Variational approximation of size-mass energies for k-dimensional currents

Antonin Chambolle, Luca Alberto Davide Ferrari, Benoît Merlet

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A. Chambolle* L. Ferrari† B. Merlet‡

In this paper we produce a Γ-convergence result for a class of energies $\mathcal{F}^k_{\varepsilon,a}$ modeled on the Ambrosio-Tortorelli functional. For the choice $k=1$ we show that $F^1_{\varepsilon,a}$ Γ-converges to a branched transportation energy whose cost per unit length is a function f^{n-1}_a depending on a parameter $a>0$ and on the codimension $n-1$. The limit cost $f_a(m)$ is bounded from below by $1+m$ so that the limit functional controls the mass and the length of the limit object. In the limit $a \downarrow 0$ we recover the Steiner energy.
We then generalize the approach to any dimension and codimension. The limit objects are now k-currents with prescribed boundary, the limit functional controls both their masses and sizes. In the limit $a \downarrow 0$, we recover the Plateau energy defined on k-currents, $k<n$. The energies $F^k_{\varepsilon,a}$ then can be used for the numerical treatment of the k-Plateau problem.

1. Introduction

Let $\Omega \subset \mathbf{R}^n$ be a convex, bounded open set. We consider vector measures $\sigma \in \mathcal{M}(\overline{\Omega}, \mathbf{R}^n)$ of the form

$$\sigma = m\,\nu\,\mathcal{H}^1 \llcorner \Sigma,$$

where Σ is a 1-dimensional rectifiable set oriented by a Borel measurable tangent map $\nu : \Sigma \to \mathbf{S}^{n-1}$ and $m : \Sigma \to \mathbf{R}_+$ is a Borel measurable function representing the multiplicity. We write $\sigma = (m, \nu, \Sigma)$ for such measures. Given a cost function $f \in C(\mathbf{R}_+, \mathbf{R}_+)$ we introduce the functional

$$\mathcal{F}(\sigma) := \begin{cases} \displaystyle\int_\Sigma f(m)\,\mathrm{d}\mathcal{H}^1 & \text{if } \sigma = (m,\nu,\Sigma), \\ +\infty & \text{otherwise in } \mathcal{M}(\overline{\Omega}, \mathbf{R}^d). \end{cases} \tag{1.1}$$

Next, given $\mathscr{S} = \{x_1, \cdots, x_{n_P}\} \subset \Omega$ a finite set of points and $c_1, \cdots, c_{n_P} \in \mathbf{R}$ such that $\sum_{j=1}^{n_P} c_j = 0$, we consider the optimization problem $\mathcal{F}(\sigma)$ for $\sigma \in \mathcal{M}(\overline{\Omega}, \mathbf{R}^n)$ satisfying

$$\nabla \cdot \sigma = \sum_{j=1}^{n_P} c_j \delta_{x_j} \qquad \text{in } \mathcal{D}'(\mathbf{R}^n). \tag{1.2}$$

The setting is similar to the one from Beckman [22] and Xia [25]. We model transport nets connecting a given set of sources $\{x_j \in \mathscr{S} : c_j > 0\}$ to a given set of wells $\{x_j \in \mathscr{S} : c_j < 0\}$ via vector valued measures. For numerical reasons, we wish to approximate the measure $\sigma = (m, \nu, \Sigma)$ by a diffuse object (a smooth vector field). For this, we introduce below a family of corresponding "diffuse" functionals $\mathcal{F}_{\varepsilon,a}$ that converge towards (1.1) in the sense of Γ-*convergence* [9, 10, 14]. This general idea has proved

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to be effective in a variety of contexts such as fracture theory, optimal partitions problems and image segmentation [3, 18, 13, 19]. More recently this tool has been used to approximate energies depending on one dimensional sets, for instance in [20] the authors take advantage of a functional similar to the one from Modica and Mortola defined on vector valued measures to approach the branched transportation problem [6]. With similar techniques approximations of the Steiner minimal tree problem ([2], [17] and [21]) have been proposed in [8, 7].

In the present paper we first extend to any ambient dimension $n \geq 2$ the *phase-field* approximation for a branched transportation energy introduced in [5] for $n = 2$. In particular the approximate functionals $\mathcal{F}_{\varepsilon,a}$ are modeled on the one from Ambrosio and Tortorelli [4]. We also extend the construction to any dimension and co-dimension. Indeed, for $1 \leq k \leq n-1$ integer, we consider k-rectifiable currents $\sigma = (\theta, e, \Sigma)$ where Σ is a countably k-rectifiable set with approximate tangent k-plane defined by a simple unit multi-vector $\xi(x) = \xi_1(x) \wedge \cdots \wedge \xi_k(x)$ and $m : \Sigma \to \mathbf{R}_+$ is a Borel measurable function (the multiplicity). The functional (1.1) extends to k-currents σ as follows,

$$\mathcal{F}(\sigma) := \begin{cases} \displaystyle\int_\Sigma f(m(x))\,\mathrm{d}\mathcal{H}^k & \text{if } \sigma = (m, \xi, \Sigma), \\ +\infty & \text{otherwise.} \end{cases}$$

Let us define the approximate functionals and describe our main results in the case $k = 1$. For our phase field approximations we relax the condition on the vector measure σ replacing it by a vector field $\sigma_\varepsilon \in L^2(\Omega, \mathbf{R}^n)$. We then need to mollify condition (1.2). Let $\rho : \mathbf{R}^n \to \mathbf{R}_+$ be a classical radial mollifier such that $\operatorname{supp} \rho \subset B_1(0)$ and $\int_{B_1(0)} \rho = 1$. For $\varepsilon > 0$, we set $\rho_\varepsilon = \varepsilon^{-n}\rho(\cdot/\varepsilon)$. We substitute for (1.2) the condition

$$\nabla \cdot \sigma_\varepsilon \;=\; \left(\sum_{j=1}^{n_P} c_j \delta_{x_j}\right) * \rho_\varepsilon = \sum_{j=1}^{n_P} c_j \rho_\varepsilon(\cdot - x_j) \qquad \text{in } \mathcal{D}'(\mathbf{R}^n). \tag{1.3}$$

Remark 1. *Notice that in* (1.2) (1.3) *the equality holds in* $\mathcal{D}'(\mathbf{R}^n)$ *and not only in* $\mathcal{D}'(\Omega)$ *so that there is no flux trough* $\partial\Omega$*.*

We also consider the functions $u \in W^{1,p}(\Omega, [\eta, 1])$ such that $u \equiv 1$ on $\partial\Omega$ where $\eta = \eta(\varepsilon)$ satisfies

$$\eta = a\,\varepsilon^n$$

for some $a \in \mathbf{R}_+$. We denote by $X_\varepsilon(\Omega)$ the set of pairs (σ, u) satisfying the above hypotheses. This set is naturally embedded in $\mathcal{M}(\overline{\Omega}, \mathbf{R}^n) \times L^2(\Omega)$. For $(\sigma, u) \in \mathcal{M}(\overline{\Omega}, \mathbf{R}^n) \times L^2(\Omega)$ we set

$$\mathcal{F}_{\varepsilon,a}(\sigma, u; \Omega) \;:=\; \begin{cases} \displaystyle\int_\Omega \left[\varepsilon^{p-n+1}|\nabla u|^p + \frac{(1-u)^2}{\varepsilon^{n-1}} + \frac{u|\sigma|^2}{\varepsilon}\right] \mathrm{d}x & \text{if } (\sigma, u) \in X_\varepsilon(\Omega), \\ +\infty & \text{in the other cases.} \end{cases} \tag{1.4}$$

Let X be the subset of $\mathcal{M}(\overline{\Omega}, \mathbf{R}^n) \times L^2(\Omega)$ consisting of those couples (σ, u) such that $u \equiv 1$ and $\sigma = (m, \nu, \Sigma)$ satisfies the constraint (1.2). Given any sequence $\varepsilon = (\varepsilon_i)_{i \in \mathbf{N}}$ of positive numbers such that $\varepsilon_i \downarrow 0$, we show that the above family of functionals Γ-converges to

$$\mathcal{F}_a(\sigma, u; \overline{\Omega}) = \begin{cases} \displaystyle\int_{\Sigma \cap \overline{\Omega}} f_a(m(x))\,\mathrm{d}\mathcal{H}^1(x) & \text{if } (\sigma, u) \in X \text{ and } \sigma = m\,\nu\,\mathcal{H}^1 \llcorner \Sigma, \\ +\infty & \text{otherwise.} \end{cases} \tag{1.5}$$

The function $f_a : \mathbf{R}_+ \to \mathbf{R}_+$ (introduced and studied in the appendix) is the minimum value of some optimization problem depending on a and on the codimension $n-1$ (we note f_a^d, with $d = n-k$ in the general case $1 \leq k \leq n-1$). In particular we prove that f_a is lower semicontinuous, subadditive, increasing, $f_a(0) = 0$ and that there exists some $c > 0$ such that

$$\frac{1}{c} \leq \frac{f_a(m)}{\sqrt{1 + a\,m^2}} \leq c \quad \text{for } m > 0.$$

The Γ-convergence holds for the topology of the weak-$*$ convergence for the sequence of measures (σ_ε) and for the strong L^2 convergence for the phase field (u_ε). For a sequence $(\sigma_\varepsilon, u_\varepsilon)$ we write $(\sigma_\varepsilon, u_\varepsilon) \to (\sigma, u)$ if $\sigma_\varepsilon \stackrel{*}{\rightharpoonup} \sigma$ and $\|u_\varepsilon - u\|_{L^2} \to 0$. In the sequel we first establish that the sequence of functionals $(\mathcal{F}_{\varepsilon,a})_\varepsilon$ is coercive with respect this topology.

Theorem 1.1. *Assume that $a > 0$. For any sequence $(\sigma_\varepsilon, u_\varepsilon) \subset \mathcal{M}(\overline{\Omega}, \mathbf{R}^n) \times L^2(\Omega)$ with $\varepsilon \downarrow 0$, such that*

$$\mathcal{F}_{\varepsilon,a}(\sigma_\varepsilon, u_\varepsilon; \Omega) \le F_0 < +\infty,$$

then there exists $\sigma \in \mathcal{M}(\overline{\Omega}, \mathbf{R}^n)$ such that, up to a subsequence, $(\sigma_\varepsilon, u_\varepsilon) \to (\sigma, 1) \in X$.

Then we prove the Γ-liminf inequality

Theorem 1.2. *Assume that $a \ge 0$. For any sequence $(\sigma_\varepsilon, u_\varepsilon) \in \mathcal{M}(\overline{\Omega}, \mathbf{R}^n) \times L^2(\Omega)$ that converges to $(\sigma, u) \in \mathcal{M}(\overline{\Omega}, \mathbf{R}^n) \times L^2(\Omega)$ as $\varepsilon \downarrow 0$ it holds*

$$\liminf_{\varepsilon \downarrow 0} \mathcal{F}_{\varepsilon,a}(\sigma_\varepsilon, u_\varepsilon; \Omega) \ge \mathcal{F}_a(\sigma, u; \overline{\Omega}).$$

We also establish the corresponding Γ-limsup inequality

Theorem 1.3. *Assume that $a \ge 0$. For any $(\sigma, u) \in \mathcal{M}(\overline{\Omega}, \mathbf{R}^n) \times L^2(\Omega)$ there exists a sequence $((\sigma_\varepsilon, u_\varepsilon)) \subset \mathcal{M}(\overline{\Omega}, \mathbf{R}^n) \times L^2(\Omega)$ such that*

$$(\sigma_\varepsilon, u_\varepsilon) \xrightarrow{\varepsilon \downarrow 0} (\sigma, u) \quad \text{in } \mathcal{M}(\overline{\Omega}, \mathbf{R}^n) \times L^2(\Omega)$$

and

$$\limsup_{\varepsilon \downarrow 0} \mathcal{F}_{\varepsilon,a}(\sigma_\varepsilon, u_\varepsilon; \Omega) \le \mathcal{F}_a(\sigma, u; \overline{\Omega}).$$

As already stated, we only considered the case $k = 1$ in this introduction. Section 4 is devoted to the extension of Theorems 1.1, 1.2 and 1.3 in the case where the 1-currents (vector measures) are replaced with k-currents.

Notice that the coercivity of the family of functionals only holds in the case $a > 0$. However, as $a \downarrow 0$ we have the important phenomena:

$$f_a \xrightarrow{a \downarrow 0} c\mathbf{1}_{(0,+\infty)} \quad \text{pointwise},$$

for some $c > 0$. As a consequence (1.5) is an approximation of $c\mathcal{H}^1(\Sigma)$ for $a > 0$ small and the minimization of (1.4) in $X_\varepsilon(\Omega)$ provides an approximation of the Steiner problem associated to the set of points $\mathscr{S}$, for a suitable choice of the weights in (1.2). In the case $k > 1$, we obtain a variational approximation of the k-Plateau problem.

Structure of the paper: In Section 2 we introduce some notation and recall some useful facts about vector measures and currents, we also anticipate the optimization problem defining the cost function f_a^d and state some results which are proved in Appendix A. In Section 3 we establish Theorems 1.1, 1.2 and 1.3. In Section 4 we extend these results to the case $1 \le k \le n-1$. In Section 5 we discuss the limit $a \downarrow 0$.

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2. Preliminaries and notation

The canonical orthonormal basis of $\mathbf{R}^n$ is denoted by the vectors $e_1, \ldots, e_n$. $\mathscr{L}^n$ denotes the Lebesgue measure in $\mathbf{R}^n$ and given an integer value k we denote with ω_k the measure of the unit ball in $\mathbf{R}^k$, i.e. $\mathscr{L}^k(B_1(0))$. For a point $x \in \mathbf{R}^n$ we note $x = (x_1; x') \in \mathbf{R} \times \mathbf{R}^{n-1}$. For any Borel-measurable set $A \subset \mathbf{R}^n$ we denote with $\mathbf{1}_A(x)$ the characteristic function of the set A

$$\mathbf{1}_A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

Given a vector space Y and its dual Y' for $\omega \in Y$ and $\sigma \in Y'$ we write $\langle \omega, \sigma \rangle$ for the dual pairing.

2.1. Measures and vector measures

We denote with $\mathcal{M}(\Omega)$ the vector space of *Radon measures* in Ω and with $\mathcal{M}(\Omega, \mathbf{R}^n) = \mathcal{M}(\Omega)^n$ the vector space of vector valued measures. For a measure $\mu \in \mathcal{M}(\Omega)$ we denote with $|\mu|$ its *total variation*, in the vector case $\mu \in \mathcal{M}(\Omega, \mathbf{R}^n)$ we write $\mu = \nu|\mu|$ where ν is a $|\mu|$-measurale map into $\mathbf{S}^{n-1}$. We say that a measure is *supported* on a Borel set E if $|\mu|(\Omega \setminus E) = 0$. For an integer $k < n$ we denote with $\mathcal{H}^k$ the k-dimensional *Hausdorff* measure as in [1]. Given a set $E \in \Omega$, such that, $\mathcal{H}^k(E)$ is finite for some k the restriction $\mathcal{H}^k \llcorner E$ defines a Radon measure in the space $\mathcal{M}(\Omega)$. A set $E \in \Omega$ is said to be *countably k-rectifiable* if up to a $\mathcal{H}^k$ negligible set N, $E \setminus N$ is contained in a countable union of C^1 k-dimensional manifolds.

2.2. Currents

We denote with $\mathcal{D}^k(\Omega)$ the vector space of compactly supported smooth k-differential forms. For a k-differential form ω its comass is defined as

$$\|\omega\| = \sup\{\langle \omega, \xi \rangle : \xi \text{ is a unit, simple } k\text{-vector}\}$$

Let $\mathcal{D}_k(\Omega)$ be the dual to $\mathcal{D}^k(\Omega)$ i.e. the space of *k-currents* with its weak-$*$ topology. We denote with ∂ the boundary operator that operates by duality as follows

$$\langle \partial\sigma, \omega \rangle = \langle \sigma, \, \mathrm{d}\omega \rangle \quad \text{ for all } (k-1)\text{-differential forms } \omega.$$

The mass of a k-current $M(\sigma)$ is the supremum of $\langle \sigma, \omega \rangle$ among all k-differential forms with comass bounded by 1. For any k-current σ such that both σ and $\partial\sigma$ are of finite mass we say that σ is a normal k-current and we write $\sigma \in N_k(\Omega)$. On the space $\mathcal{D}_k(\Omega)$ we can define the *flat norm* by

$$\mathbb{F}(\sigma) = \inf \left\{ M(R) + M(S) \; : \; \sigma = R + \partial S \text{ where } S \in \mathcal{D}_{k+1}(\Omega) \text{ and } R \in \mathcal{D}_k(\Omega) \right\},$$

which metrizes the weak-$*$ topology on currents on compact subsets of $N_k(\Omega)$. By the Radon-Nikodym theorem we can identify a k-current σ with finite mass with the vector valued measure $\nu\mu_\sigma$ where μ_σ is a finite positive valued measure and ν is a μ_σ-measurable map in the set of unitary k-vectors for the mass norm. In particular the action of σ on ω can be written as

$$\langle \sigma, \omega \rangle = \int_\Omega \langle \omega, \nu \rangle \, \mathrm{d}\mu_\sigma.$$

For a finite mass k-current the mass of σ coincides with the total variation of the measure μ_σ. A k-current σ is said to be k-rectifiable if we can associate to it a triplet (θ, ν, Σ) such that

$$\langle \sigma, \omega \rangle = \int_\Sigma \theta \langle \omega, \nu \rangle \, \mathrm{d}\mathcal{H}^k$$

where Σ is a countably k-rectifiable subset of Ω, ν at $\mathcal{H}^k$ a.e. point is a unit simple k-vector that spans the tangent plane to Σ and θ is an $L^1(\Omega, \mathcal{H}^k \llcorner \Sigma)$ function that can be assumed positive. We will denote with $R_k(\Omega)$ the space of these k-rectifiable currents. Among these we name out the subset $P_k(\Omega)$ of k-rectifiable currents for which Σ is a finite union of polyhedra, these will be called *polyhedral chains*. Finally the *flat chains* $F_k(\Omega)$ consist of the closure of $P_k(\Omega)$ in the weak-$*$ topology. By the scheme of Federer [16, 4.1.24] it holds

$$P_k(\Omega) \subset N_k(\Omega) \subset F_k(\Omega).$$

Remark 2 (1-Currents and Vector Measures)**.** *Since the vector spaces $\Lambda_1 \mathbf{R}^n$, $\Lambda^1 \mathbf{R}^n$ identify with $\mathbf{R}^n$, any vector measure $\sigma \in \mathcal{M}(\Omega, \mathbf{R}^n)$ with finite mass indentifies with a 1-current with finite mass and viceversa. The divergence operator acting on measures is defined by duality as the boundary operator for currents. In the following $\sigma \in \mathcal{M}(\Omega, \mathbf{R}^n)$ is called a rectifiable vector measure if it is 1-rectifiable as 1-current. In the same fashion we define polyhedral 1-measures.*

2.3. Functionals defined on flat chains

For $f : \mathbf{R} \mapsto \mathbf{R}^+$ an even function we define a functional

$$\begin{aligned} P_k(\Omega) \quad &\longrightarrow \quad \mathbf{R}_+, \\ P = \sum_j (m_j, \nu_j, \Sigma_j) &\longmapsto \mathcal{F}(P) = \sum_j f(m_j) \mathcal{H}^k(\Sigma_j), \end{aligned}$$

on the space of polyhedral currents. Under the assumption that f is lower semi-continuous and subadditive, $\mathcal{F}$ can be extended to a lower semi-continuous functional by relaxation

$$\begin{array}{ccc} F_k(\Omega) & \longrightarrow & \mathbf{R}_+, \\ P & \longmapsto & \mathcal{F}(P) = \inf \left\{ \liminf\limits_{P_j \to P} \mathcal{F}(P) : (P_j)_j \subset P_k(\Omega) \text{ and } P_j \to P \right\}. \end{array}$$

as shown in [23, Section 6]. Furthermore, in [12] the authors show that if $f(t)/t \to \infty$ as $t \to 0$, then $\mathcal{F}(\sigma) < \infty$ if and only if σ is rectifiable and for any such σ the functional takes the explicit form

$$\mathcal{F}(\sigma) = \int_\Sigma f(m(x)) \, \mathrm{d}\mathcal{H}^k(x) \qquad \text{if } \sigma = (m, \nu, \Sigma). \tag{2.1}$$

To conclude this subsection let us recall a sufficient condition for a flat chain to be rectifiable, proved by White in [24, Corollary 6.1].

