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ON THE DOMAIN OF IMPLICIT FUNCTIONS IN A PROJECTIVE LIMIT SETTING WITHOUT ADDITIONNAL NORM ESTIMATES

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Abstract. We examine how implicit functions on ILB-Fréchet spaces can be obtained without metric or norm estimates which are classically assumed. We obtain implicit functions defined on a domain D which is not necessarily open, but which contains the unit open ball of a Banach space. The corresponding inverse functions theorem is obtained, and we finish with an open question on the adequate (generalized) notion of differentiation, needed for the corresponding version of the Fröbenius theorem.

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Introduction

Classical inverse functions theorems, implicit functions theorems and Fröbenius theorems on Banach spaces are known to be equivalent. There exists numerous extensions to setting on Fréchet or locally convex spaces, and to our knowledge almost all proofs are based on a contraction principle. In order to obtain in the proofs a mapping which is contracting, one needs to assume conditions which are not automatically fulfilled by a mapping on Fréchet spaces, but which are automatically locally fulfilled (on an open set) by a sufficiently regular mapping on Banach spaces.For classical statements, one can see [1, 3, 4, 5, 6, 12, 13].

We analyze here how a very classical proof of the implicit function theorem can be adapted on a ILB setting, more precisely on a slightly more general framefork, that is when Fréchet spaces E_∞ considered are projective limits of a sequence of Banach spaces (E_i) without assumption of density for teh inclusion $E_{i+1} \subset E_i$, and when the functions f_∞ on Fréchet spaces are restrictions of bounded functions f_i on the sequence of Banach space. This is what one may call order 0 maps, by analogy with the order of differential operators. Then we get (Theorem 1.6) an implicit function which is defined on a domain D, which is not a priori open in the Fréchet topology, but which contains the open ball of a Banach space. This results can be adapted to some functions f_∞ for which there does not exist any extension to a Banach space. These functions have to be controlled by a family of injective maps $\{\Phi_x\}$, which explains the terminology "tame" (Theorem 1.9). As a special case of applications, we recover the maps f_∞ which extend to maps $E_i \rightarrow E_{i-r}$ (called order r maps)

We have to remark that the domain D can be very small. This is the reason why regularity results on implicit functions cannot be stated: differentiability, in a classical sense, requires open domains or at least manifolds. This leads to natural questions for the adequate setting for analysis beyond the Banach setting. Even if not open, following the same motivations as the ones of Kriegl and Michor in [8] when they consider smoothness on non open domains, the domain D may inherit some kind of generalized setting for differential calculus, such as diffeologies [7] which are used in [10]. This question is left open, because out of the scope of this work: the most adapted (generalized) framework for the extension of the regularity (i.e. differentiability) has to be determined.

We then give consequences for an inverse functions theorem, which can be stated with the same restrictions as before on the nature of teh domain D, and with an obstruction to follow the classical proof of the Fröbenius theorem from [13] where differentiation on D is explicitely needed. Finally, in last section, we show how this theorem can describe a Banach Lie gbgroup of a topological group arising in the ILH setting.

1. Implicit functions from Banach spaces to projective limits

Let $(E, ||.||_E)$ anf $(F, ||.||_F)$ be two Banach spaces. The Banach space $E \times F$ is endowed wthh ine norm $||(x,y)||_{E\times F} = \max\{||x||_E, ||y||_F\}$. We note by D_1 and D_2 the (Fréchet) differential with respect to the variables in E and F respectively. Let us first give the statement and a proof of a classical implicit function theorem on Banach spaces, for the sake of extracting key features for generalization. For this, let U be an open neighborhood of O in E, let V be an open neighborhood of 0 in F, and let

$$f : U \times V \quad \rightarrow \quad F \tag{1}$$

be a C^r-function ($r \geq 1$) in the Fréchet sense, such that

$$f(0,0) \quad = \quad 0 \tag{2}$$

and

$$D_2 f(0;0) \quad = \quad Id_F. \tag{3}$$

Theorem 1.1. *There exists a constant $c > 0$ such that, on the open ball $B(0,c) \subset E$, there is an unique map*

$$u : B(0,c) \rightarrow V$$

such that

$$\forall x \in B(0,c), \quad f(x, u(x)) = 0. \tag{4}$$

Let us remark that regularity if the function u is ignored for it is not necessary for next developments. We now divide the main arguments of the classical direct proof of this theorem into three lemmas.

