Geometric Pseudo-Spectral Method for Spatial Integration of Dynamical Systems

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1. Introduction

Port-based modelling techniques and languages have been extensively used these last decades to model, simulate and control a wide variety of lumped parameters physical systems [5] [35] [28][37]. When dealing with distributed parameters systems, first "series"-like expansions models were derived from various kind of finite differences, modal or finite elements methods. These already appear in many textbooks such as [5]. More recently a more intrinsic formulation of port-based models for distributed parameter systems with boundary energy flow have been proposed [6]. They are based on the definition of the state variables as the densities of some thermodynamical extensive variables. Their time derivatives and the distributed conjugated intensive variables form the pairs of variables which permit to extend the port-Hamiltonian formulation. Using these variables the port Hamiltonian formulation may be extended to the infinite-dimensional systems using a canonical geometric structure called Stokes-Dirac structure [6]. This modelling approach has been applied successfully to many hyperbolic systems as varied as transmission lines models [7], beam equations [8] or shallow water equations [9]. The approach has also been applied to parabolic model such as for heat and mass transport models in an adsorption column [25], for fuel cell models [11] or for Ionic Polymer-Metal Composites [32].

In this paper we shall suggest a spatial discretization scheme which preserves both the geometric structure of the port Hamiltonian model, that is the Dirac structure, and approximate the behavior of the actual system in terms of conservation of energy and other conserved quantities. This means that in the lossless case some approximated extensive variables will be conserved (as a first integral) and that in general, the dissipated power will be approximated conveniently. Finite difference schemes are often used by specialists of application domains but in very specific formulations and they seldom lead to general formulation with such desired geometric and energetic properties. Finite element methods have

already been adapted to lead to such a formulation. First applied to electromagnetism problems [17], mixed finite elements formulations have been successfully applied to various Hamiltonian formulation of distributed parameter systems such as transmission line [7], diffusion problems [26] and shallow water fluid dynamics [18].

However in many applications, pseudospectral methods are preferred because they lead to low order approximate model, with good spectral properties (in the linear case). When a polynomial basis is chosen for the approximation space, the derived pseudospectral method may be viewed as a collocation method where the collocation points are the zeros of the chosen polynomial . In this case, the reduced model is moreover stated in "natural" variables (the infinite dimensional state variables evaluated at the collocation points), making its physical meaning easy to catch [14]. These collocation (polynomial) methods are the most used in process engineering either for processes simulation or control. Indeed the nice properties of collocation methods make them also very effective to develop control laws on the reduced model [19].

In this paper, we suggest a polynomial pseudospectral method which preserves the geometric structure of port Hamiltonian models, the phenomenological laws and the conservation laws without introducing any uncontrolled numerical dissipation. Doing so, we expect nice structural dynamical properties for the obtained reduced model as well as easy implementation of passivity based or energy shaping control techniques. Mixed finite elements methods may be viewed as a particular case of the methodology developed hereafter for the case of low order polynomial approximations. In this sense, this work is a generalization of previous ones [7, 26, 18] and it provides a wider and more theoretical interpretation of implicit choices made in these earlier works. Among these choices, those related to the kernel of exact exterior derivation in exterior differential forms spaces, and to the implicit choice of boundary port variables, as well as their relation with the geometric interconnection structure and its associated power product, will be given in some detail.

The paper is organized as follows : section 2 is dedicated to recalls on the Hamiltonian formulation of open distributed parameter systems, on the Stokes-Dirac structure and illustrates the definition on an hyperbolic example. In section 3 the different polynomial approximation spaces are defined and the relations defining the Stokes-Dirac structure are restricted to these spaces. In section 4, the kernel and image representation of the Dirac structure as well as the choice of (implicit) boundary port-variables are discussed and related to previous works on boundary control systems. The discretization of phenomenological laws which are closure equations for the interconnection structure is discussed in section 5. The examples of section 2 are reduced with the proposed scheme and their power balance equations are written down. In section 6 it is shown on the transmission line example that, while preserving the geometric interconnection structure and invariant (energy) of the actual PDE model, the proposed method also share the excellent spectral behavior of the classical collocation method.

2. Port-based modelling for distributed parameter systems

In this section we recall briefly the definition of boundary Port Hamiltonian Systems, as it has been introduced in [6] and presented in more details in [31] and in [37, Chap. 4].

2.1. Canonical Stokes-Dirac structure

Port Hamiltonian systems, introduced in [36], are extensions of Hamiltonian systems which are defined with respect to a geometric structure called Dirac structure [21] which generalizes Poisson or presymplectic structures used in classical Hamiltonian systems [1] and is also extremely useful for the description of open or controlled physical systems. The reader is refered to the recent tutorial book [37] for an overview on its use in dynamical systems' modeling and control.

Definition 2.1. *[Dirac structure] Let $\mathscr{F}$ (called* space of flow variables*) and $\mathscr{E}$ (called* space of effort variables*) be two real vectors spaces endowed with the non degenerate bilinear form (*power pairing*) $\langle\cdot,\cdot\rangle : \mathscr{B} = \mathscr{F}\times\mathscr{E} \to \mathbb{R}$ where is called the (*bond space*. Define an associated* symmetric *bilinear form by symmetrization on the previous one : $\ll\cdot,\cdot\gg: \mathscr{B}\times\mathscr{B}\to\mathbb{R} : ((f_1,e_1),(f_2,e_2)) \mapsto \ll(f_1,e_1),(f_2,e_2)\gg := \langle e_1|f_2\rangle + \langle e_2|f_1\rangle$. Then the vector subspace $\mathscr{D}\subset\mathscr{B}$ is a Dirac structure with respect to the bilinear form $\ll.\gg$ if and only if $\mathscr{D}=\mathscr{D}^\perp$, where the orthogonality is defined with respect to the bilinear form $\ll\cdot,\cdot\gg$.*

The flow variables $f\in\mathscr{F}$ and the effort variables $e\in\mathscr{E}$ are said conjugate variables in the sense that their product may be computed with the pairing $\langle\cdot,\cdot\rangle$. For physical system this pairing has the dimension of power and the Dirac structure corresponds, for instance, to Kirchhoff's laws applying to the voltages and currents in an electrical circuit [36], the kinematic and static static models applying to the forces and velocities of a mechanism [35]. The definition of Dirac structure may also be related to a famous results on the admissible currents and voltages in electrical circuits as follows: the isotropy condition $\mathscr{Z}\subset\mathscr{Z}^\perp$ is equivalent to Tellegen's theorem. Actually it might be useful to consider a weaker notion than Dirac structure, satisfying only to the isotropy condition and called Tellegen structure [23, chap. 5].

