

# Convexity in a measure

Auguste HÉBERT

Univ Lyon, UJM-Saint-Etienne CNRS

UMR 5208 CNRS, F-42023, SAINT-ETIENNE, France

auguste.hebert@univ-st-etienne.fr

## Abstract

Masures are generalizations of Bruhat-Tits buildings. They were introduced to study Kac-Moody groups over ultrametric fields, which generalize reductive groups over the same fields. If  $A$  and  $A'$  are two apartments in a building, their intersection is convex (as a subset of the finite dimensional affine space  $A$ ) and there exists an isomorphism from  $A$  to  $A'$  fixing this intersection. We study this question for measures and prove that the analogous statement is true in some particular cases. We deduce a new axiomatic of measures, simpler than the one given by Rousseau.

## 1 Introduction

An important tool to study a split reductive group  $G$  over a non-archimedean local field is its Bruhat-Tits building defined by Bruhat and Tits in [BT72] and [BT84]. Kac-Moody groups are interesting infinite dimensional (if not reductive) generalizations of reductive groups. In order to study them over fields endowed with a discrete valuation, Gaussent and Rousseau introduced measures (also known as hovels) in [GR08], which are analogs of Bruhat-Tits buildings. Charignon and Rousseau generalized this construction in [Cha10], [Rou17] and [Rou16]: Charignon treated the almost split case and Rousseau suppressed restrictions on the base field and on the group. Rousseau also defined an axiomatic of measures in [Rou11]. Recently, Freyn, Hartnick, Horn and Köhl made an analog construction in the archimedean case (see [FHHK17]): to each split real Kac-Moody group, they associate a space on which the group acts, generalizing the notion of riemannian symmetric space.

Masures enable to obtain results on the arithmetic of (almost)-split Kac-Moody groups over non-archimedean local fields. Let us survey them briefly. Let  $G$  be such a group and  $\mathcal{I}$  be its measure. In [GR08], Gaussent and Rousseau use  $\mathcal{I}$  to prove a link between the Littlemann's path model and some kind of Mirković-Vilonen cycle model of  $G$ . In [GR14], Gaussent and Rousseau associate a spherical Hecke algebra  ${}^s\mathcal{H}$  to  $G$  and they obtain a Satake isomorphism in this setting. These results generalize works of Braverman and Kazhdan obtained when  $G$  is supposed affine, see [BK11]. In [BPGR16], Bardy-Panse, Gaussent and Rousseau define the Iwahori-Hecke algebra  ${}^I\mathcal{H}$  of  $G$ . Braverman Kazhdan and Patnaik had already done this construction when  $G$  is affine in [BKP16]. In [Héb17], we obtain finiteness results on  $G$  enabling to give a meaning to one side of the Gindikin-Karpelevich formula obtained by Braverman, Garland, Kazhdan and Patnaik in the affine case in [BGKP14]. In [AH17], together with Abdellatif, we define a completion of  ${}^I\mathcal{H}$  and generalize the construction of the Iwahori-Hecke algebra of  $G$ : we associate Hecke algebras to subgroups of  $G$  more general than the Iwahori subgroup, the analogue of the parahoric subgroups. In [BPGR17], Bardy-Panse,

Gaussent and Rousseau prove a Macdonald's formula for  $G$ : they give an explicit formula for the image of some basis of  ${}^s\mathcal{H}$  by the Satake isomorphism. Their formula generalizes a well-known formula of Macdonald for reductive groups (see [Mac71]) which had already been extended to affine Kac-Moody groups in [BKP16].

Despite these results some very basic questions are still open in the theory of mesures. In this paper we are interested in questions of enclosure maps and of convexity in mesures. Let us be more precise. The mesure is an object similar to the Bruhat-Tits building. This is a union of subsets called apartments. An apartment is a finite dimensional affine space equipped with a set of hyperplanes called walls. The group  $G$  acts by permuting these apartments, which are therefore all isomorphic to one of them called the standard apartment  $\mathbb{A}$ .

To define the mesure  $\mathcal{I}$  associated to  $G$ , Gaussent and Rousseau (following Bruhat and Tits) first define  $\mathbb{A}$ . Let us describe it briefly. Suppose that the field of definition is local. Let  $Q^\vee$  be the coroot lattice of  $G$  and  $\Phi$  be its set of real roots. One can consider  $Q^\vee$  as a lattice of some affine space  $\mathbb{A}$  and  $\Phi$  as a set of linear forms on  $\mathbb{A}$ . Let  $Y$  be a lattice of  $\mathbb{A}$  containing  $Q^\vee$  (one can consider  $Y = Q^\vee$  in a first approximation). Then the set  $\mathcal{M}$  of walls of  $\mathbb{A}$  is the set of hyperplanes containing an element of  $Y$  and whose direction is  $\ker(\alpha)$  for some  $\alpha \in \Phi$ . Suppose that  $G$  is reductive. Then  $\Phi$  is finite and  $\mathcal{I}$  is a building. Let  $P \subset \mathbb{A}$ . Then one defines the enclosure  $\text{cl}(P)$  of  $P$  to be the intersection of the half-spaces delimited by a wall of  $\mathcal{M}$  and containing  $P$ . A well known property of buildings is that if  $A$  is an apartment of  $\mathcal{I}$ , then  $A \cap \mathbb{A} = \text{cl}(A \cap \mathbb{A})$  and there exists an isomorphism from  $A$  to  $\mathbb{A}$  fixing  $A \cap \mathbb{A}$  (see 2.5.7 and Proposition 2.5.8 of [BT72]). We study this property in the case of mesures.

The axiomatic definition given by Rousseau in [Rou11] involves an enclosure map and weak forms of the above property. When  $G$  is no more reductive  $\mathcal{M}$  can be dense in  $\mathbb{A}$ . Consequently, Gaussent and Rousseau define the enclosure of a subset to be a filter and no more necessarily a set (which is already the case for buildings when the valuation of the base field is not discrete). Moreover, there are several natural choices of enclosure maps. In [GR08], the enclosure map involves imaginary roots and infinite intersections of half-spaces delimited by roots. In [Rou17], Rousseau considers many different enclosure maps, some involving only real roots, other involving imaginary roots, some authorizing only finite intersections, other authorizing arbitrary intersections. We prove that all these choices of enclosure maps lead to the same definition of mesure; therefore the "good" enclosure map is the bigger one, which is the one involving only real roots and finite intersections.

Similarly to buildings, we can still define a fundamental chamber  $C_f^v$  in the standard apartment  $\mathbb{A}$ . This enables to define the Tits cone  $\mathcal{T} = \bigcup_{w \in W^v} w \cdot \overline{C_f^v}$ , where  $W^v$  is the vectorial Weyl group of  $G$ . An important difference between buildings and mesures is that when  $G$  is reductive,  $\mathcal{T} = \mathbb{A}$  and when  $G$  is not reductive,  $\mathcal{T} \neq \mathbb{A}$  is only a convex cone. This defines a preorder on  $\mathbb{A}$  by saying that  $x, y \in \mathbb{A}$  satisfy  $x \leq y$  if  $y \in x + \mathcal{T}$ . This preorder extends to a preorder on  $\mathcal{I}$  - the Tits preorder - by using isomorphisms of apartments. Convexity properties in  $\mathcal{I}$  were known only on preordered pairs of points. If  $A, A'$  are apartments and contain two points  $x, y$  such that  $x \leq y$  then  $A \cap A'$  contains the segment in  $A$  between  $x$  and  $y$  and there exists an isomorphism from  $A$  to  $A'$  fixing this segment (this is Proposition 5.4 of [Rou11]).

A ray (half-line) of  $\mathcal{I}$  is said to be generic if its direction meets the interior  $\overset{\circ}{\mathcal{T}}$  of  $\mathcal{T}$ . The main result of this paper is the following theorem:

**Theorem 1.** Let  $A, B$  be apartments such that  $A \cap B$  contains a generic ray of  $A$ . Then  $A \cap B$  is a finite intersection of half-apartments of  $A$  and there exists an isomorphism from  $A$  to  $B$  fixing  $A \cap B$ .

Using this theorem, we define a new axiomatic of measures and prove that it is equivalent to the one given by Rousseau (we recall it in 2.2.2). Our axiomatic is simpler and closer to the usual geometric axiomatic of euclidean buildings. To emphasize this analogy, we first recall one of their definitions in the case where the valuation is discrete (see Section IV of [Bro89] or Section 6 of [Rou04]).

**Definition 1.1.** A euclidean building is a set  $\mathcal{I}$  equipped with a set  $\mathcal{A}$  of subsets called apartments satisfying the following axioms :

- (I0) Each apartment is a euclidean apartment (see Section 6 of [Rou04]).
- (I1) For any two faces  $F$  and  $F'$  there exists an apartment containing  $F$  and  $F'$ .
- (I2) If  $A$  and  $A'$  are apartments, their intersection is a union of faces and, for any faces  $F, F'$  in  $A \cap A'$  there exists an isomorphism from  $A$  to  $A'$  fixing  $F$  and  $F'$ .

Actually, one can replace (I2) by :

- (I2') If  $A$  and  $A'$  are apartments,  $A \cap A'$  is a finite intersection of half-apartments and there exists an isomorphism  $\phi : A \rightarrow A'$  fixing  $A \cap A'$ .

In the statement of the next theorem, we use the notion of chimney. They are some kind of thickened sector faces. A splayed chimney is a chimney containing a generic ray. We prove the following theorem:

**Theorem 2.** Let  $\mathbb{A}$  be the apartment associated to the root system of  $G$ . Let  $(\mathcal{I}, \mathcal{A})$  be a couple such that  $\mathcal{I}$  is a set and  $\mathcal{A}$  is a set of subsets of  $\mathcal{I}$  called apartments. Then  $(\mathcal{I}, \mathcal{A})$  is a measure of type  $\mathbb{A}$  in the sense of [Rou11] if and only if it satisfies the following axioms:

- (MA i) Each apartment is an apartment of type  $\mathbb{A}$ .
- (MA ii) If two apartments  $A$  and  $A'$  are such that  $A \cap A'$  contains a generic ray, then  $A \cap A'$  is a finite intersection of half-apartments and there exists an isomorphism  $\phi : A \rightarrow A'$  fixing  $A \cap A'$ .
- (MA iii) If  $\mathfrak{R}$  is the germ of a splayed chimney and  $F$  is a face or a germ of a chimney, then there exists an apartment containing  $\mathfrak{R}$  and  $F$ .

The axiom (MA iii) (very close to the axiom (MA3) of Rousseau) corresponds to the existence parts of Iwasawa, Bruhat and Birkhoff, decompositions in  $G$ , respectively for  $F$  a face and  $\mathfrak{R}$  a sector-germ,  $F$  and  $\mathfrak{R}$  two sector-germs of the same sign and  $F$  and  $\mathfrak{R}$  two opposite sector-germs. The axiom (MA ii), which implies the axiom (MA4) of Rousseau, corresponds to the unicity part of these decompositions.

The fact that if  $x, y \in \mathcal{I}$  are such that  $x \leq y$ , the segment between  $x$  and  $y$  does not depend on the apartment containing  $\{x, y\}$  was an axiom of measures (axiom (MAO)). A step of our proof of Theorem 2 is to show that (MAO) is actually a consequence of the other axioms of measures (see Proposition 5.3).

When  $G$  is of affine type, we give a simpler axiomatic:

**Theorem 3.** Suppose moreover  $G$  is affine. Let  $\mathbb{A}$  be the apartment associated to the root system of  $G$ . Let  $(\mathcal{I}, \mathcal{A})$  be a couple such that  $\mathcal{I}$  is a set and  $\mathcal{A}$  is a set of subsets of  $\mathcal{I}$  called apartments. Then  $(\mathcal{I}, \mathcal{A})$  is a measure of type  $\mathbb{A}$  in the sense of [Rou11] if and only if it satisfies the following axioms:

- (MA af i) Each apartment is an apartment of type  $\mathbb{A}$ .
- (MA af ii) If  $A$  and  $A'$  are two apartments, then  $A \cap A'$  is a finite intersection of half-apartments and there exists an isomorphism  $\phi : A \rightarrow A'$  fixing  $A \cap A'$ .
- (MA af iii) If  $\mathfrak{R}$  is the germ of a splayed chimney and  $F$  is a face or a germ of a chimney, then there exists an apartment containing  $\mathfrak{R}$  and  $F$ .

In a second (shorter) part, we study the Tits preorder in a measure associated to an affine Kac-Moody group  $G$ . Let  $\delta_{\mathbb{A}} : \mathbb{A} \rightarrow \mathbb{R}$  be the smallest imaginary root of  $G$ . We extend  $\delta_{\mathbb{A}}$  to a map  $\delta : \mathcal{I} \rightarrow \mathbb{R}$  which is affine on each apartment. We then prove that if  $x, y \in \mathcal{I}$  such that  $\delta(x) < \delta(y)$ , then  $x \leq y$ . This answers the question of the last paragraph of Section 5 of [Rou11].

Actually we do not limit our study to measures associated to Kac-Moody groups: for us a measure is a set satisfying the axioms of [Rou11] and whose apartments are associated to a root generating system (and thus to a Kac-Moody matrix). We do not assume that there exists a group acting strongly transitively on it. We do not either make any discreteness hypothesis for the standard apartment: if  $M$  is a wall, the set of walls parallel to it is not necessarily discrete; this enables to handle measures associated to split Kac-Moody groups over any ultrametric field.

The paper is organized as follows.

In Section 2, we describe the general framework and recall the definition of measures.

In Section 3 we study the intersection of two apartments  $A$  and  $B$ , without assuming that  $A \cap B$  contains a generic ray. We prove that  $A \cap B$  can be written as a union of enclosed subsets and that  $A \cap B$  is enclosed when it is convex. If  $P \subset A \cap B$ , we give a sufficient condition of existence of an isomorphism from  $A$  to  $B$  fixing  $P$ .

In Section 4, we study the intersection of two apartments sharing a generic ray and prove Theorem 1, which is stated as Theorem 4.22. The reader only interested in measures associated to affine Kac-Moody groups can skip this Section and replace Theorem 4.22 by Lemma 5.36, which is far more easy to prove.

In Section 5, we deduce new axiomatics of measures: we show Theorem 2 and Theorem 3, which correspond to Theorem 5.1 and Theorem 5.34.

In Section 6, we study the Tits preorder on a measure associated to an affine Kac-Moody group.

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## 2 General framework, Measure

In this section, we define our framework and recall the definition of measures. Then we recall some notions on measures. References for this section are [Rou11], Section 1 and 2 and Section 1 of [GR14].

## 2.1 Standard apartment

### 2.1.1 Root generating system

Let  $A$  be a **Kac-Moody matrix** (also known as generalized Cartan matrix) i.e a square matrix  $A = (a_{i,j})_{i,j \in I}$  with integers coefficients, indexed by a finite set  $I$  and satisfying:

1.  $\forall i \in I, a_{i,i} = 2$
2.  $\forall (i,j) \in I^2 | i \neq j, a_{i,j} \leq 0$
3.  $\forall (i,j) \in I^2, a_{i,j} = 0 \Leftrightarrow a_{j,i} = 0$ .

A **root generating system** of type  $A$  is a 5-tuple  $\mathcal{S} = (A, X, Y, (\alpha_i)_{i \in I}, (\alpha_i^\vee)_{i \in I})$  made of a Kac-Moody matrix  $A$  indexed by  $I$ , of two dual free  $\mathbb{Z}$ -modules  $X$  (of **characters**) and  $Y$  (of **cocharacters**) of finite rank  $\text{rk}(X)$ , a family  $(\alpha_i)_{i \in I}$  (of **simple roots**) in  $X$  and a family  $(\alpha_i^\vee)_{i \in I}$  (of **simple coroots**) in  $Y$ . They have to satisfy the following compatibility condition:  $a_{i,j} = \alpha_j(\alpha_i^\vee)$  for all  $i, j \in I$ . We also suppose that the family  $(\alpha_i)_{i \in I}$  is free in  $X$  and that the family  $(\alpha_i^\vee)_{i \in I}$  is free in  $Y$ .

Let  $\mathbb{A} = Y \otimes \mathbb{R}$ . Every element of  $X$  induces a linear form on  $\mathbb{A}$ . We will consider  $X$  as a subset of the dual  $\mathbb{A}^*$  of  $\mathbb{A}$ : the  $\alpha_i$ 's,  $i \in I$  are viewed as linear forms on  $\mathbb{A}$ . For  $i \in I$ , we define an involution  $r_i$  of  $\mathbb{A}$  by  $r_i(v) = v - \alpha_i(v)\alpha_i^\vee$  for all  $v \in \mathbb{A}$ . Its space of fixed points is  $\ker \alpha_i$ . The subgroup of  $\text{GL}(\mathbb{A})$  generated by the  $\alpha_i$  for  $i \in I$  is denoted by  $W^v$  and is called the Weyl group of  $\mathcal{S}$ . The system  $(W^v, \{r_i | i \in I\})$  is a Coxeter system. For  $w \in W^v$ , we denote by  $l(w)$  the length of  $w$  with respect to  $\{r_i | i \in I\}$ .

One defines an action of the group  $W^v$  on  $\mathbb{A}^*$  by the following way: if  $x \in \mathbb{A}$ ,  $w \in W^v$  and  $\alpha \in \mathbb{A}^*$  then  $(w.\alpha)(x) = \alpha(w^{-1}.x)$ . Let  $\Phi = \{w.\alpha_i | (w, i) \in W^v \times I\}$ ,  $\Phi$  is the set of **real roots**. Then  $\Phi \subset Q$ , where  $Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ . Let  $Q^+ = \bigoplus_{i \in I} \mathbb{N}\alpha_i$ ,  $\Phi^+ = Q^+ \cap \Phi$  and  $\Phi^- = (-Q^+) \cap \Phi$ . Then  $\Phi = \Phi^+ \sqcup \Phi^-$ . Let  $\Delta$  be the set of all roots as defined in [Kac94] and  $\Delta_{im} = \Delta \setminus \Phi$ . Then  $(\mathbb{A}, W^v, (\alpha_i)_{i \in I}, (\alpha_i^\vee)_{i \in I}, \Delta_{im})$  is a vectorial datum as in Section 1 of [Rou11].

### 2.1.2 Vectorial faces and Tits cone

Define  $C_f^v = \{v \in \mathbb{A} | \alpha_i(v) > 0, \forall i \in I\}$ . We call it the **fundamental chamber**. For  $J \subset I$ , one sets  $F^v(J) = \{v \in \mathbb{A} | \alpha_i(v) = 0 \forall i \in J, \alpha_i(v) > 0 \forall i \in I \setminus J\}$ . Then the closure  $\overline{C_f^v}$  of  $C_f^v$  is the union of the  $F^v(J)$  for  $J \subset I$ . The **positive** (resp. **negative**) **vectorial faces** are the sets  $w.F^v(J)$  (resp.  $-w.F^v(J)$ ) for  $w \in W^v$  and  $J \subset I$ . A **vectorial face** is either a positive vectorial face or a negative vectorial face. We call **positive chamber** (resp. **negative**) every cone of the shape  $w.C_f^v$  for some  $w \in W^v$  (resp.  $-w.C_f^v$ ). For all  $x \in C_f^v$  and for all  $w \in W^v$ ,  $w.x = x$  implies that  $w = 1$ . In particular the action of  $w$  on the positive chambers is simply transitive. The **Tits cone**  $\mathcal{T}$  is defined by  $\mathcal{T} = \bigcup_{w \in W^v} w.\overline{C_f^v}$ . We also consider the negative cone  $-\mathcal{T}$ . We define a  $W^v$  invariant preorder  $\leq$  (resp.  $\overset{\circ}{\leq}$ ) on  $\mathbb{A}$ , the **Tits preorder** (the **Tits open preorder**) by:  $\forall (x, y) \in \mathbb{A}^2, x \leq y \Leftrightarrow y - x \in \mathcal{T}$  (resp.  $x \overset{\circ}{\leq} y \Leftrightarrow y - x \in \overset{\circ}{\mathcal{T}} \cup \{0\}$ ).

### 2.1.3 Weyl group of $\mathbb{A}$

We now define the Weyl group  $W$  of  $\mathbb{A}$ . If  $X$  is an affine subspace of  $\mathbb{A}$ , one denotes by  $\vec{X}$  its direction. One equips  $\mathbb{A}$  with a family  $\mathcal{M}$  of affine hyperplanes called **real walls** such that:

1. For all  $M \in \mathcal{M}$ , there exists  $\alpha_M \in \Phi$  such that  $\vec{M} = \ker(\alpha_M)$ .

2. For all  $\alpha \in \Phi$ , there exists an infinite number of hyperplanes  $M \in \mathcal{M}$  such that  $\alpha = \alpha_M$ .
3. If  $M \in \mathcal{M}$ , we denote by  $r_M$  the reflexion of hyperplane  $M$  whose associated linear map is  $r_{\alpha_M}$ . We assume that the group  $W$  generated by the  $r_M$  for  $M \in \mathcal{M}$  stabilizes  $\mathcal{M}$ .

The group  $W$  is the Weyl group of  $\mathbb{A}$ . A point  $x$  is said to be **special** if every real wall is parallel to a real wall containing  $x$ . We suppose that 0 is special and thus  $W \supset W^v$ .

If  $\alpha \in \mathbb{A}^*$  and  $k \in \mathbb{R}$ , one sets  $M(\alpha, k) = \{v \in \mathbb{A} \mid \alpha(v) + k = 0\}$ . Then for all  $M \in \mathcal{M}$ , there exists  $\alpha \in \Phi$  and  $k_M \in \mathbb{R}$  such that  $M = M(\alpha, k_M)$ . If  $\alpha \in \Phi$ , one sets  $\Lambda_\alpha = \{k_M \mid M \in \mathcal{M} \text{ and } \vec{M} = \ker(\alpha)\}$ . Then  $\Lambda_{w.\alpha} = \Lambda_\alpha$  for all  $w \in W^v$  and  $\alpha \in \Phi$ .

If  $\alpha \in \Phi$ , one denotes by  $\tilde{\Lambda}_\alpha$  the subgroup of  $\mathbb{R}$  generated by  $\Lambda_\alpha$ . By 3,  $\Lambda_\alpha = \Lambda_\alpha + 2\tilde{\Lambda}_\alpha$  for all  $\alpha \in \Phi$ . In particular,  $\Lambda_\alpha = -\Lambda_\alpha$  and when  $\Lambda_\alpha$  is discrete,  $\tilde{\Lambda}_\alpha = \Lambda_\alpha$  is isomorphic to  $\mathbb{Z}$ .

One sets  $Q^\vee = \bigoplus_{\alpha \in \Phi} \tilde{\Lambda}_\alpha \alpha^\vee$ . This is a subgroup of  $\mathbb{A}$  stable under the action of  $W^v$ . Then one has  $W = W^v \ltimes Q^\vee$ .

For a first reading, the reader can consider the situation where the walls are the  $\phi^{-1}(\{k\})$  for  $\phi \in \Phi$  and  $k \in \mathbb{Z}$ . We then have  $\Lambda_\alpha = \mathbb{Z}$  for all  $\alpha \in \Phi$ , and  $Q^\vee = \bigoplus_{i \in I} \mathbb{Z} \alpha_i^\vee$ .

#### 2.1.4 Filters

**Definition 2.1.** A filter in a set  $E$  is a nonempty set  $F$  of nonempty subsets of  $E$  such that, for all subsets  $S, S'$  of  $E$ , if  $S, S' \in F$  then  $S \cap S' \in F$  and, if  $S' \subset S$ , with  $S' \in F$  then  $S \in F$ .

If  $F$  is a filter in a set  $E$ , and  $E'$  is a subset of  $E$ , one says that  $F$  contains  $E'$  if every element of  $F$  contains  $E'$ . If  $E'$  is nonempty, the set  $F_{E'}$  of subsets of  $E$  containing  $E'$  is a filter. By abuse of language, we will sometimes say that  $E'$  is a filter by identifying  $F_{E'}$  and  $E'$ . If  $F$  is a filter in  $E$ , its closure  $\bar{F}$  (resp. its convex envelope) is the filter of subsets of  $E$  containing the closure (resp. the convex envelope) of some element of  $F$ . A filter  $F$  is said to be contained in another filter  $F'$ :  $F \subset F'$  (resp. in a subset  $Z$  in  $E$ :  $F \subset Z$ ) if and only if any set in  $F'$  (resp. if  $Z$ ) is in  $F$ .

If  $x \in \mathbb{A}$  and  $\Omega$  is a subset of  $\mathbb{A}$  containing  $x$  in its closure, then the **germ** of  $\Omega$  in  $x$  is the filter  $germ_x(\Omega)$  of subsets of  $\mathbb{A}$  containing a neighborhood of  $x$  in  $\Omega$ .

A **sector** in  $\mathbb{A}$  is a set of the shape  $\mathfrak{s} = x + C^v$  with  $C^v = \pm w.C_f^v$  for some  $x \in \mathbb{A}$  and  $w \in W^v$ . A point  $u$  such that  $\mathfrak{s} = u + C^v$  is called a **base point of  $\mathfrak{s}$**  and  $C^v$  is its **direction**. The intersection of two sectors of the same direction is a sector of the same direction.

The **sector-germ** of a sector  $\mathfrak{s} = x + C^v$  is the filter  $\mathfrak{S}$  of subsets of  $\mathbb{A}$  containing an  $\mathbb{A}$ -translate of  $\mathfrak{s}$ . It only depends on the direction  $C^v$ . We denote by  $+\infty$  (resp.  $-\infty$ ) the sector-germ of  $C_f^v$  (resp. of  $-C_f^v$ ).

A ray  $\delta$  with base point  $x$  and containing  $y \neq x$  (or the interval  $]x, y] = [x, y] \setminus \{x\}$  or  $[x, y]$  or the line containing  $x$  and  $y$ ) is called **preordered** if  $x \leq y$  or  $y \leq x$  and **generic** if  $y - x \in \pm \overset{\circ}{\mathcal{T}}$ , the interior of  $\pm \mathcal{T}$ .

#### 2.1.5 Enclosure maps

Let  $\Delta = \Phi \cup \Delta_{im}^+ \cup \Delta_{im}^-$  be the set of all roots. For  $\alpha \in \Delta$ , and  $k \in \mathbb{R} \cup \{+\infty\}$ , let  $D(\alpha, k) = \{v \in \mathbb{A} \mid \alpha(v) + k \geq 0\}$  (and  $D(\alpha, +\infty) = \mathbb{A}$  for all  $\alpha \in \Delta$ ) and  $D^\circ(\alpha, k) = \{v \in \mathbb{A} \mid \alpha(v) + k > 0\}$  (for  $\alpha \in \Delta$  and  $k \in \mathbb{R} \cup \{+\infty\}$ ). If  $\alpha \in \Delta_{im}$ , one sets  $\Lambda_\alpha = \mathbb{R}$ . Let  $[\Phi, \Delta]$  be the set of sets  $\mathcal{P}$  satisfying  $\Phi \subset \mathcal{P} \subset \Delta$ .

If  $X$  is a set, one denotes by  $\mathcal{P}(X)$  the set of subsets of  $X$ . Let  $\mathcal{L}$  be the set of families  $(\Lambda'_\alpha) \in \mathcal{P}(\mathbb{R})^\Delta$  such that for all  $\alpha \in \Delta$ ,  $\Lambda_\alpha \subset \Lambda'_\alpha$  and  $\Lambda'_\alpha = -\Lambda'_{-\alpha}$ .

Let  $\mathcal{F}(\mathbb{A})$  be the set of filters of  $\mathbb{A}$ . If  $\mathcal{P} \in [\Phi, \Delta]$  and  $\Lambda' \in \mathcal{L}$ , one defines the map  $\text{cl}_{\Lambda'}^{\mathcal{P}} : \mathcal{P}(\mathbb{A}) \rightarrow \mathcal{F}(\mathbb{A})$  as follows. If  $U \subset \mathbb{A}$ ,

$$\text{cl}_{\Lambda'}^{\mathcal{P}}(U) = \{V \in \mathcal{P}(\mathbb{A}) \mid \exists (k_{\alpha}) \in \prod_{\alpha \in \mathcal{P}} (\Lambda'_{\alpha} \cup \{+\infty\}) \mid V \supset \bigcap_{\alpha \in \mathcal{P}} D(\alpha, k_{\alpha}) \supset U\}.$$

If  $\Lambda' \in \mathcal{L}$ , let  $\text{cl}_{\Lambda'}^{\#} : \mathcal{P}(\mathbb{A}) \rightarrow \mathcal{F}(\mathbb{A})$  defined as follows. If  $U \subset \mathbb{A}$ ,

$$\text{cl}_{\Lambda'}^{\#}(U) = \{V \in \mathcal{P}(\mathbb{A}) \mid \exists n \in \mathbb{N}, (\beta_i) \in \Phi^n, (k_i) \in \prod_{i=1}^n \Lambda'_{\beta_i} \mid V \supset \bigcap_{i=1}^n D(\beta_i, k_i) \supset U\}.$$

Let  $\mathcal{CL}^{\infty} = \{\text{cl}_{\Lambda'}^{\mathcal{P}} \mid \mathcal{P} \in [\Phi, \Delta] \text{ and } \Lambda' \in \mathcal{L}\}$ . An element of  $\mathcal{CL}^{\infty}$  is called an **infinite enclosure map**. Let  $\mathcal{CL}^{\#} = \{\text{cl}_{\Lambda'}^{\#} \mid \Lambda' \in \mathcal{L}\}$ . An element of  $\mathcal{CL}^{\#}$  is called a **finite enclosure map**. Although  $\mathcal{CL}^{\infty}$  and  $\mathcal{CL}^{\#}$  might not be disjoint (for example if  $\mathbb{A}$  is associated to a reductive group over a local field), we define the set of **enclosure maps**  $\mathcal{CL} = \mathcal{CL}^{\infty} \sqcup \mathcal{CL}^{\#}$  : in 2.2.1, the definition of the set of faces associated to an enclosure map  $\text{cl}$  depends on if  $\text{cl}$  is finite or not.

If  $\text{cl} \in \mathcal{CL}$ ,  $\text{cl} = \text{cl}_{\Lambda'}^{\mathcal{P}}$  with  $\mathcal{P} \in [\Phi, \Delta] \cup \{\#\}$  and  $\Lambda' \in \mathcal{L}$ , then for all  $\alpha \in \Delta$ ,  $\Lambda'_{\alpha} = \{k \in \mathbb{R} \mid \text{cl}(D(\alpha, k)) = D(\alpha, k)\}$ . Therefore  $\text{cl}^{\#} := \text{cl}_{\Lambda'}^{\#}$  is well defined. We do not use exactly the same notation as Rousseau in [Rou17] in which  $\text{cl}^{\#}$  means  $\text{cl}_{\Lambda}^{\#}$ .

If  $\Lambda' \in \mathcal{L}$ , one sets  $\mathcal{CL}_{\Lambda'} = \{\text{cl}_{\Lambda'}^{\mathcal{P}} \mid \mathcal{P} \in [\Phi, \Delta]\} \sqcup \{\text{cl}_{\Lambda'}^{\#}\}$ .

In order to simplify, the reader can consider the situation where  $\Lambda_{\alpha} = \Lambda'_{\alpha} = \mathbb{Z}$  for all  $\alpha \in \Phi$ ,  $\mathcal{P} = \Delta$  and  $\text{cl} = \text{cl}_{\Lambda}^{\Delta}$ , which is the situation of [GR14], [BPGR16] and [Héb17] for example.

An **apartment** is a root generating system equipped with a Weyl group  $W$  (i.e with a set  $\mathcal{M}$  of real walls, see 2.1.3) and a family  $\Lambda' \in \mathcal{L}$ . Let  $\mathbb{A} = (\mathcal{S}, W, \Lambda')$  be an apartment. A set of the shape  $M(\alpha, k)$ , with  $\alpha \in \Phi$  and  $k \in \Lambda'_{\alpha}$  is called a **wall** of  $\mathbb{A}$  and a set of the shape  $D(\alpha, k)$ , with  $\alpha \in \Phi$  and  $k \in \Lambda'_{\alpha}$  is called a **half-apartment** of  $\mathbb{A}$ . A subset  $X$  of  $\mathbb{A}$  is said to be enclosed if there exist  $k \in \mathbb{N}$ ,  $\beta_1, \dots, \beta_k \in \Phi$  and  $(\lambda_1, \dots, \lambda_k) \in \prod_{i=1}^k \Lambda'_{\beta_i}$  such that  $X = \bigcap_{i=1}^k D(\beta_i, \lambda_i)$  (i.e  $X = \text{cl}_{\Lambda'}^{\#}(X)$ ). As we shall see, if  $\Lambda' \in \mathcal{L}$  is fixed, the definition of masures does not depend on the choice of an enclosure map in  $\mathcal{CL}_{\Lambda'}$  and thus it will be more convenient to choose  $\text{cl}_{\Lambda'}^{\#}$ , see Theorem 5.1 and Theorem 5.2.

**Remark 2.2.** Here and in the following, we may replace  $\Delta_{im}^{+}$  by any  $W^v$ -stable subset of  $\bigoplus_{i \in I} \mathbb{R}_{+}\alpha_i$  such that  $\Delta_{im}^{+} \cap \bigcup_{\alpha \in \Phi} \mathbb{R}\alpha$  is empty. We then set  $\Delta_{im}^{-} = -\Delta_{im}^{+}$ . This is useful to include the case of almost split Kac-Moody groups, see 6.11.3 of [Rou17].

