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► **To cite this version:**

Marc Arnaudon, Anton Thalmaier, Feng-Yu Wang. Gradient Estimates on Dirichlet Eigenfunctions. 2017. hal-01625890v1

**HAL Id: hal-01625890**

**<https://hal.science/hal-01625890v1>**

Preprint submitted on 29 Oct 2017 (v1), last revised 11 Aug 2018 (v3)

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# Gradient Estimates on Dirichlet Eigenfunctions

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October 29, 2017

## Abstract

By methods of stochastic analysis on Riemannian manifolds, we derive explicit constants  $c_1(D)$  and  $c_2(D)$  for a  $d$ -dimensional compact Riemannian manifold  $D$  with boundary such that

$$c_1(D)\sqrt{\lambda}\|\phi\|_\infty \leq \|\nabla\phi\|_\infty \leq c_2(D)\sqrt{\lambda}\|\phi\|_\infty$$

holds for any Dirichlet eigenfunction  $\phi$  of  $-\Delta$  with eigenvalue  $\lambda$ . In particular, when  $D$  is convex with nonnegative Ricci curvature, this estimate holds for

$$c_1(D) = \frac{1}{de}, \quad c_2(D) = \sqrt{e} + \frac{e\sqrt{2}}{\sqrt{\pi}}.$$

**AMS subject Classification:** 35P20, 60H30, 58J65

**Keywords:** Dirichlet eigenfunction, gradient estimate, diffusion process, curvature, second fundamental form.

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AT is supported by FNR Luxembourg: OPEN scheme (project GEOMREV O14/7628746).  
FW is supported in part by NNSFC (11771326, 11431014).

# 1 Introduction

Let  $D$  be a  $d$ -dimensional compact Riemannian manifold with boundary  $\partial D$ . We write  $(\phi, \lambda) \in \text{Eig}(\Delta)$  if  $\phi$  is a Dirichlet eigenfunction of  $-\Delta$  in  $D$  with eigenvalue  $\lambda > 0$ . According to [6], there exist two constants  $c_1(D), c_2(D) > 0$  such that

$$(1.1) \quad c_1(D)\sqrt{\lambda}\|\phi\|_\infty \leq \|\nabla\phi\|_\infty \leq c_2(D)\sqrt{\lambda}\|\phi\|_\infty, \quad (\phi, \lambda) \in \text{Eig}(\Delta).$$

In this paper, by using stochastic analysis of the Brownian motion on  $D$ , we present explicit expressions of these two constants in terms of the lower bounds of  $\text{Ric}_D$  and  $\mathbb{I}_{\partial D}$  where  $\text{Ric}_D$  is the Ricci curvature on  $D$  and  $\mathbb{I}_{\partial D}$  the second fundamental form of  $\partial D$ .

**Theorem 1.1.** *Let  $K, \theta \geq 0$  be two constants such that*

$$\text{Ric}_D \geq -K, \quad \mathbb{I}_{\partial D} \geq -\theta.$$

Let

$$\alpha_0 = \frac{1}{2}((d-1)\theta + \sqrt{(d-1)K}).$$

Then, for any nontrivial  $(\phi, \lambda) \in \text{Eig}(\Delta)$ ,

$$\frac{\lambda}{\sqrt{de(\lambda+K)}} \leq \frac{\|\nabla\phi\|_\infty}{\|\phi\|_\infty} \leq \sqrt{e(\lambda+K)} + e \left( \alpha_0 + \frac{\sqrt{2\lambda}}{\sqrt{\pi}} + \alpha_0 \wedge \frac{\alpha_0^2}{\sqrt{2\pi\lambda}} \right).$$

In particular, when  $\text{Ric}_D, \mathbb{I}_{\partial D} \geq 0$ ,

$$(1.2) \quad \frac{\sqrt{\lambda}}{\sqrt{de}} \leq \frac{\|\nabla\phi\|_\infty}{\|\phi\|_\infty} \leq \sqrt{\lambda} \left( \sqrt{e} + \frac{e\sqrt{2}}{\sqrt{\pi}} \right), \quad (\phi, \lambda) \in \text{Eig}(\Delta).$$

*Proof.* This result follows from Theorem 2.1 and Theorem 3.1 below for the special case  $V = 0$ . In this case,  $\text{Ric}_D^V = \text{Ric}_D \geq -K$  is equivalent to (2.1) with  $n = d$ .  $\square$

*Remark 1.1.* Various other estimates can be obtained from our method. For instance, let  $\alpha \in \mathbb{R}$  be such that  $\frac{1}{2}\Delta\rho_{\partial D} \leq \alpha$  outside the focal set, where  $\rho_{\partial D}$  denotes the distance to boundary  $\partial D$ . Then for  $\lambda < \alpha^2/4$ ,

$$(1.3) \quad \frac{\|\nabla\phi\|_\infty}{\|\phi\|_\infty} \leq \sqrt{e(\lambda+K)} + e \left( 2 \max(\alpha, 0) + \frac{\alpha^2 e^2}{4} \sqrt{\frac{2\pi}{\lambda}} e^{-\frac{\alpha^2}{2\lambda}} \right).$$

This relies on Remark 3.1 where another estimate of the right hand side of (3.13) is given. It improves the estimate in Theorem 1.1 in the case when  $\alpha < 0$  and  $|\alpha|$  is large. See Theorem 3.5 below for case that  $k = 0$  and  $\theta < 0$ .