Proposition 2.2. *Define the spaces of flow and effort variables, denoted respectively by $\mathscr{F}$, $\mathscr{E}$ as $\mathscr{F}=\Lambda^1\times\Lambda^1\times\mathbb{R}^2 \ni (f^q,f^p;f_\partial)$ and $\mathscr{E}=\Lambda^0\times\Lambda^0\times\mathbb{R}^2\ni(e^q,e^p;e_\partial)$. The subspace $\mathscr{D}\subset\mathscr{B}=\mathscr{F}\times\mathscr{E}$*

$$\begin{array}{rcl}\mathscr{D} & = & \{(f_p,f_q,f_b,e_p,e_q,e_b)\in\mathcal{F}\times\mathcal{E}| \\ & & \left[\begin{array}{c} f_p \\ f_q \end{array}\right] = \varepsilon\left[\begin{array}{cc} 0 & d \\ d & 0 \end{array}\right]\left[\begin{array}{c} e_p \\ e_q \end{array}\right], \\ & & \left[\begin{array}{c} f_b \\ e_b \end{array}\right] = \left[\begin{array}{cc} \varepsilon & 0 \\ 0 & -1 \end{array}\right]\left[\begin{array}{c} e_{p|\partial Z} \\ e_{q|\partial Z} \end{array}\right]\Bigg\}\end{array} \tag{1}$$

where $\varepsilon \in \{-1,+1\}$ and $|_{\partial Z}$ denotes restriction to the boundary ∂Z is a Stokes-Dirac structure with respect to the symmetric pairing:

$$\langle\langle (f_1^p, f_1^q, f_{\partial,1}, e_1^p, e_1^q, e_{\partial,1}), (f_2^p, f_2^q, f_{\partial,2}, e_2^p, e_2^q, e_{\partial,2})\rangle\rangle = \int_Z (e_1^p \wedge f_2^p + e_1^q \wedge f_2^q + e_2^p \wedge f_1^p + e_2^q \wedge f_1^q) + \int_{\partial Z} (e_{\partial,1} \wedge f_{\partial,2} + e_{\partial,2} \wedge f_{\partial,1}) \tag{2}$$

This Dirac structure arizes in the formulation of systems of conservation laws for reversible but also irreversible physical systems. Thereby it arizes in both hyperbolic and parabolic Partial Differential Equations (see [37, Chap. 4]) as is now illustrated with the example of transmission line without and with dissipation through Ohm's law.

2.2. The lossless transmission line

The lossless transmission line corresponds to the canonical example of a hyperbolic system of two conservation laws. It corresponds, when eliminating one of the state variables to a, possibly nonlinear wave equation, in the same way as the p-system or the vibrating string [34]. It is well-known that the wave equation admits a Hamiltonian state space realization [33]. For the lossless transmission line, considering the the charge density $q(t,z)$, the magnetic flux density $p(t,z)$, the current $I(t,z)$ and the voltage $V(t,z)$, the dynamical model of the transmission line is:

$$\frac{\partial q(t,z)}{\partial t} = -\frac{\partial I(t,z)}{\partial z}, \qquad \frac{\partial p(t,z)}{\partial t} = -\frac{\partial V(t,z)}{\partial z} \tag{3}$$

Recalling that the charge and the magnetic flux are subject to conservation laws and hence correspond, in a thermodynamic perspective to extensive variables, whereas the current and voltage correspond to the conjugated intensive variables [37, Chap. 4], one may consider the following identification with exterior forms:

$$\begin{pmatrix} e^q(t) \\ e^p(t) \end{pmatrix} = \begin{pmatrix} I(t,z) \\ V(t,z) \end{pmatrix} \in \Lambda^0 \times \Lambda^0 \quad \text{and} \quad \begin{pmatrix} f^q(t) \\ f^p(t) \end{pmatrix} = \begin{pmatrix} \dot{q}(t,z)\, dz \\ \dot{p}(t,z)\, dz \end{pmatrix} \in \Lambda^1 \times \Lambda^1 \tag{4}$$

The dynamical equations (3) may then be written as:

$$\begin{bmatrix} f_p \\ f_q \end{bmatrix} = -\begin{bmatrix} 0 & \mathrm{d} \\ \mathrm{d} & 0 \end{bmatrix} \begin{bmatrix} e_p \\ e_q \end{bmatrix} \tag{5}$$

By considering the boundary value of the current and voltage:

$$\begin{aligned} e_\partial^0 &= e^q(0) & e_\partial^L &= e^q(L) \\ f_\partial^0 &= e^p(0) & f_\partial^L &= e^p(L) \end{aligned} \tag{6}$$

one recognize the Dirac structure defined in proposition 2.2 above.

The total electromagnetic energy of the transmission line is given by the Hamiltonian function:

$$H = H^c + H^p = \int_0^L \frac{\star q(z)}{2C(z)} \wedge q(z) + \int_0^L \frac{\star p(z)}{2L(z)} \wedge p(z) \tag{7}$$

with variational derivatives:

$$\delta_q H(q,p) = V(t,z) = e^q(t) \quad \text{and} \quad \delta_p H(q,p) = I(t,z) = e^p(t) \tag{8}$$

According to the definition in [6] the system (5) with the identification (8) and (4) define a *port-Hamiltonian system*, by: $\{\dot{p}, \dot{q}, \delta_p H(q,p), \delta_q H(q,p)\} \in \mathscr{D}$ with respect to the Stokes-Dirac structure of the theorem 2.2.

2.3. Transmission line with dissipation

We have presented the port- Hamiltonian formulation of the lossless transmission line above. Now we shall add to this model adissipative phenomenon, Ohm's law, and show how, acording to [6] one may extend the port-Hamiltonian formulation in order to encompass the dissipation. Actually Ohm's law induces a dissipative voltage: $V_R(t,\, z) = r(z)\, I(t,\, z)$ where $r(z)$ is the resitance of the medium defined over the space domain $[0, L]$. According to the previous identification with k-forms, one may also write that Ohm's law indices a 1-form: $f^r(t) = V_R(t,\, z)\, dz = r(z) \star e^p(t)$. Hence the the transmission line with dissipation may be formulated as a dissipative Hamiltonian system:

$$\begin{pmatrix} f^q(z) \\ f^p(z) \end{pmatrix} = - \underbrace{\begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix}}_{conservative} \begin{pmatrix} e^q(z) \\ e^p(z) \end{pmatrix} - \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & r(z)\star \end{pmatrix}}_{dissipative} \begin{pmatrix} e^q(z) \\ e^p(z) \end{pmatrix} \tag{9}$$

The port boundary variables are defined precisely as in the non-dissipative case as the voltage and currents at the boundary points according to (6) . The system composed of the Hamiltonian evolution equations (9) and the boundary port variables (6) is actually defined as the *dissipative boundary port-Hamiltonian system* [6][29] associated with the operator: $\begin{pmatrix} 0 & d \\ d & -r(z)\star \end{pmatrix}$.

