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Loïc Dijoux, Vincent Fontaine

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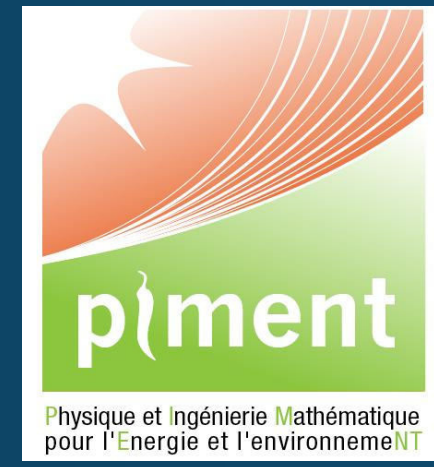
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High-order hybrid discontinuous galerkin methods for diffusion-convection equations with evanescent diffusion



Loïc DIJOUX, Vincent FONTAINE
University of La Réunion



Physical context

- Diffusion-advection equation

$$\nabla \cdot (-\alpha \nabla u + \beta u) = f \quad \text{in } \Omega$$

where α : the conductivity tensor, u : the state variable, β : velocity advection field and f : source term.

Application field:

- Ground water flow
- Heat and Mass transfert, ...

A mixed formulation

We introduce an auxiliary variable Q_d :

$$\begin{cases} Q_d = -\alpha \nabla u \\ \nabla \cdot q_d + \nabla \cdot \beta u = f \end{cases}$$

where $q_d = \alpha \nabla Q_d$. We introduce two approximate variables u_h and Q_h of respectively u and Q_d :

$$\begin{aligned} Q_h &= \sum_{i=1}^{n_q} Q_i w_i \\ u_h &= \sum_{j=1}^{n_u} u_j v_j \end{aligned}$$

where n_q and n_u are the degrees of freedom of Q_h and u_h respectively. w_i and v_j are their corresponding interpolations functions.

Approximation spaces

We define finite element spaces:

$$\mathcal{W}_h^p = \{w \in [L^2(\Omega)]^d : w|_K \in \mathcal{W}^p(K), \quad \forall K \in \mathcal{T}_h\},$$

$$\mathcal{V}_h^p = \{v \in L^2(\Omega) : v|_K \in \mathcal{V}^p(K), \quad \forall K \in \mathcal{T}_h\},$$

and add a traced finite element space:

$$\mathcal{M}_h^p = \{\mu \in L^2(\varepsilon_h) : \mu|_e \in \mathcal{M}^p(e), \quad \forall e \in \varepsilon_h\}$$

Discrete weak formulation

We impose the continuity of $\alpha \nabla Q_h + \beta u_h$ on each edge $e \in \varepsilon_h$ through one more equation and finally write :

Find $(Q_h, u_h, \lambda_h) \in \mathcal{W}_h^p \times \mathcal{V}_h^p \times \mathcal{M}_h^p$ such that :

$$\begin{aligned} (Q_h, w_h)_{\mathcal{T}_h} - (u_h, \nabla \cdot (\alpha \nabla w_h))_{\mathcal{T}_h} + \langle \lambda_h, \alpha \nabla w_h \cdot n \rangle_{\partial \mathcal{T}_h} &= -(g_D, \alpha \nabla w_h \cdot n)_{\Gamma_D} \quad \forall w_h \in \mathcal{W}_h^p \\ \langle \alpha \nabla Q_h + \beta u_h, v_h \rangle_{\partial \mathcal{T}_h} - (\alpha \nabla Q_h + \beta u_h, \nabla v_h)_{\mathcal{T}_h} &= (f, v_h)_{\mathcal{T}_h} \quad \forall v_h \in \mathcal{V}_h^p \\ \langle [\alpha \nabla Q_h + \beta u_h], \mu_h \rangle_{\varepsilon_h} &= -(g_N, \mu_h)_{\Gamma_N} \quad \forall \mu_h \in \mathcal{M}_h^p \end{aligned}$$

Definition of \hat{q}_h (see [1, 3]) on each face $e \in \partial \mathcal{T}_h$:

$$\hat{q}_h \cdot n_e + \beta u_h \cdot n_e = q_h \cdot n_e + \beta \cdot n_e \lambda_h + \tau_e (P u_h - \lambda_h)$$

where $\tau_e = \sqrt{\det(\alpha)} h^{-1} + \text{MAX}(\beta \cdot n_e, 0)$, see [2] and [1]