## 2.2 Measure

In this section, we define masures. They were introduced in [GR08] for symmetrizable split Kac-Moody groups over ultrametric fields whose residue field contains  $\mathbb{C}$ , axiomatized in [Rou11], then developed and generalized to almost-split Kac-Moody groups over ultrametric fields in [Rou16] and [Rou17].

### 2.2.1 Definitions of faces, chimneys and related notions

Let  $\mathbb{A} = (\mathcal{S}, W, \Lambda')$  be an apartment. We choose an enclosure map  $\text{cl} \in \mathcal{CL}_{\Lambda'}$ .

A **local-face** is associated to a point  $x$  and a vectorial face  $F^v$  in  $\mathbb{A}$ ; it is the filter  $F^l(x, F^v) = \text{germ}_x(x + F^v)$  intersection of  $x + F^v$  and the filter of neighborhoods of  $x$  in  $\mathbb{A}$ .



A **face**  $F$  in  $\mathbb{A}$  is a filter associated to a point  $x \in \mathbb{A}$  and a vectorial face  $F^v \subset \mathbb{A}$ . More precisely, if  $\text{cl}$  is infinite (resp.  $\text{cl}$  is finite),  $\text{cl} = \text{cl}_{\Lambda'}^{\mathcal{P}}$  with  $\mathcal{P} \in [\Phi, \Delta]$  (resp.  $\text{cl} = \text{cl}_{\Lambda'}^{\#}$ ),  $F(x, F^v)$  is the filter made of the subsets containing an intersection (resp. a finite intersection) of half-spaces  $D(\alpha, \lambda_\alpha)$  or  $D^\circ(\alpha, \lambda_\alpha)$ , with  $\lambda_\alpha \in \Lambda'_\alpha \cup \{+\infty\}$  for all  $\alpha \in \mathcal{P}$  (at most one  $\lambda_\alpha \in \Lambda_\alpha$  for each  $\alpha \in \mathcal{P}$ ) (resp.  $\Phi$ ).

There is an order on the faces: if  $F \subset \overline{F'}$  one says that “ $F$  is a face of  $F'$ ” or “ $F'$  contains  $F$ ”. The dimension of a face  $F$  is the smallest dimension of an affine space generated by some  $S \in F$ . Such an affine space is unique and is called its **support**. A face is said to be **spherical** if the direction of its support meets the open Tits cone  $\overset{\circ}{\mathcal{T}}$  or its opposite  $-\overset{\circ}{\mathcal{T}}$ ; then its pointwise stabilizer  $W_F$  in  $W^v$  is finite.

A **chamber** (or alcove) is face of the form  $F(x, C^v)$  where  $x \in \mathbb{A}$  and  $C^v$  is a vectorial chamber of  $\mathbb{A}$ .

A **panel** is a face of the form  $F(x, F^v)$ , where  $x \in \mathbb{A}$  and  $F^v$  is a vectorial face of  $\mathbb{A}$  spanning a wall.

A **chimney** in  $\mathbb{A}$  is associated to a face  $F = F(x, F_0^v)$  and to a vectorial face  $F^v$ ; it is the filter  $\mathfrak{r}(F, F^v) = \text{cl}(F + F^v)$ . The face  $F$  is the basis of the chimney and the vectorial face  $F^v$  its direction. A chimney is **splayed** if  $F^v$  is spherical, and is **solid** if its support (as a filter, i.e., the smallest affine subspace of  $\mathbb{A}$  containing  $\mathfrak{r}$ ) has a finite pointwise stabilizer in  $W^v$ . A splayed chimney is therefore solid.

A **shortening** of a chimney  $\mathfrak{r}(F, F^v)$ , with  $F = F(x, F_0^v)$  is a chimney of the shape  $\mathfrak{r}(F(x + \xi, F_0^v), F^v)$  for some  $\xi \in \overline{F^v}$ . The **germ** of a chimney  $\mathfrak{r}$  is the filter of subsets of  $\mathbb{A}$  containing a shortening of  $\mathfrak{r}$  (this definition of shortening is slightly different from the one of [Rou11] 1.12 but follows [Rou17] 3.6) and we obtain the same germs with these two definitions).

## 2.2.2 Measure

An **apartment of type**  $\mathbb{A}$  is a set  $A$  with a nonempty set  $\text{Isom}(\mathbb{A}, A)$  of bijections (called **Weyl-isomorphisms**) such that if  $f_0 \in \text{Isom}(\mathbb{A}, A)$  then  $f \in \text{Isom}(\mathbb{A}, A)$  if and only if there exists  $w \in W$  satisfying  $f = f_0 \circ w$ . We will say **isomorphism** instead of Weyl-isomorphism in the sequel. An isomorphism between two apartments  $\phi : A \rightarrow A'$  is a bijection such that ( $f \in \text{Isom}(\mathbb{A}, A)$  if and only if  $\phi \circ f \in \text{Isom}(\mathbb{A}, A')$ ). We extend all the notions that are preserved by  $W$  to each apartment. Thus sectors, enclosures, faces and chimneys are well defined in any apartment of type  $\mathbb{A}$ .

**Definition 2.3.** A measure of type  $(\mathbb{A}, \text{cl})$  is a set  $\mathcal{I}$  endowed with a covering  $\mathcal{A}$  of subsets called **apartments** such that:

(MA1) Any  $A \in \mathcal{A}$  admits a structure of apartment of type  $\mathbb{A}$ .

(MA2, cl) If  $F$  is a point, a germ of a preordered interval, a generic ray or a solid chimney in an apartment  $A$  and if  $A'$  is another apartment containing  $F$ , then  $A \cap A'$  contains the enclosure  $\text{cl}_A(F)$  of  $F$  and there exists an isomorphism from  $A$  onto  $A'$  fixing  $\text{cl}_A(F)$ .

(MA3, cl) If  $\mathfrak{R}$  is the germ of a splayed chimney and if  $F$  is a face or a germ of a solid chimney, then there exists an apartment containing  $\mathfrak{R}$  and  $F$ .

(MA4, cl) If two apartments  $A, A'$  contain  $\mathfrak{R}$  and  $F$  as in (MA3), then there exists an isomorphism from  $A$  to  $A'$  fixing  $\text{cl}_A(\mathfrak{R} \cup F)$ .

(MAO) If  $x, y$  are two points contained in two apartments  $A$  and  $A'$ , and if  $x \leq_A y$  then the two segments  $[x, y]_A$  and  $[x, y]_{A'}$  are equal.

In this definition, one says that an apartment contains a germ of a filter if it contains at least one element of this germ. One says that a map fixes a germ if it fixes at least one

element of this germ.

The main example of measure is the measure associated to an almost-split Kac-Moody group over a ultrametric field, see [Rou17].

### 2.2.3 Example: measure associated to a split Kac-Moody group over a ultrametric field

Let  $A$  be a Kac-Moody matrix and  $\mathcal{S}$  be a root generating system of type  $A$ . We consider the group functor  $\mathbf{G}$  associated to the root generating system  $\mathcal{S}$  in [Tit87] and in Chapitre 8 of [Rém02]. This functor is a functor from the category of rings to the category of groups satisfying axioms (KMG1) to (KMG 9) of [Tit87]. When  $R$  is a field,  $\mathbf{G}(R)$  is uniquely determined by these axioms by Theorem 1' of [Tit87]. This functor contains a toric functor  $\mathbf{T}$ , from the category of rings to the category of commutative groups (denoted  $\mathcal{T}$  in [Rém02]) and two functors  $\mathbf{U}^+$  and  $\mathbf{U}^-$  from the category of rings to the category of groups.

Let  $\mathcal{K}$  be a field equipped with a non-trivial valuation  $\omega : \mathcal{K} \rightarrow \mathbb{R} \cup \{+\infty\}$ ,  $\mathcal{O}$  its ring of integers and  $G = \mathbf{G}(\mathcal{K})$  (and  $U^+ = \mathbf{U}^+(\mathcal{K})$ , ...). For all  $\epsilon \in \{-, +\}$ , and all  $\alpha \in \Phi^\epsilon$ , we have an isomorphism  $x_\alpha$  from  $\mathcal{K}$  to a group  $U_\alpha$ . For all  $k \in \mathbb{R}$ , one defines a subgroup  $U_{\alpha,k} := x_\alpha(\{u \in \mathcal{K} \mid \omega(u) \geq k\})$ . Let  $\mathcal{I}$  be the measure associated to  $G$  constructed in [Rou16]. Then for all  $\alpha \in \Phi$ ,  $\Lambda_\alpha = \Lambda'_\alpha = \omega(\mathcal{K}) \setminus \{+\infty\}$  and  $\text{cl} = \text{cl}_\Lambda^\Delta$ . If moreover  $\mathcal{K}$  is local, one has (up to renormalization, see Lemma 1.3 of [GR14])  $\Lambda_\alpha = \mathbb{Z}$  for all  $\alpha \in \Phi$ . Moreover, we have:

- the fixer of  $\mathbb{A}$  in  $G$  is  $H = \mathbf{T}(\mathcal{O})$  (by remark 3.2 of [GR08])
- the fixer of  $\{0\}$  in  $G$  is  $K_s = \mathbf{G}(\mathcal{O})$ .
- for all  $\alpha \in \Phi$  and  $k \in \mathbb{Z}$ , the fixer of  $D(\alpha, k)$  in  $G$  is  $H.U_{\alpha,k}$  (by 4.2 7) of [GR08])
- for all  $\epsilon \in \{-, +\}$ ,  $U^\epsilon$  is the fixer of  $\epsilon\infty$  (by 4.2 4) of [GR08]).

If moreover,  $\mathcal{K}$  is local, with residue cardinal  $q$ , each panel is contained in  $1+q$  chambers.

The group  $G$  is reductive if and only if  $W^v$  is finite. In this case,  $\mathcal{I}$  is the usual Bruhat-Tits building of  $G$  and one has  $\mathcal{T} = \mathbb{A}$ .

## 2.3 Preliminary notions on mesures

In this subsection we recall notions on mesures introduced in [GR08], [Rou11], [Héb17] and [Héb16].

### 2.3.1 Tits preorder and Tits open preorder on $\mathcal{I}$

As the Tits preorder  $\leq$  and the Tits open preorder  $\overset{\circ}{\leq}$  on  $\mathbb{A}$  are invariant under the action of  $W^v$ , one can equip each apartment  $A$  with preorders  $\leq_A$  and  $\overset{\circ}{\leq}_A$ . Let  $A$  be an apartment of  $\mathcal{I}$  and  $x, y \in A$  such that  $x \leq_A y$  (resp.  $x \overset{\circ}{\leq}_A y$ ). Then by Proposition 5.4 of [Rou11], if  $B$  is an apartment containing  $x$  and  $y$ ,  $x \leq_B y$  (resp.  $x \overset{\circ}{\leq}_B y$ ). This defines a relation  $\leq$  (resp.  $\overset{\circ}{\leq}$ ) on  $\mathcal{I}$ . By Théorème 5.9 of [Rou11], this defines a preorder  $\leq$  (resp.  $\overset{\circ}{\leq}$ ) on  $\mathcal{I}$ . It is invariant by isomorphisms of apartments: if  $A, B$  are apartments,  $\phi : A \rightarrow B$  is an isomorphism of apartments and  $x, y \in A$  are such that  $x \leq y$  (resp.  $x \overset{\circ}{\leq} y$ ), then  $\phi(x) \leq \phi(y)$  (resp.  $\phi(x) \overset{\circ}{\leq} \phi(y)$ ). We call it the **Tits preorder on  $\mathcal{I}$**  (resp. the **Tits open preorder on  $\mathcal{I}$** ).

### 2.3.2 Retractions centered at sector-germs

Let  $\mathfrak{s}$  be a sector-germ of  $\mathcal{I}$  and  $A$  be an apartment containing it. Let  $x \in \mathcal{I}$ . By (MA3), there exists an apartment  $A_x$  of  $\mathcal{I}$  containing  $x$  and  $\mathfrak{s}$ . By (MA4), there exists an isomorphism of apartments  $\phi : A_x \rightarrow A$  fixing  $\mathfrak{s}$ . By [Rou11] 2.6,  $\phi(x)$  does not depend on the choices we made and thus we can set  $\rho_{A,\mathfrak{s}}(x) = \phi(x)$ .

The map  $\rho_{A,\mathfrak{s}}$  is a retraction from  $\mathcal{I}$  onto  $A$ . It only depends on  $\mathfrak{s}$  and  $A$  and we call it the **retraction onto  $A$  centered at  $\mathfrak{s}$** .

If  $A$  and  $B$  are two apartments, and  $\phi : A \rightarrow B$  is an isomorphism of apartments fixing some set  $X$ , one writes  $\phi : A \xrightarrow{X} B$ . If  $A$  and  $B$  share a sector-germ  $\mathfrak{q}$ , one denotes by  $A \xrightarrow{A \cap B} B$  or by  $A \xrightarrow{\mathfrak{q}} B$  the unique isomorphism of apartments from  $A$  to  $B$  fixing  $\mathfrak{q}$  (and also  $A \cap B$ ). We denote by  $\mathcal{I} \xrightarrow{\mathfrak{q}} A$  the retraction onto  $A$  fixing  $\mathfrak{q}$ . One denotes by  $\rho_{+\infty}$  the retraction  $\mathcal{I} \xrightarrow{+\infty} \mathbb{A}$  and by  $\rho_{-\infty}$  the retraction  $\mathcal{I} \xrightarrow{-\infty} \mathbb{A}$ .

### 2.3.3 Parallelism in $\mathcal{I}$ and building at infinity

Let us explain briefly the notion of parallelism in  $\mathcal{I}$ . This is done more completely in [Rou11] Section 3.

Let us begin with rays. Let  $\delta$  and  $\delta'$  be two generic rays in  $\mathcal{I}$ . By (MA3) and [Rou11] 2.2 3) there exists an apartment  $A$  containing sub-rays of  $\delta$  and  $\delta'$  and we say that  $\delta$  and  $\delta'$  are **parallel**, if these sub-rays are parallel in  $A$ . Parallelism is an equivalence relation and its equivalence classes are called **directions**. Let  $Q$  be a sector of  $\mathcal{I}$  and  $A$  be an apartment containing  $Q$ . One fixes the origin of  $A$  in a base point of  $Q$ . Let  $\nu \in Q$  and  $\delta = \mathbb{R}_+\nu$ . Then  $\delta$  is a generic ray in  $\mathcal{I}$ . By Lemma 3.2 of [Héb17], for all  $x \in \mathcal{I}$ , there exists a unique ray  $x + \delta$  of direction  $\delta$  and base point  $x$ . To obtain this ray, one can choose an apartment  $A_x$  containing  $x$  and a sub-ray  $\delta'$  of  $\delta$ , which is possible by (MA3) and [Rou11] 2.2 3), and then we take the translate of  $\delta'$  in  $A_x$  having  $x$  as a base point.

A **sector-face**  $f$  of  $\mathbb{A}$ , is a set of the shape  $x + F^v$  for some vectorial face  $F^v$  and some  $x \in \mathbb{A}$ . The germ  $\mathfrak{F} = \text{germ}_\infty(f)$  of this sector face is the filter containing the elements of the shape  $q + f$ , for some  $q \in \overline{F^v}$ . The sector-face  $f$  is said to be spherical if  $F^v \cap \overset{\circ}{\mathcal{T}}$  is nonempty. A **sector-panel** is a sector-face included in a wall and spanning this one as an affine space. A sector-panel is spherical (see [Rou11] 1). We extend these notions to  $\mathcal{I}$  thanks to the isomorphisms of apartments. Let us make a summary of the notion of parallelism for sector-faces. This is also more complete in [Rou11], 3.3.4).

If  $f$  and  $f'$  are two spherical sector-faces, there exists an apartment  $B$  containing their germs  $\mathfrak{F}$  and  $\mathfrak{F}'$ . One says that  $f$  and  $f'$  are parallel if there exists a vectorial face  $F^v$  of  $B$  such that  $\mathfrak{F} = \text{germ}_\infty(x + F^v)$  and  $\mathfrak{F}' = \text{germ}_\infty(y + F^v)$  for some  $x, y \in B$ . Parallelism is an equivalence relation. The parallelism class of a sector-face germ  $\mathfrak{F}$  is denoted  $\mathfrak{F}^\infty$ . We denote by  $\mathcal{I}^\infty$  the set of directions of spherical faces of  $\mathcal{I}$ .

By Proposition 4.7.1) of [Rou11], for all  $x \in \mathcal{I}$  and all  $\mathfrak{F}^\infty \in \mathcal{I}^\infty$ , there exists a unique sector-face  $x + \mathfrak{F}^\infty$  of direction  $\mathfrak{F}^\infty$  and with base point  $x$ . The existence can be obtained in the same way as for rays.

### 2.3.4 Distance between apartments

Here we recall the notion of distance between apartments introduced in [Héb16]. It will often enable us to make inductions and to restrict our study to apartments sharing a sector. Let  $\mathfrak{q}$  and  $\mathfrak{q}'$  be two sector germs of  $\mathcal{I}$  of the same sign  $\epsilon$ . By (MA4), there exists an apartment  $B$  containing  $\mathfrak{q}$  and  $\mathfrak{q}'$ . In  $B$ , there exists a minimal gallery between  $\mathfrak{q}$  and  $\mathfrak{q}'$  and the length

of this gallery is called the distance between  $\mathfrak{q}$  and  $\mathfrak{q}'$ . This does not depend on the choice of  $B$ . If  $A'$  is an apartment of  $\mathcal{I}$ , the distance  $d(A', \mathfrak{q})$  between  $A'$  and  $\mathfrak{q}$  is the minimal possible distance between a sector-germ of  $A'$  of sign  $\epsilon$  and  $\mathfrak{q}$ . If  $A$  and  $A'$  are apartments of  $\mathcal{I}$  and  $\epsilon \in \{-1, 1\}$ , the distance of sign  $\epsilon$  between  $A$  and  $A'$  is the minimal possible distance between a sector-germ of sign  $\epsilon$  of  $A$  and a sector-germ of sign  $\epsilon$  of  $A'$ . We denote it  $d_\epsilon(A, A')$  or  $d(A, A')$  if the sign is fixed.

Let  $\epsilon \in \{-, +\}$ . Then  $d_\epsilon$  is not a distance on the apartments of  $\mathcal{I}$  because if  $A$  is an apartment, all apartment  $A'$  containing a sector of  $A$  of sign  $\epsilon$  (and there are many of them by (MA3)) satisfies  $d_\epsilon(A, A') = 0$ .

## 2.4 Notation

Let  $X$  be a finite dimensional affine space. Let  $C \subset X$  be a convex set and  $A'$  be its support. The **relative interior** (resp. **relative frontier**) of  $C$ , denoted  $\text{Int}_r(C)$  (resp.  $\text{Fr}_r(C)$ ) is the interior (resp. frontier) of  $C$  seen as a subset of  $A'$ . A set is said to be **relatively open** if it is open in its support.

If  $X$  is an affine space and  $U \subset X$ , one denotes by  $\text{conv}(X)$  the convex hull of  $X$ . If  $x, y \in \mathbb{A}$ , we denote by  $[x, y]$  the segment of  $\mathbb{A}$  joining  $x$  and  $y$ . If  $A$  is an apartment and  $x, y \in A$ , we denote by  $[x, y]_A$  the segment of  $A$  joining  $x$  and  $y$ .

If  $X$  is a topological space and  $a \in X$ , one denotes by  $\mathcal{V}_X(a)$  the set of open neighborhoods of  $a$ .

If  $X$  is a subset of  $\mathbb{A}$ , one denotes by  $\overset{\circ}{X}$  or by  $\text{Int}(X)$  (depending on the legibility) its interior. One denotes by  $\text{Fr}(X)$  the boundary (or frontier) of  $X$ :  $\text{Fr}(X) = \overline{X} \setminus \overset{\circ}{X}$ .

If  $X$  is a topological space,  $x \in X$  and  $\Omega$  is a subset of  $X$  containing  $x$  in its closure, then the **germ** of  $\Omega$  in  $x$  is denoted  $\text{germ}_x(\Omega)$ .

We use the same notation as in [Rou11] for segments and segment-germs in an affine space  $X$ . For example if  $X = \mathbb{R}$  and  $a, b \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ ,  $[a, b] = \{x \in \overline{\mathbb{R}} \mid a \leq x \leq b\}$ ,  $[a, b[ = \{x \in \overline{\mathbb{R}} \mid a \leq x < b\}$ ,  $[a, b) = \text{germ}_a([a, b]) \dots$

## 3 General properties of the intersection of two apartments

In this section, we study the intersection of two apartments, without assuming that their intersection contains a generic ray.

In Subsection 3.1, we extend results obtained for measure on which a group acts strongly transitively to our framework.

In Subsection 3.2, we write the intersection of two apartments as a finite union of enclosed parts.

In Subsection 3.3, we use the results of Subsection 3.2 to prove that if the intersection of two apartments is convex, then it is enclosed.

In Subsection 3.4, we study the existence of isomorphisms fixing subsets of an intersection of two apartments

Let us sketch the proof of Theorem 1. The most difficult part is to prove that if  $A$  and  $B$  are apartments sharing a generic ray,  $A \cap B$  is convex. We first reduce our study to the case where  $A \cap B$  has nonempty interior. We then parametrize the frontier of  $A$  and  $B$  by a map  $\text{Fr} : U \rightarrow \text{Fr}(A \cap B)$ , where  $U$  is an open and convex set of  $A$ . The idea is then to prove that for “almost” all choices of  $x, y \in U$ , a map associated to  $\text{Fr}_{x,y} : t \in [0, 1] \mapsto \text{Fr}(tx + (1 - t)y)$

is convex. An important step in this proof is the fact that  $\text{Fr}_{x,y}$  is piecewise affine and this relies on the decomposition of Subsection 3.2. The convexity of  $A \cap B$  is obtained by using a density argument. We then conclude thanks to Subsection 3.3 and Subsection 3.4.

### 3.1 Preliminaries

In this subsection, we extend some results of [Héb17] and [Héb16], obtained for a measure on which a group acts strongly transitively to our framework.

#### 3.1.1 Splitting of apartments

The following lemma generalizes Lemma 3.2 of [Héb16] to our frameworks:

**Lemma 3.1.** *Let  $A_1$  and  $A_2$  be two distinct apartments such that  $A_1 \cap A_2$  contains a half-apartment. Then  $A_1 \cap A_2$  is a half-apartment.*

*Proof.* One identifies  $A_1$  and  $\mathbb{A}$ . By the proof of Lemma 3.2 of [Héb16],  $D = A_1 \cap A_2$  is a half-space of the shape  $D(\alpha, k)$  for some  $\alpha \in \Phi$  and  $k \in \mathbb{R}$  (our terminology is not the same as in [Héb16] in which a half-apartment is a half-space of the shape  $D(\beta, l)$ , with  $\beta \in \Phi$  and  $l \in \mathbb{R}$ , whereas now, we ask moreover that  $l \in \Lambda'_\beta$ ). Let  $F, F'$  be opposed sector-panels of  $M(\alpha, k)$ . Let  $Q$  be a sector of  $D$  dominating  $F$ ,  $\mathfrak{q}$  its germ and  $\mathfrak{F}'$  be the germ of  $F'$ . By (MA4),  $A_1 \cap A_2 \supset \text{cl}(\mathfrak{F}', \mathfrak{q})$ . But  $\text{cl}(\mathfrak{F}', \mathfrak{q}) \supset \text{cl}(D) \supset D = A_1 \cap A_2$  and thus  $k \in \Lambda'_\alpha$ :  $A_1 \cap A_2$  is a half-apartment.  $\square$

As a consequence, one can use Lemma 3.6 and Proposition 3.7 of [Héb16] in our framework. We thus have the following proposition:

**Proposition 3.2.** *Let  $A$  be an apartment,  $\mathfrak{q}$  be a sector-germ of  $\mathcal{I}$  such that  $\mathfrak{q} \not\subseteq A$  and  $n = d(\mathfrak{q}, A)$ .*

1. *One can write  $A = D_1 \cup D_2$ , where  $D_1$  and  $D_2$  are opposite half-apartments of  $A$  such that for all  $i \in \{1, 2\}$ , there exists an apartment  $A_i$  containing  $D_i$  and such that  $d(A_i, \mathfrak{q}) = n - 1$ .*
2. *There exist  $k \in \mathbb{N}$ , enclosed subsets  $P_1, \dots, P_k$  of  $A$  such that for all  $i \in \llbracket 1, k \rrbracket$ , there exist an apartment  $A_i$  containing  $\mathfrak{q} \cup P_i$  and an isomorphism  $\phi_i : A \xrightarrow{P_i} A_i$ .*

**Remark 3.3.** The choice of the Weyl group  $W$  (and thus of  $Q^\vee$ ) imposes restrictions on the walls that can delimit the intersection of two apartments. Let  $A$  be an apartment and suppose that  $A \cap \mathbb{A} = D(\alpha, k)$  for some  $\alpha \in \Phi$  and  $k \in \Lambda'_\alpha$ . Then  $k \in \frac{1}{2}\alpha(Q^\vee)$ . Indeed, let  $D = A \cap \mathbb{A}$ ,  $D_1$  be the half-apartment of  $\mathbb{A}$  opposed to  $D$  and  $D_2$  be the half-apartment of  $A$  opposed to  $D_1$ . By Proposition 2.9 2) of [Rou11]  $B = D_1 \cup D_2$  is an apartment of  $\mathcal{I}$ . Let  $f : \mathbb{A} \xrightarrow{D} A$ ,  $g : A \xrightarrow{D_2} B$  and  $h : B \xrightarrow{D_1} \mathbb{A}$ : these isomorphisms exist because two apartments sharing a half-apartment in particular share a sector, see 2.3.2. Let  $s : \mathbb{A} \rightarrow \mathbb{A}$  making the following diagram commute:

$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{f} & A \\ \downarrow s & & \downarrow g \\ \mathbb{A} & \xrightarrow{h^{-1}} & B. \end{array}$$

The map  $s$  fixes  $M(\alpha, k)$ . Moreover, if  $x \in \mathring{D}$ ,  $f(x) = x$ , thus  $g(f(x)) \in \mathring{D}_1$  and hence  $h^{-1}(g(f(x))) \in \mathring{D}_1$ :  $s \neq \text{Id}$ . The map  $s$  is an isomorphism of apartments and thus  $s \in W$ .

As  $s$  fixes  $M(\alpha, k)$ , the vectorial part  $\vec{s}$  of  $s$  fixes  $M(\alpha, 0)$ . As  $W = W^v \times Q^\vee$ ,  $s = t \circ \vec{s}$ , where  $t$  is a translation of vector  $q^\vee$  in  $Q^\vee$ . If  $y \in M(\alpha, k)$ , one has  $\alpha(s(y)) = k = \alpha(q^\vee) - k$  and therefore  $k \in \frac{1}{2}\alpha(Q^\vee)$ . This could enable to be more precise in Proposition 3.2.

### 3.1.2 A characterization of the points of $\mathbb{A}$

The aim of this subsection is to extend Corollary 4.4 of [Héb17] to our framework.

**Vectorial distance on  $\mathcal{I}$**  We recall the definition of the vectorial distance defined in Section 1.7 of [GR14]. Let  $x, y \in \mathcal{I}$  such that  $x \leq y$ . Then there exists an apartment  $A$  containing  $x, y$  and an isomorphism  $\phi : A \rightarrow \mathbb{A}$ . One has  $\phi(y) - \phi(x) \in \mathcal{T}$  and thus there exists  $w \in W^v$  such that  $\lambda = w \cdot (\phi(y) - \phi(x)) \in \overline{C_f^v}$ . Then  $\lambda$  does not depend on the choices we made, it is called the **vectorial distance between  $x$  and  $y$**  and denoted  $d^v(x, y)$ . The vectorial distance is invariant under isomorphism of apartments: if  $x, y$  are two points in an apartment  $A$  such that  $x \leq y$ , if  $B$  is an apartment and if  $\phi : A \rightarrow B$  is an isomorphism of apartments, then  $d^v(x, y) = d^v(\phi(x), \phi(y))$ .

**Image of a preordered segment by a retraction** In Theorem 6.2 of [GR08], Gaussent and Rousseau give a very precise description of the image of a preordered segment by a retraction centered at a sector-germ. However they suppose that a group acts strongly transitively on  $\mathcal{I}$ . Without this assumption, they prove a simpler property of these images. We recall it here.

Let  $\lambda \in \overline{C_f^v}$ . A  $\lambda$ -**path**  $\pi$  in  $\mathbb{A}$  is a map  $\pi : [0, 1] \rightarrow \mathbb{A}$  such that there exists  $n \in \mathbb{N}$  and  $0 \leq t_1 < \dots < t_n \leq 1$  such that for all  $i \in \llbracket 1, n-1 \rrbracket$ ,  $\pi$  is affine on  $[t_i, t_{i+1}]$  and  $\pi'(t) \in W^v \cdot \lambda$  for all  $t \in [t_i, t_{i+1}]$ .

**Lemma 3.4.** *Let  $A$  be an apartment of  $\mathcal{I}$ ,  $x, y \in A$  such that  $x \leq y$  and  $\rho : \mathcal{I} \rightarrow \mathbb{A}$  be a retraction of  $\mathcal{I}$  onto  $\mathbb{A}$  centered at a sector-germ  $\mathfrak{q}$  of  $\mathbb{A}$ . Let  $\tau : [0, 1] \rightarrow A$  defined by  $\tau(t) = (1-t)x + ty$  for all  $t \in [0, 1]$  and  $\lambda = d^v(x, y)$ . Then  $\rho \circ \tau$  is a  $\lambda$ -path between  $\rho(x)$  and  $\rho(y)$ .*

*Proof.* We rewrite the proof of the beginning of Section 6 of [GR08]. Let  $\phi : A \rightarrow \mathbb{A}$  be an isomorphism such that  $\phi(y) - \phi(x) = \lambda$ , which exists by definition of  $d^v$ . By the same reasoning as in the paragraph of [GR08] before Remark 4.6, there exist  $n \in \mathbb{N}$ , apartments  $A_1, \dots, A_n$  of  $\mathcal{I}$  containing  $\mathfrak{q}$ ,  $0 = t_1 < \dots < t_n = 1$  such that  $\tau([t_i, t_{i+1}]) \subset A_i$  for all  $i \in \llbracket 1, n-1 \rrbracket$ .

Using Proposition 5.4 of [Rou11], for all  $i \in \llbracket 1, n-1 \rrbracket$ , one chooses an isomorphism  $\psi_i : A \xrightarrow{\tau([t_i, t_{i+1}])} A_i$ . Let  $\phi_i : A_i \xrightarrow{A_i \cap \mathbb{A}} \mathbb{A}$ . For all  $t \in [t_i, t_{i+1}]$ ,  $\rho(\tau(t)) = \phi_i \circ \psi_i(\tau(t))$ . Moreover,  $\phi_i \circ \psi_i : A \rightarrow \mathbb{A}$  and by (MA1), there exists  $w_i \in W$  such that  $\phi_i \circ \psi_i = w_i \circ \phi$ . Therefore for all  $t \in [t_i, t_{i+1}]$ ,  $(\rho \circ \tau)'(t) = w_i \cdot \lambda$ , which proves that  $\rho \circ \tau$  is a  $\lambda$ -path.  $\square$

**The projection  $y_\nu$**  Let  $\nu \in C_f^v$  and  $\delta = \mathbb{R}^+ \nu$ . By paragraph “Definition of  $y_\nu$  and  $T_\nu$ ” of [Héb17], for all  $x \in \mathcal{I}$ , there exists  $y_\nu(x) \in \mathbb{A}$  such that  $x + \bar{\delta} \cap \mathbb{A} = y_\nu(x) + \bar{\delta}$ , where  $x + \bar{\delta}$  is the closure of  $x + \delta$  (defined in 2.3.3) in any apartment containing it.

**The  $Q_{\mathbb{R},+}^\vee$ -order in  $\mathbb{A}$**  One sets  $Q_{\mathbb{R},+}^\vee = \sum_{\alpha \in \Phi^+} \mathbb{R}_+ \alpha^\vee = \bigoplus_{i \in I} \mathbb{R}_+ \alpha_i$ . One has  $Q_{\mathbb{R},+}^\vee \subset \bigoplus_{i \in I} \mathbb{R}_+ \alpha_i^\vee$ . If  $x, y \in \mathbb{A}$ , one denotes  $x \leq_{Q^\vee} y$  if  $y - x \in Q_{\mathbb{R},+}^\vee$ .

The following lemma is the writing of Proposition 3.12 d) of [Kac94] in our context.

**Lemma 3.5.** *Let  $\lambda \in \overline{C_f^v}$  and  $w \in W^v$ . Then  $w.\lambda \leq_{Q^v} \lambda$ .*

If  $x \in \mathbb{A}$  and  $\lambda \in \overline{C_f^v}$ , one defines  $\pi_\lambda^a : [0, 1] \rightarrow \mathbb{A}$  by  $\pi_\lambda^a(t) = a + t\lambda$  for all  $t \in [0, 1]$ .