3. A polynomial geometric discretization scheme

In the previous section we have seen how Stokes-Dirac structures may be used in models of both hyperbolic and parabolic distributed parameters systems. This interconnection structure both guarantees power conservation (and resulting energy balance equations) and completely specifies the way all model subparts interact together. In this section we shall present a spatial discretization scheme which preserves these properties and therefore in a first instance discretize the Stokes-Dirac structure. The scheme is based on the use of polynomial pseudo-spectral methods, adapted in such a way that the restriction of the

constitutive relations of the Stokes-Dirac structure to the approximation spaces define a gain a finite-dimensional Dirac structure [21, 22]. Such an approach has already been developed using linear mixed finite elements methods [17, 7, 26]: our purpose is to generalize these results to pseudo-spectral methods in general, to develop a systematic way to construct the resulting finite-dimensional Dirac structure, to interpret this structure in term of a reduced power product, and to connect these results to implicit choices of boundary conditions and boundary control systems. We will use, for the spatial reduction, polynomial approximation bases (with Lagrange interpolation) in such a way that the reduced variables will be approximations of the distributed ones at chosen "collocation" points. Usually these points are zeros of orthogonal polynomials in order to to reduce the oscillations of the solution. A methodology to select them, according to the conditioning and dynamical stability of the reduced system, is presented in [20].

3.1. Polynomial approximation and exact differentiation

According to their definition in the proposition 2.2, the effort variables are approximated in a polynomial basis of 0-forms , with polynomials $\varphi_i(z)$, $i = 0, .., N$, and the flow variables are approximated in a polynomial basis of 1-forms with polynomials $\psi_i(z)$, $i = 0, .., N+1$:

$$e^q(z) = \sum_{i=0}^{N} e_i^q \varphi_i(z) \tag{10}$$

$$e^p(z) = \sum_{i=0}^{N} e_i^p \varphi_i(z) \tag{11}$$

$$f^q(z) = \sum_{i=0}^{N-1} f_i^q \psi_i(z) dz \tag{12}$$

$$f^p(z) = \sum_{i=0}^{N-1} f_i^p \psi_i(z) dz \tag{13}$$

The interpolating polynomials $\varphi_i(z)$ and $\psi_i(z)$ are Lagrange polynomials, respectively of degree N and $N-1$ defined as

$$\varphi_i(z) = \prod_{j=0, j\neq i}^{N} , \frac{z-\zeta_j}{\zeta_i - \zeta_j} \quad \psi_i(z) = \prod_{k=0, k\neq i}^{N-1} \frac{z-z_k}{z_i - z_k}$$

satisfying $\varphi_i(\zeta_j) = \delta_{ij}$ and $\psi_i(z_j) = \delta_{ij}$, $\zeta_j \in]0, L[$, $j = 0, ..., N$ being the interpolating points associated to the basis $\varphi_i(z)$, $i = 0, .., N$, while $z_j \in]0, L[$, $j = 0, ..., N-1$ are those of the basis $\psi_i(z)$, $i = 0, .., N+1$.
Inserting relations (10) - (13) into the differential relation of the Stokes-Dirac structure (1) , and evaluating the approximations at the collocation points z_k

(zeroes of the ψ_i Lagrange polynomials), one obtains the following matrix relations:

$$\begin{aligned} f_k^q &= \sum_{i=0}^{N} D_{k,i} e_i^p \\ f_k^p &= \sum_{i=0}^{N} D_{k,i} e_i^q \end{aligned} \tag{14}$$

where D is a $N \times (N+1)$ matrix obtained by exact differentiation

$$D_{k,i} = -\frac{d\varphi_i}{dz}(z_k) \tag{15}$$

The boundary variables are defined according to their definition in (1) by the polynomial interpolation of efforts e^p, e^q (0-forms) in the two-boundary points 0 and L.

$$\begin{aligned} e_\partial^0 &= e^q(0) = \sum_{i=0}^{N} e_i^q \varphi_i(0) \\ e_\partial^L &= e^q(L) = \sum_{i=0}^{N} e_i^q \varphi_i(L) \\ f_\partial^0 &= e^p(0) = \sum_{i=0}^{N} e_i^p \varphi_i(0) \\ f_\partial^L &= e^p(L) = \sum_{i=0}^{N} e_i^p \varphi_i(L) \end{aligned} \tag{16}$$

The two relations (14) and (16) define the restriction of the constitutive relations of the Stokes-Dirac structure to the approximation spaces of 0- and 1-forms:

$$\begin{pmatrix} f^q \\ f^p \\ f_\partial^0 \\ f_\partial^L \\ e_\partial^0 \\ e_\partial^L \end{pmatrix} = \begin{pmatrix} 0 & D \\ D & 0 \\ 0 & \varphi(0)^T \\ 0 & \varphi(L)^T \\ \varphi(0)^T & 0 \\ \varphi(L)^T & 0 \end{pmatrix} \begin{pmatrix} e^q \\ \\ e^p \end{pmatrix} \tag{17}$$

where $e_\partial^0, e_\partial^L, f_\partial^0, f_\partial^L \in \mathbb{R}$, and $f^q, f^p \in \mathbb{R}^N$ while $e^q, e^p \in \mathbb{R}^{N+1}$. The vectors $\varphi(0) \in \mathbb{R}^{N+1}$ and $\varphi(L) \in \mathbb{R}^{N+1}$ are the vectors of polynomials $\varphi_i(z)$ evaluated respectively at the boundary points $z = 0$ and $z = L$.

3.2. Power product, Stokes Theorem and reduced Dirac structure

The spaces of the discretized effort and flow variables may be endowed with a power product, the bilinear product obtained by restricting the symmetric

pairing (2) to the approximation spaces. Replacing the flow and effort variables $e_i^q, e_i^p,\ f_i^q, f_i^p,\ i = 1, 2$ by their approximations in (2), the discretization of the symmetric pairing gives:

$$\begin{pmatrix} e_1^p \\ e_1^q \\ e_1^0 \\ e_1^L \\ f_1^p \\ f_1^q \\ f_1^0 \\ f_1^L \end{pmatrix}^T \begin{pmatrix} 0 & 0 & 0 & 0 & M & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & M & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ M^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & M^T & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_2^p \\ e_2^q \\ e_2^0 \\ e_2^L \\ f_2^p \\ f_2^q \\ f_2^0 \\ f_2^L \end{pmatrix} \tag{18}$$

where M is the $(N+1) \times N$ matrix whose elements are

$$M_{i,j} = \int_0^L \varphi_i(z)\psi_j(z)dz \tag{19}$$

It's important to notice that M^T has a non-zero kernel $Ker(M^T) \subset \mathscr{E}$ due to the fact that the dimensions of the approximation spaces are different for the effort and the flow variables. As a consequence, the symmetric pairing (18) is degenerate. This is an obstruction to the definition of a Dirac structure with respect to this pairing.

