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# DOA estimation in fluctuating environments: an approximate message-passing approach

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## Abstract

In this work, we propose a new algorithm in order to perform direction-of-arrival (DOA) estimation considering waves propagating through a fluctuating medium. In this particular context, the measurements are affected by a phase noise, which can mislead the estimation. It is then important to take into account these phase uncertainties in the estimation procedures. In this paper, we propose to model the phase noise by some random distribution and combine it with a sparse-enforcing distribution on the directions of arrival. These prior models are exploited through a message passing algorithm. Our method relates to previous works in phase recovery and can be seen in particular as an extension of the so-called “prSAMP” algorithm. Once implemented, we evaluated the performance of our algorithm in terms of quality of the reconstruction in comparison with other state-of-the-art algorithms. In particular, the method proves to be more robust to additive noise and present a better behavior than conventional beamforming and recent variational Bayesian algorithms.

**Keywords:** DOA estimation, sparse representation, Bayesian estimation, variational Bayesian approximations, message passing algorithms

## 1 Introduction

Common to many applications such as SONAR, RADAR, and telecommunications, the task of direction-of-arrival (DOA) estimation aims at locating one or more sources emitting in some propagation media. Many various methods dealing with this problem have been proposed in the literature. They can roughly be distinguished by the assumptions put on the propagating medium and the sources. Thus, the beamforming approach [1], which constitutes probably the most famous approach, exploits a simple assumption of Gaussian additive noise, while the so-called “high-resolution” techniques consider in addition some assumptions over the number or the nature of the sources. This is the case of the well-known MUSIC method [2], based on strong assumptions on the number of sources and the separability of the sub-spaces where noise and signal are assumed to live. More recently, we can also mention the “compressive” beamforming approaches proposed in [3] that benefit from an explicit sparse model on the sources.

In addition to these assumptions, recent work has focused on the integration of phase noise models. The objective of such approaches is to take into account the wave front distortion occurring when waves travel through fluctuating media. This case study is observed in many application areas as in acoustics [4], space observation or communications [5].

These contributions mainly relate to recent advances in phase recovery (see e.g. [6]) and extend them in the sense that they put informative prior on the missing phases. In this respect, we can mention the work [7] which proposes a method inspired by the PhaseCut algorithm and the Bayesian approach based on a mean-field approximation.

Part of the continuity of this latter work, we propose here a novel approach, based on the “swept approximate message passing” (SwAMP) algorithm introduced in [8]. The major contribution of this work is the consideration of a corrupting phase model over our measurement, which have a strong impact on the success of classical DOA techniques due to the poor integration of the propagation model in the case of fluctuating medium.

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## 2 PROBLEM STATEMENT

In this section, we recall the Bayesian modeling introduced in [9], which we shall follow in the remaining of the paper, and introduce the estimation problem we propose to solve.

### 2.1 Observation Model

We consider an antenna composed of  $N$  sensors. Then, the received signal at sensor  $n$  can be expressed as

$$y_n = e^{j\theta_n} \left( \sum_{m=1}^M d_{nm} x_m + \mu_{z_n} \right), \quad (1)$$

where  $\theta_n$  stands for the phase noise due to the propagation through the fluctuating medium and  $\mu_{z_n}$  is an additive noise. The scalar  $d_{nm}$  is the  $n$ -th element of the steering vector  $\mathbf{d}_m = [e^{j\frac{2\pi}{\lambda}\Delta \sin(\phi_m)} \dots e^{j\frac{2\pi}{\lambda}\Delta N \sin(\phi_m)}]^T$  where the  $\phi_m$ 's are some potential angles of arrival,  $\Delta$  is the distance between two adjacent sensors and  $\lambda$  is the wavelength of the propagation waves.

At each sensor of the antenna, we thus assume that the received field is a combination of a few waves arriving from different angles  $\phi_m$ . The DOA estimation problem consists then in identifying the positions of the non-zero coefficients in  $\mathbf{x} \triangleq [x_1 \dots x_M]^T$ . To model the sparse assumption on the number of the sources, we adopt a Bernoulli-Gaussian model

$$p(x_m) = \rho_m \mathcal{CN}(x_m; 0, \sigma_x^2) + (1 - \rho_m) \delta_0(x_m), \quad (2)$$

where  $\rho_m$  is the Bernoulli parameter and equals the probability for  $x_m$  to be non-zero,  $\mathcal{CN}(x_m; 0, \sigma_x^2)$  stands for the centered circular complex Gaussian distribution with variance  $\sigma_x^2$  and  $\delta_0(x_m)$  for a Dirac distribution. The Bernoulli-Gaussian model is now widely used when considering Bayesian inference methods for sparsity-constrained problems (see *e.g.* [10]).

To model spatial fluctuations all along the antenna, we assume in addition that the phase noise obeys the following Markov model

$$p(\theta_n | \theta_{n-1}) = \mathcal{N}(\beta \theta_{n-1}, \sigma_\theta^2), \quad \forall n \in \{2, \dots, N\}, \quad (3)$$

$$p(\theta_1) = \mathcal{N}(0, \sigma_1^2), \quad (4)$$

with  $\beta \in \mathbb{R}_+$ . Note that variance  $\sigma_\theta^2$  allows us to take into account the strength of the fluctuations.

Finally, the additive noise  $\mu_{z_n}$  is assumed to be circular complex Gaussian of variance  $\sigma^2$ .

