



**HAL**  
open science

# Scaling limit of dynamical percolation on critical Erdős-Rényi random graphs

Raphaël Rossignol

► **To cite this version:**

Raphaël Rossignol. Scaling limit of dynamical percolation on critical Erdős-Rényi random graphs. 2018. hal-01623055v3

**HAL Id: hal-01623055**

**<https://hal.science/hal-01623055v3>**

Preprint submitted on 24 Jun 2018 (v3), last revised 11 Feb 2020 (v4)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Scaling limit of dynamical percolation on critical Erdős-Rényi random graphs.

Raphaël Rossignol

*Raphaël Rossignol  
 Université Grenoble Alpes  
 Institut Fourier  
 CS 40700  
 38058 Grenoble cedex 9  
 France*

*e-mail: [raphael.rossignol@univ-grenoble-alpes.fr](mailto:raphael.rossignol@univ-grenoble-alpes.fr)*

**Abstract:** Consider a critical Erdős-Rényi random graph:  $n$  is the number of vertices, each one of the  $\binom{n}{2}$  possible edges is kept in the graph independently from the others with probability  $n^{-1} + \lambda n^{-4/3}$ ,  $\lambda$  being a fixed real number. When  $n$  goes to infinity, Addario-Berry, Broutin and Goldschmidt [2] have shown that the collection of connected components, viewed as suitably normalized measured compact metric spaces, converges in distribution to a continuous limit  $\mathcal{G}_\lambda$  made of random real graphs. In this paper, we consider notably the dynamical percolation on critical Erdős-Rényi random graphs. To each pair of vertices is attached a Poisson process of intensity  $n^{-1/3}$ , and every time it rings, one resamples the corresponding edge. Under this process, the collection of connected components undergoes coalescence and fragmentation. We prove that this process converges in distribution, as  $n$  goes to infinity, towards a fragmentation-coalescence process on the continuous limit  $\mathcal{G}_\lambda$ . We also prove convergence of discrete coalescence and fragmentation processes and provide Feller-type properties associated to fragmentation and coalescence.

**MSC 2010 subject classifications:** Primary 60K35; secondary 05C80, 60F05..

**Keywords and phrases:** Erdős-Rényi, random graph, coalescence, fragmentation, dynamical percolation, scaling limit, Gromov-Hausdorff-Prokhorov distance, Feller property.

## Contents

1	Introduction . . . . .	2
2	Notations and Background . . . . .	3
	2.1 General notations . . . . .	3
	2.2 Discrete graphs and dynamical percolation . . . . .	3
	2.3 The multiplicative coalescent . . . . .	4
	2.4 Measured semi-metric spaces . . . . .	5
	2.5 The Gromov-Hausdorff-Prokhorov distance . . . . .	7
	2.6 Gluing and coalescence . . . . .	14
	2.6.1 Gluing and $\delta$ -gluing . . . . .	14
	2.6.2 The coalescence processes . . . . .	15
	2.7 $\mathbb{R}$ -graphs . . . . .	16
	2.8 Cutting, fragmentation and dynamical percolation . . . . .	17
	2.9 The scaling limit of critical Erdős-Rényi random graphs . . . . .	19
3	Main results . . . . .	19
4	Proofs of the main results for coalescence . . . . .	20
	4.1 The Coalescent on $\mathcal{N}_1$ . . . . .	20
	4.2 Structural result for Aldous' multiplicative coalescent . . . . .	23
	4.3 The Coalescent on $\mathcal{S}$ . . . . .	26
	4.4 Convergence of the coalescent on Erdős-Rényi random graphs . . . . .	30
5	Proofs of the results for fragmentation . . . . .	32
	5.1 Notations . . . . .	32
	5.2 Reduction to finite graphs . . . . .	33
	5.3 The Feller property for trees . . . . .	34

5.4	The Feller property for graphs . . . . .	39
5.5	Application to Erdős-Rényi random graphs . . . . .	48
6	Combining fragmentation and coalescence: dynamical percolation . . . . .	49
6.1	Almost Feller Property . . . . .	49
6.2	Application to Erdős-Rényi random graphs . . . . .	52
	References . . . . .	53

---

## 1. Introduction

Consider  $\mathcal{G}(n, p(\lambda, n))$ , the Erdős-Rényi random graph on  $n$  vertices inside the critical window, that is when the probability of an edge is  $p(\lambda, n) := n^{-1} + \lambda n^{-4/3}$ . The largest components are of order  $n^{2/3}$ , and their diameter is of order  $n^{1/3}$ , and this particular scaling of  $p$  with respect to  $n$  corresponds to what is called the *critical window* associated to the emergence of a giant component. In this regime, we are particularly interested in those large components, because in some sense, they contain all the complexity of the graph. There is a clean procedure to capture the behaviour of those components in the large  $n$  limit: if one assigns mass  $n^{-2/3}$  to each vertex and length  $n^{-1/3}$  to each edge, this graph converges to a random collection of real graphs  $\mathcal{G}_\lambda$  (see Theorem 2.38 below, or Theorem 24 in [2] for a more precise statement).

Now, put the following dynamic on  $\mathcal{G}(n, p(\lambda, n))$ : each pair of vertices is equipped with an independent Poisson process with rate  $\gamma_n$  and every time it rings, one *refreshes* the corresponding edge, meaning that one replaces its state by a new independent one: present with probability  $p(\lambda, n)$ , absent with probability  $1 - p(\lambda, n)$ . This procedure corresponds to *dynamical percolation* on the complete graph with  $n$  vertices, at rate  $\gamma_n$ . A natural question now is “At which rate should we refresh the edges in order to see a non trivial process in the large  $n$  limit?”. In this question, it is understood that one keeps getting interested in the same scaling as before concerning masses and lengths.

A moment of thought suggests that a good choice should be  $\gamma_n = n^{-1/3}$ . Indeed, since large components are of size  $\Theta(n^{2/3})$ , in a pair of components there are  $\Theta(n^{4/3})$  pairs of vertices which after refreshment lead to  $\Theta(n^{4/3}p(\lambda, n)) = \Theta(n^{1/3})$  edges added. Thus, choosing  $\gamma_n = \Theta(n^{-1/3})$  will lead large components to coalesce at rate  $\Theta(1)$ . Furthermore, on those large components typical distances are of order  $\Theta(n^{1/3})$ . An edge will be destroyed at rate  $\gamma_n(1 - p(\lambda, n))$  so the geometry inside such a component will be affected at rate  $\Theta(n^{1/3}\gamma_n(1 - p(\lambda, n)))$ , which is again of order  $\Theta(1)$  when  $\gamma_n = \Theta(n^{-1/3})$ . This scaling is already present in the work of Aldous [5] where he studied the evolution of the collection of rescaled masses of the components when one coalesce components at a rate proportional to the product of their mass. This is the so-called multiplicative coalescent. Of course, instead of refreshing the edges, one may decide to only add edges, or to only destroy edges. In the first case, one will assist to coalescence of components and in the second case, to fragmentation. Once again, one may ask the same question as before: what is the good rate in order to obtain a non trivial process in the large  $n$  limit, and what is this limit process. One of the main purposes of this article is to give an answer to these questions for the three cases that we just defined informally: dynamical percolation, coalescence and fragmentation. The limit processes will be dynamical percolation, coalescence and fragmentation processes acting on the limit  $\mathcal{G}_\lambda$  obtained in [2]. Furthermore, we will show that coalescence is the time-reversal of fragmentation on this limit. The method we use is to provide Feller-type properties for coalescence and fragmentation, which we hope will be useful in the future to study scaling limits of similar dynamics on other critical random graphs. Notice that the study of coalescence of graphs is a central tool in the work of [7] to show convergence of a number of critical random graphs to  $\mathcal{G}_\lambda$  (configuration models, inhomogeneous random graphs etc.).

Since an important amount of notations is needed in order to make such statements precise we will switch to the presentation of notations in section 2 and then announce the main results and outline the plan of the rest of the article in section 3.

We finish this section by mentioning a few related works. Dynamical percolation was introduced in [12], and studied in a number of subsequent works by various authors. In the context of [12], only the edges of some fixed infinite graph are resampled while in the definition above, we resample the edges of a finite complete graph. The scaling limit of dynamical percolation for critical percolation on the two dimensional triangular

lattice was obtained in [11], with techniques quite different from the one used in the present paper. More related to the present paper is the work [17], where dynamical percolation on critical Erdős-Rényi random graphs, as introduced above, is studied notably at rate 1. The authors show that the size of the largest connected component that appears during the time interval  $[0, 1]$  is of order  $n^{2/3} \log^{1/3} n$  with probability tending to one as  $n$  goes to infinity. They also study “quantitative noise-sensitivity” of the event  $A_n$  that the largest component of  $\mathcal{G}(n, p(\lambda, n))$  is of size at least  $an^{2/3}$  for some fixed  $a > 0$  (see Proposition 2.2 in [17]). The results in the present paper can be used to find the precise scaling of quantitative noise-sensitivity for events talking about the sizes of the largest components (like  $A_n$  for instance). However, we leave this question and precise statements for future work.

## 2. Notations and Background

### 2.1. General notations

If  $(X, \tau)$  is a topological space, we denote by  $\mathcal{B}(X)$  the Borel  $\sigma$ -field on  $X$ .

If  $\psi$  is a measurable map between  $(E, \mathcal{E})$  and  $(F, \mathcal{F})$ , and  $\mu$  is a measure on  $(E, \mathcal{E})$ , then we denote by  $\psi\#\mu$  the push-forward of  $\mu$  by  $\psi$ :  $\psi\#\mu(A) = \mu(\psi^{-1}(A))$  for any  $A \in \mathcal{F}$ .

We shall frequently use Poisson processes. Let  $(E, \mathcal{E}, \mu)$  be a measurable set with  $\mu$   $\sigma$ -finite, denote by  $\text{Leb}(\mathbb{R}^+)$  the Lebesgue sigma-field on  $\mathbb{R}^+$ ,  $\text{leb}_{\mathbb{R}^+}$  the Lebesgue measure on  $\mathbb{R}^+$  and let  $\gamma \geq 0$ . If  $\mathcal{P}$  is a Poisson random set with intensity  $\gamma$  on  $(E \times \mathbb{R}^+, \mathcal{E} \times \text{Leb}(\mathbb{R}^+), \mu \times \text{leb}_{\mathbb{R}^+})$  (that is with intensity measure  $\gamma\mu \otimes \text{leb}_{\mathbb{R}^+}$ ) we shall denote by  $\mathcal{P}_t$  the points of  $\mathcal{P}$  with birthtime at most  $t$ :

$$\mathcal{P}_t := \{x \in E : \exists s \leq t, (x, s) \in \mathcal{P}\}.$$

When  $(X, d)$  is a Polish space and  $D([0, \infty), X)$  is the set of càdlàg functions from  $\mathbb{R}^+$  to  $X$ , we shall always put on  $D([0, \infty), X)$  the *topology of compact convergence* (also known as topology of uniform convergence on compact sets) (see chapter V and notably section V.5 in [16]). Recall that this topology is metrizable and complete (although not separable in general), finer than Skorokhod’s topology and that a sequence  $\omega^n = (\omega^n(t))_{t \in \mathbb{R}^+}$  in  $D([0, \infty), X)$  converges in this topology to  $\omega^\infty = (\omega^\infty(t))_{t \in \mathbb{R}^+}$  if and only if for every  $T > 0$ ,

$$\sup_{t \in [0, T]} |\omega^n(t) - \omega^\infty(t)| \xrightarrow{n \rightarrow +\infty} 0.$$

Notice a slight subtlety: we shall prove convergence in distribution of a sequence of processes  $((\mathbf{X}_n(t))_{t \geq 0})_{n \geq 1}$  towards  $(\mathbf{X}(t))_{t \geq 0}$  by exhibiting couplings showing that the Lévy-Prokhorov distance between the distributions of  $\mathbf{X}_n$  and  $\mathbf{X}$  goes to zero as  $n$  goes to infinity. This implies convergence in distribution (there is no need for separability or completeness in this direction).

Finally, we shall use the notation  $\overline{\mathbb{N}} := \mathbb{N} \cup \{+\infty\}$ .

### 2.2. Discrete graphs and dynamical percolation

We will talk of a *discrete graph* to mean the usual graph-theoretic notion of an unoriented graph, that is a pair  $G = (V, E)$  with  $V$  a finite set and  $E$  a subset of  $\binom{V}{2} := \{\{u, v\} : u \neq v \in V\}$ . Often,  $E$  is seen as a point in  $\{0, 1\}^{\binom{V}{2}}$ , where 0 codes for the absence of the corresponding edge and 1 for its membership to  $E$ .

For a positive integer  $n$  and  $p \in [0, 1]$ , the *Erdős-Rényi random graph* (or Gilbert random graph)  $\mathcal{G}(n, p)$  is the random graph with vertices  $[n] := \{1, \dots, n\}$  such that each edge is present with probability  $p$ , independently from the others. Alternatively, one may see it as a Bernoulli bond percolation with parameter  $p$  on the complete graph with  $n$  vertices  $K_n = ([n], \binom{[n]}{2})$ . It amounts to put the product of Bernoulli measures with parameter  $p$  on  $\{0, 1\}^{\binom{[n]}{2}}$ .

Let  $\gamma_+$  and  $\gamma_-$  be non-negative real numbers. If  $G = ([n], E)$  is a discrete graph on  $n$  vertices, define a random process  $N_{\gamma_+, \gamma_-}(G, t) = (G, E_t)$ ,  $t \geq 0$  as follows on the set of subgraphs of the complete graph  $K_n$ . To each pair  $e \in \binom{[n]}{2}$ , we attach two Poisson processes on  $\mathbb{R}^+$ :  $\mathcal{P}_e^+$  of intensity  $\gamma_+$  and  $\mathcal{P}_e^-$  of intensity  $\gamma_-$ .

We suppose that all the  $\binom{[n]}{2}$  Poisson processes are independent. Each time  $\mathcal{P}_e^+$  rings, we replace  $E_{t-}$  by  $E_{t-} \cup \{e\}$ , and each time  $\mathcal{P}_e^-$  rings, we replace  $E_{t-}$  by  $E_{t-} \setminus \{e\}$ . The letter  $N$  is reminiscent of “noise”. If one wants to insist on the Poisson processes, we shall write  $N(G, (\mathcal{P}^+, \mathcal{P}^-)_t)$  instead of  $N_{\gamma^+, \gamma^-}(G, t)$ , with an implicit definition for the map  $N$ .

One may take only  $\mathcal{P}^+$  or only  $\mathcal{P}^-$  into account: write  $N^+(G, \mathcal{P}_t^+)$  for  $N(G, (\mathcal{P}^+, \emptyset)_t)$  and  $N^-(G, \mathcal{P}_t^+)$  for  $N(G, (\emptyset, \mathcal{P}^-)_t)$ . Then,  $N^+(G, \mathcal{P}_t^+)$  will be referred as *the discrete coalescent process of intensity  $\gamma^+$  started at  $G$*  and  $N^-(G, \mathcal{P}_t^+)$  as *the discrete fragmentation process of intensity  $\gamma^+$  started at  $G$* .

Now, *dynamical percolation of parameter  $p$  and intensity  $\gamma$* , as described in the introduction, corresponds to the process  $N_{\gamma p, \gamma(1-p)}$ , and is in its stationary state when started with  $\mathcal{G}(n, p)$  (independently of the Poisson processes used to define the dynamical percolation).

All these processes will have continuous counterparts in the scaling limit, which will be defined in sections 2.6 and 2.8.

### 2.3. The multiplicative coalescent

The main tool to analyze our coalescent and fragmentation processes will be a refinement of Aldous’ work [5] on the multiplicative coalescent. In this section, we recall what we will use of his work.

Let us define:

$$\ell_+^2 := \left\{ x \in (\mathbb{R}^+)^{\mathbb{N}^*} : \sum_{i \geq 1} x_i^2 < \infty \right\},$$

$$\ell_{\searrow}^2 := \{x \in \ell_+^2 : x_1 \geq x_2 \geq \dots\}.$$

Let  $(N_{i,j})_{i,j \in \mathbb{N}^*}$  be independent Poisson point processes on the real line with intensity 1. Denote by  $T_{i,j,n}$  the  $n$ -th jump-time of  $N_{i,j}$ . For  $x \in \ell_+^2$ , let  $\text{MG}(x, t)$  denote the multigraph (with loops) with vertex set  $\mathbb{N}$  and edge set  $\cup_{n \in \mathbb{N}} \{(i, j) \in \binom{\mathbb{N}}{2} \text{ s.t. } T_{i,j,n} \text{ or } T_{j,i,n} \leq \frac{t}{2} x_i x_j\}$ . If one forgets loops and transforms any multiple edge into a single edge,  $\text{MG}(x, t)$  becomes  $\mathcal{W}(x, t)$ , the nonuniform random graph of section 1.4 of [5]. Denoting by  $X(x, t)$  the sequence of sizes, listed in decreasing order, of the connected components of  $\mathcal{W}(x, t)$ , Aldous proved in [5], Proposition 5, that  $X(x, t)$  defines a Markov process on  $\ell_{\searrow}^2$  which possesses the Feller property. Following Aldous, we denote by  $S(x, t)$  the sum of squares of the masses of the components of  $\text{MG}(x, t)$ .

We shall use later the following lemmas.

**Lemma 2.1** ([5], Lemma 20). *For  $x$  in  $\ell_{\searrow}^2$ ,*

$$\mathbb{P}(S(x, t) > s) \leq \frac{tsS(x, 0)}{s - S(x, 0)}, \quad s > S(x, 0).$$

**Lemma 2.2** ([5], Lemma 23). *Let  $(z_i, 1 \leq i \leq n)$  be strictly positive vertex weights, and let  $1 \leq m < n$ . Consider the bipartite random graph  $\mathcal{B}$  on vertices  $\{1, 2, \dots, m\} \cup \{m+1, \dots, n\}$  defined by: for each pair  $(i, j)$  with  $1 \leq i \leq m < j \leq n$ , the edge  $(i, j)$  is present with probability  $1 - \exp(-tz_i z_j)$ , independently for different pairs. Write  $\alpha_1 = \sum_{i=1}^m z_i^2$ ,  $\alpha_2 = \sum_{i=m+1}^n z_i^2$ . Let  $(Z_i)$  be the sizes of the components of  $\mathcal{B}$ . Then,*

$$\varepsilon \mathbb{P}\left(\sum_i Z_i^2 \geq \alpha_1 + \varepsilon\right) \leq (1 + t(\alpha_1 + \varepsilon))^2 \alpha_2, \quad \varepsilon > 0.$$

**Remark 2.3.** *As noticed in [5], page 842, Lemma 2.2 extends to  $z \in \ell^2$ .*

In [5], this lemma is used in conjunction with the following one

**Lemma 2.4** ([5], Lemma 17). *Let  $\tilde{G}$  be a graph with vertex weights  $(\tilde{x}_i)$ . Let  $G$  be a subgraph of  $\tilde{G}$  (that is, each edge of  $G$  is an edge of  $\tilde{G}$ ) with vertex weights  $x_i \leq \tilde{x}_i$ . Let  $\tilde{a}$  and  $a$  be the decreasing orderings of the component sizes of  $\tilde{G}$  and  $G$ . Then*

$$\|\tilde{a} - a\|_2 \leq \sum_i \tilde{a}_i^2 - \sum_i a_i^2$$

*provided  $\sum_i a_i^2 < \infty$ .*

Finally, we shall need the following lemma.

**Lemma 2.5.** *Let  $G = (V, E)$  be a multigraph whose vertices have weights  $(x_i)_{i \in V}$ . If  $W \subset V$  and  $E' \subset E$ , let  $\text{comp}(W, E')$  denote the set of connected components of the graph  $(W, E' \cap \binom{W}{2})$  and define*

$$S(W, E') := \sum_{m \in \text{comp}(W, E')} \left( \sum_{i \in m} x_i \right)^2 .$$

*Now, let  $W \subset V$  be such that for any  $m_1 \in \text{comp}(W, E)$  and  $m_2 \in \text{comp}(V \setminus W, E)$ , there is at most one edge of  $E$  between  $m_1$  and  $m_2$ . Then, for any  $E' \subset E$*

$$S(V, E) - S(W, E) \geq S(V, E') - S(W, E') ,$$

*provided  $S(W, E) < \infty$ .*

*Proof.* For  $i$  and  $j$  in  $V$  and  $E' \subset E$ , we denote by  $i \underset{E'}{\sim} j$  the fact that  $i$  and  $j$  are distinct and connected by a path in  $(V, E')$ . The hypothesis on  $W$  implies that for any  $E' \subset E$ , two vertices of  $W$  are in the same component of  $(V, E')$  if and only if they are in the same component of  $(W, E' \cap \binom{W}{2})$ :

$$\forall i, j \in W, i \underset{E'}{\sim} j \Leftrightarrow i \underset{E' \cap \binom{W}{2}}{\sim} j . \quad (2.1)$$

Now,

$$\begin{aligned} S(V, E') &= \sum_{i \in V} x_i^2 + \sum_{\substack{i, j \in V \\ i \underset{E'}{\sim} j}} x_i x_j \\ &= S(W, E') + \sum_{i \in V \setminus W} x_i^2 + \sum_{\substack{i, j \in V \\ i \underset{E'}{\sim} j}} x_i x_j - \sum_{\substack{i, j \in W \\ i \underset{E' \cap \binom{W}{2}}{\sim} j}} x_i x_j \\ &= S(W, E') + \sum_{i \in V \setminus W} x_i^2 + \sum_{\substack{(i, j) \in V^2 \setminus W^2 \\ i \underset{E'}{\sim} j}} x_i x_j + \sum_{\substack{(i, j) \in W^2 \\ i \underset{E'}{\sim} j}} x_i x_j - \sum_{\substack{i, j \in W \\ i \underset{E' \cap \binom{W}{2}}{\sim} j}} x_i x_j \\ &= S(W, E') + \sum_{i \in V \setminus W} x_i^2 + \sum_{\substack{(i, j) \in V^2 \setminus W^2 \\ i \underset{E'}{\sim} j}} x_i x_j \end{aligned}$$

because of (2.1). The two sums on the right of the last equation are increasing in  $E'$ , and this shows the result.  $\square$

#### 2.4. Measured semi-metric spaces

The main characters in this article are the connected components of Erdős-Rényi random graphs and their continuum limit, each one undergoing the updates due to dynamical percolation. One task is therefore to define a proper space where those characters can live, and first to precise what we mean by “the connected components of a graph” seen as a single object. One option is to order the components by decreasing order of size<sup>1</sup>, as in [2], or in a size-biased way, as in [5], and thus see the collected components of a graph as a sequence of graphs. However, this order is not preserved under the process of dynamical percolation. Also, looking only at the mass to impose which graphs are pairwise compared between two collections of graphs might lead to a larger distance than what one would expect. Indeed, suppose that  $(G_1, G_2)$  and  $(G'_1, G'_2)$  are

<sup>1</sup>It requires some device to break ties, but those disappear in the continuum limit, for the Erdős-Rényi random graphs at least.

two pairs of graphs, with  $G_1$  (resp.  $G'_1$ ) having slightly larger mass than  $G_2$  (resp.  $G'_2$ ). One might have  $G_1$  close to  $G'_2$  and  $G_2$  close to  $G'_1$  in some topology (the Gromov-Hausdorff-Prokhorov topology to be defined later), but  $G_1$  far from  $G'_1$  in this topology. For all these reasons, I found it somewhat uncomfortable to work with such a topology in the dynamical context. The topology we will use will be defined in section 2.5, and the story begins with the definition of a semi-metric space.

One way to present the connected components of a graph is to consider the graph as a metric space using the usual graph distance, allowing the metric to take the value  $+\infty$  between points which are not in the same connected component, as in [8], page 1. In addition, the main difficulty in defining dynamical percolation on the continuum limit will be in defining coalescence. In this process some points will be identified, and one clear way to present this is to modify the metric and allow it to be equal to zero between different points rather than performing the corresponding quotient operation. This type of space is called a semi-metric space in [8], Definition 1.1.4, and we shall stick to this terminology.

**Definition 2.6.** A semi-metric space is a couple  $(X, d)$  where  $X$  is a non-empty set and  $d$  is a function from  $X^2$  to  $\mathbb{R}^+ \cup \{+\infty\}$  such that for all  $x, y$  and  $z$  in  $X$ :

- $d(x, z) \leq d(x, y) + d(y, z)$ ,
- $d(x, x) = 0$ ,
- $d(x, y) = d(y, x)$ .

A semi-metric space  $(X, d)$  is a **metric space** if in addition

- $d(x, y) = 0 \Rightarrow x = y$ .

A metric or semi-metric space  $(X, d)$  is said to be **finite** if  $d$  is finite.

Of course, when thinking about a semi-metric space  $(X, d)$ , one may visualize the *quotient metric space*  $(X/d, d)$  where points at null distance are identified.  $X$  and  $X/d$  are at zero Gromov-Hausdorff distance (defined in section 2.5 below). Notice that  $(X, d)$  is not necessarily a separated metric space, but  $(X/d, d)$  always is. Furthermore,  $(X, d)$  is separable if and only if  $(X/d, d)$  is separable.

**Definition 2.7.** If  $(X, d)$  is a semi-metric space, the relation  $\mathcal{R}$  defined by:

$$x\mathcal{R}y \Leftrightarrow d(x, y) < \infty$$

is an equivalent relation. Each equivalence class is called a **component** of  $(X, d)$  and  $\text{comp}(X, d)$  denotes the set of components. We denote by  $\text{diam}(X)$  the diameter of  $(X, d)$ :

$$\text{diam}(X) = \sup_{x, y \in X} d(x, y)$$

and by  $\text{supdiam}(X)$  the supremum of the diameters of its components:

$$\text{supdiam}(X) = \sup_{m \in \text{comp}(X, d)} \text{diam}(m).$$

**Definition 2.8.** A **measured semi-metric space** (*m.s-m.s*) is a triple  $\mathbf{X} = (X, d, \mu)$  where  $(X, d)$  is a semi-metric space and  $\mu$  is a measure on  $X$  defined on a  $\sigma$ -field containing the Borel  $\sigma$ -field for the topology induced by  $d$ .

An *m.s-m.s*  $(X, d, \mu)$  is said to be **finite** if  $(X, d)$  is a finite totally bounded semi-metric space and  $\mu$  is a finite measure.

Finally, we define  $\text{comp}(\mathbf{X}) := \text{comp}(X, d)$  and

$$\text{masses}(\mathbf{X}) := (\mu(m))_{m \in \text{comp}(\mathbf{X})}.$$

Notice that a finite m.s-m.s has only one component.

**Remark 2.9.** One might feel more comfortable after realizing the following. Let  $\pi$  denote the projection from  $(X, d)$  to  $X' := X/d$ ,  $\mathcal{B}'$  the Borel  $\sigma$ -field on  $X'$  and  $\mathcal{B}$  the Borel  $\sigma$ -field on  $X$ . Then,  $\pi^{-1}(\mathcal{B}') = \mathcal{B}$  and the image measure  $\pi\#\mu$  on  $X'$  is a Borel  $\sigma$ -finite measure.

## 2.5. The Gromov-Hausdorff-Prokhorov distance

In the introduction, we mentioned that  $\mathcal{G}(n, p(\lambda, n))$  converges in distribution, but we did not mention precisely the underlying topology. The main topological ingredient in [3] is the Gromov-Hausdorff-Prokhorov distance between two components of the graph, and we shall use this repeatedly. To define it, we need to recall some definitions from [3].

If  $\mathbf{X} = (X, d, \mu)$  and  $\mathbf{X}' = (X', d', \mu')$  are two measured semi-metric spaces a *correspondance*  $\mathcal{R}$  between  $X$  and  $X'$  is a measurable subset of  $X \times X'$  such that:

$$\forall x \in X, \exists x' \in X' : (x, x') \in \mathcal{R}$$

and

$$\forall x' \in X', \exists x \in X : (x, x') \in \mathcal{R}.$$

We let  $C(X, X')$  denote the set of correspondances between  $X$  and  $X'$ . The *distortion* of a correspondance  $\mathcal{R}$  is defined as

$$\text{dis}(\mathcal{R}) := \inf \left\{ \varepsilon > 0 : \forall (x, x'), (y, y') \in \mathcal{R}, \left( \begin{array}{c} d(x, y) \leq d'(x', y') + \varepsilon \\ \text{and} \\ d'(x', y') \leq d(x, y) + \varepsilon \end{array} \right) \right\}$$

The Gromov-Hausdorff distance between two semi-metric spaces  $(X, d)$  and  $(X', d')$  may be defined as:

$$d_{GH}((X, d), (X', d')) := \inf_{\mathcal{R} \in C(X, X')} \frac{1}{2} \text{dis}(\mathcal{R}).$$

We denote by  $M(X, X')$  the set of finite Borel measures on  $X \times X'$ . For  $\pi$  in  $M(X, X')$ , we denote by  $\pi_1$  (resp.  $\pi_2$ ) the first (resp. the second) marginal of  $\pi$ . For any  $\pi \in M(X, X')$ , and any finite measures  $\mu$  on  $X$  and  $\mu'$  on  $X'$  one defines:

$$D(\pi; \mu, \mu') = \|\pi_1 - \mu\| + \|\pi_2 - \mu'\|$$

where  $\|\nu\|$  is the total variation of a signed measure  $\nu$ .

The *Gromov-Hausdorff-Prokhorov distance* is defined as follows in [3].

**Definition 2.10.** *If  $\mathbf{X} = (X, d, \mu)$  and  $\mathbf{X}' = (X', d', \mu')$  are two m.s.m.s, the Gromov-Hausdorff-Prokhorov distance between them is defined as:*

$$d_{GHP}(\mathbf{X}, \mathbf{X}') = \inf_{\substack{\pi \in M(X, X') \\ \mathcal{R} \in C(X, X')}} \{D(\pi; \mu, \mu') \vee \frac{1}{2} \text{dis}(\mathcal{R}) \vee \pi(\mathcal{R}^c)\}.$$

It is not difficult to show that  $d_{GHP}$  satisfies the axioms of a semi-metric. Let us give a bit more intuition to what the Gromov-Hausdorff-Prokhorov distance measures. On a semi-metric space  $(X, \delta)$ , let us denote by  $\delta_H$  the Hausdorff distance and by  $\delta_{LP}$  the Lévy-Prokhorov distance. Let us recall their definition. For  $B \subset X$  and  $\varepsilon > 0$ , let

$$B^\varepsilon := \{x \in X : \exists y \in B, d(x, y) < \varepsilon\}.$$

Now, for  $A$  and  $B$  subsets of  $X$ ,

$$\delta_H(A, B) := \inf\{\varepsilon > 0 : A \subset B^\varepsilon \text{ and } B \subset A^\varepsilon\}$$

and for finite measures  $\mu$  and  $\nu$  on  $X$ ,

$$\delta_{LP}(\mu, \nu) := \inf\{\varepsilon > 0 : \forall B \in \mathcal{B}(X), \mu(B) \leq \nu(B^\varepsilon) + \varepsilon \text{ and } \nu(B) \leq \mu(B^\varepsilon) + \varepsilon\},$$

where  $B^\varepsilon = \{x \in X : \exists y \in B, d(x, y) < \varepsilon\}$ .

The following lemma shows that the Gromov-Hausdorff-Prokhorov distance measures how well two measured semi-metric spaces can be put in the same ambient space so that simultaneously their measures are close in Prokhorov distance and their geometry are close in the Hausdorff distance. It shows that the definitions of [2] and [1] are equivalent, and its proof is a small variation on the proof of Proposition 6 in [15], where only probability measures were considered.



**Lemma 2.11.** *If  $\mathbf{X} = (X, d, \mu)$  and  $\mathbf{X}' = (X', d', \mu')$  be two measured separable semi-metric spaces, let*

$$\tilde{d}_{GHP}(\mathbf{X}, \mathbf{X}') := \inf_{\delta} \{ \delta_H(X, X') \vee \delta_{LP}(\mu, \mu') \}$$

where the infimum is over all semi-metric  $\delta$  on the disjoint union  $X \cup X'$  extending  $d$  and  $d'$ . Then,

$$\frac{1}{2} \tilde{d}_{GHP}(\mathbf{X}, \mathbf{X}') \leq d_{GHP}(\mathbf{X}, \mathbf{X}') \leq \tilde{d}_{GHP}(\mathbf{X}, \mathbf{X}').$$

*Proof.* Let  $\varepsilon > 0$  and suppose that

$$\tilde{d}_{GHP}(\mathbf{X}, \mathbf{X}') < \varepsilon.$$

Let

$$\mathcal{R} := \{(x, x') \in X \times X' : \delta(x, x') \leq \varepsilon\},$$

so that  $\mathcal{R}$  is a correspondance in  $C(X, X')$  with distortion at most  $2\varepsilon$ . Suppose without loss of generality that  $\mu'(X') \leq \mu(X)$ . Since  $\delta_P(\mu, \mu') \leq \varepsilon$ , for any closed set  $B$  in the disjoint union  $X \cup X'$ ,

$$\begin{aligned} \mu(B) &\leq \mu'(B^\varepsilon) + \varepsilon \\ \frac{\mu(B)}{\mu(X)} &\leq \frac{\mu'(B^\varepsilon)}{\mu'(X')} + \frac{\varepsilon}{\mu(X)} \end{aligned}$$

Thus, Strassens's theorem (Theorem 11.6.2 in [9]) asserts that there exists a coupling  $\pi^0 \in M(X, X')$  such that:

$$\pi_1^0 = \frac{\mu}{\mu(X)}, \quad \pi_2^0 = \frac{\mu'}{\mu'(X')} \quad \text{and} \quad \pi^0(\mathcal{R}^c) \leq \frac{\varepsilon}{\mu(X)}.$$

Let  $\pi := \mu(X)\pi^0$ . Then,  $\pi(\mathcal{R}^c) \leq \varepsilon$  and

$$\|\pi_1 - \mu\| = 0, \quad \|\pi_2 - \mu'\| = \mu(X) - \mu'(X') \leq \varepsilon.$$

Thus,  $d_{GHP}(\mathbf{X}, \mathbf{X}') \leq \varepsilon$  and this proves the second inequality.

Suppose now that  $d_{GHP}(\mathbf{X}, \mathbf{X}') < \varepsilon$ , and let  $\mathcal{R} \in C(X, X')$  and  $\pi \in M(X, X')$  be such that

$$\text{dis}(\mathcal{R}) \leq 2\varepsilon, \quad \pi(\mathcal{R}^c) \leq \varepsilon \quad \text{and} \quad D(\pi; \mu, \mu') \leq \varepsilon.$$

Then, define a semi-metric  $\delta$  on the disjoint union  $X \cup X'$  as follows:

$$\delta(x, x') := \inf_{(y, y') \in \mathcal{R}} \{d(x, y) + \varepsilon + d'(y', x')\}$$

it is proved in [15] Proposition 6 that  $\delta$  is a semi-metric on  $X \cup X'$ , which extends  $d$  and  $d'$ . Furthermore, if  $(x, x') \in \mathcal{R}$ , then  $\delta(x, x') = \varepsilon$ . Clearly,  $\delta_H(X, X') \leq \varepsilon$  and for any Borel set  $B$  in  $X \cup X'$ ,

$$\begin{aligned} \mu(B) &\leq \|\pi_1 - \mu\| + \pi_1(B) \\ &\leq \|\pi_1 - \mu\| + \pi_2(B^\varepsilon) + \pi(\mathcal{R}^c) \\ &\leq \|\pi_1 - \mu\| + \|\pi_2 - \mu'\| + \mu'(B^\varepsilon) + \pi(\mathcal{R}^c) \\ &\leq 2\varepsilon + \mu'(B^\varepsilon) \end{aligned}$$

Thus,  $\delta_P(\mu, \mu') \leq 2\varepsilon$  and this proves the second assertion.  $\square$

It is easy to see that two m.s-m.s  $\mathbf{X}$  and  $\mathbf{X}'$  are at zero  $d_{GHP}$ -distance if and only if there are two distance and measure preserving maps  $\phi$  and  $\phi'$  such that  $\phi$  is a map from  $\mathbf{X}$  to  $\mathbf{X}'$  and  $\phi'$  a map from  $\mathbf{X}'$  to  $\mathbf{X}$ . Let  $\mathcal{C}$  denote the class of finite measured semi-metric spaces and  $\mathcal{R}$  the equivalence relation on  $\mathcal{C}$  defined by  $\mathbf{X} \mathcal{R} \mathbf{X}' \Leftrightarrow d_{GHP}(\mathbf{X}, \mathbf{X}') = 0$ . Even though there is no set of finite measured semi-metric spaces (see for instance [8] Remark 7.2.5 and above),  $\mathcal{C}/\mathcal{R}$  can be considered as a set in the sense that there exists a set of representatives of elements of  $\mathcal{C}$ .