4. Discretized Dirac structure on a reduced bond space

In order to define a Dirac structure we shall, in the first step, define a reduced space of effort variables:

$$\begin{aligned} \tilde{e}^q &= M^T e^q \\ \tilde{e}^p &= M^T e^p \end{aligned} \tag{20}$$

with $\tilde{e}^q,\ \tilde{e}^p \in \mathbb{R}^N$. Considering the space of flow variables

$$\begin{pmatrix} f^q \\ f^p \\ f_\partial^0 \\ f_\partial^L \end{pmatrix} = \tilde{f} \in \tilde{\mathscr{F}} = \mathbb{R}^{2N+2} \tag{21}$$

and using the reduced effort variables to define the space *of effort variables*

$$\begin{pmatrix} \tilde{e}^q \\ \tilde{e}^p \\ e_\partial^0 \\ e_\partial^L \end{pmatrix} = \tilde{e} \in \tilde{\mathscr{E}} = \mathbb{R}^{2N+2} \tag{22}$$

, the symmetric pairing (18) becomes the non-degenerated canonical pairing:

$$\left\langle \left\langle \begin{pmatrix} \tilde{f}_1 \\ \tilde{e}_1 \end{pmatrix}, \begin{pmatrix} \tilde{f}_2 \\ \tilde{e}_2 \end{pmatrix} \right\rangle \right\rangle = \tilde{f}_1 \sigma \tilde{e}_2 + \tilde{f}_2 \sigma \tilde{e}_1 \tag{23}$$

with σ denoting the signature matrix: $\sigma = \text{diag } (1,\, 1,\, -1,\, 1)$.

Using this non-degenerated pairing (23), we may now, in the second step, define a Dirac structure on the reduced flow and effort variables. We shall define this Dirac structure using a matrix representation, called image representation [2, 4].

Definition 4.1. *Let $\mathscr{D} \subset \mathscr{F} \times \mathscr{E}$, with $\dim \mathscr{F} = \dim \mathscr{E} = n$, endowed with a non-degenerated pairing $\langle\, ,\, \rangle$. Any two $n \times n$ matrices, denoted here E and F, and satisfying*

1. $E\sigma F^T + F\sigma E^T = 0$
2. $rank[E : F] = n$

define a Dirac structure $\mathscr{D} = \{(f, e) \in \mathscr{F} \times \mathscr{E} | f = E^T\lambda, e = F^T\lambda,\ \lambda \in \mathbb{R}^n\}$ with respect to the bilinear symmetrization of the pairing $\langle\, ,\, \rangle$. This description is called an image representation *of the Dirac structure $\mathscr{D}$ and the matrices E and F are called* structure matrices *of the image representation [2, 4].*

We will in the sequel, find the two structure matrices E and F for the reduced Dirac structure $\mathscr{D}_r$ on the flow and reduced effort variables. Therefore using the definition of the degenerated pairing (18), one obtains:

$$e_1^{pT} M f_2^p + e_1^{qT} M f_2^q + f_1^{pT} M^T e_2^p + f_1^{qT} M^T e_2^q + e_{\partial 1}^0 f_{\partial 2}^0 - e_{\partial 1}^L f_{\partial 2}^L + f_{\partial 1}^0 e_{\partial 2}^0 - f_{\partial 1}^L e_{\partial 2}^L = 0 \tag{24}$$

Replacing the flux and effort variables in (24) by their expressions in the constitutive relations (17), one gets:

$$e_1^q (MD + D^T M^T + T^0 - T^L) e_2^p + e_1^p (MD + D^T M^T + T^0 - T^L) e_2^q = 0 \tag{25}$$

where T^0 is the $(N+1) \times (N+1)$ matrix with elements $T_{ij}^0 = \varphi_i(0)\varphi_j(0)$, and T^L the $(N+1) \times (N+1)$ matrix with elements $T_{ij}^L = \varphi_i(L)\varphi_j(L)$. Since (25) holds for any (e_1^q, e_2^p) and for any (e_1^p, e_2^q), we deduce

$$MD + D^T M^T + T^0 - T^L = 0 \tag{26}$$

This result which may be seen as the "finite-dimensional" Stokes theorem, will be very useful for the sequel.

Remark 4.2. *If $T^L - T^0 = 0$ in (26), which is the case for isolated system without any energy exchange across its boundaries, we get $MD + D^T M^T = 0$. The matrix MD is thus skew-symmetric and we can recognize a well-known a Poisson structure for conservative systems.*

According to (17) we can write

$$\begin{pmatrix} f^q \\ f^p \\ f^0_\partial \\ f^L_\partial \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & D \\ D & 0 \\ 0 & \varphi(0)^T \\ 0 & \varphi(L)^T \end{pmatrix}}_{E^T} \begin{pmatrix} e^q \\ e^p \end{pmatrix} \tag{27}$$

where f^q and f^p are in $\mathbb{R}^N$ while e^q and e^p are $\mathbb{R}^{N+1}$. These projected efforts allow to complete the image representation of the reduced Dirac structure $\mathscr{D}_r$. According to (20) and (17), one has:

$$\begin{pmatrix} \tilde{e}^q \\ \tilde{e}^p \\ e^0_\partial \\ e^L_\partial \end{pmatrix} = \underbrace{\begin{pmatrix} M^T & 0 \\ 0 & M^T \\ \varphi(0)^T & 0 \\ \varphi(L)^T & 0 \end{pmatrix}}_{F^T} \begin{pmatrix} e^q \\ e^p \end{pmatrix} \tag{28}$$

Proposition 4.3. *The sub-space*

$$\mathscr{D}_r = \left\{ \left(E^T\lambda, F^T\lambda \right) \middle| \lambda \in \mathbb{R}^{2N+2} \right\}$$

is a Dirac structure in $\mathscr{F} \times \mathscr{E} = \mathbb{R}^{2N+2} \times \mathbb{R}^{2n+2}$.

Proof 4.4. *(i)* $E\sigma F^T + F\sigma E^T = 0$

$$\begin{aligned} E\sigma F^T + F\sigma E^T = &\begin{pmatrix} 0 & D^T & 0 & 0 \\ D^T & 0 & \varphi(0) & \varphi(L) \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} M^T & 0 \\ 0 & M^T \\ \varphi(0)^T & 0 \\ \varphi(L)^T & 0 \end{pmatrix} \\ &+ \begin{pmatrix} M & 0 & \varphi(0) & \varphi(L) \\ 0 & M & 0 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & D \\ D & 0 \\ 0 & \varphi(0)^T \\ 0 & \varphi(L)^T \end{pmatrix} \end{aligned} \tag{29}$$

After elementary calculation (29) gives

$$\begin{pmatrix} 0 & D^T M^T + MD + T^0 - T^L \\ D^T M^T + MD + T^0 - T^L & 0 \end{pmatrix} \tag{30}$$

According to (26) we get $E\sigma F^T + F\sigma E^T = 0$
(ii) $[E : F]$ *is full rank* $(2n + 2)$
We should show that $(D_1^T, D_2^T, ..., D_n^T, \varphi(0))$ *is an independent set. where* $D_i^T = \varphi(z_i)$ *the* N *column vectors of* $(N + 1) \times N$ D^T *matrix, the* z_i, $i = 1, 2, ..., N$ *are the* N *interpolating points of* ψ *polynomial base.*
We prove it by contradiction. Let assume that the set under consideration is

dependent. This supposes one can deduce $\varphi(0)$, and thus a polynomial set of degree N, and that only with N z_i points. Actually,

$$\varphi(z) = \alpha_0\varphi(0) + \int_0^z \sum_{i=1}^{N} \alpha_i \varphi_i'(z)dz$$

But it is known that one cannot generate polynomial of degree N with only N interpolating points, which contradicts our assumption, so $(\varphi'(z_1), \varphi'(z_2), ..., \varphi(0))$ is an independent and maximal set in $\mathbb{R}^{n+1}$ therefore it is a base. Thus,

$$\begin{pmatrix} D^T & \varphi(0) & 0 & 0 \\ 0 & 0 & D^T & \varphi(0) \end{pmatrix} \tag{31}$$

is an independent and maximal set in $\mathbb{R}^{2n+2}$ and thus, concatenate matrix $[E : F]$ is full rank, $2n+2$.

