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A conjecture which implies that there are infinitely many primes of the form $n! + 1$

Apoloniusz Tyszka

Abstract

Let $f(6) = 720$, and let $f(n+1) = f(n)!$ for every integer $n \geq 6$. For an integer $n \geq 6$, let Λ_n denote the following statement: if a system $\mathcal{S} \subseteq \{x_i! = x_j : 1 \leq i < j \leq n\} \cup \{x_i \cdot x_j = x_{j+1} : 1 \leq i < j \leq n-1\}$ has at most finitely many solutions in integers x_1, \dots, x_n greater than 3, then each such solution (x_1, \dots, x_n) satisfies $x_1, \dots, x_n \leq f(n)$. We conjecture that the statements $\Lambda_6, \dots, \Lambda_9$ are true. We prove that the statement Λ_9 implies that there are infinitely many primes of the form $n! + 1$.

Key words and phrases: prime numbers of the form $n! + 1$.

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It is conjectured that $n! + 1$ is prime for infinitely many positive integers n , see [2]. In this note, we propose a conjecture which implies this.

Lemma 1. For every integers x and y greater than 1, $x! \cdot y = y!$ if and only if $x + 1 = y$.

For an integer $n \geq 6$, let \mathcal{V}_n denote the following system of equations:

$$\left\{ \begin{array}{l} \forall i \in \{1, \dots, n-1\} \setminus \{1, 3, 5\} \quad x_i! = x_{i+1} \\ x_1! = x_6 \\ x_1 \cdot x_2 = x_3 \\ x_3 \cdot x_4 = x_5 \\ x_1 \cdot x_5 = x_6 \end{array} \right.$$

The diagram in Figure 1 illustrates the construction of the system \mathcal{V}_n .

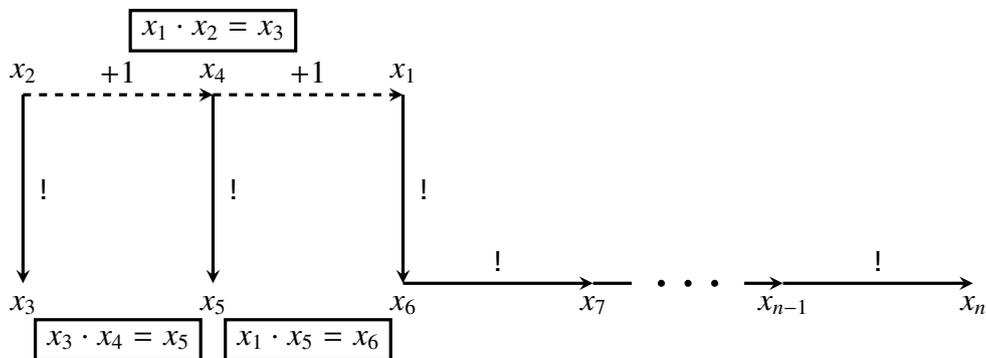


Fig. 1 Construction of the system \mathcal{V}_n

Let $f(6) = 720$, and let $f(n + 1) = f(n)!$ for every integer $n \geq 6$.

Lemma 2. For every integer $n \geq 6$, the system \mathcal{V}_n has exactly one solution in integers greater than 3, namely $(6, 4, 24, 5, 120, f(6), \dots, f(n))$.

Proof. By Lemma 1, $x_2 + 2 = x_1$. Hence, $(x_2 + 2) \cdot x_2 = x_1 \cdot x_2 = x_3 = x_2!$. Therefore, $x_2 = 4$. The rest of the proof follows from the diagram in Figure 1. \square

Let

$$T_n = \{x_i! = x_j : 1 \leq i < j \leq n\} \cup \{x_i \cdot x_j = x_{j+1} : 1 \leq i < j \leq n - 1\}$$

For an integer $n \geq 6$, let Λ_n denote the following statement: if a system $\mathcal{S} \subseteq T_n$ has at most finitely many solutions in integers x_1, \dots, x_n greater than 3, then each such solution (x_1, \dots, x_n) satisfies $x_1, \dots, x_n \leq f(n)$. We conjecture that the statements $\Lambda_6, \dots, \Lambda_9$ are true.

Theorem 1. For every statement Λ_n , the bound $f(n)$ cannot be decreased.