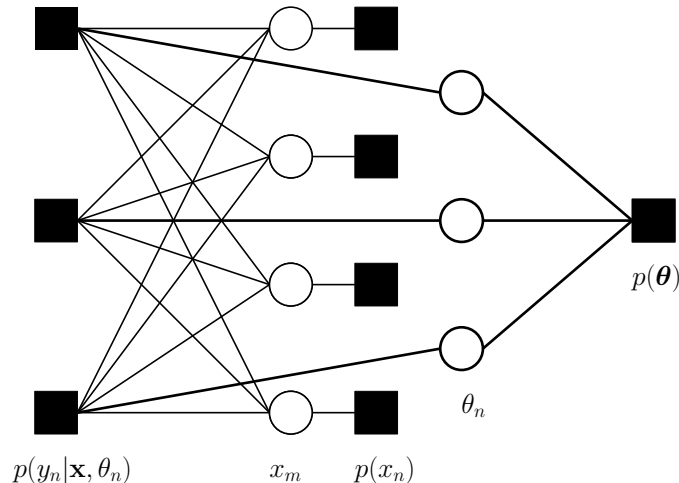


Fig. 1: Factor graph of the observation model

To illustrate model (1)-(4), Figure 1 gives the corresponding factor graph. White circles denote random variables when black squares are for the probability distribution factors.

### 2.2 Bayesian formulation of the problem

Within model (1)-(4), we focus on the following Minimum Mean Square Error (MMSE) problem:

$$\hat{\mathbf{x}} = \underset{\tilde{\mathbf{x}}}{\operatorname{argmin}} \int_x \|\mathbf{x} - \tilde{\mathbf{x}}\|_2^2 p(\mathbf{x}|\mathbf{y}) d\mathbf{x} \quad (5)$$

where  $\mathbf{y} \triangleq [y_1 \dots y_N]^T$  and

$$p(\mathbf{x}|\mathbf{y}) = \int_{\boldsymbol{\theta}} p(\mathbf{x}, \boldsymbol{\theta}|\mathbf{y}) d\boldsymbol{\theta} \quad (6)$$

with  $\boldsymbol{\theta} \triangleq [\theta_1 \dots \theta_N]^T$ .

To solve this problem, we propose to resort to a Bayesian inference strategy, that approximates the posterior distributions by distributions having suitable factorization. In [9], a mean-field approximation  $p(\mathbf{x}, \boldsymbol{\theta}|\mathbf{y}) \simeq q(\boldsymbol{\theta}) \prod_{m=1}^M q(x_m)$  was considered. In this paper, we are interested in a different type of factorization, relating to the ‘‘approximate message passing’’ (AMP) algorithms [11]. We motivate and detail our approach in the next section.

### 3 Proposed paSAMP approach

#### 3.1 Motivation and main principles of the approach

AMP algorithms have been considered since a few years as a serious solution to linear problems under sparsity-constraints. First considered in the sole case of random matrices<sup>1</sup>, they have been recently improved by the so-called ‘‘swept approximate message passing’’ (SwAMP) approach, which helps alleviate the convergence issues of AMP.

Both AMP and SwAMP have been extended to generalized but component-wise measurement models [12]. They have been then successfully applied to the phase recovery task where  $\theta_n \sim \mathcal{U}[0, 2\pi]$ ,  $\forall n \in \{1, \dots, N\}$ , given raise to the prGAMP [13] and the prSAMP [6] algorithms. In particular, the latter has proved to outperform other state-of-the-art algorithms among which the mean-field approximation [9].

The prSAMP algorithm constitutes thus a promising approach for our DOA estimation problem (5). However, in our study case, the phases  $\theta_n$ 's are considered as dependent on each other (as highlighted by figure 1). This prevents us from a direct adaptation of prSAMP.

We thus propose an iterative algorithm based on the two following mathematical derivations:

- i) extend prSAMP to a *i.i.d.* Gaussian prior on the phases,
- ii) use a mean-field approximation to estimate the (Gaussian) posterior distribution on the phases.

We detail both operations in the next two sub-sections.

#### 3.2 Extension of prSAMP to *i.i.d.* Gaussian phases

For a sake of clarity and due to space limitation, we will adopt and refer the reader to the notations of paper [8] which introduced the SwAMP algorithm described in **Algorithm 1**.

We would like to adapt this algorithm in order to fit our noise model and the prior on our signal  $\mathbf{x}$ .

The work of [8] evoke the possibility to obtain a generalized version of SwAMP by replacing  $\frac{y_n - \mu_{z_n}(t)}{\sigma^2 + \Sigma_{z_n}(t)}$  in  $g_n$  and the  $\frac{1}{\Sigma_{z_n}(t) + \sigma^2}$  term in the  $\mu_n^r(t)$  by  $g_{out}(\omega, V)$  and  $-g'_{out}(\omega, V)$ , where  $g_{out}(\omega, V)$  and  $g'_{out}(\omega, V)$  are respectively the moment of order 1 and 2 of the following pdf, assuming that the  $z_n$ 's follow Gaussian distributions of mean  $\mu_{z_n}$  and variance  $\Sigma_{z_n}$  as mixtures of Bernoulli-Gaussian distributions.

$$p(z_n|y_n, \mu_{z_n}, \Sigma_{z_n}) = \frac{p(y_n|z_n)\mathcal{CN}(z_n; \mu_{z_n}, \Sigma_{z_n})}{\int_{z'_n} p(y_n|z'_n)\mathcal{CN}(z'_n; \mu_{z_n}, \Sigma_{z_n})} = \frac{p(y_n|z_n)\mathcal{CN}(z_n; \mu_{z_n}, \Sigma_{z_n})}{Z_{nor}} \quad (7)$$

<sup>1</sup> Namely, the elements  $d_{nm}$  are assumed to be *i.i.d.* randomly distributed.