**Definition 2.12.** Let  $\mathbb{U}$  to be the universal Urysohn space and consider the set  $\mathcal{P}$  of finite measured metric subspaces of  $\mathbb{U}$ . We denote by  $\mathcal{M}$  the quotient  $\mathcal{P}/\mathcal{R}$  of  $\mathcal{P}$  by the equivalence relation  $\mathcal{R}$ , where:

$$\mathbf{X}\mathcal{R}\mathbf{X}' \Leftrightarrow d_{GHP}(\mathbf{X}, \mathbf{X}') = 0 .$$

By abuse of language, we may call  $\mathcal{M}$  the “set of equivalence classes of finite measured semi-metric spaces, equipped with the Gromov-Hausdorff-Prokhorov distance  $d_{GHP}$ ”.

$\mathcal{M}$  is a set of representatives of elements of  $\mathcal{C}$ . Indeed, since every separable metric space is isometric to a metric subspace of  $\mathbb{U}$ , every member of  $\mathcal{C}$  will be at zero  $d_{GHP}$ -distance from some element of  $\mathcal{P}$ , and even at zero  $d_{GHP}$ -distance from some compact element of  $\mathcal{P}$ . Thus, for every member  $\mathbf{X}$  of the class  $\mathcal{C}$ , there is an element  $[\mathbf{X}']$  of  $\mathcal{M}$  such that for any  $\mathbf{X}'' \in [\mathbf{X}]$ ,  $d_{GHP}(\mathbf{X}, \mathbf{X}'') = 0$ . Abusing notation, we shall denote by  $[\mathbf{X}]$  the member of  $\mathcal{M}$  whose elements are at zero  $d_{GHP}$ -distance from  $\mathbf{X}$ .

For our purpose, it is in fact not crucial to have Definition 2.12, and one could reformulate all the results in this article in terms of sequences of random variables, at the expense of much more heavy statements.

The following is shown in [1].

**Theorem 2.13.**  $(\mathcal{M}, d_{GHP})$  is a complete separable metric space.

Now, the Gromov-Hausdorff-Prokhorov distance in Definition 2.10 is too strong for our purpose when applied to m.s-m.s which have an infinite number of components: it essentially amounts to a uniform control of the  $d_{GHP}$  distance between paired components. We are interested in a weaker distance which localizes around the largest components. We shall restrict to countable unions of finite semi-metric spaces with the additional property that for any  $\varepsilon > 0$ , there are only a finite number of components whose size exceeds  $\varepsilon$ . To formulate the distance, it will be convenient to view those semi-metric spaces as a set of counting measures on  $\mathcal{M}$ .

**Definition 2.14.** For any  $\varepsilon > 0$ , let

$$\mathcal{M}_{>\varepsilon} = \{[(X, d, \mu)] \in \mathcal{M} \text{ s.t. } \mu(X) > \varepsilon\} .$$

For any counting measure  $\nu$  on  $\mathcal{M}$ , denote by  $\nu_{>\varepsilon}$  the restriction of  $\nu$  to  $\mathcal{M}_{>\varepsilon}$ . Denote by  $\mathcal{N}$  the set of counting measures  $\nu$  on  $\mathcal{M}$  such that for any  $\varepsilon > 0$ ,  $\nu_{>\varepsilon}$  is a finite measure and such that  $\nu$  does not have atoms of mass 0, that is:

$$\nu(\{[(X, d, \mu)] \in \mathcal{M} \text{ s.t. } \mu(X) = 0\}) = 0$$

When  $\mathbf{X}$  is a measured semi-metric space whose components are finite, we denote by  $\nu_{\mathbf{X}}$  the counting measure on  $\mathcal{M}$  defined by

$$\nu_{\mathbf{X}} := \sum_{m \in \text{comp}(\mathbf{X})} \delta_{[m]}$$

Abusing notations, we shall say that  $\mathbf{X} \in \mathcal{N}$  if  $\nu_{\mathbf{X}}$  belongs to  $\mathcal{N}$ , and we shall denote by  $\mathbf{X}_{>\varepsilon}$  the disjoint union of components of  $\mathbf{X}$  whose masses are larger than  $\varepsilon$ .

Notice that  $\mathbf{X}$  is in  $\mathcal{N}$  if and only if it has an at most countable number of components, each one of its components has positive mass and for any  $\varepsilon > 0$ ,  $\mathbf{X}_{>\varepsilon}$  is the disjoint union of a finite number of components, each one being totally bounded and equipped with a finite measure.

To define a Gromov-Hausdorff-Prokhorov distance between elements of  $\mathcal{N}$ , let first  $\rho_{LP}$  be the Lévy-Prokhorov distance on the set of finite measures on the metric space  $(\mathcal{M}, d_{GHP})$ . Recall that

$$\rho_{LP}(\nu, \nu') = \inf \left\{ \varepsilon > 0 : \forall B \in \mathcal{B}(\mathcal{M}), \left( \begin{array}{c} \nu(B) \leq \nu'(B^\varepsilon) + \varepsilon \\ \text{and} \\ \nu'(B) \leq \nu(B^\varepsilon) + \varepsilon \end{array} \right) \right\}$$

where  $B^\varepsilon = \{\mathbf{X} \in \mathcal{M} : d_{GHP}(\mathbf{X}, B) < \varepsilon\}$  and  $\mathcal{B}(\mathcal{M})$  is the Borel  $\sigma$ -algebra on  $(\mathcal{M}, d_{GHP})$ . Now, for  $\mathbf{X} = [(X, d, \mu)] \in \mathcal{M}$  and  $k \geq 1$ , define a function  $f_k$  by

$$f_k(\mathbf{X}) := \begin{cases} 1 & \text{if } \mu(X) \geq \frac{1}{k} \\ k(k+1) \left( \mu(X) - \frac{1}{k+1} \right) & \text{if } \mu(X) \in \left[ \frac{1}{k+1}, \frac{1}{k} \right] \\ 0 & \text{if } \mu(X) < \frac{1}{k+1} \end{cases}$$

The following distance is an analogue of the distance in Lemma 4.6 of [14] where it is used to metrize the vague topology on boundedly finite measures.

**Definition 2.15.** Let  $\nu$  and  $\nu'$  be counting measures on  $\mathcal{M}$ . Then, we define the **Gromov-Hausdorff-Prokhorov metric**  $L_{GHP}$  between  $\nu$  and  $\nu'$  as:

$$L_{GHP}(\nu, \nu') := \sum_{k \geq 1} 2^{-k} \{1 \wedge \rho_{LP}(f_k \nu, f_k \nu')\}.$$

**Remark 2.16.**  $L_{GHP}$  makes sense even between counting measures whose atoms are semi-metric spaces.

**Remark 2.17.** Notice that  $\rho_{LP}(\nu_{\mathbf{X}}, 0)$  is at most the number of connected components of  $\mathbf{X}$ . Thus

$$L_{GHP}(\nu_{\mathbf{X}}, 0) \leq 2^{2 - (\sup_{m \in \text{comp}(\mathbf{X})} \mu(m))^{-1}}.$$

One sees thus that if  $\mathbf{X}^{(n)}$  is a sequence of m.s-m.s such that

$$\sup_{m \in \text{comp}(\mathbf{X}^{(n)})} \mu(m) \xrightarrow{n \rightarrow +\infty} 0$$

and whatever the diameters of the components are, then  $\nu_{\mathbf{X}^{(n)}}$  converges to zero for  $L_{GHP}$ , which can be seen as an empty collection of measured metric spaces. Notably,  $\text{supdiam}$  is not continuous with respect to the  $L_{GHP}$ -distance.

The following proposition is analogue to similar results concerning vague convergence of boundedly finite measures (see section 4 of [14]).

**Proposition 2.18.**  $(\mathcal{N}, L_{GHP})$  is a complete separable metric space, and if  $\nu_n$ ,  $n \geq 0$  and  $\nu$  are elements of  $\mathcal{N}$ ,  $(\nu_n)_{n \geq 0}$  converges to  $\nu$  if and only if for every  $\varepsilon > 0$  such that  $\nu(\{(X, d, \mu) \in \mathcal{M} : \mu(X) = \varepsilon\}) = 0$ ,  $\rho_{LP}(\nu_{>\varepsilon}^{(n)}, \nu_{>\varepsilon})$  goes to zero as  $n$  goes to infinity.

*Proof.* The fact that  $L_{GHP}$  is a metric is left to the reader. Let  $D$  be a countable set dense in  $\{(X, d, \mu) \in \mathcal{M} : \mu(X) \neq 0\}$ . Let  $\mathcal{D} := \{\sum_{k=1}^m \delta_{\mathbf{X}_k} : m \in \mathbb{N}, \mathbf{X}_1, \dots, \mathbf{X}_m \in D\}$ . It is easy to show that  $\mathcal{D}$  is dense in  $(\mathcal{N}, L_{GHP})$ . This shows separability.

Now, suppose that  $\nu^{(n)}$  is a Cauchy sequence for  $L_{GHP}$ . Then, for any  $k \geq 1$ ,  $f_k \nu^{(n)}$  is a Cauchy sequence of finite measures for  $\rho_{LP}$ . From the completeness of the Lévy-Prokhorov distance on finite measures on a Polish space we get that for each  $k$ , there is some measure  $\nu_k$  such that

$$\rho_{LP}(f_k \nu^{(n)}, \nu_k) \xrightarrow{n \rightarrow \infty} 0.$$

Notice that if  $2 \leq k \leq l$ ,  $\nu_k = \nu_l$  on  $\mathcal{M}_{>\frac{1}{k-1}}$ . Define

$$\nu = \sup_{k \geq 0} \mathbf{1}_{M_{\frac{1}{k+1}}} \nu_{k+2}$$

so that for any  $k \geq 1$ ,  $f_k \nu = \nu_k$ . Then,  $\nu$  is an element of  $\mathcal{N}$  and

$$L_{GHP}(\nu^{(n)}, \nu) \xrightarrow{n \rightarrow \infty} 0,$$

showing the completeness of  $(\mathcal{N}, L_{GHP})$ .

Finally, suppose that  $L_{GHP}(\nu^{(n)}, \nu)$  goes to zero as  $n$  goes to infinity and let  $\varepsilon > 0$  be such that  $\nu(\{(X, d, \mu) \in \mathcal{M} : \mu(X) = \varepsilon\}) = 0$ . Then, for any  $\alpha > 0$ , let  $k$  be such that

$$\frac{1}{k} \leq \varepsilon$$

let  $N$  be such that

$$\forall n \geq N, \forall A \in \mathcal{B}(\mathcal{M}), f_k \nu(A) \leq f_k \nu^{(n)}(A^\alpha) + \alpha \text{ and } f_k \nu^{(n)}(A) \leq f_k \nu(A^\alpha) + \alpha .$$

Then, for  $n \geq N$  and  $B \in \mathcal{B}(\mathcal{M}_{>\varepsilon})$ ,

$$\begin{aligned} \nu_{>\varepsilon}(B) &\leq \nu(B \cap \mathcal{M}_{>\varepsilon+\alpha}) + \nu(\mathcal{M}_{>\varepsilon} \setminus \mathcal{M}_{>\varepsilon+\alpha}) \\ &= f_k \nu(B \cap \mathcal{M}_{>\varepsilon+\alpha}) + \nu(\mathcal{M}_{>\varepsilon} \setminus \mathcal{M}_{>\varepsilon+\alpha}) \\ &\leq f_k \nu^{(n)}((B \cap \mathcal{M}_{>\varepsilon+\alpha})^\alpha) + \alpha + \nu(\mathcal{M}_{>\varepsilon} \setminus \mathcal{M}_{>\varepsilon+\alpha}) \\ &= \nu_{>\varepsilon}^{(n)}((B \cap \mathcal{M}_{>\varepsilon+\alpha})^\alpha) + \alpha + \nu(\mathcal{M}_{>\varepsilon} \setminus \mathcal{M}_{>\varepsilon+\alpha}) \\ &\leq \nu_{>\varepsilon}^{(n)}((B)^\alpha) + \alpha + \nu(\mathcal{M}_{>\varepsilon} \setminus \mathcal{M}_{>\varepsilon+\alpha}) \end{aligned}$$

where we used the fact that  $(B \cap \mathcal{M}_{>\varepsilon+\alpha})^\alpha \subset \mathcal{M}_{>\varepsilon}$  and  $f_k$  equals 1 on  $\mathcal{M}_{>\varepsilon}$ . Also, for  $n \geq N$  and  $B \in \mathcal{B}(\mathcal{M}_{>\varepsilon})$ ,

$$\begin{aligned} \nu_{>\varepsilon}^{(n)}(B) &= f_k \nu^{(n)}(B) \\ &\leq f_k \nu(B^\alpha) + \alpha \\ &\leq \nu(B^\alpha) + \alpha \\ &\leq \nu(B^\alpha \cap \mathcal{M}_{>\varepsilon}) + \nu(\mathcal{M}_{>\varepsilon-\alpha} \setminus \mathcal{M}_{>\varepsilon}) \\ &= \nu_{>\varepsilon}((B)^\alpha) + \nu(\mathcal{M}_{>\varepsilon-\alpha} \setminus \mathcal{M}_{>\varepsilon}) \end{aligned}$$

To finish the proof, note that since  $\nu(\{(X, d, \mu) \in \mathcal{M} : \mu(X) = \varepsilon\}) = 0$ , then

$$\nu(\mathcal{M}_{>\varepsilon} \setminus \mathcal{M}_{>\varepsilon+\alpha}) + \nu(\mathcal{M}_{>\varepsilon-\alpha} \setminus \mathcal{M}_{>\varepsilon}) \xrightarrow{\alpha \rightarrow 0} 0 .$$

□

Notice that any m.s-m.s of  $\mathcal{N}$  is at zero  $L_{GHP}$ -distance of an m.s-m.s whose components are compact metric spaces. In this article, we really are interested in equivalence classes of m.s-m.s for the equivalence relation “being at zero  $L_{GHP}$ -distance”, although in order to define random processes such as coalescence and fragmentation, it will be convenient to have in mind a particular representative of such a class.

We shall always use the following lemmas to bound  $L_{GHP}$  from above.

**Lemma 2.19.** *Let  $\mathbf{X} = (X, d, \mu)$  and  $\mathbf{X}' = (X', d', \mu')$  belong to  $\mathcal{N}$  and  $\varepsilon > 0$ . Suppose there exists two injective maps*

$$\sigma : \text{comp}(\mathbf{X}_{>\varepsilon}) \rightarrow \text{comp}(\mathbf{X}') \text{ and } \sigma' : \text{comp}(\mathbf{X}'_{>\varepsilon}) \rightarrow \text{comp}(\mathbf{X})$$

such that:

$$\forall m \in \text{comp}(\mathbf{X}_{>\varepsilon}), d_{GHP}(m, \sigma(m)) \leq \alpha$$

and

$$\forall m' \in \text{comp}(\mathbf{X}'_{>\varepsilon}), d_{GHP}(m', \sigma'(m')) \leq \alpha .$$

Then,

$$L_{GHP}(\nu_{\mathbf{X}}, \nu_{\mathbf{X}'}) \leq \alpha(1 + 8\# \text{comp}(\mathbf{X}_{>\varepsilon-\alpha})) + 16\varepsilon ,$$

and if  $\varepsilon > \alpha$ , for  $p > 0$ ,

$$L_{GHP}(\nu_{\mathbf{X}}, \nu_{\mathbf{X}'}) \leq \alpha \left( 1 + 8 \frac{\sum_{m \in \text{comp}(\mathbf{X})} \mu(m)^p}{(\varepsilon - \alpha)^p} \right) + 16\varepsilon ,$$

*Proof.* Consider any  $\varepsilon_0 \geq \varepsilon$ . For a component  $m$  in  $\text{comp}(\mathbf{X}_{>\varepsilon_0})$ , the difference between the masses  $\mu(m)$  and  $\mu'(\sigma(m))$  is at most  $\alpha$ , and the same holds between  $m'$  and  $\sigma'(m')$  when  $m' \in \text{comp}(\mathbf{X}'_{>\varepsilon_0})$ . Thus  $\sigma'$  sends  $\text{comp}(\mathbf{X}'_{>\varepsilon_0})$  in  $\text{comp}(\mathbf{X}_{>\varepsilon_0-\alpha})$  and

$$\#\{m \in \text{comp}(\mathbf{X}'_{>\varepsilon_0})\} \leq \#\{m \in \text{comp}(\mathbf{X}_{>\varepsilon_0-\alpha})\}.$$

Now, let  $k$  be such that

$$\frac{1}{k+1} \geq \varepsilon.$$

Let  $B \in \mathcal{B}(\mathcal{M}_{>\frac{1}{k+1}})$ . Then, for any  $m$  in  $\text{comp}(\mathbf{X}_{>\frac{1}{k+1}}) \cap B$ ,  $\sigma(m)$  belongs to  $\text{comp}(\mathbf{X}') \cap B^\alpha$ . Notice that  $f_k$  is  $k(k+1)$ -Lipschitz.

$$\begin{aligned} f_k \nu_{\mathbf{X}}(B) &= \sum_{m \in \text{comp}(\mathbf{X}_{>\frac{1}{k+1}}) \cap B} f_k(m) \\ &\leq \sum_{m \in \text{comp}(\mathbf{X}_{>\frac{1}{k+1}}) \cap B} f_k(\sigma(m)) + \alpha k(k+1) \# \text{comp}(\mathbf{X}_{>\frac{1}{k+1}}) \\ &\leq \sum_{\substack{m' \in \text{comp}(\mathbf{X}') \\ m' \in B^\alpha}} f_k(m') + \alpha k(k+1) \# \text{comp}(\mathbf{X}_{>\frac{1}{k+1}}) \\ &= f_k \nu_{\mathbf{X}'}(B^\alpha) + \alpha k(k+1) \# \text{comp}(\mathbf{X}_{>\frac{1}{k+1}}) \end{aligned}$$

and symmetrically

$$f_k \nu_{\mathbf{X}'}(B) \leq f_k \nu_{\mathbf{X}}(B^\alpha) + \alpha k(k+1) \# \text{comp}(\mathbf{X}'_{>\frac{1}{k+1}})$$

Thus, for any  $k$  such that  $\frac{1}{k+1} \geq \varepsilon$ ,

$$\begin{aligned} \rho_{LP}(f_k \nu_{\mathbf{X}}, f_k \nu_{\mathbf{X}'}) &\leq \alpha(1 + k(k+1) \# \text{comp}(\mathbf{X}_{>\frac{1}{k+1}}) \vee \# \text{comp}(\mathbf{X}'_{>\frac{1}{k+1}})) \\ &\leq \alpha(1 + k(k+1) \# \text{comp}(\mathbf{X}_{>\varepsilon-\alpha})) \end{aligned}$$

Thus,

$$\begin{aligned} &L_{GHP}(\nu_{\mathbf{X}}, \nu_{\mathbf{X}'}) \\ &\leq \alpha \left( 1 + \# \text{comp}(\mathbf{X}_{>\varepsilon-\alpha}) \sum_{k < \frac{1}{\varepsilon}-1} 2^{-k} k(k+1) \right) + \sum_{k \geq \frac{1}{\varepsilon}-1} 2^{-k} \\ &\leq \alpha(1 + 8 \# \text{comp}(\mathbf{X}_{>\varepsilon-\alpha})) + 16\varepsilon \end{aligned}$$

□

If  $\mathbf{X}$  and  $\mathbf{X}'$  are two m.s-m.s with a finite number of finite components, one may measure their distance with  $d_{GHP}$  (using Definition 2.10), with  $L_{GHP}$  (using Definition 2.15) or with

$$1 \wedge \inf_{\sigma} \sup_{m \in \text{comp}(\mathbf{X})} d_{GHP}(m, \sigma(m)) = 1 \wedge \rho_{LP}(\nu_{\mathbf{X}}, \nu_{\mathbf{X}'})$$

where the infimum is over bijections  $\sigma$  between  $\text{comp}(\mathbf{X})$  and  $\text{comp}(\mathbf{X}')$ . Those three distances do not necessarily coincide, and the following lemma clarifies the links between them.

**Lemma 2.20.** *Let  $\mathbf{X} = (X, d, \mu)$  and  $\mathbf{X}' = (X', d', \mu')$  be two m.s-m.s in  $\mathcal{N}$  with a finite number of components.*

(i) *If  $d_{GHP}(\mathbf{X}, \mathbf{X}') < \infty$ , then there is a bijection  $\sigma$  from  $\text{comp}(\mathbf{X})$  to  $\text{comp}(\mathbf{X}')$  such that:*

$$\forall m \in \text{comp}(\mathbf{X}), d_{GHP}(m, \sigma(m)) \leq 2d_{GHP}(\mathbf{X}, \mathbf{X}'),$$

and thus,

$$L_{GHP}(\mathbf{X}, \mathbf{X}') \leq 2d_{GHP}(\mathbf{X}, \mathbf{X}')(1 + 8 \# \text{comp}(\mathbf{X})).$$

(ii) If there exists a bijection  $\sigma$  from  $\text{comp}(\mathbf{X})$  to  $\text{comp}(\mathbf{X}')$  such that:

$$\sup_{m \in \text{comp}(\mathbf{X})} d_{GHP}(m, \sigma(m)) < \infty ,$$

then,

$$d_{GHP}(\mathbf{X}, \mathbf{X}') \leq \sup_{m \in \text{comp}(\mathbf{X})} d_{GHP}(m, \sigma(m)) \# \text{comp}(\mathbf{X}) ,$$

and

$$L_{GHP}(\mathbf{X}, \mathbf{X}') \leq \sup_{m \in \text{comp}(\mathbf{X})} d_{GHP}(m, \sigma(m))(1 + 8 \# \text{comp}(\mathbf{X})) .$$

*Proof. Proof of (i).* Suppose that  $d_{GHP}(\mathbf{X}, \mathbf{X}') < \varepsilon < \infty$ . Let  $\mathcal{R} \in C(X, X')$  and  $\pi \in M(X, X')$  be such that

$$D(\pi; \mu, \mu') \vee \frac{1}{2} \text{dis}(\mathcal{R}) \vee \pi(\mathcal{R}^c) \leq \varepsilon .$$

Since  $\mathcal{R}$  has finite distortion,

$$\forall (x, x'), (y, y') \in \mathcal{R}, d(x, y) = +\infty \Leftrightarrow d'(x', y') = +\infty$$

which shows that each component  $m$  of  $\mathbf{X}$  (resp.  $\mathbf{X}'$ ) is in correspondance through  $\mathcal{R}$  with exactly one component  $\sigma(m)$  of  $\mathbf{X}'$  (resp.  $\mathbf{X}$ ).  $\sigma$  is thus a bijection, and  $\mathcal{R} \cap m \times \sigma(m) \in C(m, \sigma(m))$  and has distortion at most  $2d_{GHP}(\mathbf{X}, \mathbf{X}')$ . Furthermore,

$$\pi|_{m \times \sigma(m)}((\mathcal{R} \cap m \times \sigma(m))^c) = \pi(\mathcal{R}^c \cap m \times \sigma(m)) \leq \varepsilon .$$

Finally, for any  $A \in \mathcal{B}(m)$ ,

$$\begin{aligned} |\pi|_{m \times \sigma(m)}(A \times \sigma(m)) - \mu|_m(A)| &\leq |\pi(A \times X') - \mu(A)| + \pi(A \times \sigma(m)^c) \\ &\leq \varepsilon + \pi(\mathcal{R}^c) \\ &\leq 2\varepsilon \end{aligned}$$

and similarly, for any  $A' \in \mathcal{B}(\sigma(m))$ ,

$$|\pi|_{m \times \sigma(m)}(m \times A') - \mu'|_{\sigma(m)}(A')| \leq 2\varepsilon .$$

Thus,

$$\forall m \in \text{comp}(\mathbf{X}), d_{GHP}(m, \sigma(m)) \leq 2\varepsilon ,$$

and the consequence on  $L_{GHP}(\mathbf{X}, \mathbf{X}')$  comes from Lemma 2.19.

*Proof of (ii).* Suppose that there exists a bijection  $\sigma$  from  $\text{comp}(\mathbf{X})$  to  $\text{comp}(\mathbf{X}')$  such that:

$$\sup_{m \in \text{comp}(\mathbf{X})} d_{GHP}(m, \sigma(m)) \leq \varepsilon < \infty .$$

Then, for any  $m$ , let  $\mathcal{R}_m \in C(m, \sigma(m))$  and  $\pi_m \in M(m, \sigma(m))$  be such that

$$D(\pi_m; \mu|_m, \mu'|_{\sigma(m)}) \vee \frac{1}{2} \text{dis}(\mathcal{R}_m) \vee \pi_m(\mathcal{R}_m^c) \leq \varepsilon .$$

Let  $\pi = \sum_{m \in \text{comp}(\mathbf{X})} \pi_m$  and  $\mathcal{R} = \cup_{m \in \text{comp}(\mathbf{X})} \mathcal{R}_m$ . Then,  $\mathcal{R}$  is a correspondance between  $X$  and  $X'$ ,

$$\frac{1}{2} \text{dis}(\mathcal{R}) \leq \frac{1}{2} \sup_m \text{dis}(\mathcal{R}_m) \leq \varepsilon ,$$

$$\pi(\mathcal{R}^c) = \sum_m \pi_m(\mathcal{R}_m^c) \leq \# \text{comp}(\mathbf{X}) \varepsilon .$$

Furthermore, for any  $A \in \mathcal{B}(X)$ ,

$$\begin{aligned} |\pi(A \times X') - \mu(A)| &\leq \sum_{m \in \text{comp}(\mathbf{X})} |\pi((A \cap m) \times X') - \mu(A \cap m)| \\ &= \sum_{m \in \text{comp}(\mathbf{X})} |\pi((A \cap m) \times \sigma(m)) - \mu(A \cap m)| \\ &\leq \# \text{comp}(\mathbf{X})\varepsilon \end{aligned}$$

and symmetrically, for any  $A' \in \mathcal{B}(X')$ ,

$$|\pi(X \times A') - \mu'(A')| \leq \# \text{comp}(\mathbf{X})\varepsilon$$

Thus

$$D(\pi; \mu, \mu') \leq \# \text{comp}(\mathbf{X})\varepsilon ,$$

and we get

$$d_{GHP}(\mathbf{X}, \mathbf{X}') \leq \# \text{comp}(\mathbf{X})\varepsilon .$$

The statement on  $L_{GHP}(\mathbf{X}, \mathbf{X}')$  follows from the hypothesis and Lemma 2.19 applied for any  $\varepsilon > 0$  small enough, and letting  $\varepsilon$  go to zero.  $\square$

## 2.6. Gluing and coalescence

### 2.6.1. Gluing and $\delta$ -gluing

If  $(X, d)$  and  $(X', d')$  are two semi-metric spaces, the *disjoint union semi-metric* on  $X \cup X'$  is the semi-metric equal to  $d$  on  $X^2$ , to  $d'$  on  $(X')^2$  and to  $+\infty$  on  $(X \times X') \cup (X' \times X)$ . Gluing corresponds to identification of points which can belong to the same semi-metric space or to different semi-metric spaces. A formal definition is as follows (see also [8], pages 62–64).

**Definition 2.21.** *Let  $(X, d)$  be a semi-metric space and  $\mathcal{R}$  be an equivalence relation on  $X$ . The **gluing of  $(X, d)$  along  $\mathcal{R}$** , is the semi-metric space  $(X, d_{\mathcal{R}})$  with semi-metric defined on  $X^2$  by*

$$d_{\mathcal{R}}(x, y) := \inf \left\{ \sum_{i=1}^k d(p_i, q_i) : p_0 = x, q_k = y, k \in \mathbb{N}^* \right\}$$

where the infimum is taken over all choices of  $\{p_i\}$  and  $\{q_i\}$  such that  $(q_i, p_{i+1}) \in \mathcal{R}$  for all  $i = 1, \dots, k-1$ .

If  $(X', d')$  is another semi-metric space and  $\tilde{\mathcal{R}} \subset X \times X'$ , let  $\mathcal{R}$  be the equivalence relation generated by  $\tilde{\mathcal{R}}$  on the disjoint union  $X \cup X'$ . The **gluing of  $(X, d)$  and  $(X', d')$  along  $\tilde{\mathcal{R}}$**  is the gluing of  $(X \cup X', d'')$  along  $\mathcal{R}$ , where  $X \cup X'$  is the disjoint union of  $X$  and  $X'$  and  $d''$  is the disjoint union semi-metric.

Now, we shall define the  $\delta$ -gluing of a semi-metric space  $X$  along a subset  $\tilde{\mathcal{R}}$  of  $X^2$  as the operation of joining every couple  $(x, x') \in \tilde{\mathcal{R}}$  by a copy of the interval  $[0, \delta]$ .

**Definition 2.22.** *Let  $(X, d)$  be a semi-metric space,  $\tilde{\mathcal{R}} \subset X^2$  and  $\delta \geq 0$ . Then, the  **$\delta$ -gluing of  $(X, d)$  along  $\tilde{\mathcal{R}}$** ,  $(X_{\mathcal{R}, \delta}, d_{\mathcal{R}, \delta})$ , is the semi-metric space which is the result of gluing an isometric copy of  $[0, \delta]$  between each couple  $(x, x')$  belonging to  $\tilde{\mathcal{R}}$ .*

When  $\mathbf{X} = (X, d, \mu)$  is a measured semi-metric space, we equip the  $\delta$ -gluing of  $(X, d)$  along  $\tilde{\mathcal{R}}$  with the restriction of the measure  $\mu$  to  $X$ , we still denote this measure  $\mu$  and denote the resulting semi-metric space by  $\text{Coal}_{\delta}(\mathbf{X}, \tilde{\mathcal{R}}) = (X_{\mathcal{R}, \delta}, d_{\mathcal{R}, \delta}, \mu)$ .

**Remark 2.23.** (i) *When  $\delta = 0$ , the  $\delta$ -gluing of  $(X, d)$  along  $\tilde{\mathcal{R}}$  can be seen as the gluing of  $(X, d)$  along the equivalence relation generated by  $\tilde{\mathcal{R}}$ .*

(ii) For any  $(x, y) \in X^2$ ,

$$d_{\mathcal{R},\delta}(x, y) = \inf\{(k-1)\delta + \sum_{i=1}^k d(p_i, q_i) : p_0 = x, q_k = y, k \in \mathbb{N}^*\}.$$

where the infimum is taken over all choices of  $\{p_i\}$  and  $\{q_i\}$  such that  $(q_i, p_{i+1}) \in \mathcal{R}$  for all  $i = 1, \dots, k-1$ .

(iii) If  $\delta > 0$ , one may like to consider the space  $X$  with metric  $\delta_{\mathcal{R},\delta}$ , which corresponds to forget the interior of the intervals  $[0, \delta]$  that have been added in  $X_{\mathcal{R},\delta}$ . If  $(X, d, \mu) \in \mathcal{N}$ , for any  $k$ ,

$$L_{GHP}((X, d_{\mathcal{R},\delta}, \mu), \text{Coal}_\delta(X, \tilde{\mathcal{R}})) \leq \delta.$$

### 2.6.2. The coalescence processes

When  $(X, d, \mu)$  is a measured semi-metric space, there is a natural coalescence process (of mean-field type) which draws pairs of points  $(x, y)$  with intensity  $\mu(dx)\mu(dy)$  (and unit intensity in time) and identifies points  $x$  and  $y$ , changing the metric accordingly. To describe the process of addition of edges during the dynamical percolation on Erdős-Rényi random graph, one needs to replace the identification of  $x$  and  $y$  by the fact that the distance between  $x$  and  $y$  drops to  $1/n$  (if  $x \neq y$ ). This leads to the following definition.

**Definition 2.24.** Let  $\mathbf{X} = (X, d, \mu)$  be an m.s-m.s with  $\mu$  sigma-finite and  $\delta \geq 0$ . Let  $\mathcal{P}^+$  be a Poisson random set on  $X^2 \times \mathbb{R}^+$  of intensity measure  $\frac{1}{2}\mu^2 \times \text{leb}_{\mathbb{R}^+}$ . The **coalescence process with edge-lengths  $\delta$  started from  $\mathbf{X}$** , denoted by  $(\text{Coal}_\delta(\mathbf{X}, t))_{t \geq 0}$ , is the random process of m.s-m.s  $(\text{Coal}_\delta(\mathbf{X}, \mathcal{P}_t^+))_{t \geq 0}$ .

Notice that this process inherits the strong Markov property from the strong Markov property of the Poisson process, and the fact that for  $A, B \subset X^2$ ,  $\text{Coal}_\delta(\mathbf{X}, A \cup B) = \text{Coal}_\delta(\text{Coal}_\delta(\mathbf{X}, A), B)$ .

When  $\delta > 0$ , if one wants to keep the space fixed and change only the metric, Remark 2.23 (iii) shows that one can do so at the price of an  $L_{GHP}$ -distance at most  $\delta$ . In this paper, one wants typically to understand scaling limits of  $N^+(G_n, \mathcal{P}_t^+)$  with  $\mathcal{P}^+$  of intensity  $\gamma_n$  and  $G_n$  a discrete graph equipped with the distance  $d_n$  which is the graph distance multiplied by some  $\delta_n > 0$  going to zero as  $n$  goes to infinity. See for instance Theorem 3.1 below. If one equips  $G_n$  and  $N^+(G_n, \mathcal{P}_t^+)$  with their counting measures multiplied by  $\sqrt{\gamma_n}$ ,  $N_{\gamma_n}^+(G_n, t)$  is at  $L_{GHP}$ -distance at most  $\delta_n$  from  $(\text{Coal}_{\delta_n}((G_n, d_n, \mu_n), t))_{t \geq 0}$ , so the scaling limits will be the same. We shall want to identify the limit itself as  $(\text{Coal}(\mathcal{G}_\lambda, t))_{t \geq 0}$ , and part of our work will consist in showing that it is a nicely behaved process. In order to accomplish this task, we need to define some subsets of  $\mathcal{N}$ .

**Definition 2.25.** For  $p > 0$ , we define  $\mathcal{N}_p$  to be the set of elements  $\nu = \sum_{m \in I} \delta_m$  of  $\mathcal{N}$  such that

$$\sum_{m \in I} \mu(m)^p < \infty.$$

For  $\nu$  in  $\mathcal{N}_p$ , we let  $\text{masses}(\nu)$  to be the sequence in  $\ell_{\searrow}^p$  of masses  $\mu(m)$  listed in decreasing order and define, for  $\nu$  and  $\nu'$  in  $\mathcal{N}_p$ .

$$L_{p,GHP}(\nu, \nu') = L_{GHP}(\nu, \nu') \vee \|\text{masses}(\nu) - \text{masses}(\nu')\|_p.$$

Again we shall abuse language, saying that  $(X, d, \mu)$  is in  $\mathcal{N}_p$  when  $\nu_{\mathbf{X}} \in \mathcal{N}_p$  and write  $L_{p,GHP}(\mathbf{X}, \mathbf{X}')$  for  $L_{p,GHP}(\nu_{\mathbf{X}}, \nu_{\mathbf{X}'})$ . We let the reader check that  $(\mathcal{N}_p, L_{p,GHP})$  is a complete separable metric space.