Proof. It follows from Lemma 2 because $\mathcal{V}_n \subseteq T_n$. \square

Theorem 2. Every statement Λ_n holds true with an integer bound that depends on n .

Proof. Indeed, for every integer $n \geq 6$, the system T_n has a finite number of subsystems. \square

Lemma 3. If a prime number x is greater than 3, then $\frac{(x-1)! + 1}{x} > 3$.

Lemma 4. (Wilson's theorem, [3, p. 89]) For every integer $x \geq 2$, x is prime if and only if x divides $(x-1)! + 1$.

Let \mathcal{A} denote the following system of equations:

$$\left\{ \begin{array}{l} x_1! = x_2 \\ x_2! = x_3 \\ x_3! = x_4 \\ x_5! = x_6 \\ x_8! = x_9 \\ x_3 \cdot x_5 = x_6 \\ x_4 \cdot x_8 = x_9 \\ x_5 \cdot x_7 = x_8 \end{array} \right.$$

Lemma 1 and the diagram in Figure 2 explain the construction of the system \mathcal{A} .

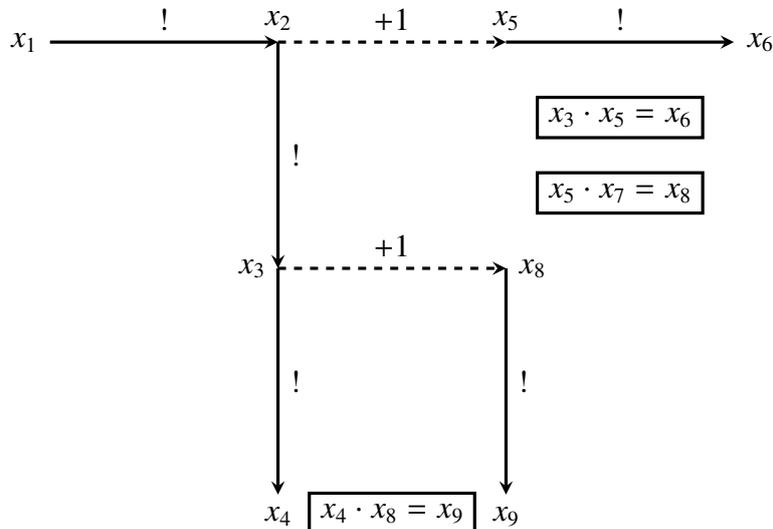


Fig. 2 Construction of the system \mathcal{A}

Lemma 5. For every integer $x_1 > 3$, the system \mathcal{A} is solvable in integers x_2, \dots, x_9 greater than 3 if and only if $x_1! + 1$ is prime. In this case, the integers x_2, \dots, x_9 are uniquely determined by the following equalities:

$$\begin{aligned}x_2 &= x_1! \\x_3 &= (x_1!)! \\x_4 &= ((x_1!)!)! \\x_5 &= x_1! + 1 \\x_6 &= (x_1! + 1)! \\x_7 &= \frac{(x_1!)! + 1}{x_1! + 1} \\x_8 &= (x_1!)! + 1 \\x_9 &= ((x_1!)! + 1)!\end{aligned}$$

Proof. By Lemmas 1 and 3, for every integer $x_1 > 3$, the system \mathcal{A} is solvable in integers x_2, \dots, x_9 greater than 3 if and only if $x_1! + 1$ divides $(x_1!)! + 1$. Hence, the claim of Lemma 5 follows from Lemma 4. \square

Theorem 3. The statement Λ_9 implies that there are infinitely many primes of the form $n! + 1$.

Proof. Harvey Dubner proved that $872! + 1$ is prime, see [1], [2], [4, p. 7], and [5]. Let $x_1 = 872$. By Lemma 5, there exists a unique tuple (x_2, \dots, x_9) of integers greater than 3 such that the tuple (x_1, x_2, \dots, x_9) solves the system \mathcal{A} . Hence,

$$x_9 = ((x_1!)! + 1)! > ((720!)! + 1)! > ((720!)!)! = ((f(6)!)!)! = f(9)$$

The statement Λ_9 and the inequality $x_9 > f(9)$ imply that the system \mathcal{A} has infinitely many solutions in integers x_1, \dots, x_9 greater than 3. This conclusion and Lemma 5 imply that $x_1! + 1$ is prime for infinitely many integers $x_1 > 3$. \square

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