Examples of HdG methods:

Methods	$\mathcal{W}_h^p(K)$	$\mathcal{V}_h^p(K)$	$\mathcal{M}_h^p(e)$	τ_e	$\ u - u_h\ _+$	$\ q - q_h^*\ _+$
RT_p	$(\mathcal{P}^p)^d + x\mathcal{P}^p$	\mathcal{P}^p	\mathcal{P}^p	0	$p+1$	$p+1$
HdG_p	$(\mathcal{P}^p)^d$	\mathcal{P}^p	\mathcal{P}^p	$kh^{-1} + \beta \cdot n_e $	$p+1$	$p+1$
HdG_p^m	$(\mathcal{P}^p)^d$	\mathcal{P}^{p+1}	\mathcal{P}^p	$kh^{-1} + \beta \cdot n_e $	$p+2$	$p+1$
HdG_p^m	$(\mathcal{P}^p)^d + x\mathcal{P}^p$	\mathcal{P}^{p+1}	\mathcal{P}^p	$\sqrt{\det(\alpha)} h^{-1} + \text{MAX}(\beta \cdot n_e, 0)$	$p+2$	$p+1$

+ : for $\beta = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Numerical results : degenerate case

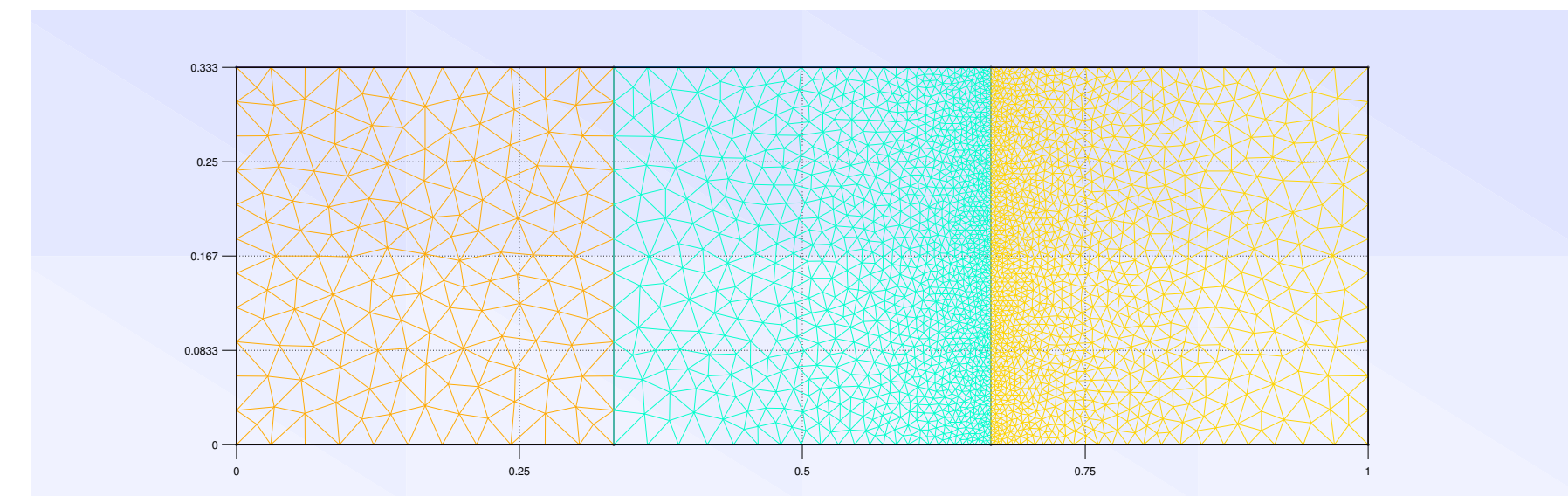
We propose to study a heterogeneous and anisotropic case with the exact solution defined in $\Omega \in (0, 1) \times (0, \frac{1}{3})$ as :

$$\begin{cases} u(x, y) = 1 & \text{for } \Omega_1 \in (0, \frac{1}{3}) \times (0, \frac{1}{3}) \\ u(x, y) = 1 & \text{for } \Omega_2 \in (\frac{1}{3}, \frac{2}{3}) \times (0, \frac{1}{3}) \\ u(x, y) = \frac{e-e^x}{e} & \text{for } \Omega_3 \in (\frac{2}{3}, 1) \times (0, \frac{1}{3}) \end{cases}$$

