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JULIA ROBINSON'S NUMBERS

PIERRE GILLIBERT AND GABRIELE RANIERI

ABSTRACT. We partially answer to a question of Vidaux and Videla by constructing an infinite family of rings of algebraic integers of totally real subfields of $\overline{\mathbb{Q}}$ whose Julia Robinson's Number is distinct from 4 and $+\infty$. Moreover the set of the Julia Robinson's Number that we construct is unbounded.

1. INTRODUCTION

Let R be the ring of the algebraic integers of a subfield of the field \mathbb{Q}^{tr} of all totally real algebraic numbers. For every $t \in \mathbb{R} \cup \{+\infty\}$, denote by R_t the set of $\alpha \in R$ such that, all conjugates β of α satisfy $0 < \beta < t$. Set $A(R) = \{t \in \mathbb{R} \cup \{+\infty\} \mid R_t \text{ is infinite}\}$. Observe that $A(R)$ is either an interval or $\{+\infty\}$. The *Julia Robinson Number* of R is defined as $\text{JR}(R) = \inf(A(R))$. Moreover R has the *Julia Robinson Property* (JRP) if $\text{JR}(R) \in A(R)$.

Robinson, in [20], generalizing her methods of [21], proves that if a ring R has the JRP, then the semi-ring $(\mathbb{N}; 0, 1, +, \cdot)$ is first-order definable in R (hence the arithmetic of R is as complicated as that of \mathbb{N} , and in particular R has an undecidable first order theory). There is a lot of examples of rings with the JRP.

If K is a totally real algebraic number field, then $\text{JR}(\mathcal{O}_K) = +\infty$ (see [20]). In particular, the first order theory of \mathcal{O}_K is undecidable. Note that Denef and Lipshitz prove a stronger result in [3, 4, 5]: if K is a totally real number field, or an extension of degree 2 of a totally real number field then \mathbb{N} is Diophantine in \mathcal{O}_K . It follows from the Matiyasevich's solution to Hilbert 10th problem [13], that the existential positive first order theory of \mathcal{O}_K is undecidable. The result is even further extended by Shlapentokh in a serie of papers [24, 25, 26, 27, 28, 29] to large subrings of K (see also Poonen [17], Poonen and Shlapentokh [18]). Mazur and Rubin prove in [11] a similar result for the algebraic integer rings of an arbitrary number fields, assuming that the Tate-Shafarevich conjecture holds. See Shlapentokh [30, 31, 32] for more results. See also Matiyasevich [12] for more on the Hilbert's 10th problem, and Moret-Bailly [14, 15] for similar problems in function fields.

On the other hands for infinite algebraic extension of \mathbb{Q} far less is known. The Julia Robinson Property seems particularly useful. Vidaux and Videla observe in [34] that if a totally real algebraic field K has the Northcott property, then $\text{JR}(\mathcal{O}_K) = +\infty$. Using the result of Bombieri and Zannier in [2], Vidaux and Videla deduce that for every positive integer d , the field generated by all totally real abelian extensions of \mathbb{Q} of degree d has its ring of algebraic integers with JR equals to $+\infty$. In particular the ring has undecidable first order theory.

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Denote by \mathbb{Z}^{tr} the ring of all totally real algebraic integers. As noted by Robinson in [20], it follows from Schur [23] or Kronecker [10] that $\text{JR}(\mathbb{Z}^{\text{tr}}) = 4$, and that \mathbb{Z}^{tr} has the JRP (also see Robinson [22] or Jarden and Videla [9]). In particular the first order theory of \mathbb{Z}^{tr} is undecidable (see [20]). In [33, Introduction], Vidaux and Videla, starting from a remark of Robinson, ask the following question

Question: Can any algebraic number (or even real number) be realized as the JR number of a ring of integers in \mathbb{Q}^{tr} ?

Then, they find a family of rings \mathcal{O} contained in \mathbb{Z}^{tr} , such that $\text{JR}(\mathcal{O})$ is distinct from 4 and $+\infty$ (see [33, Theorem 1.4] for the details). On the other hand, they do not know if at least an element of the family is the ring of the algebraic integers of its quotient field. The paper [33] is also interesting because it contains a very precise bibliography on undecidability problem over algebraic rings. Finally, very recently, Vidaux and Videla (see [34]) discovered an interesting relation between the finiteness of the Julia Robinson Number and the Northcott Property. Starting from this, Widmer defined the Northcott Number and gave interesting examples of families of algebraic rings with bounded Northcott Numbers (see [36] for the details).

In this paper we find infinitely many real numbers $t > 4$ such that t is the Julia Robinson Number of a ring of the algebraic integers of a subfield of \mathbb{Q}^{tr} . This yields the first example of ring of the algebraic integers of a totally real field with JR distinct from 4 and $+\infty$. More precisely, we prove the following result.

Theorem 8.8. *Let $t \geq 1$ be a square-free odd number. Then the following statement holds.*

- (1) *There are infinitely many fields K such that \mathcal{O}_K has Julia Robinson's number $\lceil 2\sqrt{2t} \rceil + 2\sqrt{2t}$.*
- (2) *There are infinitely many fields K such that \mathcal{O}_K has Julia Robinson's number $8t$.*

To prove Theorem 8.8, we choose $0 < a < b$ coprime integers satisfying certain conditions (see for instance Section 7) and we set $\alpha = \frac{a+i\sqrt{b^2-a^2}}{b}$. Then α is a quadratic number with absolute value equal to 1. We recall that a number field L is *CM* if it is a quadratic imaginary extension of a totally real field (or equivalently L is not contained in \mathbb{R} and there exists ι in $\text{Aut}(L/\mathbb{Q})$ such that for every embedding of L over \mathbb{C} , ι is the complex conjugation, see for instance [35, Chapter 4]). Amoroso and Nuccio [1, Proposition 2.3] prove that a field is *CM* if and only if it is generated over \mathbb{Q} by an element whose all conjugates have absolute value equal to 1. Then for every positive integer n , the field generated by an n th root of α is a *CM* field, and so it contains a totally real field of index 2. We choose a positive integer r such that, for every prime number dividing b , $v_p(b)$ is coprime with r and, for every positive integer k , we set K_{r^k} the maximal totally real field contained in the field generated by a r^k -th root of α . Let K be the union of all K_{r^k} and \mathcal{O}_K its ring of the algebraic integers. By characterizing the algebraic integers of K_{r^k} for every k , we get a very precise description of the algebraic integers of K and we compute the JR Number of K (see Theorem 7.10). Moreover \mathcal{O}_K has the JRP and so the first order theory of \mathcal{O}_K is undecidable (see Corollary 7.11).

In Section 8 we use some results of analytic number theory to get the explicit examples of Julia Robinson's number given in Theorem 8.8.

2. NOTATIONS

Let F be an algebraic extension of \mathbb{Q} , we denote by \mathcal{O}_F the ring of all algebraic integers of F . Assume that F is a number field. Let p be a prime number, β be in F^* , and P be a prime ideal of \mathcal{O}_F over p , let e be its ramification index over \mathbb{Q} and $\lambda \in \mathbb{Z}$ be the exponent of P in the factorization of the fractional ideal (β) . We define the P -adic valuation of β as $\frac{\lambda}{e}$. Then in particular, the P -adic valuation of β should not be an integer. We always denote by v_p the p -adic valuation and we always say that the P -adic valuation extends the p -adic valuation.

For every real number γ , we set $\lceil \gamma \rceil$ the smallest integer $\geq \gamma$, and $\lfloor \gamma \rfloor$ the largest integer $\leq \gamma$.

3. SOME PRELIMINARIES

Given an integer $b \geq 1$, and given $p_1^{n_1} p_2^{n_2} \dots p_s^{n_s}$ the prime decomposition of b , and $u \geq 0$ in \mathbb{Q} , we denote

$$r(b, u) = p_1^{\lceil n_1 u \rceil} p_2^{\lceil n_2 u \rceil} \dots p_s^{\lceil n_s u \rceil}.$$

The next Lemma gives some basic properties of $r(b, u)$.

Lemma 3.1. *Let $b \geq 1$ be an integer, let $u \geq 0$ be in \mathbb{Q} . Then $r(b, u)$ is an integer, and $r(b, u) b^{-u}$ is an algebraic integer. Also note that $r(b, 0) = 1$ and $r(b, 1) = b$.*

Let K be a number field and let α be an element of K . Choose an n th root of α and denote it by $\alpha^{\frac{1}{n}}$. We need to study some properties of a linear combination with coefficients in K of some powers of $\alpha^{\frac{1}{n}}$. We do this in the following Lemmas.

Lemma 3.2. *Let $n \geq 1$ be an integer, let t be an integer relatively prime to n , $-n + 1 \leq k_1, k_2 \leq n - 1$, and ℓ_1, ℓ_2 in \mathbb{Z} . Assume that $\ell_1 + \frac{k_1 t}{n} = \ell_2 + \frac{k_2 t}{n}$. Then one of the following holds:*

- (1) $k_1 = k_2$.
- (2) $k_1 < 0 < k_2$ and $k_2 = k_1 + n$.
- (3) $k_2 < 0 < k_1$ and $k_1 = k_2 + n$.

Proof. Assume that $\ell_1 + \frac{k_1 t}{n} = \ell_2 + \frac{k_2 t}{n}$. Hence $n\ell_1 + k_1 t = n\ell_2 + k_2 t$. Thus $(k_1 - k_2)t$ is a multiple of n . Then it follows, as t is relatively prime to n , that $k_1 - k_2$ is a multiple of n . Since $-2n + 2 \leq k_1 - k_2 \leq 2n - 2$, we have that $k_1 - k_2 \in \{-n, 0, n\}$.

If $k_1 - k_2 = 0$ then $k_1 = k_2$.

If $k_1 - k_2 = n$, then $k_1 = n + k_2$. Moreover, we have $k_1 = n + k_2 \geq n - n + 1 > 0$ and $k_2 = k_1 - n \leq n - 1 - n < 0$.

If $k_1 - k_2 = -n$, then $k_2 = n + k_1$ and $k_1 = -n + k_2 \leq -n + n - 1 < 0$. Similarly $k_2 = k_1 + n \geq n - 1 - n < 0$. \square

Lemma 3.3. *Let K be a number field, n be a natural number, p be a prime number, α be in K . Let $\nu_{1,p}$, and $\nu_{2,p}$ be valuations over K extending v_p to K , let $(\lambda_k \mid -n + 1 \leq k \leq n - 1)$ be a family in K . Assume that the following conditions hold:*

- (1) $\nu_{1,p}(\alpha)$ is a positive integer relatively prime to n .
- (2) $\nu_{2,p}(\alpha)$ is a negative integer relatively prime to n .
- (3)

$$\gamma = \sum_{k=-n+1}^{n-1} \lambda_k \alpha^{\frac{k}{n}} \quad \text{is an algebraic integer.}$$

(4) For all $-n + 1 \leq k \leq n - 1$, we have $\nu_{1,p}(\lambda_k) = \nu_{2,p}(\lambda_k)$ is an integer.

Then for all $-n + 1 \leq k \leq n - 1$, we have $\nu_{1,p}(\lambda_k) \geq \left\lceil \frac{-k\nu_{2,p}(\alpha)}{n} \right\rceil$, $\nu_{1,p}(\lambda_k) \geq \left\lceil \frac{-k\nu_{1,p}(\alpha)}{n} \right\rceil$, and $\nu_{1,p}(\lambda_0) \geq 0$.

Proof. By conditions (1) and (2), $K(\alpha^{\frac{1}{n}})/K$ is totally ramified over $\nu_{1,p}$ and $\nu_{2,p}$. Then, $\nu_{1,p}$ and $\nu_{2,p}$ extend in a unique way to $K(\alpha^{\frac{1}{n}})$. Denote $I = \{k \in \mathbb{Z} \mid -n + 1 \leq k \leq n - 1\}$. As γ is an algebraic integer, it follows that $\nu_{1,p}(\gamma) \geq 0$, and $\nu_{2,p}(\gamma) \geq 0$. Given $k \in I$, the following equalities hold:

$$\nu_{1,p}(\lambda_k \alpha^{\frac{k}{n}}) = \nu_{1,p}(\lambda_k) + \frac{k\nu_{1,p}(\alpha)}{n}.$$

$$\nu_{2,p}(\lambda_k \alpha^{\frac{k}{n}}) = \nu_{2,p}(\lambda_k) + \frac{k\nu_{2,p}(\alpha)}{n} = \nu_{1,p}(\lambda_k) + \frac{k\nu_{2,p}(\alpha)}{n}.$$

Denote

$$S_1 = \left\{ \nu_{1,p}(\lambda_k) + \frac{k\nu_{1,p}(\alpha)}{n} \mid k \in I \right\} = \left\{ \nu_{1,p}(\lambda_k \alpha^{\frac{k}{n}}) \mid k \in I \right\}$$

$$S_2 = \left\{ \nu_{1,p}(\lambda_k) + \frac{k\nu_{2,p}(\alpha)}{n} \mid k \in I \right\} = \left\{ \nu_{2,p}(\lambda_k \alpha^{\frac{k}{n}}) \mid k \in I \right\}.$$

Let u be the smallest element of $S_1 \cup S_2$. Assume that $u < 0$ and that $u \in S_1$. Fix k_1 such that $u = \nu_{1,p}(\lambda_{k_1}) + \frac{k_1\nu_{1,p}(\alpha)}{n}$.

Assume that for all $k \neq k_1$, we have $u < \nu_{1,p}(\lambda_k \alpha^{\frac{k}{n}})$, then $\nu_{1,p}(\gamma) = u < 0$; a contradiction. Therefore, there is $k_2 \in I$ such that $k_2 \neq k_1$ and the following equalities hold:

$$\nu_{1,p}(\lambda_{k_1}) + \frac{k_1\nu_{1,p}(\alpha)}{n} = u = \nu_{2,p}(\lambda_{k_2} \alpha^{\frac{k_2}{n}}) = \nu_{1,p}(\lambda_{k_2}) + \frac{k_2\nu_{1,p}(\alpha)}{n}.$$

Up to an exchange of k_1 and k_2 , we can assume that $k_1 < k_2$, hence it follows from Lemma 3.2 that $k_1 < 0 < k_2$ and $k_2 = k_1 + n$. As $k_2 > 0$ and $\nu_{1,p}(\alpha) > 0 > \nu_{2,p}(\alpha)$, it follows that

$$\nu_{1,p}(\lambda_{k_2}) + \frac{k_2\nu_{2,p}(\alpha)}{n} < \nu_{1,p}(\lambda_{k_2}) < \nu_{1,p}(\lambda_{k_2}) + \frac{k_2\nu_{1,p}(\alpha)}{n} = u.$$

Hence u is not the smallest element of $S_1 \cup S_2$; a contradiction. With a similar argument if $u \in S_2$, we can find another contradiction. It follows that $u \geq 0$.

For all $k \in I$, we have $\nu_{1,p}(\lambda_k) + \frac{k\nu_{1,p}(\alpha)}{n} \geq u \geq 0$. Hence $\nu_{1,p}(\lambda_k) \geq -\frac{k\nu_{1,p}(\alpha)}{n}$, therefore, as $\nu_{1,p}(\lambda_k)$ is an integer it follows that $\nu_{1,p}(\lambda_k) \geq \left\lceil \frac{-k\nu_{1,p}(\alpha)}{n} \right\rceil$. A similar argument yields $\nu_{1,p}(\lambda_k) = \nu_{2,p}(\lambda_k) \geq \left\lceil \frac{-k\nu_{2,p}(\alpha)}{n} \right\rceil$. \square

Lemma 3.4. *Let K be a number field, $n \in \mathbb{N}$, p be a prime number, $\alpha \in K$. Let ν_p be a valuation over K extending v_p to K . Assume that the following conditions hold:*

- (1) $\nu_p(\alpha)$ is an integer relatively prime to n .
- (2) For all $\lambda \in K^*$, $\nu_p(\lambda)$ is an integer.

Then $[K(\alpha^{\frac{1}{n}}) : K] = n$.

Proof. We extend ν_p to $K(\alpha^{\frac{1}{n}})$. By abuse of notation, we still denote ν_p the extension. Let $(\lambda_k \mid 0 \leq k \leq n-1)$ be a family of K such that: $\sum_{k=0}^n \lambda_k \alpha^{\frac{k}{n}} = 0$.

Note that, for all $0 \leq k \leq n$, we have $\nu_p(\lambda_k \alpha^{\frac{k}{n}}) = \nu_{1,p}(\lambda_k) + \frac{k}{n} \nu_p(\alpha)$.

Set $I = \{k \in \mathbb{N} \mid k \leq n-1 \text{ and } \lambda_k \neq 0\}$. It follows from Lemma 3.2 that for all $k \neq \ell$ in I we have $\nu_p(\lambda_k) + \frac{k}{n} \nu_p(\alpha) \neq \nu_p(\lambda_\ell) + \frac{\ell}{n} \nu_p(\alpha)$. Therefore, if I is not empty, $\nu_p(\sum_{k=0}^n \lambda_k \alpha^{\frac{k}{n}})$ is the minimum of $\nu_p(\lambda_k) + \frac{k}{n} \nu_p(\alpha)$ for $k \in I$. Therefore $\nu_p(0) \neq \infty$; a contradiction. \square

Lemma 3.5. *Let K be a field, let α be in K , let n be an integer ≥ 2 . Assume that $[K(\alpha^{\frac{1}{n}}) : K] = n$. Let $k \in \mathbb{Z}$ be such that k is not a multiple of n . Then $\text{Tr}_{K(\alpha^{\frac{1}{n}})/K}(\alpha^{\frac{k}{n}}) = 0$.*

Proof. Set $d = (k, n)$ and let u, v in \mathbb{Z} be such that $ku + nv = d$. Since $[K(\alpha^{\frac{1}{n}}) : K] = n$, we have that $X^{\frac{n}{d}} - \alpha^{\frac{k}{d}}$ is the minimal polynomial of $\alpha^{\frac{k}{d}}$ over K . It follows that $\text{Tr}_{K(\alpha^{\frac{1}{n}})/K}(\alpha^{\frac{k}{n}}) = 0$. \square

In the sequel, we are interested in the case when K is a quadratic imaginary field and $\alpha \in K$ such that α has absolute value equal to 1. In the following Lemma we apply some of the previous results in this particular case.

Lemma 3.6. *Let a, b be relatively prime in \mathbb{N} , with $a < b$. Let α be equal to $\frac{a+i\sqrt{b^2-a^2}}{b}$. Let $n \geq 1$ be an integer. Assume that $\alpha^{\frac{1}{n}}$ generates an extension of degree n over $\mathbb{Q}(i\sqrt{b^2-a^2})$. Let $(\lambda_k \mid -n+1 \leq k \leq n-1)$ be a family in \mathbb{Q} . Assume that*

$$\sum_{k=-n+1}^{n-1} \lambda_k \alpha^{\frac{k}{n}} \quad \text{is an algebraic integer.}$$

Then, for all $1 \leq k \leq n-1$, both $nb\lambda_{-k} + n\lambda_{n-k}(a + i\sqrt{b^2-a^2})$ and $nb\lambda_k + n\lambda_{k-n}(a - i\sqrt{b^2-a^2})$ are algebraic integers, and $n\lambda_0$ is an integer.