By (1.2), when  $D$  is convex with nonnegative Ricci curvature, (1.1) holds with

$$c_1(D) = \frac{1}{\sqrt{de}}, \quad c_2(D) = \sqrt{e} + \frac{e\sqrt{2}}{\sqrt{\pi}}.$$

To estimate  $c_1(D)$  and  $c_2(D)$  for positive  $K$  or  $\theta$ , let  $\lambda_1 > 0$  be the first Dirichlet eigenvalue of  $-\Delta$  on  $D$ . Then Theorem 1.1 implies that the inequalities (1.1) hold for

$$c_1(D) = \frac{\sqrt{\lambda_1}}{\sqrt{de(\lambda_1 + K)}},$$

$$c_2(D) = \frac{\sqrt{e(\lambda_1 + K)}}{\sqrt{\lambda_1}} + e\left(\frac{(d-1)\theta + \sqrt{K(d-1)}}{2\sqrt{\lambda_1}} + \frac{\sqrt{2}}{\sqrt{\pi}} + \frac{((d-1)\theta + \sqrt{K(d-1)})^2}{4\lambda_1\sqrt{2\pi}}\right).$$

This is due to the fact that the first expression is an increasing function of  $\lambda$  and the second one is a decreasing function of  $\lambda$ . Since there exist explicit lower bound estimates on  $\lambda_1$  (see [8] and references within), this gives explicit lower bounds of  $c_1(D)$  and upper bounds of  $c_2(D)$ .

The lower bound estimate of  $\|\nabla\phi\|_\infty$  will be derived by using Itô's formula for  $|\nabla\phi|^2(X_t)$  where  $X_t$  is a Brownian motion (with drift) on  $D$ , see Section 2 for details. A powerful probabilistic tool for establishing upper bound gradient estimates is the use of Bismut type formulas for the Dirichlet semigroup  $P_t^D$  on  $D$ , which gives

$$|\nabla P_t^D f(x)| \leq \frac{c(t)}{\rho_{\partial D}} \|f\|_\infty, \quad t > 0, \quad f \in \mathcal{B}_b(D),$$

where  $\rho_{\partial D}$  is the Riemannian distance to  $\partial D$  and  $c(t)$  an explicit quantity depending on the geometry of  $D$ , see [7] for details. However, as this estimate blows up at the boundary  $\partial D$ , it does not give the wanted upper bound estimate of  $\|\nabla\phi\|_\infty$  near the boundary. To achieve the goal of a uniform upper bound on  $D$ , we will construct some martingales to reduce  $\|\nabla\phi\|_\infty$  to  $\|\nabla\phi\|_{\partial D, \infty} := \sup_{\partial D} |\nabla\phi|$ , and to estimate the latter using  $\|\phi\|_\infty$ , see Section 3 for details.

In general, we will consider Dirichlet eigenfunctions for the symmetric operator  $L := \Delta + \nabla V$  on  $D$  where  $V \in C^2(D)$ . We denote by  $\text{Eig}(L)$  the set of pairs  $(\phi, \lambda)$  where  $\phi$  is a Dirichlet eigenfunction of  $-L$  on  $D$  with eigenvalue  $\lambda$ .

## 2 Lower bound estimate

In this Section we will estimate  $\|\nabla\phi\|_\infty$  from below using the following Bakry-Émery curvature-dimension condition:

$$(2.1) \quad \frac{1}{2}L|\nabla f|^2 - \langle \nabla Lf, \nabla f \rangle \geq -K|\nabla f|^2 + \frac{(Lf)^2}{n}, \quad f \in C^\infty(D),$$

where  $K \in \mathbb{R}, n \geq d$  are two constants. When  $V = 0$ , this condition with  $n = d$  is equivalent to  $\text{Ric}_D \geq -K$ .

**Theorem 2.1** (Lower bound estimate). *Assume that (2.1) holds. Then*

$$(2.2) \quad \|\nabla\phi\|_\infty^2 \geq \|\phi\|_\infty^2 \sup_{t>0} \frac{\lambda^2(e^{Kt} - 1)}{nKe^{(\lambda+K)t}}, \quad (\phi, \lambda) \in \text{Eig}(L).$$

Consequently, for  $K^+ := \max\{0, K\}$  there holds

$$(2.3) \quad \|\nabla\phi\|_\infty^2 \geq \frac{\lambda^2}{ne(\lambda + K^+)} \|\phi\|_\infty^2, \quad (\phi, \lambda) \in \text{Eig}(L).$$

*Proof.* Let  $X_t$  be the diffusion process generated by  $\frac{1}{2}L$  in  $D$ , and let

$$\tau_D := \inf\{t \geq 0 : X_t \in \partial D\}.$$

By Itô's formula, we have

$$(2.4) \quad d|\nabla\phi|^2(X_t) = \frac{1}{2}L|\nabla\phi|^2(X_t) dt + dM_t, \quad t \leq \tau_D,$$

for some martingale  $M_t$ . By the curvature dimension condition (2.1) and  $L\phi = -\lambda\phi$ , we obtain

$$\frac{1}{2}L|\nabla\phi|^2 = \frac{1}{2}L|\nabla\phi|^2 - \langle \nabla L\phi, \nabla\phi \rangle - \lambda|\nabla\phi|^2 \geq -(K + \lambda)|\nabla\phi|^2 + \frac{\lambda^2}{n}\phi^2.$$

Therefore, (2.4) gives

$$d|\nabla\phi|^2(X_t) \geq \left( \frac{\lambda^2}{n}\phi^2 - (K + \lambda)|\nabla\phi|^2 \right)(X_t) dt + dM_t, \quad t \leq \tau_D.$$

Hence, for any  $t > 0$ ,

$$\begin{aligned} e^{(K+\lambda)t} \|\nabla\phi\|_\infty^2 &\geq \mathbb{E} \left[ |\nabla\phi|^2(X_{t \wedge \tau_D}) e^{(K+\lambda)(t \wedge \tau_D)} \right] \\ &\geq \frac{\lambda^2}{n} \mathbb{E} \left[ \int_0^{t \wedge \tau_D} e^{(K+\lambda)s} \phi(X_s)^2 ds \right] \\ &= \frac{\lambda^2}{n} \mathbb{E} \left[ \int_0^t 1_{\{s < \tau_D\}} e^{(K+\lambda)s} \phi(X_s)^2 ds \right]. \end{aligned}$$

