistency indicator ii defined before, there is plethora of inconsistency indicators. Each inconsistency indicator intends to measure how far a PC matrix is from the set of consistent PC matrices, which, for 3×3 PC matrices, is a 2-dimensional manifold of matrices of the form

$$\begin{pmatrix} 1 & x & xy \\ x^{-1} & 1 & y \\ x^{-1}y^{-1} & y^{-1} & 1 \end{pmatrix}, \text{ with } (x, y) \in (\mathbb{R}_+^*)^2.$$

Unfortunately, the notion and the theory of inconsistency indicators is not actually fixed and achieved, and many competing, incompatible approaches are actually developed.

2. THE MATRIX OF HOLONOMIES AND PAIRWISE COMPARISONS

We follow first [14]. Let I be a set of indexes among \mathbb{Z} , \mathbb{N} or $\{0, \dots, n\}$ for some $n \in \mathbb{N}^*$.

Definition 2.1. Let (G, \cdot) be a group. A PC matrix is a matrix

$$A = (a_{i,j})_{(i,j) \in I^2}$$

such that:

- (1) $\forall (i, j) \in I^2, a_{i,j} \in G.$
- (2) $\forall (i, j) \in I^2, a_{j,i} = a_{i,j}^{-1}.$
- (3) $a_{i,i} = 1_G.$

The matrix A is **covariantly consistent** if

$$(2.1) \quad \forall (i, j, k) \in I^3, \quad a_{i,j} \cdot a_{j,k} = a_{i,k}.$$

Due to the contravariant composition thereafter, we will use the following notion: the PC matrix is **contravariantly consistent** if

$$(2.2) \quad \forall (i, j, k) \in I^3, \quad a_{j,k} \cdot a_{i,j} = a_{i,k}.$$

These two notions are dual, and depend on which order we require for the group multiplication. Similarly, a contravariant consistent PC matrix $A = (a_{i,j})_{(i,j) \in I^2}$ generates a covariant consistent PC matrix $B = (b_{i,j})_{(i,j) \in I^2}$ setting

$$b_{i,j} = a_{i,j}^{-1} = a_{j,i}.$$

For convenience, we use the terms **covariant PC matrix** (resp. **contravariant PC matrix**) when covariant consistency (resp. contravariant consistency) is naturally required.

Theorem 2.2.

$$\exists(\lambda_i)_{i \in I}, \quad a_{i,j} = \lambda_i^{-1} \cdot \lambda_j \Leftrightarrow A \text{ is consistent.}$$

Let $n \in \mathbb{N}^*$ and

$$\Delta_n = \left\{ (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid \left(\sum_{i=0}^n x_i = 1 \right) \wedge (\forall i \in \{0, \dots, n\}, x_i \geq 0) \right\}$$

be an n -simplex. This simplex can be generalized to the infinite dimension:

$$\Delta_{\mathbb{N}} = \left\{ (x_n)_{n \in \mathbb{N}} \in l^1(\mathbb{N}, \mathbb{R}_+^*) \mid \sum_{i=0}^{\infty} x_i = 1 \right\}$$

and

$$\Delta_{\mathbb{Z}} = \left\{ (x_n)_{n \in \mathbb{Z}} \in l^1(\mathbb{Z}, \mathbb{R}_+^*) \mid \sum_{i \in \mathbb{Z}} x_i = 1 \right\},$$

where the summation over \mathbb{Z} is done by integration with respect to the counting measure. In the sequel, Δ will denote Δ_n , $\Delta_{\mathbb{N}}$ or $\Delta_{\mathbb{Z}}$. We define a gauge $(g_i)_{i \in I} \in G^I$ with $\tilde{\gamma}_i(1) = (\gamma_i(1), g_i)$ where

$$\gamma_i = [s_0, s_1] * \dots * [s_{i-1}, s_i] \text{ if } i > 0$$

and

$$\gamma_i = [s_0, s_{-1}] * \dots * [s_{i+1}, s_i] \text{ if } i < 0.$$

We set $g_i = Hol_{(s_0, 1_G)} \gamma_i$. Let us recall that, for two paths c and c' such that $c * c'$ exists (i.e. $c(1) = c'(0)$), if $p = (c(0), e_G)$, $p' = (c'(0), e_G)$ and $h = Hol_p c$, we have: Let

$$(2.3) \quad a_{i,j} = g_j \cdot Hol_{(s_0; e_G)} (\gamma_i[s_i, s_j] \gamma_j^{-1}) \cdot g_i^{-1}.$$

In the light of these specifications, we set, for any connection $\theta \in \Omega^1(\Delta, \mathfrak{g})$,

$$A = Mat(a_{i,j})$$

and the required notion of consistency is contravariant consistency.