Proof. Set $\gamma = \sum_{k=-n+1}^{n-1} \lambda_k \alpha^{\frac{k}{n}}$. Set $K = \mathbb{Q}(i\sqrt{b^2-a^2})$ and set $L = K(\alpha^{\frac{1}{n}})$. It follows from Lemma 3.5 that

$$\text{Tr}_{L/K}(\alpha^{\frac{k}{n}}) = 0, \quad \text{for all } k \in \mathbb{Z} \text{ which is not a multiple of } n. \quad (3.1)$$

Therefore $\text{Tr}_{L/K}(\gamma) = \text{Tr}_{L/K}(\lambda_0) = n\lambda_0$. However, γ is an algebraic integer, therefore $n\lambda_0$ is an algebraic integer. As $\lambda_0 \in \mathbb{Q}$, it follows that $n\lambda_0$ is an integer.

Let $1 \leq k \leq n-1$ be an integer. The following equalities hold:

$$(b\alpha^{\frac{k}{n}})^n = b^n \alpha^k = b^n \left(\frac{a + i\sqrt{b^2-a^2}}{b} \right)^k = b^{n-k} (a + i\sqrt{b^2-a^2})^k.$$

Therefore $b\alpha^{\frac{k}{n}}$ is an algebraic integer. The following equalities hold

$$\begin{aligned} \text{Tr}_{L/K}(b\alpha^{\frac{k}{n}}\gamma) &= \text{Tr}_{L/K} \left(\sum_{\ell=-n+1}^{n-1} b\lambda_\ell \alpha^{\frac{\ell+k}{n}} \right) \\ &= \text{Tr}_{L/K}(b\lambda_{-k}) + \text{Tr}_{L/K}(b\lambda_{n-k}\alpha), \quad \text{by (3.1).} \\ &= nb\lambda_{-k} + nb\lambda_{n-k}\alpha, \quad \text{as } b\lambda_{-k} \text{ and } b\lambda_{n-k}\alpha \text{ are in } K. \\ &= nb\lambda_{-k} + n\lambda_{n-k}(a + i\sqrt{b^2-a^2}). \end{aligned}$$

However, $b\alpha^{\frac{k}{n}}\gamma$ is an algebraic integer, therefore $nb\lambda_{-k} + n\lambda_{n-k}(a + i\sqrt{b^2 - a^2})$ is an algebraic integer.

Similarly,

$$(b\alpha^{\frac{-k}{n}})^n = b^n\alpha^{-k} = b^n \left(\frac{a + i\sqrt{b^2 - a^2}}{b} \right)^{-k} = b^n \left(\frac{a - i\sqrt{b^2 - a^2}}{b} \right)^k = b^{n-k}(a - i\sqrt{b^2 - a^2})^k.$$

Hence $b\alpha^{\frac{-k}{n}}$ is an algebraic integer. The following equalities hold

$$\begin{aligned} \mathrm{Tr}_{L/K}(b\alpha^{\frac{-k}{n}}\gamma) &= \mathrm{Tr}_{L/K} \left(\sum_{\ell=-n+1}^{n-1} b\lambda_\ell \alpha^{\frac{\ell-k}{n}} \right) \\ &= \mathrm{Tr}_{L/K}(b\lambda_k) + \mathrm{Tr}_{L/K}(b\lambda_{k-n}\alpha^{-1}), \quad \text{by (3.1).} \\ &= nb\lambda_k + nb\lambda_{k-n}\alpha^{-1}, \quad \text{as } b\lambda_k \text{ and } b\lambda_{k-n}\alpha^{-1} \text{ are in } K. \\ &= nb\lambda_k + n\lambda_{k-n}(a - i\sqrt{b^2 - a^2}). \end{aligned}$$

However, $b\alpha^{\frac{-k}{n}}\gamma$ is an algebraic integer, therefore $nb\lambda_k + n\lambda_{k-n}(a - i\sqrt{b^2 - a^2})$ is an algebraic integer. \square

Let K and α be as in the previous Lemma. Since α has absolute value equal to 1, there exists $\theta \in \mathbb{R}$ such that $\alpha = \cos(\theta) + i\sin(\theta)$. In various cases, we shall need some basic trigonometric formulas, as Euler's formula, from which it follows the following result.

Lemma 3.7. *Let λ, μ, x and y in \mathbb{R} . Then:*

$$\lambda \cos(x) + \mu \cos(y) = (\lambda + \mu) \cos\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right) + (\mu - \lambda) \sin\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right)$$

Similarly the following lemma is a consequence of Euler's formula.

Lemma 3.8. *For all $\lambda, \mu, \theta, n \neq 0$, and k in \mathbb{R} the following equality holds.*

$$\begin{aligned} \left(\lambda \cos \frac{k\theta}{n} + \mu \cos \frac{(n-k)\theta}{n} \right)^2 &= \frac{1}{2} \left(\lambda^2 + \mu^2 + 2\lambda\mu \cos \theta + \right. \\ &\quad \left. + (\lambda^2 - \mu^2) \cos \frac{2k\theta}{n} + 2(\lambda\mu + \mu^2 \cos \theta) \cos \frac{(n-2k)\theta}{n} \right) \end{aligned}$$

Lemma 3.9. *Let $0 < a < b$ be relatively prime integers, set $\alpha = \frac{a+i\sqrt{b^2-a^2}}{b}$. Assume that b is even. Let $q \geq 0$ be in \mathbb{Q} . Then $r\left(\frac{b}{2}, q\right)\alpha^q$ is an algebraic integer.*

Proof. As b is even, it follows that $\frac{a+i\sqrt{b^2-a^2}}{2}$ is an algebraic integer. The following equalities hold

$$\begin{aligned} r\left(\frac{b}{2}, q\right)\alpha^q &= r\left(\frac{b}{2}, q\right) \left(\frac{a + i\sqrt{b^2 - a^2}}{b} \right)^q \\ &= r\left(\frac{2}{b}, q\right) \left(\frac{2}{b} \right)^{-q} \left(\frac{a + i\sqrt{b^2 - a^2}}{2} \right)^q. \end{aligned}$$

From Lemma 3.1 we have $r\left(\frac{2}{b}, q\right) \left(\frac{2}{b}\right)^{-q}$ is an algebraic integer, moreover $\frac{a+i\sqrt{b^2-a^2}}{2}$ is an algebraic integer. Therefore $r\left(\frac{b}{2}, q\right)\alpha^q$ is an algebraic integer. \square

Lemma 3.10. *Let $0 < a < b$ be relatively prime integers, set $\theta = \arccos(\frac{a}{b}) \in]0, \frac{\pi}{2}[$. Let $n \geq 2$ be an integer and $1 \leq k \leq n - 1$. Let $-\frac{1}{2} \leq q \leq \frac{1}{2}$ be in \mathbb{Q} . Then $2\sqrt{b} \cos(q\theta)$ and $2\sqrt{b} \sin(q\theta)$ are algebraic integers.*

Proof. We can assume that $q = \frac{u}{v}$ with u, v positive integers. The following equalities hold:

$$(\sqrt{b}e^{i\frac{u}{v}\theta})^v = b^{\frac{v}{2}}e^{iu\theta} = b^{\frac{v}{2}-u}(be^{i\theta})^u = b^{\frac{v}{2}-u} \left(a + i\sqrt{b^2 - a^2} \right)^u.$$

As $\frac{v}{2} \geq u$ it follows that $b^{\frac{v}{2}-u}$ is an algebraic integer, moreover $(a + i\sqrt{b^2 - a^2})^u$ is an algebraic integer. Therefore $\sqrt{b}e^{i\frac{u}{v}\theta}$ is an algebraic integer.

Similarly

$$(\sqrt{b}e^{-i\frac{u}{v}\theta})^v = b^{\frac{v}{2}}e^{-iu\theta} = b^{\frac{v}{2}-u}(be^{-i\theta})^u = b^{\frac{v}{2}-u} \left(a - i\sqrt{b^2 - a^2} \right)^u.$$

Hence $\sqrt{b}e^{-i\frac{u}{v}\theta}$ is algebraic integer. Therefore $2\sqrt{b} \cos(q\theta) = \sqrt{b}(e^{iq\theta} + e^{-iq\theta})$ is an algebraic integer. Similarly $2\sqrt{b} \sin(q\theta)$ is an algebraic integer. \square

Lemma 3.11. *Let $0 < a < b$ be relatively prime integers, let $\theta = \arccos(\frac{a}{b}) \in]0, \frac{\pi}{2}[$. Let $n \geq 2$ be an integer and $1 \leq k \leq n - 1$. Let λ and μ in \mathbb{Q} be such that $(\lambda + \mu)\sqrt{b+a}$ and $(\mu - \lambda)\sqrt{b+a}$ are algebraic integers. Then $4b(\lambda \cos(\frac{k\theta}{n}) + \mu \cos(\frac{(n-k)\theta}{n}))$ is an algebraic integer.*

Proof. Set:

$$\gamma = \lambda \cos\left(\frac{k\theta}{n}\right) + \mu \cos\left(\frac{(n-k)\theta}{n}\right).$$

It follows from Lemma 3.7 that

$$\gamma = (\lambda + \mu) \cos\left(\frac{\theta}{2}\right) \cos\left(\frac{(2k-n)\theta}{2n}\right) + (\mu - \lambda) \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{(2k-n)\theta}{2n}\right).$$

Note that, since $\theta = \arccos(\frac{a}{b}) \in]0, \frac{\pi}{2}[$,

$$\cos\left(\frac{\theta}{2}\right) = \sqrt{\frac{1}{2}(1 + \cos(\theta))} = \sqrt{\frac{1}{2}\left(1 + \frac{a}{b}\right)} = \sqrt{\frac{b+a}{2b}}.$$

Similarly

$$\sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{b-a}{2b}}.$$

Therefore

$$\begin{aligned} 2\sqrt{2}b\gamma &= 2\sqrt{2}b \left((\lambda + \mu) \sqrt{\frac{b+a}{2b}} \cos\left(\frac{(2k-n)\theta}{2n}\right) + (\mu - \lambda) \sqrt{\frac{b-a}{2b}} \sin\left(\frac{(2k-n)\theta}{2n}\right) \right) \\ &= (\lambda + \mu) \sqrt{b+a} 2\sqrt{b} \cos\left(\frac{(2k-n)\theta}{2n}\right) + (\mu - \lambda) \sqrt{b-a} 2\sqrt{b} \sin\left(\frac{(2k-n)\theta}{2n}\right). \end{aligned}$$

Note that $-\frac{1}{2} \leq \frac{(2k-n)\theta}{2n} \leq \frac{1}{2}$. It follows, from Lemma 3.10, that $2\sqrt{b} \cos\left(\frac{(2k-n)\theta}{2n}\right)$ and $2\sqrt{b} \sin\left(\frac{(2k-n)\theta}{2n}\right)$ are algebraic integers. Moreover, we assumed that $(\lambda + \mu)\sqrt{b+a}$ and $(\mu - \lambda)\sqrt{b-a}$ are algebraic integers, therefore $4b\gamma$ is an algebraic integer. \square

Lemma 3.12. *Let $0 < a < b$ be coprime integers. Set $\alpha = \frac{a+i\sqrt{b^2-a^2}}{b}$. Let p be a prime number dividing b and, if $p = 2$ suppose that 4 divides b . Then (p) splits in $\mathbb{Q}(\alpha)$. Let P_1 and P_2 be the prime ideals over p and $\nu_{1,p}$, respectively $\nu_{2,p}$ the associated valuations. Then, if p is odd $\nu_{1,p}(\alpha) = -\nu_{2,p}(\alpha) = v_p(b)$, where, as before, v_p is the p -adic valuation. On the other hand, if $p = 2$, $\nu_{1,2}(\alpha) = -\nu_{2,2}(\alpha) = v_2(b) - 1$.*

Proof. First observe that $\alpha\bar{\alpha} = 1$. Let R be the ring of the algebraic integers of $\mathbb{Q}(\alpha)$. Suppose first that p is odd. Since p does not divide $b^2 - a^2$, (p) is not ramified in R . Suppose that (p) divides $a + i\sqrt{b^2 - a^2}$ in R . Then, since (p) is fixed by the complex conjugation, p divides b and $\alpha\bar{\alpha} = 1$, we should have p divides $a + i\sqrt{b^2 - a^2}$ and $a - i\sqrt{b^2 - a^2}$ in R . In this case, (p) should divide $2a$ and this is not possible because p is odd and p does not divide a . This proves that $pR = P_1P_2$, with P_1 and P_2 distinct prime ideals of R and that, for every i, j in $\{1, 2\}$, $i \neq j$, if P_i divides $a + i\sqrt{b^2 - a^2}$, then P_j divides $a - i\sqrt{b^2 - a^2}$. Suppose that P_1 divides $a + i\sqrt{b^2 - a^2}$ and P_2 divides $a - i\sqrt{b^2 - a^2}$ and let r be the exponent of p in the prime factorization of b . Then, since $\alpha\bar{\alpha} = 1$, by the unique prime decomposition of fractional ideal of R with integer exponents, we get that the fractional ideal (α) generated by α can be written as

$$(\alpha) = P_1^{2r}(p)^{-r}I = P_1^rP_2^{-r}I, \quad (3.2)$$

with I fractional ideal coprime with (p) . In fact if t is the exponent of P_1 in the factorization of $a + i\sqrt{b^2 - a^2}$, since $P_2 = \bar{P}_1$ we have that t is the exponent of P_2 in the factorization of $a - i\sqrt{b^2 - a^2}$. Moreover, we have $(a + i\sqrt{b^2 - a^2})(a - i\sqrt{b^2 - a^2}) = b^2$. Since p^{2r} divides b^2 and p^{2r+1} does not divide b^2 , we have $t = 2r$, proving (3.2).

Let $\nu_{1,p}$ be respectively $\nu_{2,p}$ the valuations associated to P_1 respectively P_2 . Thus by (3.2), $r = \nu_{1,p}(\alpha) = -\nu_{2,p}(\alpha) = v_p(b)$.

Suppose now that $p = 2$. Since 4 divides b and a is coprime with b , $a^2 - b^2 \equiv 1 \pmod{8}$. Then by the reciprocity quadratic law (2) splits in the ring of the algebraic integers of $\mathbb{Q}(\alpha)$. Observe that $\frac{a+i\sqrt{b^2-a^2}}{2}$ is a root of the polynomial $x^2 - ax + b^2$ with integer coefficients. Then (2) divides $a+i\sqrt{b^2-a^2}$ and $a-i\sqrt{b^2-a^2}$. Moreover (4) does not divide $a+i\sqrt{b^2-a^2}$ and $a-i\sqrt{b^2-a^2}$ because $\frac{a+i\sqrt{b^2-a^2}}{4}$ has the trace equals to $\frac{a}{2}$, which is not an integer. Write $(2) = P_1P_2$ with P_1 and P_2 prime ideals of the ring of the algebraic integers of $\mathbb{Q}(\alpha)$. Since $\alpha\bar{\alpha} = 1$, (2) divides $a+i\sqrt{b^2-a^2}$ and (4) does not divide $a+i\sqrt{b^2-a^2}$, we get that there exists a positive integer c and ideal I_1, I_2 coprime with (2) such that

$$(\alpha) = (2)P_1^{2c}I_1(b)^{-1} = (2)P_1^{2c}I_1(2)^{-c-1}I_2^{-1} = P_1^cI_1P_2^{-c}I_2^{-1}. \quad (3.3)$$

Let $\nu_{1,2}$ be respectively $\nu_{2,2}$ the valuations associated to P_1 respectively P_2 . Thus by (3.3), $\nu_{1,2}(\alpha) = c = -\nu_{2,2}(\alpha)$ and $v_2(b) = c + 1 = \nu_{1,2}(\alpha) + 1$. \square

The following Corollary immediately follows from Lemmas 3.4, Lemma 3.5, and Lemma 3.12.

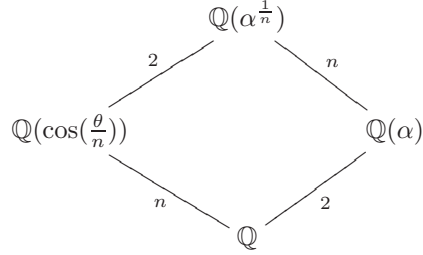
Corollary 3.13. *Let $0 < a < b$ be coprime integers. Set $\alpha = \frac{a+i\sqrt{b^2-a^2}}{b}$. Let p be a prime number dividing b . Let $n > 0$ be an integer. If p is odd we assume that $v_p(b)$ is coprime to n . If p is even we assume that 4 divides b , and that $v_2(b) - 1$*

is coprime to n . Then $[\mathbb{Q}(\alpha^{\frac{1}{n}}) : \mathbb{Q}(\alpha)] = n$, and for all integers k not multiple of n we have $\text{Tr}_{\mathbb{Q}(\alpha^{\frac{1}{n}})/\mathbb{Q}(\alpha)}(\alpha^{\frac{k}{n}}) = 0$.

Lemma 3.14. *With the hypothesis of Corollary 3.13, let $\theta = \arccos(\frac{a}{b}) \in]0, \frac{\pi}{2}[$. Then $1, \cos \frac{\theta}{n}, \cos \frac{2\theta}{n}, \dots, \cos \frac{(n-1)\theta}{n}$ is a \mathbb{Q} -basis of $\mathbb{Q}(\cos \frac{\theta}{n})$.*

Moreover, $\text{Tr}_{\mathbb{Q}(\cos \frac{\theta}{n})/\mathbb{Q}}(\cos \frac{k\theta}{n}) = 0$ for all $1 \leq k \leq n-1$.

Proof. By Corollary 3.13, $[\mathbb{Q}(\alpha^{\frac{1}{n}}) : \mathbb{Q}] = 2n$. Observe that $\cos \frac{\theta}{n} = \frac{\alpha^{\frac{1}{n}} + \overline{\alpha^{\frac{1}{n}}}}{2}$. So $X^2 - 2 \cos \frac{\theta}{n} X + 1$ is the minimal polynomial of $\alpha^{\frac{1}{n}}$ over $\mathbb{Q}(\cos \frac{\theta}{n})$, and so $[\mathbb{Q}(\cos \frac{\theta}{n}) : \mathbb{Q}] = n$, as illustrated by the following diagram.



