



**HAL**  
open science

# Hybrid finite-volume/finite-element simulations of fully-nonlinear/weakly dispersive wave propagation, breaking, and runup on unstructured grids

Andrea Gilberto Filippini, Maria Kazolea, Mario Ricchiuto

► **To cite this version:**

Andrea Gilberto Filippini, Maria Kazolea, Mario Ricchiuto. Hybrid finite-volume/finite-element simulations of fully-nonlinear/weakly dispersive wave propagation, breaking, and runup on unstructured grids. SIAM Conference on Mathematical and Computational Issues in the Geosciences, Sep 2017, Erlangen, Germany. hal-01625308

**HAL Id: hal-01625308**

**<https://hal.science/hal-01625308>**

Submitted on 20 Jul 2018

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

HYBRID FINITE-VOLUME/FINITE-ELEMENT SIMULATIONS OF  
FULLY-NONLINEAR/WEAKLY DISPERSIVE WAVE PROPAGATION,  
BREAKING, AND RUNUP ON UNSTRUCTURED GRIDS

A.G. Filippini, M. Kazolea

Mario Ricchiuto

Team CARDAMOM

INRIA Bordeaux - Sud-Ouest, France

SIAM-GS17

Erlangen, September 14th 2017

## THANKS TO ...

- ▶ R. Chassagne (Inria CARDAMOM), P. Bonneton (EPOC Bordeaux, France): tidal bore dispersive propagation
- ▶ D. Lannes (IMB Bordeaux, France), F. Marche (U. Montpellier, France): Boussinesq modelling
- ▶ R. Pedreros & S. LeRoy (BRGM, France), R. Ata (EDF Chatou, France): Tohoku tsunami data

## SOME REFERENCES

### SOURCE MATERIAL

1. MR and A.G. Filippini, J.Comput.Phys. 271, 2014
2. A.G. Filippini, M. Kazolea and MR, J.Comput.Phys. 310, 2016
3. A.G. Filippini, M. Kazolea and MR, ISOPE proc.s , 2017

see also the PhD of A.G. Filippini (December 2016, available online on [hal](#))

### ADDITIONAL REFERENCES/RELATED WORK

1. J.T. Kirby and G. Wei, J. Waterway, Port, Coastal, and Ocean Eng. 1995;
2. F. Shi, et al. Ocean Modelling, 2012
3. M. Kazolea et al., Coast.Eng. 2012 and J.Comput.Phys, 2014
4. N. Aissiouene et al, Networks and Heterogeneous media, 2016

# TOHOKU EXAMPLE

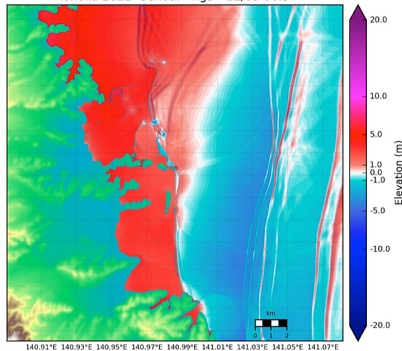


# TOHOKU EXAMPLE

Wave arrival in Sendai Bay

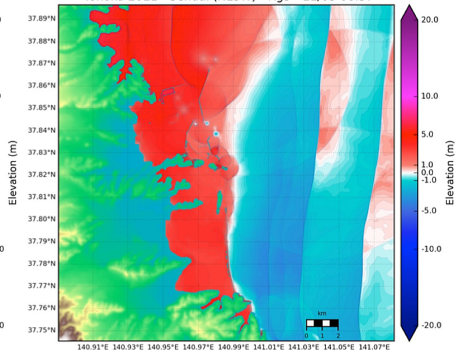
# TOHOKU EXAMPLE

Tohoku 2011 - Sendai - Rg3 - 11/03 06:57



Boussinesq

Tohoku 2011 - Sendai (NLSW) - Rg3 - 11/03 06:57



Shallow Water

## MODELLING APPROACH 1/6

Near shore: wave transformation due to interaction with **complex bathymetries**

- ▶ strong **nonlinearity** and **dispersion** ;
- ▶ refraction and diffraction ;
- ▶ **shoaling**;
- ▶ **breaking**;
- ▶ induced currents;
- ▶ run-up and inundation;



2004 Sumatra tsunami reaching the coast of Thailand



Sea waves diffracting around a peninsula

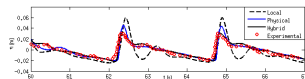
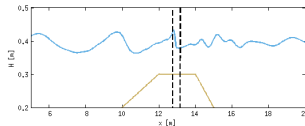
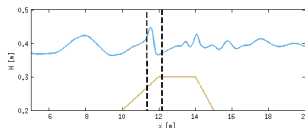


Rip current, Park beach (Coffs harbour, NSW Australia)



## NEAR SHORE HYDRODYNAMICS: MODELLING STANDPOINT

(Ribbed channel clip)



Propagation: large scales,  
dispersion, shoaling, etc  
POTENTIAL FLOW

Wave breaking:  
dissipation, vorticity  
CLOSURE MODEL

Runup/flooding :  
hydrostatic shallow water  
SHALLOW WATER

## NEAR SHORE HYDRODYNAMICS: MODELLING STANDPOINT

### Propagation vs wave breaking closure

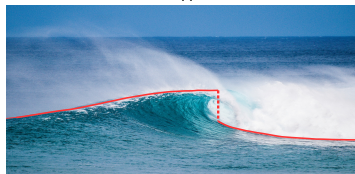
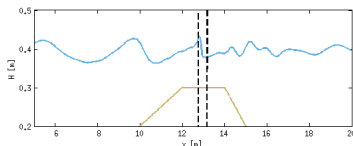
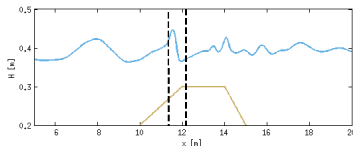
1. Potential/dispersive PDE for propagation + 3D Navier-Stokes (or SPH) for breaking/impact ;
2. Dispersive PDE for propagation + eddy viscosity to model dissipation in surf zone/breakers ;
3. Coupling dispersive PDEs with shallow water/hydrostatic limit:
  - ▶ Kirby, Grilli, et al (FUNWAVE-TVD)
  - ▶ Lynett et al USC (COULWave)
  - ▶ Delis, Kazolea, Synolakis (TUCWave) Coast.Eng. 2011, JCP 2014
  - ▶ Smit, Zijlema et al DELFT (SWASH)
  - ▶ See also:  
Tonelli, Petti Coast.Eng. 2009, Bonneton et al. JCP 2011, Coast.Eng. 2012