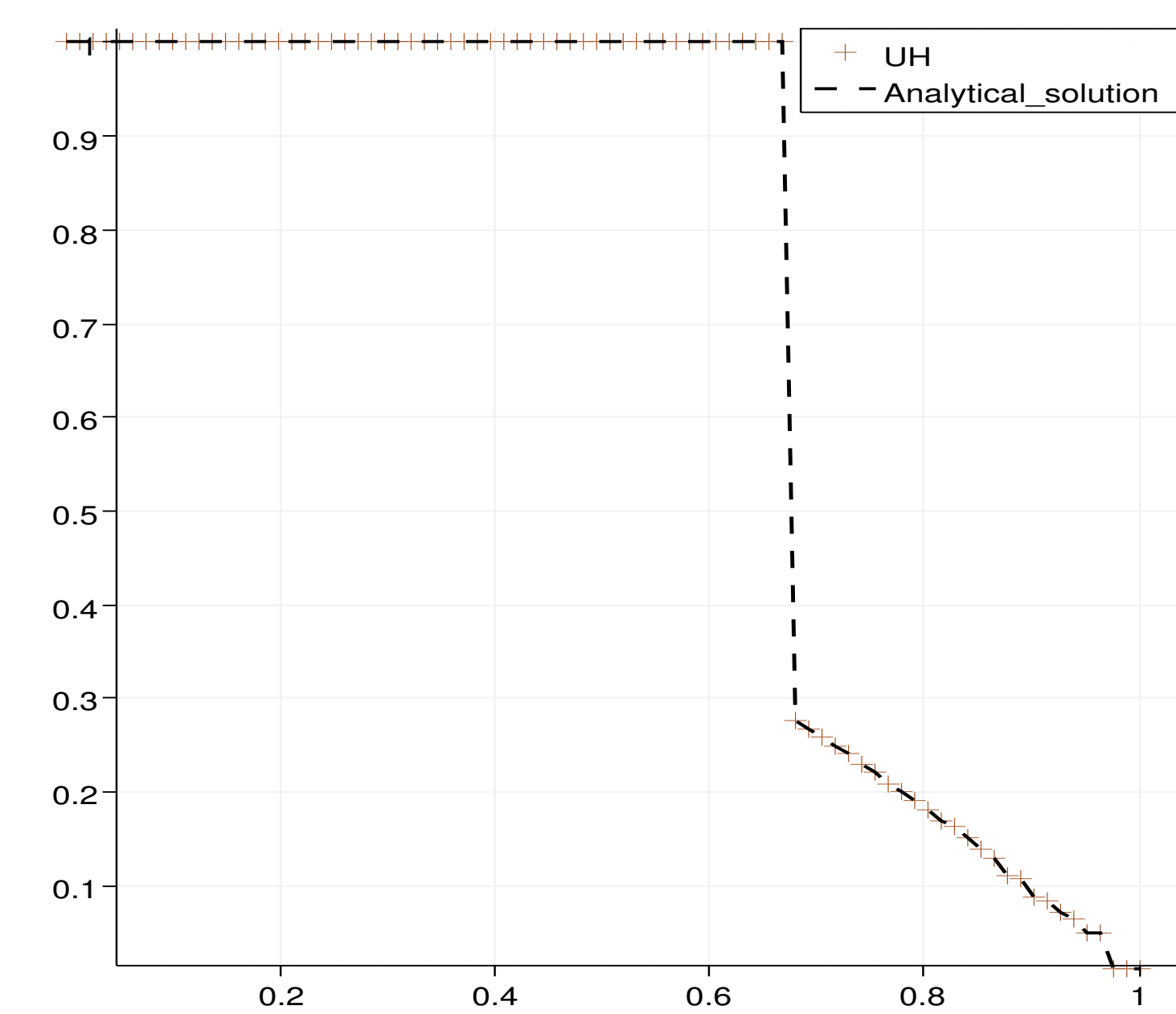
For Ω_1, Ω_3 we have the conductivity tensor $\alpha = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and for Ω_2 we have $\alpha = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. The velocity field is defined in Ω and we have $\beta = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Numerical results : degenerate case

Triangular mesh (GMSH : 7547 faces) & HdG_p^m :



Discrete solution : $u_h \in \mathcal{W}_h^2$:



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Contact

Loïc DIJOUX : loic.dijoux2@univ-reunion.fr
Vincent FONTAINE : vincent.fontaine@univ-reunion.fr