Since  $\phi|_{\partial D} = 0$  and  $L\phi = -\lambda\phi$ , by Jensen's inequality we have

$$\mathbb{E} [1_{\{s < \tau_D\}} \phi(X_s)^2] \geq (\mathbb{E}[\phi(X_{s \wedge \tau_D})])^2 = e^{-\lambda s} \phi(x)^2,$$

where  $x = X_0 \in D$  is the starting point of  $X_t$ . Then, by taking  $x$  such that  $\phi(x)^2 = \|\phi\|_\infty^2$ , we arrive at

$$\begin{aligned} e^{(K+\lambda)t} \|\nabla\phi\|_\infty^2 &\geq \frac{\lambda^2}{n} \int_0^t e^{(K+\lambda)s} e^{-\lambda s} \phi(x)^2 ds \\ &= \frac{\lambda^2 \|\phi\|_\infty^2}{n} \int_0^t e^{Ks} ds = \frac{\lambda^2 (e^{Kt} - 1)}{nK} \|\phi\|_\infty^2. \end{aligned}$$

This completes the proof of (2.2).

Since (2.1) holds for  $K^+$  replacing  $K$ , we may and do assume that  $K \geq 0$ . By taking  $t = \frac{1}{\lambda+K}$  in (2.2), we obtain

$$\|\nabla\phi\|_\infty^2 \geq \frac{\lambda^2 (e^{\frac{K}{\lambda+K}} - 1)}{nKe} \|\phi\|_\infty^2 \geq \frac{\lambda^2}{ne(\lambda+K)} \|\phi\|_\infty^2.$$

Hence (2.3) holds. □

### 3 Upper bound estimate

Let  $\text{Ric}_D^V = \text{Ric}_D - \text{Hess}_V$ .

**Theorem 3.1** (Upper bound estimate). *Let  $K_V, K_0, \theta \geq 0$  be constants such that*

$$\text{Ric}_D^V \geq -K_V, \quad \text{Ric}_D \geq -K_0, \quad \mathbb{I}_{\partial D} \geq -\theta.$$

Let

$$(3.1) \quad \alpha = \frac{1}{2} \left( (d-1)\theta + \sqrt{(d-1)K_0} + \|\nabla V\|_\infty \right).$$

Then, for any  $(\phi, \lambda) \in \text{Eig}(L)$ ,

$$\|\nabla \phi\|_\infty \leq \|\phi\|_\infty \left\{ \sqrt{e(\lambda + K_V)} + e \left( \alpha + \frac{\sqrt{2\lambda}}{\sqrt{\pi}} + \alpha \wedge \frac{\alpha^2}{\sqrt{2\pi\lambda}} \right) \right\}.$$

To prove this result, we first estimate  $\|\nabla \phi\|_\infty$  in terms of  $\|\phi\|_\infty$  and  $\|\nabla \phi\|_{\partial D, \infty}$  where  $\|f\|_{\partial D, \infty} := \|1_{\partial D} f\|_\infty$  for a function  $f$  on  $D$ .

**Lemma 3.2.** *Assume  $\text{Ric}_D^V \geq -K_V$  for some constant  $K_V \in \mathbb{R}$ . Then, for any  $(\phi, \lambda) \in \text{Eig}(L)$ ,*

$$(3.2) \quad \|\nabla \phi\|_\infty \leq e^{\frac{(\lambda + K_V)^+}{2}t} \|\nabla \phi\|_{\partial D, \infty} + \|\phi\|_\infty e^{\frac{\lambda}{2}t} \left( \frac{K_V}{1 - e^{-K_V t}} \right)^{1/2}, \quad t > 0.$$

Consequently,

$$(3.3) \quad \|\nabla \phi\|_\infty \leq e^{1/2} \left( \|\nabla \phi\|_{\partial D, \infty} + \sqrt{\lambda + K_V^+} \|\phi\|_\infty \right), \quad (\phi, \lambda) \in \text{Eig}(L).$$

*Proof.* We first recall some facts concerning the diffusion process generated by  $\frac{1}{2}L$ , see for instance [1, 3]. For any  $x \in D$ , the diffusion  $X_t$  solves the SDE

$$(3.4) \quad dX_t = \frac{1}{2} \nabla V(X_t) dt + u_t \circ dB_t, \quad X_0 = x, \quad t \leq \tau_D,$$

where  $B_t$  is a  $d$ -dimensional Brownian motion,  $u_t$  is the horizontal lift of  $X_t$  onto the orthonormal frame bundle  $O(D)$  with initial value  $u_0 \in O_x(D)$ , and

$$\tau_D := \inf\{t \geq 0 : X_t \in \partial D\}$$

is the hitting time of  $X_t$  to the boundary  $\partial D$ . Setting  $Z := \nabla V$ , we have

$$(3.5) \quad du_t = \frac{1}{2} Z^*(u_t) dt + \sum_{i=1}^d H_i(u_t) \circ dB_t^i$$

where  $Z^*(u) := h_u(Z_{\pi(u)})$  and  $H_i(u) := h_u(u e_i)$  are defined by means of the horizontal lift  $h_u: T_{\pi(u)}D \rightarrow T_u O(D)$  at  $u \in O(D)$ . Note that formally  $h_{u_t}(u_t \circ dB_t) = \sum_i h_{u_t}(u_t e_i) \circ dB_t^i = \sum_i H_i(u_t) \circ dB_t^i$ .

For  $f \in C^\infty(D)$ , let  $a := df \in \Gamma(T^*D)$ . Setting  $m_t := u_t^{-1}a(X_t)$ , we see by Itô's formula that

$$(3.6) \quad dm_t \stackrel{\text{m}}{=} \frac{1}{2}u_t^{-1}(\square a + \nabla_Z a)(X_t) dt$$

where  $\square a = \text{tr } \nabla^2 a$  denotes the so-called connection (or rough) Laplacian on 1-forms and  $\stackrel{\text{m}}{=}$  equality modulo the differential of a local martingale.