Theorem 2.1. *Let $\sigma \in F_k(\Omega)$. If $M(\sigma) + M(\partial\sigma) < \infty$ and if there exists a set $\Sigma \subset \Omega$ with finite k-dimensional Hausdorff measure such that $\sigma = \sigma \llcorner \Sigma$ then $\sigma \in R_k(\Omega)$ i.e., σ writes as (m, ν, Σ).*

In the context of vector measures the theorem writes as

Theorem 2.2. *Let $\sigma \in \mathcal{M}(\Omega, \mathbf{R}^n)$. If $|\sigma|(\Omega) + |\nabla \cdot s|(\Omega) < \infty$, $\nabla \cdot \sigma$ is at most a countable sum of Dirac masses and there exists Σ with $\mathcal{H}^1(\Sigma) < \infty$ and $\sigma = \sigma \llcorner \Sigma$ then σ is a rectifiable vector measure in the sense expressed in Subsection 2.2.*

2.4. Reduced problem results in dimension $n - k$

This subsection is devoted at introducing some notation and results corresponding to the case $k = 0$. In the sequel, these results are used to describe the energetical behaviour of the $(n-k)$-dimensional slices of the configuration $(\sigma_\varepsilon, u_\varepsilon)$. We postopone the proofs to Appendix A. We set $d = n - k$, $p > d$ and consider ε to be a sequence such that $\varepsilon \downarrow 0$. Let $B_r(0) \subset \mathbf{R}^d$ be the ball of radius r centered in the origin, we consider the functional

$$E_{\varepsilon,a}(\vartheta, u; B_r) := \int_{B_r} \left[\varepsilon^{p-d} |\nabla u|^p + \frac{(1-u)^2}{\varepsilon^d} + \frac{u|\vartheta|^2}{\varepsilon} \right] \mathrm{d}x$$

where $u \in W^{1,p}(B_r)$ is constrained to satisfy the lower bound $u \geq a\,\varepsilon^{d+1} =: \eta$ and $\vartheta \in L^2(B_r)$ is such that $\mathrm{supp}(\vartheta) \subset B_{\tilde r}$ with $0 < \tilde r < r$, $\|\vartheta\|_1 = m$. This leads to define the set

$$Y_{\varepsilon,a}(m, r, \tilde r) = \left\{ (\vartheta, u) \in L^2(B_r) \times W^{1,p}(B_r, [\eta, 1]) \,:\, \|\vartheta\|_1 = m \text{ and } \mathrm{supp}(\vartheta) \subset B_{\tilde r} \right\},$$

and the optimization problem

$$f^d_{\varepsilon,a}(m, r, \tilde r) = \inf_{Y_{\varepsilon,a}(m,r,\tilde r)} E_{\varepsilon,a}(\vartheta, u; B_r). \tag{2.2}$$

Let $f_a^d : [0,+\infty) \longrightarrow \mathbf{R}_+$ be defined as

$$f_a^d(m) = \begin{cases} \min\limits_{\hat r>0} \quad \left\{ \dfrac{a\, m^2}{\omega_d\, \hat r^d} + \ \omega_d\, \hat r^d + (d-1)\, \omega_d\, q_\infty^d(0,\hat r) \right\}, & \text{for } m>0, \\ \\ \qquad\qquad 0, & \text{for } m=0, \end{cases} \tag{2.3}$$

with

$$q_\infty^d(\xi,\hat r) := \inf\left\{ \int_{\hat r}^{+\infty} t^{d-1} \left[|v'|^p + (1-v)^2 \right]\, \mathrm{d}t \,:\, v(\hat r)=\xi \ \text{ and } \ \lim_{t\to+\infty} v(t)=1 \right\}, \tag{2.4}$$

for $\hat r>0$, $\xi\geq 0$. We have the following results

Proposition 2.1. *For any $r>\tilde r>0$, it holds*

$$\liminf_{\varepsilon\downarrow 0} f_{\varepsilon,a}^d(m,r,\tilde r) \geq f_a^d(m). \tag{2.5}$$

There exists a uniform constant $\kappa := \kappa(d,p)$ such that

$$f_a^d(m) \geq \kappa \qquad\qquad \textit{for every } m>0. \tag{2.6}$$

Proposition 2.2. *For fixed $m>0$ let r_* be the minimizing radius in the definition of $f_a^d(m)$ (2.3). For any $\delta>0$ and ε small enough there exist a function $\vartheta_\varepsilon = c\mathbf{1}_{B_{r_*\varepsilon}}$ with $c>0$ such that $\int_{B_r}\vartheta_\varepsilon = m$ and a nondecreasing radial function $u_\varepsilon : B_r \mapsto [\eta,1]$ such that $u_\varepsilon(0)=\eta$, $u_\varepsilon=1$ on ∂B_r and*

$$E_{\varepsilon,a}(\vartheta_\varepsilon, u_\varepsilon; B_r) \leq f_a^d(m)+\delta. \tag{2.7}$$

Proposition 2.3. *The function f_a^d is continuous in $(0,+\infty)$, increasing, sub-additive and $f_a^d(0)=0$.*

3. The 1-dimensional problem

3.1. Compactness

We prove the compactness Theorem 1.1 for the family of functionals $(\mathcal{F}_{\varepsilon,a})_\varepsilon$. Let us consider a family of functions $(\sigma_\varepsilon,u_\varepsilon)_{\varepsilon\downarrow 0}$, such that $(\sigma_\varepsilon,u_\varepsilon)\in X_\varepsilon(\Omega)$ and

$$\mathcal{F}_{\varepsilon,a}(\sigma_\varepsilon,u_\varepsilon;\Omega) \,\leq\, F_0.$$

As a first step we prove:

Lemma 3.1. *Assume $a>0$. There exists $C\geq 0$, only depending on Ω, F_0 and a such that*

$$\int_\Omega |\sigma_\varepsilon| \,\leq\, C, \qquad \forall\, \varepsilon. \tag{3.1}$$

As a consequence there exist a positive Radon measure $\mu\in(\mathbf{R}^n,\mathbf{R}_+)$ supported in $\overline{\Omega}$ and a vectorial Radon measure $\sigma\in\mathcal{M}(\overline{\Omega},\mathbf{R}^n)$ with $\nabla\cdot\sigma = \sum a_j\delta_{x_j}$ and $|\sigma|\leq\mu$ such that up to a subsequence

$$u_\varepsilon\to 1 \ \textit{in } L^2(\Omega), \qquad |\sigma_\varepsilon| \xrightharpoonup{\ *\ } \mu \ \textit{in } \mathcal{M}(\mathbf{R}^n), \qquad \sigma_\varepsilon \xrightharpoonup{\ *\ } \sigma \ \textit{in } \mathcal{M}(\mathbf{R}^n,\mathbf{R}^n).$$

Proof. We divide the proof into three steps.

Step 1. We start by proving the uniform bound (3.1). Let $\lambda\in(0,1]$ and let

$$\Omega_\lambda \,:=\, \{x\in\Omega \,:\, u_\varepsilon>\lambda\}\,.$$

Being σ_ε square integrable we identify the measure σ_ε with its density with respect to $\mathscr{L}^n$. Therefore splitting the total variation of σ_ε, we write

$$|\sigma_\varepsilon|(\Omega) = \int_\Omega |\sigma_\varepsilon|\,\mathrm{d}x = \int_{\Omega_\lambda} |\sigma_\varepsilon|\,\mathrm{d}x + \int_{\Omega\setminus\Omega_\lambda} |\sigma_\varepsilon|\,\mathrm{d}x.$$

We estimate each term separately. By Cauchy-Schwarz inequality we have

$$\int_{\Omega_\lambda} |\sigma_\varepsilon| \le \left(\int_{\Omega_\lambda} \frac{u_\varepsilon|\sigma_\varepsilon|^2}{\varepsilon}\right)^{1/2} \left(\int_{\Omega_\lambda} \frac{\varepsilon}{u_\varepsilon}\right)^{1/2}.$$

Since $\lambda < u_\varepsilon < 1$ on Ω_λ and $\int_{\Omega_\lambda}(u_\varepsilon|\sigma_\varepsilon|^2)/(\varepsilon)\,\mathrm{d}x$ being bounded by $\mathcal{F}_{\varepsilon,a}(\sigma_\varepsilon,u_\varepsilon)$ from the previous we get

$$\int_{\Omega_\lambda} |\sigma_\varepsilon| \le \left(\int_{\Omega_\lambda} \frac{u_\varepsilon|\sigma_\varepsilon|^2}{\varepsilon}\right)^{1/2} \sqrt{\frac{|\Omega|\varepsilon}{\lambda}} \le \sqrt{\frac{|\Omega|\;\varepsilon\;F_0}{\lambda}}\;.$$

Next, in $\Omega\setminus\Omega_\lambda$, by Young inequality, we have

$$2\int_{\Omega\setminus\Omega_\lambda} |\sigma_\varepsilon| \le \int_{\Omega\setminus\Omega_\lambda} \frac{u_\varepsilon|\sigma_\varepsilon|^2}{\varepsilon} + \int_{\Omega\setminus\Omega_\lambda} \frac{\varepsilon}{u_\varepsilon}.$$

Using $u_\varepsilon \ge \eta(\varepsilon)$, $\eta/\varepsilon^n = a$ and $(1-\lambda)^2 \le (1-u_\varepsilon)^2$ in $\Omega\setminus\Omega_\lambda$, we obtain

$$\int_{\Omega\setminus\Omega_\lambda} |\sigma_\varepsilon| \le \frac{1}{2}\int_\Omega \frac{u_\varepsilon|\sigma_\varepsilon|^2}{\varepsilon} + \frac{\varepsilon^n}{2\,\eta\,(1-\lambda)^2}\int_\Omega \frac{(1-u_\varepsilon)^2}{\varepsilon^{n-1}} \le \frac{F_0}{2} + \frac{F_0}{2\,a\,(1-\lambda)^2}.$$

Hence

$$|\sigma_\varepsilon|(\Omega) \le \frac{F_0}{2} + \frac{F_0}{2\;a\;(1-\lambda)^2} + \sqrt{\frac{|\Omega|\;\varepsilon\;F_0}{\lambda}}.$$

As $a > 0$, this yields (3.1).

Step 2. We easily see from $\int_\Omega (1-u_\varepsilon)^2 \le F_0\varepsilon^{n-1}$ that $u_\varepsilon \to 1$ in $L^2(\Omega)$ as $\varepsilon \downarrow 0$.

Step 3. The existence of the Radon measures μ and σ such that, up to extraction, $|\sigma_\varepsilon| \overset{*}{\rightharpoonup} \mu$ and $\sigma_\varepsilon \overset{*}{\rightharpoonup} \sigma$ follows from (3.1). The properties on the support of μ, on the divergence of σ and the fact that $|\sigma| \le \mu$ follow from the respective properties of σ_ε. □

We have just showed that the limit σ of a family $(\sigma_\varepsilon,u_\varepsilon)_\varepsilon$ equibounded in energy is bounded in mass. In what follows, we assume $a \ge 0$ and that σ_ε is bounded in mass. We show that the limiting σ is rectifiable.

Proposition 3.1. *Assume $a \ge 0$ and that the conclusions of Lemma 3.1 hold true. There exists a Borel subset Σ with finite length and a Borel measurable function $\nu : \Sigma \to S^{n-1}$ such that $\sigma = \nu|\sigma|\llcorner\Sigma$. Moreover, we have the following estimate,*

$$\mathcal{H}^1(\Sigma) \le C_* F_0,$$

where the constant $C_* \ge 0$ only depends on d and p.

This proposition together with Lemma 3.1 and Theorem 2.2 leads to

Proposition 3.2. *σ is a 1-rectifiable vector measure and in particular Σ is a countably $\mathcal{H}^1$-rectifiable set.*

The latter ensures that the limit couple $(\sigma,1)$ belongs to X and concludes the proof of Theorem 1.1. We now establish Proposition 3.1

Sketch of the proof: We first define Σ. Then we show in Lemma 3.3 that for $x \in \Sigma$, we have $\liminf_{\varepsilon\downarrow 0}\mathcal{F}_{\varepsilon,a}(\sigma_\varepsilon,u_\varepsilon;B(x,r_j)) \ge \kappa r_j$ for a sequence of radii $r_j \downarrow 0$ and $\kappa > 0$. The proof of the lemma is based on slicing and on the results of Appendix A. The proposition then follows from an application of the Besicovitch covering theorem.

First we introduce the Borel set

$$\widetilde{\Sigma} := \left\{ x \in \Omega : \forall r > 0,\ |\sigma|(B_r(x)) > 0 \text{ and } \exists \nu = \nu(x) \in \mathbf{S}^{n-1} \text{ such that } \nu = \lim_{r\downarrow 0} \frac{\sigma(B_r(x))}{|\sigma|(B_r(x))} \right\}.$$

We observe that by Besicovitch derivation theorem,

$$\sigma = \nu|\sigma|\llcorner\widetilde{\Sigma}.$$

Next we fix $\theta \in (0, 1/4^n)$ and define

$$\Gamma := \left\{ x \in \widetilde{\Sigma} \,:\, \exists\, r_0 > 0 \text{ such that, } \frac{|\sigma|(B_{r/4}(x))}{|\sigma|(B_r(x))} \leq \theta \ \text{ for every } r \in (0, r_0] \right\}.$$

We show that this set is $|\sigma|$-negligible.

Lemma 3.2. *We have* $|\sigma|(\Gamma) = 0$.

Proof. Let $x \in \Gamma$. Applying the inequality $|\sigma|(B_{r/4}(x)) \leq \theta|\sigma|(B_r(x))$ with $r = r_k = 4^{-k} r_0$, $k \geq 0$, we get $|\sigma|(B_{r_k}) \leq \theta^k |\sigma|(B_{r_0})$. Hence there exists $C \geq 0$ such that

$$|\sigma|(B_r(x)) \,\leq\, C r^{(\ln 1/\theta)/(\ln 4)}.$$

Noting, $\lambda = (\ln \frac{1}{\theta})/(\ln 4)$, we have by assumption $\lambda > n$. Therefore, for every $\xi > 0$ there exists $r_\xi = r_\xi(x) \in (0,1)$ such that

$$|\sigma|(B_{r_\xi}(x)) \,\leq\, \xi |B_{r_\xi}(x)|.$$

Now, for $R > 0$, we cover $\Gamma \cap B_R$ with balls of the form $B_{r_\xi(x)}(x)$. Using Besicovitch covering theorem, we have

$$\Gamma \cap B_R \,\subset\, \cup_{j=1}^{N(n)} \mathcal{B}_j$$

where $N(n)$ only depends on n and each $\mathcal{B}_j$ is a (finite or countable) disjoint union of balls of the form $B_{r_\xi(x_k)}(x_k)$. Then we get

$$|\sigma|(\Gamma \cap B_R) \,\leq\, \sum_{j=1}^{N(n)} |\sigma|(\mathcal{B}_j) \,\leq\, N(n)\xi|\mathcal{B}_j| \,\leq\, N(n)|B_{R+1}|\xi.$$

Sending ξ to 0 and then R to ∞, we obtain $|\sigma|(\Gamma) = 0$. $\square$

Set $\Sigma := \widetilde{\Sigma} \setminus \Gamma$, from Lemma 3.2, we have $\sigma = \nu|\sigma|\llcorner\Sigma$. Recall that $\mathscr{S} = \{x_1, \cdots, x_{n_P}\}$.

Lemma 3.3. *For every $x \in \Sigma \setminus \mathscr{S}$, there exists a sequence $(r_j) = (r_j(x)) \subset (0,1)$ with $r_j \downarrow 0$ such that*

$$\liminf_{\varepsilon \downarrow 0} \mathcal{F}_{\varepsilon,a}(\sigma_\varepsilon, u_\varepsilon; B(x, r_j)) \,\geq\, \sqrt{2}\,\kappa\, r_j,$$

where κ is the constant of Proposition 2.1.