## BREAKING CLOSURE: SHALLOW WATER DISSIPATION

### CLOSURE MODEL

1. Detect breaking regions
2. Remove dispersive terms
3. → shallow water shock
4. Total energy  $E = gh^2/2 + hu^2/2$
5. Dissipation :

$$\mathcal{D}_b = [\mathcal{F}_E - \sigma E] \approx \gamma_b [H]^3$$

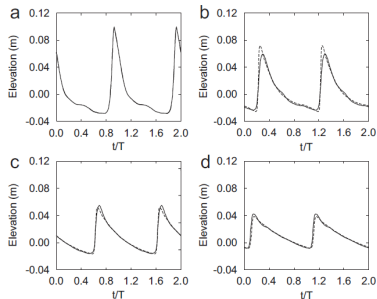
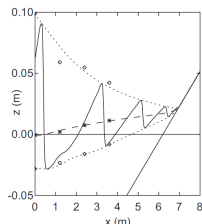


## BREAKING CLOSURE: SHALLOW WATER DISSIPATION

### CLOSURE MODEL

1. Detect breaking regions
2. Remove dispersive terms
3. → shallow water shock
4. Total energy  $E = gh^2/2 + hu^2/2$
5. Dissipation :

$$\mathcal{D}_b = [\mathcal{F}_E - \sigma E] \approx \gamma_b [H]^3$$



## BREAKING CLOSURE: SHALLOW WATER DISSIPATION

### CLOSURE MODEL

1. Detect breaking regions
2. Remove dispersive terms
3. → shallow water shock
4. Total energy  $E = gh^2/2 + hu^2/2$
5. Dissipation :

$$\mathcal{D}_b = [\mathcal{F}_E - \sigma E] \approx \gamma_b [H]^3$$



Need to handle shallow water:

- ▶ “Upwinding” / numerical dissipation
- ▶ Shock capturing in breaking regions
- ▶ etc

All the std. artillery...

## DISPERSIVE MODELS FOR PROPAGATION

Which dispersion dominates ?

1. Continuous dispersive models have a range of validity related to the **model dispersion error**
2. Discrete dispersive models have a range of validity related to the **scheme dispersion error**

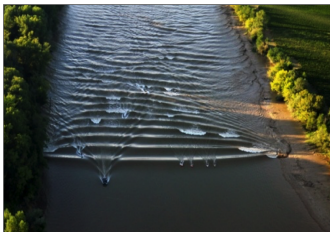
How can we exploit this knowledge to construct efficient low dispersion schemes on unstructured grids ?

# DISPERSIVE SURFACE WAVES: BOUSSINESQ APPROACH (1/8)

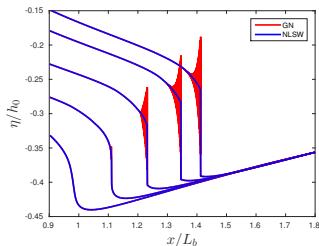
In fluid dynamics, **dispersion** of water **waves** generally refers to frequency **dispersion**, which means that **waves** of different wavelengths travel at different phase speeds. Water **waves**, in this context, are **waves** propagating on the water surface, with gravity and surface tension as the restoring forces.

[Dispersion \(water waves\) - Wikipedia](https://en.wikipedia.org/wiki/Dispersion_(water_waves))

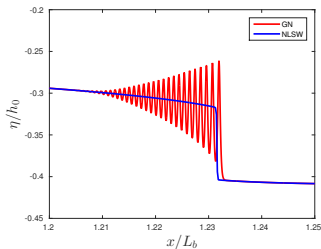
[https://en.wikipedia.org/wiki/Dispersion\\_\(water\\_waves\)](https://en.wikipedia.org/wiki/Dispersion_(water_waves))



Undular bore (Garonne river,  
Bonneton et al *J.Geophys.Res.* 2015)



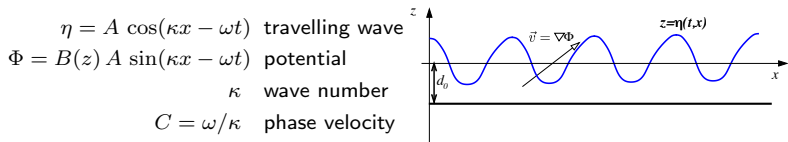
Tidal wave distortion in a converging estuary



## DISPERSIVE SURFACE WAVES: BOUSSINESQ APPROACH (2/8)

### LINEAR TRAVELLING WAVES: INCOMPRESSIBLE EULER EQUATIONS

Theory due to (G.B. Airy, *Encyclopædia Metropolitana*, 1841)



$$B(z) = \frac{g}{\omega} \frac{\cosh(\kappa(d_0 + z))}{\cosh(\kappa(d_0))}$$

$$C^2 = C_0^2 \frac{\tanh(\kappa d_0)}{\kappa d_0}$$

### PHASE RELATION AND PHASE DISPERSION

$C_0^2 = g d_0$  shallow water (hyperbolic) celerity.  $C = C(\kappa)$ !!



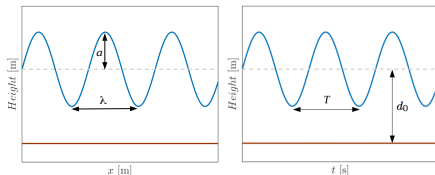
# DISPERSIVE SURFACE WAVES: BOUSSINESQ APPROACH (3/8)

$a$  wave amplitude

$\lambda = 2\pi/\kappa$  wave length

$T$  wave period

$\lambda = C(\kappa)T = C(2\pi/\lambda)T$  phase relation



## DIMENSIONLESS PARAMETERS

▶  $\mu = \frac{d_0}{\lambda} = \frac{\kappa d_0}{2\pi}$  dispersion

▶  $\epsilon = \frac{a}{d_0}$  nonlinearity

## PHYSICAL HYPOTHESES

- ▶ Long waves: smallness of  $\mu^2$
- ▶ Weak-nonlinearity: smallness of  $\epsilon = \mathcal{O}(\mu^2)$
- ▶ Full-nonlinearity:  $\epsilon = \mathcal{O}(1)$

# DISPERSIVE SURFACE WAVES: BOUSSINESQ APPROACH (4/8)

## MODELLING PRINCIPLES<sup>1</sup>

Starting from the 3D nonlinear wave equations:

1. dimensionless form ;
2. **asymptotic development** w.r.t.  $\mu^2$ :  $\nabla\Phi = \nabla\Phi_0 + \mu^2\nabla\Phi_2 + \mu^4\nabla\Phi_4 + \dots$
3. **depth averaging**: 3D  $\rightarrow$  2D
4. retain appropriate order terms ....