Denote by  $Q_t: T_x D \rightarrow T_{X_t} D$  the solution, along the paths of  $X_t$ , to the covariant ordinary differential equation

$$DQ_t = -\frac{1}{2}(\text{Ric}_D^V)^\sharp Q_t dt, \quad Q_0 = \text{id}_{T_x D}, \quad t \leq \tau_D,$$

where  $D := u_t du_t^{-1}$  and where by definition

$$(\text{Ric}_D^V)^\sharp v = \text{Ric}_D^V(\cdot, v)^\sharp, \quad v \in T_x D.$$

Thus, condition  $\text{Ric}_D^V \geq -K_V$  implies

$$(3.7) \quad |Q_t v| \leq e^{\frac{K_V}{2}t} |v|, \quad t \leq \tau_D.$$

Finally, note that for any smooth function  $f$  on  $D$ , we have by the Weitzenböck formula:

$$(3.8) \quad \begin{aligned} d(\Delta + Z)f &= d(-d^*df + (df)Z) \\ &= \Delta^{(1)}df + \nabla_Z df + \langle \nabla \cdot Z, \nabla f \rangle \\ &= (\square + \nabla_Z)(df) - \text{Ric}_D^V(\cdot, \nabla f) \\ &= (\square - \text{Ric}_D^V + \nabla_Z)(df) \end{aligned}$$

where  $\Delta^{(1)}$  denotes the Hodge-deRham Laplacian on 1-forms.

Now let  $(\phi, \lambda) \in \text{Eig}(L)$ , i.e.  $L\phi = -\lambda\phi$ , where  $L = \Delta + Z$ . For  $v \in T_x D$ , consider the process

$$n_t(v) := (d\phi)(Q_t v).$$

Then

$$n_t(v) = \langle \nabla \phi(X_t), Q_t v \rangle = \langle u_t^{-1}(\nabla \phi)(X_t), u_t^{-1}Q_t v \rangle.$$

Using (3.6), we see by Itô's formula and formula (3.8) that

$$dn_t(v) \stackrel{\text{m}}{=} \frac{1}{2}(\square d\phi + \nabla_Z d\phi)(X_t) Q_t v dt + d\phi(X_t)(DQ_t v) dt = -\frac{\lambda}{2}n_t(v) dt.$$

It follows that

$$e^{\lambda t/2} n_t(v) = e^{\lambda t/2} \langle \nabla \phi(X_t), Q_t v \rangle, \quad t \leq \tau_D,$$

is a martingale, and consequently, for any function  $h \in C^1([0, \infty); \mathbb{R})$ ,

$$h_t e^{\lambda t/2} \langle \nabla \phi(X_t), Q_t v \rangle - \int_0^t \dot{h}_s e^{\lambda s/2} \langle \nabla \phi(X_s), Q_s v \rangle ds, \quad t \leq \tau_D,$$

is a martingale as well. By the formula

$$e^{\lambda t/2} \phi(X_t) = \phi(X_0) + \int_0^t e^{\lambda s/2} \langle \nabla \phi(X_s), u_s dB_s \rangle$$

we see then that

$$N_t(v) := h_t e^{\lambda t/2} \langle \nabla \phi(X_t), Q_t v \rangle - e^{\lambda t/2} \phi(X_t) \int_0^t \langle \dot{h}_s Q_s v, u_s dB_s \rangle, \quad t \leq \tau_D,$$

is a martingale.

Now, for fixed  $t > 0$ , we take  $h \in C^1([0, t]; [0, 1])$  such that  $h_0 = 1$  and  $h_t = 0$ . Then, by the martingale property of  $\{N_{s \wedge \tau_D}(v)\}_{s \in [0, t]}$  we obtain

$$\begin{aligned} |\nabla_v \phi|(x) &= |N_0(v)| = |\mathbb{E} N_{t \wedge \tau_D}(v)| \\ &= \left| \mathbb{E} \left[ \mathbf{1}_{\{t > \tau_D\}} e^{\lambda \tau_D/2} h_{\tau_D} \langle \nabla \phi(X_{\tau_D}), Q_{\tau_D} v \rangle - \mathbf{1}_{\{t \leq \tau_D\}} e^{\lambda t/2} \phi(X_t) \int_0^t \langle \dot{h}_s Q_s v, u_s dB_s \rangle \right] \right|. \end{aligned}$$

This together with (3.7) yields

$$|\nabla \phi(x)| \leq e^{(\lambda + K_V)^+ t/2} \|\nabla \phi\|_{\partial D, \infty} + e^{\lambda t/2} \|\phi\|_{\infty} \left( \int_0^t (\dot{h}_s)^2 e^{K_V s} ds \right)^{1/2}.$$

Taking

$$h_s = \frac{e^{-K_V t} - e^{-K_V s}}{e^{-K_V t} - 1}, \quad s \in [0, t],$$

we obtain (3.2). Finally, noting that

$$\frac{K_V}{1 - e^{-K_V t}} \leq \frac{K_V^+}{1 - e^{-K_V^+ t}} \leq t^{-1} e^{K_V^+ t},$$

and taking  $t = (K_V^+ + \lambda)^{-1}$  in (3.2), we prove (3.3).  $\square$

To estimate the term  $\|\nabla \phi\|_{\partial D, \infty}$ , we shall compare  $\phi(x)$  and

$$\psi(t, x) := \mathbb{P}(\tau_D^x > t), \quad t > 0,$$

for small  $\rho_{\partial D}(x) := \text{dist}(x, \partial D)$ . Let  $P_t^D$  be the Dirichlet semigroup generated by  $\frac{1}{2}L$ . Then  $\psi(t, x) = P_t^D 1_D(x)$ , so that

$$(3.9) \quad \partial_t \psi(t, x) = \frac{1}{2} L \psi(t, \cdot)(x), \quad t > 0.$$