**Lemma 3.6.** *Let  $\lambda \in \overline{C_f^v}$  and  $a \in \mathbb{A}$ . Then the unique  $\lambda$ -path from  $a$  to  $a + \lambda$  is  $\pi_\lambda^a$ .*

*Proof.* Let  $\pi$  be a  $\lambda$ -path from  $a$  to  $a + \lambda$ . One chooses a subdivision  $0 = t_1 < \dots < t_n = 1$  of  $[0, 1]$  such that for all  $i \in \llbracket 1, n-1 \rrbracket$ , there exists  $w_i \in W^v$  such that  $\pi'_{\llbracket t_i, t_{i+1} \rrbracket}(t) = w_i.\lambda$ . By Lemma 3.5,  $w_i.\lambda \leq_{Q^v} \lambda$  for all  $i \in \llbracket 1, n-1 \rrbracket$ . Let  $h : \bigoplus_{i \in I} \mathbb{R}\alpha_i^v \rightarrow \mathbb{R}$  defined by  $h(\sum_{i \in I} u_i \alpha_i^v) = \sum_{i \in I} u_i$  for all  $(u_i) \in \mathbb{R}^I$ . Suppose that for some  $i \in \llbracket 1, n-1 \rrbracket$ ,  $w_i.\lambda \neq \lambda$ . Then  $h(w_i.\lambda - \lambda) < 0$  and for all  $j \in \llbracket 1, n-1 \rrbracket$ ,  $h(w_j.\lambda - \lambda) \leq 0$ . By integrating, we get that  $h(0) < 0$ : a contradiction. Therefore  $\pi(t) = a + t\lambda = \pi_\lambda^a(t)$  for all  $t \in [0, 1]$ , which is our assertion.  $\square$

The following proposition corresponds to Corollary 4.4 of [Héb17].

**Proposition 3.7.** *Let  $x \in \mathcal{I}$  such that  $\rho_{+\infty}(x) = \rho_{-\infty}(x)$ . Then  $x \in \mathbb{A}$ .*

*Proof.* Let  $x \in \mathcal{I}$  such that  $\rho_{+\infty}(x) = \rho_{-\infty}(x)$ . Suppose that  $x \in \mathcal{I} \setminus \mathbb{A}$ . By Lemma 3.5 a) of [Héb17], one has  $x \leq y_\nu(x)$  and  $d^v(x, y_\nu(x)) = \lambda$ , with  $\lambda = y_\nu(x) - \rho_{+\infty}(x) \in \mathbb{R}_+^* \nu$ . Let  $A$  be an apartment containing  $x$  and  $+\infty$ , which exists by (MA3). Let  $\tau : [0, 1] \rightarrow A$  be defined by  $\tau(t) = (1-t)x + ty_\nu(x)$  for all  $t \in [0, 1]$  (this does not depend on the choice of  $A$  by Proposition 5.4 of [Rou11]) and  $\pi = \rho_{-\infty} \circ \tau$ . Then by Lemma 3.4,  $\pi$  is a  $\lambda$ -path from  $\rho_{-\infty}(x) = \rho_{+\infty}(x)$  to  $y_\nu(x) = \rho_{+\infty}(x) + \lambda$ .

By Lemma 3.6,  $\pi(t) = \rho_{+\infty}(x) + t\lambda$  for all  $t \in [0, 1]$ . By Lemma 3.6 of [Héb17],  $\tau([0, 1]) \subset \mathbb{A}$ :  $x = \tau(0) \in \mathbb{A}$ : this is absurd. Therefore  $x \in \mathbb{A}$ , which is our assertion.  $\square$

### 3.1.3 Topological considerations on apartments

The following proposition generalizes Corollary 5.9 (ii) of [Héb16] to our framework.

**Proposition 3.8.** *Let  $\mathfrak{q}$  be a sector-germ of  $\mathcal{I}$  and  $A$  be an apartment of  $\mathcal{I}$ . Let  $\rho : \mathcal{I} \xrightarrow{\mathfrak{q}} \mathbb{A}$ . Then  $\rho|_A : A \rightarrow \mathbb{A}$  is continuous (for the affine topologies on  $A$  and  $\mathbb{A}$ ).*

*Proof.* Using Proposition 3.2 2, one writes  $A = \bigcup_{i=1}^n P_i$  where the  $P_i$ 's are closed sets of  $A$  such that for all  $i \in \llbracket 1, n \rrbracket$ , there exists an apartment  $A_i$  containing  $P_i$  and  $\mathfrak{q}$  and an isomorphism  $\psi_i : A \xrightarrow{P_i} A_i$ . For all  $i \in \llbracket 1, n \rrbracket$ , one denotes by  $\phi_i$  the isomorphism  $A_i \xrightarrow{\mathfrak{q}} \mathbb{A}$ . Then  $\rho|_{P_i} = \phi_i \circ \psi_i|_{P_i}$  for all  $i \in \llbracket 1, n \rrbracket$ .

Let  $(x_k) \in A^\mathbb{N}$  be a converging sequence and  $x = \lim x_k$ . Then for all  $k \in \mathbb{N}$ ,  $\rho(x_k) \in \{\phi_i \circ \psi_i(x_k) \mid i \in \llbracket 1, n \rrbracket\}$  and thus  $(\rho(x_k))$  is bounded. Let  $(x_{\sigma(k)})$  be a subsequence of  $(x_k)$  such that  $(\rho(x_{\sigma(k)}))$  converges. Maybe extracting a subsequence of  $(x_{\sigma(k)})$ , one can suppose that there exists  $i \in \llbracket 1, n \rrbracket$  such  $x_{\sigma(k)} \in P_i$  for all  $k \in \mathbb{N}$ . One has  $(\rho(x_{\sigma(k)})) = (\phi_i \circ \psi_i(x_{\sigma(k)}))$  and thus  $\rho(x_{\sigma(k)}) \rightarrow \phi_i \circ \psi_i(x) = \rho(x)$  (because  $P_i$  is closed) and thus  $(\rho(x_k))$  converges towards  $\rho(x)$ :  $\rho|_A$  is continuous.  $\square$

The following proposition generalizes Corollary 5.10 of [Héb16] to our context.

**Proposition 3.9.** *Let  $A$  be an apartment. Then  $A \cap \mathbb{A}$  is closed.*

*Proof.* By Proposition 3.7,  $A \cap \mathbb{A} = \{x \in A \mid \rho_{+\infty}(x) = \rho_{-\infty}(x)\}$ , which is closed by Proposition 3.8.  $\square$

### 3.2 Decomposition of the intersection of two apartments in enclosed subsets

The aim of this subsection is to show that  $\mathbb{A} \cap A$  is a finite union of enclosed subsets of  $\mathbb{A}$ .

We first suppose that  $A$  and  $\mathbb{A}$  share a sector. One can suppose that  $+\infty \subset A \cap \mathbb{A}$ . By Proposition 3.2, one has  $A = \bigcup_{i=1}^k P_i$ , for some  $k \in \mathbb{N}$ , where the  $P_i$ 's are enclosed and  $P_i, -\infty$  is included in some apartment  $A_i$  for all  $i \in \llbracket 1, k \rrbracket$ .

**Lemma 3.10.** *Let  $X$  be a finite dimensional affine space,  $U \subset X$  be a set such that  $U \subset \overline{\mathring{U}}$  and suppose that  $U = \bigcup_{i=1}^n U_i$ , where for all  $i \in \llbracket 1, n \rrbracket$   $U_i$  is the intersection of  $U$  and of a finite number of half-spaces. Let  $J = \{j \in \llbracket 1, n \rrbracket \mid \mathring{U}_j \neq \emptyset\}$ . Then  $U = \bigcup_{j \in J} U_j$ .*

*Proof.* Let  $j \in \llbracket 1, n \rrbracket$ . Then  $\text{Fr}(U_j) \cap \mathring{U}$  is included in a finite number of hyperplanes. Therefore, if one chooses a Lebesgue measure on  $X$ ,  $\bigcup_{i \in \llbracket 1, n \rrbracket} \mathring{U} \cap \text{Fr}(U_i)$  has measure 0 and thus  $\mathring{U} \setminus \bigcup_{i \in \llbracket 1, n \rrbracket} \text{Fr}(U_i)$  is dense in  $\mathring{U}$  and thus in  $U$ . Let  $x \in U$ . Let  $(x_k) \in (\mathring{U} \setminus \bigcup_{i \in \llbracket 1, n \rrbracket} \text{Fr}(U_i))^{\mathbb{N}}$  converging towards  $x$ . Extracting a sequence if necessary, one can suppose that for some  $i \in \llbracket 1, n \rrbracket$ ,  $x_k \in U_i$  for all  $k \in \mathbb{N}$ . By definition of the frontier,  $x_k \in \mathring{U}_i$  for all  $k \in \mathbb{N}$ . As  $U_i$  is closed in  $U$ ,  $x \in U_i$  and the lemma follows.  $\square$

**Lemma 3.11.** *Let  $i \in \llbracket 1, k \rrbracket$  such that  $A \cap \mathbb{A} \cap P_i$  has nonempty interior in  $\mathbb{A}$ . Then  $A \cap \mathbb{A} \supset P_i$ .*

*Proof.* One chooses an apartment  $A_i$  containing  $P_i, -\infty$  and  $\phi_i : A \xrightarrow{P_i} A_i$ . Let  $\psi_i : A_i \xrightarrow{A_i \cap \mathbb{A}} \mathbb{A}$  ( $\psi_i$  exists and is unique by Subsection 2.3.2). By definition of  $\rho_{-\infty}$ , if  $x \in P_i$ ,  $\rho_{-\infty}(x) = \psi_i(x)$  and thus  $\rho_{-\infty}(x) = \psi_i \circ \phi_i(x)$ .

Let  $f : A \xrightarrow{A \cap \mathbb{A}} \mathbb{A}$ . One has  $\rho_{+\infty}(x) = f(x)$  for all  $x \in A$ . By Proposition 3.7,  $A \cap \mathbb{A} \cap P_i = \{x \in P_i \mid \rho_{+\infty}(x) = \rho_{-\infty}(x)\} = P_i \cap (f - \psi_i \circ \phi_i)^{-1}(\{0\})$ .

As  $f - \psi_i \circ \phi_i$  is affine,  $(f - \psi_i \circ \phi_i)^{-1}(\{0\})$  is an affine subspace of  $A$  and as it has nonempty interior,  $(f - \psi_i \circ \phi_i)^{-1}(\{0\}) = A$ . Therefore  $P_i \subset A \cap \mathbb{A}$ .  $\square$

We recall the definition of  $x + \infty$ , if  $x \in \mathcal{I}$  (see 2.3.3). Let  $x \in \mathcal{I}$  and  $B$  be an apartment containing  $x$  and  $+\infty$ . Let  $Q$  be a sector of  $\mathbb{A}$ , parallel to  $C_f^v$  and such that  $Q \subset A \cap \mathbb{A}$ . Then  $x + \infty$  is the sector of  $A$  based at  $x$  and parallel to  $Q$ . This does not depend on the choice of  $A$ .

**Lemma 3.12.** *One has  $A \cap \mathbb{A} = \overline{\text{Int}(A \cap \mathbb{A})}$ .*

*Proof.* By Proposition 3.9,  $A \cap \mathbb{A}$  is closed and thus  $\overline{\text{Int}(A \cap \mathbb{A})} \subset A \cap \mathbb{A}$ .

Let  $x \in A \cap \mathbb{A}$ . By (MA4)  $x + \infty \subset A \cap \mathbb{A}$ . The fact that there exists  $(x_n) \in \text{Int}(x + \infty)^{\mathbb{N}}$  such that  $x_n \rightarrow x$  proves the lemma.  $\square$

**Lemma 3.13.** *Let  $J = \{i \in \llbracket 1, k \rrbracket \mid \text{Int}_{\mathbb{A}}(P_i \cap A \cap \mathbb{A}) \neq \emptyset\}$ . Then  $A \cap \mathbb{A} = \bigcup_{j \in J} P_j$ .*

*Proof.* Let  $U = A \cap \mathbb{A}$ . Then by Lemma 3.12 and Lemma 3.10,  $U = \bigcup_{j \in J} U \cap P_j$  and Lemma 3.11 completes the proof.  $\square$

We no more suppose that  $A$  contains  $+\infty$ . We say that  $\bigcup_{i=1}^k P_i$  is a **decomposition of  $A \cap \mathbb{A}$  in enclosed subsets** if:

1.  $k \in \mathbb{N}$  and for all  $i \in \llbracket 1, k \rrbracket$ ,  $P_i$  is enclosed



2.  $A \cap \mathbb{A} = \bigcup_{i=1}^k P_i$
3. for all  $i \in \llbracket 1, k \rrbracket$ , there exists an isomorphism  $\phi_i : \mathbb{A} \xrightarrow{P_i} A$ .

**Proposition 3.14.** *Let  $A$  be an apartment. Then there exists a decomposition  $\bigcup_{i=1}^k P_i$  of  $A \cap \mathbb{A}$  in enclosed subsets.*

*As a consequence there exists a finite set  $\mathcal{M}$  of walls such that  $\text{Fr}(A \cap \mathbb{A}) \subset \bigcup_{M \in \mathcal{M}} M$ .*

*If moreover  $A \cap \mathbb{A}$  is convex, one has  $A \cap \mathbb{A} = \bigcup_{j \in J} P_j$ , where  $J = \{j \in \llbracket 1, k \rrbracket \mid \text{supp}(P_j) = \text{supp}(A \cap \mathbb{A})\}$ .*

*Proof.* Let  $n \in \mathbb{N}$  and  $\mathcal{P}_n$ : “for all apartment  $B$  such that  $d(B, \mathbb{A}) \leq n$ , there exists a decomposition  $\bigcup_{i=1}^l Q_i$  of  $\mathbb{A} \cap B$  in enclosed subsets”. The property  $\mathcal{P}_0$  is true by Lemma 3.13. Let  $n \in \mathbb{N}$  and suppose that  $\mathcal{P}_n$  is true. Suppose that there exists an apartment  $B$  such that  $d(B, \mathbb{A}) = n + 1$ . Using Proposition 3.2, one writes  $B = D_1 \cup D_2$  where  $D_1, D_2$  are opposite half-apartments such that for all  $i \in \{1, 2\}$ ,  $D_i$  is included in a apartment  $B_i$  satisfying  $d(B_i, \mathbb{A}) = n$ . If  $i \in \{1, 2\}$ , one writes  $B_i \cap \mathbb{A} = \bigcup_{j=1}^{l_i} Q_j^i$ , where  $l_i \in \mathbb{N}$ , the  $Q_j^i$ ’s are enclosed and there exists an isomorphism  $\psi_j^i : B_i \xrightarrow{Q_j^i} \mathbb{A}$ . Then  $B \cap \mathbb{A} = \bigcup_{j=1}^{l_1} (D_1 \cap Q_j^1) \cup \bigcup_{j=1}^{l_2} (D_2 \cap Q_j^2)$ . If  $i \in \{1, 2\}$ , one denotes by  $f^i$  the isomorphism  $B \xrightarrow{D_i} B_i$ . Then if  $j \in \llbracket 1, l_i \rrbracket$ ,  $\psi_j^i \circ f^i$  fixes  $Q_j^i \cap D_i$  and thus  $\mathcal{P}_{n+1}$  is true.

Therefore  $A \cap \mathbb{A} = \bigcup_{i=1}^k P_i$  where the  $P_i$ ’s are enclosed. One has  $\text{Fr}(A \cap \mathbb{A}) \subset \bigcup_{i=1}^k \text{Fr}(P_i)$ , which is included in a finite union of walls.

Suppose that  $A \cap \mathbb{A}$  is convex. Let  $X = \text{supp}(A \cap \mathbb{A})$ . By Lemma 3.10 applied with  $U = A \cap \mathbb{A}$ ,

$$A \cap \mathbb{A} = \bigcup_{i \in \llbracket 1, k \rrbracket, \text{Int}_X(P_i) \neq \emptyset} P_i,$$

which completes the proof. □

### 3.3 Enclosedness of a convex intersection

In this subsection, we prove Proposition 3.22: if  $A$  is an apartment such that  $A \cap \mathbb{A}$  is convex, then  $A \cap \mathbb{A}$  is enclosed. For this we study the “gauge” of  $A \cap \mathbb{A}$ , which is a map parameterizing the frontier of  $A \cap \mathbb{A}$ .

**Lemma 3.15.** *Let  $A$  be a finite dimensional affine space,  $k \in \mathbb{N}^*$  and  $D_1, \dots, D_k$  be half-spaces of  $A$  and  $M_1, \dots, M_k$  be their hyperplanes. Then there exists  $J \subset \llbracket 1, k \rrbracket$  (maybe empty) such that  $\text{supp}(\bigcap_{i=1}^k D_i) = \bigcap_{j \in J} M_j$*

*Proof.* Let  $d \in \mathbb{N}^*$  and  $l \in \mathbb{N}$ . Let  $\mathcal{P}_{d,l}$ : “for all affine space  $X$  such that  $\dim X \leq d$  and for all half-spaces  $E_1, \dots, E_l$  of  $X$ , there exists  $J \subset \llbracket 1, l \rrbracket$  such that  $\text{supp}(\bigcap_{i=1}^l E_i) = \bigcap_{j \in J} H_j$  where for all  $j \in J$ ,  $H_j$  is the hyperplane of  $E_j$ ”.

It is clear that for all  $l \in \mathbb{N}$ ,  $\mathcal{P}_{1,l}$  is true and that for all  $d \in \mathbb{N}$ ,  $\mathcal{P}_{d,0}$  and  $\mathcal{P}_{d,1}$  is true. Let  $d \in \mathbb{N}_{\geq 2}$  and  $l \in \mathbb{N}$  and suppose that (for all  $d' \leq d - 1$  and  $l' \in \mathbb{N}$ ,  $\mathcal{P}_{d',l'}$  is true) and that (for all  $l' \in \llbracket 0, l \rrbracket$ ,  $\mathcal{P}_{d,l'}$  is true).

Let  $X$  be a  $d$  dimensional affine space,  $E_1, \dots, E_{l+1}$  be half-spaces of  $X$  and  $H_1, \dots, H_{l+1}$  be their hyperplanes. Let  $L = \bigcap_{j=1}^l E_j$  and  $S = \text{supp} L$ . Then  $E_{l+1} \cap S$  is either  $S$  or a half-space of  $S$ . In the first case,  $E_{l+1} \supset S \supset L$ , thus  $\bigcap_{i=1}^{l+1} E_i = L$  and thus by  $\mathcal{P}_{d,l}$ ,  $\text{supp}(\bigcap_{i=1}^{l+1} E_i) = \bigcap_{j \in J} H_j$  for some  $J \subset \llbracket 1, l \rrbracket$ .

Suppose that  $E_{l+1} \cap S$  is a half-space of  $S$ . Then either  $E_{l+1}^\circ \cap L \neq \emptyset$  or  $E_{l+1}^\circ \cap L = \emptyset$ . In the first case, one chooses  $x \in E_{l+1}^\circ \cap L$  and a sequence  $(x_n) \in (\text{Int}_r(L))^{\mathbb{N}}$  converging towards

$x$ . Then for  $n \gg 0$ ,  $x_n \in E_{l+1}^\circ \cap \text{Int}_r(L)$ . Consequently,  $L \cap E_{l+1}$  has nonempty interior in  $S$ . Thus  $\text{supp}(\bigcap_{i=1}^{l+1} E_i) = S$  and by  $\mathcal{P}_{d,l}$ ,  $\text{supp}(\bigcap_{i=1}^{l+1} E_i) = \bigcap_{j \in J} H_j$  for some  $J \subset \llbracket 1, l \rrbracket$ .

Suppose now that  $E_{l+1}^\circ \cap L$  is empty. Then  $L \cap E_{l+1} \subset H_{l+1}$ , where  $H_{l+1}$  is the hyperplane of  $E_{l+1}$ . Therefore  $\bigcap_{i=1}^{l+1} E_i = \bigcap_{i=1}^{l+1} (E_i \cap H_{l+1})$  and thus by  $\mathcal{P}_{d-1, l+1}$ ,  $\text{supp}(\bigcap_{i=1}^{l+1} E_i) = \bigcap_{j \in J} H_j$  for some  $J \subset \llbracket 1, l+1 \rrbracket$ . □

**Lemma 3.16.** *Let  $A$  be an apartment such that  $A \cap \mathbb{A}$  is convex. Then  $\text{supp}(A \cap \mathbb{A})$  is enclosed.*

*Proof.* Using Proposition 3.14, one writes  $A \cap \mathbb{A} = \bigcup_{i=1}^k P_i$ , where the  $P_i$ 's are enclosed and  $\text{supp}(P_i) = \text{supp}(A \cap \mathbb{A})$  for all  $i \in \llbracket 1, k \rrbracket$ . By Lemma 3.15, if  $i \in \llbracket 1, k \rrbracket$ ,  $\text{supp}(P_i)$  is a finite intersection of walls, which proves the lemma. □

**Gauge of a convex** Let  $A$  be a finite dimensional affine space. Let  $C$  be a closed and convex subset of  $A$  with nonempty interior. One chooses  $x \in \overset{\circ}{C}$  and one fixes the origin of  $A$  in  $x$ . Let  $j_{C,x} : A \rightarrow \mathbb{R}_+ \cup \{+\infty\}$  defined by  $j_{C,x}(s) = \inf\{t \in \mathbb{R}_+^* \mid s \in tC\}$ . The map  $j_{C,x}$  is called the **gauge** of  $C$  based at  $x$ . In the sequel, we will fix some  $x \in \overset{\circ}{C}$  and we will denote  $j_C$  instead of  $j_{C,x}$ . Then by Theorem 1.2.5 of [HUL12] and discussion at the end of Section 1.2 of loc cit,  $j_C(A) \subset \mathbb{R}_+$  and  $j_C$  is continuous.

The following lemma is easy to prove:

**Lemma 3.17.** *Let  $C$  be a convex closed set with nonempty interior. Fix the origin of  $A$  in a point of  $\overset{\circ}{C}$ . Then  $C = \{x \in A \mid j_C(x) \leq 1\}$  and  $\overset{\circ}{C} = \{x \in A \mid j_C(x) < 1\}$ .*

**Lemma 3.18.** *Let  $C$  be a convex closed set with nonempty interior. Fix the origin of  $A$  in  $\overset{\circ}{C}$ . Let  $U = U_C = \{s \in A \mid j_C(s) \neq 0\}$ . Let  $\text{Fr} = \text{Fr}_C : U \rightarrow \text{Fr}(C)$  defined by  $\text{Fr}(s) = \frac{s}{j_C(s)}$  for all  $s \in U$ . Then  $\text{Fr}$  is well defined, continuous and surjective.*

*Proof.* If  $s \in U$ , then  $j_C(\text{Fr}(s)) = \frac{j_C(s)}{j_C(s)} = 1$  and thus  $\text{Fr}$  takes its values in  $\text{Fr}(C)$  by Lemma 3.17. The continuity of  $\text{Fr}$  is a consequence of the one of  $j_C$ .

Let  $f \in \text{Fr}(C)$ . Then  $\text{Fr}(f) = f$  and thus  $\text{Fr}$  is surjective. □

Let  $A$  be an apartment such that  $A \cap \mathbb{A}$  is convex and nonempty. Let  $X$  be the support of  $A \cap \mathbb{A}$  in  $\mathbb{A}$ . By Lemma 3.16, if  $A \cap \mathbb{A} = X$ , then  $A \cap \mathbb{A}$  is enclosed. One now supposes that  $A \cap \mathbb{A} \neq X$ . One chooses  $x_0 \in \text{Int}_X(A \cap \mathbb{A})$  and consider it as the origin of  $\mathbb{A}$ . One defines  $U = U_{A \cap \mathbb{A}}$  and  $\text{Fr} : U \rightarrow \text{Fr}_r(A \cap \mathbb{A})$  as in Lemma 3.18. The set  $U$  is open and nonempty. Using Proposition 3.14, one writes  $A \cap \mathbb{A} = \bigcup_{i=1}^r P_i$ , where  $r \in \mathbb{N}$ , the  $P_i$ 's are enclosed and  $\text{supp}(P_i) = X$  for all  $i \in \llbracket 1, r \rrbracket$ . Let  $M_1, \dots, M_k$  be distinct walls not containing  $X$  such that  $\text{Fr}_r(A \cap \mathbb{A}) \subset \bigcup_{i=1}^k M_i$ , which exists because the  $P_i$ 's are intersections of half-spaces of  $X$  and  $A \cap \mathbb{A} \neq X$ . Let  $\mathcal{M} = \{M_i \cap X \mid i \in \llbracket 1, k \rrbracket\}$ . If  $M \in \mathcal{M}$ , one sets  $U_M = \text{Fr}^{-1}(M)$ .

**Lemma 3.19.** *Let  $U' = \{x \in U \mid \exists (M, V) \in \mathcal{M} \times \mathcal{V}_U(x) \mid \text{Fr}(V) \subset M\}$ . Then  $U'$  is dense in  $U$ .*

*Proof.* Let  $M \in \mathcal{M}$ . By Lemma 3.18,  $U_M$  is closed in  $U$ . Let  $V' \subset U$  be nonempty and open. Then  $V' = \bigcup_{M \in \mathcal{M}} U_M \cap V'$ . By Baire's Theorem, there exists  $M \in \mathcal{M}$  such that  $V' \cap U_M$  has nonempty interior and hence  $U'$  is dense in  $U$ . □

**Lemma 3.20.** *Let  $x \in U'$  and  $V \in \mathcal{V}_U(x)$  such that  $\text{Fr}(V) \subset M$  for some  $M \in \mathcal{M}$ . The wall  $M$  is unique and does not depend on  $V$ .*

*Proof.* Suppose that  $\text{Fr}(V) \subset M \cap M'$ , where  $M, M'$  are hyperplanes of  $X$ . Let  $\alpha, \alpha' \in \Phi$ ,  $k, k' \in \mathbb{R}$  such that  $M = \alpha^{-1}(\{k\})$  and  $M' = \alpha'^{-1}(\{k'\})$ . By definition of  $U$ , for all  $y \in V$ ,  $\text{Fr}(y) = \lambda(y)y$  for some  $\lambda(y) \in \mathbb{R}_+^*$ . Suppose that  $k = 0$ . Then  $\alpha(y) = 0$  for all  $y \in V$ , which is absurd because  $\alpha \neq 0$ . By the same reasoning,  $k' \neq 0$ .

If  $y \in V \setminus (\alpha^{-1}(\{0\}) \cup \alpha'^{-1}(\{0\}))$ ,  $\text{Fr}(y) = \lambda(y)y$  for some  $\lambda(y) \in \mathbb{R}_+^*$  and thus  $\text{Fr}(y) = \frac{k}{\alpha(y)}y = \frac{k'}{\alpha'(y)}y$ . As  $V \setminus (\alpha^{-1}(\{0\}) \cup \alpha'^{-1}(\{0\}))$  is dense in  $V$ ,  $k\alpha'(y) - k'\alpha(y) = 0$  for all  $y \in V$  and thus  $M$  and  $M'$  are parallel. Therefore  $M = M'$ . It remains to show that  $M$  does not depend on  $V$ . Let  $V_1 \in \mathcal{V}_U(x)$  such that  $\text{Fr}(V_1) \subset M_1$  for some  $M_1 \in \mathcal{M}$ . By the unicity we just proved applied to  $V \cap V_1$ ,  $M = M_1$ , which completes the proof.  $\square$

If  $x \in U'$ , one denotes by  $M_x$  the wall defined by Lemma 3.20.

**Lemma 3.21.** *Let  $x \in U'$  and  $D_1, D_2$  be the two half-spaces of  $X$  defined by  $M_x$ . Then  $A \cap \mathbb{A} \subset D_i$ , for some  $i \in \{1, 2\}$ .*

*Proof.* Let  $V \in \mathcal{V}_U(x)$  such that  $\text{Fr}(V) \subset M_x$ . Let us prove that  $\text{Fr}(V) = \mathbb{R}_+^*V \cap M_x$ . As  $\text{Fr}(y) \in \mathbb{R}_+^*y$  for all  $y \in V$ ,  $\text{Fr}(V) \subset \mathbb{R}_+^*V \cap M_x$ . Let  $f$  be a linear form on  $X$  such that  $M_x = f^{-1}(\{k\})$  for some  $k \in \mathbb{R}$ . If  $k = 0$ , then for all  $v \in V$ ,  $f(v) = 0$ , and thus  $f = 0$ : this is absurd and  $k \neq 0$ . Let  $a \in \mathbb{R}_+^*V \cap M_x$ . One has  $a = \lambda \text{Fr}(v)$ , for some  $\lambda \in \mathbb{R}_+^*$  and  $v \in V$ . Moreover  $f(\text{Fr}(v)) = k = f(a)$  and as  $k \neq 0$ ,  $a = \text{Fr}(v) \in \text{Fr}(V)$ . Thus  $\text{Fr}(V) = \mathbb{R}_+^*V \cap M_x$  and  $\text{Fr}(V)$  is an open set of  $M_x$ . Suppose there exists  $(x_1, x_2) \in (\overset{\circ}{D}_1 \cap A \cap \mathbb{A}) \times (\overset{\circ}{D}_2 \cap A \cap \mathbb{A})$ . Then  $\text{conv}(x_1, x_2, \text{Fr}(V)) \subset A \cap \mathbb{A}$  is an open neighborhood of  $\text{Fr}(V)$  in  $X$ . This is absurd because  $\text{Fr}$  takes its values in  $\text{Fr}_r(A \cap \mathbb{A})$ . Thus the lemma is proved.  $\square$

If  $x \in U'$ , one denotes by  $D_x$  the half-space delimited by  $M_x$  and containing  $A \cap \mathbb{A}$ .

**Proposition 3.22.** *Let  $A$  be an apartment such that  $A \cap \mathbb{A}$  is convex. Then  $A \cap \mathbb{A}$  is enclosed.*

*Proof.* If  $u \in U'$ , then  $A \cap \mathbb{A} \subset D_u$  and thus  $A \cap \mathbb{A} \subset \bigcap_{u \in U'} D_u$ .

Let  $x \in U' \cap \bigcap_{u \in U'} D_u$ . One has  $0 \in A \cap \mathbb{A}$  and thus  $0 \in D_x$ . Moreover  $\text{Fr}(x) \in M_x \cap A \cap \mathbb{A}$  and thus  $x \in [0, \text{Fr}(x)] \subset A \cap \mathbb{A}$ . Therefore  $U' \cap \bigcap_{x \in U'} D_x \subset A \cap \mathbb{A}$ .

Let  $x \in \text{Int}_X(\bigcap_{u \in U'} D_u)$ . If  $x \notin U$ , then  $x \in A \cap \mathbb{A}$ . Suppose  $x \in U$ . Then by Lemma 3.19, there exists  $(x_n) \in (U' \cap \text{Int}_X(\bigcap_{u \in U'} D_u))^{\mathbb{N}}$  such that  $x_n \rightarrow x$ . But then for all  $n \in \mathbb{N}$ ,  $x_n \in A \cap \mathbb{A}$  and by Proposition 3.9,  $x \in A \cap \mathbb{A}$ . As a consequence,  $A \cap \mathbb{A} \supset \text{Int}_X(\bigcap_{u \in U'} D_u)$  and as  $A \cap \mathbb{A}$  is closed,  $A \cap \mathbb{A} \supset \overline{\text{Int}_X(\bigcap_{u \in U'} D_u)} = \bigcap_{u \in U'} D_u$  because  $\bigcap_{u \in U'} D_u$  is closed, convex with nonempty interior in  $X$ . Thus we have proved  $A \cap \mathbb{A} = \bigcap_{u \in U'} D_u$ .

Let  $M'_1, \dots, M'_k$  be walls of  $\mathbb{A}$  such that for all  $x \in U'$ , there exists  $i(x) \in \llbracket 1, k \rrbracket$  such that  $M'_{i(x)} \cap X = M_x$ . One sets  $M'_x = M'_{i(x)}$  for all  $x \in U'$  and one denotes by  $D'_x$  the half-apartment of  $\mathbb{A}$  delimited by  $M'_x$  and containing  $D_x$ . Then  $X \cap \bigcap_{x \in U'} D'_x = A \cap \mathbb{A}$ . Lemma 3.16 completes the proof.  $\square$

### 3.4 Existence of isomorphisms of apartments fixing a convex set

In this section, we study, if  $A$  is an apartment and  $P \subset \mathbb{A} \cap A$ , the existence of isomorphisms of apartments  $\mathbb{A} \xrightarrow{P} A$ . We give a sufficient condition of existence of such an isomorphism in Proposition 3.26. The existence of an isomorphism  $A \xrightarrow{A \cap \mathbb{A}} \mathbb{A}$  when  $A$  and  $\mathbb{A}$  share a generic ray will be a particular case of this Proposition, see Theorem 4.22. In the affine case, this will be a first step to prove that for all apartment  $A$ , there exists an isomorphism  $A \xrightarrow{A \cap \mathbb{A}} \mathbb{A}$ .