---

<sup>1</sup>J. Boussinesq, *J.Math.Pures Appl.*, 1872 – M.W. Dingemans, World Scientific, 1997

# DISPERSIVE SURFACE WAVES: BOUSSINESQ APPROACH (5/8)

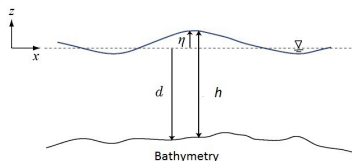
## EXAMPLE: SHALLOW WATER EQUATIONS (ZERO-TH ORDER MODEL)

$$\tilde{h}_t + \tilde{q}_x = 0$$

$$\tilde{q}_t + \epsilon(\tilde{u}\tilde{q})_x + \tilde{h}\tilde{\eta}_x = 0 \pm \mathcal{O}(\mu^2)$$

With the notation

- ▶ dimensionless depth :  $\tilde{h} = \tilde{d} + \epsilon\tilde{\eta}$
- ▶ dimensionless volume flux :  $q = \tilde{h}\tilde{u}$
- ▶ dimensionless depth averaged velocity:  $\tilde{u}$
- ▶ Red : hyperbolic shallow water equations
- ▶ Blue terms are responsible for dispersion



Nonlinear – non dispersive

# DISPERSIVE SURFACE WAVES: BOUSSINESQ APPROACH (6/8)

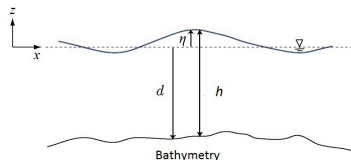
## EXAMPLE: PEREGRINE'S EQUATIONS<sup>2</sup>

$$\tilde{h}_t + \tilde{q}_x = 0$$

$$\tilde{q}_t + \epsilon(\tilde{u}\tilde{q})_x + \tilde{h}\tilde{\eta}_x = \mu^2\tilde{h}\left(\frac{\tilde{d}^2}{3}u_{txx} + \frac{\tilde{d}\tilde{d}_x}{3}\tilde{u}_{tx}\right) + \cancel{O(\mu^4, \epsilon\mu^2)}$$

With the notation

- ▶ dimensionless depth :  $\tilde{h} = \tilde{d} + \epsilon\tilde{\eta}$
- ▶ dimensionless volume flux :  $q = \tilde{h}\tilde{u}$
- ▶ dimensionless depth averaged velocity:  $\tilde{u}$
- ▶ Red : hyperbolic shallow water equations
- ▶ Blue terms are responsible for dispersion



Weakly nonlinear – weakly dispersive

<sup>2</sup>D.H. Peregrine. J.Fluid.Mech, 1967

# DISPERSIVE SURFACE WAVES: BOUSSINESQ APPROACH (7/8)

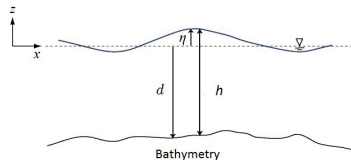
## EXAMPLE: MADSEN & SORENSEN'S ENHANCED EQUATIONS<sup>3</sup>

$$\tilde{h}_t + \tilde{q}_x = 0$$

$$\tilde{q}_t + \epsilon(\tilde{u}\tilde{q})_x + \tilde{h}\tilde{\eta}_x = \mu^2 \left( \beta \tilde{d}^2 \tilde{q}_{txx} + \frac{\tilde{d}\tilde{d}_x}{3} \tilde{q}_{tx} + B \tilde{d}^3 \tilde{\eta}_{xxx} + 2B \tilde{d}^2 \tilde{d}_x \tilde{\eta}_{xx} \right) + \cancel{O(\mu^4, \epsilon\mu^2)}$$

With the notation

- ▶ dimensionless depth :  $\tilde{h} = \tilde{d} + \epsilon\tilde{\eta}$
- ▶ dimensionless volume flux :  $q = \tilde{h}\tilde{u}$
- ▶ dimensionless depth averaged velocity:  $\tilde{u}$
- ▶ Red : hyperbolic shallow water equations
- ▶ Blue terms are responsible for dispersion



Weakly nonlinear – weakly dispersive

Phase enhancement via the tunable coeff.  $B$  and  $\beta = B + 1/3$  (cf. later)

<sup>3</sup>P.A. Madsen and O.R. Sorensen Coast.Eng., 1992

# DISPERSIVE SURFACE WAVES: BOUSSINESQ APPROACH (8/8)

## EXAMPLE: ENHANCED SERRE-GREEN-NAGHDI EQUATIONS<sup>4</sup>

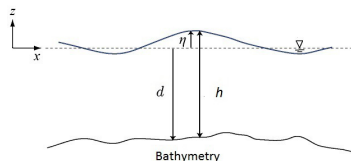
$$\tilde{h}_t + \tilde{q}_x = 0$$

$$\tilde{q}_t + \epsilon(\tilde{u}\tilde{q})_x + \tilde{h}\tilde{\eta}_x = \mu^2 \tilde{h}\tilde{\psi} \pm \mathcal{O}(\mu^4)$$

$$\tilde{\psi} = \alpha \left[ \partial_x(\tilde{h}^2 \partial_x(\tilde{u}_t + \tilde{u}\tilde{u}_x)) \right] + (\alpha - 1) \left[ \partial_x(\tilde{h}^2 \partial_{xx}\tilde{\eta}) \right] + \mathcal{Q}_\psi(\tilde{u}, \tilde{h}, \tilde{d}; \tilde{u}_x, \tilde{h}_x, \tilde{d}_x)$$

With the notation

- ▶ dimensionless depth :  $\tilde{h} = \tilde{d} + \epsilon\tilde{\eta}$
- ▶ dimensionless volume flux :  $q = \tilde{h}\tilde{u}$
- ▶ dimensionless depth averaged velocity:  $\tilde{u}$
- ▶ Red : hyperbolic shallow water equations
- ▶ Blue terms are responsible for dispersion



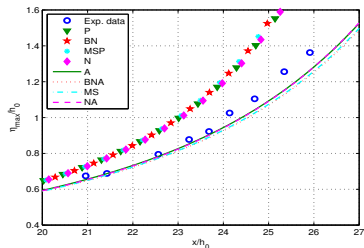
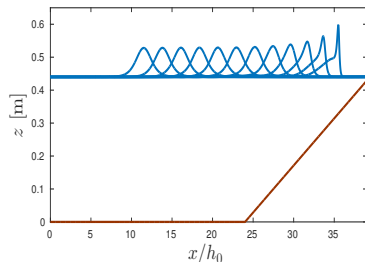
Fully nonlinear – weakly dispersive

Phase enhancement via the tunable coeff.  $\alpha$  (cf. later)

<sup>4</sup>A.E. Green J.Fluid Mech., 1976 – F. Chazel et al. J.Sci.Comp., 2011

## WHAT ARE THESE MODELS GOOD FOR: NONLINEAR BEHAVIOR

### Shoaling test, weakly nonlinear models<sup>5</sup>



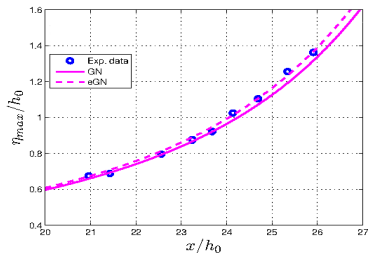
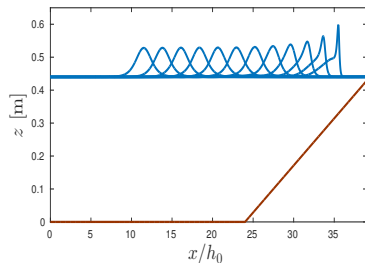
Many variations for a given asymptotic accuracy (same linear limit), e.g.