**Lemma 3.3.** *For any  $(\phi, \lambda) \in \text{Eig}(L)$ ,*

$$(3.10) \quad \|\nabla \phi\|_{\partial D, \infty} \leq \|\phi\|_{\infty} \inf_{t > 0} e^{\lambda t/2} \|\nabla \psi(t, \cdot)\|_{\partial D, \infty}.$$



*Proof.* To prove (3.10), we fix  $x \in \partial D$ . For small  $\varepsilon > 0$ , let  $x^\varepsilon = \exp_x(\varepsilon N)$ , where  $N$  is the inward unit normal vector field of  $\partial D$ . Since  $\phi|_{\partial D} = 0$  and  $\psi(t, \cdot)|_{\partial D} = 0$ , we have

$$(3.11) \quad |\nabla \phi(x)| = |N\phi(x)| = \lim_{\varepsilon \rightarrow 0} \frac{|\phi(x^\varepsilon)|}{\varepsilon}, \quad |\nabla \psi(t, \cdot)(x)| = \lim_{\varepsilon \rightarrow 0} \frac{|\psi(t, x^\varepsilon)|}{\varepsilon}.$$

Let  $X_t^\varepsilon$  be the  $L$ -diffusion starting at  $x^\varepsilon$  and  $\tau_D^\varepsilon$  its first hitting time of  $\partial D$ . Note that

$$N_t := \phi(X_{t \wedge \tau_D^\varepsilon}^\varepsilon) e^{\lambda(t \wedge \tau_D^\varepsilon)/2}, \quad t \geq 0,$$

is a martingale. Thus, for each fixed  $t > 0$ , we can estimate as follows:

$$\begin{aligned} |\nabla \phi(x)| &= \lim_{\varepsilon \rightarrow 0} \frac{|\phi(x^\varepsilon)|}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{|\mathbb{E}[\phi(X_t^\varepsilon) \mathbf{1}_{\{t < \tau_D^\varepsilon\}}] e^{\lambda(t \wedge \tau_D^\varepsilon)/2}|}{\varepsilon} \\ &\leq \|\phi\|_\infty e^{\lambda t/2} \lim_{\varepsilon \rightarrow 0} \frac{\mathbb{E}[\mathbf{1}_{\{t < \tau_D^\varepsilon\}}]}{\varepsilon} \\ &\leq \|\phi\|_\infty e^{\lambda t/2} \lim_{\varepsilon \rightarrow 0} \frac{\psi(t, x^\varepsilon)}{\varepsilon} \\ &= \|\phi\|_\infty e^{\lambda t/2} |\nabla \psi(t, \cdot)|(x). \end{aligned}$$

Taking the infimum over  $t$  gives the claim.  $\square$

We now estimate  $\|\nabla \psi(t, \cdot)\|_\infty$ . Let  $\text{cut}(D)$  be the cut-locus of  $\partial D$ , which is a zero-volume closed subset of  $D$  such that  $\rho_{\partial D} := \text{dist}(\cdot, \partial D)$  is smooth in  $D \setminus \text{cut}(D)$ .

**Proposition 3.4.** *Let  $\alpha \in \mathbb{R}$  be such that*

$$(3.12) \quad \frac{1}{2} L \rho_{\partial D}(x) \leq \alpha, \quad x \in D \setminus \text{cut}(D).$$

*Then*

$$(3.13) \quad \begin{aligned} \|\nabla \psi(t, \cdot)\|_{\partial D, \infty} &\leq \alpha + \frac{\sqrt{2}}{\sqrt{\pi t}} + \int_0^t \frac{1 - e^{-\frac{\alpha^2 s}{2}}}{\sqrt{2\pi s^3}} ds \\ &\leq \frac{\sqrt{2}}{\sqrt{\pi t}} + \min \left\{ 2\alpha^+, \alpha + \frac{\alpha^2 \sqrt{t}}{\sqrt{2\pi}} \right\}. \end{aligned}$$

*Proof.* Let  $x \in D$  and let  $X_t$  solve SDE (3.4). As shown in [5],  $(\rho_{\partial D}(X_t))_{t \leq \tau_D}$  is a semi-martingale satisfying

$$(3.14) \quad \rho_{\partial D}(X_t) = \rho_{\partial D}(x) + b_t + \frac{1}{2} \int_0^t L \rho_{\partial D}(X_s) ds - l_t, \quad t \leq \tau_D,$$

where  $b_t$  is a real-valued Brownian motion starting at 0, and  $l_t$  a non-decreasing process which increases only when  $X_t \in \text{cut}(D)$ . Hence, setting  $\varepsilon = \rho_{\partial D}(x)$ , we deduce from (3.12) and (3.14) that

$$(3.15) \quad \rho_{\partial D}(X_t(x)) \leq Y_t^\alpha(\varepsilon) := \varepsilon + b_t + \alpha t, \quad t \leq \tau_D.$$

Consequently, letting  $T^\alpha(\varepsilon)$  be the first hitting time of 0 by  $Y_t^\alpha(\varepsilon)$ , we obtain

$$(3.16) \quad \psi(t, x) \leq \mathbb{P}(t < T^\alpha(\varepsilon)).$$

On the other hand, since  $\psi(t, \cdot)$  vanishes on the boundary and is positive in  $D$ , we have for all  $y \in \partial D$

$$(3.17) \quad |\nabla \psi(t, y)| = \lim_{x \in D, x \rightarrow y} \frac{\psi(t, x)}{\rho_{\partial D}(x)}.$$

Hence, by (3.16), to prove the first inequality in (3.13) it is enough to establish that

$$(3.18) \quad \limsup_{\varepsilon \downarrow 0} \frac{\mathbb{P}(t < T^\alpha(\varepsilon))}{\varepsilon} \leq \alpha + \frac{\sqrt{2}}{\sqrt{\pi t}} + \int_0^t \frac{1 - e^{-\frac{\alpha^2 s}{2}}}{\sqrt{2\pi s^3}} ds.$$