**Lemma 3.23.** *Let  $A$  be an apartment of  $\mathcal{I}$  and  $\phi : \mathbb{A} \rightarrow A$  be an isomorphism of apartments. Let  $P \subset \mathbb{A} \cap A$  be a nonempty relatively open convex set,  $Z = \text{supp}(P)$  and suppose that  $\phi$  fixes  $P$ . Then  $\phi$  fixes  $P + (\mathcal{T} \cap \vec{Z}) \cap A$ , where  $\mathcal{T}$  is the Tits cone.*

*Proof.* Let  $x \in P + (\mathcal{T} \cap \vec{Z}) \cap A$ ,  $x = p + t$ , where  $p \in P$  and  $t \in \mathcal{T}$ . Suppose  $t \neq 0$ . Let  $L = p + \mathbb{R}t$ . Then  $L$  is a preordered line in  $\mathcal{I}$  and  $\phi$  fixes  $L \cap P$ . Moreover,  $p \leq x$  and thus by Proposition 5.4 of [Rou11], there exists an isomorphism  $\psi : \mathbb{A} \xrightarrow{[p,x]} A$ . In particular,  $\phi^{-1} \circ \psi : \mathbb{A} \rightarrow \mathbb{A}$  fixes  $L \cap P$ . But then  $\phi^{-1} \circ \psi|_L$  is an affine isomorphism fixing a nonempty open set of  $L$ : this is the identity. Therefore  $\phi^{-1} \circ \psi(x) = x = \phi^{-1}(x)$ , which shows the lemma.  $\square$

**Lemma 3.24.** *Let  $A$  be an apartment of  $\mathcal{I}$ . Let  $U \subset \mathbb{A} \cap A$  be a nonempty relatively open set and  $X = \text{supp}(U)$ . Then there exists a nonempty open subset  $V$  of  $U$  (in  $X$ ) such that there exists an isomorphism  $\phi : \mathbb{A} \xrightarrow{V} A$ .*

*Proof.* Let  $\bigcup_{i=1}^k P_i$  be a decomposition in enclosed subsets of  $A \cap \mathbb{A}$ . Let  $i \in \llbracket 1, k \rrbracket$  such that  $P_i \cap U$  has nonempty interior in  $X$  and  $\phi : \mathbb{A} \xrightarrow{P_i} A$ . Then  $\phi$  fixes a nonempty open set of  $U$ , which proves the lemma.  $\square$

**Lemma 3.25.** *Let  $A$  be an apartment of  $\mathcal{I}$  and  $\phi : \mathbb{A} \rightarrow A$  be an isomorphism. Let  $F = \{z \in A \mid \phi(z) = z\}$ . Then  $F$  is closed in  $\mathbb{A}$ .*

*Proof.* By Proposition 3.8,  $\rho_{+\infty} \circ \phi : \mathbb{A} \rightarrow \mathbb{A}$  and  $\rho_{-\infty} \circ \phi : \mathbb{A} \rightarrow \mathbb{A}$  are continuous. Let  $(z_n) \in F^{\mathbb{N}}$  such that  $(z_n)$  converges in  $\mathbb{A}$  and  $z = \lim z_n$ .

For all  $n \in \mathbb{N}$ , one has  $\rho_{+\infty}(\phi(z_n)) = z_n = \rho_{-\infty}(\phi(z_n)) \rightarrow \rho_{+\infty}(\phi(z)) = z = \rho_{-\infty}(\phi(z))$  and by Proposition 3.7,  $z = \phi(z)$ , which proves the lemma.  $\square$

**Proposition 3.26.** *Let  $A$  be an apartment of  $\mathcal{I}$  and  $P \subset \mathbb{A} \cap A$  be a convex set. Let  $X = \text{supp}(P)$  and suppose that  $\mathcal{T} \cap \vec{X}$  has nonempty interior in  $\vec{X}$ . Then there exists an isomorphism of apartments  $\phi : \mathbb{A} \xrightarrow{P} A$ .*

*Proof.* (see Figure 3.1) Let  $V \subset P$  be a nonempty open set of  $X$  such that there exists an isomorphism  $\phi : \mathbb{A} \xrightarrow{V} A$  (such a  $V$  exists by Lemma 3.24). Let us show that  $\phi$  fixes  $\text{Int}_r(P)$ .

Let  $x \in V$ . One fixes the origin of  $\mathbb{A}$  in  $x$  and thus  $X$  is a vector space. Let  $(e_j)_{j \in J}$  be a basis of  $\mathbb{A}$  such that for some subset  $J' \subset J$ ,  $(e_j)_{j \in J'}$  is a basis of  $X$  and  $(x + \mathcal{T}) \cap X \supset \bigoplus_{j \in J'} \mathbb{R}_+^* e_j$ . For all  $y \in X$ ,  $y = \sum_{j \in J'} y_j e_j$  with  $y_j \in \mathbb{R}$  for all  $j \in J'$ , one sets  $|y| = \max_{j \in J'} |y_j|$ . If  $a \in A$  and  $r > 0$ , one sets  $B(a, r) = \{y \in X \mid |y - a| < r\}$ .

Suppose that  $\phi$  does not fix  $\text{Int}_r(P)$ . Let  $y \in \text{Int}_r(P)$  such that  $\phi(y) \neq y$ . Let  $s = \sup\{t \in [0, 1] \mid \exists U \in \mathcal{V}_X([0, ty]) \mid \phi \text{ fixes } U\}$  and  $z = sy$ . Then by Lemma 3.25,  $\phi(z) = z$ .

By definition of  $z$ , for all  $r > 0$ ,  $\phi$  does not fix  $B(z, r)$ . Let  $r > 0$  such that  $B(z, 5r) \subset \text{Int}_r P$ . Let  $z_1 \in B(z, r) \cap [0, z]$  and  $r_1 > 0$  such that  $\phi$  fixes  $B(z_1, r_1)$  and  $z'_2 \in B(z, r)$  such that  $\phi(z'_2) \neq z'_2$ . Let  $r'_2 \in (0, r)$  such that for all  $a \in B(z'_2, r'_2)$ ,  $\phi(z) \neq z$ . Let  $z_2 \in B(z'_2, r'_2)$  such that for some  $r_2 \in (0, r'_2)$ ,  $B(z_2, r_2) \subset B(z'_2, r'_2)$  and such that there exists an isomorphism  $\psi : \mathbb{A} \xrightarrow{B(z_2, r_2)} A$  (such  $z_2$  and  $r_2$  exists by Lemma 3.24). Then  $|z_1 - z_2| < 3r$ .

Let us prove that  $(z_1 + \mathcal{T} \cap X) \cap (z_2 + \mathcal{T} \cap X) \cap \text{Int}_r(P)$  contains a nonempty open set  $U \subset X$ . One identifies  $X$  and  $\mathbb{R}^{J'}$  thanks to the basis  $(e_j)_{j \in J'}$ . One has  $z_2 - z_1 \in (-3, 3)^{J'}$  and thus  $(z_1 + \mathcal{T}) \cap (z_2 + \mathcal{T}) = (z_1 + \mathcal{T}) \cap (z_1 + z_2 - z_1 + \mathcal{T}) \supset z_1 + (3, 4)^{J'}$ . As  $P \supset B(z_1, 4r)$ ,  $(z_1 + \mathcal{T} \cap X) \cap (z_2 + \mathcal{T} \cap X) \cap \text{Int}_r(P)$  contains a nonempty open set  $U \subset X$ .

By Lemma 3.23,  $\phi$  and  $\psi$  fix  $U$ . Therefore,  $\phi^{-1} \circ \psi$  fixes  $U$  and as it is an isomorphism of affine space of  $A$ ,  $\phi^{-1} \circ \psi$  fixes  $X$ . Therefore  $\phi^{-1} \circ \psi(z_2) = \phi^{-1}(z_2) = z_2$  and thus  $\phi(z_2) = z_2$ : this is absurd. Hence  $\phi$  fixes  $\text{Int}_r(P)$ . By Lemma 3.25,  $\phi$  fixes  $\text{Int}_r(P) = \overline{P}$  and thus  $\phi$  fixes  $P$ , which shows the proposition.  $\square$

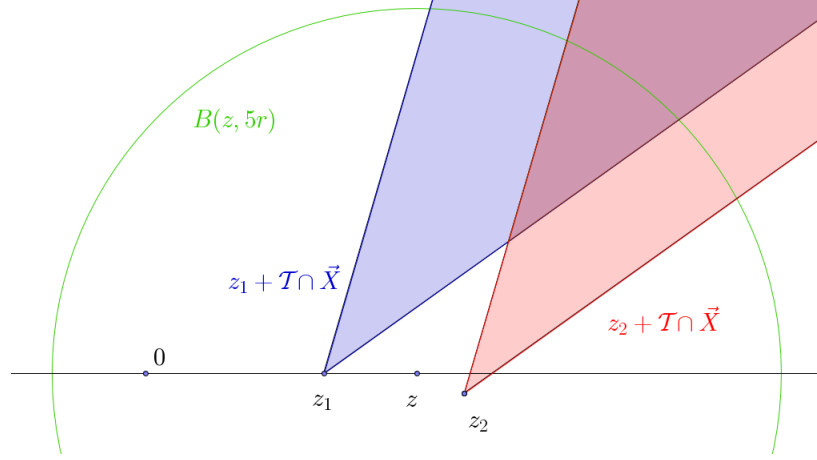


Figure 3.1 – Proof of Proposition 3.26

## 4 Intersection of two apartments sharing a generic ray

The aim of this section is to prove Theorem 4.22: let  $A$  and  $B$  be two apartments sharing a generic ray. Then  $A \cap B$  is enclosed and there exists an isomorphism  $\phi : A \xrightarrow{A \cap B} B$ .

We first reduce our study to the case where  $A \cap B$  has nonempty interior by the following lemma:

**Lemma 4.1.** *Suppose that for all apartments  $A, B$  such that  $A \cap B$  contains a generic ray and has nonempty interior,  $A \cap B$  is convex. Then if  $A_1$  and  $A_2$  are two apartments containing a generic ray,  $A_1 \cap A_2$  is enclosed and there exists an isomorphism  $\phi : A_1 \xrightarrow{A_1 \cap A_2} A_2$ .*

*Proof.* Let us prove that  $A_1 \cap A_2$  is convex. Let  $\delta$  be the direction of a generic ray shared by  $A_1$  and  $A_2$ . Let  $x_1, x_2 \in A_1 \cap A_2$  and  $\mathfrak{F}^\infty$  be the vectorial face direction containing  $\delta$ . Let  $\mathfrak{F}'^\infty$  be the vectorial face direction of  $A_1$  opposite to  $\mathfrak{F}^\infty$ . Let  $C_1$  be a chamber of  $A_1$  containing  $x_1$ ,  $C_2$  be a chamber of  $A_2$  containing  $x_2$ ,  $\mathfrak{r}_1 = \mathfrak{r}(C_1, \mathfrak{F}'^\infty) \subset A_1$ ,  $\mathfrak{r}_2 = \mathfrak{r}(C_2, \mathfrak{F}^\infty) \subset A_2$ ,  $\mathfrak{R}_1 = \text{germ}(\mathfrak{r}_1)$  and  $\mathfrak{R}_2 = \text{germ}(\mathfrak{r}_2)$ . By (MA3) there exists an apartment  $A_3$  containing  $\mathfrak{R}_1$  and  $\mathfrak{R}_2$ .

Let us prove that  $A_3$  contains  $x_1$  and  $x_2$ . One identifies  $A_1$  and  $\mathbb{A}$ . Let  $F^v = 0 + \mathfrak{F}^\infty$  and  $F'^v = 0 + \mathfrak{F}'^\infty$ . As  $A_3 \supset \mathfrak{R}_1$ , there exists  $f' \in F'^v$  such that  $A_3 \supset x_1 + f' + F'^v$ . Moreover  $A_3 \supset \mathfrak{F}^\infty$  and thus it contains  $x_1 + f' + \mathfrak{F}^\infty$ . By Proposition 4.7.1 of [Rou11]  $x_1 + f' + \mathfrak{F}^\infty = x_1 + f' + F^v$  and thus  $A_3 \ni x_1$ . As  $A_3 \supset \mathfrak{R}_2$ , there exists  $f \in F^v$  such that  $A_3 \supset x_2 + f$ . As  $A_3 \supset \mathfrak{F}'^\infty$ ,  $A_3 \supset x_2 + f + \mathfrak{F}'^\infty = x_2 + f + F'^v$  by Proposition 4.7.1 of [Rou11] and thus  $A_3 \ni x_2$ .

If  $i \in \{1, 2\}$ , each element of  $\mathfrak{R}_i$  has nonempty interior in  $A_i$  and thus  $A_i \cap A_3$  has nonempty interior. By hypothesis,  $A_1 \cap A_3$  and  $A_2 \cap A_3$  are convex. By Proposition 3.26, there exist  $\phi : A_1 \xrightarrow{A_1 \cap A_3} A_3$  and  $\psi : A_2 \xrightarrow{A_2 \cap A_3} A_3$ . Therefore  $[x_1, x_2]_{A_1} = [x_1, x_2]_{A_3} = [x_1, x_2]_{A_2}$  and thus  $A_1 \cap A_2$  is convex.

The existence of an isomorphism  $A_1 \xrightarrow{A_1 \cap A_2} A_2$  is a consequence of Proposition 3.26 because the direction  $X$  of  $A_1 \cap A_2$  meets  $\mathcal{T}$  and thus  $\vec{X} \cap \mathcal{T}$  spans  $\mathcal{T}$ .

The fact that  $A_1 \cap A_2$  is enclosed is a consequence of Proposition 3.22.  $\square$

### 4.1 Definition of the frontier maps

The aim of 4.1 to 4.5 is to prove that if  $A$  and  $B$  are two apartments containing a generic ray and such that  $A \cap B$  has nonempty interior,  $A \cap B$  is convex. There is no loss of generality

in assuming that  $B = \mathbb{A}$  and that the direction  $\mathbb{R}_+\nu$  of  $\delta$  is included in  $\pm\overline{C_f^v}$ . As the roles of  $C_f^v$  and  $-C_f^v$  are similar, one supposes that  $\mathbb{R}_+\nu \subset \overline{C_f^v}$  and that  $A \neq \mathbb{A}$ . These hypothesis run until the end of 4.5.

In this subsection, we parametrize  $\text{Fr}(A \cap \mathbb{A})$  by a map and describe  $A \cap \mathbb{A}$  using the values of this map.

**Lemma 4.2.** *Let  $V$  be a bounded subset of  $\mathbb{A}$ . Then there exists  $a \in \mathbb{R}$  such that for all  $u \in [a, +\infty[$  and  $v \in V$ ,  $v \leq u\nu$ .*

*Proof.* Let  $a \in \mathbb{R}_+^*$  and  $v \in V$ , then  $av - v = a(\nu - \frac{1}{a}v)$ . As  $\nu \in \overset{\circ}{\mathcal{T}}$  and  $V$  is bounded, there exists  $b > 0$  such that for all  $a > b$ ,  $\nu - \frac{1}{a}v \in \overset{\circ}{\mathcal{T}}$ , which proves the lemma because  $\overset{\circ}{\mathcal{T}}$  is a cone.  $\square$

**Lemma 4.3.** *Let  $y \in A \cap \mathbb{A}$ . Then  $A \cap \mathbb{A}$  contains  $y + \mathbb{R}_+\nu$ .*

*Proof.* Let  $x \in \mathbb{A}$  such that  $A \cap \mathbb{A} \supset x + \mathbb{R}_+\nu$ . The ray  $x + \mathbb{R}_+\nu$  is generic and by (MA4), if  $y \in \mathbb{A}$ ,  $A \cap \mathbb{A}$  contains the convex hull of  $y$  and  $x + [a, +\infty[\nu$ , for some  $a \gg 0$ . In particular it contains  $y + \mathbb{R}_+\nu$ .  $\square$

$$\text{Let } U = \{y \in \mathbb{A} \mid y + \mathbb{R}\nu \cap A \neq \emptyset\} = (A \cap \mathbb{A}) + \mathbb{R}\nu.$$

**Lemma 4.4.** *The set  $U$  is convex.*

*Proof.* Let  $u, v \in U$ . Let  $u' \in u + \mathbb{R}_+\nu \cap A$ . By Lemma 4.2 and Lemma 4.3, there exists  $v' \in v + \mathbb{R}_+\nu$  such that  $u' \leq v'$ . By conséquence 2) of Proposition 5.4 of [Rou11],  $[u', v'] \subset A \cap \mathbb{A}$ . By definition of  $U$ ,  $[u', v'] + \mathbb{R}\nu \subset U$  and in particular  $[u, v] \subset U$ , which is the desired conclusion.  $\square$

There are two possibilities: either there exists  $y \in \mathbb{A}$  such that  $y + \mathbb{R}\nu \subset A$  or for all  $y \in \mathbb{A}$ ,  $y + \mathbb{R}\nu \not\subset A$ . The first case is the easiest and we treat it in the next lemma.

**Lemma 4.5.** *Suppose that for some  $y \in \mathbb{A}$ ,  $y - \mathbb{R}_+\nu \subset A \cap \mathbb{A}$ . Then  $A \cap \mathbb{A} = U$ . In particular,  $A \cap \mathbb{A}$  is convex.*

*Proof.* By Lemma 4.3,  $A \cap \mathbb{A} = (A \cap \mathbb{A}) + \mathbb{R}_+\nu$ . By symmetry and by hypothesis on  $A \cap \mathbb{A}$ ,  $(A \cap \mathbb{A}) + \mathbb{R}_-\nu = A \cap \mathbb{A}$ . Therefore  $A \cap \mathbb{A} = (A \cap \mathbb{A}) + \mathbb{R}\nu = U$ .  $\square$

**Definition of the frontier** Let  $u \in U$ . Then by Lemma 4.3,  $u + \mathbb{R}\nu \cap A$  is of the form  $a + \mathbb{R}_+\nu$  or  $a + \mathbb{R}_+\nu$  for some  $a \in \mathbb{A}$ . As  $A \cap \mathbb{A}$  is closed (by Proposition 3.9), the first case cannot occur. One sets  $\text{Fr}_\nu(u) = a \in \mathbb{A} \cap A$ . One fixes  $\nu$  until the end of 4.5 and one writes  $\text{Fr}$  instead of  $\text{Fr}_\nu$ .

**Lemma 4.6.** *The map  $\text{Fr}$  takes its values in  $\text{Fr}(\mathbb{A} \cap A)$  and  $\mathbb{A} \cap A = \bigcup_{x \in U} \text{Fr}(x) + \mathbb{R}_+\nu$ .*

*Proof.* Let  $u \in U$ . Then  $\text{Fr}(u) + \mathbb{R}_+\nu = (u + \mathbb{R}\nu) \cap A$ . Thus  $\text{Fr}(u) \notin \text{Int}(A \cap \mathbb{A})$ . By Proposition 3.9,  $\text{Fr}(u) \in \text{Fr}(A \cap \mathbb{A})$  and hence  $\text{Fr}(U) \subset \text{Fr}(A \cap \mathbb{A})$ .

Let  $u \in A \cap \mathbb{A}$ . One has  $u \in A \cap (u + \mathbb{R}\nu) = \text{Fr}(u) + \mathbb{R}_+\nu$  and we deduce that  $\mathbb{A} \cap A \subset \bigcup_{x \in U} \text{Fr}(x) + \mathbb{R}_+\nu$ . The reverse inclusion is a consequence of Lemma 4.3, which finishes the proof.  $\square$

Let us sketch the proof of the convexity of  $A \cap \mathbb{A}$  (which is Lemma 4.21). If  $x, y \in \mathring{U}$ , one defines  $\text{Fr}_{x,y} : [0, 1] \rightarrow \text{Fr}(A \cap \mathbb{A})$  by  $\text{Fr}_{x,y}(t) = \text{Fr}((1-t)x + ty)$  for all  $t \in [0, 1]$ . For all  $t \in [0, 1]$ , there exists a unique  $f_{x,y}(t) \in \mathbb{R}$  such that  $\text{Fr}_{x,y}(t) = (1-t)x + ty + f_{x,y}(t)\nu$ . We prove that for “almost” all  $x, y \in \mathring{U}$ ,  $f_{x,y}$  is convex. Let  $x, y \in \mathring{U}$ . We first prove that  $f_{x,y}$  is continuous and piecewise affine. This enables to reduce the study of the convexity of  $f_{x,y}$  to the study of  $f_{x,y}$  at the points where the slope changes. Let  $\mathcal{M}$  be a finite set of walls such that  $\text{Fr}(\mathring{U}) \subset \bigcup_{M \in \mathcal{M}} M$ , which exists by Proposition 3.14. Using order-convexity, we prove that if  $\{x, y\}$  is such that for each point  $u \in ]0, 1[$  at which the slope changes,  $\text{Fr}_{x,y}(u)$  is contained in exactly two walls of  $\mathcal{M}$ , then  $f_{x,y}$  is convex. We then prove that there are “enough” such pairs and conclude by an argument of density.

## 4.2 Continuity of the frontier

In this subsection, we prove that  $\text{Fr}$  is continuous on  $\mathring{U}$ , using order-convexity.

Let  $\lambda : U \rightarrow \mathbb{R}$  such that for all  $x \in U$ ,  $\text{Fr}(x) = x + \lambda(x)\nu$ . We prove the continuity of  $\text{Fr}|_{\mathring{U}}$  by proving the continuity of  $\lambda|_{\mathring{U}}$ . For this, we begin by majorizing  $\lambda([x, y])$  if  $x, y \in \mathring{U}$  (see Lemma 4.7) by a number depending on  $x$  and  $y$ . We use it to prove that if  $n \in \mathbb{N}$  and  $a_1, \dots, a_n \in \mathring{U}$ ,  $\lambda(\text{conv}(\{a_1, \dots, a_n\}))$  is majorized and then deduce that  $\text{Fr}|_{\mathring{U}}$  is continuous (which is Lemma 4.12).

**Lemma 4.7.** *Let  $x, y \in U$ ,  $M = \max\{\lambda(x), \lambda(y)\}$  and  $k \in \mathbb{R}_+$  such that  $x + k\nu \geq y$ . Then for all  $u \in [x, y]$ ,  $\lambda(u) \leq k + M$ .*

*Proof.* By Lemma 4.3,  $x + M\nu, y + M\nu \in A$ . By hypothesis,  $x + k\nu + M\nu \geq y + M\nu$ . Let  $t \in [0, 1]$  and  $u = tx + (1-t)y$ . By order-convexity  $t(x + k\nu + M\nu) + (1-t)(y + M\nu) \in A$ . Therefore  $\lambda(u) \leq M + tk \leq M + k$ , which is our assertion.  $\square$

**Lemma 4.8.** *Let  $d \in \mathbb{N}$ ,  $X$  be a  $d$  dimensional affine space and  $P \subset X$ . One sets  $\text{conv}_0(P) = P$  and for all  $k \in \mathbb{N}$ ,  $\text{conv}_{k+1}(P) = \{(1-t)p + tp' | t \in [0, 1] \text{ and } (p, p') \in \text{conv}_k(P)^2\}$ . Then  $\text{conv}_d(P) = \text{conv}(P)$ .*

*Proof.* By induction,  $\text{conv}_k(P) = \{\sum_{i=1}^{2^k} \lambda_i p_i | (\lambda_i) \in [0, 1]^{2^k}, \sum_{i=1}^{2^k} \lambda_i = 1 \text{ and } (p_i) \in P^{2^k}\}$ . This is thus a consequence of Carathéodory’s Theorem.  $\square$

**Lemma 4.9.** *Let  $P$  be a bounded subset of  $\mathring{U}$  such that  $\lambda(P)$  is majorized. Then  $\lambda(\text{conv}_1(P))$  is majorized.*

*Proof.* Let  $M = \sup_{x \in P} \lambda(x)$  and  $k \in \mathbb{R}_+$  such that for all  $x, x' \in P$ ,  $x' + k\nu \geq x$ , which exists by Lemma 4.2. Let  $u \in \text{conv}_1(P)$  and  $x, x' \in P$  such that  $u \in [x, x']$ . By Lemma 4.7,  $\lambda(u) \leq k + M$  and the lemma follows.  $\square$

**Lemma 4.10.** *Let  $x \in \mathring{U}$ . Then there exists  $V \in \mathcal{V}_{\mathring{U}}(x)$  such that  $V$  is convex and  $\lambda(V)$  is majorized.*

*Proof.* Let  $n \in \mathbb{N}$  and  $a_1, \dots, a_n \in \mathring{U}$  such that  $V = \text{conv}(a_1, \dots, a_n)$  contains  $x$  in its interior. Let  $M \in \mathbb{R}_+$  such that for all  $y, y' \in V$ ,  $y + M\nu \geq y'$ , which is possible by Lemma 4.2. One sets  $P = \{a_1, \dots, a_n\}$  and for all  $k \in \mathbb{N}$ ,  $P_k = \text{conv}_k(P)$ . By induction using Lemma 4.9,  $\lambda(P_k)$  is majorized for all  $k \in \mathbb{N}$  and we conclude with Lemma 4.8.  $\square$

**Lemma 4.11.** *Let  $V \subset \mathring{U}$  be open, convex, bounded and such  $\lambda(V)$  is majorized by some  $M \in \mathbb{R}_+$ . Let  $k \in \mathbb{R}_+$  such that for all  $x, x' \in V$ ,  $x + k\nu \geq x'$ . Let  $a \in V$  and  $u \in \mathbb{A}$  such that  $a + u \in V$ . Then for all  $t \in [0, 1]$ ,  $\lambda(a + tu) \leq (1-t)\lambda(a) + t(M + k)$ .*

*Proof.* By Lemma 4.3,  $a + u + (M + k)\nu \in A$ . Moreover  $a + u + (M + k)\nu \geq a + M\nu$ ,  $a + M\nu \geq a + \lambda(a)\nu = \text{Fr}(a)$  and thus  $a + u + (M + k)\nu \geq \text{Fr}(a)$ .

Let  $t \in [0, 1]$ . Then by order-convexity,

$$(1 - t)(a + \lambda(a)\nu) + t(a + u + (M + k)\nu) = a + tu + ((1 - t)\lambda(a) + t(M + k))\nu \in A.$$

Therefore  $\lambda(a + tu) \leq (1 - t)\lambda(a) + t(M + k)$ , which is our assertion.  $\square$

**Lemma 4.12.** *The map  $\text{Fr}$  is continuous on  $\mathring{U}$ .*

*Proof.* Let  $x \in \mathring{U}$  and  $V \in \mathcal{V}_{\mathring{U}}(x)$  be convex, open, bounded and such that  $\lambda(V)$  is majorized by some  $M \in \mathbb{R}_+$ , which exists by Lemma 4.10. Let  $k \in \mathbb{R}_+$  such that for all  $v, v' \in V$ ,  $v + kv \geq v'$ . Let  $\|\cdot\|$  be a norm on  $\mathbb{A}$  and  $r > 0$  such that  $B(x, r) \subset V$ , where  $B(x, r) = \{u \in \mathbb{A} \mid |x - u| \leq r\}$ . Let  $S = \{u \in \mathbb{A} \mid |u - x| = r\}$ . Let  $N = M + k$ .

Let  $y \in S$  and  $t \in [0, 1]$ . By applying Lemma 4.11 with  $a = x$  and  $u = y - x$ , we get that

$$\lambda((1 - t)x + ty) \leq \lambda(x) + tN.$$

By applying Lemma 4.11 with  $a = (1 - t)x + ty$  and  $u = x - y$ , we obtain that

$$\lambda(x) = \lambda((1 - t)x + ty + t(x - y)) \leq \lambda((1 - t)x + ty) + tN.$$

Therefore for all  $t \in [0, 1]$  and  $y \in S$ ,

$$\lambda(x) - tN \leq \lambda((1 - t)x + ty) \leq \lambda(x) + tN.$$

Let  $(x_n) \in B(x, r)^{\mathbb{N}}$  such that  $x_n \rightarrow x$ . Let  $n \in \mathbb{N}$ . One sets  $t_n = \frac{|x_n - x|}{r}$ . If  $t_n = 0$ , one chooses  $y_n \in S$ . If  $t_n \neq 0$ , one sets  $y_n = x + \frac{1}{t_n}(x_n - x) \in S$ . Then  $x_n = t_n y_n + (1 - t_n)x$  and thus  $|\lambda(x_n) - \lambda(x)| \leq t_n N \rightarrow 0$ . Consequently  $\lambda|_{\mathring{U}}$  is continuous and we deduce that  $\text{Fr}|_{\mathring{U}}$  is continuous.  $\square$

### 4.3 Piecewise affineness of $\text{Fr}_{x,y}$

We now study the map  $\text{Fr}$ . We begin by proving that there exists a finite set  $\mathcal{H}$  of hyperplanes of  $\mathbb{A}$  such  $\text{Fr}$  is affine on each connected component of  $\mathring{U} \setminus \bigcup_{H \in \mathcal{H}} H$ .

Let  $\mathcal{M}$  be finite set of walls such that  $\text{Fr}(A \cap \mathbb{A})$  is included in  $\bigcup_{M \in \mathcal{M}} M$ , whose existence is provided by Proposition 3.14. Let  $r = |\mathcal{M}|$ . Let  $\{\beta_1, \dots, \beta_r\} \in \Phi^r$  and  $(l_1, \dots, l_r) \in \prod_{i=1}^r \Lambda'_{\beta_i}$  such that  $\mathcal{M} = \{M_i \mid i \in \llbracket 1, r \rrbracket\}$  where  $M_i = \beta_i^{-1}(\{l_i\})$  for all  $i \in \llbracket 1, r \rrbracket$ .

If  $i, j \in \llbracket 1, r \rrbracket$ , with  $i \neq j$ ,  $\beta_i(\nu)\beta_j(\nu) \neq 0$  and  $M_i$  and  $M_j$  are not parallel, one sets  $H_{i,j} = \{x \in \mathbb{A} \mid \frac{l_i - \beta_i(x)}{\beta_i(\nu)} = \frac{l_j - \beta_j(x)}{\beta_j(\nu)}\}$  (this definition will appear naturally in the proof of the next lemma). Then  $H_{i,j}$  is a hyperplane of  $\mathbb{A}$ . Indeed, otherwise  $H_{i,j} = \mathbb{A}$ . Hence  $\frac{\beta_j(x)}{\beta_j(\nu)} - \frac{\beta_i(x)}{\beta_i(\nu)} = \frac{l_j}{\beta_j(\nu)} - \frac{l_i}{\beta_i(\nu)}$ , for all  $x \in \mathbb{A}$ . Therefore  $\frac{\beta_j(x)}{\beta_j(\nu)} - \frac{\beta_i(x)}{\beta_i(\nu)} = 0$  for all  $x \in \mathbb{A}$  and thus  $M_i$  and  $M_j$  are parallel: a contradiction. Let  $\mathcal{H} = \{H_{i,j} \mid i \neq j, \beta_i(\nu)\beta_j(\nu) \neq 0 \text{ and } M_i \not\parallel M_j\} \cup \{M_i \mid \beta_i(\nu) = 0\}$ .

Even if the elements of  $\mathcal{H}$  can be walls of  $\mathbb{A}$ , we will only consider them as hyperplanes of  $\mathbb{A}$ . To avoid confusion between elements of  $\mathcal{M}$  and elements of  $\mathcal{H}$ , we will try to use the letter  $M$  (resp.  $H$ ) in the name of objects related to  $\mathcal{M}$  (resp.  $\mathcal{H}$ ).

**Lemma 4.13.** *Let  $M_{\cap} = \bigcup_{M \neq M' \in \mathcal{M}} M \cap M'$ . Then  $\text{Fr}^{-1}(M_{\cap}) \subset \bigcup_{H \in \mathcal{H}} H$ .*



*Proof.* Let  $x \in \text{Fr}^{-1}(M_\cap)$ . One has  $\text{Fr}(x) = x + \lambda\nu$ , for some  $\lambda \in \mathbb{R}$ . There exists  $i, j \in \llbracket 1, r \rrbracket$  such that  $i \neq j$ ,  $\beta_i(\text{Fr}(x)) = l_i$  and  $\beta_j(\text{Fr}(x)) = l_j$  and  $M_i$  and  $M_j$  are not parallel. Therefore if  $\beta_i(\nu)\beta_j(\nu) \neq 0$ ,  $\lambda = \frac{l_i - \beta_i(x)}{\beta_i(\nu)} = \frac{l_j - \beta_j(x)}{\beta_j(\nu)}$  and thus  $x \in H_{i,j}$ , and if  $\beta_i(\nu)\beta_j(\nu) = 0$ ,  $x \in M_i \cup M_j$ , which proves the lemma.  $\square$

**Lemma 4.14.** *One has  $A \cap \mathbb{A} = \overline{\text{Int}(A \cap \mathbb{A})}$ .*

*Proof.* By Proposition 3.9,  $A \cap \mathbb{A}$  is closed and thus  $\overline{\text{Int}(A \cap \mathbb{A})} \subset A \cap \mathbb{A}$ .

Let  $x \in A \cap \mathbb{A}$ . Let  $V$  be an open bounded set included in  $A \cap \mathbb{A}$ . By Lemma 4.2 applied to  $x - V$ , there exists  $a > 0$  such that for all  $v \in V$ ,  $v + av \geq x$ . One has  $V + av \subset A \cap \mathbb{A}$  and by order convexity (Conséquence 2 of Proposition 5.4 in [Rou11]),  $\text{conv}(V + av, x) \subset A \cap \mathbb{A}$ . As  $\text{conv}(V + av, x)$  is a convex set with nonempty interior, there exists  $(x_n) \in \text{Int}(\text{conv}(V + av, x))^{\mathbb{N}}$  such that  $x_n \rightarrow x$ , and the lemma follows.  $\square$

Let  $f_1, \dots, f_s$  be affine forms on  $\mathbb{A}$  such that  $\mathcal{H} = \{f_i^{-1}(\{0\}) \mid i \in \llbracket 1, s \rrbracket\}$  for some  $s \in \mathbb{N}$ . Let  $R = (R_i) \in \{\leq, \geq, <, >\}^s$ . One sets  $P_R = \mathring{U} \cap \{x \in \mathbb{A} \mid (f_i(x) R_i 0) \forall i \in \llbracket 1, s \rrbracket\}$ . If  $R = (R_i) \in \{\leq, \geq\}^s$ , one defines  $R' = (R'_i) \in \{<, >\}^s$  by  $R'_i = "<"$  if  $R_i = "\leq"$  and  $R'_i = ">"$  otherwise (one replaces large inequalities by strict inequalities). If  $R \in \{\leq, \geq\}^s$ , then  $\text{Int}(P_R) = P_{R'}$ .