Lemma 1.2. *There exists $c_0 > 0$ and $K > 0$ such that*

$$||(x,y)||_{E\times F} < c_0 \Rightarrow ||D_1 f(x,y)||_{L(E,F)} < K.$$

Proof. Since f is C^1, there exists a neighborhood of $(0,0) \in E \times F$ such that $||D_1 f(x,y)||_{L(E,F)}$ is bounded. □

Lemma 1.3. *There exists $c_1 > 0$ such that*

$$||(x,y)||_{E\times F} < c_1 \quad \Rightarrow \quad ||D_2 f(x,y) - Id_F||_{L(F)} < \frac{1}{2}. \tag{5}$$

Proof. The map f is of class C^r, with $r \geq 1$, so that, the map $D_2 f : (x, y) \in U \times V \mapsto D_2 f(x, y)(.) \in L(F)$ is of class C^{r-1}, and in particular of class C^0. By the way,

$$\exists c_1 > 0,\ ||(x, y)||_{E\times F} < c_1 \Rightarrow ||D_2 f(x, y) - D_2 f(0; 0)||_{L(F)} < \frac{1}{2}.$$

□

Lemma 1.4. *Let c_1 be teh constant of Lemma 1.3. There exists $c_2 > 0$ such that*

$$||x||_E < c_2 \quad\Rightarrow\quad ||f(x, 0)||_F < \frac{c_1}{4}. \tag{6}$$

Proof. The map f is of class C^r, with $r \geq 1$, so that it is in particular of class C^0. By the way, there exists a constant c_2 such that

$$||x - 0||_E < c_2 \Rightarrow ||f(x, 0) - f(0, 0)||_F < \frac{c_1}{4}.$$

□

Lemma 1.5. *Let $c = \min\{c_0, c_1, c_2, 1\}$. Let x such that*

$$||x|| < c.$$

Then the sequence $(u_n)_{\mathbb{N}} \in F^{\mathbb{N}}$, defined by induction by

$$\begin{cases} u_0 = 0 \\ \forall n \in \mathbb{N},\ u_{n+1} = u_n - f(x, u_n) \end{cases} \tag{7}$$

is well-defined and converges to $u(x) \in V$.

Proof. Let us assume x fixed. Let $g(x, y) = y - f(x, y)$. By the way, $u_n = g^n(u_0)$. Applying Lemma 1.3, let $(y, y') \in F^2$ such that both (x, y) and (x, y) are in $B(0, c) \subset B(0, c_1)$.

$$f(x, y) - f(x, y') = \int_0^1 D_2 f(x, ty + (1 - t)y').(y' - y)dt$$

By the way

$$\begin{aligned} ||g(x, y) - g(x, y')||_F &\leq \int_0^1 ||D_2 g(x, ty + (1 - t)y').(y' - y)||_F dt \\ &\leq \int_0^1 ||D_2 f(x, ty + (1 - t)y') - Id_F||_{L(F)}.||y' - y||_F dt \\ &\leq \frac{||y' - y||_F}{2} \end{aligned}$$

By the way, g is $\frac{1}{2}-$Lipschitz. Thus, applying Lemma 1.4, we obtain by induction:

$$||u_1 - u_0||_F < \frac{c_1}{4} \Rightarrow ||u_{n+1} - u_n||_F < \frac{c_1}{2^{n+2}}.$$

and

$$\forall n \in \mathbb{N}, |u_n||_F \leq \frac{c_1(2^n - 1)}{2^{n+2}} < c_1.$$

Hence (u_n) is converging to $u(x)$, which is in V.