It is well known that the (sub-probability) density  $f_{\alpha, \varepsilon}$  of  $T^\alpha(\varepsilon)$  is

$$(3.19) \quad f_{\alpha, \varepsilon}(s) = \frac{\varepsilon \exp\left(\frac{-(\varepsilon + \alpha s)^2}{2s}\right)}{\sqrt{2\pi s^3}},$$

which can be obtained by the reflection principle for  $\alpha = 0$  and the Girsanov transform for  $\alpha \in \mathbb{R}$ . Thus

$$(3.20) \quad \begin{aligned} \mathbb{P}(t \geq T^\alpha(\varepsilon)) &= \varepsilon \int_0^t \frac{\exp\left(\frac{-(\varepsilon + \alpha s)^2}{2s}\right)}{\sqrt{2\pi s^3}} ds \\ &= \varepsilon \exp(-\alpha\varepsilon) \int_0^t \frac{e^{-\frac{\alpha^2 s}{2}}}{\sqrt{2\pi s^3}} \exp\left(-\frac{\varepsilon^2}{2s}\right) ds \\ &= \exp(-\alpha\varepsilon) \int_0^{2t/\varepsilon^2} \frac{e^{-1/r}}{\sqrt{\pi r^3}} \exp\left(-\frac{\alpha^2 \varepsilon^2 r}{4}\right) dr, \end{aligned}$$

where we have made the change of variable  $r = 2s/\varepsilon^2$ . With the change of variable  $v = 1/r$  we easily check that

$$(3.21) \quad \int_0^\infty r^{-3/2} e^{-1/r} dr = \Gamma(1/2) = \sqrt{\pi},$$

and this allows to write

$$(3.22) \quad \mathbb{P}(t \geq T^\alpha(\varepsilon)) = \exp(-\alpha\varepsilon) \left( 1 - \int_{2t/\varepsilon^2}^\infty \frac{e^{-1/r}}{\sqrt{\pi r^3}} dr - \int_0^{2t/\varepsilon^2} \frac{e^{-1/r}}{\sqrt{\pi r^3}} \left(1 - e^{-\frac{\alpha^2 \varepsilon^2 r}{4}}\right) dr \right).$$

As  $\varepsilon \rightarrow 0$ ,

$$\int_{2t/\varepsilon^2}^\infty \frac{e^{-1/r}}{\sqrt{r^3}} dr = \int_{2t/\varepsilon^2}^\infty \frac{1}{\sqrt{r^3}} dr + o(\varepsilon) = \frac{\varepsilon\sqrt{2}}{\sqrt{t}} + o(\varepsilon),$$

and with change of variable  $s = \frac{1}{2}\varepsilon^2 r$

$$\begin{aligned} \int_0^{2t/\varepsilon^2} \frac{e^{-1/r}}{\sqrt{\pi r^3}} \left(1 - e^{-\frac{\alpha^2 \varepsilon^2 r}{4}}\right) dr &= \varepsilon \int_0^t \frac{e^{-\frac{\varepsilon^2}{2s}}}{\sqrt{2\pi s^3}} \left(1 - e^{-\frac{\alpha^2 s}{2}}\right) ds \\ &= \varepsilon \int_0^t \frac{1 - e^{-\frac{\alpha^2 s}{2}}}{\sqrt{2\pi s^3}} ds + o(\varepsilon) \end{aligned}$$

by monotone convergence. Combining these with  $e^{-\alpha\varepsilon} = 1 - \alpha\varepsilon + o(\varepsilon)$ , we deduce from (3.22) that

$$(3.23) \quad \mathbb{P}(t \geq T^\alpha(\varepsilon)) = 1 - \varepsilon \left( \alpha + \frac{\sqrt{2}}{\sqrt{\pi t}} + \int_0^t \frac{1 - e^{-\frac{\alpha^2 s}{2}}}{\sqrt{2\pi s^3}} ds \right) + o(\varepsilon)$$

which yields (3.18).

Obviously, the inequality  $1 - e^{-s} \leq s$  for  $s \geq 0$  implies

$$(3.24) \quad \int_0^t \frac{1 - e^{-\frac{\alpha^2 s}{2}}}{\sqrt{2\pi s^3}} ds \leq \int_0^t \frac{\alpha^2}{2\sqrt{2\pi s}} ds = \frac{\alpha^2 \sqrt{t}}{\sqrt{2\pi}}.$$

Moreover, we will show that

$$(3.25) \quad \int_0^t \frac{1 - e^{-\frac{\alpha^2 s}{2}}}{\sqrt{2\pi s^3}} ds = |\alpha| - \frac{\sqrt{2}}{\sqrt{\pi t}} \int_0^{|\alpha|\sqrt{t}} dr \int_r^\infty e^{-\frac{s^2}{2}} ds \leq |\alpha|,$$

which then together with (3.24) gives the second inequality in (3.13).

Thus, to finish the proof, it remains to establish (3.25). Indeed, noting that

$$\sqrt{\frac{2}{\pi t}} \int_0^{|\alpha|\sqrt{t}} \left( \int_0^\infty e^{-s^2/2} ds \right) dr = |\alpha|,$$

we see that (3.25) is equivalent to

$$(3.26) \quad \sqrt{\frac{2}{\pi t}} \int_0^{|\alpha|\sqrt{t}} \left( \int_0^r e^{-s^2/2} ds \right) dr = \int_0^t \frac{1 - e^{-\frac{\alpha^2 s}{2}}}{\sqrt{2\pi s^3}} ds.$$

Two changes of variables give

$$\sqrt{\frac{2}{\pi t}} \int_0^{|\alpha|\sqrt{t}} \left( \int_0^r e^{-s^2/2} ds \right) dr = \frac{\alpha^2}{2\sqrt{2\pi t}} \int_0^t \frac{1}{\sqrt{u}} \left( \int_0^u \frac{1}{\sqrt{v}} e^{-\frac{\alpha^2 v}{2}} dv \right) du.$$