It is easy to see that if  $(X, d, \mu)$  belongs to  $\mathcal{N}_1$ , then almost surely, for every  $t \geq 0$ ,  $\text{Coal}_{t,\delta}(X, d, \mu)$  is in  $\mathcal{N}_1$ . We even have the Feller property on  $\mathcal{N}_1$ , which will be proved in section 4.1. A consequence of the Feller property of the multiplicative coalescent in  $\ell^2$  is that if  $\mathbf{X} = (X, d, \mu)$  belongs to  $\mathcal{N}_2$ , then almost surely for every  $t \geq 0$   $\sum_{m \in \text{comp}(\text{Coal}_\delta(\mathbf{X}, t))} \mu(m)^2 < \infty$ . However, one cannot guarantee that components stay totally bounded, and thus that  $\text{Coal}_\delta(\mathbf{X}, t)$  and even  $\text{Coal}_0(\mathbf{X}, t)$  belongs to  $\mathcal{N}_2$ . One will thus have to restrict to a subclass of  $\mathcal{N}_2$ , which will fortunately contain  $\mathcal{G}_\lambda$  with probability one.



**Definition 2.26.** We define  $\mathcal{S}$  to be the class of m.s-m.s  $\mathbf{X} = (X, d, \mu)$  in  $\mathcal{N}_2$  such that

$$\forall t \geq 0, \sup_{\eta \rightarrow 0} \mathbb{P}(\text{Coal}_0(\mathbf{X}_{\leq \eta}, t)) \xrightarrow{\mathbb{P}} 0 \quad (2.2)$$

It will be shown in Lemma 4.10 that if  $\mathbf{X} \in \mathcal{S}$ , then almost surely, for any  $t \geq 0$ ,  $\text{Coal}_0(\mathbf{X}, t) \in \mathcal{N}_2$ . Of course, I suspect that  $\mathcal{S}$  has a more intrinsic definition, and that there is a convenient topology which turns it into a Polish space, however I could not prove this for the moment. Let us mention that there are elements in  $\mathcal{N}_2 \setminus \mathcal{S}$ , see Remark 4.8.

Let us finish this section by a description of the coalescence at the level of components. When  $\mathbf{X}$  and  $\mathcal{P}$  are as in Definition 2.24, one may associate to them a process of multigraphs with vertices  $\text{comp}(X)$  which we denote by  $\text{MG}(X, t)$ . It is defined as follows: there is an edge in  $\text{MG}(X, t)$  between  $m$  and  $m'$  if there is a point  $(x, y, s)$  of  $\mathcal{P}$  with  $s \leq t$ ,  $x \in m$  and  $y \in m'$ . If  $x = \text{masses}(\mathbf{X})$ , this multigraph is of course closely related to the multigraph  $\text{MG}(x, t)$  defined in section 2.3.

When  $A$  is a measurable subset of  $X$ , let  $\mathbf{A} = (A, d|_{A \times A}, \mu|_A)$ . There is an obvious coupling between  $(\text{Coal}(\mathbf{X}, t))_{t \geq 0}$  and  $(\text{Coal}(\mathbf{A}, t))_{t \geq 0}$ : just take the restriction of the poisson process  $\mathcal{P}$  to  $A^2 \times \mathbb{R}$ . We shall call it the *obvious coupling*. We shall use several times the following easy fact.

**Lemma 2.27.** Suppose that  $A$  is a union of components of  $X$ . Under the obvious coupling, if  $\text{MG}(X, t)$  is a forest, then for every  $x, y$  in  $A$  and every  $s \leq t$ , if the distance between  $x$  and  $y$  in  $(\text{Coal}(\mathbf{A}, s))$  is finite, then it is equal to the distance between  $x$  and  $y$  in  $(\text{Coal}(\mathbf{X}, s))$ .

## 2.7. $\mathbb{R}$ -graphs

We refer to [3] for background on the definitions and statements in this section.

**Definition 2.28.** An  $\mathbb{R}$ -tree is a geodesic and acyclic finite metric space. An  $\mathbb{R}$ -graph is a totally bounded geodesic finite metric space  $(G, d)$  such that there exists  $R > 0$  such that for any  $x \in G$ ,  $(B_R(x), d|_{B_R(x)})$  is an  $\mathbb{R}$ -tree, where  $B_R(x)$  is the ball of radius  $R$  and center  $x$ .

For a semi-metric space  $(X, d)$ , we shall say that it is an  $\mathbb{R}$ -graph if the quotient metric space  $(X/d, d)$  is an  $\mathbb{R}$ -graph.

**Remark 2.29.** The definition above differs slightly from Definition 2.2 in [3], where an  $\mathbb{R}$ -graph  $(X, d)$  is defined as a compact geodesic metric space such that for any  $x \in G$ , there exists  $\varepsilon = \varepsilon(x) > 0$  such that  $(B_\varepsilon(x), d|_{B_\varepsilon(x)})$  is an  $\mathbb{R}$ -tree, where  $B_\varepsilon(x)$  is the ball of radius  $\varepsilon$  and center  $x$ . When  $(X, d)$  is compact, the two definitions agree: one direction is obvious, whereas the other follows from the arguments at the beginning of section 6.1 in [3]. One advantage of working with precompact spaces instead of compact ones is that one may avoid completion to recover an  $\mathbb{R}$ -graph after fragmentation.

The degree  $d_G(x)$  of a point  $x$  in a graph  $(G, d)$  is the number of connected components of  $B_{\varepsilon(x)}(x) \setminus \{x\}$ . A *branchpoint*  $x$  is a point with  $\text{deg}_G(x) \geq 3$ . A *leaf*  $x$  is a point with degree one. We denote by  $\text{leaves}(G)$  the set of leaves of  $G$ . An  $\mathbb{R}$ -tree or an  $\mathbb{R}$ -graph is said to be *finite* if it is compact and has a finite number of leaves.

An  $\mathbb{R}$ -graph  $(G, d)$  is naturally equipped with a *length measure*, which assigns for instance its length to the image of a geodesic. We shall denote it by  $\ell_G$ , it is a sigma-finite diffuse measure.

The structure of an  $\mathbb{R}$ -graph is explained thoroughly<sup>2</sup> in [3]. The *core* of  $(G, d)$ , denoted by  $\text{core}(G)$  is the union of all simple arcs with both endpoints in embedded cycles of  $G$ . It is also the maximal compact subset of  $G$  having only points of degree at least 2 (cf. Corollary 2.5 in [3], where one needs to replace “closed” by “compact” in our precompact setting). The core of a tree is empty, that of a unicyclic graph is a cycle. When  $G$  is neither a tree nor unicyclic, there is a finite connected multigraph  $\text{ker}(G) = (k(G), e(G))$  called the *kernel* of  $G$  such that the core of  $G$  may be obtained from  $\text{ker}(G)$  by gluing along each edge an isometric copy of the interval  $[0, l]$ , for some  $l > 0$ . The *surplus* of  $G$  is defined as 0 when  $G$  is a tree, 1 when  $G$  is unicyclic, and in general as:

$$\text{surplus}(G) = |e(G)| - |k(G)| + 1,$$

<sup>2</sup>Although our definition differs slightly, the proof of Proposition 6.2 in [3] can be adapted straightforwardly.

which is then at least two.

Using the existence of the core, one gets the following equivalent definition of an  $\mathbb{R}$ -graph, where an  $\mathbb{R}$ -graph is obtained as a “tree with shortcuts”, to employ the expression of [7]. We leave the proof to the reader.

**Lemma 2.30.** *A metric space  $(X, d)$  is an  $\mathbb{R}$ -graph if and only if there exists a totally bounded  $\mathbb{R}$ -tree  $(T, d)$  and a finite set  $A \subset T^2$  such that  $(X, d)$  is isomorphic to the quotient metric space obtained from  $\text{Coal}_0((T, d), A)$ .*

**Definition 2.31.** *Let  $\mathcal{P}^{graph}$  denote the set of metric subspaces of the Urysohn space  $\mathbb{U}$  that are  $\mathbb{R}$ -graphs. Let  $\mathcal{M}^{graph}$  denote the set of equivalence classes on  $\mathcal{P}^{graph}$  under the equivalence relation of being at zero  $d_{GHP}$ -distance.*

*Define  $\mathcal{N}^{graph}$  (resp.  $\mathcal{N}_p^{graph}$ , resp.  $\mathcal{S}^{graph}$ ) from  $\mathcal{M}^{graph}$  in the same way that  $\mathcal{N}$  (resp.  $\mathcal{N}_p$ , resp.  $\mathcal{S}$ ) was defined from  $\mathcal{M}$ .*

*If  $\mathbf{X} = (X, d, \mu)$  and  $\mathbf{X}' = (X', d', \mu')$  belong to  $\mathcal{M}^{graph}$ , we let  $\text{surplus}(X)$  denote the surplus of any graph in the equivalent class of  $X$  and define*

$$d_{GHP}^{surplus}(\mathbf{X}, \mathbf{X}') := d_{GHP}(\mathbf{X}, \mathbf{X}') \vee |\text{surplus}(X) - \text{surplus}(X')|.$$

*Finally, define  $L_{GHP}^{surplus}$  and  $L_{p,GHP}^{surplus}$  in the same way that  $L_{GHP}$  and  $L_{p,GHP}$  were defined, but replacing  $d_{GHP}$  by  $d_{GHP}^{surplus}$ .*

Thanks to Lemma 2.30, it is clear that if  $\mathbf{X} \in \mathcal{N}^{graph}$  and  $\mathcal{P} \subset X^2$  is finite, then for any  $\delta \geq 0$ ,  $\text{Coal}_\delta(\mathbf{X}, \mathcal{P})$  still belongs to  $\mathcal{N}^{graph}$ . Furthermore, it will be shown in Lemma 4.10 that if  $\mathbf{X} \in \mathcal{S}^{graph}$ , then almost surely, for any  $t \geq 0$ ,  $\text{Coal}_0(\mathbf{X}, t) \in \mathcal{N}_2^{graph}$ .

Additional notations concerning  $\mathbb{R}$ -graphs will be introduced when needed, in section 5.1.

## 2.8. Cutting, fragmentation and dynamical percolation

**Definition 2.32.** *Suppose that  $\mathbf{X} = (X, d, \mu)$  is an m.s-m.s whose components are length spaces and  $\mathcal{P}^-$  is a subset of  $X$ . Then, the cut of  $\mathbf{X}$  along  $\mathcal{P}^-$ , denoted by  $\text{Frag}(\mathbf{X}, \mathcal{P}^-)$  is the m.s-m.s  $(X \setminus \mathcal{P}^-, d_{\mathcal{P}^-}^{\text{Frag}}, \mu|_{X \setminus \mathcal{P}^-})$  where*

$$d_{\mathcal{P}^-}^{\text{Frag}}(x, y) := \inf_{\gamma} \{\ell_X(\gamma)\}$$

*and the infimum is over all paths  $\gamma$  from  $x$  to  $y$  disjoint from  $\mathcal{P}^-$ .*

**Remark 2.33.** (i)  $\text{Frag}(\mathbf{X}, \emptyset)$  is the same as  $\mathbf{X}$  precisely because the components of  $(X, d)$  are length spaces.

(ii) Notice that if  $(X, d)$  is complete,  $\text{Frag}((X, d, \mu), \mathcal{P})$  is generally not complete anymore, but its completion is at zero  $L_{GHP}$ -distance from  $(X, d, \mu)$ .

**Definition 2.34.** *Let  $\mathbf{X} = (X, d, \mu)$  be an m.s-m.s whose components are length spaces. Let  $\ell$  be a diffuse  $\sigma$ -finite Borel measure on  $X$ . Let  $\mathcal{P}^-$  be a Poisson random set on  $X \times \mathbb{R}^+$  of intensity measure  $\ell \otimes \text{leb}_{\mathbb{R}^+}$ . The **fragmentation process started from  $\mathbf{X}$** , denoted by  $(\text{Frag}(\mathbf{X}, t))_{t \geq 0}$ , is the random process of m.s-m.s  $(\text{Frag}(\mathbf{X}, \mathcal{P}_t^-))_{t \geq 0}$ . When  $\mathbf{X} \in \mathcal{N}^{graph}$ , we shall always take  $\ell$  to be  $\ell_X$ , the length-measure on  $\mathbf{X}$ .*

**Remark 2.35.** (i) A similar fragmentation on the CRT is considered in [6].

(ii) Since  $\ell$  is a diffuse measure, almost surely  $\mu(\mathcal{P}_t^-) = 0$  for any  $t \geq 0$ . Thus we shall abuse notation and consider that  $\text{Frag}(\mathbf{X}, \mathcal{P}_t^-)$  is still equipped with  $\mu$ , instead of  $\mu|_{X \setminus \mathcal{P}_t^-}$ .

(iii) Notice that this process inherits the strong Markov property from the strong Markov property of the Poisson process, and the fact that for  $A, B \subset X$ ,  $\text{Frag}(\mathbf{X}, A \cup B) = \text{Frag}(\text{Frag}(\mathbf{X}, A), B)$ .

Now, one wants to define dynamical percolation on measured length spaces by performing independently and simultaneously coalescence and fragmentation. One needs to be a bit careful here: when  $(X, d)$  is a geodesic space,  $A \subset X^2$  and  $B \subset X$ , even if  $B \cap \{x \in X : \exists y \in X, (x, y) \text{ or } (y, x) \in A\} = \emptyset$ , one cannot guarantee that  $\text{Coal}_0(\text{Frag}(X, B), A)$  is the same as  $\text{Frag}(\text{Coal}_0(X, A), B)$ . Indeed, let  $X = [0, 1]$  with the

usual metric, let  $B = \{\frac{3}{2^n}, n \geq 2\}$  and  $A = \{(\frac{1}{2^{n+1}}, \frac{1}{2^n}), n \geq 1\}$ . Then, there are two components in  $\text{Coal}_0(\text{Frag}(X, B), A)$ :  $\{0\}$  and  $]0, 1] \setminus B$ , whereas there is only one component in  $\text{Frag}(\text{Coal}_0(X, A), B)$ :  $[0, 1] \setminus B$ . However, it will be shown in Lemma 4.10 that if  $\mathbf{X} \in \mathcal{S}^{graph}$ ,  $\mathcal{P}^+$  is as in Definition 2.24,  $\mathcal{P}^-$  as in Definition 2.34 then almost surely,

$$\forall t \geq 0, \text{Coal}_0(\text{Frag}(\mathbf{X}, \mathcal{P}_t^-), \mathcal{P}_t^+) = \text{Frag}(\text{Coal}_0(\mathbf{X}, \mathcal{P}_t^+), \mathcal{P}_t^-). \quad (2.3)$$

This will rely on the following property. Hereafter, we say that a path  $\gamma$  in  $\text{Coal}(X, A)$  takes a shortcut  $(a, b)$  in  $A \subset X^2$  if  $(a, b) \in A$  and  $\gamma \cap \{a, b\} \neq \emptyset$ .

**Lemma 2.36.** *Let  $(X, d)$  be a length space and  $A \subset X^2$  an equivalent relation. Suppose that for any  $(x, y) \in X^2$ , every simple rectifiable path in  $\text{Coal}(X, A)$  from  $x$  to  $y$  takes only a finite number of shortcuts in  $A$ . Then, for any  $B \subset X$  such that  $B \cap \{x \in X : \exists y \in X, (x, y) \text{ or } (y, x) \in A\} = \emptyset$ ,*

$$\text{Coal}_0(\text{Frag}(X, B), A) = \text{Frag}(\text{Coal}_0(X, A), B).$$

*Proof:* Let  $\ell_X(\gamma)$  denote the length of a path  $\gamma$  in  $X$ . Let  $d^{fragcoal}$  (resp.  $d^{coalfrag}$ , resp.  $d^{frag}$ ) denote the distance of  $\text{Frag}(\text{Coal}_0(X, A), B)$  (resp.  $\text{Coal}_0(\text{Frag}(X, B), A)$ , resp.  $\text{Frag}(X, B)$ ) on  $X \setminus B$ . We want to show that  $d^{fragcoal} = d^{coalfrag}$ . First, it is always true that  $d^{fragcoal} \leq d^{coalfrag}$ . Indeed, let  $\{p_i\}$  and  $\{q_i\}$ ,  $i = 1, \dots, k$  be such that  $(q_i, p_{i+1}) \in A$  for all  $i = 1, \dots, k-1$  and  $p_0 = x, q_k = y$ . Then, the concatenation of  $(k-1)$  paths  $\gamma_i$  in  $X$ ,  $i = 1, \dots, k-1$  such that  $\gamma_i$  goes from  $p_i$  to  $q_i$  and each path avoids  $B$  gives a path in  $\text{Coal}(X, A)$  from  $x$  to  $y$  avoiding  $B$ . Thus, for any  $x$  and  $y$  in  $X \setminus B$ ,

$$\begin{aligned} d^{fragcoal}(x, y) &\leq \inf_{\substack{k, \{p_i\}, \{q_i\} \\ \gamma_i: p_i \rightarrow q_i \\ \gamma_i \cap B = \emptyset}} \left( \sum_{i=1}^k \ell_X(\gamma_i) \right) \\ &= \inf_{k, \{p_i\}, \{q_i\}} \left( \sum_{i=1}^k d^{frag}(p_i, q_i) \right) \\ &= d^{coalfrag}(x, y). \end{aligned}$$

Let us show now that  $d^{coalfrag} \leq d^{fragcoal}$ . Let  $x$  and  $y$  be in  $X \setminus B$  and let  $\gamma$  be a rectifiable simple path from  $x$  to  $y$  in  $\text{Coal}(X, A)$  such that  $\gamma \cap B = \emptyset$ . Then,  $\gamma$  takes only a finite number of shortcuts in  $A$ . Thus, there exists  $\{p_i\}$  and  $\{q_i\}$ ,  $i = 1, \dots, k$  and paths  $\gamma_i$ ,  $i = 1, \dots, k-1$  such that  $(q_i, p_{i+1}) \in A$  and  $\gamma_i$  is a path from  $p_i$  to  $q_i$  in  $X$  and  $\gamma$  is the concatenation of  $\gamma_1, \dots, \gamma_k$ . Thus,  $\gamma_i \cap B = \emptyset$  for any  $i$  and

$$\begin{aligned} \ell_{\text{Coal}(X, A)}(\gamma) &= \sum_{i=1}^k \ell_X(\gamma_i) \\ &\geq \sum_{i=1}^k d^{frag}(p_i, q_i) \\ &\geq d^{coalfrag}(x, y) \end{aligned}$$

Taking the infimum over rectifiable simple paths  $\gamma$  from  $x$  to  $y$  in  $\text{Coal}(X, A)$  such that  $\gamma \cap B = \emptyset$  gives that  $d^{coalfrag} \leq d^{fragcoal}$ .  $\square$

**Definition 2.37.** *Let  $\mathbf{X} = (X, d, \mu)$  be an m.s.-m.s whose components are length spaces. Let  $\ell$  be a diffuse  $\sigma$ -finite Borel measure on  $X$ . Let  $\mathcal{P}^-$  be a Poisson random set on  $X \times \mathbb{R}^+$  of intensity measure  $\ell \otimes \text{leb}_{\mathbb{R}^+}$  and  $\mathcal{P}^+$  be a Poisson random set on  $X^2 \times \mathbb{R}^+$  of intensity measure  $\frac{1}{2}\mu^2 \times \text{leb}_{\mathbb{R}^+}$ . The **dynamical percolation process started from  $\mathbf{X}$** , denoted by  $(\text{CoalFrag}(\mathbf{X}, t))_{t \geq 0}$ , is the stochastic process  $(\text{Coal}_0(\text{Frag}(\mathbf{X}, \mathcal{P}_t^-), \mathcal{P}_t^+))_{t \geq 0}$ .*

Property (2.3) (when it holds !) shows that  $(\text{CoalFrag}(\mathbf{X}, t))_{t \geq 0}$  inherits the strong Markov Property from that of the Poisson process.

## 2.9. The scaling limit of critical Erdős-Rényi random graphs

The scaling limit of critical Erdős-Rényi random graphs was obtained in [2], Theorem 24, for the Gromov-Hausdorff topology, and the result is extended to Gromov-Hausdorff-Prokhorov topology in [3], Theorem 4.1. Let  $\bar{\mathcal{G}}_n, \lambda$  denote element of  $\mathcal{N}_2^{graph}$  obtained by replacing each edge of  $\mathcal{G}(n, p)$  by an isometric copy of a segment of length  $n^{-1/3}$  (notably, the distance is the graph distance divided by  $n^{1/3}$ ) and choosing as measure the counting measure on vertices divided by  $n^{2/3}$ . Theorem 4.1 in [3] and Corollary 2 in [5] easily imply the following.

**Theorem 2.38** ([2],[3]). *Let  $\lambda \in \mathbb{R}$  and  $p(\lambda, n) = \frac{1}{n} + \frac{\lambda}{n^{4/3}}$ . There is a random element  $\mathcal{G}_\lambda$  of  $\mathcal{N}_2^{graph}$  such that*

$$\bar{\mathcal{G}}_{n,\lambda} \xrightarrow[n \rightarrow \infty]{(d)} \mathcal{G}_\lambda,$$

where the convergence in distribution is with respect to the  $L_{2,GHP}^{surplus}$ -topology.

We refer to [2] for the precise definition of the limit  $\mathcal{G}_\lambda$  and to [4] for various properties of  $\mathcal{G}_\lambda$ .

## 3. Main results

The main results concerning Erdős-Rényi random graphs are the following. Notice that the fact that  $(\text{Coal}_0(\mathcal{G}_\lambda, t))_{t \geq 0}$  and  $(\text{CoalFrag}(\mathcal{G}_\lambda, t))_{t \geq 0}$  are well-defined processes in  $\mathcal{S}^{graph}$  will be part of the proofs, see section 4.3 and Remark 4.11. Recall that every convergence of a process stated in this article is with respect to the topology of compact convergence (associated to  $L_{1,GHP}$ ,  $L_{1,GHP}^{surplus}$ ,  $L_{2,GHP}$  or  $L_{1,GHP}^{surplus}$ ), see section 2.1.

**Theorem 3.1.** *Let  $\mathbf{G}^{n,\lambda,+}(t)$ ,  $t \geq 0$  be the discrete coalescence process of intensity  $n^{-4/3}$ , started at  $\mathcal{G}(n, p(\lambda, n))$ , equipped with the graph distance multiplied by  $n^{-1/3}$  and the counting measure on vertices multiplied by  $n^{-2/3}$ . Then, the sequence of processes  $\mathbf{G}^{n,\lambda,+}(\cdot)$  converges to  $\text{Coal}_0(\mathcal{G}_\lambda, \cdot)$  for  $L_{2,GHP}$  as  $n$  goes to infinity.*

**Theorem 3.2.** *Let  $\mathbf{G}^{n,\lambda,-}(t)$ ,  $t \geq 0$  be the discrete fragmentation process of intensity  $n^{-1/3}$  started at  $\mathcal{G}(n, p(\lambda, n))$ , equipped with the graph distance multiplied by  $n^{-1/3}$  and the counting measure on vertices multiplied by  $n^{-2/3}$ . Then, the sequence of processes  $\mathbf{G}^{n,\lambda,-}(\cdot)$  converges to  $\text{Frag}(\mathcal{G}_\lambda, \cdot)$  for  $L_{2,GHP}^{surplus}$  as  $n$  goes to infinity.*

**Theorem 3.3.** *Let  $\mathbf{G}^{n,\lambda}(t)$ ,  $t \geq 0$  be the dynamical percolation processes of parameter  $p(\lambda, n)$  and intensity  $n^{-1/3}$  started with  $\mathcal{G}(n, p(\lambda, n))$ , equipped with the graph distance multiplied by  $n^{-1/3}$  and the counting measure on vertices multiplied by  $n^{-2/3}$ . Then, the sequence of processes  $\mathbf{G}^{n,\lambda}(\cdot)$  converges to  $\text{CoalFrag}(\mathcal{G}_\lambda, \cdot)$  for  $L_{2,GHP}$  as  $n$  goes to infinity.*

In the course of proving those results, I tried to obtain more general results, such that one could apply the same technology to other sequences of random graphs, for instance those belonging to the basin of attraction of  $\mathcal{G}_\lambda$  (see [7] for this notion). This is reflected in what I called below Feller or almost Feller properties for coalescence (Lemma 4.13), fragmentation (Proposition 5.5), and dynamical percolation (Proposition 6.2). Note that those are variations on the Feller property, often weaker than a true Feller property in the sense that I need to add some condition in order to ensure convergence, but also a bit stronger in the sense that I added in the results the convergence of the whole process in the sense of the topology of compact convergence.

Let me describe the rest of the article. Section 4 is devoted to the proofs of the (almost) Feller property for coalescence and of Theorem 3.1. It contains notably the fact that  $(\text{Coal}_0(\mathcal{G}_\lambda, t))_{t \geq 0}$  and  $(\text{CoalFrag}(\mathcal{G}_\lambda, t))_{t \geq 0}$  are processes in  $\mathcal{S}^{graph}$ . Probably the most important work lies inside Lemma 4.5, which is a statement about the structure of the graph  $W(x, t)$  in Aldous' multiplicative coalescent. It allows notably to reduce the proof of the Feller property from  $\mathcal{N}_2$  to  $\mathcal{N}_1$ , where it is much easier to prove. Section 5 is devoted to the proofs of the Feller property for fragmentation and of Theorem 3.2. We shall also show that for  $\mathcal{G}_\lambda$ , coalescence is the time-reversal of fragmentation, see Proposition 5.13. Finally, section 6 is devoted to the proofs of the (almost) Feller property for dynamical percolation and of Theorem 3.3.

## 4. Proofs of the main results for coalescence

### 4.1. The Coalescent on $\mathcal{N}_1$

On  $\mathcal{N}_1$ , coalescence behaves very gently since there is a finite number of coalescence events in any finite time interval. Notably, for  $\mathbf{X} \in \mathcal{N}_1$ ,  $(\text{Coal}(\mathbf{X}, t))_{t \geq 0}$  is clearly càdlàg. The aim of this section is to prove Proposition 4.3, which is essentially a Feller property.

**Lemma 4.1.** *Let  $\varepsilon \in ]0; 1[$ ,  $\mathbf{X} = (X, d, \mu)$  and  $\mathbf{X}' = (X', d', \mu')$  be two m.s-m.s with a finite number of finite components.*

*Suppose that  $P = \{(x_i, y_i), 1 \leq i \leq k\}$  are pairs of points in  $X$  and  $P' = \{(x'_i, y'_i), 1 \leq i \leq k\}$  are pairs of points in  $X'$ . Suppose that there exists  $\pi \in M(X, X')$  and  $\mathcal{R} \in C(X, X')$  such that:*

$$D(\pi; \mu, \mu') \vee \pi(\mathcal{R}^c) \vee \frac{1}{2} \text{dis}(\mathcal{R}) \leq \varepsilon$$

*and that for any  $i \leq k$ ,  $(x_i, x'_i) \in \mathcal{R}$  and  $(y_i, y'_i) \in \mathcal{R}$ . Then, for any  $\delta, \delta' > 0$ ,*

$$d_{GHP}(\text{Coal}_\delta(\mathbf{X}, P), \text{Coal}_{\delta'}(\mathbf{X}', P')) \leq (2\varepsilon + |\delta - \delta'|)(k + 1)$$

*Proof.* This is essentially<sup>3</sup> Lemma 21 in [2] and Lemma 4.2 in [3], thus we leave the details to the reader.  $\square$

**Lemma 4.2.** *Let  $\varepsilon \in ]0; 1[$  and  $\delta > 0$ ,  $\mathbf{X} = (X, d, \mu)$  and  $\mathbf{X}' = (X', d', \mu')$  be two m.s-m.s with a finite number of finite components. If there exists  $\pi \in M(X, X')$  and  $\mathcal{R} \in C(X, X')$  such that:*

$$D(\pi; \mu, \mu') \vee \pi(\mathcal{R}^c) \vee \frac{1}{2} \text{dis}(\mathcal{R}) \leq \varepsilon$$

*then, one may couple two Poisson processes:  $\mathcal{P}$  of intensity  $\frac{1}{2}\mu^{\otimes 2} \otimes \text{leb}_{[0, T]}$  and  $\mathcal{P}'$  of intensity  $\frac{1}{2}(\mu')^{\otimes 2} \otimes \text{leb}_{[0, T]}$ , such that with probability larger than  $1 - T\varepsilon(10 + 8\mu(X) + 8\mu'(X')) - \sqrt{2\varepsilon + |\delta - \delta'|}$ , for any  $t \leq T$ ,*

$$d_{GHP}(\text{Coal}_\delta(\mathbf{X}, \mathcal{P}_t), \text{Coal}_{\delta'}(\mathbf{X}', \mathcal{P}'_t)) \leq (T\mu(X)^2 + 1)\sqrt{2\varepsilon + |\delta - \delta'|}.$$

*Proof.* Let  $\mathcal{P}(\mu)$  denote the distribution of a Poisson random set of intensity measure  $\mu$ . Using the coupling characterization of total variation distance and the gluing lemma (cf [18] page 23), one may construct three Poisson random sets on the same probability space,  $P$ ,  $\tilde{P}$  and  $P'$  such that:

- (i)  $P = (X_i, Y_i, t_i)_{i=1, \dots, N}$  has distribution  $\mathcal{P}(\mu^{\otimes 2} \times \text{leb}_{[0, T]})$ ,
- (ii)  $P' = (X'_i, Y'_i, t'_i)_{i=1, \dots, N'}$  has distribution  $\mathcal{P}((\mu')^{\otimes 2} \times \text{leb}_{[0, T]})$ ,
- (iii)  $P = (\tilde{X}_i, \tilde{X}'_i, \tilde{Y}_i, \tilde{Y}'_i, \tilde{t}_i)_{i=1, \dots, \tilde{N}}$  has distribution  $\mathcal{P}(\pi^{\otimes 2} \times \text{leb}_{[0, T]})$

and furthermore:

$$\begin{aligned} & \mathbb{P}[(X_i, Y_i, t_i)_{i=1, \dots, N} \neq (\tilde{X}_i, \tilde{Y}_i, \tilde{t}_i)_{i=1, \dots, \tilde{N}}] \\ & \leq \| \mathcal{P}(\mu^{\otimes 2} \times \text{leb}_{[0, T]}) - \mathcal{P}(\pi_1^{\otimes 2} \times \text{leb}_{[0, T]}) \| \end{aligned}$$

and

$$\begin{aligned} & \mathbb{P}[(X'_i, Y'_i, t'_i)_{i=1, \dots, N'} \neq (\tilde{X}'_i, \tilde{Y}'_i, \tilde{t}_i)_{i=1, \dots, \tilde{N}}] \\ & \leq \| \mathcal{P}((\mu')^{\otimes 2} \times \text{leb}_{[0, T]}) - \mathcal{P}(\pi_2^{\otimes 2} \times \text{leb}_{[0, T]}) \| . \end{aligned}$$

Now, for any  $T > 0$ ,

$$\begin{aligned} & \| \mathcal{P}(\mu^{\otimes 2} \times \text{leb}_{[0, T]}) - \mathcal{P}(\pi_1^{\otimes 2} \times \text{leb}_{[0, T]}) \| \\ & \leq 2 \| \mu^{\otimes 2} \times \text{leb}_{[0, T]} - \pi_1^{\otimes 2} \times \text{leb}_{[0, T]} \| \\ & = 2T \| \mu^{\otimes 2} - \pi_1^{\otimes 2} \| \\ & \leq 4T(\mu(X) + \pi_1(X)) \| \mu - \pi_1 \| \\ & \leq 4T\varepsilon(2\mu(X) + \varepsilon) \end{aligned}$$

<sup>3</sup> In [3], Lemma 4.2 is stated for trees and for  $\delta = 0$ .

by hypothesis. Similarly,

$$\|\mathcal{P}((\mu')^{\otimes 2} \times \text{leb}_{[0,T]}) - \mathcal{P}(\pi_2^{\otimes 2} \times \text{leb}_{[0,T]})\| \leq 4T\varepsilon(2\mu'(X') + \varepsilon).$$

Furthermore,  $(\tilde{X}_i, \tilde{X}'_i, \tilde{t}_i)_{i=1, \dots, \tilde{N}}$  and  $(\tilde{Y}_i, \tilde{Y}'_i, \tilde{t}_i)_{i=1, \dots, \tilde{N}}$  both have distribution  $\mathcal{P}(\pi \otimes \text{leb}_{[0,T]})$ . Thus,

$$\mathbb{P}(\exists i \leq \tilde{N}, (\tilde{X}_i, \tilde{X}'_i) \notin \mathcal{R}) \leq T\pi(\mathcal{R}^c) \leq T\varepsilon$$

and

$$\mathbb{P}(\exists i \leq \tilde{N}, (\tilde{Y}_i, \tilde{Y}'_i) \notin \mathcal{R}) \leq T\varepsilon.$$

Let  $\mathcal{E}$  be the event that  $N = N'$  and for any  $i$ ,  $(\tilde{X}_i, \tilde{X}'_i) \in \mathcal{R}$  and  $(\tilde{Y}_i, \tilde{Y}'_i) \in \mathcal{R}$ . Altogether, we get that  $\mathcal{E}$  has probability at least  $1 - T\varepsilon(10 + 8\mu(X) + 8\mu'(X'))$ .

Since the distortion of  $\mathcal{R}$  is at most  $2\varepsilon$ , we get using Lemma 4.1 that on the event  $\mathcal{E}$ , for any  $t \leq T$

$$d_{GHP}(\text{Coal}_\delta(\mathbf{X}, \mathcal{P}_t), \text{Coal}_{\delta'}(\mathbf{X}', \mathcal{P}'_t)) \leq (N+1)(2\varepsilon + |\delta - \delta'|)$$

Since  $N$  has distribution  $\mathcal{P}(\mu(X)^2 T)$ ,

$$\mathbb{P}\left(N \geq \frac{T\mu(X)^2}{\sqrt{2\varepsilon + |\delta - \delta'|}}\right) \leq \sqrt{2\varepsilon + |\delta - \delta'|}$$

this gives the result.  $\square$

In the Proposition below, recall from section 2.1 that convergence of processes uses the topology of compact convergence (here for the metric space  $(\mathcal{N}_1, L_{1,GHP})$ ).

**Proposition 4.3.** *Let  $\mathbf{X}^n = (X^n, d^n, \mu^n)$ ,  $n \geq 0$  be a sequence of elements in  $\mathcal{N}_1$  and  $(\delta^n)_{n \geq 0}$  a sequence of non-negative real numbers. Suppose that:*

- (a)  $(\mathbf{X}^n)_{n \geq 0}$  converges (for  $L_{1,GHP}$ ) to  $\mathbf{X}^\infty = (X^\infty, d^\infty, \mu^\infty)$  as  $n$  goes to infinity
- (b)  $\delta^n \xrightarrow[n \rightarrow \infty]{} \delta^\infty$

Then,

- (i)  $(\text{Coal}_{\delta^n}(\mathbf{X}^n, t))_{t \geq 0}$  converges in distribution to  $(\text{Coal}_\delta(\mathbf{X}^\infty, t))_{t \geq 0}$ ,
- (ii) if  $t^n \xrightarrow[n \rightarrow \infty]{} t$ ,  $\text{Coal}_{\delta^n}(\mathbf{X}^n, t^n)$  converges in distribution to  $\text{Coal}_\delta(\mathbf{X}^\infty, t)$ .