However for simulation purposes, one prefer an explicit relation (a map) defining the Dirac structure, to the implicit definition of its image representation. And actually, by partial inversion, one compute the following relation between effort and flux variables which is defined by the skew-symmetric and full rank matrix J such that:

$$\begin{pmatrix} f^q \\ -f_\partial^L \\ f^p \\ e_\partial^0 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & \begin{pmatrix} D \\ -\varphi(L) \end{pmatrix} \begin{pmatrix} M^T \\ \varphi(0) \end{pmatrix}^{-1} \\ \begin{pmatrix} D \\ \varphi(0) \end{pmatrix} \begin{pmatrix} M^T \\ \varphi(L) \end{pmatrix}^{-1} & 0 \end{pmatrix}}_{J} \begin{pmatrix} \tilde{e}^q \\ e_\partial^L \\ \tilde{e}^p \\ f_\partial^0 \end{pmatrix} \tag{32}$$

This representation is called the *input output representation*, see [23]. We can check easily that the matrix J is skew-symmetric. Actually, calculating the anti-diagonal element's sum

$$\begin{pmatrix} D \\ -\varphi(L) \end{pmatrix} \begin{pmatrix} M^T \\ \varphi(0) \end{pmatrix}^{-1} + \left(\begin{pmatrix} D \\ \varphi(0) \end{pmatrix} \begin{pmatrix} M^T \\ \varphi(L) \end{pmatrix}^{-1} \right)^T = \left(M\varphi(L)^T \right)^{-1} \left(MD + D^T M^T + T^0 - T^L \right) \begin{pmatrix} M^T \\ \varphi(0) \end{pmatrix}^{-1} = 0 \tag{33}$$

according to (26).

Remark 4.5. *Actually the input output representation (32) is a particular of choice of inputs and outputs. It may be generalized by choosing as input and output variables any linear combinations of the boundary port variables defined in (27) and (28):*

$$\begin{pmatrix} u \\ y \end{pmatrix} = \begin{pmatrix} W \\ \tilde{W} \end{pmatrix} \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} \tag{34}$$

The definition of the matrices W and $\tilde{W}$ is not further dicussed here but may be related to the definition of boundary control systems associated with the port Hamiltonian systems defined on Stokes Dirac structures in [24].

5. Discretization of the constitutive relations

In previous sections we have presented a discretization approach preserving the Dirac structure. In this section we shall complete the discretization of the Hamiltonian evolution equations by discretizing the closure equations defining the Hamiltonian function or the dissipation relation. We shall apply a reduction scheme to the discretization of conservative and dissipative closure equations in such a way thattheir finite-dimensional approximation have the same properties with respect to energy storage or energy disspation. The basic idea is thus to approximate the stored energy or the dissipated power in the previously chosen approximation spaces. In this way one generates the spatial discretization of the boundary port Hamiltonian systems, yielding a finite-dimensional port Hamiltonian system [27, chap. 6]. We shall in this section illustrate the discretization of the closure relations on the two examples presented here above: the transmission line and the adsorption-diffusion models.

5.1. Discretization of the lossless transmission line

Consider again the boundary port Hamiltonian formulation of the transmission line which is defined with respect to the Stokes-Dirac structure and generated by the Hamiltonian functional (7). Hence the closure relation is given by the expression of the Hamiltonian (7) or its variational derivative. The variational derivative appears when considering the time derivative (the instanteneous power) in the transmission line:

$$\frac{dH}{dt} = \int_0^L \frac{\star q(z,t)}{C(z)} \dot{q}(z,t) + \frac{\star p(z,t)}{L(z)} \dot{p}(z,t) \tag{35}$$

which may be identified with [6]:

$$\frac{dH}{dt} = \int_Z \delta_q H \wedge \dot{q} + \delta_p H \wedge \dot{p} = \int_Z e^q \wedge f^q + e^p \wedge f^p \tag{36}$$

The Hamiltonian $H(p,q)$ depends on the conserved quantitities, the magnetic flux 1-form p and the electrical charge 1-form q. They are approximated using the ψ polynomial base as in section 3

$$\begin{aligned} \bar{q}(z,t) &= \sum_{i=0}^{N-1} q_i(t)\psi_i(z)dz \\ \bar{p}(z,t) &= \sum_{i=0}^{N-1} p_i(t)\psi_i(z)dz \end{aligned} \tag{37}$$

Inserting (37) into (35) the approximate power can be written:

$$\frac{dH}{dt} = \bar{q}^T(t) C \dot{\bar{q}}(t) + \bar{p}^T(t) L \dot{\bar{p}}(t) \tag{38}$$

where

$$C_{ij} = \int_0^L \frac{\psi_i(z)\psi_j(z)}{C(z)} dz$$

and

$$L_{ij} = \int_0^L \frac{\psi_i(z)\psi_j(z)}{L(z)} dz$$

In addition we know that the discrete Hamiltonian function takes the form:

$$\frac{d\bar{H}}{dt} = \sum_{i=0}^{N-1} \frac{\partial \bar{H}}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial \bar{H}}{\partial p_i}\frac{dp_i}{dt} \tag{39}$$

Identifying (39) with (38) we get the approximate efforts are expressed by:

$$\begin{aligned} e_i^q(t) &= \sum_{j=0}^{N} C_{ij} q_j(t) \\ e_i^p(t) &= \sum_{j=0}^{N} L_{ij} p_j(t) \end{aligned} \tag{40}$$

The discretized boundary port Hamiltonian is then defined as the finite-dimensional port Hamiltonian system generated by the Hamiltonian:

$$H(\bar{q}, \bar{p}) = \frac{1}{2}\bar{q}^T C \bar{q} + \frac{1}{2}\bar{p}^T L \bar{p}$$

with respect to the Dirac structure given in the input-output representation (32).