□

Proof of Theorem 1.1. By Lemma 1.5 the function $x \mapsto u(x)$ exists for $||x||_E < c$.

□

We now adapt these results to the following setting. Let $(E_i)_{i\in\mathbb{N}}$ and $(F_i)_{i\in\mathbb{N}}$ be two decreasing sequences of Banach spaces, i.e. $\forall i \in \mathbb{N}$, we have $E_{i+1} \subset E_i$ and $F_{i+1} \subset F_i$, with continuous inclusion maps. We then consider U_0 and V_0 two open neighborhoods of 0 in E_0 and F_0 respectively, and a function f_0, of class C^r with the same properties as in equations (1,2,3). Let us now define, for $i \in \mathbb{N}$, $U_i = U_0 \cap E_i$ and $V_i = V_0 \cap V_i$, and let us assume that f_0 restricts to C^r-maps

$$f_i : U_i \times V_i \quad \to \quad F_i. \tag{8}$$

Let $E_\infty = \varprojlim\{E_i;\, i \in \mathbb{N}\}$, let $F_\infty = \varprojlim\{F_i;\, i \in \mathbb{N}\}$ and let $f_\infty = \varprojlim\{F_i;\, i \in \mathbb{N}\}$.

Theorem 1.6. *There exists a non-empty domain $D_\infty \subset U_\infty$, possibily non-open in U_∞, and a function $u_\infty : D_\infty \to V_\infty$ such that,*

$$\forall x \in D_\infty,\ f_\infty(x, u_\infty(x)) = 0,$$

and such that D_∞ contains the unit ball of the Banach space $B_{f_\infty} \subset E_\infty$ defined as the domain of the norm

$$||x||_{f_\infty} = \sup_{i\in\mathbb{N}} \left\{ \frac{||x||_{E_i}}{c_i} \right\}.$$

Proof. Let $i \in \mathbb{N}$. We now consider a maximal domain $D_i \subset U_i$ where there exists an unique function u_i such that

$$\forall x \in D_i,\ f_i(x, u_i(x)) = 0.$$

This domain is non empty since it contains $0 \in E_i$ and, applying Theorem 1.1, there exists a constant $c_i > 0$ such that

$$||x||_{E_i} < c_i \Rightarrow x \in D_i.$$

By the way, any maximal D_i is an open neighborhood of 0. By the way, setting D_∞ as the intersection of such a family (D_i), we get that D_∞ contains $0 \in E_\infty$. Of course, D_∞ is not a priori open in th e projective limit topology. However, let

$$B_{f_\infty} = \left\{ x \in E_\infty \mid \sup_{i\in\mathbb{N}} \left\{ \frac{||x||_{E_i}}{c_i} \right\} < +\infty \right\}.$$

This space is a Banach space for the norm

$$||.||_{f_\infty} = \sup_{i\in\mathbb{N}} \left\{ \frac{||.||_{E_i}}{c_i} \right\}.$$

Since

$$||x||_{f_\infty} < 1 \Leftrightarrow \forall i \in \mathbb{N},\ ||x||_{E_i} < c_i,$$

we get that the open ball of radius 1 in B_{f_∞} is a subset of D_∞, which ends the proof. □

Let us now extend it to a class of functions that we call **tame**. For this, we define the sequences $(E_i)_{i\in\mathbb{N}}$ and $(F_i)_{i\in\mathbb{N}}$ as before, as well as E_∞ and F_∞. We also define a similar sequence $(G_i)_{i\in\mathbb{N}}$ of Banach spaces and G_∞ the projective limit of this family.