Proof. Let $x \in \Sigma \setminus \mathscr{S}$. Without loss of generality, we assume $x = 0$ and $\nu(x) = e_1$. Let $\xi > 0$ be a small parameter to be fixed later. From the definition of Σ, there exists a sequence $(r_j) = (r_j(x)) \subset (0, d(x, \mathscr{S}))$ such that for every $j \geq 0$,

$$\sigma(B_{r_j}) \cdot e_1 \,\geq\, (1 - \xi)|\sigma|(B_{r_j}) \qquad \text{and} \qquad |\sigma|(B_{r_j/4}) \,\geq\, \theta|\sigma|(B_{r_j}). \tag{3.2}$$

Let us fix $j \geq 0$ and set, to simplify the notation, $r = r_j$ and $r_* = r/\sqrt{2}$. Recall the notation $x = (x_1, x') \in \mathbf{R} \times \mathbf{R}^{n-1}$ and define the cylinder

$$C_{r_*} := \{x : |x_1| \leq r_* \quad \text{and} \quad |x'| \leq r_*\}$$

so that $C_{r_*} \subset B_r$ and $B_{r/4} \subset C_{r_*/2}$, as shown in figure 1. Let $\chi \in C_c^\infty(\mathbf{R}^{n-1}, [0,1])$ be a radial cut-off function such that $\chi(x') = 1$ if $|x'| \leq \frac{1}{2}$ and $\chi(x') = 0$ for $|x'| \geq \frac{3}{4}$. Then, we note $\chi_{r_*}(x') = \chi(x'/r_*)$ and for $s \in [-r, r]$, we set

$$\forall s \in [-r, r], \qquad g_\varepsilon(s) \,:=\, e_1 \cdot \int_{B'_{r_*}} \sigma_\varepsilon(s, x')\, \chi_{r_*}(x') \,\,\mathrm{d}x'.$$

Since σ_ε is divergence free, $e_1 \cdot \sigma_\varepsilon(\cdot, s)$ has a meaning on the hyperplane $\{x_1 = s\}$ in the sense of trace,

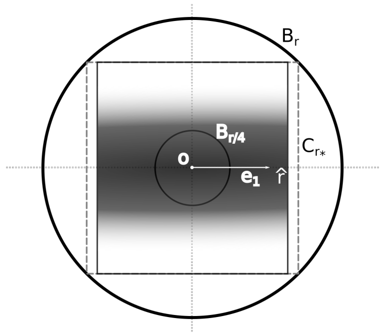

Figure 1: Illustration of the sections of B_r, $B_{r/4}$ and C_{r_*}. In grayscale we represent the level sets of the function $\chi_{r_*}(x')\mathbf{1}_{[-\hat r,\hat r]}$.

moreover, g_ε is continuous. Now, let us fix $\hat r \in [(1-\xi)r_*, r_*]$ such that $\mu(\{-\hat r,\hat r\}\times B'_r)=0$ (which holds true for a.e. $\hat r \in [(1-\xi)r_*, r_*]$) and let us define the mean value,

$$\bar g_\varepsilon \;:=\; \frac{1}{2\hat r}\int_{-\hat r}^{\hat r} g_\varepsilon(s)\ \mathrm{d}s.$$

From $\sigma_\varepsilon \overset{*}{\rightharpoonup} \sigma$, $|\sigma_\varepsilon| \overset{*}{\rightharpoonup} \mu$, we have

$$\lim_{\varepsilon\downarrow 0}\bar g_\varepsilon \;=\; \left(\frac{1}{2\hat r}\int_{(-\hat r,\hat r)\times B'_{r_*}} \chi_{r_*}(x')\ \mathrm{d}\sigma(s,x')\right)\cdot e_1 \;=:\; \overline{m}.$$

From (3.2), we see that $\overline{m}>0$ for ξ small enough. Indeed, we have

$$\begin{aligned}(1-\xi)|\sigma|(B_r) \;\le\; +2\hat r\overline{m} + \sigma(B_r)\cdot e_1 \;&=\; \int_{B_r}\left(1-\chi_{r_*}(x')\mathbf{1}_{[-\hat r,\hat r]}\right)\ \mathrm{d}\sigma(s,x')\cdot e_1\\ &\le\; 2\hat r\overline{m} + \int_{B_r}\left(1-\chi_{r_*}(x')\mathbf{1}_{[-\hat r,\hat r]}\right)\ \mathrm{d}|\sigma|(s,x')\\ &\le\; 2\hat r\overline{m} + |\sigma|(B_r) - \int_{B_r}\chi_{r_*}(x')\mathbf{1}_{[-\hat r,\hat r]}\ \mathrm{d}|\sigma|(s,x').\end{aligned}$$

Since by construction $\chi_{r_*}(x')\mathbf{1}_{[-\hat r,\hat r]} \ge \mathbf{1}_{B_{r/4}}$, using the second inequality of (3.2), we have

$$\overline{m} \;\ge\; \frac{1}{2\hat r}(\theta-\xi)|\sigma|(B_r) \;>\; 0,$$

for ξ small enough. Similarly, denoting $\Pi:\mathbf{R}^n\to\mathbf{R}^{n-1},\ (t,x')\mapsto x'$ the orthogonal projection onto the last $(n-1)$ coordinates, we deduce again from (3.2) that

$$|\Pi\sigma|(C_{r_*}) \le \frac{\sqrt{\xi}\,\overline{m}}{\theta-\xi}2\hat r. \tag{3.3}$$

Now, for ε small enough, we have $\nabla\cdot\sigma_\varepsilon = 0$ in C_{r_*}. Using this, we have for almost every $s,t\in[-\hat r,\hat r]$, with $s<t$,

$$g_\varepsilon(t)-g_\varepsilon(s) \;=\; \int_s^t\left[\int_{B'_{r_*}}\sigma_\varepsilon(x',h)\cdot\nabla'\chi_{r_*}(x')\ \mathrm{d}x'\right]\ \mathrm{d}h.$$

Integrating in s over $(-\hat r,\hat r)$, we get for almost every $t\in[-r,r]$,

$$g_\varepsilon(t)-\bar g_\varepsilon \;=\; \frac{1}{2\hat r}\int_{(-\hat r,\hat r)\times B'_{r_*}}\phi_t(x',h)\cdot\sigma_\varepsilon(x',h)\ \mathrm{d}x'\ \mathrm{d}h$$

with

$$\phi_t(h,x') \;=\; \begin{cases}(h+\hat r)\,\nabla'\chi_{r_*}(x') & \text{if } h<t,\\ (h-\hat r)\,\nabla'\chi_{r_*}(x') & \text{if } h>t.\end{cases}$$

We deduce the following convergence

$$g_\varepsilon(t) - \overline{m} \xrightarrow{\varepsilon\downarrow 0} \frac{1}{2\hat{r}} \int_{(-\hat{r},\hat{r})\times B'_{r_*}} \phi_t(h,x') \cdot \ \mathrm{d}\sigma(h,x').$$

in the $L^1(-\hat{r},\hat{r})$ topology. Using (3.3), we see that the above right hand side is bounded by $c\frac{\sqrt{\xi}}{\theta-\xi}\overline{m}$. Taking into account (3.3) and the continuity of g_ε, we conclude that

$$\liminf_{\varepsilon\downarrow 0} g_\varepsilon(t) \geq \left(1 - c\frac{\sqrt{\xi}}{\theta-\xi}\right)\overline{m} \quad \text{ for } \quad t\in[-\hat{r},\hat{r}].$$

Next, by decomposing the integral we have

$$\begin{aligned}
\mathcal{F}_{\varepsilon,a}(\sigma_\varepsilon,u_\varepsilon;B_r) &\geq \int_{-\hat{r}}^{\hat{r}}\int_{B'_{r_*}} \left[\varepsilon^{p-n+1}|\nabla u_\varepsilon|^p + \frac{(1-u_\varepsilon)^2}{\varepsilon^{n-1}} + \frac{u_\varepsilon|\sigma_\varepsilon|^2}{\varepsilon}\right] \mathrm{d}x' \ \mathrm{d}t \\
&\geq \int_{-\hat{r}}^{\hat{r}}\int_{B'_{r_*}} \left[\varepsilon^{p-n+1}|\nabla u_\varepsilon|^p + \frac{(1-u_\varepsilon)^2}{\varepsilon^{n-1}} + \frac{u_\varepsilon|\chi_{r_*}(x')\sigma_\varepsilon|^2}{\varepsilon}\right] \mathrm{d}x' \ \mathrm{d}t.
\end{aligned}$$

Let us set

$$\vartheta_\varepsilon^t(x') := |\chi_{r_*}(x')\sigma_\varepsilon(t,x')|.$$

By construction ϑ_ε^t has the properties:

- $\vartheta_\varepsilon^t \in L^1(B'_{r_*})$,
- $\liminf_{\varepsilon\downarrow 0}\int_{B'_{r_*}} \vartheta_\varepsilon^t(x')\,dx' \ \geq \liminf_{\varepsilon\downarrow 0} g_\varepsilon(t) \ \geq \ \left(1 - c\frac{\sqrt{\xi}}{\theta-\xi}\right)\overline{m} = \tilde{m} > 0$,
- $\mathrm{supp}(\vartheta_\varepsilon^t) \subset B'_{\tilde{r}}$ with $\tilde{r} := \frac{3}{4}r_* < r_*$.

By definition of the minimization problem introduced in Subsection 2.4, we have

$$\mathcal{F}_{\varepsilon,a}(\sigma_\varepsilon,u_\varepsilon;B_r) \ \geq \int_{-\hat{r}}^{\hat{r}} \left[\inf_{(\vartheta,u)\in Y_{\varepsilon,a}(\tilde{m},r,\tilde{r})} E_{\varepsilon,a}(\vartheta,u;B_r)\right] \ \mathrm{d}t = \int_{-\hat{r}}^{\hat{r}} f_\varepsilon^{\tilde{r}}\,(\tilde{m}) \ \ \mathrm{d}t.$$

Taking the infimum limit, by Fatou's lemma and equation (2.6) of Proposition 2.1 we get

$$\liminf_{\varepsilon\downarrow 0} \mathcal{F}_{\varepsilon,a}(\sigma_\varepsilon,u_\varepsilon;B_r) \geq \int_{-\hat{r}}^{\hat{r}} \liminf_{\varepsilon\downarrow 0} f_\varepsilon^{\tilde{r}}\,(\tilde{m}) \ \ \mathrm{d}t \ \geq \ 2\,\hat{r}\,\kappa.$$

The latter holds for almost every $\hat{r}\in[(1-\xi)r_*, r_*]$ and eventually, since the $r_* = r/\sqrt{2}$, we conclude

$$\liminf_{\varepsilon\downarrow 0} \mathcal{F}_{\varepsilon,a}(\sigma_\varepsilon,u_\varepsilon;B_r) \geq \ \sqrt{2}\,\kappa\,r.$$

□

The proof of Proposition 3.1 is then obtained via the Besicovitch covering theorem [15].

3.2. Γ-liminf inequality

In this subsection we prove the $\Gamma - \liminf$ inequality stated in Theorem 1.2.

Proof of Theorem 1.2. With no loss of generality we assume that $\liminf_{\varepsilon\downarrow 0}\mathcal{F}_{\varepsilon,a}(\sigma_\varepsilon,u_\varepsilon) < +\infty$ otherwise the inequality is trivial. For a Borel set $A\subset\Omega$, we define

$$H(A) \ := \ \liminf_{\varepsilon\downarrow 0} \mathcal{F}_{\varepsilon,a}(\sigma_\varepsilon,u_\varepsilon;A),$$

so that H is a subadditive set function. By assumption, the limit measure σ is 1-rectifiable; we write $\sigma = m\,\nu\,\mathcal{H}^1 \llcorner \Sigma$. Furthermore we can assume σ to be compactly supported in Ω. Consider a convex open set Ω_0 such that $\mathrm{supp}(\nabla\cdot\sigma) = \mathscr{S} \subset\subset \Omega_0 \subset\subset \Omega$ and let $h := [0,1]\times\mathbf{R}^n \to \mathbf{R}^n$ be a smooth homotopy of the indentity map on $\mathbf{R}^n$ onto a contraction of $\overline{\Omega}$ into $\overline{\Omega}_0$ such that $h(t,\cdot)$ restricted to Ω_0 is the identity map, for any $t \in [0,1]$. Let $\sigma_t = h(t,\cdot)\sharp\sigma$, indeed $\liminf_{t\downarrow 0} \mathcal{F}(\sigma_t,1) \geq \mathcal{F}(\sigma,1)$ as $\sigma_t \stackrel{*}{\rightharpoonup} \sigma$. Further $\nabla\cdot\sigma_t = \nabla\cdot\sigma$ since $h(t,\cdot)$ is the identity on $\mathscr{S}$. Now we claim that

$$\liminf_{r\downarrow 0} \frac{H\left(\overline{B(x,r)}\right)}{2r} \geq f_a(m(x)) \qquad \text{for } \mathcal{H}^1\text{-almost every } x \in \Sigma. \tag{3.4}$$

Let us fix $\lambda \geq 1$ and let us note $f_{a,\lambda}(t) := \min(f_a(t),\lambda)$. We then introduce the Radon measure

$$H'_\lambda(A) := \int_{\Sigma\cap A} f_{a,\lambda}(m)\ \mathrm{d}\mathcal{H}^1.$$

Now, let $\delta \in (0,1)$. Assuming that (3.4) holds true, there exists $\Sigma' \subset \Sigma$ with $\mathcal{H}^1(\Sigma\backslash\Sigma_0) = 0$ such that for every $x \in \Sigma_0$, there exists $r_0(x) > 0$ with

$$(1+\delta)H\left(\overline{B(x,r)}\right) \geq 2rf_{a,\lambda}(m(x)) \qquad \text{for every } r \in (0,r_0(x)).$$

By the Besicovitch differentiation Theorem, there exists $\Sigma_1 \subset \Sigma$ with $\mathcal{H}^1(\Sigma\backslash\Sigma_1) = 0$ such that for every $x \in \Sigma_1$, there exists $r_1(x) > 0$ with

$$(1+\delta)2rf_a(m(x)) \geq H'_\lambda\left(\overline{B(x,r)}\right) \qquad \text{for every } r \in (0,r_1(x)).$$

We consider the familly $\mathcal{B}$ of closed balls $\overline{B(x,r)}$ with $x \in \Sigma_0\cap\Sigma_1$ and $0 < r < \min(r_0(x),r_1(x))$ and we apply the Vitali-Besicovitch covering theorem [1, Theorem 2.19.] to the family $\mathcal{B}$ and to the Radon measure H'_λ. We obtain a disjoint family of closed balls $\mathcal{B}' \subset \mathcal{B}$ such that

$$H'_\lambda(\Omega) = H'_\lambda(\Sigma) = \sum_{\overline{B(x,r)}\in\mathcal{B}'} H'_\lambda\left(\overline{B(x,r)}\right) \leq (1+\delta)^2 \sum_{\overline{B(x,r)}\in\mathcal{B}'} H\left(\overline{B(x,r)}\right) \leq (1+\delta)^2 H(\Omega).$$

Sending λ to infinity and then δ to 0, we get the lower bound $H(\Omega) \geq \int_\Sigma f_a(m)\,d\mathcal{H}^1$ which proves the theorem.

Let us now establish the claim (3.4). Since σ is a rectifiable measure, we have for $\mathcal{H}^1$-almost every $x \in \Sigma$,

$$\frac{1}{2r}\int \varphi(x+ry)\ \mathrm{d}|\sigma|(y) \stackrel{r\downarrow 0}{\longrightarrow} m(x)\int_{\mathbf{R}} \varphi(t\nu(x))\ \mathrm{d}t \qquad \text{for every } \varphi \in C_c(\mathbf{R}^n), \tag{3.5}$$

and

$$\frac{1}{2r}\int_{B(x,r)\cap\Sigma} |\nu(y)-\nu(x)|\ \mathrm{d}|\sigma|(y) \stackrel{r\downarrow 0}{\longrightarrow} 0. \tag{3.6}$$

Let $x \in \Sigma\backslash\mathcal{S}$ be such a point. Without loss of generality, we assume $x = 0$, $\nu(0) = e_1$ and $\overline{m} := m(0) > 0$. Let $\delta \in (0,1)$. Our goal is to establish a precise lower bound for $\mathcal{F}_{\varepsilon,a}(\sigma_\varepsilon,u_\varepsilon;C)$ where C is a cylinder of the form

$$C_r^\delta := \{x \in \mathbf{R}^n : |x_1| < \delta r, |x'| < r\}.$$

For this we proceed as in the proof of Lemma 3.3, here, the rectifiability of σ simplifies the argument. Let $\chi^\delta \in C_c^\infty(\mathbf{R}^{n-1},[0,1])$ be a radial cut-off function with $\chi^\delta(x') = 1$ if $|x'| \leq \delta/2$, $\chi^\delta(x') = 0$ if $|x'| \geq \delta$. For $\varepsilon > 0$ and $r \in (0,d(0,\partial\Omega))$, we define for $s \in (-r,r)$,

$$g_\varepsilon^{\delta,r}(s) := e_1\cdot\int_{\mathbf{R}^{n-1}} \sigma_\varepsilon(s,x')\,\chi^\delta(x'/r)\ \mathrm{d}x'.$$

We also introduce the mean value

$$\overline{g_\varepsilon^{\delta,r}} := \frac{1}{2r}\int_{-r}^{r} g_\varepsilon^{\delta,r}(s)\ \mathrm{d}s.$$

From (3.5), we have for $r > 0$ small enough,

$$\overline{g_0^{\delta,r}} := \frac{1}{2r}\int_{-r}^{r} e_1\cdot\int_{\mathbf{R}^{n-1}} \sigma_\varepsilon(s,x')\,\chi^\delta(x'/r)\ \mathrm{d}x\ \mathrm{d}s \geq (1-\delta)\overline{m}.$$

For such $r > 0$, we deduce from $\sigma_\varepsilon \stackrel{*}{\rightharpoonup} \sigma$ that for $\varepsilon > 0$ small enough

$$\overline{g_\varepsilon^{\delta,r}} := \frac{1}{2r}\int_{-r}^{r} g_\varepsilon^{\delta,r}(s)\ \mathrm{d}s \ \geq\ (1-2\delta)\overline{m}. \tag{3.7}$$

We study the variation of $g_\varepsilon^{\delta,r}(s)$. Using $\nabla\cdot\sigma_\varepsilon = 0$ in C_r^δ, we compute as in the proof of Lemma 3.3,

$$g_\varepsilon^{\delta,r}(t) - \overline{g_\varepsilon^{\delta,r}} \ =\ \frac{1}{2r}\int_{(-r,r)\times B_{\delta r}} \phi_t(x',h)\cdot\sigma_\varepsilon(x',h)\ \mathrm{d}x'\ \mathrm{d}h$$

with

$$\phi_t(h,x') \ =\ \begin{cases} (h+\hat{r})\,\nabla'\chi^\delta(x'/r) & \text{if } h < t,\\ (h-\hat{r})\,\nabla'\chi^\delta(x'/r) & \text{if } h > t.\end{cases}$$

Using again the convergence $\sigma_\varepsilon \stackrel{*}{\rightharpoonup} \sigma$, we deduce

$$g_\varepsilon^{\delta,r}(t) - \overline{g_\varepsilon^{\delta,r}} \ \stackrel{\varepsilon\downarrow 0}{\longrightarrow}\ \frac{1}{2r}\int_{(-r,r)\times B_{\delta r}} \phi_t(x',h)\cdot\ \mathrm{d}\sigma(x',h),$$

in $L^1(-r,r)$. Now, since $e_1\cdot\nabla'\chi^\delta \equiv 0$, we deduce from (3.6) that the right hand side goes to 0 as $r\downarrow 0$. Hence, for $r > 0$ small enough,

$$\left|\frac{1}{2r}\int_{(-r,r)\times B_{\delta r}} \phi_t(x',h)\cdot\sigma(x',h)\ \mathrm{d}x'\ \mathrm{d}h\right| \ \leq\ \delta\overline{m}.$$

Using (3.7), we conclude that for $r > 0$ small enough and then for $\varepsilon > 0$ small enough, we have

$$g_\varepsilon^{\delta,r}(t) \ \geq\ (1-3\delta)\overline{m}, \qquad \text{for a.e. } t\in(-r,r).$$

By definition of the codimension-0 problem, we conclude that

$$\mathcal{F}_{\varepsilon,a}(\sigma_\varepsilon,u_\varepsilon;C_r^\delta) \ \geq\ 2r f_{\varepsilon,a}^{n-1}\left((1-3\delta)\overline{m}\right).$$

Sending $\varepsilon\downarrow 0$, we obtain

$$H(C_r^\delta) \ \geq\ 2r f^{n-1}\left((1-3\delta)\overline{m}\right).$$

We notice that $H(B_{\sqrt{1+\delta^2}\,r}) \geq H(C_r^\delta)$. Dividing by $2\sqrt{1+\delta^2}\,r$ and taking the liminf as $r\downarrow 0$, we get

$$\liminf_{r\downarrow 0}\frac{H(B_{\sqrt{1+\delta^2}\,r})}{2\sqrt{1+\delta^2}\,r} \ \geq\ \frac{f_a\left((1-3\delta)\overline{m}\right)}{\sqrt{1+\delta^2}}.$$

Sending δ to 0, we get (3.4) by lower semi-continuity of f_a. □

3.3. Γ-limsup inequality

Proof of Theorem 1.3.
Let us suppose $\mathcal{F}(\sigma,u;\overline{\Omega}) < +\infty$, so that in particular $u\equiv 1$. From Xia [26], we can assume σ to be supported on a finite union of compact segments and to have constant multiplicity on each of them, namely polyhedral vector measures are dense in energy. We first construct a recovery sequence for a measure σ concentrated on a segment with constant multiplicity. Then we show how to deal with the case of a polyhedral vector measures.