5.2. Discretization of the transmission line with dissipation

Here, we are interested in the discretization of the dissipative part. In order to do that, we calculate the dissipative power:

$$\int_0^L f^p(z) \wedge e^p(z) = \int_0^L e^p(z) \star r(z) e^p(z) \tag{41}$$

Since the effort variable $e^p(z)$ is approximated in polynomial base $\varphi_i(z)$ with $i = 0, ..., N$

$$\begin{aligned} \int_0^L f^p(z) \wedge e^p(z) &= \int_0^L \sum_{i=0}^{N}\sum_{j=0}^{N} e_i^p \varphi_i(z) \star r(z) e_j^p \varphi_j(z) = e^{pT} R e^p \\ &= \begin{pmatrix} \tilde{e}^p \\ f_\partial^0 \end{pmatrix}^T \begin{pmatrix} M^T \\ \varphi(0) \end{pmatrix}^{-T} R \begin{pmatrix} M^T \\ \varphi(0) \end{pmatrix}^{-1} \begin{pmatrix} \tilde{e}^p \\ f_\partial^0 \end{pmatrix} \end{aligned} \tag{42}$$

where R is a $(N+1) \times (N+1)$ positive definite matrix given:

$$R = \int_0^L \sum_{i=0}^{N}\sum_{j=0}^{N} \varphi_i(z) \star r(z) \varphi_j(z) \tag{43}$$

Thus, the finite port-Hamiltonian model of a dissipative line transmission is written according to (32)

$$\begin{pmatrix} f^q \\ -f^L_\partial \\ f^p \\ e^0_\partial \end{pmatrix} = \left(\underbrace{\begin{pmatrix} 0 & \begin{pmatrix} D \\ -\varphi(L) \end{pmatrix} \begin{pmatrix} M^T \\ \varphi(0) \end{pmatrix}^{-1} \\ \begin{pmatrix} D \\ \varphi(0) \end{pmatrix} \begin{pmatrix} M^T \\ \varphi(L) \end{pmatrix}^{-1} & 0 \end{pmatrix}}_{J} - \begin{pmatrix} 0 & 0 \\ 0 & \begin{pmatrix} M^T \\ \varphi(0) \end{pmatrix}^{-T} R \begin{pmatrix} M^T \\ \varphi(0) \end{pmatrix}^{-1} \end{pmatrix} \right) \begin{pmatrix} \tilde{e}^q \\ e^L_\partial \\ \tilde{e}^p \\ f^0_\partial \end{pmatrix} \tag{44}$$

One can write the power balance for dissipative line transmission since the matrix J est skew'symmetric

$$\begin{pmatrix} \tilde{e}^p \\ f^0_\partial \end{pmatrix}^T \begin{pmatrix} M^T \\ \varphi(0) \end{pmatrix}^{-T} R \begin{pmatrix} M^T \\ \varphi(0) \end{pmatrix}^{-1} \begin{pmatrix} \tilde{e}^p \\ f^0_\partial \end{pmatrix} = e^{pT} R e^p > 0 \tag{45}$$

6. Numerical examples: comparison with classical collocation

In previous sections, we have developed a new discretization approach based on the collocation method, and adapted it to open dynamical systems by including the boundary variables in the finite Dirac structure. This insures in particular the stability of the power balance property in the reduction of the infinite dimensional system, contrarily to many existing classical numerical methods. In this section, we consider the line transmission model with constant physical parameters (capacitance, inductance and resistance), in order to allow exact computation of the eigenvalues of the system, and use them as a reference to compare the accuracy of our geometric scheme with to existing classical methods.

6.1. Lossless Transmission Line

In this section, we will analyze the numerical properties of the proposed geometric reduction algorithm and compare them with those of classical collocation methods when applied to either closed systems (with fixed boundary conditions) or to boundary control systems. More precisely we will consider the spectral properties of an ideal transmission line and its reduced models. The eigenvalues of the model resulting from our "geometric" collocation method are obtained by diagonalizing the matrix $J \times Q$ from the input-output representation (32). When substituting the efforts from (40) and considering Dirichlet conditions

$q(0) = 0$ and $p(L) = 0$, one obtains

$$\begin{pmatrix} f^q \\ -f_\partial^L \\ f^p \\ e_\partial^0 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & \begin{pmatrix} D \\ -\varphi(L) \end{pmatrix} \begin{pmatrix} M^T \\ \varphi(0) \end{pmatrix}^{-1} \\ \begin{pmatrix} D \\ \varphi(0) \end{pmatrix} \begin{pmatrix} M^T \\ \varphi(L) \end{pmatrix}^{-1} & 0 \end{pmatrix} \begin{pmatrix} C & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{L} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{J \times Q} \begin{pmatrix} q \\ e_\partial^L \\ p \\ f_\partial^0 \end{pmatrix} \tag{46}$$

6.1.1. Classical collocation with fixed boundaries

The classical collocation considered first includes the choice of the (fixed) boundary conditions in the definition of the polynomial approximation bases. Therefore, it is not adapted to open dynamical systems. However, it allows to avoid boundary effects [14] and does not introduce any additional numerical dissipation. In that sense it could be considered as a reference scheme and comparison with our method, in the closed system case with fixed boundary conditions, is meaningful. Consider the following eigenvalue problem

$$\begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix} \begin{pmatrix} q(z) \\ p(z) \end{pmatrix} = \lambda \begin{pmatrix} q(z) \\ p(z) \end{pmatrix} \tag{47}$$

where

$$q(z) = \sum_{i=1}^{N} q_i \psi_i(z) z, \qquad p(z) = \sum_{i=1}^{N} p_i \psi_i(z)(z - L) \tag{48}$$

satisfying thus the Dirichlet constraints $q(0) = 0$, $p(L) = 0$. The approximating basis ψ used here is the same as the one shown in our geometric method, except that they are modulated to take into account the boundary constraints.

The table 1 below gives the exact eigenvalues of the ideal line transmission model and those obtained by performing the reduction by our geometric approach and by the classical collocation method with fixed boundaries with 8 interior collocation points in both cases. The first 8 pairs of eigenvalues are considered. The collocation points are the zeros of the 8th. Tchebychev polynomial in order to minimize the uniform norm of the interpolation error [20].

The spectrum of the geometric method is more accurate than the classical method's one. For instance, the first eigenvalue obtained with the geometric method is exact within the machine precision, and 10^5 order of magnitude more accurate than the one from the classical method. In both cases the eigenvalues pairs are close to the imaginary axe (real parts of the approximated eigenvalues are zeros within the machine precision). This is automatically the case for the geometric method, whatever the chosen collocation points are, since the Dirac structure is preserved and thus no numerical dissipation can occur. In the "classical" method, this is only true for zeros of orthogonal polynomials as collocation points. Otherwise, a numerical dissipation does occur.