- ▶  $\mu^2 \tilde{d} \approx \mu^2 \tilde{h}$  as  $\epsilon \mu^2 \tilde{\eta}$  is negligible (weakly nonlinear)
- ▶  $\mu^2 (\tilde{d}\tilde{u})_{xxt} \approx \mu^2 \tilde{q}_{xxt}$  as  $\epsilon \mu^2 (\tilde{\eta}\tilde{u})_{xx}$  is negligible (weakly nonlinear)
- ▶ etc.

## MODELLING ERROR (2/3)

WHAT ARE THESE MODELS GOOD FOR: NONLINEAR BEHAVIOR

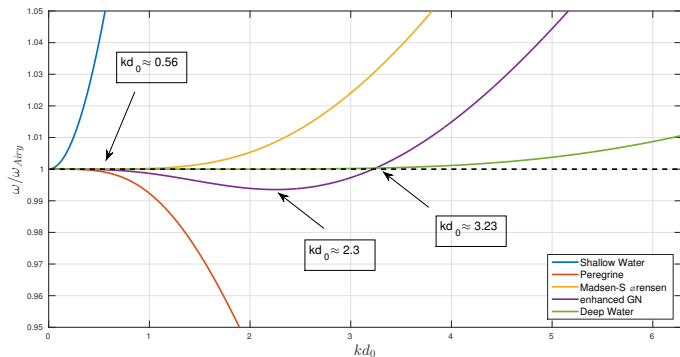
Shoaling test, fully nonlinear models





## MODELLING ERROR (3/3)

### WHAT ARE THESE MODELS GOOD FOR: PHASE BEHAVIOR



CAVEAT: MS and eGN are equivalent. The "classical" optimized MS obtained changing  $\alpha$

# CONTINUOUS AND DISCRETE

## CONTINUOUS: ERROR W.R.T. EULER EQ.S

- ▶ Range of validity in terms of reduced wave number  $\kappa d_0 = 2\pi d_0/\lambda$
- ▶ Near shore:  $(\kappa d_0)_{\max} \approx \pi$ ,  $(d_0/\lambda)_{\max} \approx 2$

## DISCRETE: ERROR W.R.T. CONTINUOUS MODEL

- ▶ Range of validity in terms of  $1/N = \Delta x/\lambda$  : pts per wavelength

Overall error w.r.t. Euler eq.s ???

# TRUNCATION ERROR HEURISTICS

## EXAMPLE: LINEARIZED SHALLOW WATER

$$\partial_t W + A \partial_x W = 0$$

with (in dimensionless form)

$$W = \begin{pmatrix} \eta \\ u \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad |A| = \text{Id}_2$$

where we recall that (all  $\tilde{\cdot}$  are removed for simplicity, and  $c_0^2 = g d_0$ )

$$\eta = \frac{\eta^{\text{dim}}}{\epsilon d_0}, \quad u = \frac{u^{\text{dim}}}{\epsilon c_0}, \quad x = \frac{x^{\text{dim}}}{\lambda}, \quad t = \frac{t^{\text{dim}}}{\lambda / c_0}$$

# TRUNCATION ERROR HEURISTICS

## EXAMPLE: LINEARIZED SHALLOW WATER

1st order upwind method

$$\partial_t W + A \frac{W_{i+1} - W_{i-1}}{2\Delta x} = \frac{\Delta x}{2} \frac{W_{i+1} - 2W_i + W_{i-1}}{\Delta x^2}$$

2nd order centered differencing

$$\partial_t W + A \frac{W_{i+1} - W_{i-1}}{2\Delta x} = 0$$

# TRUNCATION ERROR HEURISTICS

## EXAMPLE: LINEARIZED SHALLOW WATER

1st order upwind and 2nd order centered differencing: modified equation/TE

$$\partial_t W^{\text{smooth}} + A\partial_x W^{\text{smooth}} = \frac{\Delta x}{2}\partial_{xx} W^{\text{smooth}} - \frac{A\Delta x^2}{6}\partial_{xxx} W^{\text{smooth}} + \mathcal{O}(\Delta x^3)$$

$$\partial_t W^{\text{smooth}} + A\partial_x W^{\text{smooth}} = -\frac{A\Delta x^2}{6}\partial_{xxx} W^{\text{smooth}} + \mathcal{O}(\Delta x^4)$$

It looks like *"if we could see the dispersive effects of the 1st order scheme, they would be the same as those of the second order"*..

but the numerical viscosity is too high...

# TRUNCATION ERROR HEURISTICS

## EXAMPLE: LINEARIZED SHALLOW WATER

Turn down the viscosity: 3<sup>rd</sup> upwind scheme

$$\partial_t W + \left( A \frac{W_{i+1/2}^R + W_{i+1/2}^L}{2\Delta x} - \frac{W_{i+1/2}^R - W_{i+1/2}^L}{2\Delta x} \right) - \left( A \frac{W_{i-1/2}^R + W_{i-1/2}^L}{2\Delta x} - \frac{W_{i-1/2}^R - W_{i-1/2}^L}{2\Delta x} \right) = 0$$

With “*quadratically reconstructed*” left and right values at  $x_i \pm \Delta x/2$

$$W_{i+1/2}^L = W_i + \frac{W_i - W_{i-1}}{6} + \frac{W_{i+1} - W_i}{3}$$
$$W_{i+1/2}^R = W_{i+1} - \frac{W_i - W_{i-1}}{6} - \frac{W_{i+2} - W_{i+1}}{3}$$

# TRUNCATION ERROR HEURISTICS

## EXAMPLE: LINEARIZED SHALLOW WATER

3<sup>rd</sup> order upwind vs 4<sup>th</sup> order central differencing: modified eq./TE

$$\partial_t W^{\text{smooth}} + A \partial_x W^{\text{smooth}} = -\frac{\Delta x^3}{12} \partial_{xxxx} W^{\text{smooth}} - \frac{A \Delta x^4}{30} \partial_{xxxxx} W^{\text{smooth}} + \mathcal{O}(\Delta x^5)$$

$$\partial_t W^{\text{smooth}} + A \partial_x W^{\text{smooth}} = -\frac{A \Delta x^4}{30} \partial_{xxxxx} W^{\text{smooth}} + \mathcal{O}(\Delta x^6)$$

- ▶ Low dissipation:  $\mathcal{O}(\Delta x^3)$  viscosity (ok for hyperbolic + explicit time stepping)
- ▶ Same dispersion of the fourth order FD !