To show that $1, \cos \frac{\theta}{n}, \cos \frac{2\theta}{n}, \dots, \cos \frac{(n-1)\theta}{n}$ is a \mathbb{Q} -basis of $\mathbb{Q}(\cos \frac{\theta}{n})$, it is sufficient to prove that they are linearly independent over \mathbb{Q} . Assume that we are given $\lambda_0, \dots, \lambda_{n-1}$ in \mathbb{Q} such that $\sum_{k=0}^{n-1} \lambda_k \cos \frac{k\theta}{n} = 0$. The following equalities hold

$$0 = \sum_{k=0}^{n-1} \lambda_k \cos \frac{k\theta}{n} = \lambda_0 + \sum_{k=1}^{n-1} \lambda_k \frac{\alpha^{\frac{k}{n}} + \overline{\alpha^{\frac{k}{n}}}}{2} = \lambda_0 + \sum_{k=1}^{n-1} \lambda_k \frac{\alpha^{\frac{k}{n}} + \overline{\alpha} \alpha^{\frac{n-k}{n}}}{2}$$

Therefore the following equality holds

$$\lambda_0 + \sum_{k=1}^{n-1} \frac{\lambda_k + \lambda_{n-k} \overline{\alpha}}{2} \alpha^{\frac{k}{n}} = 0.$$

Thus, as $1, \alpha^{\frac{1}{n}}, \alpha^{\frac{2}{n}}, \dots, \alpha^{\frac{n-1}{n}}$ is a basis of $\mathbb{Q}(\alpha^{\frac{1}{n}})$ over $\mathbb{Q}(\alpha)$, it follows that $\lambda_k = 0$ for all k .

By Corollary 3.13, we have $\text{Tr}_{\mathbb{Q}(\alpha^{\frac{k}{n}})/\mathbb{Q}}(\alpha^{\frac{k}{n}}) = 0$. Since $\cos \frac{k\theta}{n} = \frac{\alpha^{\frac{k}{n}} + \overline{\alpha^{\frac{k}{n}}}}{2}$, we immediately get $\text{Tr}_{\mathbb{Q}(\cos \frac{\theta}{n})/\mathbb{Q}}(\cos \frac{k\theta}{n}) = 0$. \square

Lemma 3.15. *With the hypothesis of Corollary 3.13, Let $\theta = \arccos(\frac{a}{b}) \in]0, \frac{\pi}{2}[$. Let $(\lambda_k \mid 0 \leq k \leq n-1)$ be a family in \mathbb{Q} . Then the conjugates of the element $\lambda_0 + \sum_{k=1}^{n-1} \lambda_k \cos(\frac{k\theta}{n})$, counted with multiplicity in $\mathbb{Q}(\cos(\frac{\theta}{n}))$, are the elements $\lambda_0 + \sum_{k=1}^{n-1} \lambda_k \cos(\frac{k(\theta+2\pi t)}{n})$ with $0 \leq t \leq n-1$.*

Proof. Note that $\alpha = e^{i\theta} = \frac{a+i\sqrt{b^2-a^2}}{b}$, and $\overline{\alpha} = e^{-i\theta} = \alpha^{-1}$. We have $[\mathbb{Q}(\alpha^{\frac{1}{n}}) : \mathbb{Q}(\alpha)] = n$, hence for each $0 \leq t \leq n-1$ we have a unique morphism, denoted by $\varphi_t : \mathbb{Q}(\alpha^{\frac{1}{n}}) \rightarrow \mathbb{C}$ preserving $\mathbb{Q}(\alpha)$, such that $\varphi_t(\alpha^{\frac{1}{n}}) = e^{i\frac{2\pi t}{n}} \alpha^{\frac{1}{n}}$. The embedding $\mathbb{Q}(\alpha^{\frac{1}{n}}) \rightarrow \mathbb{C}$ preserving $\mathbb{Q}(\alpha)$, are $\varphi_0, \dots, \varphi_{n-1}$. Since α is an algebraic number whose Galois conjugates have absolute value equal to 1, then $\alpha^{\frac{1}{n}}$ has the same property. Then, by [1, Proposition 2.3], $\mathbb{Q}(\alpha^{\frac{1}{n}})$ is a CM field. In particular the complex conjugation is well-defined and it is independent of any embedding of

$\mathbb{Q}(\alpha^{\frac{1}{n}})$ over \mathbb{C} . The embedding $\mathbb{Q}(\alpha^{\frac{1}{n}}) \rightarrow \mathbb{C}$ are $\varphi_0, \dots, \varphi_{n-1}$, and their complex conjugates $\overline{\varphi}_0, \dots, \overline{\varphi}_{n-1}$. Let k be in \mathbb{Z} , and $0 \leq t \leq n-1$. The following equalities hold

$$\varphi_t \left(e^{\frac{ik\theta}{n}} \right) = \varphi_t \left(\alpha^{\frac{k}{n}} \right) = \varphi_t \left(\alpha^{\frac{1}{n}} \right)^k = \left(e^{i\frac{2\pi t}{n}} \alpha^{\frac{1}{n}} \right)^k = e^{i\frac{2\pi kt}{n}} e^{\frac{ik\theta}{n}} = e^{i\frac{k(\theta+2\pi t)}{n}}. \quad (3.4)$$

Therefore

$$\varphi_t \left(\cos\left(\frac{k\theta}{n}\right) \right) = \frac{1}{2} \varphi_t \left(e^{\frac{ik\theta}{n}} + e^{\frac{-ik\theta}{n}} \right) = e^{i\frac{k(\theta+2\pi t)}{n}} + e^{-i\frac{k(\theta+2\pi t)}{n}} = \cos\left(\frac{k(\theta+2\pi t)}{n}\right). \quad (3.5)$$

Set $K = \mathbb{Q}(\alpha^{\frac{1}{n}} + \overline{\alpha}^{\frac{1}{n}}) = \mathbb{Q}(\cos(\frac{\theta}{n}))$. Note that $\varphi_t|_K = \overline{\varphi}_t|_K$, for all $0 \leq t \leq n-1$. Hence the morphisms $K \rightarrow \mathbb{C}$ are $\varphi_0|_K, \dots, \varphi_{n-1}|_K$. Set $\beta = \lambda_0 + \sum_{k=1}^{n-1} \lambda_k \cos(\frac{k\theta}{n})$. Let $0 \leq t \leq n-1$. The following equalities hold

$$\begin{aligned} \varphi_t(\beta) &= \varphi_t \left(\lambda_0 + \sum_{k=1}^{n-1} \lambda_k \cos\left(\frac{k\theta}{n}\right) \right) \\ &= \lambda_0 + \sum_{k=1}^{n-1} \lambda_k \varphi_t \left(\cos\left(\frac{k\theta}{n}\right) \right) \\ &= \lambda_0 + \sum_{k=1}^{n-1} \lambda_k \cos\left(\frac{k(\theta+2\pi t)}{n}\right). \quad \square \end{aligned}$$

Lemma 3.16. *Let $0 < a < b$ be relatively prime integers, let $\theta = \arccos(\frac{a}{b}) \in]0, \pi[$. Let $n \geq 2$. Let $(\lambda_k \mid 0 \leq k \leq n-1)$ be a family in \mathbb{Q} . Let u, v in \mathbb{N}^* be such that $\frac{b+a}{u^2}$ and $\frac{b-a}{v^2}$ are square-free integers. Assume the following statements*

- 4 divides b .
- $v_2(b) - 1$ is relatively prime to n .
- If p is an odd prime number such that p divides n then p divides b .
- If p is an odd prime number dividing b then $v_p(b)$ is relatively prime to n .

Then $\lambda_0 + \sum_{k=1}^{n-1} \lambda_k \cos(\frac{k\theta}{n})$ is an algebraic integer if and only if for all $1 \leq k \leq n-1$ the following statements hold

- (1) λ_0 is an integer.
- (2) For all odd prime number p such that p divides b we have $v_p(\lambda_k) \geq \left\lceil \frac{kv_p(b)}{n} \right\rceil$.
- (3) $v_2(\lambda_k) \geq 1 + \left\lceil \frac{k(v_2(b)-1)}{n} \right\rceil$.
- (4) $u(\lambda_k + \lambda_{n-k})$ and $v(\lambda_k - \lambda_{n-k})$ are integers.

Proof. As b is even and a is relatively prime to b , it follows that $b-a$ and $b+a$ are odd, thus u and v are odd. Set $\alpha = e^{i\theta} = \frac{a+i\sqrt{b^2-a^2}}{b}$.

Assume that (1) – (4) hold. Let $1 \leq k \leq n-1$. It follows from (4) that $2uv\lambda_k$ is an integer. However by (3) we have $v_2(\lambda_k) \geq 1$, therefore $uv\frac{\lambda_k}{2}$ is an integer.

It follows from (2) and (3) that $r\left(\frac{b}{2}, \frac{k}{n}\right)$ divides $uv\frac{\lambda_k}{2}$, and so by Lemma 3.9 we have $uv\frac{\lambda_k}{2}\alpha^{\frac{k}{n}} = uv\frac{\lambda_k}{2}e^{\frac{ki\theta}{n}}$ is an algebraic integer. Therefore $uv\lambda_k \cos\left(\frac{k\theta}{n}\right) = uv\frac{\lambda_k}{2} \left(e^{\frac{ki\theta}{n}} + e^{\frac{-ki\theta}{n}} \right)$ is an algebraic integer. Similarly $uv\lambda_{n-k} \cos\left(\frac{(n-k)\theta}{n}\right)$ is an algebraic integer. Therefore $uv \left(\lambda_k \cos\left(\frac{k\theta}{n}\right) + \lambda_{n-k} \cos\left(\frac{(n-k)\theta}{n}\right) \right)$ is an algebraic integer.

Lemma 3.11 implies that $2b \left(\lambda_k \cos \left(\frac{k\theta}{n} \right) + \lambda_{n-k} \cos \left(\frac{(n-k)\theta}{n} \right) \right)$ is an algebraic integer. However $2b$ and uv are relatively prime, hence $\lambda_k \cos \left(\frac{k\theta}{n} \right) + \lambda_{n-k} \cos \left(\frac{(n-k)\theta}{n} \right)$ is an algebraic integer. As λ_0 is an integer, it follows that

$$\lambda_0 + \sum_{k=1}^{n-1} \lambda_k \cos \left(\frac{k\theta}{n} \right)$$

is an algebraic integer.

Reciprocally assume that $\lambda_0 + \sum_{k=1}^{n-1} \lambda_k \cos \left(\frac{k\theta}{n} \right)$ is an algebraic integer. The following equality holds:

$$\lambda_0 + \sum_{k=1}^{n-1} \lambda_k \cos \left(\frac{k\theta}{n} \right) = \lambda_0 + \sum_{k=1}^{n-1} \frac{\lambda_k}{2} \left(e^{\frac{ik\theta}{n}} + e^{\frac{-ik\theta}{n}} \right) = \lambda_0 + \sum_{k=1}^{n-1} \frac{\lambda_k}{2} \left(\alpha^{\frac{k}{n}} + \alpha^{\frac{-k}{n}} \right).$$

By Lemma 3.12 there are valuations $\nu_{1,2}$ and $\nu_{2,2}$ defined over $\mathbb{Q}(\alpha)$, extending the 2-adic valuation, such that $\nu_{1,2}(\alpha) = -\nu_{2,2}(\alpha) = v_2(b) - 1$. However, $v_2(b) - 1$ is relatively prime to n . Moreover, for all $x \in \mathbb{Q}(\alpha)^*$, we have that $\nu_{1,2}(x)$ is an integer. In fact since 4 divides b and a and b are coprime, by the reciprocity quadratic law, (2) is not ramified over $\mathbb{Q}(\alpha)$. Hence it follows from Lemma 3.4 that $[\mathbb{Q}(\alpha^{\frac{1}{n}}) : \mathbb{Q}(\alpha)] = n$.

As $\nu_{1,2}(\alpha) = v_2(b) - 1$ is relatively prime to n , it follows from Lemma 3.3 that $v_2(\lambda_0) \geq 0$, and the following inequalities hold

$$v_2 \left(\frac{\lambda_k}{2} \right) \geq \left\lceil \frac{k(v_2(b) - 1)}{n} \right\rceil, \quad \text{for all } 1 \leq k \leq n.$$

Therefore (3) holds.

Let p be an odd prime dividing b . Thus by Lemma 3.12, there are valuations $\nu_{1,p}$ and $\nu_{2,p}$ extending v_p to $\mathbb{Q}(\alpha)$ such that $\nu_{1,p}(\alpha) = -\nu_{2,p}(\alpha) = v_p(b)$. As $v_p(b)$ is relatively prime to n , it follows from Lemma 3.3 that $v_p(\lambda_0) \geq 0$, and the following inequalities hold

$$v_p(\lambda_k) = v_p \left(\frac{\lambda_k}{2} \right) \geq \left\lceil \frac{kv_p(b)}{n} \right\rceil, \quad \text{for all } 1 \leq k \leq n.$$

Therefore (2) holds.

It follows from Lemma 3.6 that $n\lambda_0$ is an integer, however we also have $v_p(\lambda_0) \geq 0$ for all odd primes p dividing n , and $v_2(\lambda_0) \geq 0$, therefore λ_0 is an integer. Therefore (1) holds.

By Lemma 3.6, $nb \frac{\lambda_k}{2} + n \frac{\lambda_{n-k}}{2} (a + i\sqrt{b^2 - a^2})$ is an algebraic integer (for all $1 \leq k \leq n - 1$). In particular, if $1 \leq k \leq n - 1$, then also $1 \leq n - k \leq n - 1$, and so $nb \frac{\lambda_{n-k}}{2} + n \frac{\lambda_k}{2} (a + i\sqrt{b^2 - a^2})$ is an algebraic integer.

Therefore, applying $\text{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}$, we obtain that $n(b\lambda_k + a\lambda_{n-k})$, $n(b\lambda_{n-k} + a\lambda_k)$ are integers, and $n\lambda_k i\sqrt{b^2 - a^2}$ is an algebraic integer.

We already proved that $v_2(\lambda_k) \geq 0$, and $v_2(\lambda_{n-k}) \geq 0$, hence $v_2(u(\lambda_k + \lambda_{n-k})) \geq 0$, and $v_2(v(\lambda_k - \lambda_{n-k})) \geq 0$.

Similarly if p be an odd prime number that divides n , then we have already proved that $v_p(\lambda_k) \geq 0$, and $v_p(\lambda_{n-k}) \geq 0$, hence $v_p(u(\lambda_k + \lambda_{n-k})) \geq 0$, and $v_p(v(\lambda_k - \lambda_{n-k})) \geq 0$.

Let p be a prime number that does not divides $n(b^2 - a^2)$. As $n\lambda_k i\sqrt{b^2 - a^2}$ is an algebraic integer, it follows that $n\lambda_k(b^2 - a^2)$ is an integer, hence $v_p(\lambda_k) =$

$v_p(n\lambda_k(b^2 - a^2)) \geq 0$. Similarly $v_p(\lambda_{n-k}) \geq 0$, therefore $v_p(u(\lambda_k + \lambda_{n-k})) \geq 0$, and $v_p(v(\lambda_k - \lambda_{n-k})) \geq 0$.

Let p be an odd prime number dividing $b^2 - a^2 = (b - a)(b + a)$. As b and a are relatively prime, it follows that either p divides $b - a$ or p divides $b + a$. Moreover p does not divide b , and so p does not divide n .

Assume that p divides $b - a$. As $n(b\lambda_k + a\lambda_{n-k})$, $n(b\lambda_{n-k} + a\lambda_k)$ are integers, it follows that $n(b\lambda_k + a\lambda_{n-k}) + n(b\lambda_{n-k} + a\lambda_k) = n(\lambda_k + \lambda_{n-k})(a + b)$ is an integer. As p does not divide n and p does not divide $a + b$, it follows that $v_p(u(\lambda_k + \lambda_{n-k})) = v_p(n(\lambda_k + \lambda_{n-k})(a + b)) \geq 0$.

Moreover $v_p(\lambda_k) = v_p(n\lambda_k i\sqrt{b^2 - a^2}) - v_p(\sqrt{b^2 - a^2}) \geq 0 - \frac{1}{2}v_p(b^2 - a^2) = -\frac{1}{2}v_p(b - a)$. However, $\frac{b-a}{v^2}$ is a square-free integer, thus $v_p(v) = \lfloor \frac{1}{2}v_p(b - a) \rfloor$. Therefore $v_p(v) > \frac{1}{2}v_p(b - a) - 1$. It follows that $v_p(v\lambda_k) = v_p(v) + v_p(\lambda_k) > \frac{1}{2}v_p(b - a) - 1 - \frac{1}{2}v_p(b - a) = -1$. However $v_p(v\lambda_k)$ is an integer, thus $v_p(v\lambda_k) \geq 0$. Similarly $v_p(v\lambda_{n-k}) \geq 0$, and so $v_p(v(\lambda_k - \lambda_{n-k})) \geq 0$.

With a similar argument, we can prove that if p divides $b + a$ then $v_p(u(\lambda_k + \lambda_{n-k})) \geq 0$ and $v_p(v(\lambda_k - \lambda_{n-k})) \geq 0$.

Hence, for all prime p we have $v_p(u(\lambda_k + \lambda_{n-k})) \geq 0$ and $v_p(v(\lambda_k - \lambda_{n-k})) \geq 0$. Therefore $u(\lambda_k + \lambda_{n-k})$ and $v(\lambda_k - \lambda_{n-k})$ are integers. That is (4) holds. \square

4. TRACE IN GENERAL

Given an algebraic number β in a number field K , we denote $T(\beta) = \frac{\text{Tr}_{K/\mathbb{Q}}(\beta)}{[K:\mathbb{Q}]}$. Note that T does not depend on the choice of K . In particular $T(\beta) = \frac{\text{Tr}_{\mathbb{Q}(\beta)/\mathbb{Q}}(\beta)}{[\mathbb{Q}(\beta):\mathbb{Q}]}$. Also note that $T(q) = q$ for all $q \in \mathbb{Q}$. We also denote by $\delta(\beta)$ the largest difference of two conjugates of β . Note that $\delta(\beta + q) = \delta(\beta)$ for all $q \in \mathbb{Q}$. The following Lemma is evident.