Step 1. (σ concentrated on a segment.) Assume that σ is supported on the segment $I = [0,L]\times\{0\}$ and writes as $m\cdot e_1\mathcal{H}^1\llcorner_I$. Consider m constant so that $\nabla\cdot\sigma = m(\delta_{(0,0)} - \delta_{(L,0)})$ and

$$\mathcal{F}(\sigma,1;\Omega) = f_a(m)\,\mathcal{H}^1(I) = L\,f_a(m).$$

For $\delta > 0$ fixed, we consider the profiles

$$\overline{u}_\varepsilon(t) := \begin{cases} \eta, & \text{for } 0\leq t\leq r_*\varepsilon,\\ v_\delta\left(\dfrac{t}{\varepsilon}\right), & \text{for } r_*\varepsilon\leq t\leq r,\\ 1 & \text{for } r\leq t,\end{cases} \qquad \text{and} \qquad \vartheta_\varepsilon = \frac{m\,\chi_{B'_{r_*\varepsilon}}(x')}{\omega_{n-1}\,(\varepsilon r_*)^{n-1}}$$

with r_* and v_δ, defined in Proposition 2.2 with $d = n-1$. Assume $r_* \geq 1$ and let $d(x, I)$ be the distance function from the segment I and introduce the sets

$$I_{r_*\varepsilon} := \{x \in \Omega \ : \ d(x, I) \leq r_*\varepsilon\}, \qquad \text{and} \qquad I_r := \{x \in \Omega \ : \ d(x, I) \leq r\}.$$

Set $u_\varepsilon(x) = \overline{u}_\varepsilon(d(x, I))$ and $\overline{\sigma}^1_\varepsilon = m\mathcal{H}^1 \llcorner I * \rho_\varepsilon$, where ρ_ε is the mollifier of equation (1.3). We first construct the vector measures

$$\sigma^1_\varepsilon = \overline{\sigma}^1_\varepsilon\, e_1 \qquad \text{and} \qquad \sigma^2_\varepsilon(x_1, x') = \vartheta_\varepsilon(|x'|)\, e_1.$$

Alternatively, $\sigma^2_\varepsilon = \sigma * \tilde{\rho}_\varepsilon$ for the choice $\tilde{\rho}_\varepsilon(x_1, x') = \chi_{B'_{r_*\varepsilon}}(x')/\ \omega_{n-1}(\varepsilon r_*)^{n-1}$. Let us highlight some properties of σ^1_ε and σ^2_ε. Both vector measures are radial in x', with an abuse of notation we denote $\overline{\sigma}^1_\varepsilon(x_1, s) = \overline{\sigma}^1_\varepsilon(x_1, |x'|)$. Since, both σ^1_ε and σ^2_ε are obtained trough convolution it holds $\operatorname{supp}(\sigma^1_\varepsilon) \cup \operatorname{supp}(\sigma^2_\varepsilon) \subset I_{r_*\varepsilon}$ and they are oriented by the vector e_1 therefore $|\sigma^1_\varepsilon| = \overline{\sigma}^1_\varepsilon$ and $|\sigma^2_\varepsilon| = \vartheta_\varepsilon$. Furthermore for any x_1, it holds

$$\int_{\{x_1\}\times B'_{r_*\varepsilon}} \left[\overline{\sigma}^1_\varepsilon(x_1, x') - \vartheta_\varepsilon(x')\right]\, \mathrm{d}x' = 0 \tag{3.8}$$

We construct σ_ε by interpolating between σ^1_ε and σ^2_ε. To this aim consider a cutoff function $\zeta_\varepsilon : \mathbf{R} \to \mathbf{R}_+$ satisfying

$$\begin{aligned} &\zeta_\varepsilon(t) = 1 \qquad \text{for } t \leq r_*\varepsilon \text{ or } t \geq L - r_*\varepsilon, \\ &\zeta_\varepsilon(t) = 0 \qquad \text{for } 2\, r_*\varepsilon \leq t \leq L - 2\, r_*\varepsilon, \end{aligned} \qquad \text{and} \qquad |\zeta'_\varepsilon| \leq \frac{1}{r_*\varepsilon}.$$

and set

$$\begin{cases} \sigma^3_\varepsilon \cdot e_1 = 0, \\ \sigma^3_\varepsilon \cdot e_i(x_1, x') = -\zeta'_\varepsilon(x_1)\, \dfrac{x_i}{|x'|^{n-1}} \displaystyle\int_0^{|x'|} s^{n-2}\left[\overline{\sigma}^1_\varepsilon(x_1, s) - \vartheta_\varepsilon(s)\right]\, \mathrm{d}s, \qquad \text{for } i = 2, \ldots, n. \end{cases}$$

The integral corresponds to the difference of the fluxes of σ^1_ε and σ^2_ε through the $(n-1)$-dimensional disk $\{x_1\} \times B'$. For σ^3_ε we have the following

$$\begin{aligned} \nabla \cdot \sigma^3_\varepsilon = -\zeta'_\varepsilon(x_1) \sum_{i=2}^n \Bigg[\left(\frac{1}{|x'|^{n-1}} - \frac{(n-1)x_i^2}{|x'|^{n+1}} \right) \int_0^{|x'|} s^{n-2}\left[\overline{\sigma}^1_\varepsilon(x_1, s) - \vartheta_\varepsilon(s)\right]\, \mathrm{d}s \\ + \frac{x_i^2}{|x'|^2}\left[\overline{\sigma}^1_\varepsilon(x_1, |x'|) - \vartheta_\varepsilon(|x'|)\right] \Bigg] = -\zeta'_\varepsilon(x_1)\left[\overline{\sigma}^1_\varepsilon(x_1, |x'|) - \vartheta_\varepsilon(|x'|)\right] \end{aligned} \tag{3.9}$$

Let

$$\sigma_\varepsilon = \zeta_\varepsilon\, \sigma^1_\varepsilon + (1 - \zeta_\varepsilon)\, \sigma^2_\varepsilon + \sigma^3_\varepsilon.$$

In force of equation (3.9) and from construction of σ^1_ε, σ^2_ε and ζ_ε we have

$$\begin{aligned} \nabla \cdot \sigma_\varepsilon &= \nabla \cdot (\zeta_\varepsilon \sigma^1_\varepsilon) + \nabla \cdot (1 - \zeta_\varepsilon)\sigma^2_\varepsilon + \nabla \cdot \sigma^3_\varepsilon \\ &= \zeta_\varepsilon \nabla \cdot \sigma^1_\varepsilon + \zeta'_\varepsilon(\overline{\sigma}^1_\varepsilon - \vartheta_\varepsilon) + \nabla \cdot \sigma^3_\varepsilon \\ &= \zeta_\varepsilon \nabla \cdot \sigma^1_\varepsilon = \nabla \cdot (\sigma * \rho_\varepsilon) \end{aligned}$$

In addition for any (x_1, x') such that $|x'| \geq r_*\varepsilon$ from (3.8) we derive

$$\sigma^3_\varepsilon \cdot e_i(x_1, x') = -\zeta'_\varepsilon(x_1)\, \frac{x_i}{|x'|^{n-2}} \int_0^{|x'|} s^{n-1}\left[\overline{\sigma}^1_\varepsilon(x_1, s) - \vartheta_\varepsilon(s)\right]\, \mathrm{d}s = 0$$

which justifies $\operatorname{supp}(\sigma_\varepsilon) \subset I_{r_*\varepsilon}$. Let us now prove

$$\limsup_{\varepsilon \downarrow 0} \mathcal{F}_{\varepsilon,a}(\sigma_\varepsilon, u_\varepsilon; \Omega) \leq L\, f_a(m) + C\delta.$$

We split Ω as the union of $\Omega \setminus I_r$, $C_{r,\varepsilon} := I_r \cap [2\,\varepsilon, L - 2\,\varepsilon] \times \mathbf{R}^{n-1}$ and D_ε and D'_ε, as show in figure 2,

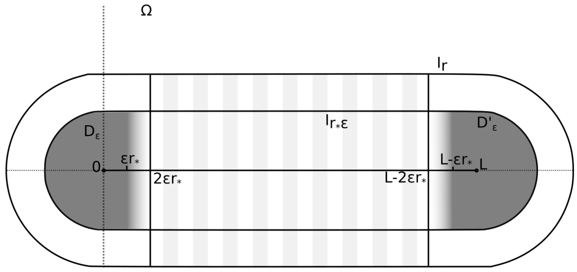

Figure 2: Illustration of the interval I and both its r and $(r_*\varepsilon)$-enlargement for $r_* \geq 1$. In grayscale we plot the levels of the function ζ_ε, whilst the striped region corresponds to the cylinder $C_{r,\varepsilon}$.

where $D_\varepsilon = \{x_1 \leq 2\, r_*\, \varepsilon\} \cap I_{r_*\varepsilon}$ and $D'_\varepsilon = \{x_1 \geq L - 2\, r_*\, \varepsilon\} \cap I_{r_*\varepsilon}$. On $\Omega \setminus I_r$ we notice that $\sigma_\varepsilon = 0$ and $u_\varepsilon = 1$ therefore

$$\mathcal{F}_{\varepsilon,a}(\sigma_\varepsilon, u_\varepsilon; \Omega \setminus I_r) = 0.$$

Observe that $|D_\varepsilon| = |D'_\varepsilon| = C\varepsilon^n$, then we have the upper bound

$$\int_{D_\varepsilon} |\sigma_\varepsilon|^2 \,\mathrm{d}x \leq 2\, \frac{m^2\, r_*^2}{\varepsilon^{n-2}} \left(\int_{B_1} \rho^2 \,\mathrm{d}x + C \right).$$

Taking into consideration this estimate we obtain

$$\mathcal{F}_{\varepsilon,a}(\sigma_\varepsilon, u_\varepsilon; D_\varepsilon) = \mathcal{F}_{\varepsilon,a}(\sigma_\varepsilon, u_\varepsilon; D'_\varepsilon) \leq \frac{(1-\eta)^2}{\varepsilon^{n-1}} \mathscr{L}^n(D_\varepsilon) + 2\, m^2\, r_*^2\, \frac{\eta}{\varepsilon^{n-2}}. \tag{3.10}$$

Finally on $C_{r,\varepsilon}$ both σ_ε and u_ε are independent of x_1 and are radial in x' then by Fubini's theorem and Proposition 2.2 we get

$$\mathcal{F}_{\varepsilon,a}(\sigma_\varepsilon, u_\varepsilon; C_{r,\varepsilon}) = \int_{2\,\varepsilon r_*}^{L-2\,\varepsilon r_*} \int_{B'_r} E_{\varepsilon,a}(\vartheta_\varepsilon, u_\varepsilon) \leq\; L\,(f_a(m) + C\,\delta).$$

Adding all together gives the desired estimate. It remains to discuss the case $r_* < 1$. From the point of view of the construction of σ_ε we need to replace the functions ζ_ε with

$$\begin{aligned} &\tilde{\zeta}_\varepsilon(t) = 1 \qquad \text{for } t \leq \varepsilon \text{ or } t \geq L - \varepsilon, \\ &\tilde{\zeta}_\varepsilon(t) = 0 \qquad \text{for } 2\,\varepsilon \leq t \leq L - 2\,\varepsilon, \end{aligned} \qquad \text{and} \qquad \left|\tilde{\zeta}'_\varepsilon\right| \leq \frac{1}{\varepsilon}.$$

This choice ensures that σ_ε has all the properties previously obtained with $r_*\,\varepsilon$ replaced by ε accordingly. Define

$$w_\varepsilon(t) := \begin{cases} \eta, & \text{for } t \leq \sqrt{3}\varepsilon \\ \dfrac{1-\eta}{r-\sqrt{3}}(t-\sqrt{3}) + \eta, & \text{for } \sqrt{3}\varepsilon \leq t \leq r. \end{cases}$$

and set

$$u_\varepsilon = \min\{\overline{u}_\varepsilon(d(x,I)), w_\varepsilon(|x|), w_\varepsilon(|x-(L;0)|)\}.$$

with these choices for u_ε and σ_ε the estimates follow analogously with small differences in the constants.

Step 2. (Case of a generic σ in polyhedral form.) Indeed, in force of the results quoted in Subsection 2.3 it is sufficient to show equation (1.3) for a polyhedral vector measure. Following the same notation introduced therein let

$$\sigma = \sum_{j=1}^{N} m_j \mathcal{H}^1 \llcorner \Sigma_j\, \nu_j.$$

With no loss of generality we can assume that the segments Σ_j intersect at most at their extremities. We consider measures σ satisfying constraint (1.2) so that if a point P belongs to $\Sigma_{j_1}, \ldots, \Sigma_{j_P}$ it must satisfy of Kirchhoff law,

$$\sum_{j_1}^{j_P} z_j\, m_j = \begin{cases} c_i, & \text{if } P \in \mathscr{S}. \\ 0, & \text{otherwise.} \end{cases} \tag{3.11}$$

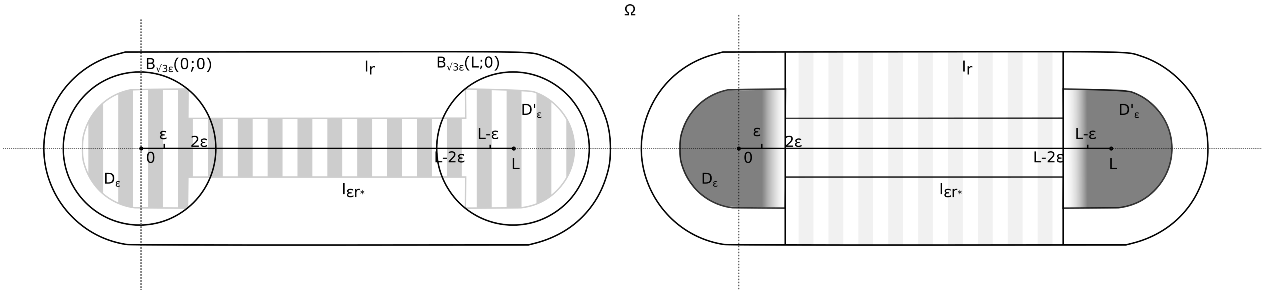

Figure 3: On the left the striped region corresponds to $\mathrm{supp}(\sigma_\varepsilon)$, remark that the balls of radius $\sqrt{3}\varepsilon$ centered respectively in $(0;0)$ and $(L;0)$ contain the modifications we have performed to satisfy the constraint. On the right we illustrate the level-lines of the cutoff function $\tilde{\zeta}_\varepsilon$ in grayscale.

where z_j, is $+1$ if P is the ending point of the segment Σ_j with respect to its orientation, and -1 if it is the starting point. Let σ_ε^j and u_ε^j be the sequences constructed above for each segment I_k and define

$$\sigma_\varepsilon = \sum_{j=1}^{N} \sigma_\varepsilon^j \qquad \text{and} \qquad u_\varepsilon = \min_j \left\{ u_\varepsilon^j \right\}.$$

Let P_j and Q_j be respectively the initial and final point of the segment Σ_j and recall that, by the construction made above, for each j

$$\nabla \cdot \sigma_\varepsilon^j = m_j \left(\delta_{P_j} - \delta_{Q_j} \right) * \rho_\varepsilon$$

then by linearity of the divergence operator, it holds

$$\nabla \cdot \sigma_\varepsilon = \sum_{j=1}^{N} \nabla \cdot \sigma_\varepsilon^k = \sum_{j=1}^{N} m_j \, \left(\delta_{P_j} - \delta_{Q_j} \right) * \rho_\varepsilon$$

and the latter satisfies constraint (1.3) in force of equation (3.11). To conclude let us prove that

$$\limsup_{\varepsilon \downarrow 0} \mathcal{F}_{\varepsilon,a}(\sigma_\varepsilon, u_\varepsilon; \Omega) \leq \sum_{j=1}^{N} f_a(m_j) \mathcal{H}^1(\Sigma_j). \tag{3.12}$$

Indeed the following inequality holds true

$$\mathcal{F}_{\varepsilon,a}(\sigma_\varepsilon, u_\varepsilon; \Omega) \leq \sum_{j=1}^{N} \mathcal{F}_{\varepsilon,a}(\sigma_\varepsilon, u_\varepsilon^j; \Omega).$$

Suppose

$$\mathrm{supp}(\sigma_\varepsilon^{j_1}) \cap \mathrm{supp}(\sigma_\varepsilon^{j_2}) \cap \cdots \cap \mathrm{supp}\, \sigma_\varepsilon^{j_P} \neq \varnothing$$

for some $j_1, \ldots, j_P$ and all ε. Let $r_*^{j_1}, \ldots, r_*^{j_P}$ be the radii introduced above for each of these measures, let $\overline{r}_* = \max\{r_*^{j_1}, \ldots, r_*^{j_P}, 1\}$, set $\overline{m} = \max\{m_{j_1}, \ldots, m_{j_P}\}$ and consider $D_{j_1}, \ldots, D_{j_P}$ as defined previously. Since

$$\left| \sum_{k=1}^{j_P} \sigma_\varepsilon^k \right|^2 \leq C \sum_{k=1}^{j_P} \left| \sigma_\varepsilon^k \right|^2$$

and $u_\varepsilon \leq u_\varepsilon^j$ for any j, we have the following inequality

$$\mathcal{F}_{\varepsilon,a}(\sigma_\varepsilon, u_\varepsilon; \mathrm{supp}(\sigma_\varepsilon^{j_1}) \cap \cdots \cap \mathrm{supp}(\sigma_\varepsilon^{j_P})) \leq C \sum_{k=j_1}^{j_P} \mathcal{F}_{\varepsilon,a}(\sigma_\varepsilon^k, u_\varepsilon^k; D_k)$$