**Algorithm 1** prSAMP Algorithm

---

**Input:**  $\mathbf{y}$ ,  $\mathbf{D}$ ,  $\sigma^2$ ,  $\sigma_x^2, T_{max}$   
**Define:**  
 $g_{out,n} \triangleq \frac{y_n - \mu_{z_n}^{m+1}(t)}{\sigma^2 + \Sigma_{z_n}^{m+1}}$   
 $g'_{out,n} \triangleq \frac{-1}{\Sigma_{z_n} + \sigma^2}$   
 $g_{in,m} \triangleq E_{X|Y}\{x_m | \mu_{x_m}, \Sigma_{x_m}\}$   
 $g'_{in,m} \triangleq var_{X|Y}\{x_m | \mu_{x_m}, \Sigma_{x_m}\}$

- 1: **while**  $t < T_{max}$  **do**
- 2:     **for**  $n = 1 \dots N$  **do**
- 3:          $\hat{z}_n(t) = \sum_{m=1}^M d_{nm} a_m(t)$
- 4:          $\Sigma_{z_n}^1(t+1) = \sum_{m=1}^M |d_{nm}|^2 v_m(t)$
- 5:          $\mu_{z_n}^1(t+1) = \hat{z}_n(t) - \Sigma_{z_n}^1(t) g_{out,n}$
- 6:     **end for**
- 7:     **for**  $m = \text{permute}[1 \dots M]$  **do**
- 8:          $\Sigma_{x_m}(t+1) = (-\sum_{n=1}^N |d_{nm}|^2 g'_{out,n})^{-1}$
- 9:          $\mu_{x_m}(t+1) = a_m(t) + \Sigma_{x_m}(t+1) \sum_{n=1}^N d_{nm} g_{out,n}$
- 10:          $v_m(t+1) = \Sigma_{x_m}(t+1) g'_{in,m}$
- 11:          $a_m(t+1) = g_{in,m}$
- 12:         **for**  $n = 1 \dots N$  **do**
- 13:              $\Sigma_{z_n}^{m+1}(t+1) =$   
                   $\Sigma_{z_n}^m(t+1) + |d_{nm}|^2 (v_m(t+1) - v_m(t))$
- 14:              $\mu_{z_n}^{m+1}(t+1) =$   
                   $\mu_{z_n}^m(t+1) + d_{nm} (a_m(t+1) - a_m(t))$   
                   $- g_{out,n}(t) (\Sigma_{z_n}^{m+1}(t+1) - \Sigma_{z_n}^m(t+1))$
- 15:         **end for**
- 16:         update  $\sigma^2$  according to [8].
- 17:         update  $[\theta_{m_n}, \Sigma_{\theta_n}]$  according to (14-15).
- 18:     **end for**
- 19: **end while**
- 20: **Output:**  $\{\hat{x}_m = a_m(T_{max})\}_m$

---

First by computing  $Z_{nor}$ :

$$Z_{nor} = \int_{z'_n} p(y_n | z'_n) \mathcal{CN}(z'_n; \mu_{z_n}, \Sigma_{z_n}) \quad (8)$$

$$= \int_{z'_n, \theta_n} p(y_n | z'_n, \theta_n) p(\theta_n) \mathcal{CN}(z'_n; \mu_{z_n}, \Sigma_{z_n}) \quad (9)$$

$$= \int_{z'_n, \theta_n} p(\theta_n) \mathcal{CN}(z'_n; y e^{-j\theta_n}, \sigma^2) \mathcal{CN}(z'_n; \mu_{z_n}, \Sigma_{z_n}) \quad (10)$$

$$= \int_{z'_n, \theta_n} p(\theta_n) \mathcal{CN}(y_n e^{-j\theta_n}; \mu_{z_n}, \Sigma_{z_n} + \sigma^2) \mathcal{CN}(z'_n; \frac{y_n e^{-j\theta_n} \Sigma_{z_n} + \mu_{z_n} \sigma^2}{\sigma^2 + \Sigma_{z_n}}, \frac{1}{\frac{1}{\sigma^2} + \frac{1}{\Sigma_{z_n}}}) \quad (11)$$

$$= \int_{\theta_n} p(\theta) \mathcal{CN}(y_n e^{-j\theta_n}; \mu_{z_n}, \Sigma_{z_n} + \sigma^2) \int_{z'_n} \mathcal{CN}(z'_n; \frac{y_n e^{-j\theta_n} \Sigma_{z_n} + \mu_{z_n} \sigma^2}{\sigma^2 + \Sigma_{z_n}}, \frac{1}{\frac{1}{\sigma^2} + \frac{1}{\Sigma_{z_n}}}) \quad (12)$$

$$= \int_{\theta_n} p(\theta_n) \mathcal{CN}(y_n e^{-j\theta_n}; \mu_{z_n}, \Sigma_{z_n} + \sigma^2) \quad (13)$$

$$= \int_{\theta_n} \frac{1}{\sqrt{2\pi(\Sigma_{z_n} + \sigma^2)}} \exp\left[-\frac{|y_n e^{-j\theta_n} - \mu_{z_n}|^2}{2(\Sigma_{z_n} + \sigma^2)}\right] p(\theta_n) \quad (14)$$

$$= \exp\left[-\frac{|y_n|^2 + |\mu_{z_n}|^2}{2(\Sigma_{z_n} + \sigma^2)}\right] \frac{1}{\sqrt{2\pi(\Sigma_{z_n} + \sigma^2)}} \int_{\theta_n} \exp\left[\frac{|y_n| |\mu_{z_n}| \cos(\arg(y_n^* \mu_{z_n}) + \theta_n)}{\Sigma_{z_n} + \sigma^2}\right] p(\theta_n) \quad (15)$$

