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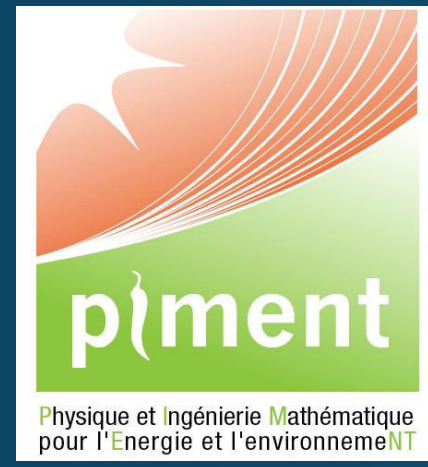
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# High-order hybrid discontinuous galerkin methods for diffusion-convection equations with evanescent diffusion



Loïc DIJOUX, Vincent FONTAINE



## Physical context

- Diffusion-advection equation

$$\nabla \cdot (-\alpha \nabla u + \beta u) = f \quad \text{in } \Omega$$

where  $\alpha$  : the conductivity tensor,  $u$  : the state variable,  $\beta$  : velocity advection field and  $f$  : source term.

Application field:

- Ground water flow
- Heat and Mass transfert, ...

## A mixed formulation

We introduce an auxiliary variable  $Q_d$  :

$$\begin{cases} Q_d = -\alpha \nabla u \\ \nabla \cdot q_d + \nabla \cdot \beta u = f \end{cases}$$

where  $q_d = \alpha \nabla Q_d$ . We introduce two approximate variables  $u_h$  and  $Q_h$  of respectively  $u$  and  $Q_d$  :

$$\begin{aligned} Q_h &= \sum_{i=1}^{n_q} Q_i w_i \\ u_h &= \sum_{j=1}^{n_u} u_j v_j \end{aligned}$$

where  $n_q$  and  $n_u$  are the degrees of freedom of  $Q_h$  and  $u_h$  respectively.  $w_i$  and  $v_j$  are their corresponding interpolations functions.

## Approximation spaces

We define finite element spaces:

$$\mathcal{W}_h^p = \{w \in [L^2(\Omega)]^d : w|_K \in \mathcal{W}^p(K), \quad \forall K \in \mathcal{T}_h\},$$

$$\mathcal{V}_h^p = \{v \in L^2(\Omega) : v|_K \in \mathcal{V}^p(K), \quad \forall K \in \mathcal{T}_h\},$$

and add a traced finite element space:

$$\mathcal{M}_h^p = \{\mu \in L^2(\varepsilon_h) : \mu|_e \in \mathcal{M}^p(e), \quad \forall e \in \varepsilon_h\}$$

## Discrete weak formulation

We impose the continuity of  $\alpha \nabla Q_h + \beta u_h$  on each edge  $e \in \varepsilon_h$  through one more equation and finally write :

Find  $(Q_h, u_h, \lambda_h) \in \mathcal{W}_h^p \times \mathcal{V}_h^p \times \mathcal{M}_h^p$  such that :

$$\begin{aligned} (Q_h, w_h)_{\mathcal{T}_h} - (u_h, \nabla \cdot (\alpha \nabla w_h))_{\mathcal{T}_h} + \langle \lambda_h, \alpha \nabla w_h \cdot n \rangle_{\partial \mathcal{T}_h} &= -(g_D, \alpha \nabla w_h \cdot n)_{\Gamma_D} \quad \forall w_h \in \mathcal{W}_h^p \\ \langle \alpha \nabla Q_h + \beta u_h, v_h \rangle_{\partial \mathcal{T}_h} - (\alpha \nabla Q_h + \beta u_h, \nabla v_h)_{\mathcal{T}_h} &= (f, v_h)_{\mathcal{T}_h} \quad \forall v_h \in \mathcal{V}_h^p \\ \langle [\alpha \nabla Q_h + \beta u_h], \mu_h \rangle_{\varepsilon_h} &= -(g_N, \mu_h)_{\Gamma_N} \quad \forall \mu_h \in \mathcal{M}_h^p \end{aligned}$$

Definition of  $\hat{q}_h$  (see [1, 3]) on each face  $e \in \partial \mathcal{T}_h$  :

$$\hat{q}_h \cdot n_e + \beta u_h \cdot n_e = q_h \cdot n_e + \beta \cdot n_e \lambda_h + \tau_e (P u_h - \lambda_h)$$

where  $\tau_e = \sqrt{\det(\alpha)} h^{-1} + \text{MAX}(\beta \cdot n_e, 0)$ , see [2] and [1]