*Proof.* Let us fix  $\varepsilon \in ]0, 1[$ . Let  $\varepsilon_0 \in ]0, \varepsilon/2[$  be such that  $\varepsilon_0 \notin \text{masses}(\mathbf{X}^\infty)$  and

$$\mu(X_{\leq \varepsilon_0}^\infty) \leq \varepsilon.$$

Proposition 2.18 shows that  $\rho_{LP}(\mathbf{X}_{>\varepsilon_0}^n, \mathbf{X}_{>\varepsilon_0}^\infty)$  goes to zero as  $n$  goes to infinity. Let  $n$  be large enough so that

$$\begin{aligned} \rho_{LP}(\mathbf{X}_{>\varepsilon_0}^n, \mathbf{X}_{>\varepsilon_0}^\infty) &\leq \left(\frac{\varepsilon \wedge \varepsilon_0}{1 + 2T\mu(X^\infty)}\right)^4 \\ |\delta^n - \delta^\infty| &\leq \left(\frac{\varepsilon \wedge \varepsilon_0}{1 + 2T\mu(X^\infty)^2}\right)^4 \end{aligned}$$

and

$$\|\text{masses}(\mathbf{X}^n) - \text{masses}(\mathbf{X}^\infty)\|_1 \leq \varepsilon.$$

Let  $k := \#\text{comp}(X_{>\varepsilon_0}^\infty)$  and notice that

$$k \leq \frac{\mu(X^\infty)}{\varepsilon_0}.$$

Lemma 2.20 shows that

$$d_{GHP}(\mathbf{X}_{>\varepsilon_0}^n, \mathbf{X}_{>\varepsilon_0}^\infty) \leq \left(\frac{\varepsilon \wedge \varepsilon_0}{1 + 2T\mu(X^\infty)^2}\right)^4 \frac{\mu(X^\infty)}{\varepsilon_0}.$$

Notice also that:

$$\mu(X_{\leq \varepsilon_0}^n) \leq \mu(X_{\leq \varepsilon_0}^\infty) + \|\text{masses}(\mathbf{X}^n) - \text{masses}(\mathbf{X}^\infty)\|_1 \leq 2\varepsilon.$$

Thus, using Lemma 4.2, one may couple the coalescence on  $X^n$  and  $X^\infty$  in such a way that with probability larger than  $1 - 2\varepsilon T$  no point of the Poisson processes touches  $X_{\leq \varepsilon_0}^\infty$  or  $X_{\leq \varepsilon_0}^n$  and with probability larger than  $1 - T\varepsilon(10 + 8\mu(X^n) + 8\mu^\infty(X^\infty)) - \sqrt{(\varepsilon \wedge \varepsilon_0)^3}$ , for any  $t \leq T$ ,

$$d_{GHP}(\text{Coal}_{\delta^n}(\mathbf{X}_{>\varepsilon_0}^n, t), \text{Coal}_\delta(\mathbf{X}_{>\varepsilon_0}^\infty, t)) \leq \sqrt{(\varepsilon \wedge \varepsilon_0)^3}.$$

Using Lemma 2.20 and Lemma 2.19, this implies that

$$L_{GHP}(\text{Coal}_{\delta^n}(\mathbf{X}^n, t), \text{Coal}_\delta(\mathbf{X}^\infty, t)) \leq C\sqrt{\varepsilon}$$

for some finite constant  $C$  depending only on  $\mu(X^\infty)$ . Furthermore, since the multigraphs  $\mathbb{M}\mathbb{G}(X^\infty, t)$  and  $\mathbb{M}\mathbb{G}(X^n, t)$  are the same for any  $t \leq T$  in this coupling,

$$\begin{aligned} & \|\text{masses}(\text{Coal}_{\delta^n}(X^n, t)) - \text{masses}(\text{Coal}_\delta(X^\infty, t))\|_1 \\ & \leq \|\text{masses}(\text{Coal}_{\delta^n}(X_{>\varepsilon_0}^n, t)) - \text{masses}(\text{Coal}_\delta(X_{>\varepsilon_0}^\infty, t))\|_1 \\ & \quad + \|\text{masses}(\text{Coal}_{\delta^n}(X_{\leq \varepsilon_0}^n, t)) - \text{masses}(\text{Coal}_\delta(X_{\leq \varepsilon_0}^\infty, t))\|_1 \\ & \leq \|\text{masses}(X_{>\varepsilon_0}^n, t) - \text{masses}(X_{>\varepsilon_0}^\infty, t)\|_1 + \mu(X_{\leq \varepsilon_0}^n) + \mu(X_{\leq \varepsilon_0}^\infty) \\ & \leq 4\varepsilon. \end{aligned}$$

This shows (i). To obtain (ii), notice that for any  $s$  and  $\eta > 0$ ,

$$\mathbb{P}(\exists t \in [s, s + \eta] : \text{Coal}_{\delta^\infty}(\mathbf{X}^\infty, t) \neq \text{Coal}_{\delta^\infty}(\mathbf{X}^\infty, s)) \leq \mu(X^\infty)^2 \eta.$$

Thus, (ii) is a simple consequence of (i). □

We shall need the following variation of Proposition 4.3 when studying simultaneous coalescence and fragmentation in section 6.

**Proposition 4.4.** *Let  $\mathbf{X}^n = (X^n, d^n, \mu^n)$ ,  $n \in \bar{\mathbb{N}}$  be a sequence of random variables in  $\mathcal{N}_1^{graph}$  and  $(\delta^n)_{n \geq 0}$  a sequence of non-negative real numbers. Suppose that:*

- (a)  $(\mathbf{X}^n)_{n \geq 0}$  converges in distribution for  $L_{1,GHP}^{surplus}$  to  $\mathbf{X}^\infty$  as  $n$  goes to infinity,
- (b)  $\delta^n \xrightarrow[n \rightarrow \infty]{} \delta$ .

Then,

- (i)  $(\text{Coal}_{\delta^n}(\mathbf{X}^n, t))_{t \geq 0}$  converges in distribution (for the topology of compact convergence associated to  $L_{1,GHP}^{surplus}$ ) to  $(\text{Coal}_\delta(\mathbf{X}^\infty, t))_{t \geq 0}$ ,
- (ii) if  $t^n \xrightarrow[n \rightarrow \infty]{} t$ ,  $\text{Coal}_{\delta^n}(\mathbf{X}^n, t^n)$  converges in distribution to  $\text{Coal}_\delta(\mathbf{X}^\infty, t)$  for  $L_{1,GHP}^{surplus}$ .

*Proof:* Notice first that when  $\mathbf{X}$  belongs to  $\mathcal{N}_1^{graph}$ , then with probability one,  $\text{Coal}_\delta(\mathbf{X}^n, t)$  is in  $\mathcal{N}_1^{graph}$  for any  $t \geq 0$ . Indeed, since  $X$  has finite mass, there is with probability one a finite number of points in the Poisson process  $\mathcal{P}_t^+$  on  $X^2$  for any  $t \geq 0$ .

Now, the proof is essentially the same as the one of Proposition 4.3, except that since  $(\mathbf{X}^n)_{n \geq 0}$  converges to  $\mathbf{X}^\infty$  for  $L_{1,GHP}^{surplus}$ , one may use  $d_{GHP}^{surplus}$  instead of  $d_{GHP}$ . The fact that the multigraphs  $\mathbb{M}\mathbb{G}(X^n, s)$  and  $\mathbb{M}\mathbb{G}(X^\infty, s)$  are the same for any  $s \leq T$ , and that no point of the Poisson processes touches  $\mathbf{X}_{\leq \varepsilon_0}^n$  or  $\mathbf{X}_{\leq \varepsilon_0}^\infty$  imply that for each component of  $\text{Coal}_{\delta^n}(X_{>\varepsilon_0}^n, s)$ , its surplus is the same as the surplus of the corresponding component in  $\text{Coal}_\delta(X_{>\varepsilon_0}^\infty, s)$ . □

#### 4.2. Structural result for Aldous' multiplicative coalescent

Recall the definition of the multigraph  $\text{MG}(x, t)$  for  $x \in \ell^2$  in section 2.3. We shall use notations analogous to Definition 2.14. For instance, for  $x \in \ell_+^2$ ,  $x_{\leq \varepsilon}$  denotes the element in  $\ell_+^2$  defined by:

$$\forall i \in \mathbb{N}, x_{\leq \varepsilon}(i) = x(i) \mathbf{1}_{x(i) \leq \varepsilon}.$$

Also, for  $i \in \mathbb{N}$ ,  $x \setminus \{i\}$  denotes the element in  $\ell^2$  defined by:

$$\forall j \in \mathbb{N}, (x \setminus \{i\})(j) = x(j) \mathbf{1}_{j \neq i}.$$

Notice that at time 0, the components of  $\text{MG}(x, 0)$  are the singletons  $\{i\}$  for  $i \in \mathbb{N}$ . Let us fix some  $\varepsilon > 0$  and say that components of  $\text{MG}(x, t)$  are significant if they are larger than  $\varepsilon$ . We shall derive three scales (at time 0), namely, Large, Medium and Small such that with high probability (as  $\varepsilon$  goes to zero), every significant component of  $\text{MG}(x, t)$  is made of a *heart* made of Large or Medium components of  $\text{MG}(x, 0)$  to which are attached *hanging trees* of small or medium components of  $\text{MG}(x, 0)$  such that the component of the trees attached to the heart are small components (see Figure 1) and the mass contained in the hanging trees is at most medium. Furthermore, these scales depend on  $x$ ,  $\varepsilon$  and  $t$  through the functions  $\alpha \mapsto \|x_{\leq \alpha}\|_2$  and  $K \mapsto \mathbb{P}(S(x, t) \geq K)$ .

**Lemma 4.5.** *Let  $x \in \ell^2(\mathbb{N})$ ,  $T \geq 0$  and  $0 < \varepsilon < 1$ . Suppose that:*

(i)  $K \geq 1$  is such that:

$$\mathbb{P}(S(x, T) \geq K) \leq \frac{\varepsilon}{100},$$

(ii)  $\varepsilon_1 \in (0, \varepsilon)$  is such that:

$$S(x_{\leq \varepsilon_1}, 0) \leq \frac{\varepsilon^2}{100(1 + T + KT^2)},$$

(iii)  $\varepsilon_2 \in (0, \varepsilon_1)$  is such that:

$$S(x_{\leq \varepsilon_2}, 0) \leq \frac{2\varepsilon_1^2 \varepsilon^2}{100(1 + T(K + 2))^2}.$$

Then with probability larger than  $1 - \varepsilon$ , the following holds for any  $t \leq T$ ,

- (a) every component of  $\text{MG}(x, t)$  of size larger than  $\varepsilon$  contains a component of  $\text{MG}(x, 0)$  of size larger than  $\varepsilon_1$ .
- (b)  $\text{MG}(x_{\leq \varepsilon_1}, t)$  is a forest.
- (c) for each component  $m$  of  $\text{MG}(x_{\leq \varepsilon_1}, t)$  and each component  $m'$  of  $\text{MG}(x_{> \varepsilon_1}, t)$ , there is at most one edge between  $m$  and  $m'$  in  $\text{MG}(x, t)$ .
- (d)  $S(x, t) - S(x_{> \varepsilon_2}, t) \leq 2\varepsilon_1^2$ .
- (e) for any component  $\{i\}$  of  $\text{MG}(x, 0)$  of size larger than  $\varepsilon_1$ , the difference between the sizes of the component containing  $i$  in  $\text{MG}(x, t)$  and the one containing  $i$  in  $\text{MG}(x_{> \varepsilon_2}, t)$  is less than  $\varepsilon_1$ .

The picture depicted before Lemma 4.5 is then a simple corollary. We state it uniformly on a convergent sequence in  $\ell_{\searrow}^2$  because it will be convenient to prove the almost Feller property on  $\mathcal{S}$ , Lemma 4.13.

**Corollary 4.6.** *Let  $x^n$  be a sequence in  $\ell_{\searrow}^2$  converging to  $x^\infty$  in  $\ell^2$ . Then, for any  $\varepsilon > 0$ , and any  $T > 0$  there exists  $\varepsilon_1$  and  $\varepsilon_2$  such that for any  $n \in \overline{\mathbb{N}}$ , with probability at least  $1 - \varepsilon$  the following holds for any  $t \in [0, T]$ :*

- (a) every significant component of  $\text{MG}(x^n, t)$  is made of a connected heart made of Large or Medium components of  $\text{MG}(x^n, 0)$  to which are attached hanging trees (each one attached by a single edge to the heart) of Small or Medium components of  $\text{MG}(x^n, 0)$  such that the components of the trees attached to the heart are Small components and the mass contained in the hanging trees is less than  $\varepsilon_1$ ,
- (b) no Medium or Small component of  $\text{MG}(x, 0)$  belongs to a cycle in  $\text{MG}(x, t)$ ,
- (c)  $S(x, t) - S(x_{> \varepsilon_2}, t) \leq 2\varepsilon_1^2$ ,



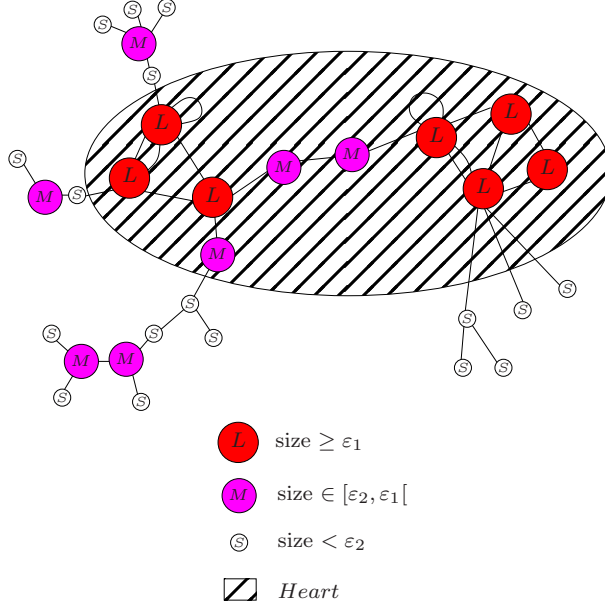


FIG 1. The structure of a significant component

where a component is significant if it has size larger than  $\varepsilon$ , Large if it has size larger than  $\varepsilon_1$ , Medium for a size in  $(\varepsilon_2, \varepsilon_1]$  and Small for a size not larger than  $\varepsilon_2$ .

The proof of Lemma 4.5 relies essentially on Aldous' analysis of the multiplicative coalescent.

*Proof.* (of Lemma 4.5)

If for some  $t \leq T$  there exists a significant component of  $\mathbb{M}\mathbb{G}(x, t)$  which does not contain any large component of  $x$ , then  $S(x_{\leq \varepsilon_1}, t) > \varepsilon^2$  and thus  $S(x_{\leq \varepsilon_1}, T) > \varepsilon^2$ . Thus Lemma 2.1 shows that the probability of (a) is larger than  $1 - \varepsilon/4$ , as soon as hypothesis (ii) of Lemma 4.5 holds.

Let  $\{i\}$  be a component of  $\mathbb{M}\mathbb{G}(x, 0)$  and define the event

$$A_i = \{\text{there exists at least two edges of } \mathbb{M}\mathbb{G}(x, t) \text{ connecting } i \text{ to a same component of } \mathbb{M}\mathbb{G}(x \setminus \{i\}, t)\}.$$

Then,

$$\begin{aligned} \mathbb{P}(A_i | \mathbb{M}\mathbb{G}(x \setminus \{i\}, t)) &\leq \sum_{m \text{ c.c. of } \mathbb{M}\mathbb{G}(x \setminus \{i\}, t)} (x_i \sum_{j \in m} x_j t)^2, \\ &= t^2 x_i^2 S(x \setminus \{i\}, t), \end{aligned}$$

thus,

$$\begin{aligned} \mathbb{P}(A_i \cap \{S(x, t) \leq K\}) &\leq \mathbb{E}[\mathbb{P}(A_i | \mathbb{M}\mathbb{G}(x \setminus \{i\}, t)) \mathbf{1}_{S(x \setminus \{i\}, t) \leq K}], \\ &\leq K t^2 x_i^2. \end{aligned} \tag{4.1}$$

We obtain thus:

$$\mathbb{P}(\cup_{i \in \mathbb{N}_s. t.} x_i \leq \varepsilon_1 A_i) \leq K t^2 S(x_{\leq \varepsilon_1}, 0) + \mathbb{P}(S(x, t) > K),$$

which shows that the probability of (b) is at least  $1 - \varepsilon/4$  as soon as hypotheses (i) and (ii) of Lemma 4.5 hold.

The proof of (c) is similar. Let

$$B = \{\text{there exists at least two edges of } \mathbb{M}\mathbb{G}(x, T) \text{ connecting a component } m \text{ of } \mathbb{M}\mathbb{G}(x_{\leq \varepsilon_1}, T) \text{ to a component } m' \text{ of } \mathbb{M}\mathbb{G}(x_{> \varepsilon_1}, T)\}.$$

Then,

$$\begin{aligned}
& \mathbb{P}(B | \mathbb{M}\mathbb{G}(x_{>\varepsilon_1}, T), \mathbb{M}\mathbb{G}(x_{\leq\varepsilon_1}, T)) \\
& \leq \sum_{m \text{ c.c. of } \mathbb{M}\mathbb{G}(x_{\leq\varepsilon_1}, T)} \sum_{m' \text{ c.c. of } \mathbb{M}\mathbb{G}(x_{>\varepsilon_1}, T)} \left( \sum_{i \in m} x_i \sum_{j \in m'} x_j T \right)^2, \\
& = T^2 S(x_{\leq\varepsilon_1}, T) S(x_{>\varepsilon_1}, T).
\end{aligned}$$

Thus,

$$\mathbb{P}(B) \leq \frac{\varepsilon}{5} + \mathbb{P}(S(x, T) \geq K) + \mathbb{P}(S(x_{\leq\varepsilon_1}, T) \geq \frac{\varepsilon}{5KT^2}),$$

which shows using Lemma 2.1 that the probability of (c) is at least  $1 - \varepsilon/4$  as soon as hypotheses (i) and (ii) of Lemma 4.5 hold.

Now, let  $Y$  be the supremum, over Large components  $\{i\}$  of  $\mathbb{M}\mathbb{G}(x, 0)$ , of the difference between the sizes of the component containing  $i$  in  $\mathbb{M}\mathbb{G}(x, t)$  and the one containing  $i$  in  $\mathbb{M}\mathbb{G}(x_{>\varepsilon_2}, t)$ . Notice that if  $Y \geq \alpha$ , then  $S(x, t) \geq S(x_{>\varepsilon_2}, t) + 2\varepsilon_1\alpha$ , which implies  $S(x, T) \geq S(x_{>\varepsilon_2}, T) + 2\varepsilon_1\alpha$  when (c) holds, thanks to Lemma 2.5. Thus points (e) and (d) will be proved if we show that  $S(x, T) > S(x_{>\varepsilon_2}, T) + 2\varepsilon_1^2$  with probability at most  $\varepsilon/4$ .

Define the event

$$C = \{S(x, T) > S(x_{>\varepsilon_2}, T) + 2\varepsilon_1^2\}.$$

Lemma 2.2 shows that:

$$\begin{aligned}
& \mathbb{P}(C \text{ and } S(x, T) \leq K \text{ and } S(x_{\leq\varepsilon_2}, T) \leq \beta) \\
& \leq (1 + T(K + 2\varepsilon_1^2))^2 \frac{\beta}{2\varepsilon_1^2}.
\end{aligned}$$

Thus,

$$\mathbb{P}(C) \leq (1 + T(K + 2\varepsilon_1^2))^2 \frac{\beta}{2\varepsilon_1^2} + \mathbb{P}(S(x_{\leq\varepsilon_2}, T) > \beta) + \mathbb{P}(S(x, T) > K),$$

which is less than  $\varepsilon/4$  if hypotheses (i), (ii) and (iii) of Lemma 4.5 hold.  $\square$

*Proof.* (of Corollary 4.6) Since  $x^n$  converges to  $x^\infty$  in  $\ell^2$ ,

$$\sup_{n \in \mathbb{N}} \|x_{\leq\varepsilon}^n\|_2 \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Also, the Feller property implies that the distributions of the sizes of  $\mathbb{M}\mathbb{G}(x^n, T)$  for  $n \in \overline{\mathbb{N}}$  form a compact family of probability measures on  $\ell^2$ . Thus,

$$\sup_{n \in \overline{\mathbb{N}}} \mathbb{P}(S(x^n, T) \geq K) \xrightarrow{K \rightarrow +\infty} 0. \quad (4.2)$$

This shows that for any  $T$  and  $\varepsilon$ , one may find  $K$ ,  $\varepsilon_1$  and  $\varepsilon_2$  such that the three hypotheses of Lemma 4.5 hold for  $x^n$  uniformly over  $n \in \overline{\mathbb{N}}$ .

Now suppose that  $x$ ,  $t$ ,  $\varepsilon$ ,  $\varepsilon_1$  and  $\varepsilon_2$  are such that (a), (b), (e) and (c) of Lemma 4.5 hold. Let  $m$  be a significant component of  $\mathbb{M}\mathbb{G}(x, t)$ . It contains a large component  $\{i\}$  of  $\mathbb{M}\mathbb{G}(x, 0)$  by point (a). Let  $\sigma(m)$  denote the component of  $\mathbb{M}\mathbb{G}(x_{>\varepsilon_2}, t)$  containing  $\{i\}$ . Point (e) implies that two large components of  $\mathbb{M}\mathbb{G}(x, 0)$  are connected in  $\mathbb{M}\mathbb{G}(x, t)$  if and only if they are connected in  $\mathbb{M}\mathbb{G}(x_{>\varepsilon_2}, t)$ , that is only through Large or Medium components. This shows that  $\sigma(m)$  does not depend on the choice of the large component  $\{i\}$  included in  $m$  and that there cannot be any large component in  $m \setminus \sigma(m)$ . Let us define the heart of  $m$  as  $\sigma(m)$ . It is made of Large or Medium components, and  $m \setminus \sigma(m)$  is a graph of medium or small components. Now if a medium component in  $m \setminus \sigma(m)$  was directly connected to some component of  $\sigma(m)$ , it would be connected to  $m$  in  $\mathbb{M}\mathbb{G}(x_{>\varepsilon_2}, t)$ , and thus would belong to  $\sigma(m)$ . Thus, the exterior boundary of  $\sigma(m)$  in  $m$  is made of small components. Point (b) shows that  $m \setminus \sigma(m)$  is a forest, and point (c) shows that each tree of this forest is attached by a single edge to  $\sigma(m)$ , and also that no Medium or Small component of  $\mathbb{M}\mathbb{G}(x, 0)$  belongs to a cycle in  $\mathbb{M}\mathbb{G}(x, t)$ .  $\square$

A useful by-product of the proof of Lemma 4.5 and Corollary 4.6 is the following simple lemma.

**Lemma 4.7.** For  $x \in \ell^2$ ,  $\varepsilon > 0$  and  $T > 0$  let  $A(x, \varepsilon, T)$  be the event that for any  $t \leq T$ :

- $\text{MIG}(x_{\leq \varepsilon}, t)$  is a forest and
- there is at most one edge between every connected component of  $\text{MIG}(x_{\leq \varepsilon}, t)$  and every component of  $\text{MIG}(x_{> \varepsilon}, t)$ .

Suppose that  $x^n$  converges to  $x^\infty$  in  $\ell^2_{\searrow}$  as  $n$  goes to infinity. Then, for any  $T > 0$

$$\inf_{n \in \mathbb{N}} \mathbb{P}(A(x^n, \varepsilon, T)) \xrightarrow{\varepsilon \rightarrow 0} 1.$$

**Remark 4.8.** Let us give an example of an m.a.m.s which is in  $\mathcal{N}_2$  but not in  $\mathcal{S}$ . Let  $I_i$ ,  $i \geq 1$  be disjoint copies of the interval  $[0, 1]$ , with its usual metric, and equip  $I_i$  with the measure  $\frac{1}{i}(\delta_0 + \delta_1)$ . Then,  $\mathbf{X} \in \mathcal{N}_2 \setminus \mathcal{S}$ . In fact, thanks to Lemma 4.5, for any  $\varepsilon > 0$  and  $t > 0$  every component of  $\text{Coal}(\mathbf{X}_{> \varepsilon}, t)$  is unbounded since it contains a forest of an infinite number (since the sizes are not in  $\ell^1$ ) of components of diameter 1.

### 4.3. The Coalescent on $\mathcal{S}$

The aim of this section is to prove the following theorem, and a Feller-like property (in Lemma 4.13).

**Theorem 4.9.** Let  $\mathbf{X} = (X, d, \mu)$  belong to  $\mathcal{S}$ . Then,  $(\text{Coal}_0(X, t))_{t \geq 0}$  defines a strong Markov process with càdlàg trajectories in  $\mathcal{S}$ .

The proof of this result is divided in three lemmas. The last one, Lemma 4.13 will be useful in its own to show convergence of discrete coalescence processes.

**Lemma 4.10.** Let  $\mathbf{X}$  be an m.s-m.s.

- (i) If  $\mathbf{X} \in \mathcal{S}$  (resp. to  $\mathcal{S}^{graph}$ ) then, almost surely, for any  $t \geq 0$ ,  $\text{Coal}_0(\mathbf{X}, t)$  belongs to  $\mathcal{N}_2$  (resp. to  $\mathcal{N}_2^{graph}$ ).
- (ii) If  $\mathbf{X} \in \mathcal{S}^{graph}$ , almost surely the commutation (2.3) holds.
- (iii) If the components of  $\mathbf{X}$  are  $\mathbb{R}$ -graphs then, almost surely, for any  $t \geq 0$ , the components of  $\text{Frag}(\mathbf{X}, t)$  are  $\mathbb{R}$ -graphs.

*Proof.* (i) Suppose that  $\mathbf{X}$  belongs to  $\mathcal{S}$  and let us show that if  $\mathbf{X} \in \mathcal{N}_2$ , then with probability one,  $\text{Coal}_0(\mathbf{X}, t)$  has totally bounded components for any  $t \geq 0$ . Let  $\alpha > 0$  be fixed and let

$$B(\varepsilon) := \{\text{supdiam}(\text{Coal}(\mathbf{X}_{\leq \varepsilon}, T)) \leq \alpha\}.$$

Since  $\mathbf{X}$  satisfies (2.2),

$$\mathbb{P}(B(\varepsilon)^c) \xrightarrow{\varepsilon \rightarrow 0} 0$$

Let  $A(\varepsilon)$  be the event that the conclusion of Corollary 4.6 holds. This corollary shows that

$$\mathbb{P}(A(\varepsilon)^c) \xrightarrow{\varepsilon \rightarrow 0} 0$$

Then, on  $A(\varepsilon) \cap B(\varepsilon)$ , we have that for any  $t \in [0, T]$ , any component of size larger than  $\varepsilon$  of  $\text{Coal}(\mathbf{X}, t)$  can be covered with a finite number of balls of radius  $2\alpha$ . Indeed, if  $m$  is such a component, one may first cover the heart with a finite number of balls of radius  $\alpha$  since the heart of  $m$  is composed of a finite number of totally bounded components of  $\mathbf{X}$  glued together, and then if we increase the radius to  $2\alpha$ , those balls will cover the whole component  $m$  because we are on  $B(\varepsilon)$ . Making  $\varepsilon$  go to zero, we see that with probability one, for any  $t \in [0, T]$  every component of  $\text{Coal}(\mathbf{X}, t)$  can be covered with a finite number of balls of radius  $2\alpha$ . Then, letting  $\alpha$  go to zero, we see that with probability one, for any  $t \in [0, T]$  every component of  $\text{Coal}(\mathbf{X}, t)$  is totally bounded, so  $\text{Coal}(\mathbf{X}, t) \in \mathcal{N}_2$ .

Notice also that if the components of  $\mathbf{X}$  are  $\mathbb{R}$ -graphs and  $m$  is a component as above, then on  $A(\varepsilon)$ ,  $m$  is an  $\mathbb{R}$ -graph. Letting  $\varepsilon$  go to zero, this shows that if  $\mathbf{X} \in \mathcal{S}^{graph}$ , then with probability one, for any  $t \in [0, T]$ ,  $\text{Coal}(\mathbf{X}, t) \in \mathcal{N}_2^{graph}$ .

- (ii) If  $\mathbf{X} \in \mathcal{S}^{graph}$ , using the same notations as above, one sees that on  $A(\varepsilon)$ , for any  $x$  and  $y$  in a component of mass larger than  $\varepsilon$ , there is only a finite number of simple paths from  $x$  to  $y$ , and every such simple path takes a finite number of shortcuts of the Poisson process  $\mathcal{P}_t^+$ . Letting  $\varepsilon$  go to zero, this holds almost surely for any component of  $\text{Coal}(\mathbf{X}, \mathcal{P}_t^+)$ . Furthermore, since  $\ell_X$  is diffuse and  $\mathcal{P}^-$  and  $\mathcal{P}^+$  are independent, almost surely one has, for any  $t$ ,

$$\mathcal{P}_t^- \cap \{x \in X : \exists y \in X, (x, y) \text{ or } (y, x) \in \mathcal{P}_t^+\} = \emptyset.$$

Thus, Lemma 2.36 shows that (ii) holds.

- (iii) Using Lemma 2.30,  $X$  is isometric to  $\text{Coal}(X', A)$  where  $X'$  is an m.s-m.s whose components are real trees, and  $A \subset \bigcup_{m \in \text{comp}(X')} m^2$  is finite on any  $m^2$ . Again, since  $\ell_X$  is diffuse, almost surely, for any  $t$ ,

$$\mathcal{P}_t^- \cap \{x \in X : \exists y \in X, (x, y) \text{ or } (y, x) \in A\} = \emptyset.$$

Thus, Lemma 2.36 shows that

$$\text{Frag}(X, \mathcal{P}_t^-) = \text{Frag}(\text{Coal}(X', A), \mathcal{P}_t^-) = \text{Coal}(\text{Frag}(X, \mathcal{P}_t^-), A).$$

The components of  $\text{Frag}(X, \mathcal{P}_t^-)$  are totally bounded  $\mathbb{R}$ -trees, thus the components of  $\text{Coal}(\text{Frag}(X, \mathcal{P}_t^-), A)$  are  $\mathbb{R}$ -graphs. □

**Remark 4.11.** If  $\mathbf{X} \in \mathcal{N}$  and  $\mathcal{P}$  is as in Definition 2.32, it may happen that  $\text{Frag}(\mathbf{X}, \mathcal{P})$  has a component of mass zero. In this case,  $\text{Frag}(\mathbf{X}, \mathcal{P})$  does not belong to  $\mathcal{N}$ , strictly speaking. However,  $\text{Frag}(\mathbf{X}, \mathcal{P})$  is at zero  $L_{GHP}$ -distance from an element of  $\mathcal{N}$ , which is  $\text{Frag}(\mathbf{X}, \mathcal{P})|_{\cup_{\varepsilon>0} \mathcal{M}_{>\varepsilon}}$ . In fact, we could have defined  $\mathcal{N}$  as the quotient of the set of counting measures on  $\mathcal{M}$  with respect to the equivalence relation defined by being at zero  $L_{GHP}$ -distance. This space is isometric to  $\mathcal{N}$  modulo the addition of components of null masses. Then  $\text{Frag}(\mathbf{X}, \mathcal{P})$  would have always belonged to  $\mathcal{N}$ . But I feel that it would have obscured the definition of  $\mathcal{N}$ . In the sequel, we shall keep in mind that components of null masses are neglected. Thus, Lemma 4.10 shows that if  $\mathbf{X}$  belongs to  $\mathcal{N}_2^{graph}$ , then almost surely, for any  $t \geq 0$ ,  $\text{Frag}(\mathbf{X}, t)$  belongs to  $\mathcal{N}_2^{graph}$  and if  $\mathbf{X}$  belongs to  $\mathcal{S}^{graph}$  then almost surely, for any  $t \geq 0$ ,  $\text{Coal}_0(\mathbf{X}, t)$  and  $\text{CoalFrag}(\mathbf{X}, t)$  belong to  $\mathcal{N}_2^{graph}$ .

**Lemma 4.12.** Let  $\mathbf{X}^n = (X^n, d^n, \mu^n)$ ,  $n \geq 0$  be a sequence of random variables in  $\mathcal{S}$  and  $(\delta^n)_{n \geq 0}$  be a sequence of non-negative real numbers. Suppose that:

- (i)  $(\mathbf{X}^n)$  converges in distribution for  $L_{2,GHP}$  to  $\mathbf{X}^\infty = (X^\infty, d^\infty, \mu^\infty)$  as  $n$  goes to infinity,
- (ii)  $\delta^n \xrightarrow[n \rightarrow \infty]{} 0$ ,
- (iii) For any  $\alpha > 0$  and any  $T > 0$ ,

$$\limsup_{n \in \mathbb{N}} \mathbb{P}(\text{supdiam}(\text{Coal}_{\delta^n}(\mathbf{X}_{\leq \varepsilon}^n, T)) > \alpha) \xrightarrow[\varepsilon \rightarrow 0]{} 0.$$

Then, with probability 1,  $\text{Coal}_0(\mathbf{X}^\infty, t)$  belongs to  $\mathcal{S}$  for any  $t \geq 0$ .

*Proof.* First, the Feller property of the multiplicative coalescent, Proposition 5 of [5], shows that  $\text{masses}(\text{Coal}_{\delta^n}(\mathbf{X}^n, T))$  converges in distribution (in  $\ell^2_{\searrow}$ ) to  $\text{masses}(\text{Coal}_0(\mathbf{X}^\infty, T))$ . Together with Prokhorov theorem and Lemma 4.7, this implies that:

$$\mathbb{P}[\text{MG}(\mathbf{X}_{\leq \varepsilon}^\infty, T) \text{ is not a forest}] \xrightarrow[\varepsilon \rightarrow 0]{} 0. \quad (4.3)$$

Notice that under the obvious coupling, when  $\text{MG}(\mathbf{X}^\infty, t+s)$  is a forest, Lemma 2.27 implies that:

$$\text{supdiam}(\text{Coal}_0(\text{Coal}_0(\mathbf{X}^\infty, t)_{\leq \eta}, s)) \leq \text{supdiam}(\text{Coal}_0(\mathbf{X}_{\leq \eta}^\infty, t+s)).$$

Thus, thanks to Lemmas 4.7 and 4.10 it is enough to show that with probability one,  $\mathbf{X}^\infty$  satisfies (2.2) for any  $t \geq 0$ .