Using Von Mises approximations [14] we approximate the cosine part considering small  $a$ :

$$\frac{1}{\sqrt{2\pi a}} e^{-\frac{1}{2a}(x-x_m)^2} \approx \frac{1}{\pi I_0(\frac{1}{a})} e^{\frac{1}{a} \cos(x-x_m)} \quad (16)$$

$$Z_{nor} = \exp \left[ -\frac{|y_n|^2 + |\mu_{z_n}|^2}{2(\Sigma_{z_n} + \sigma^2)} \right] \frac{\pi}{\sqrt{2\pi(\Sigma_{z_n} + \sigma^2)}} I_0 \left( \frac{|y_n||\mu_{z_n}|}{V + \sigma^2} \right) \quad (17)$$

$$\int_{\theta_n} \mathcal{N} \left( \theta_n; -\arg(y_n^* \mu_{z_n}), \frac{V + \sigma^2}{|y_n||\mu_{z_n}|} \right) \mathcal{N}(\theta_n, \mu_{\theta_n}, \Sigma_{\theta_n}). \quad (18)$$

$$= \exp \left[ -\frac{|y_n|^2 + |\mu_{z_n}|^2}{2(\Sigma_{z_n} + \sigma^2)} \right] \frac{\pi}{\sqrt{2\pi(\Sigma_{z_n} + \sigma^2)}} I_0 \left( \frac{|y_n||\mu_{z_n}|}{V + \sigma^2} \right) \quad (19)$$

$$\int_{\theta_n} \mathcal{N} \left( \theta_n; \mu_{\theta_n}^z, \Sigma_{\theta_n}^z \right) \mathcal{N}(-\arg(y_n^* \mu_{z_n}); \theta_n, \alpha + \sigma_{\theta_n}) \quad (20)$$

with,

$$\frac{1}{\Sigma_{\theta}^z} = \frac{1}{\alpha} + \frac{1}{\Sigma_{\theta_n}}, \quad \mu_{\theta_n}^z = \frac{-\frac{\arg(y_n^* \mu_{z_n})}{\alpha} + \frac{\mu_{\theta_n}}{\Sigma_{\theta_n}}}{\frac{1}{\alpha} + \frac{1}{\Sigma_{\theta_n}}}, \quad \alpha = \frac{\Sigma_{z_n} + \sigma^2}{|y_n||\mu_{z_n}|},$$

Finally:

$$Z_{nor} = \exp \left[ -\frac{|y_n|^2 + |\mu_{z_n}|^2}{2(\Sigma_{z_n} + \sigma^2)} \right] \frac{\pi}{\sqrt{2\pi(\Sigma_{z_n} + \sigma^2)}} I_0 \left( \frac{|y_n||\mu_{z_n}|}{V + \sigma^2} \right) \quad (21)$$

$$\frac{1}{\sqrt{2\pi(a + \sigma_{\theta_n}^2)}} \exp \left[ -\frac{|\mu_{\theta_n} - \arg(y_n^* \mu_{z_n})|^2}{2(a + \sigma_{\theta_n}^2)} \right] \quad (22)$$

Now we can compute the momentum by integrating over the realizations of  $z_m$  and then over  $\theta_m$ :

$$E_{Z|Y} \{z_n | y_n, \mu_{z_n}, \Sigma_{z_n}\} = \frac{1}{Z_{nor}} \int_{\theta_n} \int_{z_n} z_n \mathcal{N}(z_n; y_n e^{-j\theta_n}, \sigma^2) \mathcal{N}(z_n, \mu_{z_n}, \Sigma_{z_n}) \quad (23)$$

$$= \frac{1}{Z_{nor}} \int_{\theta_n} \mathcal{N}(y_n e^{-j\theta_n}; \mu_{z_n}, \Sigma_{z_n} + \sigma^2) p(\theta_n) \quad (24)$$

$$\int_{z_n} z_n \mathcal{N} \left( z_n; \left[ \frac{y_n e^{-j\theta_n} \Sigma_{z_n} + \mu_{z_n} \sigma^2}{\sigma^2 + \Sigma_{z_n}} \right], \frac{1}{\frac{1}{\sigma^2} + \frac{1}{\Sigma_{z_n}}} \right) \quad (25)$$

$$= \frac{1}{Z_{nor}} \int_{\theta_n} \mathcal{N}(y_n e^{-j\theta_n}; \mu_{z_n}, \Sigma_{z_n} + \sigma^2) p(\theta_n) \left[ \frac{y_n e^{-j\theta_n} \Sigma_{z_n} + \mu_{z_n} \sigma^2}{\sigma^2 + \Sigma_{z_n}} \right] \quad (26)$$

$$= \frac{1}{Z_{nor}} \left[ \frac{y_n \Sigma_{z_n}}{\sigma^2 + \Sigma_{z_n}} \right] \int_{\theta_n} e^{-j\theta_n} \mathcal{N}(y_n e^{-j\theta_n}; \mu_{z_n}, \Sigma_{z_n} + \sigma^2) p(\theta_n) + \left[ \frac{\mu_{z_n} \sigma^2}{\sigma^2 + \Sigma_{z_n}} \right] \quad (27)$$