Examples of HdG methods:

| Methods   | $\mathcal{W}_h^p(K)$                 | $\mathcal{V}_h^p(K)$ | $\mathcal{M}_h^p(e)$ | $\tau_e$  | $\ u - u_h\ _+$ | $\ q - q_h^*\ _+$ |
|-----------|--------------------------------------|----------------------|----------------------|---|-----------------|-------------------|
| $RT_p$    | $(\mathcal{P}^p)^d + x\mathcal{P}^p$ | $\mathcal{P}^p$      | $\mathcal{P}^p$      | 0   | $p+1$           | $p+1$             |
| $HdG_p$   | $(\mathcal{P}^p)^d$                  | $\mathcal{P}^p$      | $\mathcal{P}^p$      | $kh^{-1} +  \beta \cdot n_e $                                 | $p+1$           | $p+1$             |
| $HdG_p^m$ | $(\mathcal{P}^p)^d$                  | $\mathcal{P}^{p+1}$  | $\mathcal{P}^p$      | $kh^{-1} +  \beta \cdot n_e $                                 | $p+2$           | $p+1$             |
| $HdG_p^m$ | $(\mathcal{P}^p)^d + x\mathcal{P}^p$ | $\mathcal{P}^{p+1}$  | $\mathcal{P}^p$      | $\sqrt{\det(\alpha)} h^{-1} + \text{MAX}(\beta \cdot n_e, 0)$ | $p+2$           | $p+1$             |

+ : for  $\beta = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

## Numerical results : degenerate case

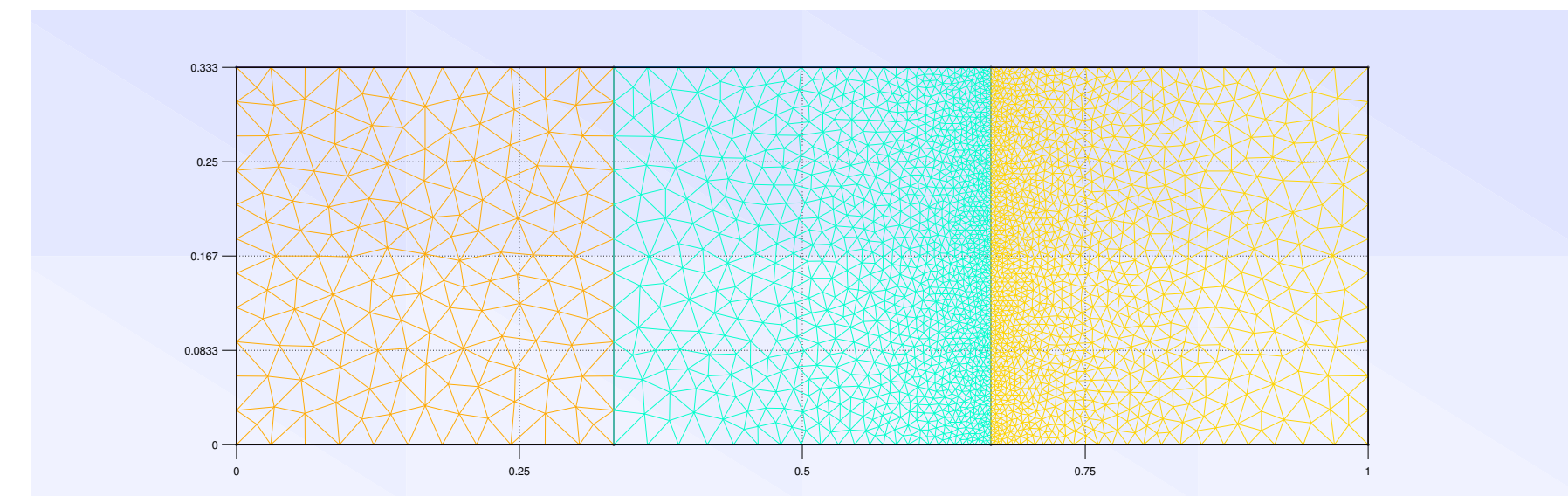
We propose to study a heterogeneous and anisotropic case with the exact solution defined in  $\Omega \in (0, 1) \times (0, \frac{1}{3})$  as :

$$\begin{cases} u(x, y) = 1 & \text{for } \Omega_1 \in (0, \frac{1}{3}) \times (0, \frac{1}{3}) \\ u(x, y) = 1 & \text{for } \Omega_2 \in (\frac{1}{3}, \frac{2}{3}) \times (0, \frac{1}{3}) \\ u(x, y) = \frac{e-e^x}{e} & \text{for } \Omega_3 \in (\frac{2}{3}, 1) \times (0, \frac{1}{3}) \end{cases}$$