Let  $P_{\mathbf{X}^\infty}$  be the distribution of  $\mathbf{X}^\infty$ . Then for  $P_{\mathbf{X}^\infty}$ -almost every  $X$  and every  $t \in [0, T]$  and  $\alpha > 0$ ,

$$\begin{aligned} & \limsup_{\varepsilon \rightarrow 0} \mathbb{P}[\text{supdiam}(\text{Coal}_0(X_{\leq \varepsilon}, t)) > \alpha] \\ = & \limsup_{\varepsilon \rightarrow 0} \mathbb{P}[\text{supdiam}(\text{Coal}_0(X_{\leq \varepsilon}, t)) > \alpha \text{ and } \text{MG}(X_{\leq \varepsilon}, T) \text{ is a forest}] \\ \leq & \limsup_{\varepsilon \rightarrow 0} \mathbb{P}[\text{supdiam}(\text{Coal}_0(X_{\leq \varepsilon}, T)) > \alpha \text{ and } \text{MG}(X_{\leq \varepsilon}, T) \text{ is a forest}] \end{aligned}$$

Thus,

$$\begin{aligned} & P_{\mathbf{X}^\infty} \{X \in \mathcal{N}_2 : \sup_{\substack{t \leq T \\ \alpha > 0}} \limsup_{\varepsilon \rightarrow 0} \mathbb{P}[\text{supdiam}(\text{Coal}_0(X_{\leq \varepsilon}, t)) > \alpha] > 0\} \\ = & \sup_{\alpha > 0} P_{\mathbf{X}^\infty} \{X : \limsup_{\varepsilon \rightarrow 0} \mathbb{P}[\text{supdiam}(\text{Coal}_0(X_{\leq \varepsilon}, T)) > \alpha] > 0\} \\ = & \sup_{\alpha > 0} \sup_{\eta > 0} \lim_{\varepsilon \rightarrow 0} P_{\mathbf{X}^\infty} \{X : \mathbb{P}[\exists \varepsilon' \in ]0, \varepsilon], \text{supdiam}(\text{Coal}_0(X_{\leq \varepsilon'}, T)) > \alpha] > \eta\} \\ \leq & \sup_{\alpha > 0} \sup_{\eta > 0} \lim_{\varepsilon \rightarrow 0} \frac{1}{\eta} \mathbb{P}[\exists \varepsilon' \in ]0, \varepsilon] : \text{supdiam}(\text{Coal}_0(\mathbf{X}_{\leq \varepsilon'}^\infty, t)) > \alpha]. \end{aligned}$$

Thus, using (4.3), it is sufficient to prove that for any  $T \geq 0$ ,  $\alpha > 0$  and  $\tilde{\varepsilon} > 0$ ,

$$\mathbb{P} \left[ \begin{array}{c} \sup_{\varepsilon' \in ]0, \varepsilon]} \text{supdiam}(\text{Coal}_0(\mathbf{X}_{\leq \varepsilon'}^\infty, t)) > \alpha \\ \text{and} \\ \text{MG}(x_{\leq \tilde{\varepsilon}}^\infty, T) \text{ is a forest} \end{array} \right] \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (4.4)$$

Let  $x^\infty := \text{masses}(\mathbf{X}^\infty)$ . Notice first that there exists a decreasing sequence of positive numbers  $(\varepsilon_p)_{p \geq 0}$  going to zero and such that:

$$\forall p \in \mathbb{N}, \mathbb{P}[\varepsilon_p \in x^\infty] = 0.$$

Fix  $\tilde{\varepsilon} \geq \varepsilon$  and choose the sequence so that  $\varepsilon_0 \leq \tilde{\varepsilon}$ . Then, using the obvious coupling and Lemma 2.27,

$$\begin{aligned} & \mathbb{P} \left[ \begin{array}{c} \sup_{\varepsilon' \in ]0, \varepsilon]} \text{supdiam}(\text{Coal}_0(\mathbf{X}_{\leq \varepsilon'}^\infty, t)) > \alpha \\ \text{and} \\ \text{MG}(x_{\leq \tilde{\varepsilon}}^\infty, T) \text{ is a forest} \end{array} \right] \\ \leq & \mathbb{P}[\text{supdiam}(\text{Coal}_0(\mathbf{X}_{\leq \tilde{\varepsilon}}^\infty, t)) > \alpha \text{ and } \text{MG}(x_{\leq \tilde{\varepsilon}}^\infty, T) \text{ is a forest}] \end{aligned}$$

Then, we have:

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \mathbb{P}(\text{supdiam}(\text{Coal}_0(\mathbf{X}_{\leq \varepsilon}^\infty, T)) > \alpha \text{ and } \text{MG}(x_{\leq \tilde{\varepsilon}}^\infty, T) \text{ is a forest}) \\ = & \lim_{m \rightarrow \infty} \mathbb{P}(\text{supdiam}(\text{Coal}_0(\mathbf{X}_{\leq \varepsilon_m}^\infty, T)) > \alpha). \end{aligned}$$

Furthermore, define  $\mathbf{X}_{m,p} := (\mathbf{X}_{\leq \varepsilon_m}^\infty)_{> \varepsilon_p}$  for  $m \leq p$ . Then,

$$\mathbb{P}(\text{supdiam}(\text{Coal}_0(\mathbf{X}_{\leq \varepsilon_m}^\infty, T)) > \alpha) = \lim_{p \rightarrow \infty} \mathbb{P}(\text{supdiam}(\text{Coal}_0(\mathbf{X}_{m,p}^\infty, T)) > \alpha)$$

Now, Proposition 4.3 implies that  $(\text{Coal}_{\delta^n}(\mathbf{X}_{m,p}^n, T))$  converges in distribution to  $(\text{Coal}_0(\mathbf{X}_{m,p}^\infty, T))$  for any  $m \leq p$ . Since we are dealing here with finite collections of m.s.m.s with positive masses, this entails that for

any  $m \leq p$ ,

$$\begin{aligned}
& \mathbb{P}(\text{supdiam}(\text{Coal}_0(\mathbf{X}_{m,p}^\infty, T)) > \alpha) \\
\leq & \limsup_{n \rightarrow \infty} \mathbb{P}(\text{supdiam}(\text{Coal}_{\delta^n}(\mathbf{X}_{m,p}^n, T)) > \alpha) \\
\leq & \limsup_{n \rightarrow \infty} \mathbb{P}(\text{supdiam}(\text{Coal}_{\delta^n}(\mathbf{X}_{m,p}^n, T)) > \alpha \text{ and } \mathbb{M}\mathbb{G}(\mathbf{X}_{\leq \tilde{\varepsilon}}^n, T) \text{ is a forest}) \\
& + \limsup_{n \rightarrow \infty} \mathbb{P}(\mathbb{M}\mathbb{G}(\mathbf{X}_{\leq \tilde{\varepsilon}}^n, T) \text{ is not a forest}) \\
\leq & \limsup_{n \rightarrow \infty} \mathbb{P}(\text{supdiam}(\text{Coal}_{\delta^n}(\mathbf{X}_{\leq \varepsilon_m}^n, T)) > \alpha \text{ and } \mathbb{M}\mathbb{G}(\mathbf{X}_{\leq \tilde{\varepsilon}}^n, T) \text{ is a forest}) \\
& + \limsup_{n \rightarrow \infty} \mathbb{P}(\mathbb{M}\mathbb{G}(\mathbf{X}_{\leq \tilde{\varepsilon}}^n, T) \text{ is not a forest}) \\
\leq & \limsup_{n \rightarrow \infty} \mathbb{P}(\text{supdiam}(\text{Coal}_{\delta^n}(\mathbf{X}_{\varepsilon_m}^n, T)) > \alpha) \\
& + \limsup_{n \rightarrow \infty} \mathbb{P}(\mathbb{M}\mathbb{G}(\mathbf{X}_{\leq \tilde{\varepsilon}}^n, T) \text{ is not a forest})
\end{aligned}$$

Then, using Lemma 4.7 and the hypothesis on  $\text{supdiam}(\text{Coal}_{\delta^n}(\mathbf{X}_{\varepsilon}^n, T))$  one sees that the right-hand side above goes to zero when we make  $m$  and then  $\tilde{\varepsilon}$  go to zero. This ends the proof of (4.4).  $\square$

At this point, we know that if  $\mathbf{X} = (X, d, \mu)$  belongs to  $\mathcal{S}$ , then the process  $(\text{Coal}_0(\mathbf{X}, t))_{t \geq 0}$  is strong Markov on  $\mathcal{S}$ .

**Lemma 4.13** (Almost Feller property). *Let  $\mathbf{X}^n = (X^n, d^n, \mu^n)$ ,  $n \geq 0$  be a sequence of random variables in  $\mathcal{N}_2$  and  $(\delta^n)_{n \geq 0}$  a sequence of non-negative real numbers. Suppose that:*

- (a)  $(\mathbf{X}^n)$  converges in distribution (for  $L_{2,GHP}$ ) to  $\mathbf{X}^\infty = (X^\infty, d^\infty, \mu^\infty)$  as  $n$  goes to infinity
- (b)  $\delta^n \xrightarrow[n \rightarrow \infty]{} 0$
- (c) For any  $\alpha > 0$  and any  $T > 0$ ,

$$\limsup_{n \in \mathbb{N}} \mathbb{P}(\text{supdiam}(\text{Coal}_{\delta^n}(\mathbf{X}_{\leq \varepsilon}^n, T)) > \alpha) \xrightarrow[\varepsilon \rightarrow 0]{} 0$$

Then,

- (i)  $(\text{Coal}_{\delta^n}(\mathbf{X}^n, t))_{t \geq 0}$  converges in distribution to  $(\text{Coal}_0(\mathbf{X}^\infty, t))_{t \geq 0}$ ,
- (ii) if  $t^n \xrightarrow[n \rightarrow \infty]{} t$ ,  $\text{Coal}_{\delta^n}(\mathbf{X}^n, t^n)$  converges in distribution to  $\text{Coal}_0(\mathbf{X}^\infty, t)$  (for  $L_{2,GHP}$ ).

*Proof.* Let  $t^n \xrightarrow[n \rightarrow \infty]{} t$  and let  $T = \sup_n t^n$ . Let us fix  $\varepsilon \in ]0, 1[$  and let  $(x^n)_{n \in \mathbb{N} \cup \{\infty\}} := (\text{masses}(\mathbf{X}^n))_{n \in \mathbb{N} \cup \{\infty\}}$ . We know that  $x^n$  converges in distribution to  $x^\infty$ . Using the Skorokhod representation theorem, Corollary 4.6 and (4.2), we obtain that there exists  $K(\varepsilon) \in ]0, +\infty[$ ,  $\varepsilon_1 \in ]0, \varepsilon[$  and  $\varepsilon_2 \in ]0, \varepsilon_1[$  such that for every  $n \in \overline{\mathbb{N}}$ , with probability larger than  $1 - \varepsilon$  the event  $\mathcal{A}_n$  holds, where  $\mathcal{A}_n$  is the event that points (a), (b) and (c) of Corollary 4.6 hold for any  $t \in [0, T]$  and  $S(x^n, T) \leq K(\varepsilon)$ .

Let  $\delta^\infty := 0$ . On this event  $\mathcal{A}_n$ , the Gromov-Hausdorff-Prokhorov distance<sup>4</sup> between a significant component of  $\text{Coal}_{\delta^n}(\mathbf{X}^n, t)$  (at any time  $t \leq T$ ) and its heart is at most  $\alpha := \delta^n + \text{supdiam}(\text{Coal}_{\delta^n}(\mathbf{X}_{\leq \varepsilon_1}^n, T)) + \varepsilon_1$ . Let  $\sigma$  be the application from  $\text{comp}(\text{Coal}_{\delta^n}(\mathbf{X}^n, t)_{> \varepsilon + \alpha})$  to  $\text{comp}(\text{Coal}_{\delta^n}(\mathbf{X}_{> \varepsilon_2}^n, t))$  which maps a component to its heart, and let  $\sigma'$  denote the application from  $\text{comp}(\text{Coal}_{\delta^n}(\mathbf{X}_{> \varepsilon_2}^n, t))_{> \varepsilon + \alpha}$  to  $\text{comp}(\text{Coal}_{\delta^n}(\mathbf{X}^n, t))$  which maps a component to the (unique, on  $\mathcal{A}_n$ ) component of  $\text{comp}(\text{Coal}_{\delta^n}(\mathbf{X}^n, t))$  which contains it. Those applications satisfy the hypotheses of Lemma 2.19, with  $\varepsilon$  replaced by  $\varepsilon + \alpha$ . This shows that on  $\mathcal{A}_n$ , we have for every time  $t \leq T$  and every  $\varepsilon'_2 \leq \varepsilon_2$ :

$$\begin{aligned}
& L_{GHP}(\text{Coal}_{\delta^n}(\mathbf{X}^n, t), \text{Coal}_{\delta^n}(\mathbf{X}_{> \varepsilon'_2}^n, t)) \\
\leq & \alpha \left( 1 + 8 \frac{S(x^n, t)}{\varepsilon^2} \right) + 16(\varepsilon + \alpha), \\
\leq & 17(\delta^n + \text{supdiam}(\text{Coal}_{\delta^n}(\mathbf{X}_{\leq \varepsilon_1}^n, T)) + \varepsilon_1) \left( 1 + \frac{8K(\varepsilon)}{\varepsilon^2} \right) + 16\varepsilon.
\end{aligned}$$

<sup>4</sup>In fact, here we could talk simply of Hausdorff-Prokhorov distance since there is a trivial embedding of one measured semi-metric space into the other.

Now, recall from Lemma 4.12 that

$$\mathbb{P}(\text{supdiam}(\text{Coal}_{\delta^\infty}(\mathbf{X}_{\leq \varepsilon}^\infty, T)) > \alpha) \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Thus, using the hypothesis on  $\text{supdiam}$ , one may choose  $\varepsilon_1$  small enough (and thus  $\varepsilon_2$  small enough) to get that for every  $n$  large enough (possibly infinite), with probability larger than  $1 - 2\varepsilon$ , we have for every time  $t \leq T$  and every  $\varepsilon'_2 \leq \varepsilon_2$ :

$$L_{GHP}(\text{Coal}_{\delta^n}(\mathbf{X}^n, t), \text{Coal}_{\delta^n}(\mathbf{X}_{> \varepsilon'_2}^n, t)) \leq 40\varepsilon.$$

Furthermore, since (c) of Corollary 4.6 holds on  $\mathcal{A}_n$ ,

$$\begin{aligned} & \| \text{masses}(\text{Coal}_{\delta^n}(\mathbf{X}^n, t)) - \text{masses}(\text{Coal}_{\delta^n}(\mathbf{X}_{> \varepsilon_2}^n, t)) \|_2^2 \\ & \leq S(x^{(n)}, t) - S(x_{> \varepsilon_2}^{(n)}, t) \leq \varepsilon, \end{aligned}$$

where the first inequality comes from Lemma 2.4. This shows that

$$\lim_{\varepsilon_2 \rightarrow 0} \lim_{N \rightarrow \infty} \sup_{\substack{n \geq N \\ n \in \mathbb{N}}} \mathbb{P}(\sup_{t \leq T} L_{2,GHP}(\text{Coal}_{\delta^n}(\mathbf{X}^n, t), \text{Coal}_{\delta^n}(\mathbf{X}_{> \varepsilon_2}^n, t)) > \varepsilon) = 0. \quad (4.5)$$

Now, let  $(\alpha_p)_{p \geq 0}$  be a decreasing sequence of positive numbers going to zero such that:

$$\forall p \in \mathbb{N}, \mathbb{P}[\alpha_p \in x^\infty] = 0.$$

For any  $p$ ,  $\mathbf{X}_{> \alpha_p}^n$  converges to  $\mathbf{X}_{> \alpha_p}^\infty$  in distribution for  $L_{1,GHP}$ . Proposition 4.3 implies that  $(\text{Coal}_{\delta^n}(\mathbf{X}_{> \alpha_p}^n, t))_{t \leq T}$  converges to  $(\text{Coal}_0(\mathbf{X}_{> \alpha_p}^\infty, t))_{t \leq T}$  for the topology of compact convergence associated to  $L_{2,GHP}$ . Together with (4.5), this proves (i).

Furthermore, Proposition 4.3 implies that  $\text{Coal}_{\delta^n}(\mathbf{X}_{> \alpha_p}^n, t^n)$  converges in distribution to  $\text{Coal}_0(\mathbf{X}_{> \alpha_p}^\infty, t)$  as  $n$  goes to infinity for  $L_{1,GHP}$ , and thus for  $L_{2,GHP}$ . Together with (4.5) we obtain that  $\text{Coal}_{\delta^n}(\mathbf{X}^n, t^n)$  converges to  $\text{Coal}_0(\mathbf{X}^\infty, t)$  in distribution for  $L_{2,GHP}$ . This proves (ii).  $\square$

**Remark 4.14.** Notice that if  $\delta_n > 0$  and  $\mathbf{X}^n \in \mathcal{N}_2 \setminus \mathcal{N}_1$ ,  $\text{Coal}_{\delta_n}(\mathbf{X}^n, t)$  is not in  $\mathcal{N}_2$  for  $t > 0$  since the components are not totally bounded. Thus, the terms ‘‘convergence in distribution for  $L_{2,GHP}$ ’’ should be understood in a larger space, where components are allowed not to be totally bounded.

However, we do not insist on this because we shall always use Lemma 4.13 with  $\mathbf{X}^n \in \mathcal{N}_1$  for any  $n \in \mathbb{N}$ , in which case  $\text{Coal}_{\delta_n}(\mathbf{X}^n, t)$  is in  $\mathcal{N}_1$  for any  $t$ .

Now, we can end the proof of Theorem 4.9. Let  $\mathbf{X} = (X, d, \mu)$  belong to  $\mathcal{S}$ . We only need to show that  $(\text{Coal}_0(\mathbf{X}, t))_{t \geq 0}$  is almost surely càdlàg. Let  $\mathbf{X}^n := \mathbf{X}_{> \frac{1}{n}}$ . Then, Lemma 4.13 shows that  $(\text{Coal}_0(\mathbf{X}^n, t))_{t \geq 0}$  converges to  $(\text{Coal}_0(\mathbf{X}, t))_{t \geq 0}$  (in the topology of compact convergence associated to  $L_{2,GHP}$ ). Since  $(\text{Coal}_0(\mathbf{X}^n, t))_{t \geq 0}$  is càdlàg, so is  $(\text{Coal}_0(\mathbf{X}, t))_{t \geq 0}$ .

#### 4.4. Convergence of the coalescent on Erdős-Rényi random graphs

In this section we prove Theorem 3.1. Recall that  $\bar{\mathcal{G}}_{n,\lambda}$  is the element of  $\mathcal{N}_2^{graph}$  obtained from  $\mathcal{G}(n, p(\lambda, n))$  by assigning to each edge a length  $n^{-1/3}$  and to each vertex a mass  $n^{-2/3}$ . We know by Theorem 2.38 that  $\bar{\mathcal{G}}_{n,\lambda}$  converges in distribution (for  $L_{2,GHP}$ ) to  $\mathcal{G}_\lambda$ . In view of Lemma 4.13, it is sufficient to prove that for any  $T$  and  $\alpha > 0$ :

$$\limsup_{n \rightarrow \infty} \mathbb{P}(\text{supdiam}(\text{Coal}_{n^{-1/3}}(\bar{\mathcal{G}}_{n,\lambda})_{\leq \varepsilon}, T)) > \alpha) \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Let us denote by  $h_{n,\lambda}$  the height process associated to the depth-first exploration process on  $\bar{\mathcal{G}}_{n,\lambda}$  as defined in [2] sections 1 and 2, and let  $\bar{h}_{n,\lambda}$  be its rescaled version:

$$\bar{h}_{n,\lambda}(x) := \frac{1}{n^{1/3}} h_{n,\lambda}(xn^{2/3}).$$

Each interval  $I$  inside an excursion interval of  $\bar{h}_{n,\lambda}$  (away from 0) corresponds to a connected subgraph of  $\bar{\mathcal{G}}(n, p(\lambda, n))$  and the diameter of this subgraph is bounded from above by  $2 \sup_{x,y \in I} |\bar{h}_{n,\lambda}(x) - \bar{h}_{n,\lambda}(y)|$ .

Let us denote by  $A_n(\varepsilon)$  the event that there is a connected component of  $\text{Coal}_{1/n}((\bar{\mathcal{G}}_{n,\lambda})_{\leq \varepsilon}, T)$  which is connected by at least two edges to a component of  $\text{Coal}_{1/n}((\bar{\mathcal{G}}_{n,\lambda})_{> \varepsilon}, T)$ . Thanks to Lemma 4.7, using Skorokhod representation theorem in  $\ell^2$  for the converging sequence masses  $(\bar{\mathcal{G}}_{n,\lambda})$ , we know that

$$\limsup_{n \rightarrow \infty} \mathbb{P}(A_n(\varepsilon)) \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Now let us consider the depth-first exploration process of  $\text{Coal}_{1/n}((\bar{\mathcal{G}}_{n,\lambda}), T)$ . When  $\mathcal{P}_T^+$  has intensity  $\gamma$ ,  $N^+(\mathcal{G}(n, p), \mathcal{P}_T^+)$  is equal in distribution to  $\mathcal{G}(n, p')$  with:

$$p' = p + (1 - p)(1 - e^{-\gamma T})$$

When  $\gamma = n^{-4/3}$ ,  $p' = p(\lambda'_n, n)$  with

$$\lambda'_n \xrightarrow{n \rightarrow \infty} \lambda + T$$

and  $\text{Coal}_{1/n}((\bar{\mathcal{G}}_{n,\lambda}), T)$  is equal, in distribution, to  $\bar{\mathcal{G}}_{n,\lambda'_n}$ . The difference between  $\lambda'_n$  and  $\lambda_n + T$  is unessential for us (for instance using Lemma 4.7,  $\text{supdiam}(\text{Coal}_{n^{-1/3}}((\bar{\mathcal{G}}_{n,\lambda})_{\leq \varepsilon}, T)$  is essentially nondecreasing in  $T$ ), so let us pursue as if  $\lambda'_n = \lambda + T$ . On the event  $A_n(\varepsilon)$ , the vertices of a connected component  $C$  of  $\text{Coal}_{1/n}((\bar{\mathcal{G}}_{n,\lambda})_{\leq \varepsilon}, T)$  are explored either consecutively (if the exploration process of the component of  $\text{Coal}_{1/n}((\bar{\mathcal{G}}_{n,\lambda}), T)$  containing  $C$  starts outside  $C$ ) either in two time intervals (which may happen if the exploration process of the component of  $\text{Coal}_{1/n}((\bar{\mathcal{G}}_{n,\lambda}), T)$  containing  $C$  starts inside  $C$ ). Thus, on  $A_n(\varepsilon)$ ,

$$\text{supdiam}((\text{Coal}_{1/n}((\bar{\mathcal{G}}_{n,\lambda})_{\leq \varepsilon}, T)) \leq 4 \sup_{\substack{x,y \in \mathbb{R}^+ \\ |x-y| \leq \varepsilon}} |\bar{h}^{n,\lambda+T}(x) - \bar{h}^{n,\lambda+T}(y)|,$$

where the supremum is restricted to couples  $(x, y)$  such that  $x$  and  $y$  belong to the same excursion of  $\bar{h}^{n,\lambda+T}$  above zero.

Thus, it only remains to prove that for any  $\alpha > 0$ ,

$$\limsup_{n \rightarrow \infty} \mathbb{P} \left( \sup_{\substack{x,y \in \mathbb{R}^+ \\ |x-y| \leq \varepsilon}} |\bar{h}^{n,\lambda+T}(x) - \bar{h}^{n,\lambda+T}(y)| > \alpha \right) \xrightarrow{\varepsilon \rightarrow 0} 0, \quad (4.6)$$

with the supremum restricted to couples  $(x, y)$  such that  $x$  and  $y$  belong to the same excursion of  $\bar{h}^{n,\lambda+T}$  above zero.

Let  $B^\lambda$  be a Brownian motion with quadratic drift, defined by  $B_t^\lambda = B_t + \lambda t - \frac{t^2}{2}$  with  $B$  a standard Brownian motion. Let  $W^\lambda$  be  $B^\lambda$  reflected above its current minimum:

$$W_t^\lambda := B_t^\lambda - \min_{0 \leq s \leq t} B_s^\lambda.$$

Then (4.6) is a consequence of the fact that  $\bar{h}^{n,\lambda+T}$  converges in distribution to  $W^\lambda$  in the uniform topology. It seems however that this convergence is not written in the literature, so in order to use only available sources, one may rest on the work done in [2] as follows. One may separate the analysis of the supremum on the  $N$  largest excursions and on the others. Let  $B_{N,n}(\varepsilon)$  be the event that the maximal height of some  $i$ -th largest component in  $\bar{\mathcal{G}}_{n,\lambda+T}$  for  $i > N$  exceeds  $\varepsilon$ . The equation p.402 below equation (24) in [2] shows that for any  $\varepsilon > 0$ ,

$$\lim_{N \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}(B_{N,n}(\varepsilon)) = 0. \quad (4.7)$$

Choose  $N$  large enough so that  $\limsup_{n \rightarrow \infty} \mathbb{P}(B_{N,n}(\varepsilon)) \leq \varepsilon$ . Then, for a fixed  $N$ , one may argue as in the proof of Theorem 24 of [2], p.398: conditionally on the sizes, the rescaled height processes associated



to those components are independent and each one converges in distribution (for the uniform topology) to a continuous excursion (a tilted Brownian excursion). Together with the convergence of the sizes and Skorokhod representation theorem, this proves that the  $N$  largest excursions of  $\bar{h}^{n,\lambda+T}$  converges as a vector in  $C([0, +\infty])^N$  to a random vector of continuous functions with bounded support. This implies that (4.6) holds when the supremum is restricted to couples  $(x, y)$  such that  $x$  and  $y$  belong to one of the  $N$  largest components of  $\bar{h}^{n,\lambda+T}$ . Finally, let  $N_n(\varepsilon)$  be the number of components of  $\bar{\mathcal{G}}_{n,\lambda+T}$  which are larger than  $\varepsilon$ . For each  $\varepsilon > 0$ ,  $(N_n(\varepsilon))_{n \geq 1}$  is a tight sequence, as follows from the convergence (in distribution) in  $\ell_{\searrow}^2$  of the sizes of  $\bar{\mathcal{G}}_{n,\lambda+T}$ . Thus, we obtain (4.6) and this ends the proof of Theorem 3.1.

## 5. Proofs of the results for fragmentation

The main goal of this section is to prove the Feller property for fragmentation on  $\mathcal{N}_2^{graph}$ , Proposition 5.5 and to apply it to prove Theorem 3.2. It is very close to the work performed in [3], which proves a continuity result for a fragmentation restricted to the core of a graph (and stopped when you get a tree). The main difference is that we want in addition to perform fragmentation on the tree part of the graphs. Another technical difference will be detailed at the beginning of section 5.4. Unfortunately, those differences force us to make substantial modifications to the arguments of [3].

### 5.1. Notations

We need to introduce a few more definitions to deal with fragmentation of  $\mathbb{R}$ -graphs. For more details, we refer to [3].

Let  $G$  be an  $\mathbb{R}$ -graph. When there is only one geodesic between  $x$  and  $y$  in  $G$ , we denote by  $[x, y]$  its image. Recall the notion of the core of  $G$  defined in section 2.7. If  $S$  is a closed connected subset of  $G$  containing  $\text{core}(G)$ , then for any  $x \in G$ , there is a unique shortest path  $\gamma_x$  going from  $x$  to  $S$ . We denote by  $p_S(x)$  the unique point belonging to  $\gamma_x \cap S$ . When  $G$  is not a tree and  $S = \text{core}(G)$ , we let  $\alpha_G(x) := p_S(x)$ .

For any  $\eta > 0$ , let

$$R_\eta(G) := \text{core}(G) \cup \{x \in G \text{ s.t. } \exists y : x \in [y, \alpha_G(y)] \text{ and } d(y, x) \geq \eta\}.$$

When  $(T, \rho)$  is a rooted  $\mathbb{R}$ -tree, we let  $R_\eta(T)$  be defined as above, with  $\alpha_G(y)$  replaced by the root  $\rho$  and  $\text{core}(G)$  replaced by  $\{\rho\}$ . Thus the definition of  $R_\eta(G)$  extends the definition of  $R_\eta(T)$  for a rooted  $\mathbb{R}$ -tree  $(T, \rho)$  in [10]. Notably, Lemma 2.6 (i) in [10] shows that for any  $\eta > 0$ ,  $R_\eta(G)$  is a finite graph.

A *multigraph with edge-lengths* is a triple  $(V, E, (\ell(e))_{e \in E})$  where  $(V, E)$  is a finite connected multigraph and for every  $e \in E$ ,  $\ell(e)$  is a strictly positive number. It may be viewed as a finite  $\mathbb{R}$ -graph with no leaf by performing on  $V$  (seen as a metric space as the disjoint union of its elements) the  $\ell(e)$ -gluing along  $e$  for each edge  $e \in E$ .

The  $\varepsilon$ -enlargement of a correspondance  $\mathcal{R} \in C(X, X')$  is defined as:

$$\mathcal{R}^\varepsilon := \{(x, x') \in X \times X' : \exists (y, y') \in \mathcal{R}, d(x, y) \vee d(x', y') \leq \varepsilon\}.$$

It is a correspondance containing  $\mathcal{R}$  with distortion at most  $\text{dis}(\mathcal{R}) + 4\varepsilon$ .

If  $\mathcal{R} \in C(X, X')$ , two Borel subsets  $A \subset X$  and  $B \subset X'$  are said to be in correspondance through  $\mathcal{R}$  if  $\mathcal{R} \cap (A \times B) \in C(A, B)$ .

Let  $\varepsilon > 0$ . If  $X$  and  $X'$  are  $\mathbb{R}$ -graphs with surplus at least 2, an  $\varepsilon$ -overlay is a correspondance  $\mathcal{R} \in C(X, X')$  with distortion less than  $\varepsilon$  and such that there exists a multigraph isomorphism  $\chi$  between the kernels  $\ker(X)$  and  $\ker(X')$  satisfying:

1.  $\forall v \in k(X), (v, \chi(v)) \in \mathcal{R}$ ,
2. For every  $e \in e(X)$ ,  $e$  and  $\chi(e)$  are in correspondance through  $\mathcal{R}$  and  $|\ell_X(e) - \ell_{X'}(\chi(e))| \leq \varepsilon$ .

If  $X$  and  $X'$  have surplus one, an  $\varepsilon$ -overlay is a correspondance with distortion less than  $\varepsilon$  such that the unique cycles of  $X$  and  $X'$  are in correspondance and the difference of their lengths is at most  $\varepsilon$ . If  $X$  and  $X'$  are trees, an  $\varepsilon$ -overlay is simply a correspondance with distortion less than  $\varepsilon$ .

We let  $\mathcal{N}_2^{tree}$  be the set of elements  $\mathbf{X} \in \mathcal{N}_2^{graph}$  whose components are trees.

## 5.2. Reduction to finite graphs

The following lemma allows to reduce the proof of the Feller property on finite graphs. This will be useful to adapt the arguments of [10].

**Lemma 5.1.** *Let  $\eta \in (0, 1]$  and  $T > 0$ . Let  $\mathbf{G}$  belong to  $\mathcal{N}_2^{graph}$ . Let  $S$  be a closed connected subset of  $\mathbf{G}$  such that  $R_\eta(\mathbf{G}) \subset S \subset \mathbf{G}$ . Suppose that for each component  $H$  of  $\mathbf{G}$ ,  $S \cap H$  is a connected  $\mathbb{R}$ -graph. Let  $\mathcal{S} := (S, d|_{S \times S}, p_S \# \mu)$ . Then, with probability at least  $1 - T\eta^{1/7}$ , for any  $t \in [0, T]$ , under the obvious coupling,*

$$L_{2,GHP}^{surplus}(\text{Frag}(\mathbf{G}, t), \text{Frag}(\mathcal{S}, t)) \leq 34\eta^{1/7} \left(1 + \sum_{H \in \text{comp}(\mathbf{G})} \mu(H)^2\right).$$

*Proof.* Let  $\mathcal{P}$  be a poisson random set of intensity measure  $\ell_G \otimes \text{leb}_{\mathbb{R}}^+$  on  $G \times \mathbb{R}^+$  and let us use it to perform the fragmentation on  $S$  and  $G$ . Define:

$$G_t^\eta := \{x \in G \setminus S \text{ s.t. } \exists y \in \mathcal{P}_t \cap (G \setminus S) \cap [x, \alpha(x)]\}.$$

Notice that a component  $m$  of  $\text{Frag}(\mathcal{S}, t)$  is endowed with the distance  $d|_{m \times m}$  and the measure  $(p_S \# \mu)|_m$ , while a component  $m$  of  $\text{Frag}(\mathbf{G}, t)$  is endowed with the distance  $d|_{m \times m}$  and the measure  $\mu|_m$ .

If  $m$  is a component of  $\text{Frag}(\mathbf{G}, t)$  such that  $m \cap S = \emptyset$  then  $m \subset G_t^\eta$ . Notably, if  $t \in [0, T]$ ,  $H \in \text{comp}(\mathbf{G})$ ,  $m$  is a component of  $\text{Frag}(\mathbf{G}, t)$  included in  $H$  and  $\mu(m) > \mu(G_T^\eta \cap H)$ , then  $m$  must intersect  $S$ . Furthermore, if  $m \cap S \neq \emptyset$  then  $m \cap S$  is a component of  $\text{Frag}(\mathcal{S}, t)$ .

Let  $H \in \text{comp}(\mathbf{G})$ . For any component  $m$  of  $\text{Frag}(H, \mathcal{P}_t)$  such that  $m \cap S \neq \emptyset$ , we claim that

$$d_{GHP}(m, m \cap S) \leq \eta \vee \mu(G_T^\eta \cap H) \quad (5.1)$$

Indeed, let  $\mathcal{R} := \{(x, p_S(x)) : x \in m\}$ , which has distortion at most  $2\eta$ , and define  $\pi := (Id \otimes p_S) \# \mu|_{m \times m \cap S}$ . Then,  $\pi(\mathcal{R}^c) = 0$  and

$$\begin{aligned} D(\pi; \mu|_m, (p_S \# \mu)|_m) &= \sup_{A \in \mathcal{B}(m \cap S)} \mu(p_S^{-1}(A) \setminus m) \\ &= \mu(p_S^{-1}(m) \setminus m) \\ &\leq \mu(G_t^\eta \cap H) \\ &\leq \mu(G_T^\eta \cap H). \end{aligned}$$

This shows (5.1). Furthermore,

$$\begin{aligned} &\|\text{masses}(\text{Frag}(\mathbf{G}, t)) - \text{masses}(\text{Frag}(\mathcal{S}, t))\|_2^2 \\ &\leq \sum_{\substack{m \in \text{Frag}(\mathbf{G}, t) \\ m \cap S \neq \emptyset}} \mu(p_S^{-1}(m) \setminus m)^2 + \sum_{\substack{m \in \text{Frag}(\mathbf{G}, t) \\ m \cap S = \emptyset}} \mu(m)^2 \end{aligned} \quad (5.2)$$

$$\begin{aligned} &\leq 2 \sum_{H \in \text{comp}(\mathbf{G})} \mu(G_t^\eta \cap H)^2 \\ &\leq 2 \sum_{H \in \text{comp}(\mathbf{G})} \mu(G_T^\eta \cap H)^2. \end{aligned} \quad (5.3)$$

Using Fubini's theorem,

$$\mathbb{E}[\mu(G_T^\eta \cap H)^2] \leq \mu(H)^2(1 - e^{-\eta T}) \leq \mu(H)^2 \eta T.$$

Thus,

$$\mathbb{P} \left( \sum_{H \in \text{comp}(\mathbf{G})} \mu(G_T^\eta \cap H)^2 \geq \eta^{6/7} \sum_{H \in \text{comp}(\mathbf{G})} \mu(H)^2 \right) \leq T\eta^{1/7}.$$

Now, let us place ourselves on the event

$$\mathcal{E} := \left\{ \sum_{H \in \text{comp}(\mathbf{G})} \mu(G_T^\eta \cap H)^2 < \eta^{6/7} \sum_{H \in \text{comp}(\mathbf{G})} \mu(H)^2 \right\}$$

and define  $\alpha := \eta^{3/7} \sqrt{1 + \sum_{H \in \text{comp}(\mathbf{G})} \mu(H)^2}$ . Notice that on  $\mathcal{E}$ , we have for any  $H \in \text{comp}(\mathbf{G})$ :

$$\mu(G_T^\eta \cap H) \leq \sqrt{\sum_{H \in \text{comp}(\mathbf{G})} \mu(G_T^\eta \cap H)^2} \leq \alpha.$$

Let  $\sigma$  assign to each component of  $\text{Frag}(\mathbf{S}, t)$  the component of  $\text{Frag}(\mathbf{G}, t)$  which contains it, and let  $\sigma'$  assign to a component  $m$  of  $\text{comp}((\text{Frag}(\mathbf{G}, t))_{>\alpha^{1/3}+\alpha})$  the component  $m \cap S$  of  $\text{comp}(\text{Frag}(\mathbf{S}, t))$ . From (5.1) we deduce that for any component  $m$  of  $\text{Frag}(\mathbf{S}, t)$ ,

$$d_{GHP}(m, \sigma(m)) \leq \alpha$$

and notice that  $m$  and  $\sigma(m)$  have the same surplus. Also, for any component  $m'$  of  $\text{Frag}(\mathbf{G}, t)$ ,

$$d_{GHP}(m', \sigma'(m')) \leq \alpha,$$

and  $m'$  and  $\sigma'(m')$  have the same surplus. According to Lemma 2.19 this shows that on the event  $\mathcal{E}$ :

$$\begin{aligned} & L_{GHP}^{surplus}(\text{Frag}(\mathbf{G}, t), \text{Frag}(\mathbf{S}, t)) \\ & \leq \alpha \left( 1 + 8 \frac{\sum_{m \in \text{comp}(\text{Frag}(\mathbf{G}, t))} \mu(m)^2}{\alpha^{2/3}} \right) + 16(\alpha^{1/3} + \alpha) \\ & \leq 17\alpha + \alpha^{1/3} \left( 16 + 8 \sum_{H \in \text{comp}(\mathbf{G})} \mu(H)^2 \right). \end{aligned}$$

And, thanks to (5.3),

$$\| \text{masses}(\text{Frag}(\mathbf{G}, t)) - \text{masses}(\text{Frag}(\mathbf{S}, t)) \|_2^2 \leq \eta^{6/7} \sum_{H \in \text{comp}(\mathbf{G})} \mu(H)^2$$

which shows the result.  $\square$

Since  $R^\eta(G)$  has finite length measure for each  $\eta > 0$  when  $G$  is a totally bounded graph, Lemma 5.1 easily implies the following continuity result.