Let  $X = \{R \in \{\leq, \geq\}^s \mid \mathring{P}_R \neq \emptyset\}$ . By Lemma 4.14 and Lemma 3.10,  $\mathring{U} = \bigcup_{R \in X} P_R$  and for all  $R \in X$ ,  $\mathring{P}_R \subset \mathbb{A} \setminus \bigcup_{H \in \mathcal{H}} H$ .

**Lemma 4.15.** *Let  $R \in X$ . Then there exists  $M \in \mathcal{M}$  such that  $\text{Fr}(P_R) \subset M$ .*

*Proof.* Let  $x \in \mathring{P}_R$ . Let  $M \in \mathcal{M}$  such that  $\text{Fr}(x) \in M$ . Let us show that  $\text{Fr}(P_R) \subset M$ . By continuity of  $\text{Fr}$  (by Lemma 4.12), it suffices to prove that  $\text{Fr}(\mathring{P}_R) \subset M$ . By connectedness of  $\mathring{P}_R$  it suffices to prove that  $\text{Fr}^{-1}(M) \cap \mathring{P}_R$  is open and closed. As  $\text{Fr}$  is continuous,  $\text{Fr}^{-1}(M) \cap \mathring{P}_R$  is closed (in  $\mathring{P}_R$ ).

Suppose that  $\text{Fr}^{-1}(M) \cap \mathring{P}_R$  is not open. Then there exists  $y \in \mathring{P}_R$  such that  $\text{Fr}(y) \in M$  and a sequence  $(y_n) \in \mathring{P}_R^{\mathbb{N}}$  such that  $y_n \rightarrow y$  and such that  $\text{Fr}(y_n) \notin M$  for all  $n \in \mathbb{N}$ . For all  $n \in \mathbb{N}$ ,  $\text{Fr}(y_n) \in \bigcup_{M' \in \mathcal{M}} M'$ , and thus, maybe extracting a subsequence, one can suppose that for some  $M' \in \mathcal{M}$ ,  $y_n \in M'$  for all  $n \in \mathbb{N}$ .

As  $\text{Fr}$  is continuous (by Lemma 4.12),  $\text{Fr}(y) \in M'$ . Thus  $\text{Fr}(y) \in M \cap M'$  and by Lemma 4.13,  $y \in \bigcup_{H \in \mathcal{H}} H$ , which is absurd by choice of  $y$ . Therefore,  $\text{Fr}^{-1}(M) \cap \mathring{P}_R$  is open, which completes the proof of the lemma.  $\square$

**Lemma 4.16.** *Let  $R \in X$  and  $M \in \mathcal{M}$  such that  $\text{Fr}(P_R) \subset M$ . Then  $\nu \notin \vec{M}$  and there exists a (unique) affine morphism  $\psi : \mathbb{A} \rightarrow M$  such that  $\text{Fr}|_{P_R} = \psi|_{P_R}$ . Moreover  $\psi$  induces an isomorphism  $\bar{\psi} : \mathbb{A}/\mathbb{R}\nu \rightarrow M$ .*

*Proof.* If  $y \in \mathring{U}$ ,  $\text{Fr}(y) = y + k(y)\nu$  for some  $k(y) \in \mathbb{R}$ . Let  $\alpha \in \Phi$  such that  $M = \alpha^{-1}(\{u\})$  for some  $u \in -\Lambda'_\alpha$ . For all  $y \in P_R$ , one has  $\alpha(\text{Fr}(y)) = \alpha(y) + k(y)\alpha(\nu) = u$  and  $\alpha(\nu) \neq 0$  because  $\alpha$  is not constant on  $P_R$ . Consequently  $\nu \notin \vec{M}$  and  $\text{Fr}(y) = y + \frac{u - \alpha(y)}{\alpha(\nu)}\nu$ . One defines  $\psi : \mathbb{A} \rightarrow M$  by  $\psi(y) = y + \frac{u - \alpha(y)}{\alpha(\nu)}\nu$  for all  $y \in \mathbb{A}$  and  $\psi$  has the desired properties.  $\square$

#### 4.4 Local convexity of $\text{Fr}_{x,y}$

Let  $M \in \mathcal{M}$  and  $\vec{M}$  be its direction. Let  $\mathcal{T}_M = \mathring{\mathcal{T}} \cap \vec{M}$  and  $D_M$  be the half-apartment containing a shortening of  $\mathbb{R}_+\nu$  and whose wall is  $M$ .

**Lemma 4.17.** *Let  $a \in \text{Fr}(\mathring{U})$  and suppose that for some  $\mathcal{K} \in \mathcal{V}_{\mathring{U}}(a)$ ,  $\text{Fr}(\mathcal{K}) \subset M$  for some  $M \in \mathcal{M}$ . Then  $\text{Fr}((a \pm \mathring{\mathcal{T}}_M) \cap \mathring{U}) \subset D_M$ .*

*Proof.* Let  $u \in \mathring{U} \cap (a - \mathring{\mathcal{T}}_M)$ ,  $u \neq a$ . Suppose  $\text{Fr}(u) \notin D_M$ . Then  $\text{Fr}(u) = u - k\nu$ , with  $k \geq 0$ . Then  $\text{Fr}(u) \leq u \leq a$  (which means that  $a - u \in \mathring{\mathcal{T}}$ ). Therefore for some  $\mathcal{K}' \in \mathcal{V}_M(a)$  such that  $\mathcal{K}' \subset \mathcal{K}$ , one has  $\text{Fr}(u) \leq u'$  for all  $u' \in \mathcal{K}'$ . As a consequence  $\mathbb{A} \cap A \supset \text{conv}(\mathcal{K}', \text{Fr}(u))$  and thus  $\text{Fr}(u') \notin M$  for all  $u' \in \mathcal{K}'$ . This is absurd and hence  $\text{Fr}(u) \in D_M$ .

Let  $v \in \mathring{U} \cap (a + \mathring{\mathcal{T}}_M)$ ,  $v \neq a$  and suppose that  $\text{Fr}(v) \notin D_M$ . Then for  $v' \in [\text{Fr}(v), v]$  near enough from  $v$ ,  $a \leq v'$ . Therefore,  $[a, v'] \subset \mathbb{A} \cap A$ . Thus for all  $t \in ]a, v[$ ,  $\text{Fr}(t) \notin D_M$ , a contradiction. Therefore  $\text{Fr}(v) \in D_M$  and the lemma follows.  $\square$

The following lemma is crucial to prove the local convexity of  $\text{Fr}_{x,y}$  for good choices of  $x$  and  $y$ . This is mainly here that we use that  $A \cap \mathbb{A}$  have nonempty interior.

$$\text{Let } H_\cap = \bigcup_{H \neq H' \in \mathcal{H}} H \cap H'.$$

**Lemma 4.18.** *Let  $x \in \mathring{U} \cap (\bigcup_{H \in \mathcal{H}} H) \setminus H_\cap$  and  $H \in \mathcal{H}$  such that  $x \in H$ . Let  $C_1$  and  $C_2$  be the half-spaces defined by  $H$ . Then there exists  $V \in \mathcal{V}_{\mathring{U}}(x)$  satisfying the following conditions:*

1. *For  $i \in \{1, 2\}$ , let  $V_i = V \cap \mathring{C}_i$ . Then  $V_i \subset P_{R_i}^\circ$  for some  $R_i \in X$ .*
2. *Let  $M$  be a wall containing  $\text{Fr}(P_{R_1})$ . Then  $\text{Fr}(V) \subset D_M$ .*

*Proof.* (see Figure 4.1) The set  $\mathring{U} \setminus \bigcup_{H \in \mathcal{H} \setminus \{H\}} H$  is open in  $\mathring{U}$ . Let  $V' \in \mathcal{V}_{\mathring{U}}(x)$  such that  $V' \cap \bigcup_{H' \in \mathcal{H} \setminus \{H\}} H' = \emptyset$  and such that  $V'$  is convex. Let  $i \in \{1, 2\}$  and  $V'_i = V' \cap \mathring{C}_i$ . Then  $V'_i \subset \mathring{U} \setminus \bigcup_{H \in \mathcal{H}} H$  and  $V'_i$  is connected. As the connected components of  $\mathring{U} \setminus \bigcup_{H \in \mathcal{H}} H$  are the  $P_R^\circ$ 's for  $R \in X$ ,  $V'$  satisfies 1.

Let  $\psi : \mathbb{A} \rightarrow M$  be the affine morphism such that  $\psi|_{P_{R_1}} = \text{Fr}|_{P_{R_1}}$  and  $\bar{\psi} : \mathbb{A}/\mathbb{R}\nu \rightarrow M$  be the induced isomorphism, which exists by Lemma 4.16. Let  $\pi : \mathbb{A} \rightarrow \mathbb{A}/\mathbb{R}\nu$  be the canonical projection. As  $C_1$  is invariant under translation by  $\nu$  (by definition of the elements of  $\mathcal{H}$ )  $\psi(C_1) = \bar{\psi}(\pi(C_1))$  is a half-space  $D$  of  $M$ . Let  $V'' = V' \cap C_1$ . Then  $\psi(V'') = \bar{\psi}(C_1) \cap \bar{\psi}(\pi(V')) \in \mathcal{V}_D(\text{Fr}(x))$ .

Let  $g : \vec{M} \rightarrow \mathbb{R}$  be a linear form such that  $D = g^{-1}([b, +\infty[)$ , for some  $b \in \mathbb{R}$ . Let  $\epsilon \in \{-1, 1\}$  such that for some  $u \in \epsilon\mathring{\mathcal{T}}_M$  one has  $g(u) > 0$ . Let  $\eta > 0$ . Then  $\text{Fr}(x + \eta u) \in x + \eta u + \mathbb{R}\nu$  and thus  $\text{Fr}(x + \eta u) = \text{Fr}(x) + \eta u + k\nu$  for some  $k \in \mathbb{R}$ . If  $\eta$  is small enough that  $x + \eta u \in V''$ ,  $k\nu = \text{Fr}(x + \eta u) - (\text{Fr}(x) + \eta u) \in \vec{M}$  and hence  $k = 0$  (by Lemma 4.16). Let  $\mathcal{K} = \psi(V'') + \mathbb{R}\nu$  and  $a = \text{Fr}(x) + \eta u$ . Then  $\mathcal{K} \in \mathcal{V}_{\mathring{U}}(a)$  and for all  $v \in \mathcal{K}$ ,  $\text{Fr}(v) \in M$ . By Lemma 4.17,  $\text{Fr}(\mathring{U} \cap (a - \epsilon\mathring{\mathcal{T}}_M)) = \text{Fr}(\mathring{U} \cap (a - \epsilon\mathring{\mathcal{T}}_M + \mathbb{R}\nu)) \subset D_M$ . Moreover,  $a - \epsilon\mathring{\mathcal{T}}_M + \mathbb{R}\nu \in \mathcal{V}_{\mathring{U}}(x)$  and thus if one sets  $V = V' \cap (a - \epsilon\mathring{\mathcal{T}}_M + \mathbb{R}\nu)$ ,  $V$  satisfies 1 and 2.  $\square$

## 4.5 Convexity of $A \cap \mathbb{A}$

Let  $\vec{\mathcal{H}} = \bigcup_{H \in \mathcal{H}} \vec{H}$  be the set of directions of the hyperplanes of  $\mathcal{H}$ .

**Lemma 4.19.** *Let  $x, y \in \mathring{U} \cap A \cap \mathbb{A}$  such that  $y - x \notin \vec{\mathcal{H}}$  and such that the line spanned by  $[x, y]$  does not meet any point of  $H_\cap$ . Then  $[x, y] \subset \mathring{U} \cap A \cap \mathbb{A}$ .*

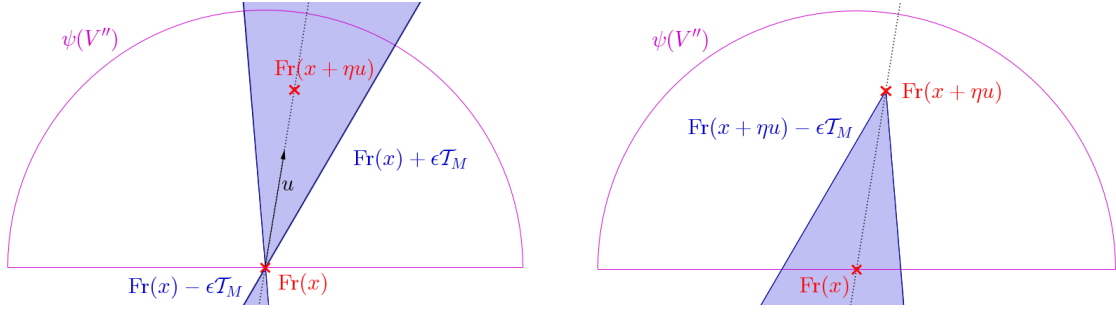


Figure 4.1 – Proof of Lemma 4.18 when  $\dim H = 2$  (the illustration is made in  $M$ )

*Proof.* Let  $\pi : [0, 1] \rightarrow \mathbb{A}$  defined by  $\pi(t) = tx + (1 - t)y$  for all  $t \in [0, 1]$  and  $g = \text{Fr} \circ \pi$ . Let  $f_1, \dots, f_s$  be affine forms on  $\mathbb{A}$  such that  $\mathcal{H} = \{f_i^{-1}(\{0\}) \mid i \in \llbracket 1, s \rrbracket\}$ . As  $y - x \notin \vec{\mathcal{H}}$ , for all  $i \in \llbracket 1, s \rrbracket$ , the map  $f_i \circ g$  is strictly monotonic and  $\pi^{-1}(\bigcup_{H \in \mathcal{H}} H)$  is finite. Therefore, there exist  $k \in \mathbb{N}$  and open intervals  $T_1, \dots, T_k$  such that  $[0, 1] = \bigcup_{i=1}^k \overline{T}_i$ ,  $T_1 < \dots < T_k$  and  $\pi(T_i) \subset \overset{\circ}{P}_{R_i}$  for some  $R_i \in X$  for all  $i \in \llbracket 1, k \rrbracket$ . For all  $t \in [0, 1]$ ,  $g(t) = \pi(t) + f(t)\nu$  for some  $f(t) \in \mathbb{R}$ . By Lemma 4.16 this equation uniquely determines  $f(t)$  for all  $t \in [0, 1]$ . By Lemma 4.12,  $f$  is continuous and by Lemma 4.16,  $f$  is affine on each  $T_i$ .

Let us prove that  $f$  is convex. Let  $i \in \llbracket 1, k - 1 \rrbracket$ . One writes  $T_i = ]a, b[$ . Then for  $\epsilon > 0$  small enough, one has  $f(b + \epsilon) = f(b) + \epsilon c_+$  and  $f(b - \epsilon) = f(b) - \epsilon c_-$ . To prove the convexity of  $f$ , it suffices to prove that  $c_- < c_+$ . Let  $M$  be a wall containing  $\text{Fr}(P_{R_i})$ . As  $\pi(b) \in \overset{\circ}{U} \cap \bigcup_{H \in \mathcal{H}} H \setminus H_\cap$ , we can apply Lemma 4.18 and there exists  $V \in \mathcal{V}_{[0,1]}(b)$  such that  $g(V) \subset D_M$ . Let  $h : \mathbb{A} \rightarrow \mathbb{R}$  be a linear map such that  $D_M = h^{-1}([a, +\infty[)$ . For  $\epsilon > 0$  small enough, one has  $h(g(b + \epsilon)) \geq a$  and  $h(g(b - \epsilon)) = a$ .

For  $\epsilon > 0$  small enough, one has

$$\begin{aligned} h(g(b + \epsilon)) &= h(\pi(b) + \epsilon(y - x) + (f(b) + \epsilon c_+)\nu) \\ &= h(g(b) + \epsilon(y - x + c_+\nu)) \\ &= a + \epsilon(h(y - x) + c_+h(\nu)) \geq a, \end{aligned}$$

and similarly,  $h(g(b - \epsilon)) = a - \epsilon(h(y - x) + c_-h(\nu)) = a$ .

Therefore  $h(y - x) + c_+h(\nu) \geq 0$ ,  $h(y - x) + c_-h(\nu) = 0$  and hence  $(c_+ - c_-)h(\nu) \geq 0$ . As  $D_M$  contains a shortening of  $\mathbb{R}_+\nu$ ,  $h(\nu) \geq 0$  and by Lemma 4.16,  $h(\nu) > 0$ . Consequently,  $c_- \leq c_+$  and, as  $i \in \llbracket 1, k - 1 \rrbracket$  was arbitrary,  $f$  is convex.

For all  $t \in [0, 1]$ ,  $f(t) \leq (1 - t)f(0) + tf(1)$ . Therefore

$$(1 - t)g(0) + tg(1) = \pi(t) + ((1 - t)f(0) + tf(1))\nu \in \pi(t) + f(t)\nu + \mathbb{R}_+\nu = g(t) + \mathbb{R}_+\nu.$$

By definition of  $\text{Fr}$ , if  $t \in [0, 1]$ ,  $(1 - t)g(0) + tg(1) \in A \cap \mathbb{A}$ . Moreover, there exist  $\lambda, \mu \geq 0$  such that  $x = g(0) + \lambda\nu$  and  $y = g(1) + \mu\nu$ . Then  $\pi(t) = (1 - t)x + ty = (1 - t)g(0) + tg(1) + ((1 - t)\lambda + t\mu)\nu \in A \cap \mathbb{A}$  and hence  $[x, y] \subset A \cap \mathbb{A}$ .  $\square$

**Lemma 4.20.** *Let  $x, y \in \text{Int}(\mathbb{A} \cap A)$  and  $\vec{\mathcal{H}} = \bigcup_{H \in \mathcal{H}} \vec{H}$ . Then there exists  $(x_n), (y_n) \in \text{Int}(A \cap \mathbb{A})^{\mathbb{N}}$  satisfying the following conditions:*

1.  $x_n \rightarrow x$  and  $y_n \rightarrow y$
2. for all  $n \in \mathbb{N}$ ,  $y_n - x_n \notin \vec{\mathcal{H}}$
3. the line spanned by  $[x_n, y_n]$  does not meet any point of  $H_\cap$ .

*Proof.* Let  $(x_n) \in (\text{Int}(A \cap \mathbb{A}) \setminus H_\cap)^\mathbb{N}$  such that  $x_n \rightarrow x$ . Let  $|\cdot|$  be a norm on  $\mathbb{A}$ . Let  $n \in \mathbb{N}$ . Let  $F$  be the set of points  $z$  such that the line spanned by  $[x_n, z]$  meets  $H_\cap$ . Then  $F$  is a finite union of hyperplanes of  $\mathbb{A}$  (because  $H_\cap$  is a finite union of spaces of dimension at most  $\dim \mathbb{A} - 2$ ). Therefore  $\mathbb{A} \setminus (F \cup x_n + \vec{\mathcal{H}})$  is dense in  $\mathbb{A}$  and one can choose  $y_n \in \mathbb{A} \setminus (F \cup x_n + \vec{\mathcal{H}})$  such that  $|y_n - y| \leq \frac{1}{n+1}$ . Then  $(x_n)$  and  $(y_n)$  satisfy the conditions of the lemma.  $\square$

**Lemma 4.21.** *The set  $A \cap \mathbb{A}$  is convex.*

*Proof.* Let  $x, y \in \text{Int}(A \cap \mathbb{A})$ . Let  $(x_n), (y_n)$  be as in Lemma 4.20. Let  $t \in [0, 1]$ . As  $\text{Int}(A \cap \mathbb{A}) \subset \overset{\circ}{U}$ , for all  $n \in \mathbb{N}$ ,  $tx_n + (1-t)y_n \in A \cap \mathbb{A}$  by Lemma 4.19. As  $A \cap \mathbb{A}$  is closed (by Proposition 3.9),  $tx + (1-t)y \in A \cap \mathbb{A}$ . Therefore  $\text{Int}(A \cap \mathbb{A})$  is convex. Consequently  $A \cap \mathbb{A} = \overline{\text{Int}(A \cap \mathbb{A})}$  (by Lemma 4.14) is convex.  $\square$

We thus have proved the following theorem:

**Theorem 4.22.** *Let  $A$  and  $B$  be two apartments sharing a generic ray. Then  $A \cap B$  is enclosed and there exists an isomorphism  $\phi : A \xrightarrow{A \cap B} B$ .*

*Proof.* By Lemma 4.21 and Lemma 4.1,  $A \cap B$  is convex. By Proposition 3.22,  $A \cap B$  is enclosed and by Proposition 3.26, there exists an isomorphism  $\phi : A \xrightarrow{A \cap B} B$ .  $\square$

## 4.6 A partial reciprocal

One says that a group  $G$  of automorphisms of  $\mathcal{I}$  acts strongly transitively on  $\mathcal{I}$  if the isomorphisms involved in (MA2) and (MA4) are induced by elements of  $G$ . For example if  $G$  is a quasi-split Kac-Moody group over a ultrametric field  $\mathcal{K}$ , it acts strongly transitively on the measure  $\mathcal{I}(G, \mathcal{K})$  associated.

We now prove a kind of weak reciprocal of Theorem 4.22 when some group  $G$  acts strongly transitively on  $\mathcal{I}$  and when  $\mathcal{I}$  is thick, which means that each panel is included in at least three chambers. This implies some restrictions on  $\Lambda'$  by Lemma 4.24 below and Remark 3.3.

**Lemma 4.23.** *Let  $P$  be an enclosed subset of  $\mathbb{A}$  and suppose that  $\overset{\circ}{P} \neq \emptyset$ . One fixes the origin of  $\mathbb{A}$  in some point of  $\overset{\circ}{P}$ . Let  $j_P$  be the gauge of  $P$  defined in Section 3.3. Let  $U = \{x \in \mathbb{A} \mid j_P(x) \neq 0\}$ . One defines  $\text{Fr} : U \rightarrow P$  as in Lemma 3.18. One writes  $P = \bigcap_{i=1}^k D_i$ , where the  $D_i$ 's are half-apartments of  $\mathbb{A}$ . Let  $j \in \llbracket 1, k \rrbracket$ ,  $M_j$  be the wall of  $D_j$  and suppose that for all open subset  $V$  of  $U$ ,  $\text{Fr}(V) \not\subseteq M_j$ . Then  $P = \bigcap_{i \in \llbracket 1, k \rrbracket \setminus \{j\}} D_i$ .*

*Proof.* Suppose that  $P \not\subseteq \bigcap_{i \in \llbracket 1, k \rrbracket \setminus \{j\}} D_i$ . Let  $V$  be a nonempty open and bounded subset included in  $\bigcap_{i \in \llbracket 1, k \rrbracket \setminus \{j\}} D_i \setminus P$ . Let  $n \in \mathbb{N}^*$  such that  $\frac{1}{n}V \subset P$ . Let  $v \in V$ . Then  $[\frac{1}{n}v, v] \cap \text{Fr}(P) = \{\text{Fr}(v)\}$ . Moreover for all  $i \in \llbracket 1, k \rrbracket \setminus \{j\}$ ,  $[\frac{1}{n}v, v] \subset \overset{\circ}{D}_i$ . As  $\text{Fr}(P) \subset \bigcup_{i \in \llbracket 1, k \rrbracket} M_i$ ,  $\text{Fr}(v) \in M_j$ : this is absurd and thus  $P = \bigcap_{i \in \llbracket 1, k \rrbracket \setminus \{j\}} D_i$ .  $\square$

**Lemma 4.24.** *Suppose that  $\mathcal{I}$  is thick. Let  $D$  be a half-apartment of  $\mathbb{A}$ . Then there exists an apartment  $A$  of  $\mathbb{A}$  such that  $D = A \cap \mathbb{A}$ .*

*Proof.* Let  $F$  be a panel of the wall of  $D$ . As  $\mathcal{I}$  is thick, there exists a chamber  $C$  dominating  $F$  and such that  $C \not\subseteq \mathbb{A}$ . By Proposition 2.9 1) of [Rou11], there exists an apartment  $A$  containing  $D$  and  $C$ . The set  $\mathbb{A} \cap A$  is a half-apartment by Lemma 3.1 and thus  $\mathbb{A} \cap A = D$ , which proves the lemma.  $\square$

**Corollary 4.25.** *Suppose that  $\mathcal{I}$  is thick and that some group  $G$  acts strongly transitively on  $\mathcal{I}$ . Let  $P$  be an enclosed subset of  $\mathbb{A}$  containing a generic ray and having nonempty interior. Then there exists an apartment  $A$  such that  $A \cap \mathbb{A} = P$ .*

*Proof.* One writes  $P = D_1 \cap \dots \cap D_k$ , where the  $D_i$ 's are half-apartments of  $\mathbb{A}$ . One supposes that  $k$  is minimal for this writing, which means that for all  $i \in \llbracket 1, n \rrbracket$ ,  $P \neq \bigcap_{j \in \llbracket 1, k \rrbracket \setminus \{i\}} D_j$ . For all  $i \in \llbracket 1, n \rrbracket$ , one chooses an apartment  $A_i$  such that  $\mathbb{A} \cap A_i = D_i$ . Let  $\phi_i : \mathbb{A} \xrightarrow{D_i} A_i$  and  $g_i \in G$  inducing  $\phi_i$ .

Let  $g = g_1 \dots g_k$  and  $A = g \cdot \mathbb{A}$ . Then  $A \cap \mathbb{A} \supset D_1 \cap \dots \cap D_k$  and  $g$  fixes  $D_1 \cap \dots \cap D_k$ . Let us show that  $A \cap \mathbb{A} = \{x \in \mathbb{A} | g \cdot x = x\}$ . By Theorem 4.22, there exists  $\phi : \mathbb{A} \xrightarrow{A \cap \mathbb{A}} A$ . Moreover  $g_{|\mathbb{A}}^{-1} \circ \phi : \mathbb{A} \rightarrow \mathbb{A}$  fixes  $D_1 \cap \dots \cap D_k$ , which has nonempty interior and thus  $g_{|\mathbb{A}}^{-1} \circ \phi = \text{Id}_{\mathbb{A}}$ , which proves that  $A \cap \mathbb{A} = \{x \in \mathbb{A} | g \cdot x = x\}$ .

Suppose that  $A \cap \mathbb{A} \supsetneq D_1 \cap \dots \cap D_k$ . Let  $i \in \llbracket 1, k \rrbracket$  such that there exists  $a \in A \cap \mathbb{A} \setminus D_i$ .

One fixes the origin of  $\mathbb{A}$  in some point of  $\overset{\circ}{P}$ , one sets  $U = \{x \in \mathbb{A} | j_P(x) \neq 0\}$  and one defines  $\text{Fr} : U \rightarrow \text{Fr}(P)$  as in Lemma 3.18. By minimality of  $k$  and Lemma 4.23, there exists a nonempty open set  $V$  of  $U$  such that  $\text{Fr}(V) \subset M_i$ .

By the same reasoning as in the proof of Lemma 3.21,  $\text{Fr}(V) \cap M_i$  is open in  $M_i$ . Consequently, there exists  $v \in \text{Fr}(V)$  such that  $v \notin \bigcup_{j \in \llbracket 1, k \rrbracket \setminus \{i\}} M_j$ . Let  $V' \in \mathcal{V}_U(v)$  such that  $V' \cap \bigcup_{j \in \llbracket 1, k \rrbracket \setminus \{i\}} M_j = \emptyset$  and such that  $V'$  is convex. Then  $V' \subset \bigcap_{j \in \llbracket 1, k \rrbracket \setminus \{i\}} \overset{\circ}{D}_j$ . Let  $V'' = \text{Fr}(V) \cap V'$ . By Theorem 4.22,  $[a, v] \subset A \cap \mathbb{A}$  and hence  $g$  fixes  $[a, v]$ . Moreover for  $u \in [a, v]$  near  $v$ ,  $u \in \bigcap_{j \in \llbracket 1, k \rrbracket \setminus \{i\}} \overset{\circ}{D}_j$ . Then  $g \cdot u = g_1 \dots g_i \cdot (g_{i+1} \dots g_k \cdot u) = g_1 \dots g_i \cdot u$ . Moreover,  $g_i \cdot u = g_{i-1}^{-1} \dots g_1^{-1} \cdot u = u$ . Therefore  $u \in D_i$ , which is absurd by choice of  $u$ .  $\square$

**Remark 4.26.** 1. In the proof above, the fact that  $P$  contains a generic ray is only used to prove that  $A \cap \mathbb{A}$  is convex and that there exists an isomorphism  $\phi : A \xrightarrow{A \cap \mathbb{A}} \mathbb{A}$ . When  $G$  is an affine Kac-Moody group and  $\mathcal{I}$  is its measure, we will see that these properties are true without assuming that  $A \cap \mathbb{A}$  contains a generic ray. Therefore, for all enclosed subset  $P$  of  $\mathbb{A}$  having nonempty interior, there exists an apartment  $A$  such that  $A \cap \mathbb{A} = P$

2. Let  $\mathbb{T}$  be a discrete homogeneous tree with valence 3 and  $x$  be a vertex of  $\mathbb{T}$ . Then there exists no pair  $(A, A')$  of apartments such that  $A \cap A' = \{x\}$ . Indeed, let  $A$  be an apartment containing  $x$  and  $C_1, C_2$  be the alcoves of  $A$  dominating  $x$ . Let  $A'$  be an apartment containing  $x$ . If  $A'$  does not contain  $C_1$ , it contains  $C_2$  and thus  $A \cap A' \neq \{x\}$ . Therefore the hypothesis “ $P$  has nonempty interior” is necessary in Corollary 4.25.

## 5 Axiomatic of measures

### 5.1 Axiomatic of measures in the general case

The aim of this section is to give an other axiomatic of measure than the one of [Rou11] and [Rou17]. For this, we mainly use Theorem 4.22.

We fix an apartment  $\mathbb{A} = (\mathcal{S}, W, \Lambda')$ . A **construction** of type  $\mathbb{A}$  is a set endowed with a covering of subsets called apartments and satisfying (MA1).

Let  $\text{cl} \in \mathcal{CL}_{\Lambda'}$ . Let (MA i)=(MA1).

Let (MA ii) : if two apartments  $A, A'$  contain a generic ray, then  $A \cap A'$  is enclosed and there exists an isomorphism  $\phi : A \xrightarrow{A \cap A'} A'$ .

Let (MA iii, cl): if  $\mathfrak{R}$  is the germ of a splayed chimney and if  $F$  is a face or a germ of a chimney, then there exists an apartment containing  $\mathfrak{R}$  and  $F$ .

It is easy to see that the axiom (MA ii) implies (MA4, cl) for all  $\text{cl} \in \mathcal{CL}_{\Lambda'}$ . If  $\text{cl} \in \mathcal{CL}_{\Lambda'}$ , (MA iii, cl) is equivalent to (MA3, cl) because each chimney is included in a solid chimney.

Let  $\mathcal{I}$  be a construction of type  $\mathbb{A}$  and  $\text{cl} \in \mathcal{CL}_{\Lambda'}$ . One says that  $\mathcal{I}$  is a **measure of type** (1, cl) if it satisfies the axioms of [Rou11]: (MA2, cl), (MA3, cl), (MA4, cl) and (MAO). One says that  $\mathcal{I}$  is a **measure of type** (2, cl) if it satisfies (MA ii) and (MA iii, cl).

The aim of the next two subsections is to prove the following theorem:

**Theorem 5.1.** *Let  $\mathcal{I}$  be a construction of type  $\mathbb{A}$  and  $\text{cl} \in \mathcal{CL}_{\Lambda'}$ . Then  $\mathcal{I}$  is a measure of type (1, cl) if and only if  $\mathcal{I}$  is a measure of type (1,  $\text{cl}^\#$ ) if and only if  $\mathcal{I}$  is a measure of type (2, cl) if and only if  $\mathcal{I}$  is a measure of type (2,  $\text{cl}^\#$ ).*

Let us introduce some other axioms and definitions. An **extended chimney** of  $\mathbb{A}$  is associated to a local face  $F^l = F^l(x, F_0^v)$  (its **basis**) and a vectorial face (its **direction**)  $F^v$ , this is the filter  $\mathfrak{r}_e(F^l, F^v) = F^l + F^v$ . Similarly to classical chimneys, we define shortenings and germs of extended chimney. We use the same vocabulary for extended chimneys as for classical: splayed, solid, full, ... We use the isomorphisms of apartments to extend these notions in constructions. Actually each classical chimney is of the shape  $\text{cl}(\mathfrak{r}_e)$  for some extended chimney  $\mathfrak{r}_e$ .

Let  $\text{cl} \in \mathcal{CL}_{\Lambda'}$ . Let (MA2', cl): if  $F$  is a point, a germ of a preordered interval or a splayed chimney in an apartment  $A$  and if  $A'$  is another apartment containing  $F$  then  $A \cap A'$  contains the enclosure  $\text{cl}_A(F)$  of  $F$  and there exists an isomorphism from  $A$  onto  $A'$  fixing  $\text{cl}_A(F)$ .