## EXAMPLE: ENHANCED LINEARIZED MADSEN-SORENSEN

$$\begin{cases} \partial_t \eta + \partial_x u = 0 \\ \partial_t u - \mu^2 \left( \frac{1}{3} + \beta \right) \partial_{xxt} u + \partial_x \eta - \mu^2 \beta \partial_{xxx} \eta = 0 \end{cases}$$



## EXAMPLE: ENHANCED LINEARIZED MADSEN-SORENSEN

$$\left\{ \begin{array}{l} \partial_t \eta + \partial_x u = 0 \\ \partial_t w + \partial_x \zeta = 0 \\ u - \mu^2 \left( \frac{1}{3} + \beta \right) \partial_{xx} u = w \\ \eta - \mu^2 \beta \partial_{xx} \eta = \zeta \end{array} \right.$$

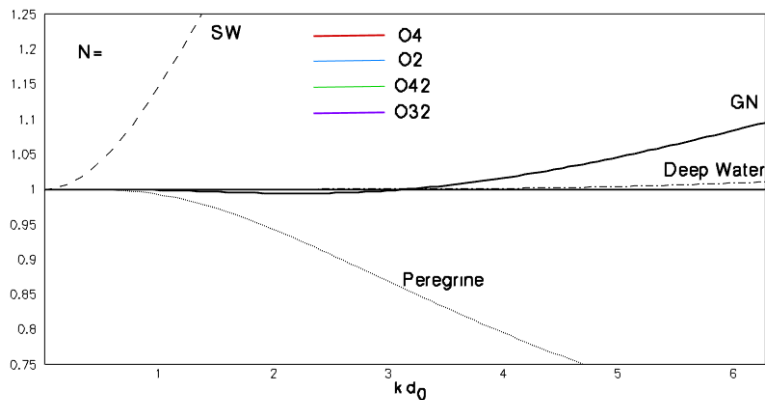
First order (“hyperbolic”) system

Overhead w.r.t. hyperbolic system

How accurate must the discretization of these red terms be ?

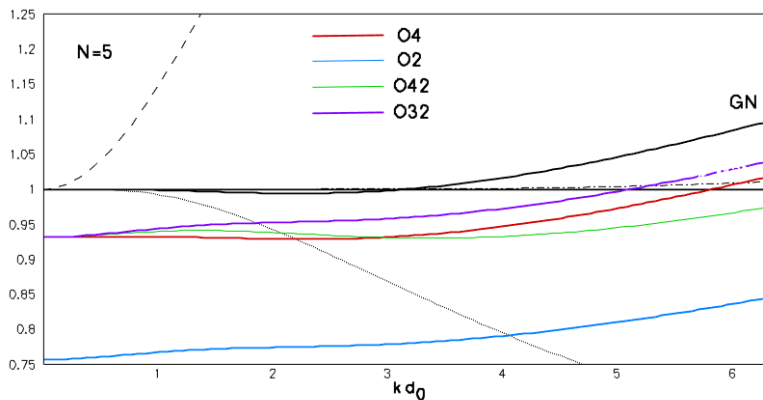
# DISPERSION ERROR HEURISTICS

## LINEARIZED MADSEN-SORENSEN (WITH GN $\alpha_{opt}$ )



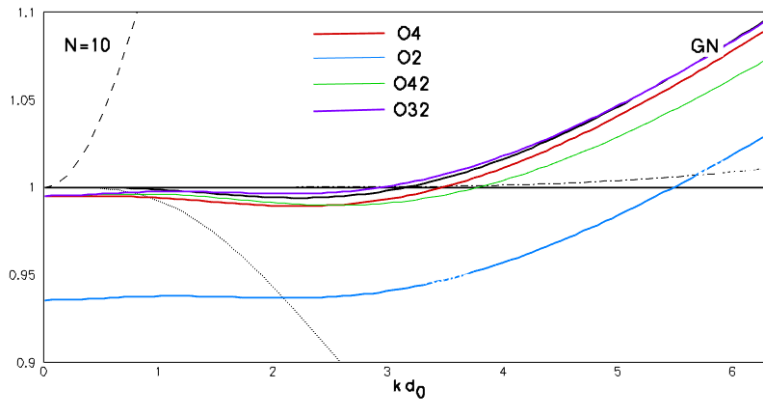
# DISPERSION ERROR HEURISTICS

## LINEARIZED MADSEN-SORENSEN (WITH GN $\alpha_{opt}$ )



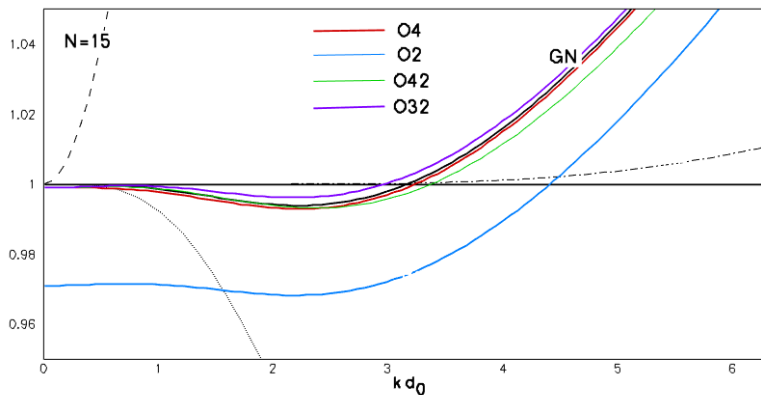
# DISPERSION ERROR HEURISTICS

## LINEARIZED MADSEN-SORENSEN (WITH GN $\alpha_{opt}$ )



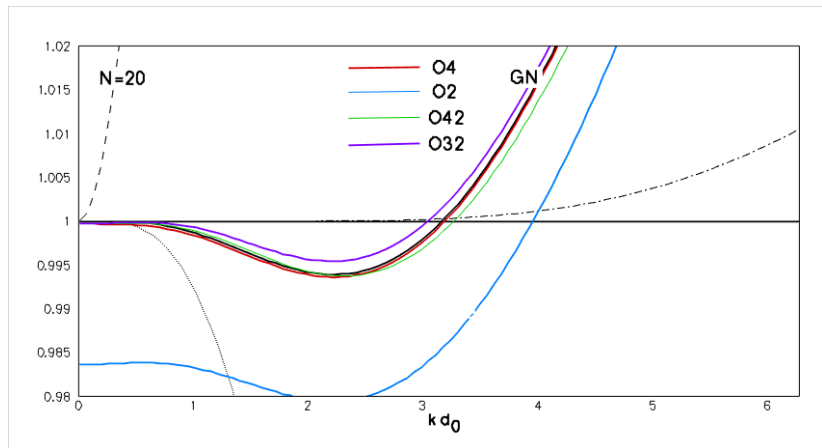
# DISPERSION ERROR HEURISTICS

## LINEARIZED MADSEN-SORENSEN (WITH GN $\alpha_{opt}$ )



# DISPERSION ERROR HEURISTICS

## LINEARIZED MADSEN-SORENSEN (WITH GN $\alpha_{opt}$ )



## LESSON LEARNED

### HYPERBOLIC OPERATOR

We **MUST** use at least a third order scheme

### ELLIPTIC OPERATOR

Waste of effort using more than second order for near shore models

### NEXT STEP

Unstructured grid generalization for the eGN model

# DISCRETIZATION OF THE (BREAKING) EGN ON UNSTRUCTURED GRIDS

## THE EGN EQUATIONS (IN 1D)

$$\tilde{h}_t + \tilde{q}_x = 0$$

$$\tilde{q}_t + \epsilon(\tilde{u}\tilde{q})_x + \tilde{h}\tilde{\eta}_x = \mu^2 \tilde{h}\tilde{\psi}$$

$$\tilde{\psi} = \alpha \left[ \partial_x(\tilde{h}^2 \partial_x(\tilde{u}_t + \tilde{u}\tilde{u}_x)) \right] + (\alpha - 1) \left[ \partial_x(\tilde{h}^2 \partial_{xx}\tilde{\eta}) \right] + \mathcal{Q}_\psi(\tilde{u}, \tilde{h}, \tilde{d}; \tilde{u}_x, \tilde{h}_x, \tilde{d}_x)$$