A first integration by parts choosing  $2(\sqrt{u} - \sqrt{t})$  as primitive of  $1/\sqrt{u}$  yields

$$\begin{aligned} \frac{\alpha^2}{2\sqrt{2\pi t}} \int_0^t \frac{1}{\sqrt{u}} \left( \int_0^u \frac{1}{\sqrt{v}} e^{-\frac{\alpha^2 v}{2}} dv \right) du &= \frac{\alpha^2}{\sqrt{2\pi}} \int_0^t \frac{1}{\sqrt{u}} e^{-\frac{\alpha^2 u}{2}} du - \frac{\alpha^2}{\sqrt{2\pi t}} \int_0^t e^{-\frac{\alpha^2 u}{2}} du \\ &= \frac{\alpha^2}{\sqrt{2\pi}} \int_0^t \frac{1}{\sqrt{u}} e^{-\frac{\alpha^2 u}{2}} du - \frac{\sqrt{2}}{\sqrt{\pi t}} \left(1 - e^{-\frac{\alpha^2 t}{2}}\right). \end{aligned}$$

A second integration by parts shows that the right hand side is equal to  $\int_0^t \frac{1 - e^{-\frac{\alpha^2 s}{2}}}{\sqrt{2\pi s^3}} ds$ .

Therefore, (3.26) holds and hence (3.25) as well.  $\square$

*Remark 3.1.* We proved that

$$\begin{aligned}
\|\nabla\psi(t, \cdot)\|_{\partial D, \infty} &\leq \alpha + \frac{\sqrt{2}}{\sqrt{\pi t}} + \int_0^t \frac{1 - e^{-\frac{\alpha^2 s}{2}}}{\sqrt{2\pi s^3}} ds \\
(3.27) \qquad \qquad \qquad &= 2 \max(0, \alpha) + \sqrt{\frac{2}{\pi t}} \int_{|\alpha|\sqrt{t}}^\infty \left( \int_r^\infty e^{-s^2/2} ds \right) dr
\end{aligned}$$

where the last equality follows from (3.25) and the observation that  $\int_0^\infty \int_r^\infty e^{-s^2/2} ds dr = 1$ . This implies that for all  $a > 1$

$$\begin{aligned}
\|\nabla\psi(t, \cdot)\|_{\partial D, \infty} &\leq 2 \max(0, \alpha) + \sqrt{\frac{2}{\pi t}} e^{-\frac{t\alpha^2}{2a}} \int_{|\alpha|\sqrt{t}}^\infty e^{-\frac{r^2(a-1)}{4a}} \left( \int_r^\infty e^{-\frac{s^2(a-1)}{4a}} ds \right) dr \\
&\leq 2 \max(0, \alpha) + \sqrt{\frac{2}{\pi t}} e^{-\frac{t\alpha^2}{2a}} \int_0^\infty e^{-\frac{r^2(a-1)}{4a}} \left( \int_0^\infty e^{-\frac{s^2(a-1)}{4a}} ds \right) dr \\
&\leq 2 \max(0, \alpha) + \frac{a}{a-1} \sqrt{\frac{2\pi}{t}} e^{-\frac{t\alpha^2}{2a}}.
\end{aligned}$$

In the case when  $4 < t\alpha^2$ , taking  $a = \frac{\alpha^2 t}{\alpha^2 t - 4}$  yields

$$\|\nabla\psi(t, \cdot)\|_{\partial D, \infty} \leq 2 \max(0, \alpha) + \frac{e^2 \alpha^2 t}{4} \sqrt{\frac{2\pi}{t}} e^{-\frac{\alpha^2 t}{2}}$$

and in particular

$$\|\nabla\psi(1/\lambda, \cdot)\|_{\partial D, \infty} \leq 2 \max(0, \alpha) + \frac{e^2 \alpha^2}{4\lambda} \sqrt{2\pi\lambda} e^{-\frac{\alpha^2}{2\lambda}}.$$

Combined with the Lemmas 3.2 and 3.3, this gives estimate (1.3) of Remark 1.1.

Finally, to estimate the constant  $\alpha$  in (3.12), we shall use the Laplacian comparison theorem to bound  $\Delta\rho_{\partial D}$  from above. See [4, 9] for the corresponding lower bound estimate.

**Theorem 3.5.** *Let  $\theta, k \in \mathbb{R}$  be such that  $\mathbb{I}_{\partial D} \geq -\theta$  and  $\text{Ric}_D \geq -(d-1)k$ . For  $t \geq 0$  let*

$$h(t) = \begin{cases} \cos(\sqrt{-k}t) + \frac{\theta}{\sqrt{-k}} \sin(\sqrt{-k}t), & \text{if } k < 0, \\ 1 + \theta t, & \text{if } k = 0, \\ \cosh(\sqrt{k}t) + \frac{\theta}{\sqrt{k}} \sinh(\sqrt{k}t), & \text{if } k > 0. \end{cases}$$

*Let  $h^{-1}(0)$  be the first zero of  $h$  (where  $h^{-1}(0) := \infty$  if  $h(t) > 0$  for all  $t \geq 0$ ). Then for any  $x \in D \setminus \text{cut}(D)$  such that  $\rho_{\partial D}(x) < h^{-1}(0)$ , there holds*

$$(3.28) \qquad \qquad \qquad \Delta\rho_{\partial D}(x) \leq (d-1) \frac{h'}{h}(\rho_{\partial D}(x)).$$

*In particular, if  $\theta, k \geq 0$  we have*

$$(3.29) \qquad \qquad \qquad \Delta\rho_{\partial D}(x) \leq (d-1)(\theta + \sqrt{k}), \quad x \in D \setminus \text{cut}(D).$$