$$= \frac{1}{Z_{nor}} \left[ \frac{y_n \Sigma_{z_n}}{\sigma^2 + \Sigma_{z_n}} \right] \exp \left[ -\frac{|y_n|^2 + |\mu_{z_n}|^2}{2(\Sigma_{z_n} + \sigma^2)} \right] \frac{1}{\sqrt{2\pi(\Sigma_{z_n} + \sigma^2)}} \quad (28)$$

$$\int_{\theta_n} \exp(-j\theta) \exp \left[ \frac{|y_n||\mu_{z_n}| \cos(\arg(y_n^* \mu_{z_n}) + \theta_n)}{\Sigma_{z_n} + \sigma^2} \right] p(\theta_n) + \left[ \frac{\mu_{z_n} \sigma^2}{\sigma^2 + \Sigma_{z_n}} \right] \quad (29)$$

$$(30)$$

Again by identifying with a Von Mises Distribution :

$$E_{Z|Y} \{z_n | y_n, \mu_{z_n}, \Sigma_{z_n}\} = \frac{1}{Z_{nor}} \left[ \frac{y_n \Sigma_{z_n}}{\sigma^2 + \Sigma_{z_n}} \right] \exp \left[ -\frac{|y_n|^2 + |\mu_{z_n}|^2}{2(\Sigma_{z_n} + \sigma^2)} \right] \frac{\pi}{\sqrt{2\pi(\Sigma_{z_n} + \sigma^2)}} I_0 \left( \frac{|y_n||\mu_{z_n}|}{V + \sigma^2} \right) \quad (31)$$

$$\int_{\theta_n} e^{-j\theta_n} \mathcal{N} \left( \theta_n; \mu_{\theta_n}^z, \Sigma_{\theta_n}^z \right) \mathcal{N}(-\arg(y_n^* \mu_{z_n}); \theta_n, a + \sigma_{\theta_n}) + \left[ \frac{\mu_{z_n} \sigma^2}{\sigma^2 + \Sigma_{z_n}} \right] \quad (32)$$

by variable change  $\theta_n + \mu_{\theta_n}^z \leftarrow \theta_n$

$$E_{Z|Y} \{z_n | y_n, \mu_{z_n}, \Sigma_{z_n}\} = \frac{1}{Z_{nor}} \left[ \frac{y_n \Sigma_{z_n}}{\sigma^2 + \Sigma_{z_n}} \right] \exp \left[ -\frac{|y_n|^2 + |\mu_{z_n}|^2}{2(\Sigma_{z_n} + \sigma^2)} \right] \frac{\pi}{\sqrt{2\pi(\Sigma_{z_n} + \sigma^2)}} I_0 \left( \frac{|y_n||\mu_{z_n}|}{V + \sigma^2} \right) \quad (33)$$

$$\frac{1}{\sqrt{2\pi(a + \sigma_{\theta_n}^2)}} \exp \left[ -\frac{|\mu_{\theta_n} - \arg(y_n^* \mu_{z_n})|^2}{2(a + \sigma_{\theta_n}^2)} \right] \int_{\theta_n} \exp(-j(\theta_n + \mu_{\theta_n}^z)) \quad (34)$$

$$\frac{1}{\sqrt{2\pi\Sigma_{\theta_n}^z}} \exp \left( -\frac{\theta_n^2}{2\Sigma_{\theta_n}^z} \right) + \left[ \frac{\mu_{z_n} \sigma^2}{\sigma^2 + \Sigma_{z_n}} \right] \quad (35)$$

thanks to Von Mises identification :

$$E_{Z|Y}\{z_n|y_n, \mu_{z_n}, \Sigma_{z_n}\} = \frac{1}{Z_{nor}} \left[ \frac{y_n \Sigma_{z_n}}{\sigma^2 + \Sigma_{z_n}} \right] \exp \left[ -\frac{|y_n|^2 + |\mu_{z_n}|^2}{2(\Sigma_{z_n} + \sigma^2)} \right] \frac{\pi}{\sqrt{2\pi(\Sigma_{z_n} + \sigma^2)}} I_0 \left( \frac{|y_n| |\mu_{z_n}|}{V + \sigma^2} \right) \quad (36)$$

$$\frac{\exp(-j\mu_{\theta_n}^z)}{\sqrt{2\pi(a + \sigma_{\theta_n}^2)}} \exp \left[ -\frac{|\mu_{\theta_n} - \arg(y_n^* \mu_{z_n})|^2}{2(a + \sigma_{\theta_n}^2)} \right] \mathbf{R}_0 \left( \frac{1}{\Sigma_{\theta}^z} \right) + \left[ \frac{\mu_{z_n} \sigma^2}{\sigma^2 + \Sigma_{z_n}} \right] \quad (37)$$

After simplification with  $Z_{nor}$  we obtain:

$$E_{Z|Y}\{z_n|y_n, \mu_{z_n}, \Sigma_{z_n}\} = \frac{y_n \Sigma_{z_n} e^{-j\mu_{\theta_n}^z}}{\sigma_n^2 + V} \mathbf{R}_0 \left( \frac{1}{\Sigma_{\theta}^z} \right) + \frac{\mu_{z_n} \sigma^2}{\sigma^2 + \Sigma_{z_n}}, \quad (38)$$

By similar method we obtain :

$$var_{Z|Y}\{z_n|y_n, \mu_{z_n}, \Sigma_{z_n}\} = \frac{|y_n \Sigma_{z_n} e^{-j\mu_{\theta_n}^z} + \mu_{z_n} \sigma^2|^2}{|\sigma^2 + \Sigma_{z_n}|^2} \mathbf{R}_0 \left( \frac{1}{\Sigma_{\theta}^z} \right) + \frac{\Sigma_{z_n} \sigma^2}{\sigma^2 + \Sigma_{z_n}} - E_{Z|Y}\{z_n|y_n, \mu_{z_n}, \Sigma_{z_n}\}^2, \quad (39)$$

$\mu_{\theta_n}$  (resp.  $\Sigma_{\theta_n}$ ) is the marginalized posterior mean (resp. variance) of the phase noise  $\theta_n$  as discussed in the next section, and  $\mathbf{R}_0(\cdot) = \frac{I_1(\cdot)}{I_0(\cdot)}$  where  $I_n(\cdot)$  is the modified Bessel function of the first kind at order  $n$ .