With the notation

- ▶ dimensionless depth :  $\tilde{h} = \tilde{d} + \epsilon\tilde{\eta}$
- ▶ dimensionless volume flux :  $q = \tilde{h}\tilde{u}$  with
- ▶  $\tilde{u}$  the (dimensionless) depth averaged velocity
- ▶ Red terms provide the hyperbolic shallow water equations
- ▶ Blue terms are responsible for dispersion

Fully nonlinear – weakly dispersive  
Phase enhancement via the tunable coeff.  $\alpha$  (cf. later)



## THE EGN EQUATIONS (IN 2D)

$$\partial_t h + \nabla \cdot \mathbf{q} = 0;$$

$$\left( \mathcal{I} + \alpha \mathbb{T}_h \right) \left( \partial_t \mathbf{q} + \nabla \cdot \left( \frac{\mathbf{q} \otimes \mathbf{q}}{h} \right) + gh \nabla \eta \right) - \mathbb{T}_h (gh \nabla \eta) + h \mathcal{Q}(\mathbf{u}) = 0.$$

With the notation

- ▶ depth :  $h = d + \eta$
- ▶ (vector) volume flux :  $\mathbf{q} = \tilde{h} \mathbf{u}$  with
- ▶  $\mathbf{q}$  the (dimensionless) depth averaged velocity vector
- ▶  $\mathbb{T}_h$  the dispersive elliptic operator given by

$$\mathbb{T}_h(\cdot) = h \mathcal{T} \left( \frac{\cdot}{h} \right)$$

where  $\mathcal{T}$  is a self adjoint operator<sup>6</sup>:

$$\mathcal{T}(\cdot) = S^*(S(\cdot)), \quad S(\cdot) = \frac{h}{\sqrt{3}} \nabla \cdot (\cdot) - \frac{\sqrt{3}-1}{2} \nabla b \cdot (\cdot)$$

<sup>6</sup> Alvarez-Samaniego and Lannes, *Indiana Univ. Math J.*, 2008

# DISCRETIZATION OF THE (BREAKING) EGN ON UNSTRUCTURED GRIDS

## THE EGN EQUATIONS

We recast the system in two independent steps:

$$\begin{aligned} \partial_t h + \nabla \cdot \mathbf{q} &= 0; && \rightarrow \text{hyperbolic step} \\ \partial_t \mathbf{q} + \nabla \cdot \left( \frac{\mathbf{q} \otimes \mathbf{q}}{h} \right) + gh \nabla \eta &= h \boldsymbol{\psi}; \\ (\mathcal{I} + \alpha \mathcal{T})(\boldsymbol{\psi}) - \mathcal{T}(g \nabla \eta) + \mathcal{Q}(\mathbf{u}) &= 0. \rightarrow \text{elliptic step} \end{aligned}$$

This reformulation aims at exploiting the self-adjoint character of  $\mathcal{T}$

# DISCRETIZATION OF THE (BREAKING) EGN ON UNSTRUCTURED GRIDS

## SOLUTION ALGORITHM

At each time-step  $n$ :

1. elliptic step is solved :  $\psi = (\mathcal{I} + \alpha\mathcal{T})^{-1}(RHS)$  using  $(h^n, \mathbf{q}^n)$  ;
2. shallow water solver + non-hydrostatic term  $\psi \rightarrow (h^{n+1}, \mathbf{q}^{n+1})$  .

## ELLIPTIC STEP

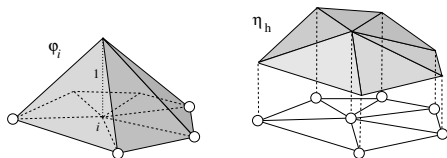
- ▶ Continuous  $P^1$  Galerkin FE exploiting the self-adjoint character of  $\mathcal{T}^7$

$$\int_{\Omega} \boldsymbol{\nu} \cdot \boldsymbol{\psi}_h + \alpha \int_{\Omega} S(\boldsymbol{\nu}) S(\boldsymbol{\psi}_h) = \text{RHS}(\eta_h, h_h, \mathbf{u}_h, b_h) ;$$

- ▶ Linear system :

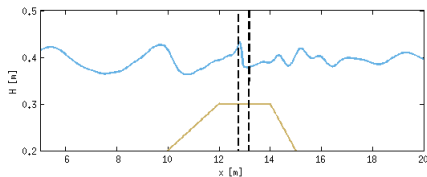
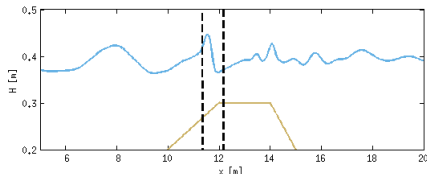
$$(\mathbb{M}_{\mathbb{H}}^{\mathbb{G}} + \alpha \mathbb{T}) \boldsymbol{\Psi} = \mathbb{T} \boldsymbol{\delta}_h - \mathbb{Q} \quad \text{with } \boldsymbol{\delta}_h \text{ the } L^2 \text{ projection of } g \nabla \eta$$

- ▶ Coercivity of  $\mathcal{I} + \alpha \mathcal{T} \rightarrow$  invertibility of  $(\mathbb{M}_{\mathbb{H}}^{\mathbb{G}} + \alpha \mathbb{T})$   
(block SPD + diagonally dominant) ;



## ELLIPTIC STEP

- ▶ Propagation: add  $h\psi$  to the rhs
- ▶ Wave breaking:
  1. Flag nodes
  2. Agglomerate elements and enlarge breaking region in wave direction
  3. Set  $\psi$  to zero: breakers as shallow water shocks<sup>9</sup>
- ▶ Wave breaking detection<sup>10</sup>
  - ▶ either  $|\eta_t| > \gamma\sqrt{gH}$  with  $\gamma \in [0.4, 0.6]$
  - ▶ or  $\|\nabla\eta\| > \text{tg}\theta$  with  $\theta \in [15, 30]^\circ$
  - ▶ and  $\text{Fr} > \text{Fr}_{cr}$