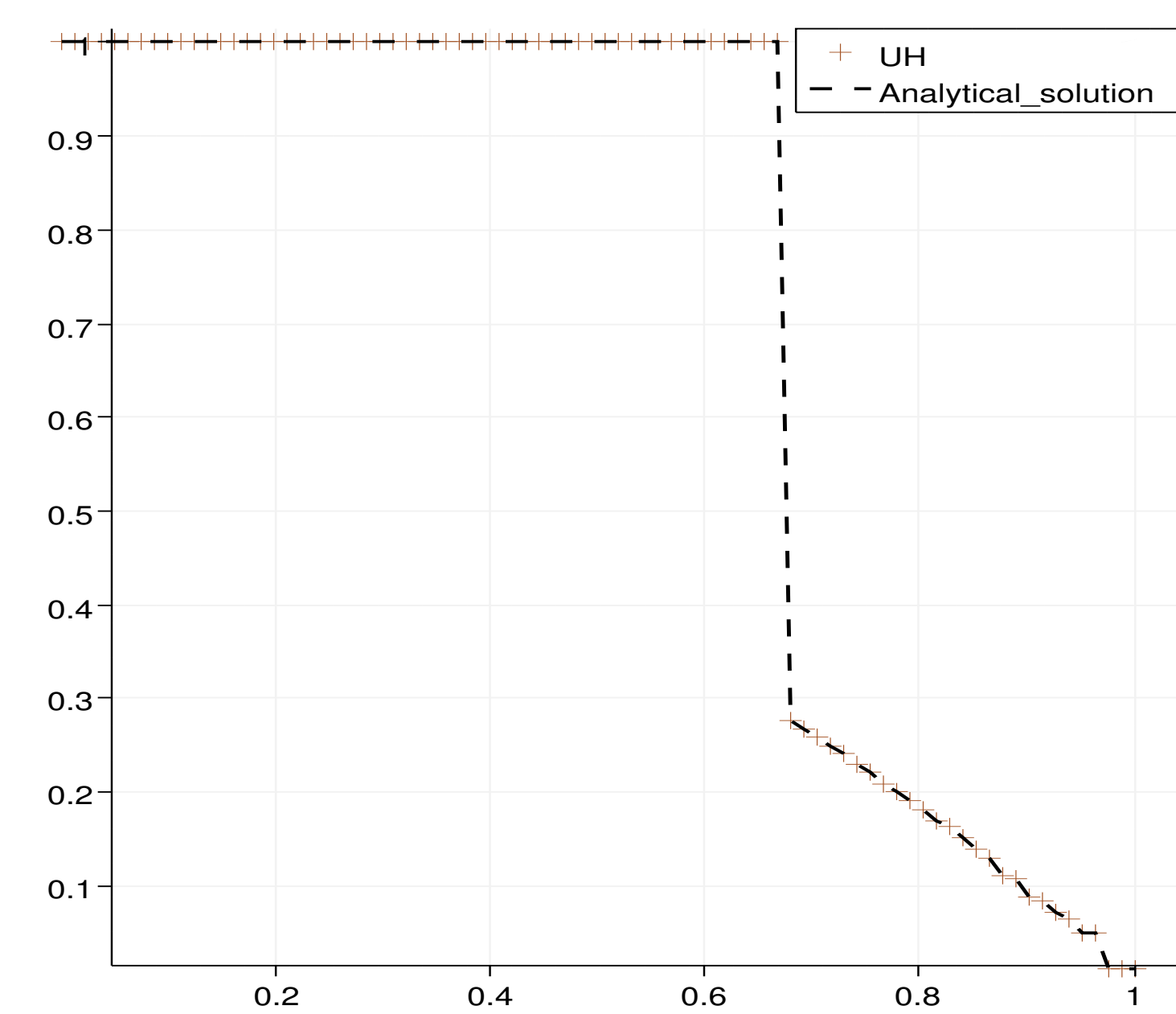
For  $\Omega_1, \Omega_3$  we have the conductivity tensor  $\alpha = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  and for  $\Omega_2$  we have  $\alpha = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ . The velocity field is defined in  $\Omega$  and we have  $\beta = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ .

## Numerical results : degenerate case

Triangular mesh (GMSH : 7547 faces) &  $HdG_p^m$  :



Discrete solution :  $u_h \in \mathcal{W}_h^2$  :



## Bibliography

### References

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