Lemma 4.1. *Let γ be a totally real algebraic number. If $T(\gamma) = 0$, then γ has a conjugate ≥ 0 and a conjugate (maybe the same) ≤ 0 .*

Lemma 4.2. *Let β be a totally real algebraic number. Then β has a conjugate $\geq T(\beta)$ and a conjugate (maybe the same) $\leq T(\beta)$.*

Proof. Set $\gamma = \beta - T(\beta)$. As $T(\gamma) = 0$ it follows from Lemma 4.1 that there is $\varphi: \mathbb{Q}(\beta) \rightarrow \mathbb{R}$ such that $\varphi(\gamma) \geq 0$, hence $\varphi(\beta) = \varphi(\gamma + T(\beta)) = \varphi(\gamma) + T(\beta) \geq T(\beta)$. Similarly β has a conjugate smaller than $T(\beta)$. \square

Lemma 4.3. *Let β be a totally real algebraic number. Then β has a conjugate $\geq \sqrt{T(\beta^2)}$, or a conjugate $\leq -\sqrt{T(\beta^2)}$.*

Proof. Lemma 4.2 implies that β^2 has a conjugate $\geq T(\beta^2)$. Hence we have a conjugate γ of β such that $\gamma^2 \geq T(\beta^2)$, therefore $\gamma \geq \sqrt{T(\beta^2)}$ or $\gamma \leq -\sqrt{T(\beta^2)}$. \square

Lemma 4.4. *Let β be a totally real algebraic number such that $T(\beta) = 0$. Then $\delta(\beta) \geq 2\sqrt{T(\beta^2)}$.*

Proof. We can assume that $\beta \neq 0$. We see from Lemma 4.2 that β^2 has a conjugate $\geq T(\beta^2)$. Therefore β has a conjugate $\geq \sqrt{T(\beta^2)}$ or a conjugate $\leq -\sqrt{T(\beta^2)}$. In the second case, we can change β to $-\beta$, and so assume that β has a conjugate $\geq \sqrt{T(\beta^2)}$.

Let γ_1 be the largest conjugate of β , and γ_2 be the smallest conjugate of β . From Lemma 4.2 it follows that $\gamma_1 > 0$, and $\gamma_2 < 0$. Let $u \in \mathbb{Q}$ be such that $u > \gamma_1$. Set $v = \frac{u^2 - T(\beta^2)}{2u}$. As $u > \gamma_1 \geq \sqrt{T(\beta^2)}$, it follows that $v > 0$.

The following equalities hold:

$$\begin{aligned}
T((\beta - v)^2) &= T(\beta^2) - 2vT(\beta) + v^2 \\
&= T(\beta^2) + v^2 && \text{as } T(\beta) = 0. \\
&= T(\beta^2) + \left(\frac{u^2 - T(\beta^2)}{2u}\right)^2 \\
&= T(\beta^2) + \frac{u^4 - 2u^2T(\beta^2) + T(\beta^2)^2}{4u^2} \\
&= \frac{u^4 + 2u^2T(\beta^2) + T(\beta^2)^2}{4u^2} \\
&= \left(\frac{u^2 + T(\beta^2)}{2u}\right)^2.
\end{aligned}$$

It follows from Lemma 4.3 that $\beta - v$ has a conjugate $\geq \frac{u^2 + T(\beta^2)}{2u}$ or a conjugate $\leq -\frac{u^2 + T(\beta^2)}{2u}$.

If $\beta - v$ has a conjugate $\geq \frac{u^2 + T(\beta^2)}{2u}$, it follows that γ_1 the largest conjugate of β satisfies

$$\gamma_1 \geq v + \frac{u^2 + T(\beta^2)}{2u} = \frac{u^2 - T(\beta^2)}{2u} + \frac{u^2 + T(\beta^2)}{2u} = u,$$

which contradicts $u > \gamma_1$. Therefore $\beta - v$ has a conjugate $\leq -\frac{u^2 + T(\beta^2)}{2u}$. It follows that γ_2 the smallest conjugate of β satisfies

$$\gamma_2 \leq v - \frac{u^2 + T(\beta^2)}{2u} = \frac{u^2 - T(\beta^2)}{2u} - \frac{u^2 + T(\beta^2)}{2u} = -\frac{T(\beta^2)}{u}.$$

This inequality is true for all $u \in \mathbb{Q}$ larger than γ_1 . It follows that $\gamma_2 \leq -\frac{T(\beta^2)}{\gamma_1}$.

Thus $\gamma_1 - \gamma_2 \geq \gamma_1 + \frac{T(\beta^2)}{\gamma_1} \geq 2\sqrt{T(\beta^2)}$. \square

Lemma 4.5. *Let β be a totally real algebraic number, set $r = T((\beta^2 - T(\beta^2))^2)$. If $T(\beta) = 0$ and $T(\beta^3) = 0$ then $\delta(\beta) \geq 2\sqrt{2\sqrt{r}}$.*

Proof. Denote by γ_1 the largest conjugate of β and by γ_2 the smallest conjugate of β . If $|\gamma_2| \geq |\gamma_1|$, then we can change β to $-\beta$, and so assume that $\gamma_1 = |\gamma_1| \geq |\gamma_2|$, in particular γ_1^2 is the largest conjugate of β^2 . Note that $T(\beta^2 - T(\beta^2)) = 0$, hence it follows from Lemma 4.4 that $\delta(\beta^2 - T(\beta^2)) \geq 2\sqrt{T((\beta^2 - T(\beta^2))^2)} = 2\sqrt{r}$. However all conjugates of β^2 are positives, hence $\gamma_1^2 \geq \delta(\beta^2) = \delta(\beta^2 - T(\beta^2)) \geq 2\sqrt{r}$, it follows that $\gamma_1 \geq \sqrt{2\sqrt{r}}$.

Consider the following function

$$\begin{aligned}
f: \mathbb{R} &\rightarrow \mathbb{R} \\
x &\mapsto x - \gamma_1 + \sqrt{2\sqrt{r} + 4x^2T(\beta^2)}.
\end{aligned}$$

Note that $f(0) = -\gamma_1 + \sqrt{2\sqrt{r}} \leq 0$ and that $\lim_{x \rightarrow \infty} f(x) = +\infty$. Moreover f is continuous, therefore we can pick $c \geq 0$, such that $f(c) = 0$. Set $s = r + 4c^2T(\beta^2)$.

Hence

$$c = \gamma_1 - \sqrt{2\sqrt{r} + 4c^2T(\beta^2)} = \gamma_1 - \sqrt{2\sqrt{s}}. \quad (4.1)$$

Let $d \in \mathbb{Q}$ be such that $d > c$. Set $\rho = (\beta - d)^2 - T((\beta - d)^2)$. Note that $T(\rho) = 0$, moreover the following equalities hold

$$\begin{aligned} \rho &= \beta^2 - 2d\beta + d^2 - T(\beta^2 - 2d\beta + d^2) \\ &= \beta^2 - 2d\beta + d^2 - (T(\beta^2) + 2dT(\beta) + d^2) \\ &= \beta^2 - T(\beta^2) - 2d\beta. \end{aligned}$$

Thus

$$\begin{aligned} T(\rho^2) &= T((\beta^2 - T(\beta^2) - 2d\beta)^2) \\ &= T((\beta^2 - T(\beta^2))^2) + 4d^2T(\beta^2) - 4dT(\beta^3) + 4dT(\beta^2)T(\beta) \\ &= r + 4d^2T(\beta^2) \\ &\geq s. \end{aligned}$$

Therefore, by Lemma 4.4, we have $\delta(\rho) \geq 2\sqrt{s}$, thus $\delta((\beta - d)^2) = \delta(\rho) \geq 2\sqrt{s}$. However all conjugates of $(\beta - d)^2$ are positives, hence there is a conjugate γ_3 of β such that $(\gamma_3 - d)^2 \geq 2\sqrt{s}$. Assume that $\gamma_3 - d \geq \sqrt{2\sqrt{s}}$, then, as $d > c$ it follows from (4.1) that $\gamma_3 \geq d + \sqrt{2\sqrt{s}} > c + \sqrt{2\sqrt{s}} = \gamma_1$; a contradiction, as γ_1 is the largest conjugate of β . Therefore $\gamma_3 - d \leq -\sqrt{2\sqrt{s}}$, hence $\gamma_2 \leq \gamma_3 \leq d - \sqrt{2\sqrt{s}}$. Therefore, for all $d \in \mathbb{Q}$ such that $d > c$, we have $\gamma_2 \leq d - \sqrt{2\sqrt{s}}$. Hence $\gamma_2 \leq c - \sqrt{2\sqrt{s}} = \gamma_1 - 2\sqrt{2\sqrt{s}}$, thus $\gamma_1 - \gamma_2 \geq \gamma_1 - (\gamma_1 - 2\sqrt{2\sqrt{s}}) = 2\sqrt{2\sqrt{s}} \geq 2\sqrt{2\sqrt{r}}$. \square

5. TRACE IN THE FIELDS FOR THE EXAMPLES

In this section we fix $0 < a < b$ relatively prime integers. We assume that if 2 divides b , then 4 divides b . Denote $\theta = \arccos(\frac{a}{b})$. We consider R a set of positive integers such that for all $n, m \in R$ the following statements hold

- $\gcd(m, n) \in R$.
- For all s that divides n we have $s \in R$.
- Let p be an odd prime that divides b , then $v_p(b)$ is relatively prime to n .
- If b is even then $v_2(b) - 1$ is relatively prime to n .

Denote $Q = \{\frac{k}{n} \mid n \in R \text{ and } 1 \leq k \leq n\}$. Note that it follows from Corollary 3.13 that

$$T\left(\cos\left(\frac{k\theta}{n}\right)\right) = 0, \text{ for all } n \in R, \text{ and all } 1 \leq k \leq n.$$

For all $x, y \in \mathbb{Q}$ we set $N(x, y) = x^2 + y^2 + 2xy\frac{a}{b}$.

Lemma 5.1. *Let n be in R . Let $(\lambda_k \mid 0 \leq k \leq n-1)$ be a family in \mathbb{Q} . Then*

$$\begin{aligned} T\left(\left(\lambda_0 + \sum_{k=1}^{n-1} \lambda_k \cos\left(\frac{k\theta}{n}\right)\right)^2\right) &= \lambda_0^2 + \frac{1}{2} \sum_{k=1}^{n-1} \lambda_k^2 + \lambda_k \lambda_{n-k} \frac{a}{b} \\ &= \lambda_0^2 + \frac{1}{2} \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} N(\lambda_k, \lambda_{n-k}). \end{aligned}$$

Proof. Set $\alpha = e^{i\theta}$. Set $K = \mathbb{Q}(e^{\frac{i\theta}{n}})$, note that for every $1 \leq k, \ell \leq n-1$, the following equalities hold:

$$\begin{aligned} \cos\left(\frac{k\theta}{n}\right) \cos\left(\frac{\ell\theta}{n}\right) &= \frac{1}{4} \left(e^{\frac{ki\theta}{n}} + e^{\frac{-ki\theta}{n}} \right) \left(e^{\frac{\ell i\theta}{n}} + e^{\frac{-\ell i\theta}{n}} \right) \\ &= \frac{1}{4} \left(e^{\frac{(k+\ell)i\theta}{n}} + e^{\frac{(k-\ell)i\theta}{n}} + e^{\frac{(-k+\ell)i\theta}{n}} + e^{\frac{(-k-\ell)i\theta}{n}} \right). \end{aligned}$$

Therefore,

$$\mathrm{Tr}_{\mathbb{Q}(\alpha^{\frac{1}{n}})/\mathbb{Q}(\alpha)} \left(\cos\left(\frac{k\theta}{n}\right) \cos\left(\frac{\ell\theta}{n}\right) \right) = \begin{cases} \frac{n}{2} + \frac{na}{2b} & \text{if } k = \ell = \frac{n}{2} \\ \frac{n}{2} & \text{if } k = \ell \neq \frac{n}{2} \\ \frac{na}{2b} & \text{if } k = n - \ell \neq \frac{n}{2} \\ 0 & \text{otherwise.} \end{cases}$$

Hence:

$$\mathrm{Tr}_{\mathbb{Q}(\cos(\frac{\theta}{n}))/\mathbb{Q}} \left(\cos\left(\frac{k\theta}{n}\right) \cos\left(\frac{\ell\theta}{n}\right) \right) = \begin{cases} \frac{n}{2} + \frac{na}{2b} & \text{if } k = \ell = \frac{n}{2} \\ \frac{n}{2} & \text{if } k = \ell \neq \frac{n}{2} \\ \frac{na}{2b} & \text{if } k = n - \ell \neq \frac{n}{2} \\ 0 & \text{otherwise.} \end{cases}$$

Moreover $\mathrm{Tr}_{\mathbb{Q}(\cos(\frac{\theta}{n}))/\mathbb{Q}}(\cos(\frac{k\theta}{n})) = 0$ for all $1 \leq k \leq n-1$. The Lemma follows directly. \square

Remark 5.2. Given $\lambda, \mu \in \mathbb{Q}$, $n \in \mathbb{R}$, and $1 \leq k < \frac{n}{2}$. It follows from Lemma 5.1 that

$$N(\lambda, \mu) = 2T \left(\left(\lambda \cos\left(\frac{k\theta}{n}\right) + \mu \cos\left(\frac{(n-k)\theta}{n}\right) \right)^2 \right).$$

Moreover if $2 \in \mathbb{R}$, then

$$N(\lambda, \lambda) = 2T \left(\left(\lambda \cos\left(\frac{\theta}{2}\right) \right)^2 \right).$$

6. SOME ANALYSIS

In this section we give various estimation of maximal value of functions that will be necessary to the calculation of the Julia Robinson Number in the next section.

Lemma 6.1. *Let λ, μ be real numbers. Then the maximum value, with $x \in \mathbb{R}$, of $\lambda \cos x + \mu \sin x$ is $\sqrt{\lambda^2 + \mu^2}$. The minimum value is $-\sqrt{\lambda^2 + \mu^2}$.*

Proof. A rotation of the angle x , of the vector (λ, μ) , has first coordinate $\lambda \cos x + \mu \sin x$. Hence the maximum possible value is the length of the vector (λ, μ) , which is $\sqrt{\lambda^2 + \mu^2}$. \square

Lemma 6.2. *Let λ, μ, θ, q be real numbers. Then the maximum value, for $x \in \mathbb{R}$, of $\lambda \cos(q\theta + x) + \mu \cos((1-q)\theta - x)$ is $\sqrt{\lambda^2 + \mu^2 + 2\lambda\mu \cos(\theta)}$.*

Proof. For all $x \in \mathbb{R}$, denote $f(x) = \lambda \cos(q\theta + x) + \mu \cos((1-q)\theta - x)$. It follows, from Lemma 3.7, that

$$f(x) = (\lambda + \mu) \cos\left(\frac{\theta}{2}\right) \cos\left(\frac{1-2q}{2}\theta + x\right) + (\mu - \lambda) \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{1-2q}{2}\theta + x\right)$$

Therefore, by Lemma 6.1, the maximum value of f is

$$V = \sqrt{(\lambda + \mu)^2 \cos^2\left(\frac{\theta}{2}\right) + (\mu - \lambda)^2 \sin^2\left(\frac{\theta}{2}\right)}.$$

The following equalities hold

$$\begin{aligned} V^2 &= (\lambda^2 + \mu^2) \left(\cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right) \right) + 2\lambda\mu \left(\cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right) \right) \\ &= \lambda^2 + \mu^2 + 2\lambda\mu \cos(\theta). \end{aligned}$$

Therefore the maximum value is $V = \sqrt{\lambda^2 + \mu^2 + 2\lambda\mu \cos(\theta)}$. \square

Corollary 6.3. *Let λ, μ be real numbers, and $\varepsilon > 0$. Set*

$$A = \max\left(1, \left\lceil 2\pi \frac{|\lambda| + |\mu|}{\varepsilon} \right\rceil\right).$$

Let θ be a real number. Set

$$s = \lambda^2 + \mu^2 + 2\lambda\mu \cos(\theta).$$

Then for all $n \geq A$, and all $1 \leq k \leq n - 1$ with $\gcd(k, n) = 1$, there is $t \in \mathbb{Z}$ such that

$$\lambda \cos \frac{k(\theta + 2\pi t)}{n} + \mu \cos \frac{(n-k)(\theta + 2\pi t)}{n} \geq \sqrt{s} - \varepsilon.$$

Proof. For each $t \in \mathbb{Z}$, we set

$$\begin{aligned} \alpha_t &= \lambda \cos\left(\frac{k(\theta + 2\pi t)}{n}\right) + \mu \cos\left(\frac{(n-k)(\theta + 2\pi t)}{n}\right) \\ &= \lambda \cos\left(\frac{k\theta}{n} + \frac{2\pi kt}{n}\right) + \mu \cos\left(\frac{(n-k)\theta}{n} - \frac{2\pi kt}{n}\right). \end{aligned}$$

Let r be an integer such that $rk \equiv 1 \pmod{n}$. By Lemma 6.2, we can pick $x \in \mathbb{R}$ such that

$$\sqrt{s} = \lambda \cos\left(\frac{k\theta}{n} + x\right) + \mu \cos\left(\frac{(n-k)\theta}{n} - x\right). \quad (6.1)$$

Set $q = \lfloor \frac{nx}{2\pi} \rfloor$. Set $t = rq$, hence $kt = krq \equiv q \pmod{n}$. The following equalities hold

$$\alpha_t = \lambda \cos\left(\frac{k\theta}{n} + \frac{2\pi q}{n}\right) + \mu \cos\left(\frac{(n-k)\theta}{n} - \frac{2\pi q}{n}\right). \quad (6.2)$$

The following inequalities hold

$$0 \leq x - \frac{2\pi q}{n} = x - \frac{2\pi}{n} \left\lfloor \frac{nx}{2\pi} \right\rfloor \leq \frac{2\pi}{n} \leq \frac{\varepsilon}{|\lambda| + |\mu|}. \quad (6.3)$$

However, $|\cos(a) - \cos(b)| \leq |a - b|$ for all $a, b \in \mathbb{R}$. Therefore, subtracting (6.1) and (6.2), and comparing the argument of the cosine function in (6.3), we obtain

$$|\alpha_t - \sqrt{s}| \leq (|\lambda| + |\mu|) \frac{\varepsilon}{|\lambda| + |\mu|} = \varepsilon.$$

It follows that $\alpha_t \geq \sqrt{s} - \varepsilon$. \square

Lemma 6.4. *Let γ be a real number, $\ell \neq 0$ be an integer, and $d > 1$ be an integer. Assume that $d \nmid \ell$. Then $\sum_{t=0}^{d-1} \cos\left(\gamma + \frac{t\ell}{d} 2\pi\right) = 0$.*

Proof. Set $x = e^{i\frac{\ell}{d}2\pi}$. Note that $x^d = 1$. The following equalities hold:

$$\begin{aligned} \sum_{t=0}^{d-1} \cos\left(\gamma + \frac{t}{d}2\pi\right) &= \Re \sum_{t=0}^{d-1} \exp\left(i\gamma + i\frac{t\ell}{d}2\pi\right) \\ &= \Re \left(e^{i\gamma} \sum_{t=0}^{d-1} x^t \right) \\ &= \Re \left(e^{i\gamma} \frac{x^d - 1}{x - 1} \right) \\ &= 0 \end{aligned} \quad \square$$

Notation 6.5. For $a \in \mathbb{R}$ and $b \in \mathbb{R}^*$, we denote $R_b(a)$ the only real number $c \in [-\frac{b}{2}, \frac{b}{2}[$ such that $a - c \in b\mathbb{Z}$. Note that for all $t \in \mathbb{R}^*$ we have $R_{tb}(ta) = tR_b(a)$. We will use the particular case $R_{2\pi}(\frac{a}{b}2\pi) = \frac{R_b(a)}{b}2\pi$

Note that if a, b are integers, then $R_b(a)$ is an integer. Also observe that if a, a', b are integers, then $|R_b(aa')| \leq |R_b(a)| \times |R_b(a')|$.