Another distribution we have to compute is  $p_{x|R}(x|\mu_{x_m}, \Sigma_{x_m})$ , is the *a-posteriori* distribution of  $\mathbf{x}$  regarding propagation of the Gaussian fields propagated by the model, the calculation of the momentum defined as  $g_{in}$  and  $g'_{in}$  will follow the works presented in [10] to redefine the generic function proposed in SwAMP.

$$E_{X|Y}(x_m|\mu_{x_m}, \Sigma_{x_m}) = \frac{\rho \sqrt{2\pi\nu^2}}{Z_{nor}} e^{-\frac{|m_x - \mu_{x_m}|^2}{2(\sigma^2 + \Sigma_{x_m})}} \tilde{M} \quad (40)$$

$$var_{X|Y}(x_m|\mu_{x_m}, \Sigma_{x_m}) = \frac{\rho \sqrt{2\pi\nu^2}}{Z_{nor}} e^{-\frac{|m_x - \mu_{x_m}|^2}{2(\sigma^2 + \Sigma_{x_m})}} |\tilde{M}^2 + \nu^2| - E_{X|Y}(x_m|\mu_{x_m}, \Sigma_{x_m})^2 \quad (41)$$

with

$$Z_{nor} = \rho \sqrt{2\pi\nu^2} e^{-\frac{|m_x - \mu_{x_m}|^2}{2(\sigma^2 + \Sigma_{x_m})}} + (1 - \rho) e^{-\frac{|\mu_{x_m}|^2}{2\Sigma_{x_m}}}, \quad (42)$$

$$\tilde{M} = \frac{\sigma^2 \mu_{x_m} + \Sigma_{x_m} m_x}{\Sigma_{x_m} + \sigma^2}, \quad \nu^2 = \frac{\sigma^2 \Sigma_{x_m}}{\Sigma_{x_m} + \sigma^2}. \quad (43)$$

### 3.3 Mean-field approximation for the phase noise

To simplify the computation of the posterior distribution of the phase noise  $\boldsymbol{\theta}$ , we propose in this step to resort to a mean-field approximation. Thus, we define

$$q(\boldsymbol{\theta}) \propto p(\boldsymbol{\theta}) \exp \left( \int_{\mathbf{x}} p(\mathbf{x}|R; \mu^r) \log p(\mathbf{y}|\mathbf{x}, \boldsymbol{\theta}) d\mathbf{x} \right). \quad (44)$$

Particularized to the previous derivations, we get, in a similar way as in [9]

$$q(\boldsymbol{\theta}) = \mathcal{N}(\boldsymbol{\theta}; \boldsymbol{\mu}_{\theta}, \Sigma_{\theta}), \quad (45)$$

$$\text{where } \Sigma_{\theta}^{-1} = \Lambda_{\theta}^{-1} + \text{diag} \left( \frac{2}{\sigma^2} |\boldsymbol{\eta}| \right), \quad (46)$$

$$\boldsymbol{\mu}_{\theta} = \Sigma_{\theta} \left( \text{diag} \left( \frac{2}{\sigma^2} |\boldsymbol{\eta}| \right) \arg(\boldsymbol{\eta}) \right), \quad (47)$$

$\boldsymbol{\eta} \triangleq [\eta_1 \dots \eta_N]^T$  with  $\eta_n = y_n E_{Z|Y}^*\{z_n|y_n, \mu_{z_n}, \Sigma_{z_n}\}$ , and  $\Lambda_{\theta}^{-1}$  is the precision matrix attached to the prior distribution (4) on  $\boldsymbol{\theta}$ , *i.e.*

$$\Lambda_{\theta}^{-1} = \begin{pmatrix} \frac{1}{\sigma_1^2} + \frac{\beta^2}{\sigma_{\theta}^2} & -\frac{\beta}{\sigma_{\theta}^2} & 0 & 0 \\ -\frac{\beta}{\sigma_{\theta}^2} & \frac{1+\beta^2}{\sigma_{\theta}^2} & \ddots & 0 \\ 0 & \ddots & \ddots & -\frac{\beta}{\sigma_{\theta}^2} \\ 0 & 0 & -\frac{\beta}{\sigma_{\theta}^2} & \frac{1}{\sigma_{\theta}^2} \end{pmatrix}. \quad (48)$$

Note that the distribution  $q(\boldsymbol{\theta})$  being Gaussian, the marginals  $q(\theta_n)$  used in the prSAMP-step of the algorithm come straightforwardly as

$$q(\theta_n) = \mathcal{N}(\theta_n; \mu_{\theta_n}, \Sigma_{\theta_n}) \quad (49)$$

where  $\mu_{\theta_n}$  (resp.  $\Sigma_{\theta_n}$ ) is the  $n$ th element in  $\boldsymbol{\mu}_{\theta}$  (resp. in the diagonal of  $\Sigma_{\theta_n}$ ).