**Proposition 5.2.** *Let  $\mathbf{G}$  belong to  $\mathcal{N}_2^{graph}$ . Then,  $\text{Frag}(\mathbf{G}, t)$  converges in probability (for  $L_{2,GHP}$ ) to  $\mathbf{G}$  when  $t$  goes to zero.*

### 5.3. The Feller property for trees

The following lemma is a slight extension of Lemma 6.3 in [10] designed to take measures into account.

**Lemma 5.3.** *Let  $\mathbf{T} = (T, d, \mu)$  be a measured finite real tree,  $\rho \in T$  and  $\varepsilon > 0$ . There exists  $\delta > 0$  (depending on  $\mathbf{T}$ ,  $\rho$  and  $\varepsilon$ ) such that if  $\mathbf{T}' = (T', d', \mu')$  is a measured finite real tree and  $d_{GHP}^{root}((\mathbf{T}, \rho), (\mathbf{T}', \rho')) < \delta$ , then there exist subtrees  $S \subset T$  and  $S' \subset T'$  such that  $\rho \in S$ ,  $\rho' \in S'$  and:*

- (i)  $d_H(S, T) < \varepsilon$  and  $d_H(S', T') < \varepsilon$ ,
- (ii) there is a bijective measurable map  $\psi : S \rightarrow S'$  that preserves length measure and has distortion at most  $\varepsilon$ ,
- (iii)  $\psi(\rho) = \rho'$ ,

(iv) the length measure of the set of points  $a \in S$  such that  $\{b \in S : \psi(a) \leq b\} \neq \psi(\{b \in S : a \leq b\})$  (that is, the set of points  $a$  such that the subtree above  $\psi(a)$  is not the image under  $\psi$  of the subtree above  $a$ ) is less than  $\varepsilon$ .

(v) there is a correspondance  $\mathcal{R} \in C(S, S')$  and a measure  $\pi \in M(S, S')$  such that:

- (a)  $\forall x \in S (x, \psi(x)) \in \mathcal{R}$
- (b)  $\pi(\mathcal{R}^c) \leq \varepsilon$
- (c)  $D(\pi; p_S \# \mu, p_{S'} \# \mu') \leq \varepsilon$
- (d)  $\text{dis}(\mathcal{R}) \leq 2\varepsilon$ .

*Proof.* Suppose that  $d_{GHP}^{root}((\mathbf{T}, \rho), (\mathbf{T}', \rho')) < \delta$  ( $\delta$  will be chosen small enough later). Then, there exists a correspondance  $\mathcal{R}_0 \in C(T, T')$  and a measure  $\pi_0 \in M(T, T')$  such that:

- (a)  $(\rho, \rho') \in \mathcal{R}_0$
- (b)  $\pi_0(\mathcal{R}_0^c) \leq \delta$
- (c)  $D(\pi_0; \mu, \mu') \leq \delta$
- (d)  $\text{dis}(\mathcal{R}) \leq 2\delta$ .

Now, one performs the proof of Lemma 6.2 in [10] and shall use their notations. First, let  $f(\rho) := \rho'$  and then for each  $x \in T$ , one chooses  $f(x) \in T'$  such that  $(x, f(x)) \in \mathcal{R}_0$  (notice that this can be done in a measurable way). Then, letting  $x_1, \dots, x_n$  to be the leaves of  $\mathbf{T}$  one defines  $x'_i = f(x_i)$  and let  $T''$  be the subtree of  $T'$  spanned by  $\rho', x'_1, \dots, x'_n$ . Finally,  $\bar{f}(x)$  is defined to be the closest point from  $f(x)$  on  $T''$ . Notice that  $x'_i = \bar{f}(x_i)$ . The proof of Lemma 6.2 in [10] shows that  $T''$  has leaves  $x'_1, \dots, x'_n$  (and root  $\rho' = \bar{f}(\rho)$ ), that  $d_H(T, T'') < 3\delta$  and that the function  $\bar{f}$  from  $T$  to  $T''$  has distortion at most  $8\delta$ . It is easy to see that

$$\forall x \in T, d'(\bar{f}(x), f(x)) \leq 4\delta. \quad (5.4)$$

Then, they take  $y_1 \in [\rho, x_1]$  and  $y'_1 \in [\rho', x'_1]$  such that  $d(\rho, y_1) = d'(\rho', y'_1) = d(\rho, x_1) \wedge d'(\rho', x'_1)$  and define  $\psi$  from  $S_1 := [\rho, y_1]$  to  $S'_1 := [\rho', y'_1]$  in the obvious way. The proof then proceeds inductively, defining  $z_{k+1}$  (resp.  $z'_{k+1}$ ) as the closest point from  $x_{k+1}$  on  $S_k$  (resp. from  $x'_{k+1}$  on  $S'_k$ ), letting  $y_{k+1} \in ]z_{k+1}, y_{k+1}]$  and  $y'_{k+1} \in ]z'_{k+1}, y_{k+1}]$  be such that

$$d(z_{k+1}, y_{k+1}) = d'(z'_{k+1}, y'_{k+1}) = d(z_{k+1}, x_{k+1}) \wedge d'(z'_{k+1}, x'_{k+1})$$

defining  $\psi$  from  $]z_{k+1}, y_{k+1}]$  to  $]z'_{k+1}, y'_{k+1}]$  in the obvious way and gluing  $]z_{k+1}, y_{k+1}]$  to  $S_k$  to get  $S_{k+1}$  (resp.  $]z'_{k+1}, y'_{k+1}]$  to  $S'_k$  to get  $S'_{k+1}$ ). Finally,  $S := S_n$  and  $S' := S'_n$ . They prove then that:

$$\text{dis}(\psi) < 280\delta, d_H(S, T) < 56\delta, d_H(S', T') < 58\delta,$$

which shows that  $\psi, S$  and  $S'$  satisfy (i) – (iii) above if  $\delta$  is chosen small enough. Meanwhile, they show that for any  $k$ ,  $d(x_k, y_k) \vee d'(x'_k, y'_k) \leq 12\delta$  (see inequality (6.28) in [10]).

Now, let us show that:

$$\forall x \in S, d'(\bar{f}(x), \psi(x)) \leq 56\delta. \quad (5.5)$$

Let  $x \in ]z_k, y_k]$ , then  $d'(\psi(x), y'_k) = d(x, y_k)$  (recall that  $\psi(y_k) = y'_k$ ). Then,  $|d(x, y_k) - d(x, x_k)| \leq 12\delta$  and  $|d'(\psi(x), y'_k) - d'(\psi(x), x'_k)| \leq 12\delta$ . Since  $\bar{f}$  has distortion at most  $8\delta$ ,  $|d(x, x_k) - d'(\bar{f}(x), x'_k)| \leq 8\delta$ . We get

$$|d'(\psi(x), x'_k) - d'(\bar{f}(x), x'_k)| \leq 32\delta.$$

Let  $z$  be the closest point to  $\bar{f}(x)$  on  $[\rho', x'_k]$ . Then,

$$\begin{aligned} d'(\bar{f}(x), z) &= \frac{1}{2}[d'(\bar{f}(x), \rho') + d'(\bar{f}(x), x'_k) - d'(\rho', x'_k)] \\ &\leq \frac{3}{2} \text{dis}(\bar{f}) + \frac{1}{2}[d(x, \rho) + d(x, x_k) - d(\rho, x_k)] \\ &\leq 12\delta, \end{aligned}$$

since  $x \in [\rho, x_k]$ . Finally, since  $\psi(x) \in [\rho', x'_k]$ ,

$$\begin{aligned} d'(\bar{f}(x), \psi(x)) &= d'(\bar{f}(x), z) + d'(z, \psi(x)) \\ &= d'(\bar{f}(x), z) + |d'(x'_k, \psi(x)) - d'(x'_k, z)| \\ &\leq 2d'(\bar{f}(x), z) + |d'(x'_k, \psi(x)) - d'(x'_k, \bar{f}(x))| \\ &\leq 24\delta + 32\delta . \end{aligned}$$

This shows (5.5).

Now, let  $\mathcal{R}$  be defined by:

$$\mathcal{R} := \left\{ (x, x') \in S \times S' : \exists (y, y') \in \mathcal{R}_0, \begin{pmatrix} d(x, y) \leq 100\delta \\ \text{and} \\ d'(x', y') \leq 100\delta \end{pmatrix} \right\} ,$$

and define  $\pi := (p_S \otimes p_{S'}) \# \pi_0$ . It remains to prove point (v). First, recall that  $(x, f(x)) \in \mathcal{R}_0$  for any  $x \in T$ . Thus (v)(a) is satisfied thanks to (5.5) and (5.4). This shows also that  $\mathcal{R}$  is a correspondance on  $S \times S'$ .

Then,

$$\text{dis}(\mathcal{R}) \leq \text{dis}(\mathcal{R}_0) + 400\delta$$

which is less than  $\varepsilon$  and shows (v)(d) if  $\delta$  is chosen small enough. Since  $d_H(S, T) \vee d_H(S', T') < 58\delta$ , we see that for any  $x \in T$  and  $x' \in T'$ ,

$$d(x, p_S(x)) < 58\delta \text{ and } d(x', p_{S'}(x')) < 58\delta .$$

Thus, if  $(x, x') \in \mathcal{R}_0$ , then  $(p_S(x), p_{S'}(x')) \in \mathcal{R}$  and this gives

$$\pi(\mathcal{R}^c) \leq \pi_0(\mathcal{R}_0^c) \leq \delta ,$$

which shows (v)(b) if  $\delta$  is chosen small enough. Finally, since  $\pi = (p_S \otimes p_{S'}) \# \pi_0$  one sees that

$$D(\pi; p_S \# \mu, p_{S'} \# \mu') \leq D(\pi_0, \mu, \mu') < \delta .$$

□

Now, let us prove the Feller property.

**Proposition 5.4.** *Let  $(\mathbf{X}^{(n)})_{n \geq 0}$  be a sequence in  $\mathcal{N}_2^{\text{tree}}$  converging to  $\mathbf{X}$  (in the  $L_{2, \text{GHP}}$  metric) and  $t \geq 0$ . Then  $(\text{Frag}(\mathbf{X}^{(n)}, s))_{s \geq 0}$  converges in distribution to  $(\text{Frag}(\mathbf{X}, s))_{s \geq 0}$  for  $L_{2, \text{GHP}}$ .*

*Proof.* First, we argue that one may without loss of generality suppose that  $\mathbf{X}^{(n)}$  and  $\mathbf{X}$  contain a single component. Indeed, fix  $\varepsilon > 0$ . Since  $\text{masses}(\mathbf{X}^{(n)})$  converges to  $\text{masses}(\mathbf{X})$  in  $\ell^2$ , one may choose  $\varepsilon' \notin \text{masses}(\mathbf{X})$  such that:

$$\| \text{masses}(\mathbf{X}_{\leq \varepsilon'} ) \|_2^2 \vee \sup_{n \in \mathbb{N}} \| \text{masses}(\mathbf{X}_{\leq \varepsilon'}^{(n)}) \|_2^2 \leq \varepsilon .$$

Then, since  $\mathbf{X}_{> \varepsilon'}^{(n)}$  converges to  $\mathbf{X}_{> \varepsilon'}$  as  $n$  goes to infinity, they have the same number of components for  $n$  large enough. Call this number  $K$ . One may list them as follows: let  $T_i^{(n)}$  (resp.  $T_i$ ),  $i = 1, \dots, K$  be the components of  $\mathbf{X}^{(n)}$  (resp. of  $\mathbf{X}$ ) such that for any  $i$ ,  $T_i^{(n)}$  converges to  $T_i$ . Then, for any coupling between  $(\text{Frag}(\mathbf{X}, s))_{s \in [0, t]}$  and  $(\text{Frag}(\mathbf{X}^{(n)}, s))_{s \in [0, t]}$ , one has:

$$\begin{aligned} & \| \text{masses}(\text{Frag}(\mathbf{X}, s)) - \text{masses}(\text{Frag}(\mathbf{X}^{(n)}, s)) \|_2^2 \\ & \leq \sum_{i=1}^K \| \text{masses}(\text{Frag}(T_i, s)) - \text{masses}(\text{Frag}(T_i^{(n)}, s)) \|_2^2 \\ & \quad + \| \text{masses}(\mathbf{X}_{\leq \varepsilon'} ) \|_2^2 + \| \text{masses}(\mathbf{X}_{\leq \varepsilon'}^{(n)}) \|_2^2 \end{aligned}$$

and

$$L_{GHP}(\text{Frag}(\mathbf{X}, s), \text{Frag}(\mathbf{X}^{(n)}, s)) \leq \sum_{i=1}^K L_{GHP}(\text{Frag}(T_i, s), \text{Frag}(T_i^{(n)}, s)) + 16\varepsilon.$$

Thus, it is sufficient to prove that for any fixed  $i$  and  $n$ , one may find a coupling such that

$$\sup_{s \in [0, t]} L_{GHP}(\text{Frag}(T_i^{(n)}, s), \text{Frag}(T_i, s)) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0.$$

In the sequel, we suppose that  $\mathbf{X}^{(n)} =: \mathbf{T}^{(n)}$  (resp.  $\mathbf{X} =: \mathbf{T}$ ) contains a single component.

Now, Lemma 5.1 implies that it is enough to consider  $\mathbf{T}^{(n)}$  and  $\mathbf{T}$  to be finite trees. Let us fix  $\varepsilon > 0$ . We know (cf. Proposition 2.1 in [3]) that we can take  $\rho$  (resp.  $\rho^{(n)}$ ) a root in  $T$  (resp.  $T^{(n)}$ ) such that  $d_{GHP}^{root}((\mathbf{T}, \rho), (\mathbf{T}^{(n)}, \rho^{(n)}))$  goes to zero as  $n$  goes to infinity. For  $n$  large enough,  $d_{GHP}^{root}((\mathbf{T}, \rho), (\mathbf{T}^{(n)}, \rho^{(n)}))$  is small enough so that one may apply Lemma 5.3.

Let us call  $(\mathbf{T}', \rho') = (\mathbf{T}^{(n)}, \rho^{(n)})$  for such a large  $n$ , in order to lighten the notations. Notice that one may suppose that  $\mu'(T') \leq \mu(T) + \varepsilon$ . Let  $S, S', \psi, \mathcal{R}$  and  $\pi$  be as in Lemma 5.3. Define

$$\mathbf{S} := (S, d|_{S \times S}, p_S \sharp \mu)$$

and

$$\mathbf{S}' := (S', d|_{S' \times S'}, p_{S'} \sharp \mu').$$

Let  $t > 0$ . Lemma 5.1 ensures that with probability at least  $1 - 2t\varepsilon^{1/7}$ , for any  $s \in [0, t]$ ,

$$L_{2,GHP}(\text{Frag}(\mathbf{T}, s), \text{Frag}(\mathbf{S}, s)) \leq 7\varepsilon^{1/7}(1 + \mu(T))^4,$$

and

$$L_{2,GHP}(\text{Frag}(\mathbf{T}', s), \text{Frag}(\mathbf{S}', s)) \leq 7\varepsilon^{1/7}(1 + \mu'(T'))^4 \leq 7\varepsilon^{1/7}(1 + \mu(T) + \varepsilon)^4.$$

For any  $z \in S$  (resp.  $z' \in S'$ ) we let  $S_z$  (resp.  $S'_{z'}$ ) be the subtree above  $z$  (resp. above  $z'$ ):

$$S_z := \{x \in S : z \in [\rho, x]\}.$$

Let us define

$$\text{Bad} := \{a \in S : S_{\psi(a)} \neq \psi(S_a)\}$$

so that Lemma 5.3 ensures that  $\ell_S(\text{Bad}) \leq \varepsilon$ .

Now, let  $\mathcal{P}$  be a Poisson random set of intensity  $\ell_S \otimes \text{leb}_{\mathbb{R}^+}$  on  $S \times \mathbb{R}^+$ . Then, for any  $s$ ,  $\psi(\mathcal{P}_s)$  is a Poisson random set of intensity  $s\ell_S$  on  $S$  (since  $\psi$  is a measure-preserving bijection), and we want to show that for any  $s \leq t$ , the fragmentation of  $\mathbf{S}$  along  $\mathcal{P}_s$ ,  $\text{Frag}(\mathbf{S}, \mathcal{P}_s)$  and that of  $\mathbf{S}'$  along  $\psi(\mathcal{P}_s)$ ,  $\text{Frag}(\mathbf{S}', \psi(\mathcal{P}_s))$  are close in  $L_{GHP}$ -distance with large probability.

Notice first that  $\text{Frag}(\mathbf{S}, \mathcal{P}_s)$  and  $\text{Frag}(\mathbf{S}', \psi(\mathcal{P}_s))$  have the same number of components. If  $m$  is a component of  $\text{Frag}(\mathbf{S}, \mathcal{P}_s)$ , it can be written as  $S_{z_s} \setminus \bigcup_{i=1}^k S_{z_{i,s}}$  for some points  $z_s, z_{1,s}, \dots, z_{k,s}$  in  $\mathcal{P}_s \cup \{\rho\}$  (we identify  $S_z \setminus \{z\}$  and  $S_z$  since it is at zero  $d_{GHP}$ -distance). If  $\mathcal{P}_t \cap \text{Bad} = \emptyset$ , then for any  $s \leq t$ ,  $\psi(m) = \psi(S_{z_s}) \setminus \bigcup_{i=1}^k \psi(S_{z_{i,s}})$  and this is a component of  $\text{Frag}(\mathbf{S}', \psi(\mathcal{P}_s))$ .

Thus, let us place ourselves on the event  $\mathcal{E}_1 := \{\mathcal{P}_t \cap \text{Bad} = \emptyset\}$  and define  $\sigma$  (which depends on  $s$ ) to be the bijection from  $\text{Frag}(\mathbf{S}, \mathcal{P}_s)$  to  $\text{Frag}(\mathbf{S}', \psi(\mathcal{P}_s))$  which maps a component  $m$  to  $\psi(m)$ . Since  $\mathcal{R}$  contains the couples  $(x, \psi(x))$  for  $x \in S$ ,  $\mathcal{R}|_{m \times \psi(m)}$  is a correspondance between  $m$  and  $\psi(m)$  with distortion at most  $\varepsilon$ . Furthermore  $\pi|_{m \times \psi(m)}$  is a measure on  $m \times \psi(m)$  which satisfies

$$\pi|_{m \times \psi(m)}(\mathcal{R}|_{m \times \psi(m)}^c) \leq \pi(\mathcal{R}^c) \leq \varepsilon.$$

It remains to bound  $D(\pi|_{m \times \psi(m)}; (p_S \sharp \mu)|_m, (p_{S'} \sharp \mu')|_{\psi(m)})$  from above. For any Borel subset  $A$  of  $m$ ,

$$\begin{aligned} & |\pi|_{m \times \psi(m)}(A \times \psi(m)) - p_S \sharp \mu(A)| \\ & \leq |\pi(A \times S') - p_S \sharp \mu(A)| + \pi(A \times S') - \pi(A \times \psi(m)) \\ & \leq |\pi(A \times S') - p_S \sharp \mu(A)| + \pi(\{(x, x') \in S \times S' : x \in m, x' \notin \psi(m)\}). \end{aligned}$$

A symmetric inequality holds for  $A'$  Borel subset of  $\psi(m)$ , and we get:

$$D(\pi|_{m \times \psi(m)}; \mu|_m, \mu'|_{\psi(m)}) \leq D(\pi; p_S \# \mu, p_{S'} \# \mu') + \pi(m \times \psi(m)^c) + \pi(m^c \times \psi(m)).$$

Now, notice that for any  $x \in S$ ,

$$x \in m \text{ and } x' \notin \psi(m) \Rightarrow [\psi(x), x'] \cap \psi(\mathcal{P}_t) \neq \emptyset$$

and

$$x \notin m \text{ and } x' \in \psi(m) \Rightarrow [\psi(x), x'] \cap \psi(\mathcal{P}_t) \neq \emptyset,$$

where  $[\psi(x), x']$  is the geodesic between  $\psi(x)$  and  $x'$ . Thus,

$$\pi(m \times \psi(m)^c) + \pi(m^c \times \psi(m)) \leq \pi\{(x, x') \in S \times S' : [\psi(x), x'] \cap \psi(\mathcal{P}_t) \neq \emptyset\}.$$

Let us denote by  $\mathcal{E}_2$  the event

$$\mathcal{E}_2 := \{\pi\{(x, x') \in S \times S' : [\psi(x), x'] \cap \psi(\mathcal{P}_t) \neq \emptyset\} \leq \sqrt{\varepsilon}\}.$$

On  $\mathcal{E}_1 \cap \mathcal{E}_2$ , we get, for any  $s \leq t$  and any component  $m$  of  $\text{Frag}(\mathbf{S}, \mathcal{P}_s)$ :

$$D(\pi|_{m \times \psi(m)}; p_S \# \mu|_m, p_{S'} \# \mu'|_{\psi(m)}) \leq \varepsilon + \sqrt{\varepsilon}.$$

Furthermore,

$$\begin{aligned} & \| \text{masses}(\text{Frag}(\mathbf{S}, \mathcal{P}_s)) - \text{masses}(\text{Frag}(\mathbf{S}', \psi(\mathcal{P}_s))) \|_2^2 \\ & \leq \sum_{m \in \text{comp}(\text{Frag}(\mathbf{S}, \mathcal{P}_s))} (p_S \# \mu(m) - p_{S'} \# \mu'(\psi(m)))^2 \\ & \leq \sup_{m \in \text{comp}(\text{Frag}(\mathbf{S}, \mathcal{P}_s))} |p_S \# \mu(m) - p_{S'} \# \mu'(\psi(m))| \\ & \quad \times \sum_{m \in \text{comp}(\text{Frag}(\mathbf{S}, \mathcal{P}_s))} p_S \# \mu(m) + p_{S'} \# \mu'(\psi(m)) \\ & \leq \sup_{m \in \text{comp}(\text{Frag}(\mathbf{S}, \mathcal{P}_s))} D(\pi|_{m \times \psi(m)}; p_S \# \mu|_m, p_{S'} \# \mu'|_{\psi(m)}) (\mu(T) + \mu'(T')) \\ & \leq (\varepsilon + \sqrt{\varepsilon}) (2\mu(T) + \varepsilon) \end{aligned}$$

Thus, on  $\mathcal{E}_1 \cap \mathcal{E}_2$ , we obtain:

$$L_{2,GHP}(\text{Frag}(\mathbf{S}, \mathcal{P}), \text{Frag}(\mathbf{S}', \psi(\mathcal{P}))) \leq (\varepsilon + \sqrt{\varepsilon}) \vee \sqrt{(\varepsilon + \sqrt{\varepsilon})(2\mu(T) + \varepsilon)}.$$

It remains to bound from above the probability of  $(\mathcal{E}_1 \cap \mathcal{E}_2)^c$ . Since  $Bad$  has length measure at most  $\varepsilon$ ,

$$\mathbb{P}(\mathcal{E}_1^c) \leq t\varepsilon.$$

Notice that since  $\mathcal{R}$  contains  $(x, \psi(x))$  for any  $x \in S$  and has distortion less than  $2\varepsilon$ ,

$$\pi\{(x, x') \in S \times S' : d'(\psi(x), x') > 2\varepsilon\} \leq \pi(\mathcal{R}^c) < \varepsilon$$

Then, using Fubini's theorem,

$$\begin{aligned} & \mathbb{E}[\pi\{(x, x') \in S \times S' : [\psi(x), x'] \cap \psi(\mathcal{P}) \neq \emptyset\}] \\ & \leq \varepsilon + \mathbb{E}[\pi\{(x, x') \in S \times S' : [\psi(x), x'] \cap \psi(\mathcal{P}) \neq \emptyset \text{ and } d'(\psi(x), x') \leq 2\varepsilon\}] \\ & = \varepsilon + \int_{S \times S'} \mathbb{P}([\psi(x), x'] \cap \psi(\mathcal{P}) \neq \emptyset) \mathbf{1}_{d'(\psi(x), x') \leq 2\varepsilon} d\pi(x, x') \\ & \leq 2t\varepsilon\pi(S \times S') \\ & \leq 2t\varepsilon(\mu(T) + \varepsilon). \end{aligned}$$

Thus, by Markov inequality,

$$\mathbb{P}(\mathcal{E}_2^c) \leq 2t\sqrt{\varepsilon}(\mu(T) + \varepsilon),$$

which ends the proof.  $\square$

#### 5.4. The Feller property for graphs

We now want to prove the analog of Proposition 5.4 for graphs. However, this cannot be true without strengthening the metric  $L_{GHP}$ . For instance, consider the situation depicted in Figure 2. There  $G_n$  converges to  $G$  for  $d_{GHP}$ , but the probability that  $a$  is separated from  $b$  in  $G_n$  when fragmentation occurs (until a fixed time  $t > 0$ ) is asymptotically 0, whereas the probability that this event occurs in  $G$  is strictly positive. However, if we impose that the surplus of  $G_n$  converges to the surplus of  $G$ , such a situation cannot happen anymore, and one may recover the Feller property.

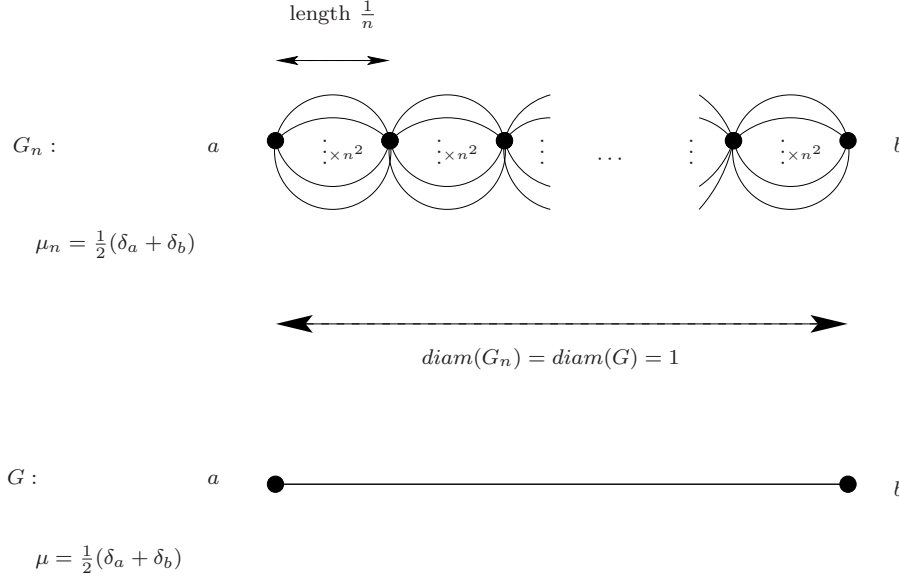


FIG 2.  $G_n$  is composed of  $n$  graphs in series each one made of  $n^2$  intervals of length  $1/n$  in parallel.  $G_n$  converges to  $G$  for  $d_{GHP}$  when  $n$  goes to infinity, but  $\text{Frag}(G_n, t)$  will not converge to  $\text{Frag}(G, t)$  for  $t > 0$ .

Let us notice that this problem was treated a bit differently in [3]: they recover continuity (in probability) of fragmentation by imposing that  $G_n$  and  $G$  live on some common subspace  $\mathcal{A}_r$  some  $r > 0$ , where  $\mathcal{A}_r$  contains the graphs which have surplus and total length of the core bounded from above and minimal edge length of the core bounded from below (see section 6.4 in [3] for a precise statement). When one wants to have Feller-type properties, this seems to us less natural than imposing convergence of the surplus. In fact, the work below shows that if  $G_n$  converges to  $G$  in the Gromov-Hausdorff topology while having the same surplus for  $n$  large enough, then there is some  $r > 0$  such that for  $n$  large enough,  $G_n$  and  $G$  belong to  $\mathcal{A}_r$ . The converse statement is also true and is a consequence of Proposition 6.5 in [3].

**Proposition 5.5.** *Let  $(\mathbf{G}^{(n)})_{n \geq 0}$  be a sequence in  $\mathcal{N}_2^{graph}$  converging to  $\mathbf{G}$  in the  $L_{2,GHP}^{surplus}$  metric. Then, for any  $t \geq 0$ ,  $\text{Frag}(\mathbf{G}^{(n)}, t)_{t \geq 0}$  converges in distribution to  $(\text{Frag}(\mathbf{G}, t))_{t \geq 0}$  (for  $L_{2,GHP}^{surplus}$ ).*

To prove this result, we first notice that the proof of section 5.3 extends to the case where one replace trees by graphs having the same core.

**Lemma 5.6.** *Let  $\mathbf{G} = (G, d, \mu)$  be a measured finite  $\mathbb{R}$ -graph which is not a tree. Let  $(\mathbf{G}^{(n)})_{n \geq 0}$  be a sequence of measured finite  $\mathbb{R}$ -graphs such that for each  $n$ , there is a correspondence  $\mathcal{R}^{(n)} \in C(G, G^{(n)})$ , a measure  $\pi^{(n)} \in M(G, G^{(n)})$  and a homeomorphism  $\psi^{(n)} : \text{core}(G) \rightarrow \text{core} G^{(n)}$  such that:*

- $\psi^{(n)}$  preserves the length-measure,
- $\forall x \in \text{core}(G^{(n)}) (x, \psi^{(n)}(x)) \in \mathcal{R}^{(n)}$ ,
- $\text{dis}(\mathcal{R}^{(n)} \vee \pi^{(n)}((\mathcal{R}^{(n)})^c) \vee D(\pi^{(n)}, \mu, \mu^{(n)}) \xrightarrow[n \rightarrow \infty]{} 0$ .

Then, the sequence of processes  $\text{Frag}(\mathbf{G}^{(n)}, \cdot)$  converges in distribution to  $\text{Frag}(\mathbf{G}, \cdot)$  for  $L_{2,GHP}^{surplus}$ .



*Proof.* It is a straightforward extension of the arguments of section 5.3, replacing roots by cores and using  $\psi^{(n)}$  to map fragmentation on  $\text{core}(G^{(n)})$  to fragmentation on  $\text{core}(G)$ .  $\square$

To prepare the proof of Proposition 5.5, we shall need the following lemmas.

**Lemma 5.7.** *Let  $(G, d)$  and  $(G', d')$  be  $\mathbb{R}$ -graphs and  $\mathcal{R} \in C(G, G')$ . Let  $(a, a') \in \mathcal{R}$ ,  $(b, b') \in \mathcal{R}$  and  $(c, c') \in \mathcal{R}$ . Suppose that  $a$  belongs to a geodesic between  $b$  and  $c$ . Let  $\gamma_{a', b'}$  (resp.  $\gamma_{a', c'}$ ) be a geodesic from  $a'$  to  $b'$  (resp. from  $a'$  to  $c'$ ). Then,*

$$\forall a'' \in \gamma_{a', b'} \cap \gamma_{a', c'}, \quad d'(a'', a') \leq 3 \text{dis}(\mathcal{R}).$$

*Proof.* Let  $a'' \in \gamma_{a', b'} \cap \gamma_{a', c'}$ . Then,

$$\begin{aligned} d'(a', a'') &= d'(a', b') + d'(a', c') - d'(a'', b') - d'(a'', c') \\ &\leq d'(a', b') + d'(a', c') - d'(b', c') \\ &\leq d(a, b) + d(a, c) - d(b, c) + 3 \text{dis}(\mathcal{R}) \\ &= 3 \text{dis}(\mathcal{R}) \end{aligned}$$

where we used the triangular inequality in the second step and the fact that  $a$  belongs to a geodesic between  $b$  and  $c$  in the last step.  $\square$

The following should be compared to Proposition 5.6 in [3].

**Lemma 5.8.** *Let  $G$  be an  $\mathbb{R}$ -graph and  $\varepsilon > 0$ . There exists  $\delta$  depending on  $\varepsilon$  and  $G$  such that if  $G'$  is an  $\mathbb{R}$ -graph with the same surplus as  $G$  and if  $\mathcal{R}_0 \in C(G, G')$  is such that  $\text{dis}(\mathcal{R}_0) < \delta$ , then there exists an  $\varepsilon$ -overlay  $\mathcal{R} \in C(G, G')$  containing  $\mathcal{R}_0$ .*

*Proof.* If  $G$  has surplus 0, there is nothing to prove. In the sequel, we suppose that  $G$  has surplus at least 2, the easier proof for unicyclic  $G$  is left to the reader. Furthermore, to lighten notations and make the argument clearer, we shall suppose that the vertices of  $\ker(G)$  are of degree 3, leaving the adaptation to the general case to the reader.

Let  $\eta := \min_{e \in e(G)} \ell(e)$ . One may view  $\text{core}(G)$  as a multigraph with edge-lengths. However, not all the edges of this graph correspond to geodesics in  $G$ . Divide each edge of  $\text{core}(G)$  into five pieces of equal length, introducing thus four new vertices of degree 2 for each edge (all degrees will be relative to the core). The new graph obtained satisfies the following:

- (i) all the edges remain of length larger than  $\eta/5$ ,
- (ii) every edge  $e$  is the unique geodesic between its two endpoints, and for any path  $\gamma$  which does not contain  $e$ ,  $\ell(\gamma) - \ell(e) > \eta/5$ ,
- (iii) for every three vertices  $a, b, c$  such that  $b \sim a$  and  $a \sim c$ ,  $a$  belongs to a geodesic between  $a$  and  $c$ .

Let us call  $\tilde{\text{core}}(G)$  this new graph (it is indeed a graph, not merely a multigraph), which has the same surplus as  $G$ , and write  $x_1, \dots, x_n$  for its vertices, which are of degree 2 or 3.