Lemma 6.6. *Let $\lambda, \mu, \theta_1, \theta_2, \rho$ be real numbers, and u be an integer. If $R_{2\pi}(u\rho) \neq uR_{2\pi}(\rho)$ and $|R_{2\pi}(u\rho)| < \frac{\pi}{2}$, then there is $|r| \leq \max(1, |u| - 1)$ such that $\lambda \cos(\theta_1 + r\rho) + \mu \cos(\theta_2 - r\rho) \geq 0$.*

Proof. We cannot have $u = 0$ or $u = 1$. Assume that $u = -1$. It follows, from $R_{2\pi}(u\rho) \neq uR_{2\pi}(\rho)$, that $\rho = \pi$. Hence $\lambda \cos(\theta_1 + \rho) + \mu \cos(\theta_2 - \rho) = -(\lambda \cos(\theta_1) + \mu \cos(\theta_2))$. Hence $r = 0$, or $r = 1$ satisfies the required condition.

We now assume that $u \geq 2$. We can assume that $R_{2\pi}(\rho) = \rho$, that is $\rho \in [-\pi, \pi[$. Set $\lambda' = \lambda \cos(\theta_1) + \mu \cos(\theta_2)$, and $\mu' = \mu \sin(\theta_2) - \lambda \sin(\theta_1)$. Given $r \in \mathbb{Z}$, denote $f(r) = \lambda \cos(\theta_1 + r\rho) + \mu \cos(\theta_2 - r\rho)$, hence the following equalities hold

$$\begin{aligned} f(r) &= \lambda(\cos(\theta_1) \cos(r\rho) - \sin(\theta_1) \sin(r\rho)) + \mu(\cos(\theta_2) \cos(r\rho) + \sin(\theta_2) \sin(r\rho)) \\ &= \lambda' \cos(r\rho) + \mu' \sin(r\rho). \end{aligned}$$

If $\lambda' \geq 0$, then $f(0) = \lambda' \geq 0$, as required. Assume that $\lambda' < 0$. We first want to find s such that $R_{2\pi}(s\rho) = s\rho$ and $s\rho \notin [-\frac{\pi}{2}, \frac{\pi}{2}[$.

If $\rho \notin [-\frac{\pi}{2}, \frac{\pi}{2}[$, then $s = 1$ satisfies the desired conditions. Assume that $\rho \in [-\frac{\pi}{2}, \frac{\pi}{2}[$. As $u\rho \notin [-\pi, \pi[$, we can take the smallest $s \geq 1$ such that $s\rho \notin [-\frac{\pi}{2}, \frac{\pi}{2}[$. From the minimality we have $(s - 1)\rho \in [-\frac{\pi}{2}, \frac{\pi}{2}[$, however $\rho \in [-\frac{\pi}{2}, \frac{\pi}{2}[$, hence $s\rho \in [-\pi, \pi[$. Thus $R_{2\pi}(s\rho) = s\rho$, in particular $|s| \leq |u| - 1$.

Note that $\cos(s\rho) = \cos(-s\rho) \leq 0$. Moreover $\sin(s\rho) = -\sin(-s\rho)$. Hence $f(s) = \lambda' \cos(s\rho) + \mu' \sin(s\rho) \geq 0$ or $f(-s) = \lambda' \cos(s\rho) - \mu' \sin(s\rho) \geq 0$.

The proof is similarly if $u \leq -2$. □

Lemma 6.7. *Let $a \in \mathbb{Z}$ and $b > 0$ an integer, then there exists $u \in \mathbb{Z} \setminus \{0\}$ such that $|u| \leq \sqrt{b}$ and $|R_b(u)| \leq \sqrt{b}$.*

Proof. If $b = 1$ the it is easy. Assume that $b \geq 2$. Consider the following system with indeterminate u, v in \mathbb{R} .

$$\begin{cases} |u| \leq \sqrt{b} \\ |ua + vb| \leq \sqrt{b}. \end{cases}$$

The set of solutions is a parallelogram of height $2\sqrt{b}$, and base $2\frac{\sqrt{b}}{b}$. Hence the surface is 4. Therefore, by Minkowski's Theorem (see for instance [8, Lemma, p. 137]), there is a solution $(u, v) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$.

Thus we have $|u| \leq \sqrt{b}$ and $|R_b(ua)| \leq |ua + vb| \leq \sqrt{b}$. Moreover if $u = 0$, then $v \neq 0$ and $b \leq |vb| = |ua + vb| \leq \sqrt{b}$; a contradiction. Therefore $u \neq 0$ as required. \square

Lemma 6.8. *Let $1 \leq k, \ell < n$ be integers such that $\gcd(k, \ell, n) = 1$, $k \neq \ell$, and $k \neq n - \ell$. Then one of the following statement holds.*

- (1) $1 < \gcd(k, n) \leq \sqrt{n}$ or $1 < \gcd(\ell, n) \leq \sqrt{n}$.
- (2) There are non-zero integers r, u such that $|u| \leq \sqrt{n}$, $rk \equiv 1 \pmod{n}$, $u(R_n(r\ell)) \neq R_n(ur\ell)$, and $R_n(ur\ell) \leq \sqrt{n}$.
- (3) There are non-zero integers r, u such that $|u| \leq \sqrt{n}$, $r\ell \equiv 1 \pmod{n}$, $u(R_n(rk)) \neq R_n(urk)$, and $R_n(urk) \leq \sqrt{n}$.

Proof. Note that $k, \ell, n - k, n - \ell$ are all distinct in $\{1, \dots, n\}$, thus $n \geq 5$. As $\gcd(k, \ell, n) = 1$, it follows that $\gcd(\ell, n) \times \gcd(k, n)$ divides n

Assume that $\gcd(k, n) > \sqrt{n}$. Then $\gcd(\ell, n) \leq \frac{n}{\gcd(k, n)} < \sqrt{n}$. If $\gcd(\ell, n) > 1$ then (1) holds. Assume that $\gcd(\ell, n) = 1$. Let r be an inverse of ℓ modulo n , thus $r\ell \equiv 1 \pmod{n}$. Set $u = \frac{n}{\gcd(k, n)}$, so $|u| = u \leq \sqrt{n}$ and $R_n(urk) = R_n(\frac{rk}{\gcd(k, n)}n) = 0$. As $r\ell \equiv 1 \pmod{n}$ and $k < n$ it follows that $n \nmid rk$, thus $R_n(rk) \neq 0$. So $u(R_n(rk)) \neq R_n(urk)$, therefore (3) holds.

Similarly if $\gcd(\ell, n) > \sqrt{n}$, then (1) or (3) holds. Hence we can assume that $\gcd(k, n) \leq \sqrt{n}$ and $\gcd(\ell, n) \leq \sqrt{n}$. If $\gcd(k, n) > 1$, or $\gcd(\ell, n) > 1$ then (1) holds. So we can assume that $\gcd(k, n) = 1 = \gcd(\ell, n)$.

Let $1 \leq r < n$ be the inverse of k modulo n , and $1 \leq q < n$ be the inverse of ℓ modulo n . First assume that $|R_n(r\ell)| > \sqrt{n}$. It follows from Lemma 6.7 that there is $u \in \mathbb{Z} \setminus \{0\}$ such that $|u| \leq \sqrt{n}$ and $|R_n(ur\ell)| \leq \sqrt{n}$. So $|u(R_n(r\ell))| = |u||R_n(r\ell)| > \sqrt{n} \geq |R_n(ur\ell)|$, thus (2) holds. Similarly $|R_n(qk)| > \sqrt{n}$ then (3) holds. Therefore we can assume that $|R_n(r\ell)| \leq \sqrt{n}$ and $|R_n(qk)| \leq \sqrt{n}$.

Assume that $|R_n(r\ell)| \geq \sqrt{n - \sqrt{n}}$. Set $u = R_n(r\ell)$, note that $|u| \leq \sqrt{n}$, moreover $n \geq u(R_n(r\ell)) = (R_n(r\ell))^2 \geq \sqrt{n - \sqrt{n}}^2 = n - \sqrt{n}$, and so $|R_n(ur\ell)| \leq \sqrt{n}$. Moreover $u(R_n(r\ell)) \geq n - \sqrt{n} > \sqrt{n}$, hence $u(R_n(r\ell)) \neq R_n(ur\ell)$, therefore (2) holds. Similarly if $|R_n(r\ell)| \geq \sqrt{n - \sqrt{n}}$ then (3) holds.

We can assume that $|R_n(r\ell)| < \sqrt{n - \sqrt{n}}$ and $|R_n(qk)| < \sqrt{n - \sqrt{n}}$. Thus $|R_n(r\ell)| \times |R_n(qk)| < n - \sqrt{n}$. However, $rkq\ell \equiv 1 \pmod{n}$, hence $|R_n(r\ell)| \times |R_n(qk)| \equiv \pm 1 \pmod{n}$. Therefore $|R_n(r\ell)| \times |R_n(qk)| = 1$. It follows that $|R_n(r\ell)| = |R_n(qk)| = 1$, so that $r\ell \equiv \pm 1 \pmod{n}$, thus $k \equiv kr\ell \equiv \pm \ell \pmod{n}$. Therefore $k = \ell$ or $k = n - \ell$; a contradiction. \square

Theorem 6.9. *Let $\lambda_1, \mu_1, \lambda_2, \mu_2$ be real numbers and $\varepsilon > 0$. Set*

$$A = \max \left(17, \left[2\pi \frac{|\lambda_1| + |\lambda_2| + |\mu_1| + |\mu_2|}{\varepsilon} \right]^2 \right).$$

Let θ_1, θ_2 be real numbers. Set

$$s = \min(\lambda_1^2 + \mu_1^2 + 2\lambda_1\mu_1 \cos(\theta_1), \lambda_2^2 + \mu_2^2 + 2\lambda_2\mu_2 \cos(\theta_2)).$$

Then for all $n \geq A$, and all $1 \leq k, \ell \leq n-1$ with $k \neq \ell$, $k \neq n-\ell$, and $\gcd(k, \ell, n) = 1$, there is $t \in \mathbb{Z}$ such that

$$\lambda_1 \cos \frac{k(\theta_1 + 2\pi t)}{n} + \mu_1 \cos \frac{(n-k)(\theta_1 + 2\pi t)}{n} + \lambda_2 \cos \frac{\ell(\theta_2 + 2\pi t)}{n} + \mu_2 \cos \frac{(n-\ell)(\theta_2 + 2\pi t)}{n} > \sqrt{s} - \varepsilon.$$

Proof. Let x be as in Lemma 6.2, that is

$$\lambda_1 \cos \left(\frac{k\theta_1}{n} + x \right) + \mu_1 \cos \left(\frac{(n-k)\theta_1}{n} - x \right) = \sqrt{\lambda_1^2 + \mu_1^2 + 2\lambda_1\mu_1 \cos(\theta)} \geq \sqrt{s}. \quad (6.4)$$

Let n be $\geq A$, let $1 \leq k, \ell \leq n-1$ be with $k \neq \ell$, $k \neq n-\ell$, and $\gcd(k, \ell, n) = 1$. For all $j \in \mathbb{Z}$, we denote

$$\begin{aligned} \alpha_j &= \lambda_1 \cos \frac{k(\theta_1 + 2\pi j)}{n} + \mu_1 \cos \frac{(n-k)(\theta_1 + 2\pi j)}{n} \\ \beta_j &= \lambda_2 \cos \frac{\ell(\theta_2 + 2\pi j)}{n} + \mu_2 \cos \frac{(n-\ell)(\theta_2 + 2\pi j)}{n}. \end{aligned}$$

One of the statement of Lemma 6.8(1), (2), or (3) holds. Assume that Lemma 6.8(1) holds. By permuting k and ℓ , we can assume $1 < \gcd(k, n) \leq \sqrt{n}$. Set $d = \gcd(k, n)$. Let $r \in \mathbb{Z}$ such that $kr \equiv d \pmod{n}$. Set $q = \lfloor \frac{nx}{2\pi d} \rfloor$. The following equalities hold:

$$\begin{aligned} \alpha_{rq} &= \lambda_1 \cos \frac{k(\theta_1 + 2\pi rq)}{n} + \mu_1 \cos \frac{(n-k)(\theta_1 + 2\pi rq)}{n} \\ &= \lambda_1 \cos \left(\frac{k\theta_1}{n} + \frac{2\pi krq}{n} \right) + \mu_1 \cos \left(\frac{(n-k)\theta_1}{n} - \frac{2\pi krq}{n} \right) \\ &= \lambda_1 \cos \left(\frac{k\theta_1}{n} + \frac{2\pi dq}{n} \right) + \mu_1 \cos \left(\frac{(n-k)\theta_1}{n} - \frac{2\pi dq}{n} \right). \end{aligned}$$

Note that the following inequalities hold:

$$0 \leq x - \frac{2\pi dq}{n} = x - \frac{2\pi d}{n} \left\lfloor \frac{nx}{2\pi d} \right\rfloor \leq \frac{2\pi d}{n} \leq \frac{2\pi}{\sqrt{n}} \leq \frac{2\pi}{\sqrt{A}} \leq \frac{\varepsilon}{|\lambda_1| + |\mu_1|}.$$

However, $|\cos(a) - \cos(b)| \leq |a - b|$ for all $a, b \in \mathbb{R}$. It follows that

$$\alpha_{rq} \geq \lambda_1 \cos \left(\frac{k\theta_1}{n} + x \right) + \mu_1 \left(\cos \frac{(n-k)\theta_1}{n} - x \right) - (|\lambda_1| + |\mu_1|) \frac{\varepsilon}{|\lambda_1| + |\mu_1|} = \sqrt{s} - \varepsilon.$$

Set $r_1 = \frac{n}{d}$ and $r_2 = \frac{k}{d}$. So $kr_1 = nr_2$. For all $t \in \mathbb{Z}$ the following equalities hold:

$$\begin{aligned} \alpha_{rq+tr_1} &= \lambda_1 \cos \frac{k(\theta_1 + 2(rq + tr_1)\pi)}{n} + \mu_1 \cos \frac{(n-k)(\theta_1 + 2(rq + tr_1)\pi)}{n} \\ &= \lambda_1 \cos \left(\frac{k(\theta_1 + 2\pi pq)}{n} + \frac{2\pi ktr_1}{n} \right) + \mu_1 \cos \left(\frac{(n-k)(\theta_1 + 2\pi rq)}{n} - \frac{2\pi ktr_1}{n} \right) \\ &= \lambda_1 \cos \left(\frac{k(\theta_1 + 2\pi rq)}{n} + 2\pi tr_2 \right) + \mu_1 \cos \left(\frac{(n-k)(\theta_1 + 2\pi rq)}{n} - 2\pi tr_2 \right) \\ &= \alpha_{rq}. \end{aligned}$$

Set $\gamma_1 = \frac{\ell(\theta_2 + 2\pi r q)}{n}$ and $\gamma_2 = \frac{(n-\ell)(\theta_2 + 2\pi r q)}{n}$. For all $t \in \mathbb{Z}$ the following equalities hold:

$$\begin{aligned} \beta_{r q + t r_1} &= \lambda_2 \cos\left(\frac{\ell(\theta_2 + 2\pi r q)}{n} + t \frac{2\pi r_1 \ell}{n}\right) + \mu_2 \cos\left(\frac{(n-\ell)(\theta_2 + 2\pi r q)}{n} - t \frac{2\pi r_1 \ell}{n}\right) \\ &= \lambda_2 \cos\left(\gamma_1 + t \frac{\ell}{d} 2\pi\right) + \mu_2 \cos\left(\gamma_2 - t \frac{\ell}{d} 2\pi\right). \end{aligned}$$

Note that $\gcd(d, \ell) = 1$. Then, we can deduce from Lemma 6.4 that $\sum_{t=0}^{d-1} \beta_{pr+tr_1} = 0$, hence there is $1 \leq t \leq d-1$ such that $\beta_{pr+tr_1} \geq 0$. Therefore

$$\alpha_{r q + t r_1} + \beta_{r q + t r_1} \geq \alpha_{r q} \geq \sqrt{s} - \varepsilon.$$

Now assume that Lemma 6.8(2) holds. Let r, u be non-zero integers such that $|u| \leq \sqrt{n}$, $r k \equiv 1 \pmod{n}$, $u R_n(r\ell) \neq R_n(ur\ell)$, and $R_n(ur\ell) \leq \sqrt{n}$. Set $q = \lfloor \frac{nx}{2\pi} \rfloor$. For each $t \in \mathbb{Z}$ we have

$$\alpha_{r(q+t)} = \lambda_1 \cos\left(\frac{k\theta_1}{n} + \frac{2\pi q}{n} + \frac{2\pi t}{n}\right) + \mu_1 \cos\left(\frac{(n-k)\theta_1}{n} - \frac{2\pi q}{n} - \frac{2\pi t}{n}\right)$$

If $|t| \leq |u| - 1$ then the following inequalities hold

$$\left| \frac{2\pi q}{n} + \frac{2\pi t}{n} - x \right| \leq \frac{2\pi}{n} + \frac{2\pi|t|}{n} \leq \frac{2\pi|u|}{n} \leq \frac{2\pi}{\sqrt{A}} \leq \frac{\varepsilon}{|\lambda_1| + |\mu_1|}.$$

As before it follows that

$$\alpha_{r(q+t)} \geq \lambda_1 \cos\left(\frac{k\theta_1}{n} + x\right) + \mu_1 \left(\cos\frac{(n-k)\theta_1}{n} - x\right) - \varepsilon.$$

We obtain the following inequality

$$\alpha_{r(q+t)} \geq \sqrt{s} - \varepsilon, \quad \text{for all } |t| \leq |u| - 1. \quad (6.5)$$

Note that

$$\beta_{r(q+t)} = \lambda_2 \cos\left(\frac{\ell(\theta_2 + 2\pi r q)}{n} + t \frac{2\pi r \ell}{n}\right) + \mu_2 \left(\cos\frac{(n-\ell)(\theta_2 + 2\pi r q)}{n} - t \frac{2\pi r \ell}{n}\right).$$

Set $\gamma_1 = \frac{\ell(\theta_2 + 2\pi r q)}{n}$, $\gamma_2 = \frac{(n-\ell)(\theta_2 + 2\pi r q)}{n}$, and $\rho = \frac{2\pi r \ell}{n}$. Therefore the following equality holds.

$$\beta_{r(q+t)} = \lambda_2 \cos(\gamma_1 + t\rho) + \mu_2 \cos(\gamma_2 - t\rho). \quad (6.6)$$

As $u R_n(r\ell) \neq R_n(ur\ell)$, we have $u R_{2\pi}(\frac{r\ell}{n} 2\pi) \neq R_{2\pi}(u \frac{r\ell}{n} 2\pi)$, therefore $u R_{2\pi}(\rho) \neq R_{2\pi}(u\rho)$. As $R_n(ur\ell) \leq \sqrt{n}$ we have $R_{2\pi}(u\rho) = R_{2\pi}(u \frac{r\ell}{n} 2\pi) = \frac{R_n(ur\ell)}{n} 2\pi \leq \frac{2\pi}{\sqrt{n}} < \frac{\pi}{2}$. Hence by Lemma 6.6 there is $|t| < |u|$ such that

$$\beta_{r(q+t)} = \lambda_2 \cos(\theta_1 + t\rho) + \mu_2 \cos(\theta_2 - t\rho) \geq 0.$$

Therefore by (6.5) we have $\alpha_{r(q+t)} + \beta_{r(q+t)} \geq \sqrt{s} - \varepsilon$. \square

With a similar and simpler proof, we obtain the following variation of Theorem 6.9.