Definition 1.7. *Let $U_0 \times V_0$ be an open neighborhood of 0 in $E_0 \times F_0$ and let $U_\infty = U_0 \cap E_\infty$ and $V_\infty = V_0 \cap F_\infty$. Let us fix $\{\Phi_x\}_{x\in U_\infty}$ be a family of injective maps from $G_\infty to F_\infty$. A map*

$$f : U_\infty \times V_\infty \to G_\infty$$

is Φ***-tame*** *if and only if*

$$f_\infty = \Phi_x \circ f : U_\infty \times V_\infty \rightarrow V_\infty$$

extends to C^r-*maps* $(r \geq 1)$

$$f_i : U_i \times V_i \rightarrow V_i$$

Example 1.8. *If there exists a linear isomorphism* $A : E_0 \rightarrow E_1$ *which restricts to isomorphisms* $E_i \rightarrow E_{i+1}$*, setting* $\Phi_x = A^r$*, the family of tame maps are exactly the family of functions* f_∞ *which extend to* C^r-*maps* $E_i \rightarrow E_{i-r}$*.*

Theorem 1.9. *Let* f *be a* $\Phi-$*tame map, such that,* $\forall i \in \mathbb{N}$*,*

$$D_2 f_i(0,0) = Id_{F_i}.$$

Then there exists a non-empty domain $D_\infty \subset U_\infty$*, possibily non-open in* U_∞*, and a function* $u_\infty : D_\infty \rightarrow V_\infty$ *such that,*

$$\forall x \in D_\infty,\ f(x, u_\infty(x)) = 0,$$

Moreover, there exists a sequence of positive real numbers (c_i) *such that* D_∞ *contains the unit ball of the Banach space* $B_{f,\Phi} \subset E_\infty$ *defined as the domain of the norm*

$$||x||_{f,\Phi} = \sup_{i\in\mathbb{N}} \left\{ \frac{||x||_{E_i}}{c_i} \right\}.$$

Proof. We apply Theorem 1.6 to $\Phi \circ f$. Then, since $\forall x, \Phi_x$ is injective, $\Phi_x \circ f(x, .) = 0 \Leftrightarrow f(x, .) = 0.$ □

2. Tentatives for inverse functions and Frobenius theorem

2.1. **"Local" inverse theorem.** Let $(E_i)_{i\in\mathbb{N}}$ be an decreasing sequence of Banach spaces, i.e. $\forall i \in \mathbb{N}$, we have $E_{i+1} \subset E_i$ with continuous inclusion maps and let E_∞ be the projective limit of teh family $(E_i)_{i\in\mathbb{N}}$. Let U_0 be an open neighborhood of 0 in E_0, and define for $i \in \mathbb{N} \cup \{\infty\}$, $U_i = U_0 \cap E_i$. Let V_0 be an open neighborhood of 0 in E_0, and define for $i \in \mathbb{N} \cup \{\infty\}$, $V_i = V_0 \cap E_i$. Let $f_\infty : U_\infty \rightarrow V_\infty$ be a C^r-map $(r \geq 1)$ such that $f(0) = 0$, which extends to C^r-maps $f_i : U_i \rightarrow V_i$ and such that $Df_i(0) = Id_{E_i}$.

Theorem 2.1. *There exists a domain* $D \subset U_\infty$*, which contains the open unit ball of a Banach space* $B_{f_\infty} \subset E_\infty$*, with norm defined by a sequence* (k_i) *of positive numbers by*

$$||.||_{f_\infty} = \sup_{i\in\mathbb{N}} \left\{ \frac{||.||_{E_i}}{k_i} \right\}$$

such that $f_\infty|_D$ *is a bijection* $D \subset U_\infty \rightarrow f_\infty(D) \subset V_\infty$*.*