Let  $G'$  be an  $\mathbb{R}$ -graph with the same surplus as  $G$  and  $\mathcal{R}_0 \in C(G, G')$ . Let  $x'_1, \dots, x'_n$  be elements of  $G'$  such that  $(x_i, x'_i) \in \mathcal{R}_0$ . Now, we shall build a subgraph of  $G'$  by mapping recursively edges adjacent to a given vertex in  $\tilde{\text{core}}(G)$  to a geodesic in  $G'$ . Suppose for instance that  $x_1$  has degree 3 (the argument is analogous for vertices with degree 2). Let  $x_i, x_j$  and  $x_k$  be its neighbours in  $\tilde{\text{core}}(G)$ , with  $i < j < k$ . Choose a geodesic  $\gamma_{x'_1, x'_i}$  between  $x'_1$  and  $x'_i$ , then choose a geodesic  $\gamma$  between  $x'_1$  and  $x'_j$ , and let  $z_1^1$  be the point of  $\gamma_{x'_1, x'_i} \cap \gamma$  which is the furthest of  $x'_1$  (see Figure 3). Let us call  $\gamma_{z_1^1, x'_j}$  the subpath of  $\gamma$  from  $z_1^1$  to  $x'_j$ . Notice that the path using  $\gamma_{x'_1, x'_i}$  from  $x'_1$  to  $z_1^1$  and  $\gamma$  from  $z_1^1$  to  $x'_j$  is a geodesic. Finally choose a geodesic  $\gamma$  between  $x'_1$  and  $x'_k$  and let  $z_1^2$  be the point of  $(\gamma_{x'_1, x'_i} \cup \gamma_{z_1^1, x'_j}) \cap \gamma$  which is the furthest of  $x'_1$ . Let us call  $\gamma_{z_1^2, x'_k}$  the subpath of  $\gamma$  from  $z_1^2$  to  $x'_k$ . Let  $S'_1 := \gamma_{x'_1, x'_i} \cup \gamma_{z_1^1, x'_j} \cup \gamma_{z_1^2, x'_k}$ . Define  $x''_1$  be the one between  $z_1^1$  and  $z_1^2$  which is the furthest from  $x'_1$ . If  $x_1$  is of degree 2, there is only one point  $z_1^1$  defined and  $x''_1$  is this one.

Then, we proceed similarly for  $r = 2, \dots, n$ : we inspect the neighbours of  $x_r$ . Notice that we do not need to choose a new geodesic between  $x'_r$  and a neighbour  $x'_j$  for  $j < r$ , we just keep the one already built. Doing this, we obtain  $S'_r$  the union of the geodesics chosen going from  $x'_r$  to the points associated to the neighbours

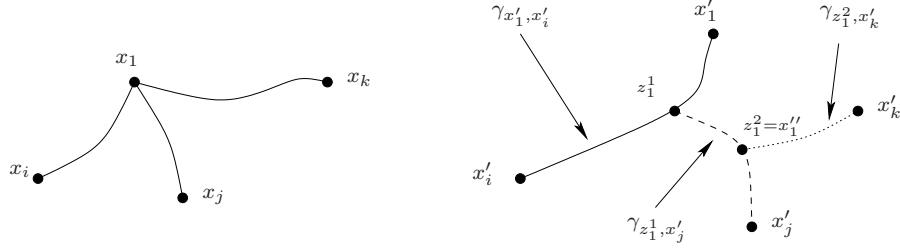


FIG 3. One maps  $\text{core}(G)$  to  $\text{core}(G')$  by first mapping the neighborhood of each vertex of  $\text{core}(G)$  to a subset of  $G'$ . Here  $x''_1$  is a vertex of  $\ker(G')$ .

of  $x_r$ , we get two points  $z_r^1$  and  $z_r^2$  if  $x_r$  is of degree 3 and only one point  $z_r^1$  if  $x_r$  is of degree 1. We define  $x''_r$  to be the one between  $z_r^1$  and  $z_r^2$  which is the furthest from  $x'_r$ .

Finally, let  $S' = \cup_{i=1}^n S'_i$ , with all the vertices  $z_i^b$  and  $x'_i$  which is a graph with edge-lengths (notice that the edges have pairwise disjoint interiors). Some edge-lengths might be zero. Thanks to point (ii) above and Lemma 5.7, we know that:

$$d'(z_i^1, x'_i) \leq 3 \text{dis}(\mathcal{R}_0),$$

and when  $x'_i$  is of degree 3,

$$d'(z_i^2, x'_i) \leq 3 \text{dis}(\mathcal{R}_0).$$

Thus, for any  $b, b' \in \{1, 2\}$  and any  $i \neq j$ ,

$$\begin{aligned} d'(z_i^b, z_j^{b'}) &\geq d'(x'_i, x'_j) - 6 \text{dis}(\mathcal{R}_0) \\ &\geq \eta - 7 \text{dis}(\mathcal{R}_0). \end{aligned}$$

Thus, if  $\text{dis}(\mathcal{R}_0) < \eta/7$ , two points  $z_i^b$  and  $z_j^{b'}$  are always distinct. This shows that  $S'$  has the same surplus as  $\text{core}(G)$ . Since  $G'$  has the same surplus as  $G$ , we deduce that  $S'$  contains  $\text{core}(G')$ . Let  $S''$  be the subgraph of  $S'$  spanned by  $x''_1, \dots, x''_n$ , in the sense that we forget the vertices  $z_i^1$  when  $x_i$  is of degree 3, and we remove the semi-open path going from  $x'_i$  to  $z_i^1$ . Notice that  $S''$  has positive edge-lengths and its edges have pairwise disjoint interiors.  $S''$  has the same surplus as  $S'$ , so it contains again  $\text{core}(G')$ . But all the vertices in  $S''$  have degree 2 or 3, so  $S'' = \text{core}(G')$  as a set.

Now, consider  $S''$  as a graph with edge-lengths and with vertices  $x''_i$ ,  $i = 1, \dots, r$ . The map  $\chi_0$  from  $\text{c\o{r}e}(G)$  to  $S''$  which maps  $x_i$  to  $x''_i$  is a graph isomorphism, and from the computation above, for any edge  $e$  of  $\text{c\o{r}e}(G)$ ,

$$|\ell(e) - \ell'(\chi_0(e))| \leq 6 \text{dis}(\mathcal{R}_0).$$

We shall denote by  $[x''_i, x''_j]$  the path in  $S''$  from  $x''_i$  to  $x''_j$  and which does not contain any other vertex  $x''_k$  for  $k \notin \{i, j\}$ . Then, let  $\gamma$  be a path from  $x''_i$  to  $x''_j$  which does not contain  $[x''_i, x''_j]$ . Using point (ii) above, we see that

$$\ell'([x''_i, x''_j]) \leq \ell'(\gamma) - \frac{\eta}{5} + 8 \text{dis}(\mathcal{R}_0)$$

Thus, if  $\text{dis}(\mathcal{R}_0) < \eta/80$ ,  $[x''_i, x''_j]$  is the unique geodesic between  $i$  and  $j$  geodesic for any  $i$  and  $j$  and every path  $\gamma$  from  $x''_i$  to  $x''_j$  which does not contain  $[x''_i, x''_j]$  satisfies:

$$\ell'(\gamma) > \ell'([x''_i, x''_j]) + \frac{\eta}{10}. \quad (5.6)$$

Let us define  $\mathcal{R}'$  by adding to  $\mathcal{R}_0$  the couples  $(x_i, x''_i)$  for  $i = 1, \dots, r$ . Then,  $\text{dis}(\mathcal{R}') \leq 7 \text{dis}(\mathcal{R}_0)$ . Let  $\mathcal{R}$  be the  $3 \text{dis}(\mathcal{R}_0)$ -enlargement of  $\mathcal{R}'$ . It has distortion at most  $19 \text{dis}(\mathcal{R}_0)$ . Let  $x$  belong to an edge  $[x_i, x_j]$  of  $\text{c\o{r}e}(G)$  and let  $x'$  be such that  $(x, x') \in \mathcal{R}_0$ . Let  $\gamma_{x', x'_i}$  (resp.  $\gamma_{x', x'_j}$ ) be a geodesic between  $x'$  and  $x'_i$  (resp. between  $x'$  and  $x'_j$ ). Then, let  $\gamma$  be the path from  $x''$  to  $x''_j$  obtained by concatenating  $\gamma_{x', x'_i}$  and  $\gamma_{x', x'_j}$ . We

have

$$\begin{aligned}
\ell'(\gamma) &\leq d'(x''_i, x') + d'(x', x''_j), \\
&\leq d(x_i, x) + d(x, x_j) + 2 \operatorname{dis}(\mathcal{R}'), \\
&= d(x_i, x_j) + 2 \operatorname{dis}(\mathcal{R}'), \\
&\leq \ell'([x''_i, x''_j]) + 3 \operatorname{dis}(\mathcal{R}').
\end{aligned}$$

Thus, if  $\operatorname{dis}(\mathcal{R}') \leq 7 \operatorname{dis}(\mathcal{R}_0) < \frac{\eta}{10}$ , we deduce from (5.6) that  $\gamma$  contains  $[x''_i, x''_j]$ . Thus, defining  $x''$  to be the furthest point to  $x'$  on  $\gamma_{x', x''_i} \cap \gamma_{x', x''_j}$ , we see that  $x''$  belongs to the geodesic  $[x''_i, x''_j]$ . Lemma 5.7 ensures that  $d'(x', x'') \leq 3 \operatorname{dis}(\mathcal{R}')$ . Thus,  $(x, x'') \in \mathcal{R}$ . Similarly, one shows that for every  $x''$  in  $[x''_i, x''_j]$  there is an  $x$  in  $[x_i, x_j]$  such that  $x \in \mathcal{R}$ . We have shown that for each edge  $e$  of  $\operatorname{core}(G)$ ,  $e$  and  $\chi_0(e)$  are in correspondance via  $\mathcal{R}$ .

Now, notice that the multigraph with edge-lengths  $S''$  obtained by keepin only vertices of degree 3 is  $\operatorname{core}(G')$  seen as a multigraph with edge-lengths. The isomorphism  $\chi_0$  induces an isomorphism  $\chi$  between  $\operatorname{core}(G)$  and  $\operatorname{core}(G')$  (by restricting  $\chi_0$  on vertices of degree 3), and we have (since every edge of  $\operatorname{core}(G)$  was divided into five parts):

$$|\ell(e) - \ell'(\chi(e))| \leq 30 \operatorname{dis}(\mathcal{R}_0).$$

Furthermore, the same correspondance  $\mathcal{R}$  as before is suitable to have that for each edge  $e$  of  $\operatorname{core}(G)$ ,  $e$  and  $\chi(e)$  are in correspondance via  $\mathcal{R}$ .

This ends the proof by taking  $\operatorname{dis}(\mathcal{R}_0) \leq \delta$  for  $\delta$  small enough, namely less than  $\frac{\varepsilon}{40} \wedge \frac{\eta}{80}$ .  $\square$

**Lemma 5.9.** *Let  $(G, d)$  and  $(G', d')$  be  $\mathbb{R}$ -graphs and  $\mathcal{R} \in C(G, G')$ . Suppose that  $\operatorname{core}(G)$  and  $\operatorname{core}(G')$  are in correspondance through  $\mathcal{R}$ . Let  $(v, v')$  and  $(x, x') \in \mathcal{R}$  with  $v \in \operatorname{core}(G)$  and  $v' \in \operatorname{core}(G')$ . Then,*

$$d(\alpha_G(x), v) \leq d'(\alpha_{G'}(x'), v') + 5 \operatorname{dis}(\mathcal{R}).$$

*Proof.* Since  $\operatorname{core}(G)$  and  $\operatorname{core}(G')$  are in correspondance through  $\mathcal{R}$ , one may find  $y \in \operatorname{core}(G)$  and  $y' \in \operatorname{core}(G')$  such that:

$$(y, \alpha_{G'}(x')) \in \mathcal{R} \text{ and } (\alpha_G(x), y') \in \mathcal{R}.$$

Let us distinguish two cases.

- $d(y, v) \geq d(\alpha_G(x), v)$ . Then,

$$d(\alpha_G(x), v) \leq d(y, v) \leq d'(\alpha_{G'}(x'), v') + \operatorname{dis}(\mathcal{R})$$

and the result follows.

- $d(y, v) < d(\alpha_G(x), v)$ . Then,

$$\begin{aligned}
d(x, v) &= d(x, \alpha_G(x)) + d(\alpha_G(x), v) \\
&\geq d(x, \alpha_G(x)) + d(y, v) \\
&\geq d'(x', y') + d'(\alpha_{G'}(x'), v') - 2 \operatorname{dis}(\mathcal{R}) \\
&= d'(x', \alpha_{G'}(x')) + d'(\alpha_{G'}(x'), y') + d'(\alpha_{G'}(x'), v') - 2 \operatorname{dis}(\mathcal{R}) \\
&= d'(x', v') + d'(\alpha_{G'}(x'), y') - 2 \operatorname{dis}(\mathcal{R}) \\
&\geq d(x, v) + d'(\alpha_{G'}(x'), y') - 3 \operatorname{dis}(\mathcal{R}).
\end{aligned}$$

Thus,

$$d'(\alpha_{G'}(x'), y') \leq 3 \operatorname{dis}(\mathcal{R})$$

which implies:

$$d(y, \alpha_G(x)) \leq 4 \operatorname{dis}(\mathcal{R})$$

Finally,

$$\begin{aligned}
d(\alpha_G(x), v) &\leq d(\alpha_G(x), y) + d(y, v) \\
&\leq 4 \operatorname{dis}(\mathcal{R}) + d(y, v) \\
&\leq 5 \operatorname{dis}(\mathcal{R}) + d'(\alpha_{G'}(x'), v')
\end{aligned}$$

□

Let us introduce some notations for the following lemmas (see Figure 4).

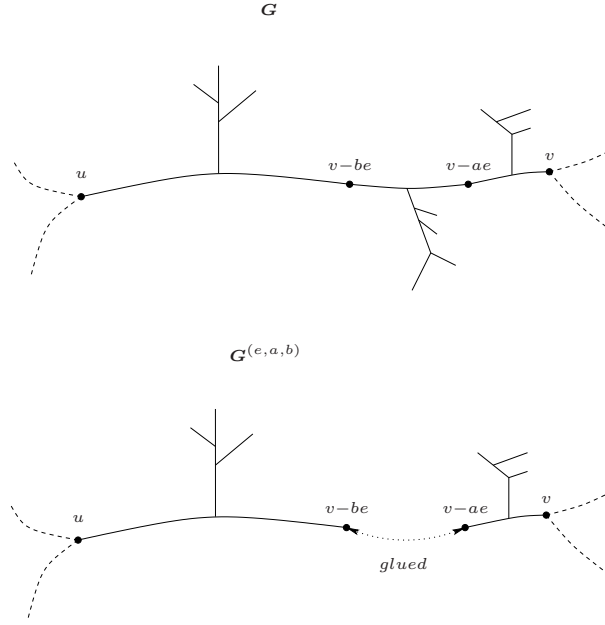


FIG 4.  $\mathbf{G}^{(e,a,b)}$  is the  $(a, b)$ -shortening of  $G$  along  $e = (u, v)$ .

**Definition 5.10.** For any graph  $G$ , for each oriented edge  $e = (u, v) \in \ker(G)$  and each  $\eta \in [0, \ell(e)]$ , we denote by  $v - \eta e$  the point at distance  $\eta$  from  $v$  on the edge  $(u, v)$ , on  $\text{core}(G)$ . For  $a < b \in [0, \ell(e)]$  let  $]v - be, v - ae[$  be the open oriented arc between  $v - be$  and  $v - ae$  in  $(u, v)$ .

We define  $\mathbf{G}^{(e,a,b)}$  the  $(a, b)$ -shortening along  $e$  as the measured  $\mathbb{R}$ -graph  $(H, d_H, \mu_H)$  obtained from  $G$  as follows:

- $H = G \setminus \alpha_G^{-1}(]v - be, v - ae[)$ ,
- $d_H$  is obtained from  $(H, d|_{H \times H})$  by gluing it along  $(v - be, v - ae)$ ,
- $\mu_H$  is the restriction of  $\mu$  on  $H$ .

Notice that  $\mathbf{G}^{(e,a,b)}$  has the same surplus as  $G$ .

**Lemma 5.11.** Let  $G$  be an  $\mathbb{R}$ -graph, and define:

$$\gamma_G(\eta) := \sup_{e=(u,v) \in \ker(G)} \text{diam}(\alpha_G^{-1}(]v - \eta e, v[)).$$

Then,

$$\gamma_G(\eta) \xrightarrow{\eta \rightarrow 0} 0.$$

*Proof.* Suppose on the contrary that  $\gamma_G(\eta) \xrightarrow{\eta \rightarrow 0} \gamma > 0$ . Then, one may find a subsequence of couples  $(x_n, y_n)_{n \in \mathbb{N}}$  in  $\alpha_G^{-1}(]v - e, v[)$  such that:

$$d(\alpha_G(x_n), v) \vee d(\alpha_G(y_n), v) \xrightarrow{n \rightarrow \infty} 0,$$

$$\forall n \in \mathbb{N}, d(x_n, y_n) \geq \gamma,$$

and

$$d(\alpha_G(x_n), v) \wedge d(\alpha_G(y_n), v) > 0.$$

Let  $z_n \in \{x_n, y_n\}$  be such that  $d(z_n, v) = d(x_n, v) \vee d(y_n, v)$ . Up to extracting a subsequence, one may also suppose that  $d(\alpha_G(z_n), v)$  is strictly decreasing and that for any  $n$ ,  $d(\alpha_G(z_n), v) < \gamma/4$ . This implies that for  $n \neq m$ ,

$$\begin{aligned} d(z_n, z_m) &\geq d(z_n, \alpha_G(z_n)) \\ &= d(z_n, v) - d(\alpha_G(z_n), v) \\ &\geq \frac{\gamma}{2} - d(\alpha_G(z_n), v) \\ &\geq \frac{\gamma}{4}. \end{aligned}$$

This contradicts the precompactness of  $G$ . □

**Lemma 5.12.** *Let  $\mathbf{G} = (G, d, \mu)$  be a measured  $\mathbb{R}$ -graph with surplus at least one, let  $e$  be an edge of  $\text{core}(G)$  and  $a < b \in [0, \ell(e)]$ . Let:*

$$\tilde{\gamma}_G(\varepsilon) := \sum_{e=(u,v) \in \ker(G)} \mu(\alpha_G^{-1}(]v - \varepsilon e, v[)).$$

*Then, under the natural coupling between  $\text{Frag}(\mathbf{G}, \cdot)$  and  $\text{Frag}(\mathbf{G}^{(e,a,b)}, \cdot)$  we have, with probability at least  $1 - t(b-a)$ , for any  $s \in [0, t]$ ,*

$$L_{2,GHP}^{\text{surplus}}(\text{Frag}(\mathbf{G}, s), \text{Frag}(\mathbf{G}^{(e,a,b)}, s)) \leq (\tilde{\gamma}_G(b) \vee 3\gamma_G(b))(3 + 4\mu(G))$$

*Proof.* Let  $e = (u, v)$ , and  $\mathcal{P}$  be a Poisson random set of intensity  $\ell_G \times \text{leb}_{\mathbb{R}^+}$  on  $(G, d)$ . Then,  $\mathcal{P}' := \mathcal{P} \setminus \alpha_G^{-1}(]v - be, v - ae[) \times \mathbb{R}^+$  is a Poisson random set of intensity  $\ell_{G'} \times \text{leb}_{\mathbb{R}^+}$  on  $G' \times \mathbb{R}^+$  with  $G' := G \setminus \alpha_G^{-1}(]v - be, v - ae[)$ . Let  $t > 0$  be fixed and let  $\mathcal{E}$  denote the event

$$\mathcal{E} := \{\mathcal{P}_t \cap ]v - be, v - ae[ = \emptyset\},$$

and let us suppose that  $\mathcal{E}$  holds. Let  $\varepsilon > 0$  be such that

$$\varepsilon \geq \mu(\alpha_G^{-1}(]v - be, v - ae[)).$$

Let us take  $s \leq t$  and let  $m$  be a component of  $\text{Frag}(G, \mathcal{P}_s)$ . Then,

- if  $m \subset \alpha_G^{-1}(]v - be, v - ae[)$ , then  $\mu(m) \leq \varepsilon$ ,
- if  $m \cap \alpha_G^{-1}(]v - be, v - ae[) = \emptyset$ , then  $m$  is a component of  $\text{Frag}(G', \mathcal{P}'_s)$ ,
- if  $m \cap \alpha_G^{-1}(]v - be, v - ae[) \neq \emptyset$  but  $m \not\subset \alpha_G^{-1}(]v - be, v - ae[)$ , then  $m$  is the unique component of  $\text{Frag}(G, \mathcal{P}_s)$  which intersects  $]v - be, v - ae[$ , and  $m \setminus \alpha_G^{-1}(]v - be, v - ae[)$  is a component of  $\text{Frag}(G', \mathcal{P}'_s)$ .

This shows that the application

$$\sigma : \begin{cases} \text{comp}(\text{Frag}(G, \mathcal{P}_s)_{>\varepsilon}) & \rightarrow & \text{comp}(\text{Frag}(G', \mathcal{P}'_s)) \\ m & \mapsto & m \setminus \alpha_G^{-1}(]v - be, v - ae[) \end{cases}$$

is well defined and injective. This shows also that the application  $\sigma'$  from  $\text{comp}(\text{Frag}(G', \mathcal{P}'_s)_{>\varepsilon})$  to  $\text{comp}(\text{Frag}(G, \mathcal{P}_s)_{>\varepsilon})$  which maps  $m'$  to the unique  $m$  which contains it is well defined and injective.

Now, let  $m \in \text{comp}(\text{Frag}(G, \mathcal{P}_s)_{>\varepsilon})$  and let

$$m' := \sigma(m) = m \setminus \alpha_G^{-1}(]v - be, v - ae[).$$

Let

$$\mathcal{R}_m := \{(x, x) : x \in m'\} \cup \{(x, v) : x \in m \cap \alpha_G^{-1}(]v - be, v - ae[)\}$$

and  $\pi_m$  be the measure in  $M(m, m')$  defined by:

$$\pi_m(C) = \mu(\{x \in m' : (x, x) \in C\}).$$

Let  $d'$  be the distance on  $m'$ . Notice that for any  $x, y$  in  $m'$ ,

$$|d(x, y) - d'(x, y)| \leq b$$

Thus,

$$\text{dis}(\mathcal{R}_m) \leq b + 2 \text{diam}(\alpha_G^{-1}([v - be, v - ae])) \leq 3\gamma_G(b).$$

Also,

$$\pi_m(\mathcal{R}_m^c) = \mu|_{m'}(\{x \in m' : (x, x) \in \mathcal{R}_m^c\}) = 0,$$

For  $A$  a Borel subset of  $m$ ,

$$\pi(A \times m') = \mu(A \cap m')$$

and for  $A'$  a Borel subset of  $m'$ ,

$$\pi(m \times A') = \mu(A')$$

Thus,

$$D(\pi; \mu|_m, \mu|_{m'}) \leq \mu(\alpha_G^{-1}([v - be, v - ae])) \leq \tilde{\gamma}_G(b).$$

Using Lemma 2.19 with

$$\alpha = \tilde{\gamma}_G(b) \vee 3\gamma_G(b)$$

and

$$\varepsilon = \sqrt{\alpha} + \alpha$$

we have shown that as soon as  $\mathcal{E}$  holds, for any  $s \in [0, t]$ ,

$$\begin{aligned} & L_{GHP}(\text{Frag}(\mathbf{G}, \mathcal{P}_s), \text{Frag}(\mathbf{G}^{(e,a,b)}, \mathcal{P}'_s)) \\ & \leq 17(\tilde{\gamma}_G(b) \vee 3\gamma_G(b)) + (16 + \mu(G))\sqrt{\tilde{\gamma}_G(b) \vee 3\gamma_G(b)}. \end{aligned}$$

Furthermore,

$$\|\text{masses}(\text{Frag}(\mathbf{G}, \mathcal{P}_s)) - \text{masses}(\mathbf{G}^{(e,a,b)}, \mathcal{P}'_s)\|_2^2 \leq 2\tilde{\gamma}_G(b)^2.$$

Also, for any  $m$  in  $\text{comp}(\text{Frag}(G, \mathcal{P}_s)_{>\varepsilon})$ ,  $m$  and  $\sigma(m)$  have the same surplus (recall the gluing in Definition 5.10). The same is true for  $m'$  and  $\sigma'(m')$ . Thus,

$$\begin{aligned} & L_{2,GHP}^{\text{surplus}}(\text{Frag}(\mathbf{G}, \mathcal{P}_s), \text{Frag}(\mathbf{G}^{(e,a,b)}, \mathcal{P}'_s)) \\ & \leq \left[ 17(\tilde{\gamma}_G(b) \vee 3\gamma_G(b)) + (16 + \mu(G))\sqrt{\tilde{\gamma}_G(b) \vee 3\gamma_G(b)} \right] \vee 2\tilde{\gamma}_G(b)^2. \end{aligned}$$

Finally, notice that  $\mathcal{E}$  has probability at least  $\exp^{-t(b-a)} \geq 1 - t(b-a)$ .  $\square$

Now, we shall prove Proposition 5.5. Let us explain the idea of the proof. If  $G^{(n)}$  is close enough from  $G$ , Lemma 5.8 shows that their cores are homomorphic multigraphs with edges having almost the same length. One may then shorten some edges of the core of  $G$  and other edges of the core of  $G^{(n)}$  in such a way that the two cores become homeomorphic as metric spaces with a length measure. Lemma 5.12 shows that one does not lose too much doing this. Finally, Lemma 5.6 shows then that the fragmentation on the two graphs are close.

*Proof.* (of Proposition 5.5)

The argument at the beginning of the proof of Proposition 5.4 shows that it is sufficient to prove the result when  $G^{(n)}$  and  $G$  have a single component. Let  $\mathbf{G} = (G, d, \mu)$  be a measured  $\mathbb{R}$ -graph and let  $\varepsilon > 0$ . We want to show that  $\text{Frag}(\mathbf{G}', t)$  converges in distribution to  $\text{Frag}(\mathbf{G}, t)$  when  $\mathbf{G}'$  converges to  $\mathbf{G}$  while having the same surplus.

Let  $\delta < \delta(\varepsilon, G)$  be given by Lemma 5.8 and let  $\mathbf{G}'$  be such that  $d_{GHP}(\mathbf{G}, \mathbf{G}') < \delta$  (we will take  $\delta$  small enough later). Thus, there is a correspondance  $\mathcal{R}_0 \in C(G, G')$  and a measure  $\pi_0 \in M(G, G')$  such that:

$$\text{dis}(\mathcal{R}_0) \vee \pi_0(\mathcal{R}_0^c) \vee D(\pi_0; \mu, \mu') < \delta$$

Lemma 5.8 shows that there exists an  $\varepsilon$ -overlay  $\mathcal{R} \in C(G, G')$  containing  $\mathcal{R}_0$ . Let us denote by  $\chi$  the multigraph isomorphism from  $\ker(G)$  to  $\ker(G')$  given by this overlay. For any edge  $e \in \ker(G)$ ,  $|\ell(e) - \ell'(\chi(e))| < \varepsilon$ .

We define two graphs  $\tilde{G}$  and  $\tilde{G}'$  obtained from  $G$  and  $G'$  as follows. For each oriented edge  $e = (u, v) \in \ker(G)$ , denoting  $(u', v') = \chi(e)$ ,

- if  $\ell(e)$  is smaller than  $\ell'(e')$  by an amount  $\eta$ , we replace  $G'$  by its  $(6\varepsilon - \eta, 6\varepsilon)$ -shortening along  $e'$  (cf. Definition 5.10),
- if  $\ell'(e')$  is smaller than  $\ell(e)$  by an amount  $\eta$ , we replace  $G$  by its  $(6\varepsilon - \eta, 6\varepsilon)$ -shortening along  $e$ .

Let us denote by  $(\tilde{G}, \tilde{d})$  and  $(\tilde{G}', \tilde{d}')$  the resulting  $\mathbb{R}$ -graphs, let  $\tilde{\mu} := \mu|_{\tilde{G}}$ ,  $\tilde{\mu}' := \mu'|_{\tilde{G}'}$ , and define  $\tilde{G} := (\tilde{G}, \tilde{d}, \tilde{\mu})$ ,  $\tilde{G}' := (\tilde{G}', \tilde{d}', \tilde{\mu}')$ .

Recalling the notation in Lemma 5.11, let

$$\kappa := \gamma_G(11\varepsilon) + 12\varepsilon$$

and define  $\mathcal{R}_1$  the  $\kappa$ -enlargement of  $\mathcal{R}$ . We will show that

$$\tilde{G} \text{ and } \tilde{G}' \text{ are in correspondance through } \mathcal{R}_1. \quad (5.7)$$

If  $x \in \tilde{G}$  and  $(x, x') \in \mathcal{R}$  with  $x' \notin \tilde{G}'$ , then,  $x' \in \alpha_{G'}^{-1}(|v' - 6\varepsilon e', v' - (6\varepsilon - \eta)e'|)$  for some edge  $e' = (u', v')$  of  $\ker(G')$  and  $\eta < \varepsilon$ . Lemma 5.9 shows that

$$0 < 6\varepsilon - \eta - 5 \operatorname{dis}(\mathcal{R}) \leq d(\alpha_G(x), v) \leq 6\varepsilon + 5 \operatorname{dis}(\mathcal{R}) \leq 11\varepsilon. \quad (5.8)$$

and thus

$$d(x, v) \leq \gamma_G(11\varepsilon) + 11\varepsilon.$$

Thus,

$$d'(x', v') \leq \gamma_G(11\varepsilon) + 12\varepsilon \leq \kappa. \quad (5.9)$$

This shows that  $(x, v') \in \mathcal{R}_1$ . Now, let  $x' \in \tilde{G}'$  and  $(x, x') \in \mathcal{R}$  with  $x \notin \tilde{G}$ . Then,  $x \in \alpha_G^{-1}(|v - 6\varepsilon e, v - (6\varepsilon - \eta)e'|)$  for some edge  $e = (u, v)$  of  $\ker(G)$  and  $\eta < \varepsilon$ . Notice that:

$$d(x, v) \leq \gamma_G(6\varepsilon) + 6\varepsilon,$$

and

$$d'(x', v') \leq d(x, v) + \operatorname{dis}(\mathcal{R}) \leq \gamma_G(6\varepsilon) + 7\varepsilon \leq \kappa.$$

Thus  $(v, x') \in \mathcal{R}_1$ . This ends the proof of (5.7).

Notice that

$$\operatorname{dis}(\mathcal{R}_1) \leq \varepsilon + 4\kappa.$$

Let  $\mathcal{R}_2 := \mathcal{R}_1|_{\tilde{G} \times \tilde{G}'} \in C(\tilde{G}, \tilde{G}')$ . Let  $K$  be the number of edges in  $\ker(G)$ . Notice that

$$\forall (x, y) \in \tilde{G}, |d(x, y) - \tilde{d}(x, y)| < K\varepsilon$$

and

$$\forall (x', y') \in \tilde{G}', |d'(x', y') - \tilde{d}'(x', y')| < K\varepsilon.$$

Thus,

$$\operatorname{dis}(\mathcal{R}_2) < (K + 13)\varepsilon + 4\gamma_G(11\varepsilon). \quad (5.10)$$

Clearly, there exists a homeomorphism  $\psi$  from  $\operatorname{core}(\tilde{G})$  to  $\operatorname{core}(\tilde{G}')$  which preserves the length-measure. For each oriented edge  $e = (u, v) \in \ker(G)$ , denoting  $(u', v') = \chi(e)$ ,  $\psi$  satisfies  $\psi(v) = v'$ . Furthermore, since  $e$  and  $e'$  are in correspondance through the overlay  $\mathcal{R}$ , we have, for each  $x \in [u, v]$ , that there exists  $x' \in [u', v']$  such that:

$$|d(x, u) - d'(x', u')| < \varepsilon.$$

If furthermore  $x \in \text{core}(\tilde{G})$ , we know that  $d(x, u) = d'(\psi(x), u')$ , so

$$|d'(\psi(x), u') - d'(x', u')| < \varepsilon .$$

Since  $x'$  and  $\psi(x)$  belong to  $[u', v']$ ,

$$d'(\psi(x), x') = |d'(\psi(x), u') - d'(x', u')| < \varepsilon ,$$

which shows that for every  $x \in \text{core}(\tilde{G})$ ,

$$(x, \psi(x)) \text{ belongs to } \mathcal{R}_2 , \quad (5.11)$$

the restriction to  $\tilde{G} \times \tilde{G}'$  of the  $\kappa$ -enlargement of  $\mathcal{R}$ .

Now, let  $\pi := \pi_0|_{\tilde{G} \times \tilde{G}'} \in M(\tilde{G}, \tilde{G}')$ . First,

$$\pi(\mathcal{R}_2^c) = \pi(\mathcal{R}_1^c) \leq \pi_0(\mathcal{R}_0^c) < \varepsilon . \quad (5.12)$$

Then,

$$D(\pi; \tilde{\mu}, \tilde{\mu}') \leq 2D(\pi_0; \mu, \mu') + \mu(G \setminus \tilde{G}) \vee \mu'(G' \setminus \tilde{G}') .$$

Now, define

$$\tilde{\gamma}_G(\varepsilon) := \sum_{e=(u,v) \in \ker(G)} \mu(\alpha_G^{-1}(\lceil v - \varepsilon e, v \rceil))$$

which goes to zero as  $\varepsilon$  goes to zero. We have

$$\mu(G \setminus \tilde{G}) \leq \tilde{\gamma}_G(6\varepsilon) .$$

Furthermore, recall inequality 5.8 which shows that if  $x' \in G' \setminus \tilde{G}'$ , then for every  $x \in G$  such that  $(x, x') \in \mathcal{R}$ ,

$$x \in \bigcup_{e=(u,v) \in \ker(G)} \alpha_G^{-1}(\lceil v - 11\varepsilon e, v \rceil) .$$

Thus,

$$\begin{aligned} \mu'(G' \setminus \tilde{G}') &\leq \pi_0(G \times (G' \setminus \tilde{G}')) + D(\pi_0; \mu, \mu') \\ &\leq \pi_0((G \times (G' \setminus \tilde{G}')) \cap \mathcal{R}) + \pi_0(\mathcal{R}^c) + \varepsilon \\ &\leq \pi_0 \left( \bigcup_{e=(u,v) \in \ker(G)} \alpha_G^{-1}(\lceil v - 11\varepsilon e, v \rceil) \times G' \right) + 2\varepsilon \\ &\leq \mu \left( \bigcup_{e=(u,v) \in \ker(G)} \alpha_G^{-1}(\lceil v - 11\varepsilon e, v \rceil) \right) + D(\pi_0; \mu, \mu') + 2\varepsilon \\ &\leq \tilde{\gamma}_G(11\varepsilon) + 3\varepsilon . \end{aligned} \quad (5.13)$$

Thus,

$$D(\pi; \tilde{\mu}, \tilde{\mu}') \leq 5\varepsilon + \tilde{\gamma}_G(11\varepsilon) . \quad (5.14)$$

Gathering (5.11), (5.10), (5.12) and (5.14) shows that one may apply Lemma 5.6, in the sense that there is a function  $f_G(\varepsilon)$  going to zero as  $\varepsilon$  goes to zero such that the Lévy-Prokhorov distance (for the topology of compact convergence associated to  $L_{2,GHP}^{surplus}$ ) between the distributions of  $(\text{Frag}(\tilde{G}, s))_{s \in [0, t]}$  and  $(\text{Frag}(\tilde{G}', s))_{s \in [0, t]}$  is less than  $f_G(\varepsilon)$ .