And by inequality (3.10) follows

$$\mathcal{F}_{\varepsilon,a}(\sigma_\varepsilon, u_\varepsilon; \mathrm{supp}(\sigma_\varepsilon^{j_1}) \cap \cdots \cap \mathrm{supp}(\sigma_\varepsilon^{j_P})) \leq C \left(\frac{(1-\eta)^2}{\varepsilon^{n-1}} \sum_{k=j_1}^{j_P} \mathscr{L}^n(D_k) + 2\, \overline{m}^2\, \overline{r}_*^2 \, \frac{\eta}{\varepsilon^{n-2}} \right).$$

Which vanishes as $\varepsilon \downarrow 0$. Let us remark that the intersection $\operatorname{supp}(\sigma_\varepsilon^{j_1}) \cap \operatorname{supp}(\sigma_\varepsilon^{j_2}) \cap \cdots \cap \operatorname{supp} \sigma_\varepsilon^{j_P}$ is non empty for any ε only if the segments $\Sigma_{j_1}, \ldots, \Sigma_{j_P}$ have a common point. Since we are considering a polyhedral vector measure composed by N segments the worst case scenario is that we have $2N$ intersections in which at most N segments intersects. We conclude

$$\mathcal{F}_{\varepsilon,a}(\sigma_\varepsilon, u_\varepsilon; \Omega) \leq \sum_{j=1}^{N} \mathcal{F}_{\varepsilon,a}(\sigma_\varepsilon^j, u_\varepsilon^j; \Omega) + C(N) \left(\frac{(1-\eta)^2}{\varepsilon^{n-1}} \sum_{k=j_1}^{j_P} \mathscr{L}^n(D_k) + 2\,\overline{m}^2\, \overline{r}_*^2 \frac{\eta}{\varepsilon^{n-2}} \right)$$

which, passing to the limit, yields inequality (3.12). $\square$

4. The k-dimensional problem

4.1. Setting

Let $\sigma_0 \in P_k(\Omega)$ a polyhedral k-current with finite mass and let $\mathscr{S} := \operatorname{supp}(\partial \sigma_0)$ be compactly contained in Ω. We want to minimize a functional of the type (2.1) where the set of candidates ranges among all currents $\mathcal{D}_k(\overline{\Omega})$ such that

$$\partial \sigma = \partial \sigma_0 \qquad \text{in } \mathcal{D}^k(\mathbf{R}^n).$$

Let us introduce a parameter $\eta = \eta(\varepsilon)$ which satisfies

$$\eta(\varepsilon) = a\varepsilon^{n-k+1} \quad \text{ for } \quad a \in \mathbf{R}_+$$

and let $X_\varepsilon(\Omega)$ be the set of couples $(\sigma_\varepsilon, u_\varepsilon)$ where $u_\varepsilon \in W^{1,p}(\Omega, [\eta, 1])$ and has trace 1 on $\partial\Omega$ and σ_ε is of finite mass with density absolutely continuous with respect to $\mathscr{L}^n$. In this case we identify the current σ_ε with its $L^1(\Omega, \Lambda_k(\mathbf{R}^n))$ density. Furthermore as in equation (1.3) given a convolution kernel ρ_ε we impose the constraint

$$\partial \sigma_\varepsilon = (\partial \sigma_0) * \rho_\varepsilon \qquad \text{in } \mathcal{D}^k(\mathbf{R}^n).$$

For $(\sigma_\varepsilon, u_\varepsilon) \in \mathcal{D}_k(\overline{\Omega}) \times L^2(\Omega)$ let

$$\mathcal{F}_{\varepsilon,a}^k(\sigma_\varepsilon, u_\varepsilon; \Omega) := \begin{cases} \displaystyle\int_\Omega \left[\varepsilon^{p-n+k} |\nabla u_\varepsilon|^p + \frac{(1-u_\varepsilon)^2}{\varepsilon^{n-k}} + \frac{u_\varepsilon |\sigma_\varepsilon|^2}{\varepsilon} \right] \mathrm{d}x, & \text{if } (\sigma_\varepsilon, u_\varepsilon) \in X_\varepsilon(\Omega), \\ +\infty, & \text{otherwise.} \end{cases}$$

Let us denote with X the set of couples (σ, u) such that σ is a k-rectifiable current satisfying (4.1) and $u \equiv 1$. In this section we show that for any sequence $\varepsilon \downarrow 0$ the Γ-limit of the family $(\mathcal{F}_{\varepsilon,a}^k)_{\varepsilon \in \mathbf{R}_+}$ is the functional

$$\mathcal{F}_a^k(\sigma, u; \overline{\Omega}) = \begin{cases} \displaystyle\int_{\operatorname{supp} \sigma} f_a^{n-k}(m(x))\, \mathrm{d}\mathcal{H}^k(x), & \text{if } (\sigma, u) \in X \\ +\infty, & \text{otherwise in } \mathcal{M}(\Omega, \mathbf{R}^n) \times L^2(\Omega) \end{cases}$$

Where the function $f_a^{n-k} : \mathbf{R}_+ \to \mathbf{R}_+$ is the function obtained in Appendix A for the choice $d = n-k$ and is endowed with the same properties stated for f in Section 1. In particular under the assumption $p > n-k$ we first prove a compactness theorem.

Theorem 4.1. *Assume that $a > 0$. For any sequence $\varepsilon \downarrow 0$, $(\sigma_\varepsilon, u_\varepsilon) \in \mathcal{D}_k(\overline{\Omega}) \times L^2(\Omega)$ such that*

$$\mathcal{F}_{\varepsilon,a}^k(\sigma_\varepsilon, u_\varepsilon; \Omega) \leq F_0 < +\infty$$

then $u_\varepsilon \to 1$ and there exists a rectifiable k-current $\sigma \in \mathcal{D}_k(\overline{\Omega})$ such that, up to a subsequence, $\sigma_\varepsilon \overset{*}{\rightharpoonup} \sigma$ and $(\sigma, 1) \in X$.

Then we show the Γ-convergence result, namely

Theorem 4.2. *Assume that $a \geq 0$.*

1. *For any $(\sigma, u) \in \mathcal{D}_k(\overline{\Omega}) \times L^2(\Omega)$ and any sequence $(\sigma_\varepsilon, u_\varepsilon) \in \mathcal{D}_k(\overline{\Omega}) \times L^2(\Omega)$ such that $(\sigma_\varepsilon, u_\varepsilon) \to (\sigma, u)$ it holds*

$$\liminf_{\varepsilon \downarrow 0} \mathcal{F}_{\varepsilon,a}^k(\sigma_\varepsilon, u_\varepsilon; \Omega) \geq \mathcal{F}_a^k(\sigma, u; \overline{\Omega}).$$

2. For any couple $(\sigma, u) \in \mathcal{D}_k(\overline{\Omega}) \times L^2(\Omega)$ there exists a sequence $(\sigma_\varepsilon, u_\varepsilon) \in \mathcal{D}_k(\overline{\Omega}) \times L^2(\Omega)$ such that $(\sigma_\varepsilon, u_\varepsilon) \to (\sigma, u)$ and

$$\limsup_{\varepsilon\downarrow 0} \mathcal{F}^k_{\varepsilon,a}(\sigma_\varepsilon, u_\varepsilon; \Omega) \le \mathcal{F}^k_a(\sigma, u; \overline{\Omega}).$$

4.2. Compactness and k-rectifiability

Proof of Proposition 4.1. By the same procedure of Lemma 3.1 we obtain

$$|\sigma_\varepsilon|(\Omega) \le \frac{F_0}{2} + \frac{F_0}{2\,a\,(1-\lambda)^2} + \sqrt{\frac{|\Omega|\,\varepsilon\,F_0}{\lambda}} \tag{4.1}$$

and

$$\int_\Omega (1-u_\varepsilon)^2 \le \varepsilon^{n-k}\,F_0.$$

Therefore by the weak compactness of $\mathcal{D}_k(\Omega)$ we obtain the existence of a limit k-current σ a limit measure μ and a subsequence ε such that $\sigma_\varepsilon \overset{*}{\rightharpoonup} \sigma$, $|\sigma_\varepsilon| \overset{*}{\rightharpoonup} \mu$. As in the 1-dimensional case it is still necessary to prove the rectifiability of the limit current. This is obtained by showing that the support of σ is of finite size.

Step 1. (Preliminaries and good representative for $v \in \Lambda_k(\mathbf{R}^n)$.) Let us introduce the set

$$\mathcal{I} := \{I = (i_1, \dots, i_k) : 1 \le i_1 < i_2 < \cdots < i_k \le n\}, \qquad e_I = e_{i_1} \wedge \cdots \wedge e_{i_k}$$

So that $\Lambda_k(\mathbf{R}^n)$ is the Euclidean space with basis $\{e_I\}_{I\in\mathcal{I}}$. Let $v \in \Lambda_k(\mathbf{R}^n)$ and consider the problem

$$a_0 = \max\{a \in \mathbf{R} \,:\, v = af_1 \wedge \cdots \wedge f_k + t \,:\, (f_1, \dots, f_n) \text{ orthonormal basis, } t \in (f_1 \wedge \cdots \wedge f_k)^\perp\}.$$

Notice that $a_0 \ge 1/\sqrt{|\mathcal{I}|}$. Assume that the optimum for the preceding problem is obtained with $(f_1, \dots, f_n) = (e_1, \dots, e_n)$. We note

$$v = a_0 e_{I_0} + \sum_{i\in\mathcal{I}_1} a_I e_I + \sum_{I\in\mathcal{J}} a_I e_I$$

with

$$I_0 = e_1\wedge\cdots\wedge e_k, \qquad \mathcal{I}_1 := \{I = (i_1, \dots, i_k) \in \mathcal{I} \,:\, 1 \le i_1 < \cdots < i_{k-1} \le k < i_k \le n\}, \qquad \mathcal{J} := \mathcal{I}\backslash(\mathcal{I}_1 \cap I_0).$$

We claim that $a_I = 0$ for $I \in \mathcal{I}_1$. Indeed, let $I_1 = (e_1, \dots, e_{l-1}, e_{l+1}, \dots, e_k, e_h) \in \mathcal{I}_1$ and for $\phi \in \mathbf{R}$, let e^ϕ be orthonormal base defined as

$$e_i = e_i^\phi \qquad \text{for } i \ne \{l, h\}, \qquad e_l = \cos(\phi)e_l^\phi - \sin(\phi)e_h^\phi, \qquad e_h = \sin(\phi)e_l^\phi + \cos(\phi)e_h^\phi.$$

In this basis

$$v = \big(a_0\cos(\phi) + a_{I_1}(-1)^{k-l}\sin(\phi)\big)\, e^\phi_{I_0} + t^\phi, \quad \text{with} \quad w^\phi \in (e^\phi)^\perp.$$

By optimality of $(e_1, \dots, e_n)$ we deduce $a_{I_1} = 0$ which proves the claim. Hence we write

$$v = a_0 e_{I_0} + t, \quad \text{with } t \in \operatorname{span}\{e_I \,:\, I \in \mathcal{J}\}. \tag{4.2}$$

Now we let Let $\theta \in (0, 1/4^n)$ and Σ be the set of points for which there exists a sequence $r_j \downarrow 0$ such that

$$\frac{\sigma(B_{r_j}(x))}{|\sigma|(B_{r_j}(x))} \longrightarrow w(x) \in S\Lambda_k(\mathbf{R}^n) \quad \text{and} \quad \frac{|\sigma|(B_{r_j/4}(x))}{|\sigma|(B_{r_j}(x))} \ge \theta.$$

In particular w is a $|\sigma|$-measurable map and we have $\sigma = w\,|\sigma|\llcorner\Sigma$.

Step 2. (Flux of σ_ε trough a small $(n-k)$-disk.) Consider a point $x \in \Sigma\backslash\mathscr{S}$, with no loss of generality we assume $x = 0$. Let $v = w(0)$, up to a change of basis, by equation (4.2) we write

$$v = a_0 e_{I_0} + t, \quad \text{with } t \in \operatorname{span}\{e_I \,:\, I \in \mathcal{J}\}.$$

Let j sufficiently small, such that $B_{r_j} \cap \mathscr{S} = \varnothing$ and

$$\sigma(B_{r_j}) \cdot v \geq (1-\xi)|\sigma|(B_{r_j}). \tag{4.3}$$

Set, to simplify notation, $r_j = r$ and $r_* = r/\sqrt{2}$. For $x \in \mathbf{R}^n$ we write $(x', x'') \in \mathbf{R}^k \times \mathbf{R}^{n-k}$ for the usual decomposition and denote B'_r, B''_r the k-dimensional and the $(n-k)$-dimensional ball respectively. Let $\chi \in C^\infty(B''_1)$ be a radial cut-off function with $\chi(x'') = 1$ for $|x''| \leq 1/2$ and $\chi(x'') = 0$ for $|x''| \geq 3/4$. Set $\chi_{r_*}(x'') = \chi(x''/r_*)$, then since σ_ε is a L^1 function for $\varepsilon > 0$ we can define

$$g_\varepsilon(x') := \int_{B''_{r_*}} \chi_{r_*}(x'')\langle \sigma_\varepsilon, e_{I_0}\rangle \, \mathrm{d}x'' = \int_{B''_{r_*}} \chi_{r_*}(x'')\sigma^0_\varepsilon \, \mathrm{d}x'' \tag{4.4}$$

for any $x' \in B'_{r_*}$. Let us compute $\partial_l g_\varepsilon(x')$ for $l \in \{1, \dots, k\}$. Since $\partial\sigma_\varepsilon = 0$ in B_r, it holds $\langle \sigma_\varepsilon, \, \mathrm{d}\omega\rangle = 0$ for any smooth $(k-1)$-differential form $\omega \in \mathcal{D}^{k-1}(B_r)$. Choosing ω of the form

$$\omega = \beta(x) \, \mathrm{d}x_1 \wedge \dots \, \mathrm{d}x_{l-1} \wedge \, \mathrm{d}x_{l+1} \wedge \dots \wedge \, \mathrm{d}x_k \tag{4.5}$$

we obtain

$$\mathrm{d}\omega = (-1)^{l-1}\partial_l\beta(x) \, \mathrm{d}x_1 \wedge \dots \wedge \, \mathrm{d}x_k + (-1)^{k-1} \sum_{h=k+1}^{d} \partial_h\beta(x) \, \mathrm{d}x_1 \wedge \dots \, \mathrm{d}x_{l-1} \wedge \, \mathrm{d}x_{l+1} \wedge \dots \wedge \, \mathrm{d}x_k \wedge \, \mathrm{d}x_h.$$

Denote $\sigma^I_\varepsilon = \langle \sigma, e^I\rangle$, then imposing $\langle \sigma_\varepsilon, \, \mathrm{d}\omega\rangle = 0$ for every $\beta \in C^\infty_c(B_r)$ in (4.5) yields

$$(-1)^{k-l}\partial_l\sigma^0_\varepsilon + \sum_{\substack{h\in\{k+1,\dots,d\} \\ I=(1,\dots,l-1,l+1,\dots,k,h)}} \partial_h\sigma^I_\varepsilon = 0.$$

Hence,

$$\partial_l g_\varepsilon(x') = \frac{(-1)^{k-l}}{r_*} \sum_{\substack{h\in\{k+1,\dots,d\} \\ I=(1,\dots,l-1,l+1,\dots,k,h)}} \int_{B''_{r_*}} \partial_h\chi_{r_*}(x'')\sigma^I_\varepsilon \, \mathrm{d}x''. \tag{4.6}$$

Let us introduce the notation

$$\sigma^{\mathcal{I}_1}_\varepsilon := \sum_{I\in\mathcal{I}_1} \sigma^I_\varepsilon \, e_I,$$

denoting with ∇' the gradient with respect to x', equation (4.6) rewrites as

$$\nabla' g_\varepsilon(x') = \frac{1}{r_*} \int_{B''_{r_*}} Y\left(\frac{x}{r_*}\right) \sigma^{\mathcal{I}_1}_\varepsilon \, \mathrm{d}x''. \tag{4.7}$$

Where Y is smooth and compactly supported in B''_1 and with values into the linear maps : $\mathrm{span}\{e_I : I \in \mathcal{I}_1\} \to \mathbf{R}^k$. Let us prove that, for some $\hat{r}$, the functions g_ε converge in BV-$*$ to some g. First for a.e. choice of $\hat{r} \in [(1-\xi)r_*, r_*]$ it must hold $\mu(\partial B'_{r_*} \times B''_{r_*}) = 0$ so that

$$g_\varepsilon(x') = \int_{B''_{r_*}} \chi_{r_*}(x'')\langle \sigma_\varepsilon, e_{I_0}\rangle \, \mathrm{d}x'' \xrightarrow{\varepsilon\downarrow 0} \int_{B''_{r_*}} \chi_{r_*}(x'') \, \mathrm{d}\langle \sigma, e_{I_0}\rangle =: g(x'). \tag{4.8}$$

Secondly we define the mean value

$$\overline{g} := \frac{1}{|B'_{\hat{r}}|} \int_{B'_{\hat{r}}} g(x') \, \mathrm{d}x' = \frac{1}{|B'_{\hat{r}}|} \int_{B'_{\hat{r}}} \left[\int_{B''_{r_*}} \chi_{r_*}(x'') \, \mathrm{d}\sigma^0\right] \, \mathrm{d}x'.$$

and taking advantage of (4.3) and the definition of Σ, we see that

$$\overline{g} \geq \left(\frac{\theta}{\sqrt{|\mathcal{I}|}} - \xi\right) \frac{|\sigma|(B_r)}{|B'_{\hat{r}}|} > 0.$$

On the other hand, denoting $\Pi : \mathbf{R}^n \to \mathbf{R}^{n-k}$, $x \mapsto x''$, from (4.2), we have

$$|\Pi\sigma|(B'_{\hat r} \times B''_{r_*}) \le \sqrt{3\xi}\left(\frac{\theta}{\sqrt{|\mathcal{I}|}} - \xi\right) |B'_{\hat r}|\,\overline{g}.$$

Now from (4.7) - - (4.8) and the latter we obtain

$$\langle D'g, \phi\rangle = \frac{1}{r_*}\int_{B'_{\hat r}\times B''_{r_*}} \phi(x')\, Y\left(\frac{x''}{r_*}\right)\, \mathrm{d}\sigma^{\mathcal{I}_1} \quad \text{ and } \quad |D'g|(B'_{\hat r}) \le \frac{C\,|B'_{\hat r}|\,\sqrt{\xi}\,\overline{g}}{r_*}.$$

Finally from Poincaré - Wirtinger inequality and the convergence $g_\varepsilon \to g$ in $L^1(B'_{\hat r})$ is easy to show that for any sufficiently small ε the sets

$$A_\varepsilon = \left\{x \in B_{\hat r}\ :\ g_\varepsilon(x) \ge \frac{\overline{g}}{8}\right\}$$

are such that $|A_\varepsilon| \ge |B'_{\hat r}|/2$.