Proof. We apply Theorem 1.6 to $g(x,y) = x - f_\infty(y)$, for $(x,y) \in V_\infty \times U_\infty$. Indeed, we define a C^r- map $g : V_\infty \times U_\infty \rightarrow E_\infty$ which extends to the maps

$$g_i : (x,y) \in V_i \times U_i \mapsto x - f_i(y) \in E_i.$$

We have that $D_2 g_i(0;0) = Df_i(0) = Id_{E_i}$, so that there exists a domain $D_\infty \subset V_\infty$, and a sequence (c_i) of positive real numbers such that D_∞ contains the unit open ball of the Banach space $B_{g_\infty} \subset E_\infty$ with norm $||.||_{g_\infty} = \sup_{i\in\mathbb{N}} \left\{ \frac{||.||_{E_i}}{c_i} \right\}$ and a function $u_\infty : D_\infty \rightarrow U_\infty$ such that

$$\forall x \in D_\infty,\ x - f_\infty(u_\infty(x)) = 0.$$

We set $D = u_\infty(D_\infty)$. Since each f_i is a C^0-map, there exists a sequence (k_i) of positive numbers such that

$$||x||_{E_i} < k_i \Rightarrow ||f_i(x)||_{E_i} < c_i.$$

By the way, $\forall x \in U_\infty$,

$$\sup_{i\in\mathbb{N}}\left\{\frac{||x||_{E_i}}{k_i}\right\} < 1 \Rightarrow \sup_{i\in\mathbb{N}}\left\{\frac{||f_\infty(x)||_{E_i}}{c_i}\right\} < 1 \Rightarrow f_\infty(x) \in D_\infty \Rightarrow x = u_\infty \circ f_\infty(x) \in D.$$

□

2.2. **An obstruction for a Frobenius theorem.** A setting for an adapted Frobenius theorem would be teh following: *Let*

$$f_i : O_i \to L(E_i, F_i), \quad i \in -\mathbb{N}$$

be a collection of smooth maps satisfying the following condition:

$$i > j \Rightarrow f_j|_{O_i} \textit{ restricts as a linear map to } f_i$$

and such that,

$$\begin{array}{c}\forall (x,y) \in O_i, \forall a, b \in E_i \\ (D_1 f_i(x,y)(a)(b) + (D_2 f_i(x,y))(f_i(x,y)(a))(b) = \\ (D_1 f_i(x,y)(b)(a) + (D_2 f_i(x,y))(f_i(x,y)(b))(a)\end{array}$$

(this condition is the analogous of the Frobenius condition in a Banach setting, that we call the ILH Frobenius condition).

Then, $\forall (x_0, y_0) \in O_\infty$, *there exists Frölicher space* D *that contains* (x_0, y_0) *and a smooth map* $J : D \to \mathbf{F}$ *such that (conditions linked to differentiability of J)*

Let us now try to adapt the classical proof in e.g. [13], with the help of theorem 1.6. We can assume with no restriction that $f(0;0) = 0$. We consider

$$G_i = C^1_b([0,1], F_i) = \{\gamma \in C^1([0,1], F_i) | \gamma(0) = 0\}$$

and

$$H_i = C^0([0,1], F_i),$$

endowed with their usual topologies. Obviously, if $i < j$, the injections $G_j \subset G_i$ and $H_j \subset H_i$ are continuous.

Let us consider B_0 an open ball of E_0 centered in x_0, B'_0 an open ball of F_0 centered in y_0, B''_0 an open ball of G_0 centered in 0. We set $B_i = B_0 \cap E_i$, $B'_i = B'_0 \cap F_i$ and $B''_i = B''_0 \cap G_i$. Then, we define, for $i \in \mathbb{N} \cup \{\infty\}$,

$$g_i : B_i \times B'_i \times B''_i \to H_i$$

$$g(x,y,\gamma)(t) = \mathfrak{d}\gamma dt(t) - f_i(t(x - x_0) + x_0, y + \gamma(t)).(x - x_0).$$

We then apply Theorem 1.6 to $\int_0^{(.)} \circ g_\infty : B_\infty \times B'_\infty \times B''_\infty \subset (E_\infty \times E_\infty) \times G_\infty \to G_\infty$. There exists a domain D_∞ such that we can define the function α_∞ as the unique function such that

$$\left\{\begin{array}{l}\alpha_\infty(x_0) = 0 \\ g_i(x, y, \alpha_\infty(x,y)) = 0, \quad \forall (x,y) \in D_\infty\end{array}\right. .$$

Since we set $J(x,y) = y + \alpha_\infty(x,y)(1)$ Uniqueness follows from Theorem 1.6.