Theroretical values	Geometric method	Classical method
± 0.32063745754047i	0.00000000000000 ± 0.32063745754047i	0.00000000000000 ± 0.32063745744583i
± 0.96191237262140i	0.00000000000000 ± 0.96191238151097i	0.00000000000000 ± 0.96191420177938i
± 1.60318728770233i	0.00000000000000 ± 1.60321563740732i	0.00000000000000 ± 1.60338416589471i
± 2.24446220278326i	0.00000000000000 ± 2.24771134219675i	0.00000000000000 ± 2.25154046253161i
± 2.88573711786419i	0.00000000000000 ± 2.94830702957369i	0.00000000000000 ± 3.00613251338085i
± 3.52701203294513i	0.00000000000000 ± 3.98374558826630i	0.00000000000000 ± 4.38041246765385i
± 4.16828694802606i	0.00000000000000 ± 6.38606332778657i	0.00000000000000 ± 8.49771153485658i
± 4.80956186310699i	0.00000000000000 ±18.76179259703309i	0.00000000000000 ± 31.96188912930034i

Table 1: The spectrum of an ideal transmission line model: theoretical values (1^{st} column), values computed using the proposed geometric collocation method (2^{nd} column) and obtained by the classical collocation scheme (3^{rd} column). The transmission line parameters values are an inductance $\mathcal{L} = 2$, a capacitance $C = 3$ and a resistance $R = 0$

6.1.2. classical collocation with free boundaries

The usual way of discretizing open dynamical systems using classical collocation scheme is to introduce additional relations between the reduced variables which are given by the boundary conditions. In our case, using the same approximating basis ψ as previously, these equations could be of the form $q(0,t) = \sum_{i=1}^{N} \psi_i(0) q_i = u_1(t)$ and $p(L,t) = \sum_{i=1}^{N} \psi_i(L) p_i = u_2(t)$ where arbitrary inputs values are given for $q(0,t)$ and $p(L,t)$. This would lead to the corresponding eigenvalue problem:

$$\begin{cases} \begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix} \begin{pmatrix} q(z) \\ p(z) \end{pmatrix} = \lambda \begin{pmatrix} q(z) \\ p(z) \end{pmatrix} & (49) \\ q(0) = \sum_{i=1}^{N} \psi_i(0) q_i = 0 & (50) \\ p(L) = \sum_{i=1}^{N} \psi_i(L) p_i = 0 & (51) \end{cases}$$

The table 2 below gives the spectrum for such an "open" ideal line transmission line. The same collocation points as previously are used. The eigenvalues obtained with the classical collocation method are now much less accurate than those obtained with the geometric method. In addition, a significative numerical dissipation appears for the 3 last eigenvalues. Finally an unstable mode in the reduced model obtained by the classical collocation method shows that the classical method leads to stability problems where the geometric scheme allows to choose the collocation points freely according to accuracy and conditioning considerations only.

Theoretical values	Geometric method	Classical collocation
± 0.32063745754047i	0.00000000000000 ± 0.32063745754047i	0.00000000000000 ± 0.28966097209118i
± 0.96191237262140i	0.00000000000000 ± 0.96191238151097i	0.00000000000000 ± 0.95732943443983i
± 1.60318728770233i	0.00000000000000 ± 1.60321563740732i	0.00000000000000 ± 1.62271610834325i
± 2.24446220278326i	0.00000000000000 ± 2.24771134219675i	0.00000000000000 ± 2.34315842499615i
± 2.88573711786419i	0.00000000000000 ± 2.94830702957369i	0.00000000000000 ± 2.96231787828937i
± 3.52701203294513i	0.00000000000000 ± 3.98374558826630i	1.08122738779532 ± 3.73134482798125i
± 4.16828694802606i	0.00000000000000 ± 6.38606332778657i	-1.08122738779532 ± 3.73134482798126i
± 4.80956186310699i	0.00000000000000 ±18.76179259703309i	-3.23331090942184 ± 0.00000000000000

Table 2: The spectrum of an ideal transmission line model with free boundaries: theoretical values (1st column), values computed using the proposed geometric collocation method (2nd column) and obtained by the classical collocation scheme (3rd column). The parameters values are the same as in table 1.

6.2. Transmission line with dissipation

Let now analyze the line transmission example with physical dissipation, and compare its spectrum obtained using the developed geometric method with both classical method with "fixed boundaries" and the one with "free boundaries", still considering Tchebychev collocation points.
The eigenvalues calculated with the geometric collocation method can be obtained considering (44), when replacing the effort variables $\tilde{e}^q$ and $\tilde{e}^p$ by their expression depending respectively on the stat variables q and p. Thus, the eigenvalue problem may come down to diagonalizing the following matrix:

$$\begin{pmatrix} f^q \\ -f^L_\partial \\ f^p \\ e^0_\partial \end{pmatrix} = \left(\underbrace{\begin{pmatrix} 0 & \begin{pmatrix} D \\ -\varphi(L) \end{pmatrix} \begin{pmatrix} M^T \\ \varphi(0) \end{pmatrix}^{-1} \\ \begin{pmatrix} D \\ \varphi(0) \end{pmatrix} \begin{pmatrix} M^T \\ \varphi(L) \end{pmatrix}^{-1} & 0 \end{pmatrix}}_{J} - \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & \begin{pmatrix} M^T \\ \varphi(0) \end{pmatrix}^{-T} R \begin{pmatrix} M^T \\ \varphi(0) \end{pmatrix}^{-1} \end{pmatrix}}_{\mathcal{R}} \right) \underbrace{\begin{pmatrix} C & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{L} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{Q} \begin{pmatrix} q \\ e^L_\partial \\ p \\ f^0_\partial \end{pmatrix} \tag{52}$$

where the boundary Dirichlet conditions are conditions are considered ($q(0) = 0$, $p(L) = 0$).

6.2.1. Classical collocation with fixed boundaries

The eigenvalue problem considered here is the one in (47) when the dissipation is taken into account. In this case, the problem can be written simply:

$$\begin{pmatrix} 0 & d \\ d & -r \end{pmatrix} \begin{pmatrix} q(z) \\ p(z) \end{pmatrix} = \lambda \begin{pmatrix} q(z) \\ p(z) \end{pmatrix} \tag{53}$$

where r is the dissipation of the line transmission. In this instance, it is a constant parameter as well as for the capacitance and the inductance.
The stat variables are still approximated on the same polynomial base as in the lossless case:

$$q(z) = \sum_{i=1}^{N} q_i \psi_i(z) z, \qquad p(z) = \sum_{i=1}^{N} p_i \psi_i(z)(z - L) \tag{54}$$

in such way that the Dirichlet condition are satisfied ($q(0) = 0$, $p(L) = 0$).
The table 3 below shows a comparison between the exact eigenvalues of the dissipative line transmission, with those obtained using our geometric method and the classical one considered here, either using Tchebychev collocation points.
As we can see, the classical method conserves the real part of the eigenvalues, when our geometric method introduces an additional numerical dissipation. But this dissipation goes toward stability. In fact, the conservation of the real part with the classical method is due to the fact that the matrix of dissipation resulting from the discretization is diagonal, when the one resulting from our method, noticed $\mathcal{R}$ is full.
However, one can notice that the accuracy of the imaginary part of the eigenvalues obtained with our method, is still better within the classical one. Of course, this accuracy decreases as well as for the real part, since we move away from the real axe. Obviously, this collocation method with fixed boundaries offers conserves better the real part of the spectrum, but it is so obvious that such method is not adapted to open dynamical systems.