*Proof.* The proof of (3.28) is adapted from [10, Theorem 1.2.2] where the corresponding Hessian upper bound is presented. For fixed  $x \in D \setminus \text{cut}(D)$ , let  $p$  be the orthogonal projection of  $x$  on  $\partial M$ , which is the unique point on  $\partial D$  such that  $\text{dist}(x, p) = \rho := \rho_{\partial D}(x)$ . Then

$$\gamma(s) := \exp_p(sN), \quad s \in [0, \rho],$$

is the minimal geodesic in  $D$  linking  $p$  and  $x$ . Let  $X_0(0) = N(p)$ , and  $\{X_i(0)\}_{1 \leq i \leq d-1}$  be an orthonormal basis of  $T_p \partial D$ . For  $0 \leq i \leq d-1$ , let

$$X_i(s) = //_{p \rightarrow \gamma(s)} X_i(0), \quad s \in [0, \rho],$$

be the parallel transport of  $X_i(0)$  along the geodesic  $\gamma$ . Moreover, for any  $1 \leq i \leq d-1$ , let  $\{J_i(s)\}_{s \in [0, \rho]}$  be the Jacobi field along  $\gamma$  such that  $J_i(\rho) = X_i(\rho)$  and

$$\langle \dot{J}_i(0), U \rangle = -\mathbb{I}_{\partial D}(J_i(0), U), \quad U \in T_p \partial D.$$

By the second variational formula (see e.g. page 321 in [2]), we have

$$\begin{aligned} \Delta \rho_{\partial D}(x) &= \sum_{i=1}^{d-1} \text{Hess}_{\rho_{\partial D}}(X_i, X_i)(\rho_{\partial D}(x)) \\ &= - \sum_{i=1}^{d-1} \mathbb{I}_{\partial D}(J_i(0), J_i(0)) \\ &\quad + \sum_{i=0}^{d-1} \int_0^{\rho_{\partial D}(x)} \left( |\dot{J}_i(s)|^2 - \langle \mathcal{R}(X_0(s), J_i(s))X_0(s), J_i(s) \rangle \right) ds \end{aligned} \tag{3.30}$$

where  $\mathcal{R}$  is the curvature tensor. Define

$$\tilde{J}_i(s) = \frac{h(s)}{h(\rho_{\partial D}(x))} X_i(s), \quad s \in [0, \rho_{\partial D}(x)], \quad 0 \leq i \leq d-1.$$

Then  $\tilde{J}_i(\rho_{\partial D}(x)) = J_i(\rho_{\partial D}(x)) = X_i$ , and by  $\mathbb{I}_{\partial D} \geq -\theta$ ,

$$\langle \dot{\tilde{J}}_i(0), \tilde{J}_i(0) \rangle = \frac{\theta}{h(\rho_{\partial D}(x))^2} \geq -\mathbb{I}_{\partial D}(\tilde{J}_i(0), \tilde{J}_i(0)), \quad 1 \leq i \leq d-1.$$

Hence, by the index lemma (see the first displayed formula on page 322 in [2]), and using the lower bound conditions on  $\mathbb{I}_{\partial D}$  and  $\text{Ric}_D$ , we deduce from (3.30) that

$$\begin{aligned} \Delta \rho_{\partial D}(x) &\leq - \sum_{i=1}^{d-1} \mathbb{I}_{\partial D}(\tilde{J}_i(0), \tilde{J}_i(0)) \\ &\quad + \sum_{i=0}^{d-1} \int_0^{\rho_{\partial D}(x)} \left( |\dot{\tilde{J}}_i(s)|^2 - \langle \mathcal{R}(X_0(s), \tilde{J}_i(s))X_0(s), \tilde{J}_i(s) \rangle \right) ds \\ &\leq \frac{1}{h(\rho_{\partial D}(x))^2} \left( -(\text{tr } \mathbb{I}_{\partial D})(p) + \int_0^{\rho_{\partial D}(x)} \{h'(s)^2 - h(s)^2 \text{Ric}(X_0(s), X_0(s))\} ds \right) \\ &\leq \frac{d-1}{h(\rho_{\partial D}(x))^2} \left( -\theta + \int_0^{\rho_{\partial D}(x)} \{h'(s)^2 + kh(s)^2\} ds \right) \\ &= (d-1) \frac{h'}{h}(\rho_{\partial D}(x)), \end{aligned}$$

where in the last step we used the facts that  $(hh')(0) = \theta$  and  $h'' = kh$ , and the latter implies

$$(h')^2 + kh^2 = (hh')' - hh'' + kh^2 = (hh')'.$$

Thus (3.28) holds. When  $\theta, k \geq 0$ , we have  $h^{-1}(0) = \infty$  and

$$\begin{aligned} \frac{h'(t)}{h(t)} &= \frac{\sqrt{k} \sinh(\sqrt{kt}) + \theta \cosh(\sqrt{kt})}{\cosh(\sqrt{kt}) + \frac{\theta}{\sqrt{k}} \sinh(\sqrt{kt})} \\ &\leq \frac{\sqrt{k} \cosh(\sqrt{kt}) + \theta \cosh(\sqrt{kt})}{\cosh(\sqrt{kt})} = \sqrt{k} + \theta. \end{aligned}$$

Then (3.29) follows from (3.28). □

We are now ready to prove Theorem 3.1.

*Proof of Theorem 3.1.* By Theorem 3.5 with  $k = \frac{K_0}{d-1}$ , condition (3.12) holds for  $\alpha$  as given in (3.1). Applying Lemmas 3.2, 3.3 and Proposition 3.4 with  $t = s = \frac{1}{\lambda}$ , we obtain

$$\begin{aligned} \|\nabla\phi\|_\infty &\leq e^{1/2} \left( \sqrt{\lambda + K_V} \|\phi\|_\infty + \|\nabla\phi\|_{\partial D, \infty} \right) \\ &\leq \|\phi\|_\infty \left\{ \sqrt{e(\lambda + K_V)} + e \left( \alpha + \frac{\sqrt{2\lambda}}{\sqrt{\pi}} + \alpha \wedge \frac{\alpha^2}{\sqrt{2\pi\lambda}} \right) \right\}. \end{aligned}$$

The result follows by substitution. □

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