Let (MA2'', cl): if  $F$  is a solid chimney in an apartment  $A$  and if  $A'$  is an other apartment containing  $F$  then  $A \cap A'$  contains the enclosure  $\text{cl}_A(F)$  of  $F$  and there exists an isomorphism from  $A$  onto  $A'$  fixing  $\text{cl}_A(F)$ .

The axiom (MA2, cl) is a consequence of (MA2', cl), (MA2'', cl) and (MA ii).

Let (MA iii'): if  $\mathfrak{R}$  is the germ of a splayed extended chimney and if  $F$  is a local face or a germ of an extended chimney, then there exists an apartment containing  $\mathfrak{R}$  and  $F$ .

Let  $\mathcal{I}$  be a construction. Then  $\mathcal{I}$  is said to be a **measure of type** 3 if it satisfies (MA ii) and (MA iii').

In order to prove Theorem 5.1, we will in fact prove the following stronger theorem:

**Theorem 5.2.** *Let  $\text{cl} \in \mathcal{CL}_{\Lambda'}$  and  $\mathcal{I}$  be a construction of type  $\mathbb{A}$ . Then  $\mathcal{I}$  is a measure of type (1, cl) if and only if  $\mathcal{I}$  is a measure of type (2, cl) if and only if  $\mathcal{I}$  is a measure of type 3.*

The proof of this theorem will be divided in two steps. In the first step, we prove that (MAO) is a consequence of variants of (MA1), (MA2), (MA3) and (MA4) (see Proposition 5.3 for a precise statement). This uses paths but not Theorem 4.22. In the second step, we prove the equivalence of the three definitions. One implication relies on Theorem 4.22.

### 5.1.1 Dependency of (MAO)

The aim of this subsection is to prove the following proposition:

**Proposition 5.3.** *Let  $\mathcal{I}$  be a construction of type  $\mathbb{A}$  satisfying (MA2'), (MA iii') and (MA4). Then  $\mathcal{I}$  satisfies (MAO).*

We now fix a construction  $\mathcal{I}$  of type  $\mathbb{A}$  satisfying (MA2'), (MA iii') and (MA4). To prove proposition above, the key step is to prove that if  $B$  is an apartment and if  $x, y \in \mathbb{A} \cap B$  is such that  $x \leq_{\mathbb{A}} y$ , the image by  $\rho_{-\infty}$  of the segment of  $B$  joining  $x$  to  $y$  is a  $(y-x)^{++}$ -path.

Let  $a, b \in \mathbb{A}$ . An  $(a, b)$ -**path** of  $\mathbb{A}$  is a continuous piecewise linear map  $[0, 1] \rightarrow \mathbb{A}$  such that for all  $t \in [0, 1)$ ,  $\pi'(t)^+ \in W^v \cdot (b-a)$ . When  $a \leq b$ , the  $(a, b)$ -paths are the  $(b-a)^{++}$ -paths defined in 3.1.2.

Let  $A$  be an apartment an  $\pi : [0, 1] \rightarrow A$  be a map. Let  $a, b \in A$ . One says that  $\pi$  is an  $(a, b)$ -**path** of  $A$  if there exists  $\Upsilon : A \rightarrow \mathbb{A}$  such that  $\Upsilon \circ \pi$  is a  $(\Upsilon(a), \Upsilon(b))$ -path of  $\mathbb{A}$ .

**Lemma 5.4.** *Let  $A$  be an apartment and  $a, b \in A$ . Let  $\pi : [0, 1] \rightarrow A$  be a  $(a, b)$ -path in  $A$  and  $f : A \rightarrow B$  be an isomorphism of apartments. Then  $f \circ \pi$  is a  $(f(a), f(b))$ -path.*

*Proof.* Let  $\Upsilon : A \rightarrow \mathbb{A}$  be an isomorphism such that  $\Upsilon \circ \pi$  is a  $(\Upsilon(a), \Upsilon(b))$ -path in  $\mathbb{A}$ . Then  $\Upsilon' = \Upsilon \circ f^{-1} : B \rightarrow \mathbb{A}$  is an isomorphism,  $\Upsilon' \circ f \circ \pi$  is a  $(\Upsilon'(f(a)), \Upsilon'(f(b)))$ -path in  $\mathbb{A}$  and we get the lemma.  $\square$

The following lemma slightly improves Proposition 2.7 1) of [Rou11]. We recall that if  $A$  is an affine space and  $x, y \in A$ ,  $[x, y)$  means the germ  $germ_x([x, y])$ ,  $(x, y]$  means  $germ_y([x, y])$ , ..., see 2.4.

**Lemma 5.5.** *Let  $\mathfrak{R}$  be the germ of a splayed extended chimney,  $A$  be an apartment of  $\mathcal{I}$  and  $x^-, x^+ \in A$  such that  $x^- \leq_A x^+$ . Then there exists a subdivision  $z_1 = x^-, \dots, z_n = x^+$  of  $[x^-, x^+]_A$  such that for all  $i \in \llbracket 1, n-1 \rrbracket$  there exists an apartment  $A_i$  containing  $[z_i, z_{i+1}]_A \cup \mathfrak{R}$  such that there exists an isomorphism  $\phi_i : A \xrightarrow{[z_i, z_{i+1}]_{A_i}} A_i$ .*

*Proof.* Let  $u \in [x^-, x^+]$ . By (MA iii'), applied to  $(x^-, u)$  and  $[u, x^+)$  there exist apartments  $A_u^-$  and  $A_u^+$  containing  $\mathfrak{R} \cup (x^-, u]$  and  $\mathfrak{R} \cup [u, x^+)$  and by (MA2'), there exist isomorphisms  $\phi_u^+ : A \xrightarrow{(x^-, u]} A_u^-$  and  $\phi_u^- : A \xrightarrow{[u, x^+)} A_u^+$ . For all  $u \in [x^-, x^+]$  and  $\epsilon \in \{-, +\}$ , one chooses a convex set  $V_u^\epsilon \in [u, x^\epsilon)$  such that  $V_u^\epsilon \subset A \cap A_u^\epsilon$  and  $V_u^\epsilon$  is fixed by  $\phi_u^\epsilon$ . If  $u \in [x^-, x^+]$ , one sets  $V_u = \text{Int}_{[x^-, x^+]_A}(V_u^+ \cup V_u^-)$ . By compactness of  $[x^-, x^+]$ , there exists a finite set  $K$  and a map  $\epsilon : K \rightarrow \{-, +\}$  such that  $[x^-, x^+] = \bigcup_{k \in K} V_k^{\epsilon(k)}$  and the lemma follows.  $\square$

Let  $\mathfrak{q}$  be a sector-germ. Then  $\mathfrak{q}$  is an extended chimney. Let  $A$  be an apartment containing  $\mathfrak{q}$ . The axioms (MA2'), (MA iii') and (MA4) enable to define a retraction  $\rho : \mathcal{I} \xrightarrow{\mathfrak{q}} \mathbb{A}$  as in 2.6 of [Rou11].

**Lemma 5.6.** *Let  $A$  and  $B$  be two apartments,  $\mathfrak{q}$  be a sector-germ of  $B$  and  $\rho : \mathcal{I} \xrightarrow{\mathfrak{q}} B$ . Let  $x, y \in A$  such that  $x \leq_A y$ ,  $\tau : [0, 1] \rightarrow A$  mapping each  $t \in [0, 1]$  on  $(1-t)x +_A ty$  and  $f : A \rightarrow B$  be an isomorphism. Then  $\rho \circ \tau$  is a  $(f(x), f(y))$ -path of  $B$ .*

*Proof.* By Lemma 5.5, there exist  $k \in \mathbb{N}$ ,  $t_1 = 0 < \dots < t_k = 1$  such that for all  $i \in \llbracket 1, k-1 \rrbracket$ , there exists an apartment  $A_i$  containing  $\tau([t_i, t_{i+1}]) \cup \mathfrak{q}$  such that there exists an isomorphism  $\phi_i : A \xrightarrow{\tau([t_i, t_{i+1}])} A_i$ .

If  $i \in \llbracket 1, k-1 \rrbracket$ , one denotes by  $\psi_i$  the isomorphism  $A_i \xrightarrow{\mathfrak{q}} B$ . Then for  $t \in [t_i, t_{i+1}]$ , one has  $\rho(\tau(t)) = \psi_i \circ \phi_i(\tau(t))$ . Let  $\Upsilon : B \rightarrow \mathbb{A}$  be an isomorphism. By (MA1), for all  $i \in \llbracket 1, k \rrbracket$ , there exists  $w_i \in W$  such that  $\Upsilon \circ \psi_i \circ \phi_i = w_i \circ \Upsilon \circ f$ .

Let  $i \in \llbracket 1, k-1 \rrbracket$  and  $t \in [t_i, t_{i+1}]$ . Then

$$\Upsilon \circ \rho \circ \tau(t) = \Upsilon \circ \psi_i \circ \phi_i \circ \tau(t) = (1-t)w_i \circ \Upsilon \circ f(x) + tw_i \circ \Upsilon \circ f(y).$$

Therefore  $\rho \circ \tau$  is a  $(f(x), f(y))$ -path in  $B$ .  $\square$

**Lemma 5.7.** *Let  $\lambda \in \overline{C_f^v}$  and  $\pi : [0, 1] \rightarrow \mathbb{A}$  be a  $\lambda$ -path. Then  $\pi(1) - \pi(0) \leq_{Q^v} \lambda$ .*

*Proof.* By definition, there exists  $k \in \mathbb{N}$ ,  $(t_i) \in [0, 1]^k$  and  $(w_i) \in (W^v)^k$  such that  $\sum_{i=1}^k t_i = 1$  and  $\pi(1) - \pi(0) = \sum_{i=1}^k t_i \cdot w_i \cdot \lambda$ . Therefore  $\pi(1) - \pi(0) - \lambda = \sum_{i=1}^k t_i (w_i \cdot \lambda - \lambda)$  and thus  $\pi(1) - \pi(0) - \lambda \leq_{Q^v} 0$  by Lemma 3.5.  $\square$

**Lemma 5.8.** *Let  $x, y \in \mathbb{A}$  such that  $x \leq_{\mathbb{A}} y$  and  $B$  be an apartment containing  $x, y$ . Let  $\tau_B : [0, 1] \rightarrow B$  defined by  $\tau_B(t) = (1-t)x +_B ty$ ,  $\mathfrak{s}$  be a sector-germ of  $\mathbb{A}$  and  $\rho_{\mathfrak{s}} : \mathcal{I} \xrightarrow{\mathfrak{s}} \mathbb{A}$ . Then  $x \leq_B y$  and  $\pi_{\mathbb{A}} := \rho_{\mathfrak{s}} \circ \tau_B$  is an  $(x, y)$ -path of  $\mathbb{A}$ .*

*Proof.* Maybe changing the choice of  $\overline{C_f^v}$ , one can suppose that  $y - x \in \overline{C_f^v}$ . Let  $\mathfrak{q}$  be a sector-germ of  $B$ ,  $\rho_B : \mathcal{I} \xrightarrow{\mathfrak{q}} B$  and  $\tau_{\mathbb{A}} : [0, 1] \rightarrow \mathbb{A}$  defined by  $\tau_{\mathbb{A}}(t) = (1-t)x + ty$ . Let  $\phi : \mathbb{A} \rightarrow B$ . By Lemma 5.6,  $\pi_B := \rho_B \circ \tau_{\mathbb{A}}$  is a  $(\phi(x), \phi(y))$ -path of  $B$  from  $x$  to  $y$ . Therefore  $x \leq_B y$ . Let  $\psi = \phi^{-1} : B \rightarrow \mathbb{A}$ . Composing  $\phi$  by some  $w \in W^v$  if necessary, one can suppose that  $\psi(y) - \psi(x) \in \overline{C_f^v}$ .

By Lemma 5.6,  $\pi_{\mathbb{A}}$  is a  $(\psi(x), \psi(y))$ -path of  $\mathbb{A}$ . By Lemma 5.7, we deduce that  $y - x \leq_{Q^v} \psi(y) - \psi(x)$ .

By Lemma 5.4,  $\psi \circ \pi_B$  is an  $(x, y)$ -path of  $\mathbb{A}$  from  $\psi(x)$  to  $\psi(y)$ . By Lemma 5.7, we deduce that  $\psi(y) - \psi(x) \leq_{Q^v} y - x$ . Therefore  $x - y = \psi(x) - \psi(y)$  and  $\pi_{\mathbb{A}}$  is an  $(x, y)$ -path of  $\mathbb{A}$ .  $\square$

If  $x, y \in \mathcal{I}$ , one says that  $x \leq y$  if there exists an apartment  $A$  containing  $x, y$  and such that  $x \leq_A y$ . By Lemma 5.8, this does not depend on the choice of  $A$ : if  $x \leq y$  then for all apartment  $B$  containing  $x, y$ , one has  $x \leq_B y$ . However, one does not know yet that  $\leq$  is a preorder: the proof of Théorème 5.9 of [Rou11] uses (MAO).

The following lemma is Lemma 3.6 of [Héb17]:

**Lemma 5.9.** *Let  $\tau : [0, 1] \rightarrow \mathcal{I}$  be a segment such that  $\tau(0) \leq \tau(1)$ , such that  $\tau(1) \in \mathbb{A}$  and such that there exists  $\nu \in \overline{C_f^v}$  such that  $(\rho_{-\infty} \circ \tau)' = \nu$ . Then  $\tau([0, 1]) \subset \mathbb{A}$  and thus  $\rho_{-\infty} \circ \tau = \tau$ .*

*Proof.* Let  $A$  be an apartment such that  $\tau$  is a segment of  $A$ . Then  $\tau$  is increasing for  $\leq_A$  and thus  $\tau$  is increasing for  $\leq$ . Let  $x, y \in A$  such that  $\tau(t) = (1-t)x + ty$  for all  $t \in [0, 1]$ . Let us first prove that  $\tau$  is increasing for  $\leq$ . It suffices to prove that  $x \leq y$ . By (MA iii'), there exists  $u \in ]0, 1]$  such that there exists an apartment  $A$  containing  $\tau([0, u])$  and  $-\infty$ . Let  $\phi : A \xrightarrow{-\infty} \mathbb{A}$ . One has

$$\phi(\tau(u)) = \rho_{-\infty}(\tau(u)) = \rho_{-\infty}(\tau(0)) + u\nu = \phi(\tau(0)) + u\nu,$$

thus  $\phi(\tau(u)) \geq \phi(\tau(0))$  and hence  $\tau(u) \geq \tau(0)$ . As  $\tau$  is a segment of  $A$ , it suffices to prove that there exists  $u > 0$  such that  $\tau(u) \geq \tau(0)$ . Therefore  $\tau$  is increasing for  $\leq$ .

Suppose that  $\tau([0, 1]) \not\subset \mathbb{A}$ . Let  $u = \sup\{t \in [0, 1] \mid \tau(t) \notin \mathbb{A}\}$ . Let us prove that  $\tau(u) \in \mathbb{A}$ . If  $u = 1$ , this is our hypothesis. Suppose  $u < 1$ . Then by (MA2') applied to  $]\tau(u), \tau(1))$ ,  $\mathbb{A}$  contains  $\text{cl}_A(]\tau(u), \tau(1))$  and thus  $\mathbb{A}$  contains  $\tau(u)$ .

By (MA iii'), there exists an apartment  $B$  containing  $\tau((0, u]) \cup -\infty$  and by (MA4), there exists an isomorphism  $\phi : B \xrightarrow{\tau(u) - \overline{C_f^v}} \mathbb{A}$ . For all  $t \in [0, u]$ , near enough from  $u$ ,  $\phi(\tau(t)) = \rho_{-\infty}(\tau(t))$ . By hypothesis, for all  $t \in [0, u]$ ,  $\rho_{-\infty}(\tau(t)) \in \tau(u) - \overline{C_f^v}$ . Therefore for  $t$  near enough from  $u$ ,  $\phi(\tau(t)) = \tau(t) \in \mathbb{A}$ : this is absurd by choice of  $u$  and thus  $\tau([0, 1]) \subset \mathbb{A}$ .  $\square$

We can now prove Proposition 5.3:  $\mathcal{I}$  satisfies (MAO).



*Proof.* Let  $x, y \in \mathbb{A}$  such that  $x \leq_{\mathbb{A}} y$  and  $B$  be an apartment containing  $\{x, y\}$ . We suppose that  $y - x \in \overline{C_f^v}$ . Let  $\pi_{\mathbb{A}} : [0, 1] \rightarrow \mathbb{A}$  mapping each  $t \in [0, 1]$  on  $\rho_{-\infty}((1 - t)x +_B ty)$ . By Lemma 5.8,  $\pi_{\mathbb{A}}$  is an  $(x, y)$ -path from  $x$  to  $y$ . By Lemma 3.6,  $\pi_{\mathbb{A}}(t) = x + t(y - x)$  for all  $t \in [0, 1]$ . Then by Lemma 5.9,  $\pi_{\mathbb{A}}(t) = (1 - t)x +_B ty$  for all  $t \in [0, 1]$ . In particular  $[x, y] = [x, y]_B$  and thus  $\mathcal{I}$  satisfies (MAO).  $\square$

### 5.1.2 Equivalence of the axiomatics

As each chimney or face contains an extended chimney or a local face of the same type, if  $\text{cl} \in \mathcal{CL}_{\Lambda'}$ , (MA iii, cl) implies (MA iii'). Therefore a measure of type (2, cl) is also a measure of type 3.

If  $A$  is an apartment and  $F$  is a filter of  $A$ , then  $\text{cl}_A(F) \subset \text{cl}_A^{\#}(F)$ . Therefore for all  $\text{cl} \in \mathcal{CL}_{\Lambda'}$ , (MA2',  $\text{cl}^{\#}$ ) implies (MA2', cl) and (MA iii,  $\text{cl}^{\#}$ ) implies (MA iii, cl).

**Lemma 5.10.** *Let  $\text{cl} \in \mathcal{CL}_{\Lambda'}$  and  $\mathcal{I}$  be a measure of type (1, cl). Then  $\mathcal{I}$  is a measure of type (2, cl).*

*Proof.* By Theorem 4.22,  $\mathcal{I}$  satisfies (MA ii). By conséquence 2.2 3) of [Rou11],  $\mathcal{I}$  satisfies (MA iii, cl).  $\square$

By abuse of notation if  $\mathcal{I}$  is a measure of any type and if  $\mathfrak{q}, \mathfrak{q}'$  are adjacent sectors of  $\mathcal{I}$ , we denote by  $\mathfrak{q} \cap \mathfrak{q}'$  the maximal face of  $\overline{\mathfrak{q}} \cap \overline{\mathfrak{q}'}$ . This has a meaning by Section 3 of [Rou11] for measures of type 1 and by (MA ii) for measures of type 2 and 3.

**Lemma 5.11.** *Let  $\mathcal{I}$  be a measure of type 3. Let  $A$  be an apartment, and  $\mathcal{X}$  be a filter of  $A$  such that for all sector-germ  $\mathfrak{s}$  of  $\mathcal{I}$ , there exists an apartment containing  $\mathcal{X}$  and  $\mathfrak{s}$ . Then if  $B$  is an apartment containing  $\mathcal{X}$ ,  $B$  contains  $\text{cl}^{\#}(\mathcal{X})$  and there exists an isomorphism  $\phi : A \xrightarrow{\text{cl}^{\#}(\mathcal{X})} B$ .*

*Proof.* Let  $\mathfrak{q}$  and  $\mathfrak{q}'$  be sector-germs of  $A$  and  $B$  of the same sign. By (MA iii'), there exists an apartment  $C$  containing  $\mathfrak{q}$  and  $\mathfrak{q}'$ . Let  $\mathfrak{q}_1 = \mathfrak{q}, \dots, \mathfrak{q}_n = \mathfrak{q}'$  be a gallery of sector-germs from  $\mathfrak{q}$  to  $\mathfrak{q}'$  in  $C$ . One sets  $A_1 = A$  and  $A_{n+1} = B$ . By hypothesis, for all  $i \in \llbracket 2, n \rrbracket$  there exists an apartment  $A_i$  containing  $\mathfrak{q}_i$  and  $\mathcal{X}$ . For all  $i \in \llbracket 1, n - 1 \rrbracket$ ,  $\mathfrak{q}_i \cap \mathfrak{q}_{i+1}$  is a splayed chimney and  $A_i \cap A_{i+1} \supset \mathfrak{q}_i \cap \mathfrak{q}_{i+1}$ . Therefore  $A_i \cap A_{i+1}$  is enclosed and there exists  $\phi_i : A_i \xrightarrow{A_i \cap A_{i+1}} A_{i+1}$ . The set  $A_n \cap A_{n+1}$  is also enclosed and there exists  $\phi_n : A_n \xrightarrow{A_n \cap A_{n+1}} A_{n+1}$ .

If  $i \in \llbracket 1, n + 1 \rrbracket$ , one sets  $\psi_i = \phi_{i-1} \circ \dots \circ \phi_1$ . Then  $\psi_i$  fixes  $A_1 \cap \dots \cap A_i$ .

Let  $i \in \llbracket 1, n \rrbracket$  and suppose that  $A_1 \cap \dots \cap A_i$  is enclosed in  $A$ . As  $\psi_i$  fixes  $A_1 \cap \dots \cap A_i$ , one has  $A_1 \cap \dots \cap A_i = \psi_i(A_1 \cap \dots \cap A_i)$  is enclosed in  $A_i$ . Moreover,  $A_i \cap A_{i+1}$  is enclosed in  $A_i$  and thus  $A_1 \cap \dots \cap A_{i+1}$  is enclosed in  $A_i$ . Consequently  $A_1 \cap \dots \cap A_{i+1} = \psi_i^{-1}(A_1 \cap \dots \cap A_{i+1})$  is enclosed in  $A$ . Let  $X = A_1 \cap \dots \cap A_{n+1}$ . By induction,  $X$  is enclosed in  $A$  and  $\phi := \psi_n$  fixes  $X$ . As  $X \supset \mathcal{X}$ ,  $X \in \text{cl}^{\#}(\mathcal{X})$  and we get the lemma.  $\square$

**Lemma 5.12.** *Let  $\mathcal{I}$  be a measure of type 3. Then for all  $\text{cl} \in \mathcal{CL}_{\Lambda'}$ ,  $\mathcal{I}$  satisfies (MA iii, cl).*

*Proof.* Each face is included in the finite enclosure of a local face and each chimney is included in the finite enclosure of an extended chimney. Thus by Lemma 5.11, applied when  $\mathcal{X}$  is a local face and a germ of a chimney,  $\mathcal{I}$  satisfies (MA iii,  $\text{cl}^{\#}$ ). Consequently for all  $\text{cl} \in \mathcal{CL}_{\Lambda'}$ ,  $\mathcal{I}$  satisfies (MA iii, cl), hence (MA3, cl) and the lemma is proved.  $\square$

**Lemma 5.13.** *Let  $\mathcal{I}$  be a measure of type 3 and  $\text{cl} \in \mathcal{CL}_{\Lambda'}$ . Then  $\mathcal{I}$  satisfies (MA2', cl).*

*Proof.* If  $A$  is an apartment and  $F$  is a filter of  $A$ , then  $\text{cl}(F) \subset \text{cl}^\#(F)$ . Therefore it suffices to prove that  $\mathcal{I}$  satisfies (MA2',  $\text{cl}^\#$ ). We conclude the proof by applying Lemma 5.11 applied when  $\mathcal{X}$  is a point, a germ of a preordered segment.  $\square$

Using Proposition 5.3, we deduce that a measure of type 2 or 3 satisfies (MAO), as (MA4) is a consequence of (MA ii).

**Lemma 5.14.** *Let  $\mathcal{I}$  be a measure of type 3. Let  $\mathfrak{r}$  be a chimney of  $\mathbb{A}$ ,  $\mathfrak{r} = \mathfrak{r}(F^l, F^v)$ , where  $F^l$  (resp.  $F^v$ ) is a local face (resp. vectorial face) of  $A$ . Let  $\mathfrak{R}^\# = \text{germ}_\infty(\text{cl}^\#(F^l, F^v))$ . Let  $A$  be an apartment containing  $\mathfrak{r}$  and  $\mathfrak{R}^\#$  and such that there exists  $\phi : \mathbb{A} \xrightarrow{\mathfrak{R}^\#} A$ . Then  $\phi : \mathbb{A} \xrightarrow{\mathfrak{r}} A$ .*

*Proof.* One can suppose that  $F^v \subset \overline{C}_f^v$ . Let  $U \in \mathfrak{R}^\#$  such that  $U$  is enclosed,  $U \subset A \cap \mathbb{A}$  and such that  $U$  is fixed by  $\phi$ . One writes  $U = \bigcap_{i=1}^k D(\beta_i, k_i)$ , with  $\beta_1, \dots, \beta_k \in \Phi$  and  $(k_1, \dots, k_r) \in \prod_{i=1}^r \Lambda'_{\beta_i}$ .

Let  $\xi \in F^v$  such that  $U \in \text{cl}(F^l + F^v + \xi)$ . Let  $J = \{i \in \llbracket 1, k \rrbracket \mid \beta_i(\xi) \neq 0\}$ . As for all  $i \in \llbracket 1, r \rrbracket$ ,  $D(\beta_i, k_i) \supset n\xi$  for  $n \gg 0$ , one has  $\beta_i(\xi) > 0$  for all  $i \in J$ . One has  $U - \xi = \bigcap_{i=1}^k D(\beta_i, k_i + \beta_i(\xi))$ . Let  $\lambda \in [1, +\infty[$  such that for all  $i \in J$ , there exists  $\tilde{k}_i \in \Lambda'_{\beta_i}$  such that  $k_i + \beta_i(\xi) \leq \tilde{k}_i \leq k_i + \lambda\beta_i(\xi)$ . Let  $\tilde{U} = \bigcap_{i=1}^k D(\beta_i, \tilde{k}_i)$ . Then  $U - \xi \subset \tilde{U} \subset U - \lambda\xi$ . Therefore,  $\tilde{U} \in \mathfrak{r}$ . Let  $V' \in \mathfrak{r}$  such that  $V' \subset A \cap \mathbb{A}$  and such that  $V' + F^v \subset V'$ . Then  $V := \tilde{U} \cap V' \in \mathfrak{r}$ . Let  $v \in V$  and  $\delta \subset F^v$  be the ray based at 0 and containing  $\xi$ . By the proof of Proposition 5.4 of [Rou11] (which uses only (MA1), (MA2'), (MA3), (MA4) and (MAO)), there exists  $g_v : \mathbb{A} \xrightarrow{v+\delta} A$ . As  $V \subset U - \lambda\xi$ , there exists a shortening  $\delta'$  of  $v + \delta$  included in  $U$ . Then  $g_v^{-1} \circ \phi : \mathbb{A} \rightarrow \mathbb{A}$  fixes  $\delta'$ . Consequently,  $g_v^{-1} \circ \phi$  fixes the support of  $\delta'$  and thus  $\phi$  fixes  $v$ :  $\phi$  fixes  $V$ . Therefore  $\phi$  fixes  $\mathfrak{r}$  and the lemma follows.  $\square$

**Lemma 5.15.** *Let  $\mathcal{I}$  be a measure of type 3 and  $\text{cl} \in \mathcal{CL}_{\Lambda'}$ . Then  $\mathcal{I}$  satisfies (MA2'',  $\text{cl}$ ).*

*Proof.* Let  $\mathfrak{r} = \text{cl}(F^l, F^v)$  be a solid chimney of an apartment  $A$  and  $A'$  be an apartment containing  $\mathfrak{r}$ . One supposes that  $A = \mathbb{A}$ . Let  $\mathfrak{r}^\# = \text{cl}^\#(F^l, F^v)$  (resp.  $\mathfrak{r}_e = F^l + F^v$ ) and  $\mathfrak{R}^\#$  (resp.  $\mathfrak{R}_e$ ) be the germ of  $\mathfrak{r}^\#$  (resp.  $\mathfrak{r}_e$ ). By Lemma 5.11 applied with  $\mathcal{X} = \mathfrak{R}_e$ , there exists  $\phi : A \xrightarrow{\mathfrak{R}^\#} A'$ . By Lemma 5.14,  $\phi$  fixes  $\mathfrak{r}$  and thus  $\mathcal{I}$  satisfies (MA2'',  $\text{cl}$ ).  $\square$

We can now prove Theorem 5.2: let  $\text{cl} \in \mathcal{CL}_{\Lambda'}$ . By Lemma 5.10, a measure of type (1,  $\text{cl}$ ) is also a measure of type (2,  $\text{cl}$ ) and thus it is a measure of type 3. By Lemma 5.12, Lemma 5.13 and Lemma 5.15, a measure of type 3 is a measure of type (1,  $\text{cl}$ ) which concludes the proof of the theorem.

## 5.2 Friendly pairs in $\mathcal{I}$

Let  $\mathbb{A} = (\mathbb{A}, W, \Lambda')$  be an apartment. Let  $\mathcal{I}$  be a measure of type  $\mathbb{A}$ . We now use the finite enclosure  $\text{cl} = \text{cl}_{\Lambda'}^\#$ , which makes sense by Theorem 5.1. A family  $(F_j)_{j \in J}$  of filters in  $\mathcal{I}$  is said to be **friendly** if there exists an apartment containing  $\bigcup_{j \in J} F_j$ . In this section we obtain friendliness results for pairs of faces, improving results of Section 5 of [Rou11]. We will use it to give a very simple axiomatic of measures in the affine case. These kinds of results also have an interest on their own: the definitions of the Iwahori-Hecke algebra of [BPGR16] and of the parahorical Hecke algebras of [AH17] relies on the existence of apartments containing pairs of faces.

If  $x \in \mathcal{I}$ ,  $\epsilon \in \{-, +\}$  and  $A$  is an apartment, one denotes by  $\mathcal{F}_x$  (resp.  $\mathcal{F}^\epsilon$ ,  $\mathcal{F}^\epsilon(A)$ ,  $\mathcal{C}_x$ , ...) the set of faces of  $\mathcal{I}$  based at  $x$  (resp. and of sign  $\epsilon$ , and included in  $A$ , the set of chambers of  $\mathcal{I}$  based at  $x$ , ...). If  $\mathcal{X}$  is a filter, one denotes by  $\mathcal{A}(\mathcal{X})$  the set of apartments containing  $\mathcal{X}$ .

**Lemma 5.16.** *Let  $A$  be an apartment of  $\mathcal{I}$ ,  $a \in A$  and  $C_1, C_2 \in \mathcal{C}_a(A)$ . Let  $\mathcal{D}_a$  be the set of half-apartments of  $A$  whose wall contains  $a$ . Suppose that  $C_1 \neq C_2$ . Then there exists  $D \in \mathcal{D}_a$  such that  $D \supset C_1$  and  $D \not\supset C_2$ .*

*Proof.* Let  $C_1^v$  and  $C_2^v$  be vectorial chambers of  $A$  such that  $C_1 = F(a, C_1^v)$  and  $C_2 = F(a, C_2^v)$ . Suppose that for all  $D \in \mathcal{D}_a$  such that  $D \supset C_1$ , one has  $D \supset C_2$ . Let  $X \in C_1$ . There exists half-apartments  $D_1, \dots, D_k$  and  $\Omega \in \mathcal{V}_A(a)$  such that  $X \supset \bigcap_{i=1}^k D_i^\circ \supset \Omega \cap (a + C_1^v)$ .

Let  $J = \{j \in \llbracket 1, k \rrbracket \mid D_j \notin \mathcal{D}_a\}$ . For all  $j \in J$ , one chooses  $\Omega_j \in \mathcal{V}_A(a)$  such that  $D_j^\circ \supset \Omega_j$ . If  $j \in \llbracket 1, k \rrbracket \setminus J$ ,  $D_j \supset C_1$ , thus  $D_j \supset C_2$  and hence  $D_j^\circ \supset C_2$ . Therefore, there exists  $\Omega_j \in \mathcal{V}_A(a)$  such that  $D_j^\circ \supset \Omega_j \cap (x + C_2^v)$ . Hence,  $X \supset \bigcap_{j=1}^k D_j^\circ \supset (\bigcap_{j=1}^k \Omega_j) \cap (x + C_2^v)$ , thus  $X \in C_2$  and  $C_1 \supset C_2$ .

Let  $D \in \mathcal{D}_a$  such that  $D \supset C_2$ . Suppose that  $D \not\supset C_1$ . Let  $D'$  be the half-apartment opposite to  $D$ . Then  $D' \supset C_1$  and therefore  $D' \supset C_2$ : this is absurd. Therefore for all  $D \in \mathcal{D}_a$  such that  $D \supset C_2$ ,  $D \supset C_1$ . By the same reasoning we just did, we deduce that  $C_2 \supset C_1$  and thus  $C_1 = C_2$ . This is absurd and the lemma is proved.  $\square$

The following proposition improves Proposition 5.1 of [Rou11]. It is the analogue of the axiom (II) of buildings (see the introduction).