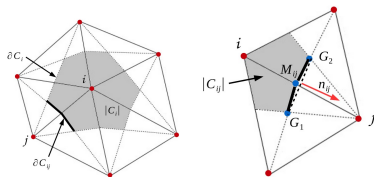


<sup>9</sup> P. Bonneton Ocean Eng., 2007

<sup>10</sup> Kazolea, Delis and Synolakis, JCP 2014

# DISCRETIZATION OF THE (BREAKING) EGN ON UNSTRUCTURED GRIDS

## HYPERBOLIC STEP



- ▶ Node - centered FV scheme

- ▶ 
$$\frac{\partial \mathbf{U}_i}{\partial t} + \frac{1}{|C_i|} \int_{\partial \Omega} \left( \mathbf{F} \hat{n}^x + \mathbf{G} \hat{n}^y \right) = \frac{1}{|C_i|} \int_{\Omega} \left( \mathbf{S}_b + h \Psi \right) ;$$

- ▶ Roe's Riemann solver + Harten-Hyman entropy fix

- ▶ High order reconstruction: weighted least squares<sup>11</sup> (quadratic or cubic) ;

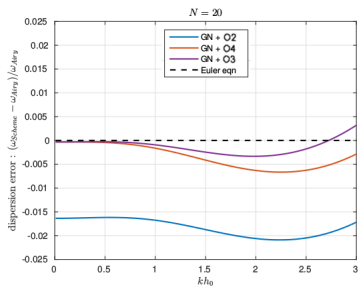
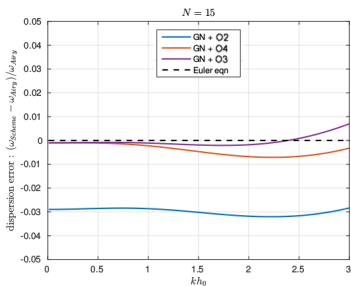
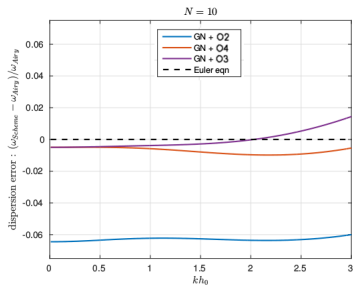
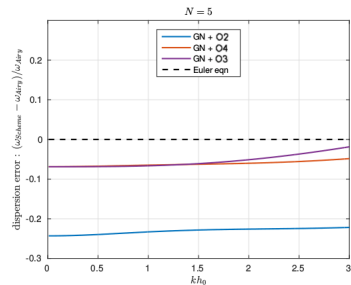
- ▶ Well-balanced treatment of topography, wet/dry fronts, etc<sup>12</sup>

<sup>11</sup> Ollivier-Gooch et al. AIAA J. 2009; Wang et al JCP 2017

<sup>12</sup> Bermudez&Vazquez, CAF 1994; Hubbard&Garcia-Navarro, JCP 2000; Brufau et al. IJNMF 2002; Castro, Math.&Computer Mod. 2005; etc.etc.

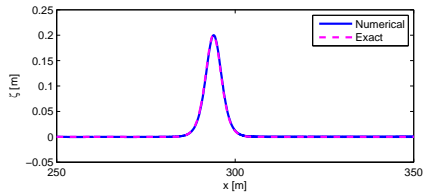
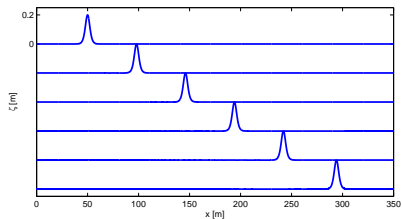
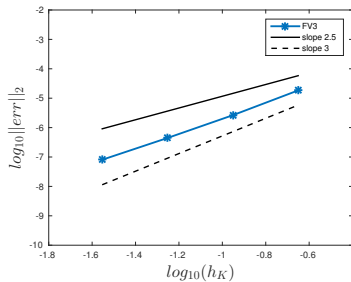
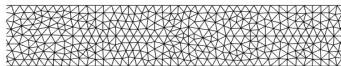
# DISCRETIZATION OF THE (BREAKING) eGN ON UNSTRUCTURED GRIDS

## DISPERSION ANALYSIS OF THE SCHEME



# BENCHMARKING: PROPAGATION TEST 1

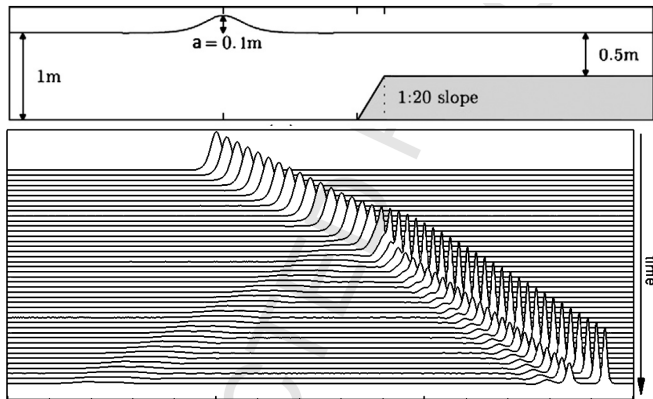
TEST DESCRIPTION:  $a_0 = 0.2$  [M],  $h_0 = 1$  [M]





# BENCHMARKING: PROPAGATION TEST 2

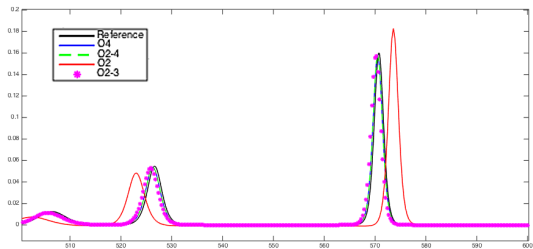
SHOALING ON A SHELVE



# BENCHMARKING: PROPAGATION TEST 2

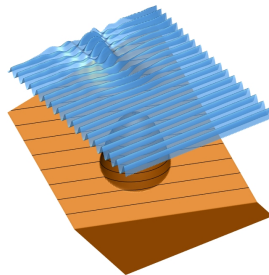
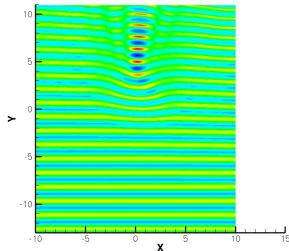
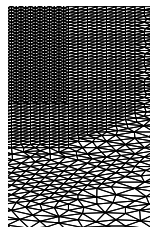
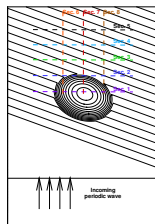
SHOALING ON A SHELF