Theorem 6.10. *Let $\lambda_1, \mu_1, \lambda_2$ be real numbers, and $\varepsilon > 0$. Set*

$$A = \max\left(1, 2 \left\lceil 2\pi \frac{|\lambda_1| + |\mu_1|}{\varepsilon} \right\rceil\right).$$

Let θ_1, θ_2 be in \mathbb{R} . Set

$$s = \lambda_1^2 + \mu_1^2 + 2\lambda_1\mu_1 \cos(\theta_1).$$

Then for all $n \geq A$ even, and all $1 \leq k, \ell \leq n-1$ with $\gcd(k, \frac{n}{2}) = 1$ there is $t \in \mathbb{Z}$ such that

$$\lambda_1 \cos \frac{k(\theta_1 + 2\pi t)}{n} + \mu_1 \cos \frac{(n-k)(\theta_1 + 2\pi t)}{n} + \lambda_2 \cos \frac{\theta_2 + 2\pi t}{2} > \sqrt{s} - \varepsilon.$$

7. DETERMINATION OF THE JULIA ROBINSON NUMBER

Let $0 < a < b$ be coprime integers, let $\theta = \arccos(\frac{a}{b}) \in]0, \frac{\pi}{2}[$. Denote $\alpha = e^{i\theta} = \frac{a+i\sqrt{b^2-a^2}}{b}$. Let $r \geq 2$ be an integer. Assume the following:

- 4 divides b ;
- $v_2(b) - 1$ is coprime to r ;
- r divides b ;
- For all odd prime p dividing b , we have $v_p(b)$ coprime to r .

We fix such a, b, r in the whole section. Subsequently if (a, b, r) satisfies those conditions we say that (a, b, r) is good.

Given $n \geq 1$, denote $L_n = \mathbb{Q}(e^{\frac{ia}{n}})$, and $K_n = \mathbb{Q}(\cos(\frac{\theta}{n}))$. Moreover, set

$$L = \bigcup_{k \in \mathbb{N}} L_{r^k} \quad \text{and} \quad K = \bigcup_{k \in \mathbb{N}} K_{r^k}.$$

We say that K is the totally real field associated to (a, b, r) . Denote by N the smallest common multiple of $v_2(b) - 1$, and all $v_p(b)$ with p odd prime.

Let u be the square factor of $b+a$, and v be the square factor of $b-a$, that is $\frac{b+a}{u^2}$ (respectively $\frac{b-a}{v^2}$) are square-free. Note that the conditions in the following notation are similar to (2)-(4) in Lemma 3.16

Notation 7.1. Let $0 \leq k \leq N-1$. Denote by S_k the set of all pairs $(x, y) \in \mathbb{Q}^2$ that satisfies the following conditions:

- (1) For all odd prime number p such that p divides b we have $v_p(x) \geq \frac{k+1}{N} v_p(b)$ and $v_p(y) \geq \frac{N-k}{N} v_p(b)$;
- (2) We have $v_2(x) \geq 1 + \frac{k+1}{N}(v_2(b) - 1)$ and $v_2(y) \geq 1 + \frac{N-k}{N}(v_2(b) - 1)$;
- (3) $u(x+y)$ and $v(x-y)$ are integers.

We denote by $U = S_1 \cup \dots \cup S_{N-1}$.

Notation 7.2. As before we consider the following map

$$N: \mathbb{R}^2 \rightarrow \mathbb{R}$$

$$(x, y) \mapsto N(x, y) = x^2 + y^2 + 2xy \frac{a}{b}.$$

Set $s = \inf \left(\sqrt{N(x, y)} \mid 0 \leq k \leq N-1 \text{ and } (x, y) \in S_k \right)$. The goal of this section is to prove that the Julia Robinson Number of \mathcal{O}_K is $\lceil s \rceil + s$.

Note that $(x, y) \in S_k$ if and only if $(y, x) \in S_{N-k-1}$. We say that $(\theta, K, s, U, N: \mathbb{R}^2 \rightarrow \mathbb{R})$ are the *objects associated to* (a, b, r) .

Lemma 7.3. *Let $1 \leq k < N$. Then S_k is a discrete \mathbb{Z} -submodule of \mathbb{Q}^2 .*

Proof. By the well-known property that for every valuation v over a field F and $\alpha, \beta \in F$, $v(\alpha + \beta) \geq \min(v(\alpha), v(\beta))$, we immediately have that if (x_1, y_1) and (x_2, y_2) satisfy the conditions (1) and (2) of Notation 7.1, then $(x_1 + x_2, y_1 + y_2)$ also satisfies those conditions. Finally, it is clear that if (x_1, y_1) and (x_2, y_2) satisfy

the conditions (3) of Notation 7.1, then $(x_1 + x_2, y_1 + y_2)$ also. Thus S_k is a subgroup of \mathbb{Q}^2 .

Observe that condition (3) in Notations 7.1 implies that if $(x, y) \in S_k$, then $2uvx \in \mathbb{Z}$ and $2uvy \in \mathbb{Z}$. Then S_k is contained in the discrete sub- \mathbb{Z} -module generated by $(1/2uv, 0)$ and $(0, 1/2uv)$. In particular it is a discrete \mathbb{Z} -module. \square

Lemma 7.4. *Let $0 \leq k \leq N - 1$. Let $x, y \in \mathbb{Q}$. Then for all $q \in]\frac{k}{N}, \frac{k+1}{N}[\cap \mathbb{Q}$, we have $(x, y) \in S_k$ if and only if the following three conditions hold:*

- (1) *For all odd prime number p such that p divides b we have $v_p(x) \geq qv_p(b)$ and $v_p(y) \geq (1 - q)v_p(b)$;*
- (2) *We have $v_2(x) \geq 1 + q(v_2(b) - 1)$ and $v_2(y) \geq 1 + (1 - q)(v_2(b) - 1)$;*
- (3) *$u(x + y)$ and $v(x - y)$ are integers.*

Proof. Let $q \in]\frac{k}{N}, \frac{k+1}{N}[\cap \mathbb{Q}$. Let p be an odd prime number dividing b and let (x, y) be in S_k .

Assume that there is an integer ℓ such that $\frac{kv_p(b)}{N} < \ell < \frac{(k+1)v_p(b)}{N}$, then $k < \frac{N}{v_p(b)}\ell < k+1$. However, $v_p(b)|N$, hence $\frac{N}{v_p(b)}\ell$ is an integer; a contradiction. Therefore there is no integer in $] \frac{kv_p(b)}{N}, \frac{(k+1)v_p(b)}{N} [$. Note that $qv_p(b) \in] \frac{kv_p(b)}{N}, \frac{(k+1)v_p(b)}{N} [$. As $v_p(x)$ is an integer it follows that $v_p(x) \geq qv_p(b)$ if and only if $v_p(x) \geq \frac{k+1}{N}v_p(b)$. Similarly $v_p(y) \geq (1 - q)v_p(b)$ if and only if $v_p(y) \geq \frac{N-k}{N}v_p(b)$. Therefore the condition Lemma 7.4(1) and Notation 7.1(1) are equivalent.

Similarly Lemma 7.4(2) and Notation 7.1(2) are equivalent. The last condition is the same. \square

Lemma 7.5. *Let $n \geq 1$ be an integer dividing a power of r . Let $\lambda_0, \lambda_1, \dots, \lambda_{n-1}$ be in \mathbb{Q} . The following statement are equivalent:*

- (1) $\lambda_0 + \sum_{\ell=1}^{n-1} \lambda_\ell \cos(\frac{\ell\theta}{n})$ is in \mathcal{O}_K .
- (2) $\lambda_0 \in \mathbb{Z}$, and for all $1 \leq \ell \leq n - 1$ we have $(\lambda_\ell, \lambda_{n-\ell}) \in S_k$, where $0 \leq k \leq N - 1$ such that $\frac{\ell}{n} \in]\frac{k}{N}, \frac{k+1}{N}[$.

Proof. Assume (1). Observe that N is the product of $v_2(b) - 1$ and all $v_p(b)$ for $p|b$ odd primes, which are each coprime to r , thus N and r are coprime. As n divides a power of r it follows that n and N are coprime, therefore for all $1 \leq \ell \leq n - 1$ and all $0 \leq k \leq N$ we have $\frac{\ell}{n} \neq \frac{k}{N}$.

By Lemma 3.16 we have $\lambda_0 \in \mathbb{Z}$, and for all $1 \leq \ell \leq n - 1$ the following statement holds.

- (i) For all odd prime number p dividing b , $v_p(\lambda_\ell) \geq \left\lceil \frac{\ell v_p(b)}{n} \right\rceil$;
- (ii) $v_2(\lambda_\ell) \geq 1 + \left\lceil \frac{\ell(v_2(b)-1)}{n} \right\rceil$;
- (iii) $u(\lambda_\ell + \lambda_{n-\ell})$ and $v(\lambda_\ell + \lambda_{n-\ell})$ are integers.

Let $1 \leq \ell \leq n - 1$. Let $0 \leq k \leq N - 1$ be such that $\frac{\ell}{n} \in]\frac{k}{N}, \frac{k+1}{N}[$. As $\frac{\ell}{n} \neq \frac{k}{N}$, we have $\frac{\ell}{n} \in]\frac{k}{N}, \frac{k+1}{N}[$. Hence it follows from Lemma 7.4 that $(\lambda_\ell, \lambda_{n-\ell}) \in S_k$. Therefore (2) holds.

Assume (2). Let $1 \leq \ell \leq n - 1$. Let $0 \leq k \leq N - 1$ such that $\frac{\ell}{n} \in]\frac{k}{N}, \frac{k+1}{N}[$. As $(\lambda_\ell, \lambda_{n-\ell}) \in S_k$ it follows from Lemma 7.4 that (i), (ii), and (iii) holds. By Lemma 3.16, we have that $\lambda_0 + \sum_{\ell=1}^{n-1} \lambda_\ell \cos(\frac{\ell\theta}{n})$ is an algebraic integer, moreover it belongs to K , therefore (1) holds. \square

Lemma 7.6. *Let M be a discrete \mathbb{Z} -submodule of \mathbb{Q}^2 . Then*

- (1) For all $C \in \mathbb{R}$ there are only finitely many $(x, y) \in M$ such that $N(x, y) \leq C$.
- (2) If M is not trivial, then there is $(x, y) \in M \setminus \{(0, 0)\}$ such that, for all $(x', y') \in M \setminus \{(0, 0)\}$ we have $N(x, y) \leq N(x', y')$.

Proof. Let C be a real number and consider the affine conic with equation $X^2 + Y^2 + 2XY\frac{a}{b} = C$. Since $0 < a < b$ by hypothesis, the locus of its zeros in \mathbb{R}^2 is an ellipse. Then the set $E_C = \{(x, y) \in \mathbb{R}^2 \mid N(x, y) \leq C\}$ is compact. Since M is discrete, there are only finitely many $(x, y) \in M$ such that $N(x, y) \leq C$.

By hypothesis M is discrete, then every point of M is isolated. Then take $C_1 \in \mathbb{R}$ such that E_{C_1} only contains $(0, 0)$ and $C_2 \in \mathbb{R}$ such that E_{C_2} contains at least a point of M distinct from $(0, 0)$ (observe that C_2 exists because M is not trivial). Then the set B given by E_{C_2} minus the interior part of E_{C_1} is a compact, and so it contains a finite number of elements of M . Take $(x, y) \in B$ such that $N(x, y) \leq N(w, z)$ for every $(w, z) \in B$. Then $(x, y) \neq (0, 0)$ and $N(x, y) \leq N(x', y')$ for every $(x', y') \in M \setminus \{(0, 0)\}$, because if $(x', y') \notin B$, then $N(x', y') \geq C_2 \geq N(x, y)$. \square

Lemma 7.7. *Let (λ, μ) be in S_k . Then there are infinitely many $q \in \mathbb{Q} \cap [0, 1]$ such that $\lambda \cos(q\theta) + \mu \cos((1 - q)\theta) \in \mathcal{O}_K$. Moreover we can choose q such that the denominator of q is any large enough divisor of a power of r .*

Proof. Take coprime integers t and $n = r^e$ such that $|\frac{2k+1}{2N} - \frac{t}{n}| < \frac{1}{2N}$, Note that there are infinitely many t, n that satisfy this condition. In particular $\frac{t}{n}$ is in the interval $]\frac{k}{N}, \frac{k+1}{N}[$. It follows that $\frac{n-t}{n} \in]\frac{N-k-1}{N}, \frac{N-k}{N}[$, moreover $(\mu, \lambda) \in S_{N-k-1}$. Therefore by Lemma 7.5 we have $\lambda \cos(\frac{t}{n}\theta) + \mu \cos((\frac{n-t}{n})\theta) \in \mathcal{O}_K$. \square

Corollary 7.8. *Denote by t the product of all odd primes (without counting multiplicity) dividing b . The following statements hold.*

- (1) s^2 is an integer, and $8t \mid s^2$.
- (2) $s \geq 2\sqrt{2}$.
- (3) $s \leq 4t$.
- (4) $s^2 \leq 8t\frac{b-a}{v^2}$ and $s^2 \leq 8t\frac{b+a}{v^2}$. In particular if $b - a$ is a square then $s^2 = 8t$.
- (5) If $(b - a)$ and $(b + a)$ are square-free and $b \geq 2t + a$, then $s = 4t$.

Proof. It follows from Lemma 7.3 and Lemma 7.6 that there is $0 \leq k \leq N - 1$ and $(\lambda, \mu) \in S_k \setminus \{(0, 0)\}$ such that $s = \sqrt{N(\lambda, \mu)}$. So $s^2 = \lambda^2 + \mu^2 + 2\lambda\mu\frac{a}{b}$.

Let $q \neq \frac{1}{2}$ be as in Lemma 7.7, that is $\lambda \cos(q\theta) + \mu \cos((1 - q)\theta) \in \mathcal{O}_K$. So $T((\lambda \cos(q\theta) + \mu \cos((1 - q)\theta))^2) = \frac{1}{2}N(\lambda, \mu) = \frac{1}{2}s^2$ is an integer.

Let p be an odd prime number dividing b . It follows from the definition of S_k that

$$v_p(\lambda) \geq \left\lceil (k+1)\frac{v_p(b)}{N} \right\rceil \geq 1 \quad \text{and} \quad v_p(\mu) \geq \left\lceil (N-k)\frac{v_p(b)}{N} \right\rceil \geq 1. \quad (7.1)$$

Therefore the following statement holds

$$v_p(\lambda\mu) = v_p(\lambda) + v_p(\mu) \quad (7.2)$$

$$\geq \left\lceil (k+1)\frac{v_p(b)}{N} \right\rceil + \left\lceil (N-k)\frac{v_p(b)}{N} \right\rceil \quad (7.3)$$

$$\geq \left\lceil (N+1)\frac{v_p(b)}{N} \right\rceil \quad (7.4)$$

$$\geq v_p(b) + 1. \quad (7.5)$$

Hence $v_p(2\lambda\mu\frac{a}{b}) \geq 1$, therefore

$$v_p(s^2) = v_p(\lambda^2 + \mu^2 + 2\lambda\mu\frac{a}{b}) \geq \min(v_p(\lambda^2), v_p(\mu^2), v_p(2\lambda\mu\frac{a}{b})) \geq 1 = v_p(8t). \quad (7.6)$$

Similarly the following statement hold

$$v_2(\lambda) \geq \left\lceil 1 + (k+1)\frac{v_2(b)-1}{N} \right\rceil \geq 2 \quad \text{and} \quad v_2(\mu) \geq \left\lceil 1 + (N-k)\frac{v_2(b)-1}{N} \right\rceil \geq 2. \quad (7.7)$$

Therefore

$$v_2(\lambda\mu) \geq \left\lceil 1 + (k+1)\frac{v_2(b)-1}{N} \right\rceil + \left\lceil 1 + (N-k)\frac{v_2(b)-1}{N} \right\rceil \quad (7.8)$$

$$\geq \left\lceil 2 + (N+1)\frac{v_2(b)-1}{N} \right\rceil \quad (7.9)$$

$$\geq 2 + v_2(b). \quad (7.10)$$

So $v_2(2\lambda\mu\frac{a}{b}) \geq 3$, therefore

$$v_2(s^2) = v_2(\lambda^2 + \mu^2 + 2\lambda\mu\frac{a}{b}) \geq 3 = v_2(8t). \quad (7.11)$$

It follows from (7.6) and (7.11) that $8t|s^2$, so (1) holds. In particular $8 \leq s^2$, so $s \geq 2\sqrt{2}$, that is (2) holds.

Let p be an odd prime number dividing b . Note that $v_p(4t) = 1 = \left\lceil \frac{v_p(b)}{N} \right\rceil$ and $v_2(4t) = 2 = \left\lceil 1 + \frac{v_2(b)-1}{N} \right\rceil$, and $4tu$ and $4tv$ are integers (as $4t$ is an integer) thus $(4t, 0) \in S_0$. Therefore $s \leq \sqrt{N(4t, 0)} = 4t$, so (3) holds.