**Proposition 5.17.** *Let  $\{x, y\}$  be a friendly pair in  $\mathcal{I}$ .*

1. *Let  $A \in \mathcal{A}(\{x, y\})$  and  $\delta$  be a ray of  $A$  based at  $x$  and containing  $y$  (if  $y \neq x$ ,  $\delta$  is unique) and  $F_x \in \mathcal{F}_x$ . Then  $(\delta, F_x)$  is friendly. Moreover, there exists  $A' \in \mathcal{A}(\delta \cup F_x)$  such that there exists an isomorphism  $\phi : A \xrightarrow{\delta} A'$ .*
2. *Let  $(F_x, F_y) \in \mathcal{F}_x \times \mathcal{F}_y$ . Then  $(F_x, F_y)$  is friendly.*

*Proof.* We begin by proving 1. Let  $C_x$  be a chamber of  $\mathcal{I}$  containing  $F_x$ . Let  $C$  be a chamber of  $A$  based at  $x$  and having the same sign as  $C_x$ . By Proposition 5.1 of [Rou11], there exists an apartment  $B$  containing  $C_x$  and  $C$ . Let  $C_1 = C, \dots, C_n = C_x$  be a gallery in  $B$  from  $C$  to  $C_x$ . If  $i \in \llbracket 1, n \rrbracket$ , one sets  $\mathcal{P}_i$ : “there exists an apartment  $A_i$  containing  $C_i$  and  $\delta$  such that there exists an isomorphism  $\phi : A \xrightarrow{\delta} A_i$ ”. The property  $\mathcal{P}_1$  is true by taking  $A_1 = A$ . Let  $i \in \llbracket 1, n-1 \rrbracket$  such that  $\mathcal{P}_i$  is true. If  $C_{i+1} = C_i$ , then  $\mathcal{P}_{i+1}$  is true. Suppose  $C_i \neq C_{i+1}$ . Let  $A_i$  be an apartment containing  $C_i$  and  $\delta$ . By Lemma 5.16, there exists a half-apartment  $D$  of  $A$  whose wall contains  $x$  and such that  $C_i \subset D$  and  $C_{i+1} \not\subset D$ . As  $C_i$  and  $C_{i+1}$  are adjacent, the wall  $M$  of  $D$  is the wall separating  $C_i$  and  $C_{i+1}$ . By (MA2), there exists an isomorphism  $\phi : B \xrightarrow{C_i} A_i$ . Let  $M' = \phi(M)$  and  $D_1, D_2$  be the half-apartments of  $A_i$  delimited by  $M'$ . Let  $j \in \{1, 2\}$  such that  $D_j \supset \delta$ . By Proposition 2.9 1) of [Rou11], there exists an apartment  $A_{i+1}$  containing  $D_j$  and  $C_{i+1}$ . Let  $\psi_i : A \xrightarrow{\delta} A_i$  and  $\psi : A_i \xrightarrow{D_j} A_{i+1}$ . Then  $\psi \circ \psi_i : A \xrightarrow{\delta} A_{i+1}$ . Therefore  $\mathcal{P}_{i+1}$  is true. Consequently,  $\mathcal{P}_n$  is true, which proves 1.

Let us prove 2, which is very similar to 1. As a particular case of 1, there exists an apartment  $A'$  containing  $F_x$  and  $y$ . Let  $C_y$  be a chamber of  $\mathcal{I}$  containing  $F_y$ . Let  $C$  be a chamber of  $A'$  based at  $y$  and of the same sign as  $F_y$ . Let  $C_1 = C, \dots, C_n = C_y$  be a gallery of chambers from  $C$  to  $C_y$  (which exists by Proposition 5.1 of [Rou11]). By the same reasoning as above, for all  $i \in \llbracket 1, n \rrbracket$ , there exists an apartment containing  $F_x$  and  $C_i$ , which proves 2.  $\square$

### 5.3 Existence of isomorphisms fixing preordered pairs of faces

A filter  $\mathcal{X}$  of an apartment  $A$  is said to be **intrinsic** if for all apartment  $B \in \mathcal{A}(\mathcal{X})$  (which means that  $B$  contains  $\mathcal{X}$ ),  $\text{conv}_A(\mathcal{X}) \subset A \cap B$  and there exists an isomorphism  $\phi : A \xrightarrow{\text{conv}_A(\mathcal{X})} B$ .

The aim of this subsection is to prove the theorem below. It improves Proposition 5.2, Proposition 5.5 of [Rou11] and Proposition 1.10 of [BPGR16]. We will not use it in the simplification of the axioms of measures in the affine case.

**Theorem 5.18.** *Let  $x, y \in \mathcal{I}$  such that  $x \leq y$  and  $(F_x, F_y) \in \mathcal{F}_x \times \mathcal{F}_y$ . Then  $F_x \cup F_y$  is intrinsic.*

To prove this our main tool will be Lemma 5.20, which establishes that under some conditions, if  $P$  is included in two apartments and is fixed by an isomorphism of apartments, the convex hull of  $P$  is also included in these apartments. We first treat the case where one of the faces is a chamber and the other one is spherical.

#### 5.3.1 Convex hull of a set fixed by an isomorphism

The following lemma is stated in the proof of Proposition 5.4 of [Rou11]:

**Lemma 5.19.** *Let  $x, y \in \mathcal{I}$  such that  $x \leq y$  and  $x \neq y$  and  $A, B \in \mathcal{A}(\{x, y\})$ . Let  $\delta_A$  be the ray of  $A$  based at  $x$  and containing  $y$  and  $\delta_B$  be the ray of  $B$  based at  $y$  and containing  $x$ . Then there exists an apartment containing  $L = \delta_A \cup \delta_B$  and in this apartment,  $L$  is a line.*

If  $A$  is an affine space and  $\mathcal{X}, \mathcal{X}'$  are two segment-germs, infinite intervals or rays, one says that  $\mathcal{X}$  and  $\mathcal{X}'$  are parallel if the line spanned by  $\mathcal{X}$  is parallel to the line spanned by  $\mathcal{X}'$ .

The following lemma is a kind of reciprocal of Proposition 3.26.

If  $P \subset \mathbb{A}$  and  $k \in \mathbb{N}$ ,  $\text{conv}_k(P)$  was defined in Lemma 4.8.

**Lemma 5.20.** *Let  $A$  be an apartment. Suppose that there exists  $P \subset A \cap \mathbb{A}$  satisfying the following conditions:*

1. *there exists  $u \in \overset{\circ}{\mathcal{T}}$  such that for all  $x \in P$ , there exists  $U_x \in \mathcal{V}_{[-1,1]}(0)$  such that  $x + U_x \cdot u \subset P$*
2. *there exists an isomorphism  $\phi : \mathbb{A} \xrightarrow{P} A$ .*

*Then  $\overline{\text{conv}(P)} \subset A \cap \mathbb{A}$  and  $\phi$  fixes  $\overline{\text{conv}(P)}$ .*

*Proof.* If  $k \in \mathbb{N}$ , one sets  $\mathcal{P}_k$ : “ $\text{conv}_k(P) \subset A \cap \mathbb{A}$  and  $\text{conv}_k(P)$  satisfies 1 and 2”. Let us prove that  $\mathcal{P}_k$  is true for all  $k \in \mathbb{N}$ . As if  $k \in \mathbb{N}$ ,  $\text{conv}_{k+1}(P) = \text{conv}_1(\text{conv}_k(P))$ , it suffices to prove that  $\mathcal{P}_1$  is true.

Let  $x, y \in P$ . Let us prove that  $[x, y] \subset A \cap \mathbb{A}$ . If  $\epsilon \in \{-, +\}$ , one denotes by  $\delta_{\mathbb{A}}^{\epsilon\infty}$  the direction of  $\mathbb{R}_{\epsilon}^* \nu$  and by  $\delta_A^{\epsilon\infty}$  the direction of  $\phi(\mathbb{R}_{\epsilon}^* u)$ . Let  $x' \in (x + \delta_{\mathbb{A}}^{+\infty}) \setminus \{x\}$  (resp.  $y' \in (y + \delta_{\mathbb{A}}^{+\infty}) \setminus \{y\}$ ) such that  $[x, x']$  (resp.  $[y, y']$ ) is included in  $A$  and fixed by  $\phi$ . By Lemma 5.19,  $L_x = (x + \delta_{\mathbb{A}}^{+\infty}) \cup (x' + \delta_A^{-\infty})$  and  $L_y = (y + \delta_{\mathbb{A}}^{+\infty}) \cup (y' + \delta_A^{-\infty})$  are lines of some apartments of  $\mathcal{I}$ .

Let  $\mathfrak{F}_{\mathbb{A}}^{+\infty}$  (resp.  $\mathfrak{F}_{\mathbb{A}}^{-\infty}$ ) be the face of  $\mathcal{I}^{\infty}$  containing  $\delta_{\mathbb{A}}^{+\infty}$  (resp.  $\delta_{\mathbb{A}}^{-\infty}$ ). By (MA3), there exists an apartment  $A'$  containing the germs of  $x + \mathfrak{F}_{\mathbb{A}}^{+\infty}$  and of  $y + \mathfrak{F}_{\mathbb{A}}^{-\infty}$ . Then  $A'$  contains

shortenings  $x'' + \delta_{\mathbb{A}}^{+\infty}$  and  $y'' + \delta_{\mathbb{A}}^{-\infty}$  of  $x + \delta_{\mathbb{A}}^{+\infty}$  and of  $y + \delta_{\mathbb{A}}^{-\infty}$ . Then  $A'$  contains  $(x'' + \delta_{\mathbb{A}}^{+\infty}) + \delta_{\mathbb{A}}^{-\infty}$  and  $(y'' + \delta_{\mathbb{A}}^{-\infty}) + \delta_{\mathbb{A}}^{+\infty}$ .

Let  $A_x$  be an apartment containing  $L_x$ . The apartment  $A_x$  contains  $\delta_{\mathbb{A}}^{+\infty}$  and  $\delta_{\mathbb{A}}^{-\infty}$ , and  $\delta_{\mathbb{A}}^{+\infty}$  is opposite to  $\delta_{\mathbb{A}}^{-\infty}$  in  $A_x$ . Therefore  $(x'' + \delta_{\mathbb{A}}^{+\infty}) + \delta_{\mathbb{A}}^{-\infty}$  is a line of  $A_x$  and as  $x'' + \delta_{\mathbb{A}}^{+\infty} \subset L_x$ ,  $(x'' + \delta_{\mathbb{A}}^{+\infty}) + \delta_{\mathbb{A}}^{-\infty} = L_x$ . By the same reasoning,  $A' \supset L_y$ . In particular,  $A'$  contains  $x$  and  $y$ . By Theorem 4.22,  $[x, y] = [x, y]_{A'} = [x, y]_A$ . Therefore, for all  $x, y \in P$ ,  $[x, y] \subset A \cap \mathbb{A}$ .

Consequently if  $x, y \in P$  and  $U \in \mathcal{V}_{[-1,1]}(0)$  is convex such that  $x + U.u \subset P$ ,  $y + U.u \subset P$ , then  $\text{conv}_1(x + U.u, y + U.u) = \text{conv}(x + U.u, y + U.u) \subset A \cap \mathbb{A}$ . By Proposition 3.26, there exists  $\psi : \mathbb{A} \xrightarrow{\text{conv}(x+U.u, y+U.u)} \mathbb{A}$ . The isomorphism  $\phi^{-1} \circ \psi : \mathbb{A} \rightarrow \mathbb{A}$  fixes  $x + U.u \cup y + U.u$ , hence it fixes its support; it fixes in particular  $\text{conv}(x + U.u, y + U.u)$ . Thus  $\phi$  fixes  $\text{conv}_1(P)$ :  $\text{conv}_1(P)$  satisfies 2. Moreover  $\text{conv}_1(P)$  satisfies 1 and it follows that for all  $k \in \mathbb{N}$ ,  $\mathcal{P}_k$  is true.

By Lemma 4.8,  $\text{conv}(P) \subset A \cap \mathbb{A}$  and  $\phi$  fixes  $\text{conv}(P)$ . We conclude by using Proposition 3.9 and Lemma 3.25.  $\square$

**Lemma 5.21.** *Let  $A$  be an apartment and  $\mathfrak{q}, \mathfrak{q}'$  be opposite sector germs of  $A$ . Then  $A$  is the unique apartment containing  $\mathfrak{q}$  and  $\mathfrak{q}'$ .*

*Proof.* One identifies  $A$  and  $\mathbb{A}$ . Let  $Q = 0 + \mathfrak{q}$  and  $Q' = 0 + \mathfrak{q}'$ . One has  $Q' = -Q$ . Let  $(e_j)_{j \in J}$  be a basis of  $\mathbb{A}$  such that  $\sum_{j \in J} \mathbb{R}_+^* e_j \subset Q$  and  $(e_j^*)$  be the dual basis of  $\mathbb{A}$ . Let  $A'$  be an apartment containing  $\mathfrak{q}$  and  $\mathfrak{q}'$ . Let  $M \in \mathbb{R}_+^*$  such that  $A' \supset \pm\{x \in \mathbb{A} \mid \forall j \in J, e_j^*(x) \geq M\}$ . Let  $a \in \mathbb{A}$ . There exists  $b \in \{x \in \mathbb{A} \mid \forall j \in J, e_j^*(x) \geq M\}$  such that  $a \in b + \mathfrak{q}'$ . Therefore  $a \in A'$  and  $A' \supset \mathbb{A}$ . By (MA ii), there exists an isomorphism  $\phi : \mathbb{A} \xrightarrow{\mathbb{A} \cap A'} A'$ . Therefore  $\mathfrak{q}, \mathfrak{q}'$  are two opposite sector-germs of  $A'$  and by symmetry,  $\mathbb{A} \supset A'$ . Thus  $\mathbb{A} = A'$ .  $\square$

**Proposition 5.22.** *Let  $A$  and  $B$  be two apartments such that  $A \cap B$  has nonempty interior. Then  $A \cap B$  is convex if and only if there exists an apartment  $A'$  such that  $d_+(A, A') = d_-(A', B) = 0$  and such that  $A \cap B \subset A'$ .*

*Proof.* The implication  $\Leftarrow$  is a corollary of Theorem 4.22. Suppose that  $A \cap B$  is convex. One identifies  $A$  and  $\mathbb{A}$ . By Proposition 3.26, there exists an isomorphism  $\phi : \mathbb{A} \xrightarrow{\mathbb{A} \cap B} B$ . Let  $\nu \in C_f^v$  and  $\delta_{\mathbb{A}}^{+\infty}$  (resp.  $\delta_B^{-\infty}$ ) be the direction of  $\mathbb{R}_+^* \nu$  (resp. of  $\phi(\mathbb{R}_+^* \nu)$ ). By Lemma 5.19, if  $x \in \text{Int}(\mathbb{A} \cap B)$ ,  $L_x = x + \delta_{\mathbb{A}}^{+\infty} \cup x + \delta_B^{-\infty}$  is a line of some apartment  $A_x$ . One has  $\text{cl}(x + \delta_{\mathbb{A}}^{+\infty}) \supset x + C_f^v$  and  $\text{cl}(x + \delta_B^{-\infty}) \supset \phi(x - C_f^v)$ . Moreover,  $x + C_f^v$  and  $\phi(x - C_f^v)$  are opposite in  $A_x$  and thus  $A_x = \text{conv}(+\infty, \text{germ}_\infty(\phi(x - C_f^v)))$ . In particular  $A_x$  is an apartment containing  $+\infty$  and  $\text{germ}_\infty(\phi(-C_f^v))$ . By Lemma 5.21,  $A_x$  does not depend on  $x$  and we denote it  $A'$ . If  $x \in \text{Int}(\mathbb{A} \cap B)$ ,  $x \in L_x \subset A_x = A'$  and thus  $\text{Int}(\mathbb{A} \cap B) \subset A'$ . By Proposition 3.9,  $\mathbb{A} \cap B \subset A'$ , which proves the proposition.  $\square$

### 5.3.2 Case of a pair chamber-spherical face

**Lemma 5.23.** *Let  $A$  be an apartment,  $x, y \in A$  such that  $x \leq y$  and  $C \in \mathcal{C}_x^+(A)$ . Then if  $B \in \mathcal{A}(C \cup [x, y])$ , there exists  $C' \in \mathcal{C}_x^+(A)$  such that  $C' \subset B$ ,  $\overline{C'} \supset [x, y]$  and such that there exists a sector  $Q$  of  $A$  based at  $x$  and satisfying  $\overline{Q} \supset [x, y]$  and  $C' = \text{germ}_x(Q)$ .*

*Proof.* One identifies  $A$  and  $\mathbb{A}$ . We call a subset of  $\mathbb{A}$  a face of  $\mathcal{T}_x$  if it is of the shape  $x + F^v$  for some positive vectorial face  $F^v$ . Let  $X \in C$  such that  $X$  is open and convex. Let  $z' \in X$  such that  $x \prec z'$ . Let  $y' \in [x, y]$  such that  $y' \prec z'$  and  $z \in X$  such that  $y' \prec z$  and such that  $z - y'$  is not included in a direction of a wall of  $\mathbb{A}$ . By Proposition 4.4.13 of [Bar96], there exists a finite number  $F_1, \dots, F_k$  of faces of  $\mathcal{T}_x$  such that  $[z, y']$  is included in  $F_1 \cup \dots \cup F_k$ .

One identifies  $[0, 1]$  and  $[z, y']$ . Then maybe renumbering and decreasing  $k$ , one can suppose that there exists  $t_1 = 0 = z < t_2 < \dots < t_{k+1} = 1 = y'$  such that for all  $i \in \llbracket 1, k \rrbracket$ ,  $[t_i, t_{i+1}] \subset F_i$ . As  $[t_k, t_{k+1}]$  is not included in any wall,  $F_k$  is necessarily a chamber. Let  $F_k^v$  be the vectorial face such that  $F_k = x + F_k^v$ . By order convexity,  $[z, y'] \subset \mathbb{A} \cap B$  and thus  $\frac{1}{2}(t_k + t_{k+1}) \in \mathbb{A} \cap B$ . As  $x \leq \frac{1}{2}(t_k + t_{k+1})$ ,  $[x, \frac{1}{2}(t_k + t_{k+1})] \subset \mathbb{A} \cap B$  and by Proposition 5.4 of [Rou11],  $\text{cl}([x, \frac{1}{2}(t_k + t_{k+1})]) \subset \mathbb{A} \cap B$ . Let  $Q$  be the sector containing the ray based at  $x$  and containing  $\frac{1}{2}(t_k + t_{k+1})$ . As  $\text{cl}([x, \frac{1}{2}(t_k + t_{k+1})])$  contains the chamber  $C' = F(x, F_k^v)$ , we get the lemma.  $\square$

**Lemma 5.24.** *Let  $A$  be an apartment,  $x \in A$ ,  $Q$  a sector of  $A$  based at  $x$  and  $y \in A$  such that  $x \leq y$  and  $[x, y] \subset \overline{Q}$ . Let  $C' = \text{germ}_x(Q) \in \mathcal{C}_x^+(A)$ . Then if  $B \in \mathcal{A}(C' \cup [x, y])$ ,  $A \cap B$  contains  $\text{conv}_A(C', [x, y])$ .*

*Proof.* Let  $X \in C'$  such that  $X \subset A \cap B$  and  $z \in X \cap Q$ . Let  $\delta_A^x$  be the ray of  $A$  based at  $x$  and containing  $z$  and  $\delta_B^z$  be the ray of  $B$  based at  $z$  and containing  $[x, z]$ . By Lemma 5.19,  $L = \delta_A^x \cup \delta_B^z$  is a line of some apartment  $A'$ . By Theorem 4.22,  $A \cap A'$  is enclosed. As the enclosure of  $\delta_A^x$  contains  $\overline{Q}$ ,  $A' \supset [x, y] \cup C'$  and  $\text{conv}_A(C', [x, y]) = \text{conv}_{A'}(C', [x, y])$ . Using Theorem 4.22 again, we get that  $\text{conv}_{A'}(C', [x, y]) = \text{conv}_B(C', [x, y])$  and the lemma follows.  $\square$

**Lemma 5.25.** *Let  $A$  be an affine space and  $U, V \subset A$  be two convex subsets. Then  $\text{conv}_1(U, V) = \text{conv}(U, V)$ .*

*Proof.* The inclusion  $\text{conv}_1(U, V) \subset \text{conv}(U, V)$  is clear. Let  $a \in \text{conv}(U, V)$ . There  $k, l \in \mathbb{N}$  exists  $u_1, \dots, u_k \in U$ ,  $v_{k+1}, \dots, v_{k+l} \in V$  and  $(\lambda_i) \in [0, 1]^{k+l}$  such that  $\sum_{i=1}^{k+l} \lambda_i = 1$  and  $a = \sum_{i=1}^k \lambda_i u_i + \sum_{i=k+1}^{k+l} \lambda_i v_i$ . Let  $t = \sum_{i=1}^k \lambda_i$ . If  $t \in \{0, 1\}$ ,  $a \in U \cup V$ . Suppose  $t \notin \{0, 1\}$ . Then  $a = t \sum_{i=1}^k \frac{\lambda_i}{t} u_i + (1-t) \sum_{i=k+1}^{k+l} \frac{\lambda_i}{(1-t)} v_i \in \text{conv}_1(U, V)$ , which proves the lemma.  $\square$

**Lemma 5.26.** *Let  $x, y \in \mathcal{I}$  such that  $x \leq y$ ,  $(F_x, C_y) \in \mathcal{F}_x^+ \times \mathcal{C}_y^+$  with  $F_x$  spherical. Then  $F_x \cup C_y$  is intrinsic.*

*Proof.* Let  $A, B \in \mathcal{A}(F_x, C_y)$ . Let  $X \in C_y$  be convex, open, such that  $X \subset A \cap B$  and such that there exists an isomorphism  $\psi : A \xrightarrow{X} B$ . Let  $z \in \overset{\circ}{X}$  such that  $y \overset{\circ}{<} z$ . Then  $x \overset{\circ}{<} z$ . Let  $X' \in F_x$  such that  $X' \subset A \cap B$ ,  $X'$  is convex, relatively open and such that there exists an isomorphism  $\psi' : A \xrightarrow{X'} B$ . Let  $U \in \mathcal{V}_{X'}(x)$  be convex and  $V \in \mathcal{V}_A(z)$  be convex and open and such that for all  $(u, v) \in U \times V$ ,  $u \overset{\circ}{<} v$ . Let  $P = \text{conv}(U, V) = \text{conv}_1(U, V)$ . Then  $P \subset A \cap B$  and by Proposition 3.26 there exists an isomorphism  $\phi : A \xrightarrow{P} B$ . As  $P \cap V$  has nonempty interior,  $\phi = \psi$ . As  $\phi|_U = \psi'|_U$ ,  $\phi$  fixes  $X'$ . Therefore  $\phi$  fixes  $\overline{\text{Int}_r(X') \cup \text{Int}(X)}$ . As  $F_x$  is spherical, one can apply Lemma 5.20 and we deduce that  $\overline{\text{conv}_A(\text{Int}_r(X'), \text{Int}(X))} \subset A \cap B$  and is fixed by  $\phi$  and the lemma follows.  $\square$

**Lemma 5.27.** *Let  $A$  be an apartment,  $x, y \in \mathcal{I}$  such that  $x \leq y$  and  $Q$  be a sector of  $A$  based at  $x$  and such that  $[x, y] \subset \overline{Q}$ . Let  $C_x = F(x, F^l(x, Q))$  and  $F_y \in \mathcal{F}_y$  spherical. Then  $C_x \cup F_y$  is intrinsic.*

*Proof.* Let  $B \in \mathcal{A}(C_x \cup F_y)$ . Let  $X_1 \in \text{conv}_A(C_x, [x, y])$  such that  $X_1$  is convex and  $X_1 \subset A \cap B$ , which exists by Lemma 5.24. Let  $P_1 = \overset{\circ}{X}_1$ . By Proposition 3.26, there exists an isomorphism  $\phi_1 : A \xrightarrow{P_1} B$ .

If  $F_y$  is positive, one sets  $\overset{\circ}{R} = \overset{\circ}{<}$  and if  $F_y$  is negative, one sets  $\overset{\circ}{R} = \overset{\circ}{>}$ . Let  $X_2 \in F_y$  such that  $X_2 \subset A \cap B$ , such that there exists  $\phi_2 : A \xrightarrow{X_2} B$  and such that  $X_2$  is convex and

$P_2 = \text{Int}_r(X_2)$ . Let  $z \in P_2$  such that  $y \overset{\circ}{R} z$ . Let  $(U, V)$  be such that  $V \in \mathcal{V}_{P_2}(z)$  is convex and relatively open and  $U \subset P_1$  is open and convex and such that for all  $u \in U$  and  $v \in V$ ,  $u \overset{\circ}{R} v$ . Then  $P_3 = \text{conv}(U, V) = \text{conv}_1(U, V)$  is included in  $A \cap B$ . By Proposition 3.26, there exists  $\phi_3 : A \xrightarrow{P_3} B$ .

The map  $\phi_1$  fixes  $P_1$ ,  $\phi_3$  fixes  $P_3$  and  $P_1 \cap P_3$  has nonempty interior. Therefore  $\phi_1 = \phi_3$ . As  $\phi_{3|V} = \phi_{2|V}$ ,  $\phi_{3|\text{supp}(V)} = \phi_{2|\text{supp}(V)}$  and  $\phi_3$  fixes  $P_2$ . Consequently,  $\phi_1$  fixes  $P_1 \cup P_2$ .

One identifies  $A$  and  $\mathbb{A}$ . Let  $u_1 \in \text{supp}(\vec{P}_2) \cap \vec{\mathcal{T}}$ , which exists because  $F_y$  is spherical. As  $P_2$  is relatively open and  $P_1$  is open,  $P_1 \cup \overline{P_2}$  satisfies conditions 1 and 2 of Lemma 5.20 (with  $u = u_1$  and  $P = P_1 \cup P_2$ ). Consequently  $\overline{\text{conv}(P_1 \cup P_2)} \subset A \cap B$  and  $\phi_1$  fixes  $\overline{\text{conv}(P_1 \cup P_2)}$ . Moreover,  $P_1 \in C_x$  and

$$\overline{\text{conv}(P_1 \cup P_2)} \supset \text{conv}(P_1 \cup \overline{P_2}) = \text{conv}(P_1 \cup X_2) \supset \text{conv}(C_x \cup F_y).$$

Therefore  $C_x \cup F_y$  is intrinsic, which is our assertion.  $\square$

**Lemma 5.28.** *Let  $x, y \in \mathcal{I}$  such that  $x \leq y$ ,  $C_x \in \mathcal{C}_x^+$  and  $F_y \in \mathcal{F}_y$  spherical. Then  $C_x \cup F_y$  is intrinsic.*

*Proof.* Let  $A, B \in \mathcal{A}(C_x \cup F_y)$ , which exist by Proposition 5.17. Let  $Q$  be a sector of  $A$  based at  $x$  such that  $[x, y] \subset \overline{Q}$  and such that  $C' = F(x, F^l(x, Q)) \subset A \cap B$ , which is possible by Lemma 5.23. Let  $X_2 \in F_y$  (resp.  $X' \in C'$ ) such that  $X_2$  is relatively open, included in  $A \cap B$  and such that there exists  $\phi : A \xrightarrow{X' \cup X_2} B$ , which exists by Lemma 5.27. Let  $X \in C$  open, included in  $A \cap B$  and such that there exists an isomorphism  $\psi : A \xrightarrow{X \cup X'} B$ , which is possible (reducing  $X'$  if necessary) by Proposition 5.2 of [Rou11]. Then  $\phi = \psi$  and we conclude with Lemma 5.20.  $\square$

### 5.3.3 Conclusion

In order to deduce Theorem 5.18 from Lemmas 5.26 and 5.28, we first prove that if  $C$  and  $C'$  are chambers of the same sign dominating some face  $F$ , there exists a gallery of chambers dominating  $F$  from  $C$  to  $C'$ , which is Lemma 5.32.

**Lemma 5.29.** *Let  $C^v$  (resp.  $F^v$ ) be a positive chamber (resp. positive face) of  $\mathbb{A}$ . Then  $\text{conv}(C^v, F^v)$  contains a generic ray  $\delta$  based at 0 such that  $\text{cl}(\delta) \supset F^v$ .*

*Proof.* Let  $(a, b) \in C^v \times F^v$  such that  $b - a$  is not included in any wall direction of  $\mathbb{A}$  and such that  $\text{cl}(\mathbb{R}_+^* b) = F^v$ . By Proposition 4.4.13 of [Bar96], there exists a finite number  $F_1^v, \dots, F_k^v$  of vectorial faces such that  $[a, b]$  is included in  $F_1^v \cup \dots \cup F_k^v$ . One identifies  $[0, 1]$  and  $[a, b]$ . Then maybe renumbering and decreasing  $k$ , one can suppose that there exists  $t_1 = 0 = a < t_2 < \dots < t_{k+1} = 1 = b$  such that for all  $i \in \llbracket 1, k \rrbracket$ ,  $[t_i, t_{i+1}] \subset F_i^v$ . As  $[t_k, t_{k+1}]$  is not included in any wall,  $F_k^v$  is necessarily a chamber. Let  $\delta = \mathbb{R}_+^*(t_k + t_{k+1})$ . Then  $\text{cl}(\delta) = \overline{F_k^v} \supset F^v$ , which is our assertion.  $\square$

**Lemma 5.30.** *Let  $x \in \mathbb{A}$ ,  $\epsilon \in \{-, +\}$  and  $C, C' \in \mathcal{C}_x^\epsilon$ . Suppose that  $C \supset C'$ . Then  $C = C'$ .*

*Proof.* Suppose that  $C \neq C'$ . Then there exists  $X \in C' \setminus C$ . There exists disjoint finite sets  $J, J^\circ$  and a family  $(D_j)_{j \in J \cup J^\circ}$  of half-apartments such that  $X \supset \bigcap_{j \in J} D_j \cap \bigcap_{j \in J^\circ} D_j^\circ$  and such that  $D_j \in C'$  for all  $j \in J$  and  $D_{j'}^\circ \in C'$  for all  $j' \in J^\circ$ . There exists  $j \in J \cup J^\circ$  such that  $D_j$  or  $D_j^\circ$  is not in  $C$ . Let  $D_j'$  be the half-apartment of  $\mathbb{A}$  opposite to  $D_j$ . Then  $D_j' \in C \setminus C'$ : a contradiction. Thus  $C = C'$ .  $\square$

**Lemma 5.31.** *Let  $a \in \mathbb{A}$  and  $F = F(a, F^v)$  be a positive face based at  $a$ , with  $F^v$  a vectorial face of  $\mathbb{A}$ . Let  $C$  be a positive chamber based at  $a$  and dominating  $F$ . Then there exists a vectorial chamber  $C^v$  dominating  $F^v$  such that  $C = F(a, C^v)$ .*

*Proof.* Let  $\mathcal{D}$  (resp.  $\mathcal{D}_a$ ) be the set of half-apartments of  $\mathbb{A}$  (resp. whose wall contains  $a$ ). Let  $C^{v'}$  be a vectorial chamber such that  $C = F(a, C^{v'})$ . Let  $\delta \subset \text{conv}(C^{v'}, F^v)$  such that  $\text{cl}(\delta)$  is a vectorial chamber  $C^v$  containing  $F^v$ , which exists by Lemma 5.29. Let us prove that  $C = F(a, C^v)$ .

Let  $X' \in C$ . Then there exists disjoint finite sets  $J$  and  $J^\circ$  and a family  $(D_j) \in \mathcal{D}^{J \cup J^\circ}$  such that if  $X = \bigcap_{j \in J} D_j \cap \bigcap_{k \in J^\circ} D_k^\circ$ ,  $X' \supset X \supset F^l(a, C^{v'})$ . One has  $\overline{X} \in \overline{C}$  and thus  $\overline{X} \supset F$ . Moreover, as  $X$  is nonempty,  $\overline{X} = \bigcap_{j \in J} D_j \cap \bigcap_{k \in J^\circ} D_k$ . Let  $L = \{j \in J \cup J^\circ \mid D_j \in \mathcal{D}_a\}$ . One has  $X = \overline{\Omega} \cap \bigcap_{l \in L} D_l$ , where  $\Omega \in \mathcal{V}_{\mathbb{A}}(a)$  is a finite intersection of open half-apartments. Let  $l \in L$ . Then  $D_l \supset a + C^{v'} \cup a + F^v$  and thus  $D_l \supset a + \text{conv}(C^{v'}, F^v)$ . Therefore  $D_l \supset a + \delta$  and hence  $D_l \supset a + C^v$ . As  $C^v$  does not meet any direction of a wall of  $\mathbb{A}$ ,  $a + C^v \cap M_l = \emptyset$ , where  $M_l$  is the wall of  $D_l$ . Consequently,  $D_l^\circ \supset a + C^v$  and thus  $X \in F(a, C^v)$ . Therefore  $F(a, C^v) \subset F(a, C^{v'}) = C$ . By Lemma 5.30,  $C = F(a, C^v)$  and the lemma follows.  $\square$

**Type of a vectorial face** Let  $F^v$  be a positive vectorial face. Then  $F^v = w.F^v(J)$  for some  $w \in W^v$  and  $J \subset I$  (see 2.1.2). The **type** of  $F^v$  is  $J$ . This does not depend on the choice of  $w$  by Section 1.3 of [Rou11].

If  $x \in \mathcal{I}$  and  $F \in \mathcal{F}_x$ , we denote by  $\mathcal{C}_F$  the set of chambers of  $\mathcal{I}$  dominating  $F$ .