## FINAL SOLUTION



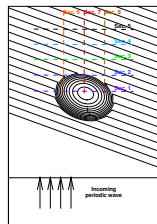
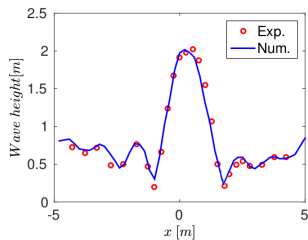
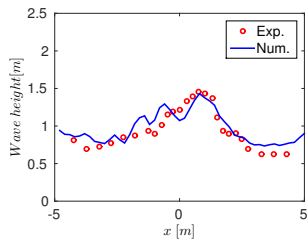
# BENCHMARKING: DIFFRACTION ON AN ELLIPTIC SHOAL

- ▶ Range :  $a/h_0 = 0.0515$ ,  $T = 1$  [s] ;
- ▶ Energy transfer to higher harmonics;
- ▶ Experiments: (*Berkhoff et al.*, 1982)
- ▶ Adapted mesh 88760 nodes ;

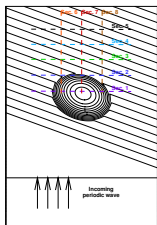
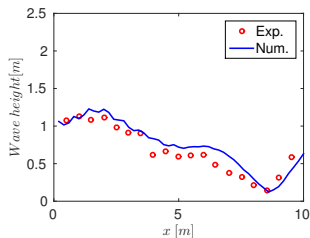
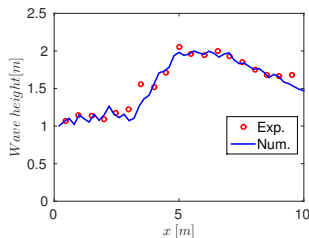


# BENCHMARKING: DIFFRACTION ON AN ELLIPTIC SHOAL

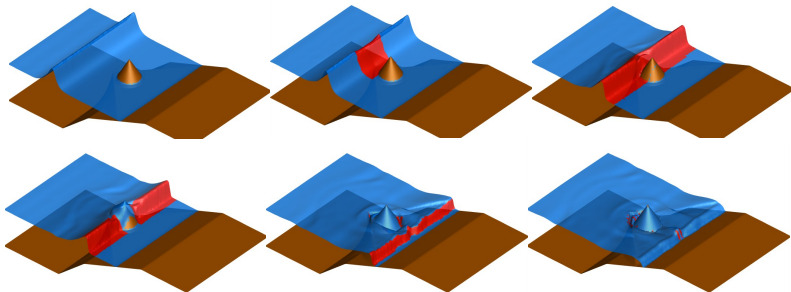
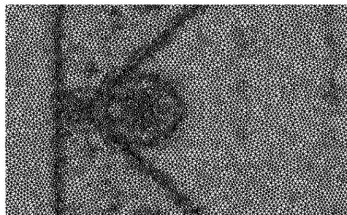
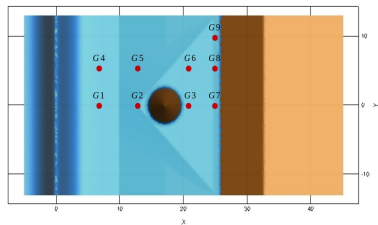
## Section 2 and Section 4



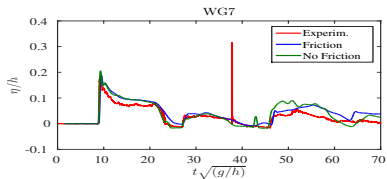
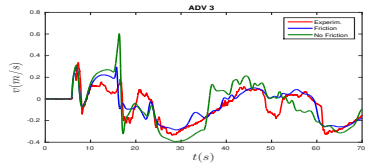
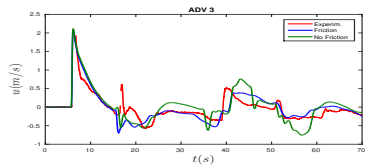
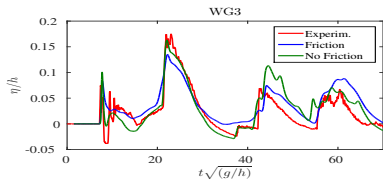
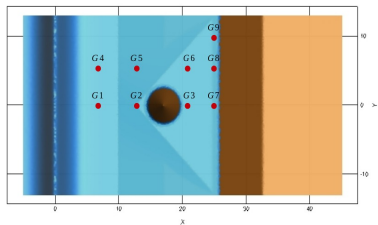
## Section 7 and Section 8



# BENCHMARKING: OVERTOPPING ON A THREE DIMENSIONAL REEF



# BENCHMARKING: OVERTOPPING ON A THREE DIMENSIONAL REEF

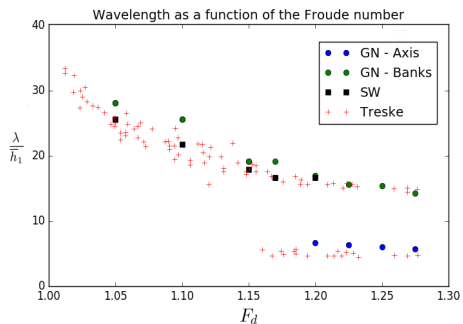


# APPLICATION: STUDY OF UNDULAR BORES AND TRESKE EXPERIMENTS



Undular bore (Garonne river,  
Bonneton et al *J.Geophys.Res.* 2015)

# APPLICATION: STUDY OF UNDULAR BORES AND TRESKE EXPERIMENTS

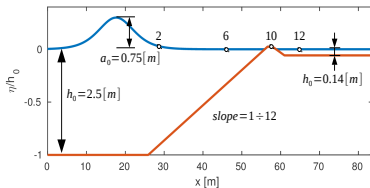


Comparison with experiments by Treske



## THE SHORT OF IT ...

- ▶ Interaction modelling/discretization error for near shore Boussinesq
- ▶ First (hyperbolic) order system vs second order elliptic operator :
  - High order on hyperbolic : third order (at least) for good dispersion
  - Elliptic component: can be treated with a second order method
  - Unstructured grid eGN: high(er) order FV + P1 FEM for elliptic part
- ▶ Wave breaking: revert to SW + shock capturing

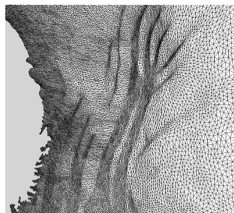


## PERSPECTIVES AND OPEN ISSUES

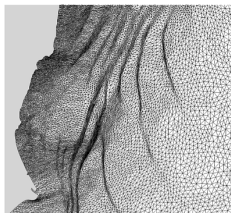
### A FEW ONGOING/FORESEEN EXTENSIONS

- ▶ Systematic orders/CPU time investigation
- ▶ Other methods, in particular DG and RD
- ▶ Implicit time integration + energy conserving in space and time ?
- ▶ Wave breaking via PDE based eddy viscosity
- ▶ Study of deep water/non-hydrostatic (multi-layer) models
- ▶ Moving meshes and adaptation  
(cf. parallel session on rupture based tsunami simulation)

Iwate



Fukushima



Miyagi