Theoretical values	Geometric method	Classical collocation
-0.25 ± 0.20076946774351i	-0.25000000000000 ± 0.20076946774351i	-0.25000000000000 ± 0.20076946759238i
-0.25 ± 0.92885704637588i	-0.25000000223150 ± 0.92885705500815i	-0.25000000000000 ± 0.92885894062816i
-0.25 ± 1.58357490490673i	-0.25000393136670 ± 1.58360306757729i	-0.25000000000000 ± 1.58377422110662i
-0.25 ± 2.23049559060822i	-0.25027657287508 ± 2.23374307917011i	-0.25000000000000 ± 2.23761803139344i
-0.25 ± 2.87488760013659i	-0.25340619431893 ± 2.93754399174322i	-0.25000000000000 ± 2.99571906026006i
-0.25 ± 3.51814068515455i	-0.26792900064430 ± 3.97546301847245i	-0.25000000000000 ± 4.37327261747736i
-0.25 ± 4.16078310911352i	-0.32855775887639 ± 6.37948731547473i	-0.25000000000000 ± 8.494033278109671
-0.25 ± 4.80305999494626i	-1.09100301027535 ± 18.72003621870064i	-0.25000000000000 ± 31.96091138740705i

Table 3: Spectrum of a dissipative transmission line model: in (1st column) the theoretical values. In the 2nd column the corresponding eigenvalues calculated using geometric collocation method.In the 3rd column the eigenvalues obtained by classical collocation. The transmission line parameters values are an inductance $\mathcal{L} = 2$, a capacitance $C = 3$ and a resistance $r = 1$

6.2.2. Classical collocation with free boundaries

Look at now the classical collocation with free boundaries, much more adapted to open dynamical systems. The eigenvalue problem considered here is the one

as in the lossless case, when a physical dissipation is added. This leads to the following system:

$$\begin{cases} \begin{pmatrix} 0 & d \\ d & -r \end{pmatrix} \begin{pmatrix} q(z) \\ p(z) \end{pmatrix} = \lambda \begin{pmatrix} q(z) \\ p(z) \end{pmatrix} & (55) \\ q(0) = \sum_{i=1}^{N} \psi_i(0) q_i = 0 & (56) \\ p(L) = \sum_{i=1}^{N} \psi_i(L) p_i = 0 & (57) \end{cases}$$

where r is the dissipation. The table 4 below compares the eigenvalues obtained using our geometric method with those obtained with the classical collocation with free boundaries. One can notice that the real part as well as the imaginary part of the eigenvalues resulting from the geometric method are much more accurate than those obtained with the classical one. In addition, the two last couples of eigenvalues obtained with the classical method are instable while the original system is passive.

Theoretical values	Geometric method	Classical collocation
-0.25± 0.20076946774351i	-0.25000000000000 ± 0.20076946774351i	-0.24930216251772 ± 0.14747653502614i
-0.25± 0.92885704637588i	-0.25000000223150 ± 0.92885705500815i	-0.25129919304432 ± 0.92378036197850i
-0.25± 1.58357490490673i	-0.25000393136670 ± 1.58360306757729i	-0.24566489648452 ± 1.60387844859800i
-0.25± 2.23049559060822i	-0.25027657287508 ± 2.23374307917011i	-0.24951676276364 ± 2.32979926140971i
-0.25± 2.87488760013659i	-0.25340619431893 ± 2.93754399174322i	-0.29829313524034± 2.94739244163835i
-0.25± 3.51814068515455i	-0.26792900064430 ± 3.97546301847245i	-1.26039743395064 ± 3.72996412500374i
-0.25± 4.16078310911352i	-0.32855775887639 ± 6.37948731547473i	0.90547685940890 ± 3.72080660109945i
-0.25± 4.80305999494626i	-1.09100301027535 ± 18.72003621870064i	3.33926057044720 ± 0.000000000000000

Table 4: Spectrum of a dissipative transmission line model: in (1^{st} column) the theoretical values. In the 2^{nd} column the corresponding eigenvalues calculated using geometric collocation method.In the 3^{rd} column the eigenvalues obtained by classical collocation.

6.3. Numerical study of rate convergence

The table 5 compares the eigenvalues of the lossless transmission line model obtained using 8 interior collocation points and those callculated with 16 collocation points, within the geometric collocation method developed in the present paper. As a first comment, one can notice that the first pairs of eigenvalues converge much more fast than the last ones. In view of the results exposed within this table, on can notice that the accuracy of the spectrum obtained using the geometric method increase highly as the number of the collocation

points increases. Actually, when doubling the number of the collocation points from 8 to 16, we capture the firs three couples of eigenvalues within machine precision, whereas one can only capture the first couple when we use 8 collocation points considering the same precision. In addition, when we increase from 8 to 16 collocation points, the accuracy of 4th pair of eigenvalues for instance, increase with 10^8 order of magnitude, and the same holds true for the 5th pair. In the light of all these observations, one can infer that the convergence of this geometric method is at least exponential.

Theroretical values	8 collocation points	16 collocation points
± 0.32063745754047i	0.00000000000000 ± 0.32063745754047i	0.00000000000000 ± 0.32063745744583i
± 0.96191237262140i	0.00000000000000 ± 0.96191238151097i	0.00000000000000 ± 0.96191237262140i
± 1.60318728770233i	0.00000000000000 ± 1.60321563740732i	0.00000000000000 ± 1.60318728770233i
± 2.24446220278326i	0.00000000000000 ± 2.24771134219675i	0.00000000000000 ± 2.24446220278361ii
± 2.88573711786419i	0.00000000000000 ± 2.94830702957369i	0.00000000000000 ± 2.88573711861816i
± 3.52701203294513i	0.00000000000000 ± 3.98374558826630i	0.00000000000000 ± 3.52701228702717i
± 4.16828694802606i	0.00000000000000 ± 6.38606332778657i	0.00000000000000 ± 4.16831018776371i
± 4.80956186310699i	0.00000000000000 ±18.76179259703309i	0.00000000000000 ± 4.81033320045241i

Table 5: The spectrum of an ideal transmission line model: theoretical values (1^{st} column), values computed using the proposed geometric collocation method with 8 Tchebychev collocation points (2^{nd} column) and the first 8 eigenvalues obtained with the same geometric collocation method using 16 Tchebychev collocation points (3^{rd} column). The transmission line parameters values are an inductance $\mathcal{L} = 2$, a capacitance $C = 3$ and a resistance $R = 0$

7. CONCLUSIONS AND FUTURE WORKS

In this paper we discussed the geometric discretization of port-based distributed parameters models. We have chosen the transmission line model as paradigmatic example. We have chosen different approximation spaces according to the degrees of the approximated differential forms. Doing this both the exterior derivative and the boundary operator may be discretized exactly. We have shown that the Dirac structure is preserved from the infinite to the finite dimension using a discrete version of the Stokes theorem. This Dirac structure is completed with the discretization of the constitutive relations describing the energy of the system. The resulting reduced model has the structure of a finite-dimensional port hamiltonian system. As a consequence of the conservation of the geometric structure after discretization, we have seen that the spectral properties of the discrete system are better preserved and more accurate than with classical collocation methods, from the numerical precision point of view as well as the stability one. In view of the many advantages that this method does offer, we are currently extending these results to the two dimensional case,

still within the port-Hamiltonian formalism. We are also interested in investigating the controllability and observability of the system resulting from the discretization, since the scheme developed in this paper seems to preserve the stability of the infinite system.

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