**Algorithm 2** paSAMP Algorithm

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**Input:**  $\mathbf{y}$ ,  $\mathbf{D}$ ,  $\sigma^2$ ,  $\rho$ ,  $\sigma_x^2$ ,  $\boldsymbol{\mu}_\theta$ ,  $\boldsymbol{\Sigma}_\theta$ ,  $T_{max}$   
**Define:**  
 $g_{out,n} \triangleq \frac{1}{\Sigma_{z_n}} (E_{Z|Y,P}\{z_n|y_n, \mu_{z_n}, \Sigma_{z_n}\} - \mu_{z_n})$   
 $g'_{out,n} \triangleq \frac{1}{\Sigma_{z_n}} (\frac{var_{Z|Y,P}\{z_n|y_n, \mu_{z_n}, \Sigma_{z_n}\}}{\Sigma_{z_n}} - 1)$   
 $g_{in,m} \triangleq E_{X|Y}\{x_m|\mu_{x_m}, \Sigma_{x_m}\}$   
 $g'_{in,m} \triangleq var_{X|Y}\{x_m|\mu_{x_m}, \Sigma_{x_m}\}$

- 1: **while**  $t < T_{max}$  **do**
- 2:   **for**  $n = 1 \dots N$  **do**
- 3:      $\hat{z}_n(t) = \sum_{m=1}^M d_{nm} a_m(t)$
- 4:      $\Sigma_{z_n}^1(t+1) = \sum_{m=1}^M |d_{nm}|^2 v_m(t)$
- 5:      $\mu_{z_n}^1(t+1) = \hat{z}_n(t) - \Sigma_{z_n}^1(t) g_{out,n}$
- 6:   **end for**
- 7:   **for**  $m = \text{permute}[1 \dots M]$  **do**
- 8:      $\Sigma_{x_m}(t+1) = (-\sum_{n=1}^N |d_{nm}|^2 g'_{out,n})^{-1}$
- 9:      $\mu_{x_m}(t+1) = a_m(t) + \Sigma_{x_m}(t+1) \sum_{n=1}^N d_{nm} g_{out,n}$
- 10:      $v_m(t+1) = \Sigma_{x_m}(t+1) g'_{in,m}$
- 11:      $a_m(t+1) = g_{in,m}$
- 12:     **for**  $n = 1 \dots N$  **do**
- 13:        $\Sigma_{z_n}^{m+1}(t+1) =$   
          $\Sigma_{z_n}^m(t+1) + |d_{nm}|^2 (v_m(t+1) - v_m(t))$
- 14:        $\mu_{z_n}^{m+1}(t+1) =$   
          $\mu_{z_n}^m(t+1) + d_{nm} (a_m(t+1) - a_m(t))$   
          $- g_{out,n}(t) (\Sigma_{z_n}^{m+1}(t+1) - \Sigma_{z_n}^m(t+1))$
- 15:     **end for**
- 16:     update  $\sigma^2$  according to [8].
- 17:     update  $[\theta_{m_n}, \Sigma_{\theta_n}]$  according to (14-15).
- 18:   **end for**
- 19: **end while**
- 20: **Output:**  $\{\hat{x}_m = a_m(T_{max})\}_m$

---

### 3.4 Additive noise estimation

In order to refine our estimation we propose to estimate  $\sigma^2$ , the second order momentum of the additive noise, according to the maximum likelihood criterion of the posterior distribution, we finally have to find :

$$\hat{\sigma}^2 = \underset{\sigma^2}{\operatorname{argmax}} \int_{\mathbf{z}, \boldsymbol{\theta}} p(\mathbf{z}, \boldsymbol{\theta} | \mathbf{y}) \log(p(\mathbf{y}, \mathbf{z}, \boldsymbol{\theta}; \sigma^2)) d\mathbf{z} d\boldsymbol{\theta} \quad (50)$$

According to [9]

$$\hat{\sigma}^2 = \frac{1}{N} \left( \mathbf{y}^H \mathbf{y} - 2\mathcal{R}(\bar{\mathbf{y}}^H \mathbf{E}_{Z|Y}) + \mathbf{E}_{Z|Y}^H \mathbf{E}_{Z|Y} + \mathbf{var}_{Z|Y}^H \mathbf{var}_{Z|Y} \right) \quad (51)$$

with

$$\bar{\mathbf{y}} = \left[ y_n e^{-j\mu_{\theta_n}} \mathbf{R}_0 \left( \frac{1}{\Sigma_\theta} \right) \right]_{n=\{1, \dots, N\}}, \quad (52)$$

$$\mathbf{E}_{Z|Y} = [E_{Z|Y}^*\{z_1|y_1, \omega_1, V^2\} \dots E_{Z|Y}^*\{z_N|y_N, \mu_{z_n}, \Sigma_{z_n}\}]^T.$$

In the following, we refer to the proposed two-procedure as ‘‘paSAMP’’ for ‘‘phase-aware SwAMP algorithm’’. The pseudo-code of paSAMP is presented in **Algorithm 2** with the references to the intermediary variables  $(\mu_{z_n}, \Sigma_{z_n}, \mu_{x_n}, \Sigma_{x_n})$  introduced above.

Once implemented, paSAMP will return  $\mathbf{a}$  and  $\mathbf{v}$ , respectively the mean and variance of  $p(\mathbf{x}|\mathbf{y})$ , with  $\mathbf{a}$  the reconstructed vector  $\mathbf{x}$ .

## 4 Numerical Experiment

In this section, we confront our method to extensive synthetic experiments and evaluate its performance in comparison to other state-of-the-art algorithms.