**Lemma 5.32.** *Let  $x \in \mathcal{I}$  and  $F \in \mathcal{F}_x^+$ . Then if  $C, C' \in \mathcal{C}_F$ , there exists a gallery  $C = C_1, \dots, C_n = C'$  of elements of  $\mathcal{C}_F$  from  $C$  to  $C'$ .*

*Proof.* By Proposition 5.1 of [Rou11] (or Proposition 5.17), there exists  $A \in \mathcal{A}(C \cup C')$ . One identifies  $A$  and  $\mathbb{A}$ . One writes  $F = F(x, F^v)$ , where  $F^v$  is a positive vectorial face of  $\mathbb{A}$ . By Lemma 5.31, one can write  $C = F(x, C^v)$  and  $C' = F(x, C'^v)$ , where  $C^v$  and  $C'^v$  are two positive vectorial chambers dominating  $F^v$ . Let  $J \subset I$  be the type of  $F^v$ . There exists  $w \in W^v$  such that  $C'^v = w.C^v$ . Then  $w.F^v$  is the face of  $C'^v$  of type  $J$ :  $w.F^v = F^v$ . By Section 1.3 of [Rou11], the stabilizer  $W_{F^v}$  of  $F^v$  is conjugated to  $W^v(J) = \langle r_j \mid j \in J \rangle$ . As  $(W^v(J), \{r_j \mid j \in J\})$  is a Coxeter system, there exists a gallery  $C_1^v = C^v, \dots, C_n^v = C'^v$  from  $C^v$  to  $C'^v$  such that  $F^v \subset \overline{C_i^v}$  for all  $i \in \llbracket 1, n \rrbracket$ . We set  $C_i = F(x, C_i^v)$  for all  $i \in \llbracket 1, n \rrbracket$  and  $C_1, \dots, C_n$  has the desired property.  $\square$

If  $x \in \mathcal{I}$ ,  $\epsilon \in \{-, +\}$  and  $C, C' \in \mathcal{C}_x^\epsilon$  are different and adjacent, one denotes by  $C \cap C'$  the face between  $C$  and  $C'$  in any apartment containing  $C$  and  $C'$ . This is well defined by Proposition 5.1 and Proposition 5.2 of [Rou11].

**Lemma 5.33.** *Let  $a, b \in \mathcal{I}$ ,  $\epsilon \in \{-, +\}$  and  $C_a \in \mathcal{C}_a$ . Suppose that for all  $F_b \in \mathcal{F}_b^\epsilon$  spherical,  $C_a \cup F_b$  is intrinsic. Then for all  $F_b \in \mathcal{F}_b^\epsilon$ ,  $C_a \cup F_b$  is intrinsic.*

*Proof.* Let  $F_b \in \mathcal{F}_b^\epsilon$ . Let  $A, B \in \mathcal{A}(C_a \cup F_b)$ . Let  $C_b^A \in \mathcal{C}_{F_b}^\epsilon(A)$  and  $C_b^B \in \mathcal{C}_{F_b}^\epsilon(B)$ . Let  $C_1 = C_b^A, \dots, C_n = C_b^B$  be a gallery of chambers of  $\mathcal{C}_{F_b}^\epsilon$  from  $C_b^A$  to  $C_b^B$ , which exists by Lemma 5.32. One sets  $A_1 = A$  and  $A_n = B$ . By Proposition 5.17, for all  $i \in \llbracket 2, n-1 \rrbracket$ , there exists an apartment  $A_i$  containing  $C_a$  and  $C_i$ . For all  $i \in \llbracket 1, n-1 \rrbracket$ ,  $A_i \cap A_{i+1} \supset C_a \cup C_i \cap C_{i+1}$  and by hypothesis,  $\text{conv}_{A_i}(C_i \cap C_{i+1}, C_a) \subset A_i \cap A_{i+1}$  and there exists an isomorphism  $\phi_i : A_i \rightarrow A_{i+1}$  fixing  $\text{conv}_{A_i}(C_i \cap C_{i+1}, C_a)$ . In particular,  $\text{conv}_{A_i}(C_a, F_b) = \text{conv}_{A_{i+1}}(C_a, F_b) \subset A_i \cap A_{i+1}$  and  $\phi_i$  fixes  $\text{conv}_{A_i}(C_a, F_b)$ . By induction  $A \cap B$  contains  $\text{conv}_A(C_a, F_b)$  and  $\phi = \phi_{n-1} \circ \dots \circ \phi_1 : A \xrightarrow{\text{conv}_A(C_a, F_b)} B$  and the lemma is proved.  $\square$



We now prove Theorem 5.18:

Let  $x, y \in \mathcal{I}$  such that  $x \leq y$ . Let  $(F_x, F_y) \in \mathcal{F}_x \times \mathcal{F}_y$ . Then  $F_x \cup F_y$  is intrinsic.

*Proof.* We have four cases to treat, depending on the signs of  $F_x$  and  $F_y$ . The case where  $F_x$  is negative and  $F_y$  is positive is a particular case of Proposition 5.5 of [Rou11].

Suppose that  $F_x$  and  $F_y$  are positive. Let  $(F_x, F_y) \in \mathcal{F}_x^+ \times \mathcal{F}_y^+$  and  $A, B \in \mathcal{A}(F_x \cup F_y)$ . Let  $(C_x^A, C_y^B) \in \mathcal{C}_{F_x}(A) \times \mathcal{C}_{F_y}(B)$ . Let  $A' \in \mathcal{A}(C_x^A \cup C_y^B)$ . By Lemma 5.26, Lemma 5.28 and Lemma 5.33,  $C_x^A \cup F_y$  and  $F_x \cup C_y^B$  are intrinsic. Therefore there exists  $\phi : A \xrightarrow{\text{conv}_A(C_x^A, F_y)} A'$  and  $\psi : A' \xrightarrow{\text{conv}_{A'}(F_x, C_y^B)} B$ . Then  $\text{conv}_A(F_x, F_y) = \text{conv}_{A'}(F_x, F_y) = \text{conv}_B(F_x, F_y)$  and  $\psi \circ \phi : A \xrightarrow{F_x \cup F_y} B$ . Consequently  $F_x \cup F_y$  is intrinsic.

Suppose  $F_x$  and  $F_y$  are negative. We deduce the fact that  $F_x \cup F_y$  is intrinsic from the previous case by exchanging the signs. Indeed, let  $\mathbb{A}'$  be the vectorial space  $\mathbb{A}$  equipped with a structure of apartment of type  $-\mathbb{A}$ : the Tits cone  $\mathcal{T}'$  of  $\mathbb{A}'$  is  $-\mathcal{T}$ , ... We obtain a measure  $\mathcal{I}'$  of type  $\mathbb{A}'$  with underlying set  $\mathcal{I}$ . Let  $(F_x, F_y) \in \mathcal{F}_x^- \times \mathcal{F}_y^-$ . One has  $y \leq' x$  and  $F_x$  and  $F_y$  are positive for  $\mathcal{I}'$ . Therefore  $F_x \cup F_y$  is intrinsic in  $\mathcal{I}'$  and thus in  $\mathcal{I}$ .

Suppose that  $F_x$  is positive and  $F_y$  is negative. Let  $A, B \in \mathcal{A}(F_x, F_y)$  and  $(C_x^A, C_y^B) \in \mathcal{C}_{F_x}(A) \times \mathcal{C}_{F_y}(B)$ . By Lemma 5.28 and Lemma 5.33,  $C_x^A \cup F_y$  is intrinsic. As the roles of  $\mathcal{T}$  and  $-\mathcal{T}$  are similar,  $F_x \cup C_y^B$  is also intrinsic and by the reasoning above,  $F_x \cup F_y$  is intrinsic.  $\square$

## 5.4 Axiomatic of mesures in the affine case

In this section, we study the particular case of measure associated to irreducible affine Kac-Moody matrix  $A$ , which means that  $A$  satisfies condition (aff) of Theorem 4.3 of [Kac94].

Let  $\mathcal{S}$  be a generating root system associated to an irreducible and affine Kac-Moody matrix and  $\mathbb{A} = (\mathcal{S}, W, \Lambda')$  be an apartment. By Section 1.3 of [Rou11], one has  $\mathring{\mathcal{T}} = \{v \in \mathbb{A} \mid \delta(v) > 0\}$  for some imaginary root  $\delta \in Q^+ \setminus \{0\}$  and  $\mathcal{T} = \mathring{\mathcal{T}} \cup \mathbb{A}_{in}$ , where  $\mathbb{A}_{in} = \bigcap_{i \in I} \ker(\alpha_i)$ .

We fix an apartment  $\mathbb{A}$  of affine type.

Let (MA af i)=(MA1).

Let (MA af ii) : let  $A$  and  $B$  be two apartments. Then  $A \cap B$  is enclosed and there exists  $\phi : A \xrightarrow{A \cap B} B$ .

Let (MA af iii)=(MA iii).

The aim of this subsection is to prove the following theorem:

**Theorem 5.34.** *Let  $\mathcal{I}$  be a construction of type  $\mathbb{A}$  and  $\text{cl} \in \mathcal{CL}_\Lambda$ . Then  $\mathcal{I}$  is a measure for  $\text{cl}$  if and only if  $\mathcal{I}$  satisfies (MA af i), (MA af ii) and (MA af iii, cl) if and only if  $\mathcal{I}$  satisfies (MA af i), (MA af ii) and (MA af iii, cl $^\#$ ).*

**Remark 5.35.** Actually, we do not know if this axiomatic is true for mesures associated to indefinite Kac-Moody groups. We do not know if the intersection of two apartments is always convex in a measure.

The fact that we can exchange (MA af iii, cl $^\#$ ) and (MA af iii, cl) for all  $\text{cl} \in \mathcal{CL}_\Lambda$  follows from Theorem 5.2. The fact that a construction satisfying (MA af ii) and (MA af iii, cl $^\#$ ) is a measure is clear and does not use the fact that  $\mathbb{A}$  is associated to a Kac-Moody matrix. It remains to prove that a measure of type  $\mathbb{A}$  satisfies (MA af ii), which is the aim of this section.

**Lemma 5.36.** *Let  $A$  and  $B$  be two apartments such that there exist  $x, y \in A \cap B$  such that  $x \leq y$  and  $x \neq y$ . Then  $A \cap B$  is convex.*

*Proof.* One identifies  $A$  and  $\mathbb{A}$ . Let  $a, b \in \mathbb{A} \cap B$ . If  $\delta(a) \neq \delta(b)$ , then  $a \leq b$  or  $b \leq a$  and  $[a, b] \subset B$  by (MAO). Suppose  $\delta(a) = \delta(b)$ . As  $\delta(x) \neq \delta(y)$ , one can suppose that  $\delta(a) \neq \delta(x)$ . Then  $[a, x] \subset B$ . Let  $(a_n) \in [a, x]^{\mathbb{N}}$  such that  $\delta(a_n) \neq \delta(a)$  for all  $n \in \mathbb{N}$  and  $a_n \rightarrow a$ . Let  $t \in [0, 1]$ . Then  $ta_n + (1-t)b \in B$  for all  $n \in \mathbb{N}$  and by Proposition 3.9,  $ta + (1-t)b \in B$ :  $\mathbb{A} \cap B$  is convex.  $\square$

**Lemma 5.37.** *Let  $A$  and  $A'$  be two apartment of  $\mathcal{I}$ . Then  $A \cap A'$  is convex. Moreover, if  $x, y \in A \cap A'$ , there exists an isomorphism  $\phi : A \xrightarrow{[x, y]_A} A'$ .*

*Proof.* Let  $x, y \in A \cap A'$  such that  $x \neq y$ . Let  $C_x$  be a chamber of  $A$  based at  $x$  and  $C_y$  be a chamber of  $A'$  based at  $y$ . Let  $B$  be an apartment containing  $C_x$  and  $C_y$ , which exists by Proposition 5.17. By Lemma 5.36,  $A \cap B$  and  $A' \cap B$  are convex and by Proposition 3.26, there exist isomorphisms  $\psi : A \xrightarrow{A \cap B} B$  and  $\psi' : B \xrightarrow{A' \cap B} A'$ . Therefore  $[x, y]_A = [x, y]_B = [x, y]_{A'}$ . Moreover,  $\phi = \psi' \circ \psi$  fixes  $[x, y]_A$  and the lemma is proved.  $\square$

**Theorem 5.38.** *Let  $A$  and  $B$  be two apartments. Then  $A \cap B$  is enclosed and there exists an isomorphism  $\phi : A \xrightarrow{A \cap B} B$ .*

*Proof.* The fact that  $A \cap B$  is enclosed is a consequence of Lemma 5.37 and Proposition 3.22. By Proposition 3.14, there exist  $l \in \mathbb{N}$ , enclosed subsets  $P_1, \dots, P_l$  of  $A$  such that  $\text{supp}(A \cap B) = \text{supp}(P_j)$  and isomorphisms  $\phi_j : A \xrightarrow{P_j} B$  for all  $j \in \llbracket 1, l \rrbracket$ . Let  $x \in \text{Int}_r(P_1)$  and  $y \in A \cap B$ . By Lemma 5.37, there exists  $\phi_y : A \xrightarrow{[x, y]} B$ . Then  $\phi_y^{-1} \circ \phi_1$  fixes a neighborhood of  $x$  in  $[x, y]$  and thus  $\phi_1$  fixes  $y$ , which proves the theorem.  $\square$

## 6 A comparison criterion for the Tits preorder

### 6.1 Standard apartment in the affine case

We now suppose that the Kac-Moody matrix involved in the definition of  $\mathbb{A}$  is indecomposable and affine. Then there exists  $\delta_{\mathbb{A}} \in \bigoplus_{i \in I} \mathbb{N}^* \alpha_i$  such that  $\check{\mathcal{T}} = \delta^{-1}(\mathbb{R}_+^*)$  and  $\delta_{\mathbb{A}}$  is  $W^v$ -invariant. One has  $\mathcal{T} = \check{\mathcal{T}} \cup \mathbb{A}_{in}$ . In this section we study the Tits preorder on  $\mathcal{I}$ .

**Lemma 6.1.** *Let  $w \in W$  such that  $w$  fixes a point of  $\mathbb{A}$ . Then  $\delta_{\mathbb{A}} \circ w = \delta_{\mathbb{A}}$ .*

*Proof.* One embeds  $W$  in  $W_{\mathbb{A}} = W^v \rtimes \mathbb{A}$ . Let  $a \in \mathbb{A}$  such that  $w(a) = a$ . Let  $\tau$  be the translation of  $\mathbb{A}$  sending  $a$  to 0. Let  $w' = \tau \circ w \circ \tau^{-1} \in W_{\mathbb{A}}$ . Then  $w' = \tau_1 \circ w_1$ , where  $\tau_1$  is a translation of  $\mathbb{A}$  and  $w_1 \in W^v$ . As  $w'(0) = 0$ ,  $\tau_1$  is the identity and  $w' \in W^v$ . Let  $x \in \mathbb{A}$ . Then  $w(x) = a + w'(x) - w'(a)$ . As  $\delta_{\mathbb{A}}$  is  $W^v$ -invariant,  $\delta(w(x)) = \delta(x) + \delta(a) - \delta(a) = \delta(x)$ , which is our assertion.  $\square$

**Proposition 6.2.** *1. There exists a unique map  $\delta : \mathcal{I} \rightarrow \mathbb{R}$  such that  $\delta|_{\mathbb{A}} = \delta_{\mathbb{A}}$  and such that the restriction of  $\delta$  to each apartment is an affine form.*

*2. Let  $\mathfrak{q}$  be a sector-germ of  $\mathbb{A}$  and  $\rho_{\mathfrak{q}} = \rho_{\mathfrak{q}, \mathbb{A}} : \mathcal{I} \rightarrow \mathbb{A}$  be the retraction on  $\mathbb{A}$  centered at  $\mathfrak{q}$ . Then  $\delta = \delta_{\mathbb{A}} \circ \rho_{\mathfrak{q}}$ .*

*Proof.* Suppose that there exist  $\delta_1, \delta_2 : \mathcal{I} \rightarrow \mathbb{R}$  satisfying 1. Let  $x \in \mathcal{I}$ . By (MA3), there exists an apartment  $A$  containing  $x$  and  $+\infty$ . As  $\delta_1|_A, \delta_2|_A$  coincide on a nonempty open set and are affine,  $\delta_1(x) = \delta_2(x)$ . Therefore,  $\delta_1 = \delta_2$ .

Let  $\mathfrak{q}$  be a sector-germ of  $\mathbb{A}$  and  $\delta_{\mathfrak{q}} = \delta \circ \rho_{\mathfrak{q}}$ . Let  $A$  be an apartment of  $\mathcal{I}$ . By Proposition 3.7 of [Héb16], there exists  $n \in \mathbb{N}$  and  $P_1, \dots, P_n$  non-empty closed convex subsets of  $A$  such

that  $A = \bigcup_{i=1}^n P_i$ , where for all  $i \in \llbracket 1, n \rrbracket$ , there exists an apartment  $A_i$  containing  $\mathfrak{q} \cup P_i$ . If  $i \in \llbracket 1, n \rrbracket$ , let  $\psi_i : A_i \xrightarrow{\mathbb{A} \cap A_i} \mathbb{A}$  and  $\phi_i : A \xrightarrow{A \cap A_i} A_i$ . Let  $i \in \llbracket 1, n \rrbracket$  and  $x \in P_i$ . Then  $\rho_{\mathfrak{q}}(x) = \psi_i(x) = \psi_i \circ \phi_i(x)$ .

Let us prove that for all  $i, j \in \llbracket 1, n \rrbracket$ ,  $\delta_{\mathbb{A}} \circ \psi_i \circ \phi_i = \delta_{\mathbb{A}} \circ \psi_j \circ \phi_j$ . Let  $i, j \in \llbracket 1, n \rrbracket$ . Let  $x \in P_i$  and  $y \in P_j$ . One identifies  $[x, y]$  and  $[0, 1]$ . Then there exists  $k \in \llbracket 1, n \rrbracket$ , a map  $\sigma : \llbracket 1, k \rrbracket \rightarrow \llbracket 1, n \rrbracket$  and  $t_1 = 0 < t_2 < \dots < t_k = 1$  such that for all  $l \in \llbracket 1, k-1 \rrbracket$ ,  $[t_l, t_{l+1}] \subset P_{\sigma(l)}$ . Let  $l \in \llbracket 1, k-1 \rrbracket$ . There exists  $w \in W$  such that  $\psi_{\sigma(l+1)} \circ \phi_{\sigma(l+1)} = w \circ \psi_{\sigma(l)} \circ \phi_{\sigma(l)}$ . Moreover if  $a = \psi_{\sigma(l+1)} \circ \phi_{\sigma(l+1)}(t_{l+1}) = \psi_{\sigma(l)} \circ \phi_{\sigma(l)}(t_{l+1})$ ,  $w$  fixes  $a$ . By Lemma 6.1,  $\delta_{\mathbb{A}}(\psi_{\sigma(l)} \circ \phi_{\sigma(l)}) = \delta_{\mathbb{A}}(\psi_{\sigma(l+1)} \circ \phi_{\sigma(l+1)})$ . By induction, we get that  $\delta_{\mathbb{A}} \circ \psi_i \circ \phi_i = \delta_{\mathbb{A}} \circ \psi_j \circ \phi_j$ . Therefore, for all  $i \in \llbracket 1, n \rrbracket$ ,  $\delta_{\mathfrak{q}|A} = \delta_{\mathbb{A}} \circ \psi_i \circ \phi_i$ . In particular,  $\delta_{\mathfrak{q}}$  is an affine form on each apartment and we get 2 and 1.  $\square$

The aim of the sequel is to prove the following theorem:

**Theorem 6.3.** *Let  $x, y \in \mathcal{I}$ . Then  $x \overset{\circ}{<} y$  if and only if  $\delta(x) < \delta(y)$ .*

The fact that if  $x \overset{\circ}{<} y$ ,  $\delta(x) < \delta(y)$  is already known by results of [Rou11]. It remains to prove that if  $\delta(x) < \delta(y)$ , then  $x \overset{\circ}{<} y$ . For this we proceed in two steps. First we prove that if  $A$  and  $B$  are apartments containing  $+\infty$ , there exist  $n \in \mathbb{N}^*$  and a sequence  $A_1, \dots, A_n$  of apartments containing  $+\infty$  such that  $A_1 = A$ ,  $A_n = B$  and  $A_i \cap A_{i+1}$  is a half-apartment for all  $i \in \llbracket 1, n-1 \rrbracket$ . This enables to define a distance  $d$  on the set of apartments containing  $+\infty$  (this step does not use the fact that the Kac-Moody matrix defining  $\mathbb{A}$  is affine). We then make an induction on  $d(A, B)$  where  $A$  (resp.  $B$ ) is an apartment containing  $x \cup +\infty$  (resp.  $y \cup +\infty$ ).

## 6.2 Distance between apartments containing $+\infty$

One denotes by  $\mathcal{A}(+\infty)$  the set of apartments containing  $+\infty$ .

**Lemma 6.4.** *Let  $A \in \mathcal{A}(+\infty)$  and  $\mathfrak{q}$  be a negative sector-germ of  $\mathcal{I}$ . Then there exists  $n \in \mathbb{N}^*$  and a sequence  $(A_i) \in \mathcal{A}(+\infty)^n$  such that  $A_1 = A$ ,  $A_n \supset \mathfrak{q}$  and  $A_i \cap A_{i+1}$  is a half-apartment for all  $i \in \llbracket 1, n-1 \rrbracket$ .*

*Proof.* Let  $\mathfrak{q}'$  be a negative sector-germ of  $A$ . By (MA3), there exists an apartment  $B$  containing  $\mathfrak{q}$  and  $\mathfrak{q}'$ . Let  $\mathfrak{q}_1 = \mathfrak{q}', \dots, \mathfrak{q}_n = \mathfrak{q}$  be a gallery of sector-germs from  $\mathfrak{q}'$  to  $\mathfrak{q}$ . Let  $A_1 = A$ . Let  $i \in \llbracket 1, n-1 \rrbracket$  and suppose that we have constructed  $(A_j) \in \mathcal{A}(+\infty)^i$  such that for all  $j \in \llbracket 1, i-1 \rrbracket$ ,  $A_j \cap A_{j+1}$  is a half-apartment and  $A_j \supset \mathfrak{q}_j$ . By Lemma 3.6 of [Héb16], there exist two half-apartments  $D_1, D_2$  of  $A_i$  such that for all  $k \in \{1, 2\}$ ,  $D_k \cup \mathfrak{q}_{i+1}$  is included in an apartment  $B_k$ . Let  $k \in \{1, 2\}$  such that  $D_k \supset +\infty$ . One sets  $A_{i+1} = B_k$ . By induction, we get a sequence  $(A_i)$  satisfying the desired property.  $\square$

**Proposition 6.5.** *Let  $A, B \in \mathcal{A}(+\infty)$ . Then there exists  $n \in \mathbb{N}^*$  and a sequence  $(A_i) \in \mathcal{A}(+\infty)^n$  such that  $A_1 = A$ ,  $A_n = B$  and for all  $i \in \llbracket 1, n-1 \rrbracket$ ,  $A_i \cap A_{i+1}$  is a half-apartment.*

*Proof.* We apply Lemma 6.4 taking  $\mathfrak{q}$  to be the sector-germ of  $B$  opposite to  $+\infty$  and conclude with Lemma 5.21.  $\square$

**Distance on  $\mathcal{A}(+\infty)$**  Let  $A, B \in \mathcal{A}(+\infty)$ . One defines  $d(A, B) \in \mathbb{N}$  as the minimal possible  $k \in \mathbb{N}$  such that there exists a sequence  $(A_i) \in \mathcal{A}(+\infty)^{k+1}$  such that  $A_1 = A$ ,  $A_{k+1} = B$  and  $A_i \cap A_{i+1}$  is a half-apartment for all  $i \in \llbracket 1, k-1 \rrbracket$ . This is well-defined by Proposition 6.5 and it is easy to see that it is a distance on  $\mathcal{A}(+\infty)$ .

### 6.3 Induction on the distance

**Lemma 6.6.** *Let  $x, y \in \mathbb{A}$  such that  $x \dot{<} y$  and  $w \in W$  such that  $w$  fixes a point of  $\mathbb{A}$ . Then  $x \dot{<} w(y)$ .*

*Proof.* By Lemma 6.1  $\delta_{\mathbb{A}}(w(y)) = \delta_{\mathbb{A}}(y)$ , thus  $\delta_{\mathbb{A}}(x) < \delta_{\mathbb{A}}(w(y))$  and hence  $x \dot{<} w(y)$ .  $\square$

**Lemma 6.7.** *Let  $A, B \in \mathcal{A}(+\infty)$  such that  $A \cap B$  is a half-apartment  $D_1$  and  $\phi : A \xrightarrow{D_1} B$ . Let  $x, y \in A$  such that  $x \dot{<} y$ . Then  $x \dot{<} \phi(y)$  and  $\phi(x) \dot{<} y$ .*

*Proof.* If  $\phi(y) = y$  this is clear. If  $\phi(x) = x$ , then  $x = \phi(x) \dot{<} \phi(y)$  by invariance of the Tits preorder. Suppose that  $\phi(y) \neq y$  and  $\phi(x) \neq x$ . Let  $D_2$  be the half-apartment of  $A$  opposed to  $D_1$ ,  $D_3 = \phi(D_2)$  and  $M$  be the wall of  $D_1$ . Let  $A_1 = D_2 \cup D_3$ . By Proposition 2.9 2 of [Rou11],  $A_1$  is an apartment of  $\mathcal{I}$ . One has  $B = D_1 \cup D_3$ . Let  $s : A_1 \rightarrow A_1$  be the reflexion of  $A_1$  fixing  $M$ . Then by Lemma 3.4 of [Héb16], the following diagram is commutative:

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow \phi_2 & & \downarrow \phi_1 \\ A_1 & \xrightarrow{s} & A_1 \end{array} ,$$

where  $\phi_2 : A \xrightarrow{D_2} A_1$  and  $\phi_1 : A \xrightarrow{D_1} B$ . By hypothesis,  $x, y \in D_2$  and thus  $\phi(x) = s(x)$  and  $\phi(y) = s(y)$ .

Let  $\psi : A \rightarrow \mathbb{A}$  be an isomorphism. Let  $s' = \psi \circ s \circ \psi^{-1} : \mathbb{A} \rightarrow \mathbb{A}$ . Then  $s' \in W$  and  $s'$  fixes  $\psi(M)$ . By Lemma 6.6,  $\psi(x) \dot{<} s'(\psi(y)) = \psi(s(y))$ . Therefore  $x \dot{<} s(y) = \phi(y)$ .

It remains to prove that  $\phi(x) \dot{<} y$ . By applying the result we just proved with  $\phi^{-1} : B \rightarrow A$  instead of  $\phi$ , we get that  $x \dot{<} \phi^{-1}(y)$ . Therefore  $\phi(x) \dot{<} \phi(\phi^{-1}(y)) = y$  and the lemma is proved.  $\square$

We can now prove Theorem 6.3: Let  $x, y \in \mathcal{I}$ . Then  $x \dot{<} y$  if and only if  $\delta(x) < \delta(y)$ .

*Proof.* By Corollaire 2.8 of [Rou11], for all  $x, y \in \mathcal{I}$ , if  $x \dot{<} y$ , then  $\rho(x) \dot{<} \rho(y)$  (resp.  $\rho(y) \dot{<} \delta(x)$ ) (there is no need to assume that  $\mathbb{A}$  is of affine type) and thus  $\delta(x) < \delta(y)$ .

If  $x, y \in \mathcal{I}$ , one sets

$$d(x, y) = \min\{d(A, B) \mid (A, B) \in \mathcal{A}(+\infty)^2, A \ni x \text{ and } B \ni y\}$$

( $d$  is not a distance on  $\mathcal{I}$ ). If  $n \in \mathbb{N}$ , one sets  $\mathcal{P}_n$ :

$$\forall (x, y) \in \mathcal{I}^2 \mid d(x, y) \leq n, \delta(x) < \delta(y) \Rightarrow x \dot{<} y.$$

The property  $\mathcal{P}_0$  is true because  $\delta$  determines the Tits preorder on  $\mathbb{A}$ . Let  $n \in \mathbb{N}$  such that  $\mathcal{P}_n$  is true and  $x, y \in \mathcal{I}$  such that  $d(x, y) = n + 1$  and  $\delta(x) < \delta(y)$ . Let  $A, B \in \mathcal{A}(+\infty)$  such that  $d(A, B) = n + 1$ ,  $A \ni x$  and  $B \ni y$ .

Let  $z \in A$  such that  $\delta(x) < \delta(z) < \delta(y)$ . Let  $A' \in \mathcal{A}(+\infty)$  such that  $d(A, A') = 1$  and  $d(A', B) = n$ , and  $\phi : A \xrightarrow{A \cap A'} A'$ . One has  $\delta(\phi(z)) = \delta(z)$  and thus  $\phi(z) \dot{<} y$  by  $\mathcal{P}_n$ . Moreover,  $\phi(x) \dot{<} \phi(z)$  (by  $\mathcal{P}_0$ ) and hence by Lemma 6.7,  $x = \phi^{-1}(\phi(x)) \dot{<} \phi(z)$ . As  $\dot{<}$  is a transitive (by Théorème 5.9 of [Rou11]), we deduce that  $x \dot{<} y$  and thus  $\mathcal{P}_{n+1}$  is true, which proves the theorem.  $\square$

**Action of  $\mathbb{A}_{in}$  on  $\mathcal{I}$**  Let  $\mathbb{A}_{in} = \bigcap_{i \in I} \ker \alpha_i$ . We define an action of  $\mathbb{A}_{in}$  on  $\mathcal{I}$  as follows. If  $x \in \mathcal{I}$ , one chooses an apartment  $A$  containing  $x$  and an isomorphism  $\phi : \mathbb{A} \rightarrow A$ . If  $u \in \mathbb{A}_{in}$ , one sets  $x + u = \phi(\phi^{-1}(x) + u)$ . By (MA af ii) (using the fact that each half-apartment containing  $\phi^{-1}(x)$  contains  $\phi^{-1}(x) + \mathbb{A}_{in}$  and that  $W^v$  fixes pointwise  $\mathbb{A}_{in}$ ), this action is well defined.

**Corollary 6.8.** *For all  $x, y \in \mathcal{I}$  such that  $x + \mathbb{A}_{in} \neq y + \mathbb{A}_{in}$ . Then:*

1.  $x \leq y$  if and only if  $\delta(x) < \delta(y)$
2.  $x$  and  $y$  are not comparable for  $\leq$  if and only if  $\delta(x) = \delta(y)$ .

For  $x, y \in \mathcal{I}$ , one denotes  $x$  NC  $y$  if  $x$  and  $y$  are not comparable for  $\leq$  or  $x + \mathbb{A}_{in} = y + \mathbb{A}_{in}$ .

**Corollary 6.9.** 1. *The relation NC is an equivalence relation on  $\mathcal{I}$  whose classes are the level sets of  $\delta$ .*

2. *Let  $x, y \in \mathcal{I}$ . Let  $A$  be an apartment containing  $x$ . Then  $x$  NC  $y$  if and only if there exists  $(x_n^+) \in A^{\mathbb{N}}$  and  $(x_n^-) \in A^{\mathbb{N}}$  such that  $x_n^+ \rightarrow x$ ,  $x_n^- \rightarrow x$  and for all  $n \in \mathbb{N}$ ,  $x_n^- \dot{<} y \dot{<} x_n^+$ .*
3. *Let  $x, y, z \in \mathcal{I}$  such that  $y$  NC  $z$ . Then  $x \dot{<} y$  if and only if  $x \dot{<} z$ .*
4. *Let  $x, y \in \mathcal{I}$ . Then  $x \dot{<} y$  if and only if  $\rho_{+\infty}(x) \dot{<} \rho_{+\infty}(y)$ .*

**Remark 6.10.** • Point 4 of Corollary 6.9 is specific to the affine case. Indeed suppose that  $G$  is associated to an indefinite Kac-Moody group. By the lemma of Section 2.9 of [GR14] and Proposition 5.8 c) of [Kac94], for all  $i \in I$ ,  $\alpha_i^\vee \in \mathbb{A} \setminus \overline{\mathcal{T}}$  and as  $r_i \cdot \alpha_i^\vee = -\alpha_i^\vee$ , we deduce that  $\alpha_i^\vee \in \mathbb{A} \setminus (\overline{\mathcal{T}} \cup \overline{-\mathcal{T}})$ .

Let  $x = \alpha_i^\vee$ . Let us show that there exists  $y \in \mathbb{A}$  satisfying  $x \dot{<} y$  but not  $x \dot{<} r_i(y)$ . Let  $u \in \mathcal{T}$  and  $y_n = x + \frac{1}{n}u$  for all  $n \in \mathbb{N}^*$ . One has  $r_i(y_n) - x = -2\alpha_i^\vee + \frac{1}{n}r_i(u)$  for all  $n \in \mathbb{N}^*$  and thus  $r_i(y_n) - x \notin \mathcal{T}$  for  $n$  large enough. Thus by the same reasoning as in the proof of Lemma 6.7, there exists an apartment  $A \in \mathcal{A}(+\infty)$  such that if  $\phi : \mathbb{A} \xrightarrow{A \cap \mathbb{A}} A$ , one has  $x \dot{<} y_n$  but not  $x \dot{<} r_i(y_n) = \rho_{+\infty}(\phi(y_n))$  for  $n$  large enough.

- NC is not an equivalence relation in the indefinite case. Indeed, let  $u \in \mathcal{T}$ ,  $x = 0$ ,  $y_n = \alpha_i^\vee + \frac{1}{n}u$  and  $z_n = \frac{2}{n}u$ . For  $n$  large enough,  $x$  NC  $y_n$  and  $y_n$  NC  $z_n$  but  $x < z_n$ .

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