Step 3. (Conclusion.) Set $\vartheta_\varepsilon(x', x'') = |\chi_{r_*}(x'')\sigma^0_\varepsilon|$ and observe that for fixed x' by construction

$$\int_{B_{r_*}} \vartheta_\varepsilon(x', x'')\, \mathrm{d}x'' = g_\varepsilon(x').$$

Therefore for any $x' \in A_\varepsilon$ it holds $\int_{B_{r_*}} \vartheta_\varepsilon(x', x'')\ \mathrm{d}x'' \ge \overline{g}/8$. Furthermore $\mathrm{supp}(\vartheta_\varepsilon(x')) \subset B'_{\tilde r}$ with $\tilde r := \frac{3}{4}r_* < r_*$. Now, by Fubini

$$\begin{aligned}\mathcal{F}^k_{\varepsilon,a}(\sigma_\varepsilon, u_\varepsilon; B_r) &\ge \int_{A_\varepsilon}\int_{B''_r}\left[\varepsilon^{p-n+k}|\nabla u_\varepsilon|^p + \frac{(1-u_\varepsilon)^2}{\varepsilon^{n-k}} + \frac{u_\varepsilon|\sigma_\varepsilon|^2}{\varepsilon}\right]\ \mathrm{d}x''\ \mathrm{d}x' \\ &\ge \int_{A_\varepsilon}\int_{B''_{r_*}}\left[\varepsilon^{p-n+k}|\nabla u_\varepsilon|^p + \frac{(1-u_\varepsilon)^2}{\varepsilon^{n-k}} + \frac{u_\varepsilon|\vartheta_\varepsilon(x',x'')|^2}{\varepsilon}\right]\ \mathrm{d}x''\ \mathrm{d}x'\end{aligned}$$

With the notation introduced in Subsection 2.4 and by defintion of A_ε

$$\mathcal{F}_{\varepsilon,a}(\sigma_\varepsilon, u_\varepsilon; B_r) \ \ge\ \int_{A_\varepsilon}\inf_{(\vartheta,u)\in\overline{Y}_{\varepsilon,a}(m,r)^{\tilde r}(\overline{g}/8,r)} E^k_{\varepsilon,a}(\vartheta,u)\ \mathrm{d}x' = \int_{A_\varepsilon} f^{\tilde r}_\varepsilon\,(\overline{g}/8)\ \ \mathrm{d}x' = f^{\tilde r}_\varepsilon\,(\overline{g}/8)\,|A_\varepsilon|.$$

Taking the infimum limit, by Proposition 2.1, in particular equation (2.6) we get

$$\liminf_{\varepsilon\downarrow 0}\mathcal{F}^k_{\varepsilon,a}(\sigma_\varepsilon, u_\varepsilon; B_r) \ge \liminf_{\varepsilon\downarrow 0} f^{\tilde r}_\varepsilon\,(\overline{g}/8)\,|A_\varepsilon| \ \ge\ \kappa\ \frac{|B'_{\hat r}|}{2}. \tag{4.9}$$

Recall that the latter stands for a.e. $\hat r \in [(1-\xi)r_*, r_*]$ and $r_* = r/\sqrt{2}$ thus we may rewrite

$$\liminf_{\varepsilon\downarrow 0}\mathcal{F}^k_{\varepsilon,a}(\sigma_\varepsilon, u_\varepsilon; B_r) \ \ge\ \kappa\ \frac{\omega_k\ r^k}{2^{1+k/2}}.$$

As in Lemma 3.3 we conclude applying Besicovitch theorem to obtain $\mathcal{H}^k(\Sigma) < +\infty$. Finally, thanks to the latter and equation (4.1), Theorem 2.1 applies and σ is a k-rectifiable current. $\square$

4.3. Γ-liminf inequality

Proof of item 1) of Theorem 4.2. With no loss of generality we assume that $\liminf_{\varepsilon\downarrow 0}\mathcal{F}^k_{\varepsilon,a}(\sigma_\varepsilon, u_\varepsilon) < +\infty$ otherwise the inequality is trivial. For a Borel set $A \subset \Omega$, we define

$$H^k(A) \ :=\ \liminf_{\varepsilon\downarrow 0}\mathcal{F}^k_{\varepsilon,a}(\sigma_\varepsilon, u_\varepsilon; A),$$

so that H^k is a subadditive set function. By assumption, the limit current σ is k-rectifiable; we write $\sigma = m\,\nu\,\mathcal{H}^k \llcorner \Sigma$. We claim that

$$\liminf_{r\downarrow 0}\frac{H^k\left(\overline{B(x,r)}\right)}{\omega_k\, r^k} \ \ge\ f^{n-k}_a(m(x)) \qquad \text{for } \mathcal{H}^k\text{-almost every } x \in \Sigma. \tag{4.10}$$

Assuming the latter the proof is achieved as in Theorem 1.2. To establish the claim (4.10) we restrict our attention to a single point and we assume $x = 0$, $m = m(0)$ and $\nu(0) = e_1 \wedge \cdots \wedge e_k$ then for any $\xi > 0$ there exists $r_0 = r(\xi)$ such that

$$\langle \sigma, e_1 \wedge \cdots \wedge e_k \rangle(B_r) \geq (1-\xi)|\sigma|(B_r) \quad \text{and} \quad (1-\xi)\, m \leq \frac{|\sigma|(B_r)}{\omega_k r^k} \leq (1+\xi)\, m, \qquad \text{for } r \leq r_0. \tag{4.11}$$

Let δ be an infinitesimal quantity and set, for $r < r_0$, $\hat{r} = \sqrt{1-\delta^2}\, r$ and $\tilde{r} = \delta r$ and define the cylinder

$$C_{\delta,r}(e_1, \wedge \cdots \wedge e_n) = C_{\delta,r} := \left\{ (x'; x'') \in \mathbf{R}^k \times \mathbf{R}^{n-k} : |x'| \leq \hat{r} \text{ and } |x''| \leq \tilde{r} \right\}.$$

Let $\chi(x'')$ be the radial cutoff introduced in the previous proposition and set $\chi_{\tilde{r}}(x'') = \chi(x''/\tilde{r})$, $\sigma^0_\varepsilon = \langle \sigma_\varepsilon, e_1 \wedge \cdots \wedge e_k \rangle$ and for any $x' \in B'_{\hat{r}}$ set

$$g_\varepsilon(x') := \int_{B''_{\tilde{r}}} \chi_{\tilde{r}}(x'')\, \mathrm{d}\langle \sigma_\varepsilon, e_{I_0} \rangle = \int_{B''_{\tilde{r}}} \chi_{\tilde{r}}(x'')\, \mathrm{d}\sigma^0_\varepsilon,$$

as in equation (4.4). Up to a smaller choice for r_0 we can assume $B_r \cap \mathscr{S} = \varnothing$ therefore $\partial\sigma \llcorner B_r = 0$, and from equations (4.4) - (4.7) it holds

$$\nabla' g_\varepsilon(x') = \frac{1}{\tilde{r}} \int_{B''_{\tilde{r}}} Y\left(\frac{x}{\tilde{r}}\right)\, \mathrm{d}\sigma^{\mathcal{I}_1}_\varepsilon.$$

For a.e. choice of δ it holds $|\sigma|(\partial B'_{\hat{r}} \times B''_{\tilde{r}}) = 0$ therefore, for any such choice, γ_ε converges in $BV(B_{\hat{r}})$ to

$$g(x') := \int_{B''_{\tilde{r}}} \chi_{\tilde{r}}(x'')\, \mathrm{d}\sigma^0 \qquad \text{and} \qquad \langle D'g, \phi \rangle = \frac{1}{\tilde{r}} \int_{B'_{\hat{r}} \times B''_{\tilde{r}}} \phi(x')\, Y\left(\frac{x''}{\tilde{r}}\right)\, \mathrm{d}\sigma^{\mathcal{I}_1}.$$

Now we use (4.11) to improve the estimates on $\overline{g}$ and $|D'g|$. Indeed, for δ sufficiently small, $\tilde{r} < \hat{r}/2$ therefore $B_{\tilde{r}} \subset B'_{\hat{r}} \times B''_{\tilde{r}}$ and

$$\lim_{\varepsilon \downarrow 0} \overline{g}_\varepsilon \geq (1-\xi) \frac{1}{|B''_{\hat{r}}|} \int_{B'_{\hat{r}} \times B''_{\tilde{r}}} \chi_{r_*}(x')\, \mathrm{d}|\sigma| \geq (1-\xi)^2 m.$$

and denoting $\Pi : \mathbf{R}^n \to \mathbf{R}^{n-k}$, $x \mapsto x''$ we have

$$|\Pi\sigma|(C_r) \leq (1+\xi)\sqrt{3\xi}\; |B'_r|\; m \quad \text{and} \quad |D'g|(B'_{\hat{r}}) \leq \frac{C\, |B'_{\hat{r}}|\, \sqrt{\xi}\, m}{\tilde{r}}.$$

Choose r sufficiently small then by Poincaré - Wirtinger inequality there exists a set A of almost full measure in $B_{\hat{r}}$ such that $g_\varepsilon(x') \geq (1-\xi)^2\, m$, and following the proof of the previous lemma (Step 3) up to equation (4.9) we get

$$\liminf_{\varepsilon \downarrow 0} \mathcal{F}^k_{\varepsilon,a}(\sigma_\varepsilon, u_\varepsilon; B_r) \geq \liminf_{\varepsilon \downarrow 0} f^{n-k}_{\varepsilon,a}\left((1-\xi)^2\, m, r, \tilde{r}\right) |A|.$$

Since ξ and δ are arbitrary and $|A|$ can be chosen arbitrary close to $|B_{\hat{r}}|$ applying Proposition 2.1 with $d = n - k$ to the latter we conclude

$$\liminf_{\varepsilon \downarrow 0} \mathcal{F}^k_{\varepsilon,a}(\sigma_\varepsilon, u_\varepsilon; B_r) \geq f^{n-k}_a(m)\, \omega_k r^k.$$

□

4.4. Γ-limsup inequality

For the lim-sup inequality, we start by approximating σ with a polyhedral current: given $\delta > 0$, there exists a k polyhedral current $\tilde{\sigma}$ satisfying $\partial\tilde{\sigma} = \partial\sigma_0$ and with $\mathbb{F}(\tilde{\sigma} - \sigma) < \delta$ and $\mathcal{F}_a(\tilde{\sigma}) < \mathcal{F}_a(\sigma) + \varepsilon$. This result of independent interest is established in [11]. A similar result has been proved recently by Colombo et al. in [12, Prop. 2.6] (see also [23, Section 6]). The authors build an approximation of a k-rectifiable current in flat norm and in energy but their construction creates new boundaries and can not ensure the condition $\partial\sigma = \partial\sigma_0$.

Proof of item 2) of Theorem 4.2:
By [11, Theorem 1.1 and Remark 1.6] we can assume that σ is a polyhedral current. We show how to produce the approximating $(\sigma_\varepsilon, u_\varepsilon)$ for σ supported on a single $k-$dimensional simplex Q. We assume with no loss of generality that $Q \subset \mathbf{R}^k$, and that σ writes as

$$m\, \mathcal{H}^k \llcorner_Q \wedge (e_1 \wedge \cdots \wedge e_k).$$

For $\delta > 0$ fixed, we consider the optimal profiles

$$\overline{u}_\varepsilon(t) := \begin{cases} \eta, & \text{for } 0 \le t \le r_*\varepsilon, \\ v_\delta\left(\dfrac{t}{\varepsilon}\right), & \text{for } r_*\varepsilon \le t \le r, \\ 1 & \text{for } r \le t, \end{cases} \qquad \text{and} \qquad \vartheta_\varepsilon = \frac{m\, \chi_{B''_{r_*\varepsilon}}(x'')}{\omega_{n-k}\, (\varepsilon r_*)^{n-k}}$$

with r_* and v_δ, defined in Proposition 2.2 for the choice $d = n-k$. We denote ∂Q the relative boundary of Q and given a set S we write $d(x, S)$ for the distance function from S. Recall that we use the notation S_t for the t-enlargement of the set S and S' to denote its projection into $\mathbf{R}^k$. We first assume, as did for the case $k = 1$, $r_* \ge 1$, and introduce ζ_ε a 0-form depending on the first k variables x', satisfying

$$\zeta_\varepsilon(x') = 1, \qquad \text{for } x' \in (\partial Q)'_{r_*\varepsilon} := \{x \in \Omega \ :\ d(x', \partial Q) \le r_*\varepsilon\},$$
$$\zeta_\varepsilon(x') = 0, \qquad \text{for } x' \in \Omega \setminus (\partial Q)'_{2r_*\varepsilon},$$
$$|\, \mathrm{d}\zeta_\varepsilon| \le \frac{1}{r_*\varepsilon}.$$

Then we proceed by steps, first set $\overline{\sigma}^1_\varepsilon := (|\sigma| * \rho_\varepsilon)$

$$\sigma^1_\varepsilon = \overline{\sigma}^1_\varepsilon e_1 \wedge \cdots \wedge e_k \qquad \text{and} \qquad \sigma^2_\varepsilon(x', x'') = \vartheta_\varepsilon(|x''|) \wedge (e_1 \wedge \cdots \wedge e_k).$$

and observe that $\mathrm{supp}(\sigma^1_\varepsilon) \cup \mathrm{supp}(\sigma^2_\varepsilon) \subset Q_{r_*\varepsilon}$, both σ^1_ε and σ^2_ε are radial in x'' and with a small abuse of notation we denote $\overline{\sigma}^1_\varepsilon(x', s) = \overline{\sigma}^1_\varepsilon(x', |x''|)$, finally for any x'

$$\int_{\{x'\} \times B''_{r_*\varepsilon}} [\overline{\sigma}^1_\varepsilon(x', |x''|) - \vartheta_\varepsilon(|x''|)]\, \mathrm{d}x'' = 0.$$

Now we take advantage of ζ_ε in order to interpolate between σ^1_ε and σ^2_ε, note that such interpolation may affect the boundary of the new current therefore we first introduce σ^3_ε which corrects this defect. In particular set

$$\sigma^3_\varepsilon(x', x'') = -\sum_{i=k+1}^{n} \left[\frac{x_i}{|x''|^{n-k}} \int_0^{|x''|} s^{n-k-1} \left[\overline{\sigma}^1_\varepsilon(x', s)\vartheta_\varepsilon(s)\right] \llcorner\, \mathrm{d}\zeta_\varepsilon\, \mathrm{d}s \right] \wedge e_i,$$

and

$$\sigma_\varepsilon = \sigma^1_\varepsilon \llcorner \zeta_\varepsilon + \sigma^2_\varepsilon \llcorner (1 - \zeta_\varepsilon) + \sigma^3_\varepsilon.$$

With this choice by a calculation similar to equation (3.9) it holds

$$\partial\sigma_\varepsilon = -\partial\sigma * \rho_\varepsilon \llcorner \zeta_\varepsilon - \sigma^1_\varepsilon \llcorner\, \mathrm{d}\zeta_\varepsilon - \underbrace{\partial\sigma^2_\varepsilon \llcorner (1 - \zeta_\varepsilon)}_{=0} + \sigma^2_\varepsilon \llcorner\, \mathrm{d}\zeta_\varepsilon + \partial\sigma^3_\varepsilon = (\partial\sigma) * \rho_\varepsilon.$$

On the other hand the phase-field is simply defined as $u_\varepsilon(x) = \overline{u}_\varepsilon(d(x, Q))$. In the case $r_* < 1$ we need to modify the construction. For σ_ε it is sufficient to replace every occurrence of ζ_ε with $\tilde{\zeta}_\varepsilon$, which satisfies

$$\tilde{\zeta}_\varepsilon(x') = 1, \qquad \text{for } x' \in (\partial Q)'_\varepsilon := \{x \in \Omega \ :\ d(x', \partial Q) \le \varepsilon\},$$
$$\tilde{\zeta}_\varepsilon(x') = 0, \qquad \text{for } x' \in \Omega \setminus (\partial Q)'_{2\varepsilon},$$
$$\left|\, \mathrm{d}\tilde{\zeta}_\varepsilon \right| \le \frac{1}{\varepsilon}.$$

Now let

$$w_\varepsilon(t) := \begin{cases} \eta, & \text{for } t \le \sqrt{3}\varepsilon, \\ \dfrac{1-\eta}{r - \sqrt{3}}(t - \sqrt{3}) + \eta, & \text{for } \sqrt{3}\varepsilon \le t \le r. \end{cases}$$

and set

$$u_\varepsilon = \min\{\overline{u}_\varepsilon(d(x, Q)), w_\varepsilon(d(x, \partial Q))\}.$$

Remark 3. *Given a polyhedral current σ such that $\partial\sigma = \partial\sigma_0$ we perform our construction on each simplex and define σ_ε as the sum of these elements. The linearity of the boundary operator grants that $\partial\sigma_\varepsilon = \partial\sigma_0 * \rho_\varepsilon$. The phase field is chosen as the pointwise minimum of the local phase fields. Finally the estimation for the Γ-limsup inequality is achieved in the same manner as Theorem 1.3.*

□

5. Discussion about the results

By Lemma A.4 for any fixed $d = n - k$ the cost function f_a^d pointwise converges as $a \downarrow 0$ to the function

$$f(m) = \begin{cases} \kappa, & \text{for } m > 0, \\ 0, & \text{if } m = 0, \end{cases}$$

where κ is the constant value obtained in Proposition 2.1 and depends on d. This condition is sufficient to prove that the family of functionals $\mathcal{F}_a^k$, parametrized in a, Γ-converges to the functional

$$\mathcal{F}^k(\sigma;\Omega) := \begin{cases} \kappa\, \mathcal{H}^k(\Sigma \cap \Omega), & \text{for } \sigma = m\,\nu\,\mathcal{H}^k \llcorner \Sigma, \\ +\infty, & \text{otherwise.} \end{cases}$$

As a matter of fact for any sequence $\sigma_a \overset{*}{\rightharpoonup} \sigma$ in $\mathcal{D}_k(\Omega)$ it holds

$$\liminf_{a\downarrow 0} \mathcal{F}_a^k(\sigma;\Omega) \geq \mathcal{F}^k(\sigma;\Omega)$$

since $f_a^d(m) \geq \kappa$. On the other hand setting $\sigma_a := \sigma$ we construct a recovery sequence for any σ and obtain the Γ-limsup inequality

$$\limsup_{a\downarrow 0} \mathcal{F}_a^k(\sigma_a;\Omega) = \limsup_{a\downarrow 0} \mathcal{F}_a^k(\sigma;\Omega) = \mathcal{F}^k(\sigma;\Omega).$$

This allows to interpret our result as an approximation of the Plateau problem in any dimension and co-dimension.