Set $k = \frac{N-1}{2}$. Set $x = 2r\left(\frac{b}{2}, \frac{1}{2}\right)$. Note that $v_2(x) = 1 + \left\lceil \frac{v_2(b)-1}{2} \right\rceil$. As $v_2(b) - 1$ is relatively prime to b , it follows that $v_2(b) - 1$ is odd, thus $v_2(x) = 1 + \frac{v_2(b)}{2}$. Note that $N \geq v_2(b) - 1$ thus the following inequality holds

$$\frac{N+1}{2N}(v_2(b)-1)+1 = \frac{v_2(b)-1}{2} + \frac{v_2(b)-1}{2N} + 1 \leq \frac{v_2(b)-1}{2} + \frac{1}{2} + 1 = v_2(x). \quad (7.12)$$

Let p be an odd prime number dividing b . Then $v_p(x) = \left\lceil \frac{v_p(b)}{2} \right\rceil = \frac{v_p(b)+1}{2}$, as $v_p(b)$ is odd. As $N \geq v_p(b)$, the following inequality holds

$$\frac{N+1}{2N}v_p(b) = \frac{v_p(b)}{2} + \frac{v_p(b)}{2N} \leq \frac{v_p(b)}{2} + \frac{1}{2} = v_p(x). \quad (7.13)$$

As v is relatively prime to b and $\frac{k+1}{N} = \frac{N-k}{N} = \frac{N+1}{2N}$, it follows from (7.12) and (7.13) that $(\frac{x}{v}, -\frac{x}{v})$ satisfies the condition (1) and (2) of Notation 7.1. Moreover $u(\frac{x}{v} - \frac{x}{v}) = 0$ and $v(\frac{x}{v} + \frac{x}{v}) = 2x$ are integers, hence $(\frac{x}{v}, -\frac{x}{v}) \in S_k$. Therefore the following inequality holds

$$s \leq N \left(\frac{x}{v}, -\frac{x}{v} \right) = 2\frac{x^2}{v^2} \left(1 - \frac{a}{b} \right). \quad (7.14)$$

However, as $v_2(x) = 1 + \frac{v_2(b)}{2}$ and for all odd prime p dividing b we have $v_p(x) = \frac{v_p(b)+1}{2}$, and for all other prime p we have $v_p(x) = 0$, it follows that $x^2 = 4tb$. It follows from (7.14) that $s \leq 2\frac{4tb}{v^2}\frac{b-a}{b} = 8t\frac{b-a}{v^2}$.

A similar proof, considering $(\frac{x}{u}, \frac{x}{u}) \in S_k$, yields $s \leq 8t\frac{b+a}{u^2}$. Hence (4) holds.

If $b \geq 2t + a$, $b - a$ and $b + a$ are square-free then $u = v = 1$. Thus $\lambda + \mu$ and $\lambda - \mu$ are integers. However as seen in (7.7), we have $v_2(\lambda) \geq 1$, and $v_2(\mu) \geq 1$.

Therefore λ and μ are integers. Moreover it follows from (7.1) and (7.7) that $4t$ divides λ and μ .

If $\lambda\mu \geq 0$ then $s^2 = N(\lambda, \mu) = \lambda^2 + \mu^2 + 2\lambda\mu\frac{a}{b} \geq \max(\lambda^2, \mu^2)$ however $4t$ divides both λ and μ , and $(\lambda, \mu) \neq (0, 0)$, thus $s \geq 4t$.

Assume that $\lambda\mu < 0$. The following statement holds

$$N(\lambda, \mu) = \lambda^2 + \mu^2 + 2\lambda\mu\frac{a}{b} = (\lambda + \mu)^2 - 2\lambda\mu\left(1 - \frac{a}{b}\right) \geq (\lambda + \mu)^2.$$

However $4t|\lambda + \mu$, so if $\lambda + \mu \neq 0$ we have $s = \sqrt{N(\lambda, \mu)} \geq 4t$.

Now assume that $\lambda + \mu = 0$, that is $\lambda = -\mu$. The following holds

$$N(\lambda, \mu) = \lambda^2 + \lambda^2 - 2\lambda^2\frac{a}{b} = 2\lambda^2\left(1 - \frac{a}{b}\right).$$

It follows from (7.5) that $v_p(\lambda^2) = v_p(\lambda\mu) \geq v_p(b) + 1 = v_p(4bt)$, similarly (7.10) implies $v_2(\lambda^2) = v_2(\lambda\mu) = 2 + v_2(b) = v_2(4tb)$. Therefore $4tb$ divides λ^2 , so $N(\lambda, \mu) \geq 8tb\left(1 - \frac{a}{b}\right) = 8t(b-a) \geq 16t^2 = (4t)^2$, and so $s = \sqrt{N(\lambda, \mu)} \geq 4t$.

However, from (2) we also have $s \leq 4t$, thus $s = 4t$. Therefore (5) holds. \square

Corollary 7.9. *There are infinitely many $\beta \in \mathcal{O}_K$ such that all conjugates of β are in $[0, \lceil s \rceil + s]$. In particular the Julia Robinson Number of \mathcal{O}_K is at most $\lceil s \rceil + s$.*

Proof. By Lemma 7.3, for every k between 0 and $N-1$, S_k is a discrete \mathbb{Z} -module. Then, by Lemma 7.6, there exists k and $(\lambda, \mu) \in S_k$ such that $N(\lambda, \mu) = s$. By Lemma 7.7, there exist infinitely many $q \in [0, 1] \cap \mathbb{Q}$ such that $\gamma_q = \lambda \cos(q\theta) + \mu \cos((1-q)\theta) \in \mathcal{O}_K$. From Lemma 3.15 we see that the conjugates of γ_q are the $\lambda \cos(q\theta + 2\pi qt) + \mu \cos((1-q)\theta - 2\pi qt)$, with $t \in \mathbb{Z}$. Hence by Lemma 6.2, every conjugate γ'_q of γ satisfies

$$|\gamma'_q| \leq \sqrt{\lambda^2 + \mu^2 + 2\lambda\mu \cos(\theta)} = \sqrt{N(\lambda, \mu)} = s.$$

Hence every conjugate of $\lceil s \rceil + \gamma_q$ is positive, and it is smaller than $\lceil s \rceil + s$. This proves that the Julia Robinson Number of \mathcal{O}_K is at most $\lceil s \rceil + s$. \square

Theorem 7.10. *The Julia Robinson Number of \mathcal{O}_K is $\lceil s \rceil + s$. Moreover \mathcal{O}_K has the Julia Robinson Property.*

Proof. From Corollary 7.9 we already know that the Julia Robinson Number of \mathcal{O}_K is at most $\lceil s \rceil + s$. Moreover there are infinitely many $\beta \in \mathcal{O}_K$ such that all conjugates of β are in $[0, \lceil s \rceil + s]$, thus to prove that \mathcal{O}_K has the Julia Robinson Property, we only have to prove that the Julia Robinson Number of \mathcal{O}_K is $\lceil s \rceil + s$.

Let $\varepsilon > 0$ be a real number. We can assume that $\varepsilon \leq 1$, and if s is not an integer we also assume that ε is smaller than the fractional part of s . In particular

$$q - s + \varepsilon < 0, \text{ for all integer } q < s. \quad (7.15)$$

Denote by S the set of all pairs $(\lambda, \mu) \in \bigcup_{k=0}^{N-1} S_k$ such that $N(\lambda, \mu) \leq 2s^2 + s + 1$. We see from Lemma 7.6 that S is finite. Let B be a power of r such that $B \geq 17$ and

$$B \geq \left\lceil 4\pi \frac{|\lambda| + |\mu|}{\varepsilon} \right\rceil^2, \text{ for all } (\lambda, \mu) \in S.$$

Set

$$X = \left\{ \lambda \cos \frac{t\theta}{d} + \mu \cos \frac{(d-t)\theta}{d} \mid d \leq B, 1 \leq t \leq d-1, \text{ and } (\lambda, \mu) \in S \right\}$$

and

$$X' = \left\{ \lambda \cos \frac{\theta}{2} \mid (\lambda, \lambda) \in S \right\}.$$

In particular $0 \in X$, and so $X \subseteq X + X$. Set $Y = \{1, 2, \dots, 2\lceil s \rceil - 1\} + X + (X \cup X')$.

Let $\gamma \in \mathcal{O}_K$ be such that all conjugates of γ are in $[0, \lceil s \rceil + s]$. There is $n \in \mathbb{N}$ such that $\beta \in K_n = \mathbb{Q}(\cos(\frac{\theta}{n}))$. By Lemma 3.14 we can write $\gamma = \lambda_0 + \sum_{t=1}^{n-1} \lambda_t \cos \frac{t\theta}{n}$, with $\lambda_0, \lambda_1, \dots, \lambda_{n-1} \in \mathbb{Q}$. From Lemma 3.16 we see that $\lambda_0 \in \mathbb{Z}$, and for all $1 \leq t \leq n-1$ we have $(\lambda_t, \lambda_{n-t})$ in some S_k , with $0 \leq k \leq N-1$. So, by construction of s , either $(\lambda_t, \lambda_{n-t}) = (0, 0)$ or $N(\lambda_t, \lambda_{n-t}) \geq s^2$. Set $\beta = \gamma - \lambda_0 = \sum_{t=1}^{n-1} \lambda_k \cos \frac{t\theta}{n}$. It follows from Lemma 3.14 that $T(\beta) = 0$.

As $\gamma = \lambda_0 + \beta$ has no negative conjugates, it follows from Lemma 4.1 that $\lambda_0 > 0$, similarly we have $\lambda_0 < 2\lceil s \rceil$. Moreover the maximal difference of two conjugates of β is $\delta(\beta) = \delta(\gamma) \leq \lceil s \rceil + s$. The trace of β^2 is computed in Lemma 5.1, that is $T(\beta^2) = \frac{1}{2} \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} N(\lambda_k, \lambda_{n-k})$. Moreover from Lemma 4.4 we have $\delta(\beta) \geq 2\sqrt{T(\beta^2)}$. Therefore we obtain the following inequality

$$2\sqrt{\frac{1}{2} \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} N(\lambda_k, \lambda_{n-k})} \leq \lceil s \rceil + s \leq 2s + 1.$$

Thus

$$2 \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} N(\lambda_k, \lambda_{n-k}) \leq (2s + 1)^2. \quad (7.16)$$

Let $1 \leq k \leq n-1$. If $N(\lambda_k, \lambda_{n-k}) > 2s^2 + s + 1$ then from (7.16) we have

$$4s^2 + 2s + 2 = 2(2s^2 + s + 1) < 2N(\lambda, \mu) \leq (2s + 1)^2 = 4s^2 + 2s + 1;$$

a contradiction. Therefore $N(\lambda_k, \lambda_{n-k}) \leq 2s^2 + s + 1$. Hence we have $(\lambda_k, \lambda_{n-k}) \in S$, for all $1 \leq k \leq n-1$.

Let C be the number of $k \in \{1, \dots, \lfloor \frac{n}{2} \rfloor\}$ such that $(\lambda_k, \lambda_{n-k}) \neq (0, 0)$. For each such k we have $N(\lambda_k, \lambda_{n-k}) \geq s^2$. Thus by (7.16) we have $2Cs^2 \leq (2s + 1)^2$, so $C \leq 2 + \frac{1}{s} + \frac{1}{2s^2}$. However, from Lemma 7.8 we have $s \geq 2\sqrt{2}$. As C is an integer it follows that $C \leq 2$.

If $C = 0$, then $\beta = 0$, and so $\gamma = \lambda \in Y$. If $C = 1$ then we can write $\beta = \lambda \cos \frac{\theta}{2}$ with $(\lambda, \lambda) \in S$, or $\beta = \lambda \cos \frac{k\theta}{n} + \mu \cos \frac{k\theta}{n}$, with $1 \leq k \leq n-1$, and $(\lambda, \mu) \in S$. In the first case we see that $\gamma = \lambda_0 + \beta \in Y$. We now treat the second case. We can assume that $\gcd(k, n) = 1$.

Assume that $n \geq B$. Thus $n \geq A$, so from Corollary 6.3 and Lemma 3.15 we obtain a conjugate β' of β such that $\beta' \geq \sqrt{\lambda^2 + \mu^2 + 2\lambda\mu \cos \theta} - \varepsilon = N(\lambda, \mu) - \varepsilon \geq s - \varepsilon$. Similarly we have a conjugate β'' of β such that $\beta'' \leq -s + \varepsilon$.

If $\lambda_0 \geq \lceil s \rceil$, then a conjugate of γ is $\lambda_0 + \beta' \geq \lceil s \rceil + s - \varepsilon$; a contradiction. If $\lambda_0 < \lceil s \rceil$, then $\lambda_0 < s$, so a conjugate of γ is $\lambda_0 + \beta'' \leq \lambda_0 - s + \varepsilon$ which is negative by (7.15); a contradiction. Therefore $n \leq B$, and so $\gamma \in Y$.

If $C = 2$, once again simplifying fractions, then we can write $\beta = \lambda_1 \cos \frac{k\theta}{n} + \mu_1 \cos \frac{(n-k)\theta}{n} + \lambda_2 \cos \frac{\theta}{2}$ with $(\lambda_2, \lambda_2) \in S$, $(\lambda_1, \mu_1) \in S$, and $\gcd(k, \frac{n}{2}) = 1$, or $\beta = \lambda_1 \cos \frac{k\theta}{n} + \mu_1 \cos \frac{(n-k)\theta}{n} + \lambda_2 \cos \frac{\ell\theta}{n} + \mu_2 \cos \frac{(n-\ell)\theta}{n}$, with $1 \leq k < \ell < n - \ell \leq n-1$, $(\lambda_1, \mu_1) \in S$, $(\lambda_2, \mu_2) \in S$, and $\gcd(k, \ell, n) = 1$. In both case, assuming $n \geq B$, using respectively Theorem 6.10 and Theorem 6.9 we have conjugates β' and β''

of β such that $\beta' \geq s - \varepsilon$, and $\beta'' \leq -s + \varepsilon$. As before we reach a contradiction. Therefore $n \leq B$ and so $\gamma = \lambda_0 + \beta \in Y$. \square

As \mathcal{O}_K has the Julia Robinson Property, from the result of Robinson in [20] we obtain the following Corollary of Theorem 7.10.

Corollary 7.11. *The first order theory of \mathcal{O}_K is undecidable.*

The following lemma express that there are not many “simple” square roots of rationals in K . It will be used to prove that many of the constructed fields are distincts.

Lemma 7.12. *Let $(\lambda, \mu) \in \mathbb{Q} \setminus \{(0, 0)\}$, let $n \geq 2$ be an integer dividing a power of r , let $1 \leq k \leq n - 1$. Then $\left(\lambda \cos \frac{k\theta}{n} + \mu \cos \frac{(n-k)\theta}{n}\right)^2 \in \mathbb{Q}$ if and only if $2k = n$.*

Proof. Set $\beta = \lambda \cos \frac{k\theta}{n} + \mu \cos \frac{(n-k)\theta}{n}$. Note that $\left(\cos \frac{\theta}{2}\right)^2 = \frac{1+\cos\theta}{2} = \frac{b+a}{2b}$ belongs to \mathbb{Q} . Thus if $n = 2k$, then the following statement holds

$$\beta^2 = \left((\lambda + \mu) \cos \frac{\theta}{2}\right)^2 \in \mathbb{Q}.$$

Assume that $n \neq 2k$, hence $k \neq n - k$. We can assume that $k < n - k$. It follows from Lemma 3.8 that

$$\beta^2 = \frac{1}{2} \left(\lambda^2 + \mu^2 + 2\lambda\mu \frac{a}{b} + (\lambda^2 - \mu^2) \cos \frac{2k\theta}{n} + 2(\lambda\mu + \mu^2 \frac{a}{b}) \cos \frac{(n-2k)\theta}{n} \right). \tag{7.17}$$

Note that if $(\lambda^2 - \mu^2) = 0$ then $\lambda = \pm\mu \neq 0$, hence $(\lambda\mu + \mu^2 \frac{a}{b}) = \mu^2(\frac{a}{b} \pm 1) \neq 0$. Assume that $2k \neq n - 2k$, it follows from Lemma 3.14 that $1, \cos \frac{2k\theta}{n}, \cos \frac{n-2k\theta}{n}$ are \mathbb{Q} -linearly independent, thus $\beta^2 \notin \mathbb{Q}$.

Assume that $2k = n - 2k$, thus $4k = n$, so n is even, and so r is even. Moreover from (7.17) we have

$$\beta^2 = \frac{1}{2} \left(\lambda^2 + \mu^2 + 2\lambda\mu \frac{a}{b} + (\lambda^2 - \mu^2 + 2\lambda\mu + 2\mu^2 \frac{a}{b}) \cos \frac{\theta}{2} \right).$$

Assume that $\beta^2 \in \mathbb{Q}$. However from Lemma 3.14 we see that $1, \cos \frac{\theta}{2}$ are linearly independent, thus $\lambda^2 - \mu^2 + 2\lambda\mu + 2\mu^2 \frac{a}{b} = 0$. Hence λ is a root of the polynomial $\Lambda^2 + 2\mu\Lambda + 2\mu^2 \frac{a}{b} - \mu^2$. Therefore the discriminant $\Delta = 4\mu^2 - 4(2\mu^2 \frac{a}{b} - \mu^2) = 8\mu^2 - 8\mu^2 \frac{a}{b} = 8\mu^2 \frac{a+b}{b}$ is a square. Therefore $2\frac{a+b}{b}$ is a square, thus $v_2(2\frac{a+b}{b}) = 1 - v_2(b)$ is even. However r and $v_2(b) - 1$ are relatively prime, and r is even; a contradiction. \square

8. EXAMPLE OF JULIA ROBINSON'S NUMBER

In this section we construct particular a and b such that we can explicitly compute the number s (see beginning of Section 7 and Notation 7.2). We obtain the examples of Julia Robinson Numbers in Theorem 8.8. We basically consider two family of cases. For one we choose a, b such that $b - a = 1$ (which is a square). The other family is such that $b - a, b + a$ are large and square-free, the existence of such pairs (a, b) relies on the fact that the density of square-free integers is large.

Notation 8.1. Given $x \geq 0, k \geq 1$, and i relatively prime to k , denote by $Q(x; i, k)$ the number of integer $j \geq 0$, such that $j \leq x$ and $j \equiv i \pmod{k}$.

The following theorem was proved by Prachar [19] (see also Hooley [7]).

Theorem 8.2. *Let $k \geq 1$ be an integer and i relatively prime to k . Then we have the following equivalence*

$$Q(x; i, k) \sim_{x \rightarrow \infty} x \frac{6}{\pi^2 k} \prod_{p|k} \frac{p^2}{p^2 - 1}.$$

Lemma 8.3. *Let $k \geq 1$ be an integer, let $1 > \varepsilon > 0$ be a real number. Then there is an integer C such that for all b multiple of k if $b \geq C$ then there is $1 \leq a \leq \varepsilon b$ such that $a \equiv 1 \pmod{k}$, and $b - a$ and $b + a$ are square-free.*

Proof. Set $l = \frac{6}{\pi^2 k} \prod_{p|k} \frac{p^2}{p^2 - 1}$. Note that $kl \geq \frac{3}{5}$. From Theorem 8.2 we have

$$\lim_{x \rightarrow \infty} \frac{Q(x; 1, k)}{x} = l \quad \text{and} \quad \lim_{x \rightarrow \infty} \frac{Q(x; -1, k)}{x} = l.$$

Set $3\eta = \varepsilon(l - \frac{3}{5k})$. Pick $A \geq 0$ such that for all $x \geq A$ and $i = \pm 1$ we have $|\frac{Q(x; i, k)}{x} - l| \leq \eta$.