We consider the problem of identifying the directions of arrival of  $K$  plane waves from a antenna composed of  $N = 256$  sensors. We assume that the angles of the  $K$  incident waves can be written as  $\phi_k = \frac{\pi}{2} + i_k \frac{\pi}{50}$  with



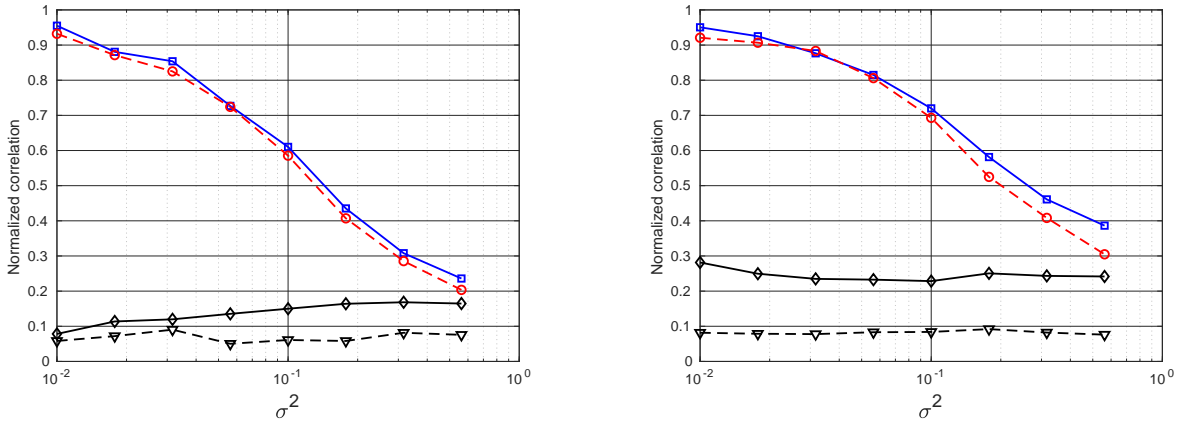


Fig. 2: Evolution of the (averaged) normalized correlation as a function of the variance  $\sigma^2$  for  $K = 2$  (left) and  $K = 5$  (right), Comparison of the performance of Conventional Beam Forming (triangles black), prSAMP (diamonds black), paVBEM (circles red) and paSAMP (squares blue). Performance shows that paSAMP provides better results and successfully integrates the phase noisy observation model.

$i_k \in [1, 50]$ . A set of  $M = 50$  steering vectors is defined from a set of angles  $\{\phi_i = -\pi + i\frac{\pi}{50}\}_{i \in \{1, \dots, 50\}}$  and the choice of the parameter  $\Delta/\lambda = 4$ . For each of the  $K$  incident waves, the coefficient  $x_{i_k}$  is initialized with  $m_x = 0.5 + j0.5$  and  $\sigma_x^2 = 0.1$ . Finally, we set the following parameters for the phase Markov model :  $\sigma_0^2 = 10$ ,  $\sigma_\theta^2 = 0.1$  and  $\beta = 0.8$ . This corresponds to a situation where we have a high uncertainty of the initial value but a physical link between two space-consecutive angle measurements is taken into account.

We compare the performance of the 4 different algorithms: *i*) the standard beamforming introduced in [1] (dashed black curve, triangle mark); *ii*) the so-called “prSAMP” algorithm proposed in [6] as a solution to the phase retrieval problem (continuous black curve, diamond mark); *iii*) the “paVBEM” procedure proposed in [9] exploiting the same prior models (dashed red curve, circle mark); *iv*) the “paSAMP” algorithm described in Section 3 (continuous blue curve, square mark). To quantify the performance of these procedures, we consider the normalized correlation between the ground truth  $\mathbf{x}$  and its reconstruction  $\hat{\mathbf{x}}$ , that is  $\frac{|\mathbf{x}^H \hat{\mathbf{x}}|}{\|\mathbf{x}\| \|\hat{\mathbf{x}}\|}$ , as a function of the additive noise variance  $\sigma^2$ . This quantity is averaged over 100 realizations for each point of simulation.

The results achieved by the 4 procedures are presented in Figure 2, resp. for  $K = 2$  (left) and  $K = 5$  (right) sources. In both cases, we see that the conventional beamforming and the “prSAMP” algorithm fail to reconstruct  $\mathbf{x}$  properly. This observation illustrates the benefits of carefully accounting for the phase noise in the type of fluctuating configurations. Moreover, we can also notice the superiority of “paSAMP” over its mean-field homologue “paVBEM”, especially in presence of a strong additive noise. This comes in the continuity of previous work [6], where “prSAMP” proved to outperform “prVBEM”. Finally, it is interesting to compare the performance of both “paSAMP” and “paVBEM” algorithms with regard to the number of sources. Both achieve better performance when confronting to  $K = 5$  sources than to  $K = 2$  sources. As mentioned in [9], this behavior is typical for the phase retrieval problems, where the loss information on the phases can be compensated by a larger number of sources.

## 5 CONCLUSION

We have presented in this paper a novel Belief Propagation able to perform DOA in a corrupted phase noisy environment. This approach relied on Bethe approximations and strong assumption over the sparsity on  $\mathbf{x}$  and on the structure of the covariance matrix of the multiplicative noise. We confronted it to conventional Beamforming and paVBEM which is known to have good performances with this kind of model. Regarding the good performances of this algorithms, we hope to include further assessment like the estimation of the physical parameter, multi-frequential estimation and taking temporal evolution of the fluctuations into account.

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