On the other hand, inequality (5.9) shows that

$$\gamma_{G'}(\varepsilon) \leq 2[\tilde{\gamma}_G(11\varepsilon) + 12\varepsilon]$$



and inequality (5.13) shows that

$$\tilde{\gamma}_{G'}(\varepsilon) \leq \tilde{\gamma}_G(11\varepsilon) + 3\varepsilon.$$

Then, Lemma 5.12 shows that there is a function  $f_G(\varepsilon)$  going to zero as  $\varepsilon$  goes to zero such that the Lévy-Prokhorov distance between the distributions of  $(\text{Frag}(G, s))_{s \in [0, t]}$  and  $(\text{Frag}(\tilde{G}, s))_{s \in [0, t]}$  is less than  $f_G(\varepsilon)$  and the Lévy-Prokhorov distance (for the topology of compact convergence associated to  $L_{2, GHP}^{\text{surplus}}$ ) between the distributions of  $(\text{Frag}(G', s))_{s \in [0, t]}$  and  $\text{Frag}(\tilde{G}', s)_{s \in [0, t]}$  is less than  $f_G(\varepsilon)$ . This ends the proof of Proposition 5.5.  $\square$

### 5.5. Application to Erdős-Rényi random graphs

In this section, we prove Theorem 3.2. Let us first compare the discrete fragmentation process and the continuous one. Let  $\mathcal{P}^-$  be a Poisson process driving the discrete fragmentation on  $G^{(n)} := G(n, p(\lambda, n))$ . Recall that  $N^-(G^{(n)}, \mathcal{P}_t^-)$  for the state of this process at time  $t$ , seen as member of  $\mathcal{N}_2^{\text{graph}}$ . Let  $\mathcal{Q}^-$  be a Poisson process of intensity  $\ell_n \otimes \text{leb}_{\mathbb{R}^+}$  on  $K_n \times \mathbb{R}^+$  where  $K_n$  is the complete graph on  $n$  vertices seen as an  $\mathbb{R}$ -graph where the edge lengths are  $\delta_n = n^{-1/3}$  and  $\ell_n$  is its length measure. Then, one may suppose that  $\mathcal{P}^-$  is obtained as follows:

$$\mathcal{P}^- = \{(e, t) : \exists x \in K_n, (x, t) \in \mathcal{Q}^-\}.$$

Then, for any  $t$ ,  $N^-(G^{(n)}, \mathcal{P}_t^-)$  is at  $L_{2, GHP}$ -distance at most  $n^{-1/3}$  from  $\text{Frag}(G^{(n)}, \mathcal{Q}_t^-)$  (cf. for instance Propositions 3.4 in [3]). Recall that by Theorem 2.38,  $\bar{\mathcal{G}}_{n, \lambda}$  (which is  $G^{(n)}$  with edge length  $\delta_n$  and vertex weights  $n^{-2/3}$ ) converges in distribution to  $\mathcal{G}_\lambda$  for  $L_{2, GHP}^{\text{surplus}}$ . Thus Proposition 5.5 implies that  $(\text{Frag}(G^{(n)}, \mathcal{Q}_t^-))_{t \geq 0}$ , and thus  $(N^-(G^{(n)}, \mathcal{P}_t^-))_{t \geq 0}$  converges to  $(\text{Frag}(\mathcal{G}_\lambda, t))_{t \geq 0}$  as  $n$  goes to infinity (in the topology of compact convergence associated to  $L_{2, GHP}^{\text{surplus}}$ ). This shows Theorem 3.2.

An interesting consequence of this result is the fact that on  $(\mathcal{G}_\lambda)_{\lambda \in \mathbb{R}}$ , fragmentation is the time-reversal of coalescence.

**Proposition 5.13.** *For any  $\lambda \in \mathbb{R}$  and  $s \in \mathbb{R}^+$ ,  $(\mathcal{G}_\lambda, \text{Coal}(\mathcal{G}_\lambda, s))$  and  $(\text{Frag}(\mathcal{G}_{\lambda+s}, s), \mathcal{G}_{\lambda+s})$  have the same distribution.*

*Proof.* Take  $\mathcal{P}_t^+$  of intensity  $\gamma = n^{-4/3}$ . Notice that the states of the edges are independent and identically distributed in  $(\mathcal{G}(n, p), N^+(\mathcal{G}(n, p), \mathcal{P}_t^+))$ . Let  $(X, Y)$  be the joint distribution of the state of one edge. Denoting by 0 the state “absent” and 1 the state “present”, it is easy to compute this distribution:

$$\begin{aligned} \mathbb{P}((X, Y) = (0, 0)) &= (1-p)e^{-\gamma t} & \mathbb{P}((X, Y) = (0, 1)) &= (1-p)(1-e^{-\gamma t}) \\ \mathbb{P}((X, Y) = (1, 0)) &= 0 & \mathbb{P}((X, Y) = (1, 1)) &= p \end{aligned}$$

Now, take  $\mathcal{P}_t^-$  of intensity  $\mu = n^{-1/3}$  and let  $(X', Y')$  be the joint distribution of the state of one edge in  $(N^-(G(n, p'), \mathcal{P}_t^-), G(n, p'))$ . Then,

$$\begin{aligned} \mathbb{P}((X', Y') = (0, 0)) &= (1-p') & \mathbb{P}((X', Y') = (0, 1)) &= p'(1-e^{-\mu t'}) \\ \mathbb{P}((X', Y') = (1, 0)) &= 0 & \mathbb{P}((X', Y') = (1, 1)) &= p'e^{-\mu t'} \end{aligned}$$

Thus, if one chooses

$$t = \frac{1}{\gamma} \ln \frac{1-p}{1-p'} \quad \text{and} \quad t' = \frac{1}{\mu} \ln \frac{p'}{p},$$

then  $(\mathcal{G}(n, p), N^+(\mathcal{G}(n, p), \mathcal{P}_t^+))$  and  $(N^-(G(n, p'), \mathcal{P}_{t'}^-), G(n, p'))$  have the same distribution. Now, take  $p = p(\lambda, n)$ ,  $p' = p(\lambda + s, n)$ . We have:

$$t = n^{4/3} \ln \left( 1 + \frac{s}{n^{4/3}(1-p')} \right) \xrightarrow{n \rightarrow \infty} s.$$

We consider that  $\mathcal{G}(n, p)$  is equipped with edge lengths  $n^{-1/3}$  and vertex weight  $n^{-2/3}$ . Thus Theorem 3.1 shows that  $(\mathcal{G}(n, p), N^+(\mathcal{G}(n, p), \mathcal{P}_t^+))$  converges in distribution to  $(\mathcal{G}_\lambda, \text{Coal}(\mathcal{G}_\lambda, s))$ . Also,

$$t' = n^{1/3} \ln \frac{1 + \frac{\lambda+s}{n^{1/3}}}{1 + \frac{\lambda}{n^{1/3}}} \xrightarrow{n \rightarrow \infty} s$$

thus Theorem 3.2 and Proposition 5.2 show that  $(N^-(G(n, p'), \mathcal{P}_{t'}^-), G(n, p'))$  converges in distribution to  $(\text{Frag}(\mathcal{G}_{\lambda+s}, s), \mathcal{G}_{\lambda+s})$ . Thus  $(\mathcal{G}_\lambda, \text{Coal}(\mathcal{G}_\lambda, s))$  and  $(\text{Frag}(\mathcal{G}_{\lambda+s}, s), \mathcal{G}_{\lambda+s})$  have the same distribution.  $\square$

Notice a curious fact: in [6], Theorem 3, it is shown that the sizes of the components of a fragmentation on the CRT are the time-reversal (after an exponential time-change) of the standard *additive* coalescent. It would be interesting to make a direct link between additive and multiplicative coalescent in the context of fragmentation on  $\mathcal{G}_\lambda$ .

## 6. Combining fragmentation and coalescence: dynamical percolation

### 6.1. Almost Feller Property

For an  $\mathbb{R}$ -graph  $G$  with length measure  $\ell_G$ , define

$$\text{suplength}(G) := \sup\{\ell_G(\gamma) : \gamma \text{ injective path in } G\}.$$

For a member  $\mathbf{G}$  of  $\mathcal{N}^{graph}$ , we let

$$\text{suplength}(\mathbf{G}) := \sup_{m \in \text{comp}(\mathbf{G})} \text{suplength}(m).$$

The following lemma is a simple variation on the proof of (4.4).

**Lemma 6.1.** *Let  $\mathbf{X}^n = (X^n, d^n, \mu^n)$ ,  $n \geq 0$  be a sequence of random variables in  $\mathcal{N}_2^{graph}$  and  $(\delta^n)_{n \geq 0}$  be a sequence of non-negative real numbers. Suppose that:*

- (i)  $(\mathbf{X}^n)$  converges in distribution (for  $L_{2,GHP}$ ) to  $\mathbf{X}^\infty = (X^\infty, d^\infty, \mu^\infty)$  as  $n$  goes to infinity
- (ii)  $\delta^n \xrightarrow[n \rightarrow \infty]{} \delta$
- (iii) For any  $\alpha > 0$  and any  $T > 0$ ,

$$\limsup_{n \in \mathbb{N}} \mathbb{P}(\text{suplength}(\text{Coal}_{\delta^n}(\mathbf{X}_{\leq \varepsilon}^n, T)) > \alpha) \xrightarrow[\varepsilon \rightarrow 0]{} 0$$

Then, for any  $\alpha > 0$  and any  $T > 0$ ,

$$\mathbb{P}(\text{suplength}(\text{Coal}_\delta(\mathbf{X}_{\leq \varepsilon}^\infty, T)) > \alpha) \xrightarrow[\varepsilon \rightarrow 0]{} 0.$$

*Proof.* The situation is simpler than in the proof of (4.4), since  $\text{suplength}$  is non-decreasing under coalescence. Using the notations of the proof of (4.4),

$$\begin{aligned} & \mathbb{P}(\text{suplength}(\text{Coal}_0(\mathbf{X}_{\leq \varepsilon_m}^\infty, T)) > \alpha) \\ &= \lim_{p \rightarrow \infty} \mathbb{P}(\text{suplength}(\text{Coal}_0(\mathbf{X}_{m,p}^\infty, T)) > \alpha) \end{aligned}$$

Now, Proposition 4.3 implies that  $(\text{Coal}_{\delta^n}(\mathbf{X}_{m,p}^n, T))$  converges in distribution to  $(\text{Coal}_0(\mathbf{X}_{m,p}^\infty, T))$  for any  $m \leq p$ . Thus, for any  $m \leq p$ ,

$$\begin{aligned} & \mathbb{P}(\text{suplength}(\text{Coal}_0(\mathbf{X}_{m,p}^\infty, T)) > \alpha) \\ & \leq \limsup_{n \rightarrow \infty} \mathbb{P}(\text{suplength}(\text{Coal}_{\delta^n}(\mathbf{X}_{m,p}^n, T)) > \alpha) \\ & \leq \limsup_{n \rightarrow \infty} \mathbb{P}(\text{suplength}(\text{Coal}_{\delta^n}(\mathbf{X}_{\leq \varepsilon_m}^n, T)) > \alpha), \end{aligned}$$

which goes to zero when  $m$  goes to infinity.  $\square$

**Proposition 6.2.** *Let  $(\mathbf{X}^{(n)})_{n \geq 0}$  be a sequence of random variables in  $\mathcal{N}_2^{graph}$  converging in distribution to  $\mathbf{X}^{(\infty)}$  in the  $L_{2,GHP}^{surplus}$  metric. Suppose also that for any  $\alpha > 0$  and any  $T \geq 0$ ,*

$$\lim_{\varepsilon_1 \rightarrow 0} \limsup_{n \rightarrow +\infty} \mathbb{P}(\text{suplength}(\text{Coal}_0(\mathbf{X}_{\leq \varepsilon_1}^{(n)}, T)) > \alpha) = 0. \quad (6.1)$$

*Then, the sequence of processes  $\text{CoalFrag}(\mathbf{X}^{(n)}, \cdot)$  converges in distribution to  $\text{CoalFrag}(\mathbf{X}^{(\infty)}, \cdot)$  in the topology of compact convergence associated to  $L_{2,GHP}$ .*

*Proof.* First, we will reduce the problem on  $\mathcal{N}_1^{graph}$  using a variation on the proof of Lemma 4.13. Let us study first  $\text{Frag}(\text{Coal}_{\delta^n}(\mathbf{X}^{(n)}, \mathcal{P}_t^+), \mathcal{P}_t^-)$  with  $\mathcal{P}^+$  and  $\mathcal{P}^-$  as in Definitions 2.24 and 2.34. Let us fix  $\varepsilon > 0$  and  $0 \leq t \leq T$ . Any component of size at least  $\varepsilon$  in  $\text{Frag}(\text{Coal}_{\delta^n}(\mathbf{X}^{(n)}, \mathcal{P}_t^+), \mathcal{P}_t^-)$  has to belong to a component of size at least  $\varepsilon$  in  $\text{Coal}_{\delta^n}(\mathbf{X}^{(n)}, \mathcal{P}_t^+)$ . Let  $x^n := \text{masses}(\mathbf{X}^n)$  for  $n \in \overline{\mathbb{N}}$ .

As in the proof of Lemma 4.13, we obtain that there exists  $K(\varepsilon)$ ,  $\varepsilon_1 \in ]0, \varepsilon[$  and  $\varepsilon_2 \in ]0, \varepsilon_1[$  such that for every  $n \in \overline{\mathbb{N}}$ , with probability larger than  $1 - \varepsilon$  the event  $\mathcal{A}_n$  holds, where  $\mathcal{A}_n$  is the event that points (a), (b) and (c) of Corollary 4.6 hold for any  $t \in [0, T]$  and  $S(x^n, T) \leq K(\varepsilon)$ .

Let us place ourselves on  $\mathcal{A}_n$ . Then, for a significant component at time  $t$ , notice that fragmentation on the hanging trees of components does change neither the mass neither the distance in the heart of a component. Thus, the same proof as that of Lemma 4.13 shows that on  $\mathcal{A}_n$ , we have for every time  $t \leq T$  and every  $\varepsilon'_2 \leq \varepsilon_2$ :

$$\begin{aligned} & L_{GHP}(\text{Frag}(\text{Coal}_{\delta^n}(\mathbf{X}^n, \mathcal{P}_t^+), \mathcal{P}_t^-), \text{Frag}(\text{Coal}_{\delta^n}(\mathbf{X}_{>\varepsilon'_2}^n, \mathcal{P}_t^+), \mathcal{P}_t^-)) \\ & \leq 17(\delta^n + \text{supdiam}(\text{Frag}(\text{Coal}_{\delta^n}(\mathbf{X}_{\leq\varepsilon_1}^n, \mathcal{P}_T^+), \mathcal{P}_T^-)) + \varepsilon_1) \left(1 + \frac{8K(\varepsilon)}{\varepsilon^2}\right) \\ & \quad + 16\varepsilon, \end{aligned}$$

for some constant  $C$ . A slight difference occurs here:

$$t \mapsto \text{supdiam}(\text{Frag}(\text{Coal}_{\delta^n}(\mathbf{X}_{\leq\varepsilon_1}^n, \mathcal{P}_T^+), \mathcal{P}_T^-))$$

is not necessarily nonincreasing. However, the supremum of the lengths of injective paths clearly decreases (non-strictly) under fragmentation. Thus, on  $\mathcal{A}_n$ ,

$$\begin{aligned} & L_{GHP}(\text{Frag}(\text{Coal}_{\delta^n}(\mathbf{X}^n, \mathcal{P}_t^+), \mathcal{P}_t^-), \text{Frag}(\text{Coal}_{\delta^n}(\mathbf{X}_{>\varepsilon'_2}^n, \mathcal{P}_t^+), \mathcal{P}_t^-)) \\ & \leq 17(\delta^n + \text{suplength}(\text{Coal}_{\delta^n}(\mathbf{X}_{\leq\varepsilon_1}^n, \mathcal{P}_T^+)) + \varepsilon_1) \left(1 + \frac{8K(\varepsilon)}{\varepsilon^2}\right) + 16\varepsilon. \end{aligned}$$

Let  $V = \text{comp}(\text{Frag}(\mathbf{X}^n, \mathcal{P}_t^-))$  and  $W = \text{comp}(\text{Frag}(\mathbf{X}_{>\varepsilon'_2}^n, \mathcal{P}_t^-)) \subset V$ . Let  $E'$  denote the set of edges on  $V$  such that  $i \sim j$  if and only if  $i$  and  $j$  are at finite distance in  $\text{Frag}(\text{Coal}_{\delta^n}(\mathbf{X}^n, \mathcal{P}_t^+), \mathcal{P}_t^-)$ . Let  $E$  denote the set of edges on  $V$  such that  $i \sim j$  if and only if  $i$  and  $j$  are at finite distance in  $\text{Coal}_{\delta^n}(\mathbf{X}^n, \mathcal{P}_t^+)$ . Define:

$$x^n(t) := \text{masses}(\text{Frag}(\text{Coal}_{\delta^n}(\mathbf{X}^n, \mathcal{P}_t^+), \mathcal{P}_t^-))$$

and

$$x_{>\varepsilon}^n(t) := \text{masses}(\text{Frag}(\text{Coal}_{\delta^n}(\mathbf{X}_{>\varepsilon}^n, \mathcal{P}_t^+), \mathcal{P}_t^-)).$$

Lemma 2.4 shows that

$$\begin{aligned} & \|x^{(n)}(t) - x_{>\varepsilon'_2}^{(n)}(t)\|_2^2 \\ & \leq \|x^{(n)}(t)\|_2^2 - \|x_{>\varepsilon'_2}^{(n)}(t)\|_2^2 \end{aligned}$$

then, using Lemma 2.5,

$$\begin{aligned} & \|x^{(n)}(t) - x_{>\varepsilon'_2}^{(n)}(t)\|_2^2 \\ & \leq S(x^{(n)}, t) - S(x_{>\varepsilon_2}^{(n)}, t) \\ & \leq \varepsilon \end{aligned}$$

since (c) of Corollary 4.6 holds on  $\mathcal{A}_n$ . Now, let us take  $\delta_n = 0$ . Define:

$$\mathbf{X}^n(t) := \text{Frag}(\text{Coal}_0(\mathbf{X}^n, \mathcal{P}_t^+), \mathcal{P}_t^-) = \text{CoalFrag}(\mathbf{X}^n, t)$$

and

$$\mathbf{X}_{>\varepsilon_1}^n(t) := \text{Frag}(\text{Coal}_0(\mathbf{X}_{>\varepsilon_1}^n, \mathcal{P}_t^+), \mathcal{P}_t^-) = \text{CoalFrag}(\mathbf{X}_{>\varepsilon_1}^n, t)$$

Using the hypothesis on suplength and Lemma 6.1, we get that for any  $\varepsilon > 0$ ,

$$\lim_{\varepsilon_1 \rightarrow 0} \sup_{n \in \mathbb{N}} \mathbb{P} \left[ \sup_{t \in [0, T]} L_{2, GHP}(\mathbf{X}^n(t), \mathbf{X}_{> \varepsilon_1}^n(t)) > \varepsilon \right] = 0. \quad (6.2)$$

Thus, it is sufficient to show the Theorem for  $\mathbf{X}^n$  converging to  $\mathbf{X}$  in  $L_{1, GHP}$  with  $\mathbf{X}^n$  and  $\mathbf{X}$  being m.s.m.s with a finite number of finite components which are finite  $\mathbb{R}$ -graphs. We shall only sketch the proof, since it is a variation on the arguments of the proofs of Propositions 4.4 and 5.5. For any  $n$  large enough, the proof of Proposition 5.5 show that one may couple a Poisson process  $\mathcal{P}^{-,n}$  on  $X^n \times \mathbb{R}^+$  with intensity measure  $\ell_{X^n} \otimes \text{leb}_{\mathbb{R}^+}$  with a Poisson process  $\mathcal{P}^-$  on  $X \times \mathbb{R}^+$  with intensity  $\ell_X \otimes \text{leb}_{\mathbb{R}^+}$  and one may find  $\pi^n \in M(X, X^n)$  and  $\mathcal{R}^n \in C(X, X^n)$  such that there is an event  $\mathcal{E}_n$  in the  $\sigma$ -algebra of  $(\mathcal{P}_t^{-,n}, \mathcal{P}_t^-)$  and a sequence  $\varepsilon_n$  such that:

- (i)  $\mathbb{P}(\mathcal{E}_n^c) \leq \varepsilon_n$
- (ii)  $\varepsilon_n \xrightarrow{n \rightarrow \infty} 0$
- (iii) on  $\mathcal{E}_n$  for any  $s \leq t$ ,  $\mathcal{R}^n \cap (X \setminus \mathcal{P}_s^-) \times (X^n \setminus \mathcal{P}_s^{-,n}) \in C(X \setminus \mathcal{P}_s^-, X^n \setminus \mathcal{P}_s^{-,n})$  and

$$D(\pi|_{(X \setminus \mathcal{P}_s^-) \times (X^n \setminus \mathcal{P}_s^{-,n}); \mu|_{X \setminus \mathcal{P}_s^-}, \mu_{X^n \setminus \mathcal{P}_s^{-,n}}}) \vee \pi^n((\mathcal{R}^n)^c) \vee \text{dis}_s(\mathcal{R}^n) \leq \varepsilon_n$$

where  $\text{dis}_s$  is the distortion of  $\mathcal{R}$  as a correspondance between the semi-metric spaces  $\text{Frag}(X, \mathcal{P}_s^-)$  and  $\text{Frag}(X^n, \mathcal{P}_s)$ .

Then, one may use the proof of Lemma 4.2 to couple a Poisson process  $\mathcal{P}^{+,n}$  on  $(X^n)^2 \times \mathbb{R}^+$  with intensity measure  $\frac{1}{2}(\mu^n)^\otimes \otimes \text{leb}_{\mathbb{R}^+}$  with a Poisson process  $\mathcal{P}^+$  on  $(X^n)^2 \times \mathbb{R}^+$  with intensity  $\frac{1}{2}(\mu^n)^\otimes \otimes \text{leb}_{\mathbb{R}^+}$  in such a way that there is an event  $\mathcal{E}'_n$ , a sequence  $\varepsilon'_n$  such that:

- (i)  $\mathbb{P}(\mathcal{E}'_n^c) \leq \varepsilon_n$
- (ii)  $\varepsilon_n \xrightarrow{n \rightarrow \infty} 0$
- (iii) on  $\mathcal{E}'_n$ , for any  $s \leq t$ ,

$$D(\pi|_{(X \setminus \mathcal{P}_s^-) \times (X^n \setminus \mathcal{P}_s^{-,n}); \mu|_{X \setminus \mathcal{P}_s^-}, \mu_{X^n \setminus \mathcal{P}_s^{-,n}}}) \vee \pi^n((\mathcal{R}^n)^c) \vee \text{dis}'_s(\mathcal{R}^n) \leq \varepsilon'_n$$

where  $\text{dis}'_s$  is the distortion of  $\mathcal{R}$  as a correspondance between the semi-metric spaces  $\text{Coal}(\text{Frag}(X, \mathcal{P}_s^-), \mathcal{P}_s^+)$  and  $\text{Coal}(\text{Frag}(X^n, \mathcal{P}_s), \mathcal{P}_s^+)$ .

Using Lemma 2.20, this ends the proof of the convergence in the sense of  $L_{GHP}$ . Convergence of the sizes in  $L^1$  is disposed of noticing, for instance, that if  $\mathbf{X}$  and  $\mathbf{X}^n$  have the same, finite, number of components,

$$\begin{aligned} & \left\| \text{masses}(\text{Coal}(\text{Frag}(\mathbf{X}, \mathcal{P}_t^-), \mathcal{P}_t^+)) \right. \\ & \quad \left. - \text{masses}(\text{Coal}(\text{Frag}(\mathbf{X}^n, \mathcal{P}_t^{-,n}, \mathcal{P}_t^{+,n})) \right\|_1 \\ & \leq \left\| \text{masses}(\text{Frag}(\mathbf{X}, \mathcal{P}_t^-)) - \text{masses}(\text{Frag}(\mathbf{X}^n, \mathcal{P}_t^{-,n})) \right\|_1, \end{aligned}$$

and one may thus use Proposition 5.5. □

**Remark 6.3.** In Proposition 6.2, the initial convergence is in  $L_{2, GHP}^{\text{surplus}}$  and the conclusion is in  $L_{2, GHP}$ . This is unavoidable since convergence in  $L_{2, GHP}^{\text{surplus}}$  does not prevent the sequence  $\mathbf{X}^{(n)}$  of having components with masses going to zero but positive surplus. These components can at positive time be glued to large components, augmenting their surplus significantly. One could recover  $L_{2, GHP}^{\text{surplus}}$  in the conclusion if one added to (6.1) the following condition

$$\lim_{\varepsilon_1 \rightarrow 0} \limsup_{n \rightarrow +\infty} \mathbb{P} \left( \sup_{m \in \text{comp}(\mathbf{X}_{\leq \varepsilon_1}^{(n)})} \{\text{surplus}(m)\} \neq 0 \right) = 0.$$

Notice however that this condition is not satisfied by the connected components of critical Erdős-Rényi random graphs.

## 6.2. Application to Erdős-Rényi random graphs

Now, we want to prove Theorem 3.3. Intuitively, the dynamical percolation process on the complete graph  $K_n$  should be very close to the process  $\text{CoalFrag}(K_n, \cdot)$ , but such a statement needs some care, essentially because  $N^+$  and  $N^-$  do not commute: some pairs of vertices might be affected by the two Poisson processes  $\mathcal{P}^+$  and  $\mathcal{P}^-$  in a time interval  $[0, T]$ . Furthermore, the typical number of such edges is of order  $n^{1/3}$ . It turns out that those edges will not be important for the  $L_{GHP}$ -metric, but it requires to adapt the proof of Proposition 6.2.

*Proof.* (of Theorem 3.3). Let  $G^\infty = \mathcal{G}_\lambda$ . Let  $p = p(\lambda, n)$ , let  $\mathbf{G}^{(n)}$  be the graph  $\mathcal{G}(n, p)$  seen as a measured  $\mathbb{R}$ -graph, with edge-lengths  $\delta_n := (1-p)n^{-1/3} \sim n^{-1/3}$  and measure the counting measure times  $\sqrt{pn^{-1/3}} \sim n^{-2/3}$ . Let  $\mathcal{P}^+$  (of intensity  $pn^{-1/3}$ ) and  $\mathcal{P}^-$  (of intensity  $(1-p)n^{-1/3}$ ) be the two Poisson processes driving the dynamical percolation on  $G^{(n)}$ . Let us write

$$G^{(n)}(t) := N(G^{(n)}, (\mathcal{P}^+, \mathcal{P}^-)_t)$$

and

$$G_{>\varepsilon_1}^{(n)}(t) := N(G_{>\varepsilon_1}^{(n)}, (\mathcal{P}^+, \mathcal{P}^-)_t).$$

for the state of this process at time  $t$ , seen as a member of  $\mathcal{N}_2^{graph}$ . Let us fix  $\varepsilon > 0$  and  $0 \leq t \leq T$ . Any component of size at least  $\varepsilon$  in  $N(G^{(n)}, (\mathcal{P}^+, \mathcal{P}^-)_t)$  has to belong to a component of size at least  $\varepsilon$  in  $N(G^{(n)}, (\mathcal{P}^+, \emptyset)_t)$ , which is nothing else but  $\text{Coal}_{\delta_n}(G^{(n)}, \mathcal{P}_t^+)$ . Now, we claim that

$$\limsup_{n \in \mathbb{N}} \mathbb{P}(\text{suplength}(\text{Coal}_{\delta_n}(\mathbf{G}_{\leq \varepsilon}^n, T)) > \alpha) \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (6.3)$$

Indeed, if  $G$  is a discrete graph with height function bounded from above by  $h$  and surplus bounded from above by  $s$ ,

$$\text{suplength}(G) \leq 2h(1+s).$$

Thus, (6.3) is a consequence of (4.7) and the fact that the maximal surplus in  $\mathbf{G}^{(n)}$  form a tight sequence (see for instance sections 13 and 14 in [13]). Then, (6.3) and Lemma 6.1 show that for any  $\alpha > 0$  and  $T > 0$ ,

$$\mathbb{P}(\text{suplength}(\text{Coal}_0(\mathbf{G}_{\leq \varepsilon}^\infty, T)) > \alpha) \xrightarrow{\varepsilon \rightarrow 0} 0.$$

The arguments leading to (6.2) show that:

$$\lim_{\varepsilon_1 \rightarrow 0} \limsup_{n \in \mathbb{N}} \mathbb{P}[\sup_{t \in [0, T]} L_{2, GHP}(G^{(n)}(t), G_{>\varepsilon_1}^{(n)}(t)) > \varepsilon] = 0.$$

and

$$\lim_{\varepsilon_1 \rightarrow 0} \mathbb{P}[\sup_{t \in [0, T]} L_{2, GHP}(\text{CoalFrag}(G^\infty, t), \text{CoalFrag}(G_{>\varepsilon_1}^\infty, t)) > \varepsilon] = 0.$$

Thus, it is sufficient to show that for any  $\varepsilon_1 > 0$ ,  $(G_{>\varepsilon_1}^{(n)}(t))_{t \geq 0}$  converges to  $(\text{CoalFrag}(G_{>\varepsilon_1}^\infty, t))_{t \geq 0}$  in the topology of compact convergence associated to  $L_{2, GHP}$ . Let  $Y_n$  denote the number of discrete coalescence events of  $\mathcal{P}_T^+$  occurring on  $G_{>\varepsilon_1}^{(n)}$ . Since the masses of  $G_{>\varepsilon_1}^{(n)}$  form a tight sequence,  $(Y_n)$  is a tight sequence. Since  $\delta_n$  goes to zero, the probability that  $\mathcal{P}_T^-$  touches an edge from  $\mathcal{P}_T^+$  in  $G_{>\varepsilon_1}^{(n)}$  goes to zero as  $n$  goes to infinity. Thus, with probability going to one, for any  $t \in [0, T]$   $G_{>\varepsilon_1}^{(n)}(t) = \tilde{G}_{>\varepsilon_1}^{(n)}(t)$  where

$$\tilde{G}_{>\varepsilon_1}^{(n)}(t) = N(\text{Coal}_{\delta_n}(G_{>\varepsilon_1}^{(n)}, \mathcal{P}_t^+), (\emptyset, \mathcal{P}^-)_t).$$

Furthermore, since  $Y_n$  is a tight sequence and  $\delta_n$  goes to zero,

$$\sup_{t \in [0, T]} L_{2, GHP}(\tilde{G}_{>\varepsilon_1}^{(n)}(t), N(\text{Coal}_0(G_{>\varepsilon_1}^{(n)}, \mathcal{P}_t^+), (\emptyset, \mathcal{P}^-)_t)) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0.$$

Let  $\mathcal{Q}^-$  be a Poisson process of intensity  $\ell_n \otimes \text{leb}_{\mathbb{R}^+}$  on  $K_n \times \mathbb{R}^+$  where  $K_n$  is the complete graph on  $n$  vertices seen as an  $\mathbb{R}$ -graph where the edge lengths are  $\delta_n$  and  $\ell_n$  is its length measure. Then, one may suppose that  $\mathcal{P}^-$  is obtained as follows:

$$\mathcal{P}^- = \{(e, t) : \exists x \in K_n, (x, t) \in \mathcal{Q}^-\}.$$

Then, for any  $t$ ,  $N(\text{Coal}_0(G_{>\varepsilon_1}^{(n)}, \mathcal{P}_t^+), (\emptyset, \mathcal{P}^-)_t)$  is at  $L_{2,GHP}$ -distance at most  $\delta_n$  from  $\text{Frag}(\text{Coal}_0(G_{>\varepsilon_1}^{(n)}, \mathcal{P}_t^+), \mathcal{Q}_t^-)$  (cf. for instance Proposition 3.4 in [3]). Altogether, we get:

$$\sup_{t \in [0, T]} L_{2,GHP}(N(G_{>\varepsilon_1}^{(n)}(t), \text{Frag}(\text{Coal}_0(G_{>\varepsilon_1}^{(n)}, \mathcal{P}_t^+), \mathcal{Q}_t^-)) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0,$$

and  $(\text{Frag}(\text{Coal}_0(G_{>\varepsilon_1}^{(n)}, \mathcal{P}_t^+), \mathcal{Q}_t^-))_{t \geq 0}$  is distributed as  $\text{CoalFrag}(G_{>\varepsilon_1}^{(n)}, t)_{t \geq 0}$ . Now Proposition 6.2 shows that the sequence of processes  $\text{CoalFrag}(G_{>\varepsilon_1}^{(n)}, \cdot)$  converges to  $\text{CoalFrag}(G_{>\varepsilon_1}^\infty, \cdot)$  for the topology of compact convergence associated to  $L_{2,GHP}$ , which finishes the proof.  $\square$

## References

- [1] Romain Abraham, Jean-François Delmas, and Patrick Hoscheit. A note on the Gromov-Hausdorff-Prokhorov distance between (locally) compact metric measure spaces. *Electron. J. Probab.*, 18:21, 2013.
- [2] L. Addario-Berry, N. Broutin, and C. Goldschmidt. The continuum limit of critical random graphs. *Probab. Theory Related Fields*, 152(3-4):367–406, 2012.
- [3] L. Addario-Berry, N. Broutin, C. Goldschmidt, and G. Miermont. The scaling limit of the minimum spanning tree of the complete graph. *ArXiv e-prints*, January 2013.
- [4] Louigi Addario-Berry, Nicolas Broutin, and Christina Goldschmidt. Critical random graphs: limiting constructions and distributional properties. *Electron. J. Probab.*, 15:741–775, 2010.
- [5] David Aldous. Brownian excursions, critical random graphs and the multiplicative coalescent. *Ann. Probab.*, 25(2):812–854, 1997.
- [6] David Aldous and Jim Pitman. The standard additive coalescent. *Ann. Probab.*, 26(4):1703–1726, 1998.
- [7] S. Bhamidi, N. Broutin, S. Sen, and X. Wang. Scaling limits of random graph models at criticality: Universality and the basin of attraction of the Erdős-Rényi random graph. *ArXiv e-prints*, November 2014.
- [8] D. Burago, Yu. Burago, and S. Ivanov. *A course in metric geometry*. Providence, RI: American Mathematical Society (AMS), 2001.
- [9] R.M. Dudley. *Real analysis and probability. Repr.* Cambridge: Cambridge University Press, repr. edition, 2002.
- [10] Steven N. Evans, Jim Pitman, and Anita Winter. Rayleigh processes, real trees, and root growth with re-grafting. *Probab. Theory Relat. Fields*, 134(1):81–126, 2006.
- [11] C. Garban, G. Pete, and O. Schramm. The scaling limits of near-critical and dynamical percolation. *ArXiv e-prints*, May 2013.
- [12] Olle Häggström, Yuval Peres, and Jeffrey E. Steif. Dynamical percolation. *Ann. Inst. Henri Poincaré, Probab. Stat.*, 33(4):497–528, 1997.
- [13] Svante Janson, Donald E. Knuth, Tomasz Łuczak, and Boris Pittel. The birth of the giant component. *Random Struct. Algorithms*, 4(3):233–358, 1993.
- [14] Olav Kallenberg. *Random measures, theory and applications*. Cham: Springer, 2017.
- [15] Grégory Miermont. Tessellations of random maps of arbitrary genus. *Ann. Sci. Éc. Norm. Supér. (4)*, 42(5):725–781, 2009.
- [16] David Pollard. *Convergence of stochastic processes*. Springer-Verlag, 1984.
- [17] M. I. Roberts and B. Sengul. Exceptional times of the critical dynamical Erdős-Rényi graph. *ArXiv e-prints*, October 2016.
- [18] Cédric Villani. *Optimal transport. Old and new*. Berlin: Springer, 2009.