A. Reduced problem in dimension $n - k$

A.1. Auxiliary problem

In this appendix we show the results previously enunciated in Subsection 2.3, with the notation introduced therein let us define the auxiliary set

$$\overline{Y}_{\varepsilon,a}(m,r) = \left\{(\vartheta,u) \in L^2(B_r) \times W^{1,p}(B_r,[\eta,1]) \,:\, \|\vartheta\|_1 = m \text{ and } u_{|\partial B_r} \equiv 1\right\},$$

and the associated minimization problem

$$\overline{f}^d_{\varepsilon,a}(m,r) = \inf_{\overline{Y}_{\varepsilon,a}(m,r)} E_{\varepsilon,a}(\vartheta,u;B_r). \tag{A.1}$$

First we show that both $f^d_{\varepsilon,a}(m,r,\tilde r)$ and $\overline{f}^d_{\varepsilon,a}(m,r)$ are bounded by the same constant as $\varepsilon \downarrow 0$ and that the value of the second term is achieved by a radially symmetric couple of $\overline{Y}_{\varepsilon,a}(m,r)$. These two facts are then used to show that for each m the limit values of $\overline{f}^d_{\varepsilon,a}(m,r)$ and $f^d_{\varepsilon,a}(m,r,\tilde r)$ as $\varepsilon \downarrow 0$ are equal and independent of the choices $(r,\tilde r)$ to the extent that $0 < \tilde r < r$. Let us start by showing the first two properties.

Lemma A.1. *For each ε, $m > 0$ and $r > 0$*

a) there exists a constant $C = C(m) \leq C_0\sqrt{1+m^2}$ such that

$$f^d_{\varepsilon,a}(m,r,\tilde r) < C \qquad \text{and} \qquad \overline{f}^d_{\varepsilon,a}(m,r) < C.$$

b) Both the problem defined in equation (2.2) *and equation* (A.1) *admit a minimizer. Moreover among the minimizers of $E_{\varepsilon,a}$ in $\overline{Y}_{\varepsilon,a}(m,r)$ it is possible to choose a radially symmetric couple $(\vartheta_\varepsilon, u_\varepsilon)$ such that u_ε is radially non-decreasing and ϑ_ε is radially non-increasing.*

Proof. a) To show the bound it is sufficient to define

$$u_\varepsilon(x) := \begin{cases} \eta & \text{if } |x| < r_1\varepsilon, \\ \eta + \dfrac{1-\eta}{(r_2 - r_1)\varepsilon}(|x| - r_1\varepsilon) & \text{if } r_1\varepsilon \le |x| < r_2\varepsilon, \\ 1 & \text{if } r_2\varepsilon \le |x| < r, \end{cases} \qquad \vartheta_\varepsilon(x) := \begin{cases} \dfrac{m}{|B_{r_1\varepsilon}|} & \text{if } |x| < r_1\varepsilon, \\ 0 & \text{if } r_1\varepsilon \le |x| < r. \end{cases}$$

Evaluating the energy we get, for any choice of $r_1 < r_2 < r$,

$$E_{\varepsilon,a}(u_\varepsilon, \vartheta_\varepsilon) \le \frac{a\, m^2}{\omega_d\ r_1^d} + \omega_d \left[r_1^d + \frac{1}{(r_2 - r_1)^2} \left(\frac{r_2^d - r_1^d}{d}\, r_2^2 - \frac{r_2^{d+1} - r_1^{d+1}}{d+1}\, 2\, r_2 + \frac{r_2^{d+2} - r_1^{d+2}}{d+2} \right) \right].$$

As soon as $r_1\varepsilon < \tilde{r}$, we have $(\vartheta_\varepsilon, u_\varepsilon) \in Y_{\varepsilon,a}(m,r,\tilde{r}) \cap \overline{Y}_{\varepsilon,a}(m,r)$. Choosing $r_1 = (\sqrt{a}m)^{1/d}$ and $r_2 = (1 + \sqrt{a}m)^{1/d}$, we get

$$\max\{f^d_{\varepsilon,a}(m,r,\tilde{r}), \overline{f}^d_{\varepsilon,a}(m,r)\} \le C_0\sqrt{1+m^2}.$$

b) To show the existence of minimizers for both minimization problems we use the direct method of the Calculus of Variation. The lower semicontinuity of the integral with integrand $u|\vartheta|^2$ is ensured by Ioffe's theorem [1, theorem 5.8]. Now given any minimizing couple $(\hat{\vartheta}_\varepsilon, \hat{u}_\varepsilon) \in \overline{Y}_{\varepsilon,a}(m,r)$, let ϑ_ε be the decreasing Steiner rearrangement of $\hat{\vartheta}_\varepsilon$ and u_ε the increasing rearrangement of $\hat{u}_\varepsilon$. Indeed, since $\hat{u}_\varepsilon$ has range in $[\eta, 1]$, we still have $u_\varepsilon\,_{|\partial B_r} \equiv 1$. Polya's Szego and Hardy-Littlewood's inequalities ensure

$$E_{\varepsilon,a}(\vartheta_\varepsilon, u_\varepsilon) \le E_{\varepsilon,a}(\hat{\vartheta}_\varepsilon, \hat{u}_\varepsilon)$$

□

Let us prove the asymptotic equivalence of the values $f^d_{\varepsilon,a}(m,r,\tilde{r})$ and $\overline{f}^d_{\varepsilon,a}(m,r)$ as $\varepsilon \downarrow 0$.

Lemma A.2 (Equivalence of the two problems)**.** *For any $\tilde{r} < r$ and $m > 0$ it holds*

$$|f^d_{\varepsilon,a}(m,r,\tilde{r}) - \overline{f}^d_{\varepsilon,a}(m,r)| \overset{\varepsilon\downarrow 0}{\longrightarrow} 0$$

Proof. Step 1: [$f^d_{\varepsilon,a}(m,r,\tilde{r}) \le \overline{f}^d_{\varepsilon,a}(m,r) + O(1)$]
Consider for each ε the radially symmetric and monotone couple $(\vartheta_\varepsilon, u_\varepsilon) \in \overline{Y}_{\varepsilon,a}(m,r)$ as introduced in the previous lemma. Take $\xi \in (\eta, 1)$ and let us set

$$r_\xi := \sup\{t \in (0,r) : u_\varepsilon(t) \le \xi\} \quad \text{with } r_\xi = 0 \text{ if the set is empty.} \tag{A.2}$$

By Cauchy-Schwartz inequality it holds

$$C \ge \frac{\int_{B_r \setminus B_{r_\xi}} u_\varepsilon |\vartheta_\varepsilon|^2\,\mathrm{d}x}{\varepsilon} \ge \xi\, \frac{\left(\int_{B_r \setminus B_{r_\xi}} |\vartheta_\varepsilon|\,\mathrm{d}x\right)^2}{\omega_d\, r^d \varepsilon}.$$

Let us define $\Delta_\xi := \int_{B_r \setminus B_{r_\xi}} |\vartheta_\varepsilon|$, the latter ensures that $\Delta_\xi \in o(\varepsilon^{\alpha/2})$. Let us now set $\hat{\vartheta}_\varepsilon = \left(\frac{m\vartheta_\varepsilon}{\int_{B_{r_\xi}} \vartheta_\varepsilon} \right) \mathbf{1}_{B_{r_\xi}}$ which is not null for ε small. We have $(\hat{\vartheta}_\varepsilon, u_\varepsilon) \in Y_{\varepsilon,a}(m,r,\tilde{r})$ if and only if $r_\xi \le \tilde{r}$. Indeed, this holds as

$$C \ge \int_{B'_{r_\xi}} \frac{(1-u_\varepsilon)^2}{\varepsilon^d}\,\mathrm{d}x \ge \omega_d\, (1-\xi)^2 \left(\frac{r_\xi}{\varepsilon}\right)^d,$$

which ensures that $r_\xi = O(\varepsilon)$. Finally let us evaluate the energy

$$\begin{aligned}
E_{\varepsilon,a}(\hat\vartheta_\varepsilon, u_\varepsilon) &= \int_{B_r} \left[\varepsilon^{p-d}|\nabla u_\varepsilon|^p + \frac{(1-u_\varepsilon)^2}{\varepsilon^d} + \frac{u_\varepsilon|\hat\vartheta_\varepsilon|^2}{\varepsilon}\right] \mathrm{d}x \\
&= \int_{B_r} \left[\varepsilon^{p-d}|\nabla u_\varepsilon|^p + \frac{(1-u_\varepsilon)^2}{\varepsilon^d}\right] \mathrm{d}x + \int_{B_{r_\xi}} \frac{u_\varepsilon\, m^2\, |\vartheta_\varepsilon|^2}{\varepsilon(\int_{B_{r_\xi}} \vartheta_\varepsilon)^2} \mathrm{d}x \\
&\le \frac{m^2\,\omega_d}{\left(\int_{B_{r_\xi}} \vartheta_\varepsilon\right)^2} E_{\varepsilon,a}(\vartheta_\varepsilon, u_\varepsilon) = [1 + O(1)] E_{\varepsilon,a}(\vartheta_\varepsilon, u_\varepsilon).
\end{aligned}$$

Passing to the infimum we get

$$f^d_{\varepsilon,a}(m, r, \tilde r) \le \overline{f}^d_{\varepsilon,a}(m, r) + O(1). \tag{A.3}$$

Step 2: [$\overline{f}^d_{\varepsilon,a}(m, r) \le f^d_{\varepsilon,a}(m, r, \tilde r) + O(1)$]
Consider a minimizing couple $(\vartheta_\varepsilon, u_\varepsilon)$ such that

$$f^d_{\varepsilon,a}(m, r, \tilde r) = E_{\varepsilon,a}(\vartheta_\varepsilon, u_\varepsilon).$$

Let χ be a smooth cutoff function such that $\chi(x) = 1$ if $|x| \le \tilde r$ and $\chi(x) = 0$ if $|x| > \frac{r+\tilde r}{2}$ and set $v_\varepsilon = \chi u_\varepsilon + (1-\chi)$. By construction $(\vartheta_\varepsilon, v_\varepsilon) \in \overline{Y}_{\varepsilon,a}(m, r)$, furthermore, since $u_\varepsilon \in (0, 1]$, it holds that $u_\varepsilon \le v_\varepsilon$ and $(1-u_\varepsilon)^2 \ge (1-v_\varepsilon)^2$. Moreover as $v_\varepsilon \equiv u_\varepsilon$ on $B_{\tilde r}$ we have $\int_{B_r} u_\varepsilon|\vartheta_\varepsilon|^2\, \mathrm{d}x = \int_{B_r} v_\varepsilon|\vartheta_\varepsilon|^2\, \mathrm{d}x$. Eventually, we estimate the gradient component of the energy as follows

$$\begin{aligned}
\int_{B_r} \varepsilon^{p-d}|\nabla v_\varepsilon|^p\, \mathrm{d}x &= \int_{B_r} \varepsilon^{p-d}|\chi\nabla u_\varepsilon + (u_\varepsilon - 1)\nabla\chi|^p\, \mathrm{d}x \\
&\le \int_{B_r} \varepsilon^{p-d}(|\nabla u_\varepsilon| + |\nabla\chi|)^p\, \mathrm{d}x \\
&\le \int_{B_r} \varepsilon^{p-d}|\nabla u_\varepsilon|^p\, \mathrm{d}x + C(r,\chi)\left(E^{1-1/p}_{\varepsilon,a}(\vartheta_\varepsilon, v_\varepsilon)\varepsilon^{\frac{p-d}{p}} + \varepsilon^{p-d}\right)
\end{aligned}$$

where we have used the inequality $(|a| + |b|)^p \le |a|^p + C_p(|a|^{p-1}|b| + |b|^p)$ and Holder inequality. We get

$$\overline{f}^d_{\varepsilon,a}(m, r) \le E_{\varepsilon,a}(\vartheta_\varepsilon, v_\varepsilon) \le E_{\varepsilon,a}(\vartheta_\varepsilon, u_\varepsilon) + O(\varepsilon^{\frac{p-d}{p}}) = f^{\tilde r}_\varepsilon(m, r) + O(1) \tag{A.4}$$

Step 3: Combining inequalities (A.3) and (A.4) we obtain $f^d_{\varepsilon,a}(m, r, \tilde r) - \overline{f}^d_{\varepsilon,a}(m, r) = o(1)$. □

A.2. Study of the transition energy

Given two values $r_1 < r_2$ let us introduce the functional

$$\mathcal{G}^d(v; (r_1, r_2)) := \int_{r_1}^{r_2} t^{d-1}\left[|v'|^p + (1-v)^2\right]$$

and for any triplet $(\xi,\ r_1,\ r_2) \in [0, 1] \times \mathbf{R}^+ \times \mathbf{R}^+$ we set

$$q^d(\xi, r_1, r_2) := \inf\left\{\mathcal{G}^d(v; (r_1, r_2))\, \mathrm{d}t\ :\ v \in W^{1,p}(r_1, r_2),\ v(r_1) = \xi \text{ and } v(r_2) = 1\right\}. \tag{A.5}$$

This value represents the cost of the transition from ξ to 1 in the ring $B_{r_2} \setminus \overline{B}_{r_1}$. We will say that a function v is admissible for the triplet (ξ, r_1, r_2) if it is a competitor in the above minimization problem. Let us investigate the properties of the function introduced.

Lemma A.3. *For any fixed triplet $(\xi, r_1, r_2) \in [0, 1] \times \mathbf{R}_+ \times \mathbf{R}_+$ the infimum in equation (A.5) is a minimum. Moreover there is a unique function achieving the minimum which is nondecreasing with range in the interval $[\xi, 1]$. Finally the function g satisfies the following properties*

1. $r_2 \mapsto q^d(\xi, r_1, r_2)$ *is nonincreasing,*

2. $r_1 \mapsto q^d(\xi, r_1, r_2)$ *is nondecreasing,*

3. $\xi \mapsto q^d(\xi, r_1, r_2)$ *is nonincreasing, and* $g(1, r_1, r_2) = 0$.

Recalling the definition (2.4) *of* q^d_∞*, we have* $q^d_\infty(\xi, \hat{r}) = q^d(\xi, r_1, \infty)$*, and* $q^d_\infty(0,0) > 0$*. Furthermore for any* $r > 0$ *the map* $\xi \mapsto q^d_\infty(\xi, r)$ *is convex and continuous on* $(0, +\infty)$.

Proof. Let $(\xi, r_1, r_2) \in [0,1] \times \mathbf{R}_+ \times \mathbf{R}_+$, the infimum is actually a minimum by means of the direct method of the calculus of variations. Such minimum is absolutely continuous on the interval (r_1, r_2) by Morrey's inequality and is unique since $\mathcal{G}^d(v; (r_1, r_2))$ is strictly convex in v. Let $v \in W^{1,p}(r_1, r_2)$ be a minimizer of (A.5) set

$$\overline{v} = \min\{\max(v, \xi), 1\}$$

then $\mathcal{G}^d(\overline{v}; (r_1, r_2)) \leq \mathcal{G}^d(v; (r_1, r_2))$ if $v \neq \overline{v}$. As a consequence for every minimizer of (A.5) we have $\xi \leq v \leq 1$. Similarly setting

$$\overline{v}(s) = \max\{v(t) : r_1 \leq t \leq s\}$$

we have $\mathcal{G}^d(\overline{v}; (r_1, r_2)) \leq \mathcal{G}^d(v; (r_1, r_2))$ if $v \neq \overline{v}$. Hence v is nondecreasing. Let us now study the

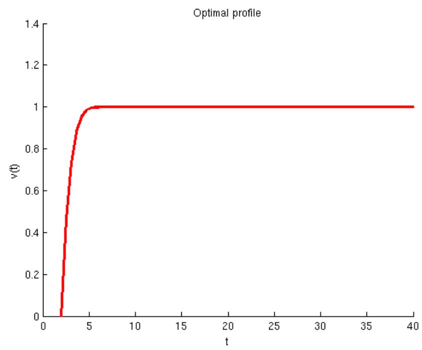

Figure 4: Graph obtained by a numerical optimization of problem (A.5), for the choice of the parameters $p = 3$, $d = 2$, $r_1 = 2$, $r_2 = 40$ and $\xi = 0$.

monotonicity of g. To do so let v be the minimizer for (ξ, r_1, r_2):

1. Let $\overline{r}_2 > r_2$ and let us extend v by 1 on the interval $(r_2, \overline{r}_2)$. We have

$$q^d(\xi, r_1, r_2) = \mathcal{G}^d(v; (r_1, r_2)) = \mathcal{G}^d(v; (r_1, \overline{r}_2)) \geq q^d(\xi, r_1, \overline{r}_2).$$

Hence $r_2 \mapsto g$ is nonincreasing.

2. Let $0 < \overline{r}_1 < r_1$ and set $\Delta = r_1^d - \overline{r}_1^d > 0$ and $\overline{r}_2 = (r_2^d - \Delta)^{\frac{1}{d}} < r_2$. Define the diffeomorphism

$$\begin{aligned} \phi : (r_1, r_2) &\longrightarrow (\overline{r}_1, \overline{r}_2), \\ s &\longmapsto \left[s^d - \Delta\right]^{1/d}. \end{aligned}$$

Let v be the minimizer of (A.5) and $\overline{v}(s) = v \circ \phi(s)$. Let us remark that $\phi'(s) = s^{d-1}/\phi(s)^{d-1}$, thus it holds

$$\begin{aligned} q^d(\xi, r_1, r_2) &= \int_{r_1}^{r_2} t^{d-1} \left[|v'|^p + (1-v)^2\right] \,\mathrm{d}t = \int_{\overline{r}_1}^{\overline{r}_2} \phi(s)^{d-1} \left[\frac{|\overline{v}'|^p}{|\phi'(s)|^p} + (1-\overline{v})^2\right] \phi(s)' \,\mathrm{d}s \\ &= \int_{\overline{r}_1}^{\overline{r}_2} s^{d-1} \left[\left(1 + \frac{\Delta}{s^d - \Delta}\right)^{\frac{pd}{d}} |\overline{v}'|^p + (1-\overline{v})^2\right] \,\mathrm{d}s \geq q^d(\xi, \overline{r}_1, \overline{r}_2) \geq q^d(\xi, \overline{r}_1, r_2). \end{aligned}$$

Therefore $r_1 \mapsto q^d$ is nondecreasing.

3. Let $0 \le \xi < \overline{\xi} \le 1$ and v the absolutely continuous, nondecreasing minimizer of problem $q^d(\xi, r_1, r_2)$. Then there exists $\overline{r} \in (r_1, r_2)$ for which $v(\overline{r}) = \overline{\xi}$. Hence
$$q^d(\xi, r_1, r_2) \ge \mathcal{G}^d(v; (\overline{r}, r_2)) \ge g(\overline{\xi}, \overline{r}, r_2) \ge g(\overline{\xi}, r_1, r_2).$$
Hence, $\xi \mapsto q^d$ is nonincreasing. Finally, for $\xi = 1$ consider the constant function $v \equiv 1$ to get $g(1, r_1, r_2) = 0$.