Proposition 2.3. *A is a PC matrix.*

Proof. This follows from holonomy in “reverse orientation”. □

Let $\gamma_{i,j,k} = \gamma_i * [s_i, s_j] * [s_j, s_k] * [s_k, s_i] * \gamma_i^{-1}$ be the loop based on s_i along the border of the oriented 2-vertex $[s_i, s_j, s_k]$, where $*$ is the composition of paths. Contravariant consistency seems to fit naturally with flatness of connections:

$$\begin{aligned} \forall i, j, k, \quad & a_{i,k} = a_{j,k} \cdot a_{i,j} \\ \Leftrightarrow & a_{k,i} \cdot a_{j,k} \cdot a_{i,j} = a_{i,i} = 1_G \\ \Leftrightarrow & Hol(\gamma_{i,j,k}) = 1_G \end{aligned}$$

By fixing an **indicator map**, defined in [11] as

$$In : G \rightarrow \mathbb{R}$$

to $In(1_G) = 0$, we get a generalization of the **inconsistency indicator** by setting

$$ii_{In} = \sup \{In(Hol(\gamma_{i,j,k})) | (i, j, k) \in I^3\}.$$

For example, if d is a left-invariant distance on G , a natural indicator map can be:

$$In : g \mapsto d(1_G, g^{-1}).$$

Such definitions extend to triangularized manifolds along the lines of [15], where one can see that the notion of holonomy on a manifold can be discretized, inserting "gaps" (i.e. 0 entries) on a larger matrix gathering all the PC matrices over the simplexes of the triangulation, along the lines of [14]. We then recover the discretization of connections via holonomies described in [17]. When $G = \mathbb{R}_+^*$ In (classical) pairwise comparisons, matrix coefficients are scalar, which enable to simplify the settings obtained. The indicator ii appears as the distance of the holonomy of the loop $[s_i s_{i+1} \dots s_{i+n} s_i]$ to identity.

3. CORRESPONDANCES AND OPEN QUESTIONS

We give here the following table of correspondences, that we comment. We first highlight how the notions on PC matrices correspond to geometric objects.

PC matrices	discretized gauge theories	continuum gauge theories
consistency	0-holonomy	0-curvature
consistency in 3×3 matrices	0-holonomy on the border of a 2-simplex Δ_2	$\Omega(X, Y) = 0$ with X, Y tangent to Δ_2
Koczkodaj's inconsistency indicator ii	$\sup d(Hol(\partial\Delta_2), 1_G)$ when Δ_2 is any 2-simplex of the triangulation	$\sup_M \ \Omega\ $
Minimization of inconsistency	?	Minimization of the curvature norm

Minimization of inconsistency is an important feature for applications of PC matrices. For decision making, it consists in adapting slightly the parameters of the studied situation in order to make "approximately consistent" choices. In order to check if the adapted PC matrix is "approximately consistent", the criteria is given by the chosen inconsistency indicator, e.g. Koczkodaj's ii , which needs to have a value $\epsilon \geq 0$ small enough. Again for Koczkodaj's ii , it is trivial to see that the sets

$$V_\epsilon = ii_3^{-1}[0; \epsilon[$$

form a filter base of open neighborhoods of the set of consistent PC matrices. The analog for continuum gauge theories is considering

$$U_\epsilon = \left\{ \theta \in C(P) \mid \sup_M \|\Omega\| < \epsilon \right\}.$$

This defines a filter base of open neighborhoods of the set of 0-curvature connections. This leads to the following open problem.

On one hand, the procedures developed to get consistencizations of PC matrices are specific to the case $G = \mathbb{R}_+^*$. However, there exists many methods for minimizing functionals, and one may wonder whether minimization of the curvature norm have

a physical meaning. The corresponding physical quantity would be the Yang-Mills action functional, but there we get here an average value instead of a supremum. This questions the choice of the supremum in the formula

$$ii_3(A) = \sup \{ii_3(B) | B \text{ is a } 3 \times 3 \text{ PC submatrix of } A\}.$$

On the other hand, considering flat connections, or only connections such that $\Omega = d\theta$ is a common feature in gauge fixing. Since minimal states represent stable solutions, consistencization may appear as a way to equilibrium.

Let us now reverse the perspective and consider second quantization. We now analyze how the Feynman-like integration (i.e. cylindrical integration for discretized theories) may arise in PC matrices.

Continuum integration	discretized integration	PC matrices
heuristic Lebesgue measure	Lebesgue measure (finite dimension, Whitney discretization)	?
integral to be studied	Product measure on $G^{n \times n}$ (G compact), on the discretization by holonomies	Measures on $n \times n$ PC matrices

If G is compact, we can assume that it is of volume 1. In this case, there is a convergence at the continuum limit to an integral. this si the approach suggested in [17], which is an alternative approach to the classical integration with respect to the heuristic Lebesgue integral, see e.g. [3]. On one hand, for both cases, the possible interpretations in terms of applications of PC matrices are not investigated. Heuristically, measures on PC matrices may arise when evaluations are random, or subject to measurement errors. On the other hand, PC matrices may furnish an interpretation in terms of information theory of Feynman type integration.

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