Set $C = \frac{A+1}{1-\varepsilon}$. Let $b \geq C$, thus $b - \varepsilon b - 1 \geq A$. Therefore we have

$$\left| \frac{Q(b - \varepsilon b; i, k)}{b - \varepsilon b} - l \right| \leq \eta.$$

It follows that

$$|Q(b - \varepsilon b; i, k) - (b - \varepsilon b)l| \leq (b - \varepsilon b)\eta. \quad (8.1)$$

Similarly

$$|Q(b; i, k) - bl| \leq b\eta \quad (8.2)$$

and

$$|Q(b + \varepsilon b; i, k) - (b + \varepsilon b)l| \leq (b + \varepsilon b)\eta. \quad (8.3)$$

From (8.1) and (8.2) we have

$$|Q(b; i, k) - Q(b - \varepsilon b; i, k) + \varepsilon bl| \leq 2b - \varepsilon b\eta \leq 3b\eta.$$

$$Q(b; i, k) - Q(b - \varepsilon b; i, k) \geq \varepsilon bl - 3b\eta = \varepsilon b(l - \frac{3}{5k}) = \varepsilon b \frac{3}{5k}. \quad (8.4)$$

Similarly from (8.2) and (8.3) we deduce

$$Q(b + \varepsilon b; i, k) - Q(b; i, k) \geq \varepsilon bl - 3b\eta = \varepsilon b \frac{3}{5k}. \quad (8.5)$$

Consider the set $X_i = \{b + ijk + i \mid j \text{ integer and } 0 \leq j \leq \frac{\varepsilon b - 1}{k}\}$. Hence $X_1 \subseteq [b, b + \varepsilon b]$, moreover X_1 is the set of all integers in $[b, b + \varepsilon b]$ congruent to 1 modulo k . Similarly $X_{-1} \subseteq [b - \varepsilon b, b]$, moreover X_{-1} is the set of all integers in $[b - \varepsilon b, b]$ congruent to -1 modulo k .

From (8.4) we have, in the interval $[b - \varepsilon b, b]$, at least $Q(b; -1, k) - Q(b - \varepsilon b; -1, k) \geq \varepsilon b \frac{3}{5k}$ square-free integers congruent to -1 modulo k . However all integer in $[b - \varepsilon b, b]$ congruent to -1 modulo k are in X_{-1} . Therefore there are at least $\lceil \varepsilon b \frac{3}{5k} \rceil$ square-free integers in X_{-1} . However X_{-1} has $\lfloor \frac{\varepsilon b - 1}{k} \rfloor$ elements, therefore the proportion of square-free elements in X_{-1} is at least $\varepsilon b \frac{3}{5k} \frac{k}{\varepsilon b} = \frac{3}{5}$. Similarly, by (8.5), the proportion of square-free elements in X_1 is at least $\frac{3}{5}$.

Note that $X_1 \rightarrow X_{-1}$, $r \mapsto 2b - r$ is a bijection. Therefore there is a square-free $r \in X_1$ such that $2b - r$ is square-free. Set $a = r - b$, thus $b + a = r$ is square-free,

and $b - a = 2b - r$ is square-free. Moreover as $r \in X_1$, we have $0 < r - b \leq \varepsilon b$. Also note that $a \equiv r \equiv 1 \pmod k$. \square

We shall give some explicit examples of finite Julia Robinson's Number distinct from 4. On the other hand we also want to show, for every example of Julia Robinson's Number j obtained, that there exist infinitely many rings of the algebraic integers of totally real fields having Julia Robinson's Number j . We will use the following lemma.

Lemma 8.4. *Let (a_1, b_1, r_1) be good; let (a_2, b_2, r_2) be good. Denote by K_1 (resp., K_2) the totally real field associated to (a_1, b_1, r_1) (resp., (a_2, b_2, r_2)). Assume that r_1, r_2 are even. If $(b_1 + a_1)(b_2 + a_2)b_1b_2$ is not a square then $K_1 \neq K_2$.*

Proof. Let $(\theta_i, K_i, U_i, s_i, N_i)$ be the objects associated to (a_i, b_i, r_i) , for $i = 1, 2$. Assume that $K_1 = K_2$, then $\mathcal{O}_{K_1} = \mathcal{O}_{K_2}$, thus $\text{JR}(\mathcal{O}_{K_1}) = \text{JR}(\mathcal{O}_{K_2})$, so by Lemma 7.10 we have $\lceil s_1 \rceil + s_1 = \lceil s_2 \rceil + s_2$, and so $s_1 = s_2$.

Claim 1. *There are $n_1 \geq 2$ a power of two, $n_2 \geq 2$ an integer, $1 \leq k_1 < \frac{n_1}{2}$, $1 \leq k_2 \leq n_2 - 1$ integers, $\lambda_1, \mu_1, \lambda_2, \mu_2$ in \mathbb{Q} such that*

$$\lambda_1 \cos \frac{k_1 \theta_1}{n_1} + \mu_1 \cos \frac{(n_1 - k_1) \theta_1}{n_1} = \lambda_2 \cos \frac{k_2 \theta_2}{n_2} + \mu_2 \cos \frac{(n_2 - k_2) \theta_2}{n_2} \neq 0. \quad (8.6)$$

Proof of Claim. As the infimum in the definition of s_1 (cf Notation 7.2) is a minimum (by Lemma 7.6), there is $(\lambda_1, \mu_1) \in U_1$ such that $s_1^2 = N_1(\lambda_1, \mu_1)$. It follows from Lemma 7.7 that there are an odd integer k_1 and a power of two $n_1 \geq 4$ such that $1 \leq k_1 < n_1$ and $\beta = \lambda_1 \cos \frac{k_1 \theta_1}{n_1} + \mu_1 \cos \frac{(n_1 - k_1) \theta_1}{n_1} \in \mathcal{O}_{K_1}$. Up to permuting λ_1 and μ_1 , we can assume that $k_1 < \frac{n_1}{2}$.

As $\beta \in \mathcal{O}_{K_1} = \mathcal{O}_{K_2}$, there is $m \geq 1$, and $\eta_0, \eta_1 \dots \eta_{m-1}$ in \mathbb{Q} such that

$$\lambda_1 \cos \frac{k_1 \theta_1}{n_1} + \mu_1 \cos \frac{(n_1 - k_1) \theta_1}{n_1} = \eta_0 + \sum_{\ell=1}^{n_2} \eta_\ell \cos \frac{\ell \theta_2}{n_2}.$$

Therefore applying T to both side of the equality, and multiplying by two, we obtain the following equality by Lemma 5.1.

$$N_1(\lambda_1, \mu_1) = 2\eta_0^2 + \sum_{\ell=0}^{\lfloor \frac{n_2}{2} \rfloor} N_2(\eta_\ell, \eta_{n_2-\ell})$$

However for each $1 \leq \ell \leq \frac{n_2}{2}$ we have either $\eta_\ell = 0 = \eta_{n_2-\ell}$ or $N_2(\eta_\ell, \eta_{n_2-\ell}) \geq s_2^2 = s_1^2 = N_1(\lambda_1, \mu_1)$. As $\beta \notin \mathbb{Q}$, it follows that there is $1 \leq p \leq n_2 - 1$ such that $\eta_p \neq 0$. Therefore for all $0 \leq \ell \leq n_2 - 1$, if $\ell \neq p$ and $\ell \neq n_2 - p$ we have $\lambda_\ell = 0$, thus we obtain

$$\lambda_1 \cos \frac{k_1 \theta_1}{n_1} + \mu_1 \cos \frac{(n_1 - k_1) \theta_1}{n_1} = \eta_p \cos \frac{p \theta_2}{n_2} + \eta_{n_2-p} \cos \frac{(n_2 - p) \theta_2}{n_2}. \quad \square \text{ Claim 1.}$$

Claim 2. *There are x, y in \mathbb{Q} such that*

$$x \cos \frac{\theta_1}{2} = y \cos \frac{\theta_2}{2} \neq 0.$$

Proof of Claim. Let $n_1, n_2, k_1, k_2, \lambda_1, \mu_1, \lambda_2, \mu_2$ as in Claim 1 with n_1 minimal. We can assume that $k_2 \leq \frac{n_2}{2}$. From Lemma 7.12 and (8.6), we have $2k_1 = n_1$ if and only if $2k_2 = n_2$. Assume that $n_1 \geq 4$. Squaring both side we obtain from Lemma 3.8

$$\begin{aligned} & \frac{1}{2}(\lambda_1^2 + \mu_1^2 + 2\lambda_1\mu_1 \frac{a_1}{b_1}) + (\lambda_1^2 + \mu_1^2) \cos \frac{2k_1\theta_1}{n_1} + 2(\lambda_1\mu_1 + \mu_1^2 \frac{a_1}{b_1}) \cos \frac{(n_1 - 2k_1)\theta_1}{n_1} \\ &= \frac{1}{2}(\lambda_2^2 + \mu_2^2 + 2\lambda_2\mu_2 \frac{a_2}{b_2}) + (\lambda_2^2 + \mu_2^2) \cos \frac{2k_2\theta_2}{n_2} + 2(\lambda_2\mu_2 + \mu_2^2 \frac{a_2}{b_2}) \cos \frac{(n_2 - 2k_2)\theta_2}{n_2} \end{aligned} \quad (8.7)$$

From Lemma 3.14, taking the trace we obtain

$$\frac{1}{2}(\lambda_1^2 + \mu_1^2 + 2\lambda_1\mu_1 \frac{a_1}{b_1}) = \frac{1}{2}(\lambda_2^2 + \mu_2^2 + 2\lambda_2\mu_2 \frac{a_2}{b_2}).$$

Hence

$$\begin{aligned} & (\lambda_1^2 + \mu_1^2) \cos \frac{2k_1\theta_1}{n_1} + 2(\lambda_1\mu_1 + \mu_1^2 \frac{a_1}{b_1}) \cos \frac{(n_1 - 2k_1)\theta_1}{n_1} \\ &= (\lambda_2^2 + \mu_2^2) \cos \frac{2k_2\theta_2}{n_2} + 2(\lambda_2\mu_2 + \mu_2^2 \frac{a_2}{b_2}) \cos \frac{(n_2 - 2k_2)\theta_2}{n_2} \end{aligned} \quad (8.8)$$

However, by Lemma 7.12, the number (8.7) is irrational, hence the number in (8.8) is not 0; a contradiction with the minimality of n_1 . Therefore $n_1 = 2$ and $k_1 = 1$, so $2k_1 = n_1$, thus $2k_2 = n_2$, the result follows for $x = \lambda_1 + \mu_1$, and $y = \lambda_2 + \mu_2$. \square Claim 2.

As $\cos \frac{\theta_1}{2} = \sqrt{\frac{b_1+a_1}{2b_1}}$ and $\cos \frac{\theta_2}{2} = \sqrt{\frac{b_2+a_2}{2b_2}}$, it follows from Claim 2 that $(b_1 + a_1)(b_2 + a_2)b_1b_2$ is a square; a contradiction. Therefore K_1 and K_2 are distincts. \square

The following lemma is an immediate consequence of Hensel's Lemma (see for instance [16, Theorem 5.6]).

Lemma 8.5. *Let $p \geq 3$ be a prime, let q be relatively prime to p . Let $n \geq 1$ be an integer. Assume that q is a square modulo p . Then q is a square modulo p^n .*

Lemma 8.6. *Let t be an odd number. Then there are infinitely many prime number p such that $p \equiv 2 \pmod{t}$, $p \equiv 3 \pmod{4}$, and $2t$ is a square modulo p .*

Proof. Given p, q we denote by $\left(\frac{p}{q}\right)$ the Legendre symbol. Set

$$\varepsilon = \prod_{\substack{q|t \\ q \text{ prime}}} (-1)^{\frac{q-1}{2}} \left(\frac{2}{q}\right).$$

As t is odd, it follows that $\varepsilon = \pm 1$. If $\varepsilon = 1$ pick p a prime number such that $p \equiv 2 \pmod{t}$ and $p \equiv 7 \pmod{8}$. If $\varepsilon = -1$ pick p a prime number such that $p \equiv 2 \pmod{t}$ and $p \equiv 3 \pmod{8}$. In both case there are infinitely many primes satisfying those conditions by Dirichlet's theorem (cf. [6]), and in both case we have $p \equiv 3 \pmod{4}$. Moreover by Gauss's quadratic reciprocity we have $\left(\frac{2}{p}\right) = \varepsilon$.

The following equalities hold

$$\begin{aligned}
 \left(\frac{2t}{p}\right) &= \left(\frac{2}{p}\right) \prod_{\substack{q|t \\ q \text{ prime}}} \left(\frac{q}{p}\right), && \text{by multiplicativity.} \\
 &= \left(\frac{2}{p}\right) \prod_{\substack{q|t \\ q \text{ prime}}} (-1)^{\frac{p-1}{2} \frac{q-1}{2}} \left(\frac{p}{q}\right), && \text{by Gauss's quadratic reciprocity.} \\
 &= \left(\frac{2}{p}\right) \prod_{\substack{q|t \\ q \text{ prime}}} (-1)^{\frac{q-1}{2}} \left(\frac{2}{q}\right), && \text{as } \frac{p-1}{2} \text{ is odd and } p \equiv 2 \pmod{t}. \\
 &= \varepsilon^2 \\
 &= 1
 \end{aligned}$$

Therefore $2t$ is a square modulo p . □

Lemma 8.7. *Let $t \geq 1$ be a square-free odd number. There are infinitely many couples of positive integers (m_i, n_i) such that, for all i , $m_i - 2$ and n_i are both relatively prime to $2t$ and, for all $i \neq j$, $(2^{m_i} t^{n_i} - 1)(2^{m_j} t^{n_j} - 1)$ is not a square.*

Proof. Assume that we have $(m_1, n_1), \dots, (m_k, n_k)$ (for $1 \leq i \leq k$), such that, for every $i \neq j$, $(2^{m_i} (2t)^{n_i} - 1)(2^{m_j} (2t)^{n_j} - 1)$ is not a square.

It follows from Lemma 8.6 that there is a prime p such that $p \equiv 2 \pmod{t}$, $p \equiv 3 \pmod{4}$, $2t$ is a square modulo p and, for all $1 \leq i \leq k$, p does not divide $2^{m_i} t^{n_i} - 1$. In particular p is relatively prime to $2t$.

Pick $u \geq 1$ odd such that $(2t)^{\frac{p-1}{2}} < p^{u+1}$, hence $(2t)^{\frac{p-1}{2}} \not\equiv 1 \pmod{p^{u+1}}$. Note that Lemma 8.5 implies that $2t$ is a square modulo p^{u+1} . Let x be such that $2t \equiv x^2 \pmod{p^{u+1}}$, and so x is prime to p . Therefore $x^{p-1} \equiv (x^2)^{\frac{p-1}{2}} \equiv (2t)^{\frac{p-1}{2}} \pmod{p^{u+1}}$, so $x^{p-1} \not\equiv 1 \pmod{p^{u+1}}$. However the multiplicative group $(\mathbb{Z}/p^{u+1}\mathbb{Z})^*$ is cyclic of order $(p-1)p^u$, hence $x^{(p-1)p^u} \equiv 1 \pmod{p^{u+1}}$.

Note that $(p^u + 1)^p \equiv 1 \pmod{p^{u+1}}$. As the subgroup of $(\mathbb{Z}/p^{u+1}\mathbb{Z})^*$ generated by x^{p-1} is not trivial and of order dividing p^u , it follows that this subgroup contains $p^u + 1$. Therefore there is $c \geq 1$ such that $x^{(p-1)c} \equiv p^u + 1 \pmod{p^{u+1}}$. As $x^{(p-1)p^u} \equiv 1 \pmod{p^{u+1}}$ we have $x^{(p-1)(c+p^u)} \equiv x^{(p-1)c} \equiv p^u + 1 \pmod{p^{u+1}}$. Therefore as p is odd, we can assume that c is odd.

Set $v = \frac{p-1}{2}c$. Note that $2^v t^v = (2t)^{\frac{p-1}{2}c} \equiv x^{(p-1)c} \equiv p^u + 1 \pmod{p^{u+1}}$. As $(p-1)p^u$ is even and relatively prime to t we can pick w_0 and w_1 odds such that $w_0 - 2$ is prime to t , w_1 is prime to t , $w_0 \equiv v \pmod{(p-1)p^u}$, and $w_1 \equiv v \pmod{(p-1)p^u}$. Therefore $2^{w_0} t^{w_1} \equiv 2^v t^v \equiv p^u + 1 \pmod{p^{u+1}}$, so p^u divides $2^{w_0} t^{w_1} - 1$ and p^{u+1} does not divide $2^{w_0} t^{w_1} - 1$.

Note that $v_p(2^{w_0} t^{w_1} - 1) = u$ is odd, moreover p does not appear in any prime decomposition of the $2^{m_i} t^{n_i} - 1$. Therefore for all $1 \leq i \leq k$, $(2^{m_i} t^{n_i} - 1)(2^{w_0} t^{w_1} - 1)$ is not a square. □

Theorem 8.8. *Let $t \geq 1$ be a square-free odd number. Then the following statement holds.*

- (1) *There are infinitely many fields K such that \mathcal{O}_K has Julia Robinson's number $\lceil 2\sqrt{2t} \rceil + 2\sqrt{2t}$.*

- (2) *There are infinitely many fields K such that \mathcal{O}_K has Julia Robinson's number $8t$.*

Proof. By Lemma 8.4 and Lemma 8.7, we can find infinitely many good (a, b, r) , such that $b = 2^{m-1}t^n$, with $m-2$ and n relatively prime to $2t$, $a = b-1$ and $r = 2t$, and all the totally real associated fields are distinct.

Fix such (a, b, r) . Let K be the totally real field associated to (a, b, r) , let s as in Notation 7.2. Therefore by Corollary 7.8(4) we have $s^2 = 8t$, thus $s = 2\sqrt{2t}$. Therefore by Theorem 7.10 the Julia Robinson's number of \mathcal{O}_K is $\lceil 2\sqrt{2t} \rceil + 2\sqrt{2t}$. So (1) holds.

Set $r = k = 2t$. Let C as in Lemma 8.3, for $\varepsilon = \frac{1}{2}$. Consider $n \geq 1$ an odd integer such that n is relatively prime to t and $2^{n+1}t \geq \max(C, 4t)$. Set $b = 2^{n+1}t$. Note that there are infinitely many possibilities for such b . By Lemma 8.3 there is $1 \leq a \leq \frac{1}{2}b$ such that $a \equiv 1 \pmod{k}$, and $b-a, b+a$ are square-free. As $k|b$, it follows that a is relatively prime to b .

Consider K the field constructed in Section 7. Let s as in Notation 7.2. As $b-a$ and $b+a$ are square-free, and $b \geq \frac{b}{2} + \frac{b}{2} \geq 2t + a$ by Corollary 7.8(3) we have $s = 4t$. Therefore by Theorem 7.10 the Julia Robinson's number of \mathcal{O}_K is $\lceil 4t \rceil + 4t = 8t$. Again, by Lemma 8.4, by choosing distinct b , we obtain distinct fields. So (2) holds. \square

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