Indeed, in view of the monotonicity, for every r_1 and r_2 we have
$$g(0, r_1, r_2) \ge g(0, 0, +\infty) = q^d_\infty(0, 0).$$
Let us show $q^d_\infty(0, 0) > 0$. As a matter of facts, taken the minimizer v for the problem (2.4), there exists $r \in (0, +\infty)$ such that $v(r) = 1/2$ and we have
$$q^d_\infty(0, 0) \ge \int_0^r t^{d-1} \left[|v'|^p + (1-v)^2\right] \, \mathrm{d}t = \int_0^r t^{d-1}|v'|^p \, \mathrm{d}t + \frac{r^d}{4\, d}.$$
A direct evaluation gives
$$\min\left\{\int_0^r t^{d-1}|v'|^p \, \mathrm{d}t \,:\, v(r) = 0 \text{ and } v(r) = 1/2\right\} = \frac{c}{r}$$
and we obtain the estimate
$$q^d_\infty(0, 0) \ge \frac{c}{r} + \frac{r^d}{4\, d} > 0.$$
Lastly, let us show that for any r the function $q^d_\infty(\cdot, r)$ is convex. Consider two values ξ_1, $\xi_2 \in (0, 1)$ and the associated minimizers v_1, v_2 for the respective energy $q^d_\infty(\cdot, r)$. Indeed, for any $\lambda \in (0, 1)$ the function $\lambda v_1 + (1-\lambda)v_2$ is a competitor for the minimization problem $q^d_\infty(\lambda\xi_1 + (1-\lambda)\xi_2, r)$, therefore it holds
$$\begin{aligned} q^d_\infty(\lambda\xi_1 + (1-\lambda)\xi_2, r) &\le \int_r^\infty t^{d-1} \left[|\lambda v_1 - (1-\lambda)v_2|^p + (1 - \lambda v_1 + (1-\lambda)v_2)^2\right] \, \mathrm{d}t \\ &\le \lambda q^d_\infty(\xi_1, r) + (1-\lambda) q^d_\infty(\xi_2, r). \end{aligned}$$
Thus $q^d_\infty(\cdot, r)$ is continuous in the open interval $(0, 1)$. To show the continuity in 0 let ξ be small and $v = \operatorname{argmin} q^d_\infty(\xi, r)$. Set
$$h(t) := \begin{cases} \dfrac{1}{1-\sqrt{\xi}}(t - \xi), & t < \sqrt{\xi}, \\ t, & t \ge \sqrt{\xi}. \end{cases}$$
and observe that $h \circ v$ is a competitor for the problem $q^d_\infty(0, r)$. Then
$$\begin{aligned} q^d_\infty(0, r) &\le \int_r^\infty t^{d-1} \left[|(h \circ v)'|^p + (1 - h \circ v^2\right] \, \mathrm{d}t \\ &\le \frac{1}{(1-\sqrt{\xi})^p} \, q^d_\infty(\xi, r) + \int_r^\infty t^{d-1} \left[(1 - h\circ v)^2 - (1-v)^2\right] \, \mathrm{d}t \end{aligned}$$
Let us estimate the second addend in the latter. By the definition of f we have
$$\begin{aligned} \int_r^\infty t^{d-1} \left[(1 - h\circ v)^2 - (1-v)^2\right] \, \mathrm{d}t &= \int_{\{v < \sqrt{\xi}\}} t^{d-1} \left[(1 - h \circ v - v)^2 (v - h \circ v)^2\right] \, \mathrm{d}t \\ &\le 4\xi \int_{\{v < \sqrt{\xi}\}} t^{d-1} \, \mathrm{d}t \\ &\le \frac{4\xi}{(1-\sqrt{\xi})^2} \, q^d_\infty(\xi, r). \end{aligned}$$
Since $q^d_\infty(\cdot, r)$ is monotone we have
$$|q^d_\infty(0, r) - q^d_\infty(\xi, r)| \le \max\left\{\frac{1 - (1-\sqrt{\xi})^p}{(1-\sqrt{\xi})^p} \; , \; \frac{4\xi}{(1-\sqrt{\xi})^2}\right\} \kappa,$$
which shows that $q^d_\infty(\cdot, r)$ is continuous in 0. $\square$

A.3. Proof of Proposition 2.1

We show that

$$\liminf_{\varepsilon\downarrow 0} \overline{f}^{d}_{\varepsilon,a}(m,r) \geq f^d_a(m)$$

then equation (2.5) easily follows from Lemma A.2. For $m=0$ set $\vartheta=0$ and $u=1$, then $(\vartheta,u)\in Y_{\varepsilon,a}(0,r)$ for any radius r and $E_{\varepsilon,a}(\vartheta,u;B_r)=0$ for each ε. Now suppose $m>0$ and let $\xi\in(\eta,1)$. Consider the radially symmetric and monotone minimizing couple $(\vartheta_\varepsilon,u_\varepsilon)$ of Lemma A.1 and r_ξ introduced in equation (A.2). Let us split the set of integration in the two sets B_{r_ξ} and $B_r\setminus B_{r_\xi}$, we obtain

$$\overline{f}^{d}_{\varepsilon,a}(m,r) = E_{\varepsilon,a}(\vartheta_\varepsilon,u_\varepsilon) \geq \underbrace{\int_{B_r\setminus B_{r_\xi}} \left[\varepsilon^{p-d}|\nabla u_\varepsilon|^p + \frac{(1-u_\varepsilon)^2}{\varepsilon^d}\right] \mathrm{d}x}_{a_\varepsilon} + \underbrace{\int_{B_{r_\xi}} \frac{(1-u_\varepsilon)^2}{\varepsilon^d}\,\mathrm{d}x + \int_{B_r} \frac{u_\varepsilon|\vartheta_\varepsilon|^2}{\varepsilon}\,\mathrm{d}x}_{b_\varepsilon}\,. \tag{A.6}$$

We deal with each addend separately. First observe that by Cauchy-Schwarz inequality, it holds

$$\frac{m^2}{\int_{B_r}\frac{1}{u_\varepsilon}\,\mathrm{d}x} \;\leq\; \int_{B_r} u_\varepsilon\vartheta_\varepsilon^2\,\mathrm{d}x.$$

Plugging the latter in the term b_ε of (A.6) we have

$$b_\varepsilon \geq \int_{B_{r_\xi}} \frac{(1-u_\varepsilon)^2}{\varepsilon^d}\,\mathrm{d}x + \frac{m^2}{\varepsilon\left(\int_{B_r\setminus B_{r_\xi}}\frac{1}{u_\varepsilon}\,\mathrm{d}x + \int_{B_{r_\xi}}\frac{1}{u_\varepsilon}\,\mathrm{d}x\right)}$$

taking into account $\eta\leq u_\varepsilon\leq\xi$ in B_{r_ξ}, $\xi\leq u_\varepsilon\leq 1$ in $B_r\setminus B_{r_\xi}$ and $\eta = a\,\varepsilon^{d+1}$ we obtain

$$b_\varepsilon \geq \omega_d(1-\xi)^2\left(\frac{r_\xi}{\varepsilon}\right)^d + \frac{m^2}{\dfrac{\omega_d}{a}\left(\dfrac{r_\xi}{\varepsilon}\right)^d + \omega_d\dfrac{\varepsilon r^d}{\xi}}. \tag{A.7}$$

Since $b_\varepsilon \leq \overline{f}^{d}_{\varepsilon,a}(m,r)\leq C(m)$ we deduce that r_ξ/ε belongs to a fixed compact subset $K=K(m,\xi)$ of $(0,+\infty)$. Up to extracting a subsequence, which we do not relabel, we can suppose r_ξ/ε to converge to some $\hat{r}>0$. Let us now consider the term a_ε. Let v_ε be the radial profile of u_ε

$$a_\varepsilon = \int_{B_r\setminus B_{r_\xi}} \left[\varepsilon^{p-d}|\nabla u_\varepsilon|^p + \frac{(1-u_\varepsilon)^2}{\varepsilon^d}\right] \mathrm{d}x = (d-1)\,\omega_d \int_{r_\xi/\varepsilon}^{r/\varepsilon} t^{d-1}\left[|v'_\varepsilon|^p + (1-v_\varepsilon)^2\right]\,\mathrm{d}t.$$

With the notation introduced in Subsection A.2 and Lemma A.3 therein we deduce

$$\liminf_{\varepsilon\downarrow 0} a_\varepsilon \geq (d-1)\,\omega_d \liminf_{\varepsilon\downarrow 0} q^d\left(\xi;(r_\xi/\varepsilon, r/\varepsilon)\right) \geq (d-1)\,\omega_d\, q^d_\infty(\xi,\hat{r}),$$

where q^d_∞ has been defined in (2.4). Combining inequality (A.7) and the latter we get

$$\lim_{\varepsilon\downarrow 0} \overline{f}^{d}_{\varepsilon,a}(m,r) \geq (d-1)\,\omega_d\, q^d_\infty(\xi,\hat{r}) + (1-\xi)^2\,\omega_d\,\hat{r}^d + \frac{a\,m^2}{\omega_d\,\hat{r}^d}.$$

Sending ξ to 0 we have, by continuity (Lemma A.3) $q^d_\infty(\xi,\hat{r})\to q^d_\infty(0,\hat{r})$. Then taking the infimum in $\hat{r}$, we obtain

$$\liminf_{\varepsilon\downarrow 0} \overline{f}^{d}_{\varepsilon,a}(m,r) \geq \min_{\hat{r}}\left\{(d-1)\,\omega_d\, q^d_\infty(0,\hat{r}) + \;\omega_d\,\hat{r}^d + \frac{a\,m^2}{\omega_d\,\hat{r}^d}\right\}.$$

Again by Lemma A.3 the function $q^d_\infty(0,\hat{r})$ is nondecreasing in $\hat{r}$, and $q^d_\infty(0,0)>0$ therefore setting

$$\kappa := (d-1)\,\omega_d\, q^d_\infty(0,0) \leq f^d_a(m)$$

we conclude the proof of Proposition 2.1.

A.4. Proof of Proposition 2.2

Let $\delta > 0$, by Lemma A.3 for ε sufficiently small

$$q^d(\eta; (r_*, r/\varepsilon)) \leq q^d_\infty(0, r_*) + \delta.$$

Let

$$v_\delta(t) = \operatorname{argmin}\left\{\mathcal{G}^d\left(v; \left(r_*, \frac{r}{\varepsilon}\right)\right)\ \mathrm{d}t\ :\ v\,(r_*) = \eta \text{ and } v\left(\frac{r}{\varepsilon}\right) = 1\right\}.$$

and set

$$u_\varepsilon(t) := \begin{cases} \eta & \text{for } 0 \leq t \leq r_*\varepsilon \\ v_\delta\left(\dfrac{t}{\varepsilon}\right) & \text{for } r_*\varepsilon \leq t \leq r \end{cases}$$

Set $\vartheta_\varepsilon(s)$ to be constant equal to $\frac{m}{\omega_d\,(\varepsilon r_*)^d}$ on the ball $B_{\varepsilon r_*}$ and zero outside. Indeed, the couple $(\vartheta_\varepsilon, u_\varepsilon(|x|))$ belongs to $\overline{Y}_{\varepsilon,a}(m,r)$. That is because u_ε is greater then η and attains value 1 at the border of B_r and

$$\int_{B_r} \vartheta_\varepsilon(x)\ \mathrm{d}x = \frac{m}{\omega_d(\varepsilon r_*)^d}\ \omega_d(\varepsilon r_*)^d = m.$$

Let us show that the couple $(\vartheta_\varepsilon, u_\varepsilon)$ defined satisfy inequality (2.7). Taking advantage of the radial symmetry of the functions we get

$$\begin{aligned} E_{\varepsilon,a}(\vartheta_\varepsilon, u_\varepsilon) = \int_{\varepsilon r_*}^{r} t^{d-1}\left[\varepsilon^{p+d}|u'_\varepsilon|^p + \frac{(1-u_\varepsilon)}{\varepsilon^d}\right]\ \mathrm{d}t \\ + \frac{(1-\eta)^2}{\varepsilon^d}\ \omega_d\ (\varepsilon r_*)^d + \frac{\eta}{\varepsilon}\left(\frac{m}{\omega_d\ (\varepsilon r_*)^d}\right)^2\ \omega_d\ (\varepsilon r_*)^d. \end{aligned}$$

By simplifying the expression and considering the change of variable $s = \frac{t}{\varepsilon}$ in the latter it holds

$$\begin{aligned} E_{\varepsilon,a}(\vartheta_\varepsilon, u_\varepsilon) &= (d-1)\ \omega_d \int_{r_*}^{\frac{r}{\varepsilon}} s^{d-1}\left[|v'_\delta|^p + (1-v_\delta)\right]\ \mathrm{d}s + (1-\eta)^2\ \omega_d\ r_\delta^d + \frac{\eta}{\varepsilon^{d+1}}\frac{m^2}{\omega_d\ r_*^d} \\ &\leq (d-1)\ \omega_d\ q^d\left(\eta; (r_*, r/\varepsilon)\right) + (1-\eta)^2\ \omega_d\ r_*^d + \frac{\eta}{\varepsilon}\frac{m^2}{\omega_d\ r_*^d} \end{aligned}$$

Then, by Lemma A.3 for ε sufficiently small we have

$$E_{\varepsilon,a}(\vartheta_\varepsilon, u_\varepsilon) \leq \frac{a\,m^2}{\omega_d\ r_*^d} +\ \omega_d\ r_*^d + (d-1)\ \omega_d\ q^d_\infty(0, r_*) + (d-1)\omega_{d-1}\delta = f^d_a(m) + C\delta,$$

which ends the proof of Proposition 2.2.

A.5. Proof of Proposition 2.3

Propositions 2.1, 2.2 and lemma A.2 ensure that

$$f^d_a(m) = \lim_{\varepsilon\downarrow 0}\overline{f}^d_{\varepsilon,a}(m,r) = \lim_{\varepsilon\downarrow 0} f^d_{\varepsilon,a}(m,r,\tilde{r})$$

independently of the choices for r and $\tilde{r} < r$. For the sake of clarity we introduce

$$T(m,r) := \left\{\frac{a\,m^2}{\omega_d\ r^d} +\ \omega_d\ r^d + (d-1)\ \omega_d\ q^d_\infty(0,r)\right\}$$

and recall that $f^d_a(m) = \min_r T(m,r)$ for $m > 0$ and $f^d_a(0) = 0$, see (2.3).

Proof.
Let us prove the continuity of f^d_a on $(0,+\infty)$. For $m_1, m_2 \in (0,+\infty)$ and for $i = 1,2$ let r_i be such that $f^d_a(m_i) = T(m_i, r_i)$. On one hand comparing with $r = 1$ it holds

$$\frac{m_i^2}{\omega_{d-1}\,r_i^d} \leq f^d_a(m_i) \leq T(m_i, 1) \tag{A.8}$$

on the other hand analougusly we have

$$\omega_{d-1}\, r_i^d \leq f_a^d(m_i) \leq T(m_i,1).$$

Consequently $\omega_{d-1}\, r_i^d$ belongs to the compact set $[m_i/T(m_i,1), T(m_i,1)]$. Now remark that

$$f_a^d(m_1) \leq T(m_1,r_2) = f_a^d(m_2) + T(m_1,r_2) - T(m_2,r_2)$$

thus

$$|f_a^d(m_1) - f_a^d(m_2)| \leq |T(m_1,r_2) - T(m_2,r_2)| \leq \frac{|m_1^2 - m_2^2|}{\omega_{d-1}\min\{r_1^d . r_2^d\}}$$

and taking into account inequality (A.8) we have

$$|f_a^d(m_1) - f_a^d(m_2)| \leq (m_1+m_2)\max\left\{\frac{T(m_1,1)}{m_1^2}, \frac{T(m_2,1)}{m_2^2}\right\}|m_1 - m_2|.$$

Observing that $T(\cdot,1)$ is continuous we conclude that f_a^d is continuous on $(0,+\infty)$.
Next, we see that f_a^d is non decreasing. Let $0 < m_1 < m_2$ and $r > 0$. Let $(\vartheta, u) \in \overline{Y}_{\varepsilon,a}(m_2,r)$ such that $E_{\varepsilon,a}(\vartheta,u;B_r) = \overline{f}^d_{\varepsilon,a}(m_2,r)$. Set $\overline{\vartheta} = m_1\vartheta/m_2$ and remark that the couple $(\overline{\vartheta}, u)$ belongs to $\overline{Y}_{\varepsilon,a}(m_1,r)$. Therefore we have the following set of inequalities

$$\overline{f}^d_{\varepsilon,a}(m_1,r) \leq E_{\varepsilon,a}(\overline{\vartheta},u;B_r) = E_{\varepsilon,a}\left(\frac{m_1\vartheta}{m_2}, u; B_r\right) < E_{\varepsilon,a}(\vartheta,u;B_r) = \overline{f}^d_{\varepsilon,a}(m_2,r).$$

Passing to the limit as $\varepsilon \downarrow 0$ we obtain

$$f_a^d(m_1) \leq f_a^d(m_2).$$

Let us now prove the sub-additivity. For a radius r consider the competitors $(\vartheta_j, u_j) \in \overline{Y}_{\varepsilon,a}(m_j,r)$ for $j = 1,2$. Consider the ball B_{2r+1} centered in the origin and two points x_1, x_2 such that the balls $B_r(x_1)$, $B_r(x_2)$ are disjoint and contained in B_{2r+1}. Set

$$\overline{\vartheta}(x) := \begin{cases} \vartheta_1(x-x_1), & x \in B_r(x_1), \\ \vartheta_2(x-x_2), & x \in B_r(x_2), \\ 0, & \text{otherwise}, \end{cases} \qquad \text{and} \qquad \overline{u}(x) := \begin{cases} u_1(x-x_1), & x \in B_r(x_1), \\ u_2(x-x_2), & x \in B_r(x_2), \\ 1, & \text{otherwise}, \end{cases}$$

and observe that the couple $(\overline{\vartheta}, \overline{u})$ belongs to $\overline{Y}(m_1+m_2, 2r+1)$. Being the balls $B_r(x_j)$ disjoint we have

$$\begin{aligned}\overline{f}^d_{\varepsilon,a}(m_1+m_2, r_1+r_2) &\leq E_{\varepsilon,a}(\vartheta_1(x-x_1), u_1(x-x_1); B_r(x_1)) + E_{\varepsilon,a}(\vartheta_2(x-x_2), u_2(x-x_2); B_r(x_2)) \\ &= \overline{f}^d_{\varepsilon,a}(m_1,r) + f^d_{\varepsilon,a}(m_2,r).\end{aligned}$$

Passing to the limit as $\varepsilon \downarrow 0$, and recalling that it is independent of the choice of the radius, we get

$$f_a^d(m_1+m_2) \leq f_a^d(m_1) + f_a^d(m_2).$$

□

We conclude the appendix by showing that

Lemma A.4. *For any sequence $a_i \downarrow 0$ it holds*

$$f_{a_i}^d \longrightarrow \kappa \mathbf{1}_{(0,\infty)}$$

pointwise.

Proof. We have already shown that $f_a^d(m) \geq \kappa$ for $m > 0$. For $m > 0$ choose $\hat{r} = (\sqrt{a}m)^{1/d}$, then by definition it holds

$$\kappa \leq f_a^d(m) \leq (d-1)\,\omega_d\, q_\infty^d(0, (\sqrt{a}m)^{1/d}) + \omega_d\sqrt{a}m + \frac{\sqrt{a}m}{\omega_d}.$$

Finally simply recall that $(d-1)\,\omega_d\, q_\infty^d(0,0) = \kappa$ and that $q_\infty^d(0,\cdot)$ is continuous. □

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