Open problem: We are now facing a theorical impossibility. Classical theory of differentiation is valid for functions on open domains. We need here to consider

$D_1 J$, which is here defined on D_∞ which is not a priori open. There exists numerous extensions of the classical theory of differentiation, one of them is used in [10] based on [7]. Which one is better for this setting?

3. An application of the implicit functions theorem on $\mathcal{L}_\infty$

We consider here a sequence of Banach spaces (E_i) as before, and we asssume also that $\forall i$, E_{i+1} is dense in E_i. Following [2, 12], we consider the set of linear maps $E_\infty \rightarrow E_\infty$ which extend to bounded linear maps $E_i \rightarrow E_i$. Let us note it as $\mathcal{L}_\infty$, and $G\mathcal{L}_\infty = \bigcap_{i\in\mathbb{N}} GL(E_i)$ is a group known as a topological group [2], and [12] quotes "natural differentiation rules" that are identified in [9, 11] as generating a smooth Lie group for generalized differentiation on Frölicher or diffeological spaces. Let $i \in \mathbb{N}$, we define

$$\mathcal{L}_i = \left\{ a \in L(E_i) \,\middle|\, a|_{E_\infty} \in \mathcal{L}_\infty \right\}.$$

We equip these spaces with the norms

$$||a||_i = \max \left\{ ||a||_{L(E_{i-r})} \,\middle|\, 0 \leq r \leq i \right\}.$$

We apply Theorem 1.6 to the map

$$f_\infty : (a,b) \in \mathcal{L}_\infty^2 \mapsto (Id + a)(Id + b) - Id,$$

for the sequence of Banach spaces $(\mathcal{L}_i)$ with projective limit $\mathcal{L}_\infty$. We already know that the maximal domain D_∞ of the implicit function obtained will be

$$D_\infty \supset \left\{ a \in \mathcal{L}_\infty \,\middle|\, Id + a \in G\mathcal{L}_\infty \right\}$$

and the implicit function will be

$$u_\infty : a \in D_\infty \mapsto (Id + a)^{-1} - Id$$

where $(Id + a)^{-1}$ is the left inverse of $Id + a$. But the main question about $G\mathcal{L}_\infty$ is the most adequate structure for it: it behaves like a Lie group [12, 9], but does not carry a priori charts which allows us only to consider it as a topological group [2]. Applying Theorem 1.6, there exists a Banach subspace B of $\mathcal{L}_\infty$ defined by the norm

$$||a|| = \sup_{i \in \mathbb{N}} \left\{ \frac{||a||_i}{c_i} \right\}.$$

But we easily show that each L_i is a Banach algebra, so that, $c_i = 1$ since its group of the units contains te open ball of radius 1 centrered at Id. By the way,

$$||a|| = \sup_{i \in \mathbb{N}} \left\{ ||a||_{L(E_i)} \right\},$$

and B is a Banach algebra, with group of the units $GL(B) \subset G\mathcal{L}_\infty$ which is a Banach Lie group.

We finish with the special case when (E_i) is a ILH sequence (i.e. a sequence of Hilbert spaces with bounded and dense inclusion, see [12]) and when there exists a self-adjoint, positive (unbounded) operator Q such that

$$(Q^i a, b)_{E_0} = (a,b)_{E_i}.$$

In this case, there exists $(e_k)_{k\in\mathbb{N}}$ an orthonormal base in E_0 of eigenvectors of Q in E_∞ with is also orthogonal in E_i. In this case, the orthogonal projections

$$a \mapsto (e_k, a)_{E_0} e_k$$

restrict to operators in B, which shows that B is an infinite dimensional Banach algebra.

Open question: There is a natural right action of $GL(B)$ on $G\mathcal{L}_\infty$ by composition. What is the structure of $G\mathcal{L}_\infty/GL(B)$?

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