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# DEFORMATIONS OF $\mathbb{A}^1$ -CYLINDRICAL VARIETIES

ADRIEN DUBOULOZ AND TAKASHI KISHIMOTO

**ABSTRACT.** An algebraic variety is called  $\mathbb{A}^1$ -cylindrical if it contains an  $\mathbb{A}^1$ -cylinder, i.e. a Zariski open subset of the form  $Z \times \mathbb{A}^1$  for some algebraic variety  $Z$ . We show that the generic fiber of a family  $f : X \rightarrow S$  of normal  $\mathbb{A}^1$ -cylindrical varieties becomes  $\mathbb{A}^1$ -cylindrical after a finite extension of the base. This generalizes the main result of [6] which established this property for families of smooth  $\mathbb{A}^1$ -cylindrical affine surfaces. Our second result is a criterion for existence of an  $\mathbb{A}^1$ -cylinder in  $X$  which we derive from a careful inspection of a relative Minimal Model Program run from a suitable smooth relative projective model of  $X$  over  $S$ .

## INTRODUCTION

An algebraic variety is called  $\mathbb{A}^1$ -cylindrical (or affine-ruled or  $\mathbb{A}^1$ -ruled) if it contains an  $\mathbb{A}^1$ -cylinder, i.e. a Zariski open subset of the form  $Z \times \mathbb{A}^1$  for some algebraic variety  $Z$ . Such  $\mathbb{A}^1$ -cylinders appear naturally in many recent problems and questions related to the geometry of algebraic varieties, both affine and projective [16, 5, 6, 7, 8, 1, 2, 3, 17, 18, 19, 24, 25]. Clearly, there are only two  $\mathbb{A}^1$ -cylindrical smooth complex curves: the affine line  $\mathbb{A}^1$  and the projective line  $\mathbb{P}^1$ . As a consequence of classical classification results, every smooth projective surface of negative Kodaira dimension is  $\mathbb{A}^1$ -cylindrical, and the same holds true for smooth affine surfaces by a deep result of Miyanishi-Sugie and Fujita [22]. But it is still an open problem to find a complete and effective characterization of which complex surfaces, possibly singular, contain  $\mathbb{A}^1$ -cylinders [16]. The situation in higher dimension is even more elusive, some natural class of examples of  $\mathbb{A}^1$ -cylindrical varieties are known, especially in relation with the study of additive group actions on affine varieties, but for instance the question whether every smooth rational projective variety is  $\mathbb{A}^1$ -cylindrical is still totally open.

A natural way to try to produce new  $\mathbb{A}^1$ -cylindrical varieties from known ones is to consider algebraic families  $f : X \rightarrow S$  of such varieties. One hopes that the fiberwise  $\mathbb{A}^1$ -cylinders could arrange themselves continuously to form a global relative  $\mathbb{A}^1$ -cylinder in the total space  $X$ , in the form of a cylinder  $U \simeq Z \times \mathbb{A}^1$  in  $X$  for some  $S$ -variety  $Z$ , whose restriction to a general closed fiber of  $f : X \rightarrow S$  is equal to the initially prescribed  $\mathbb{A}^1$ -cylinder in it. For families of relative dimension one, it is a classical fact [14] that a smooth fibration  $f : X \rightarrow S$  whose general closed fibers are isomorphic to  $\mathbb{A}^1$  indeed restricts to trivial  $\mathbb{A}^1$ -bundle  $Z \times \mathbb{A}^1$  over a dense open subset  $Z$  of  $S$ . But on the other hand, the existence of nontrivial conic bundles  $f : X \rightarrow S$  shows that it is in general too much to expect that fiberwise cylinders are restrictions of global ones. Indeed, for such a nontrivial conic bundle, the general closed fibers are isomorphic to  $\mathbb{P}^1$ , hence are  $\mathbb{A}^1$ -cylindrical, but the generic fiber of  $f : X \rightarrow S$  is a nontrivial form of  $\mathbb{P}^1$  over the function field  $K$  of  $S$ : the latter does not contain any open subset isomorphic to  $\mathbb{A}_K^1$ , which prevents in turn the existence of a global  $\mathbb{A}^1$ -cylinder in  $X$  over an  $S$ -variety. Nevertheless, such an  $\mathbb{A}^1$ -cylinder exists after extending the scalars to a suitable quadratic extension of  $K$ , leading to the conclusion that the total space of any smooth family  $f : X \rightarrow S$  of  $\mathbb{A}^1$ -cylindrical varieties of dimension one always contain a relative  $\mathbb{A}^1$ -cylinder, possibly after an étale extension of the base  $S$ .

A similar property is known to hold for certain families of relative dimension 2. More precisely, it was established in [10, Theorem 3.8] and [6, Theorem 7] by different methods, involving respectively the study of log-deformations of suitable relative projective models and the geometry of smooth affine surfaces of negative Kodaira dimension defined over non closed fields, that for smooth families  $f : X \rightarrow S$  of complex  $\mathbb{A}^1$ -cylindrical affine surfaces, there exists an étale morphism  $T \rightarrow S$  such that  $X_T = X \times_S T$  contains an  $\mathbb{A}^1$ -cylinder  $U \simeq Z \times \mathbb{A}^1$  over a  $T$ -variety  $Z$ . The first main result of this article consists of a generalization of this property to arbitrary families  $f : X \rightarrow S$  of normal algebraic varieties defined over an uncountable base field, namely:

**Theorem 1.** *Let  $k$  be an uncountable field of characteristic zero and let  $f : X \rightarrow S$  be dominant morphism between geometrically integral algebraic  $k$ -varieties. Suppose that for general closed points  $s \in S$ , the fiber  $X_s$  contains*

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an  $\mathbb{A}^1$ -cylinder  $U_s \simeq Z_s \times \mathbb{A}^1$  over a  $\kappa(s)$ -variety  $Z_s$ . Then there exists an étale morphism  $T \rightarrow S$  such that  $X_T = X \times_S T$  contains an  $\mathbb{A}^1$ -cylinder  $U \simeq Z \times \mathbb{A}^1$  over a  $T$ -variety  $Z$ .

We next turn to the problem of finding effective conditions on the fiberwise  $\mathbb{A}^1$ -cylinders which ensure that a global relative  $\mathbb{A}^1$ -cylinder exists, without having to take any base change. The question is quite subtle already in the case of fibrations of relative dimension 2, as illustrated on the one hand by smooth del Pezzo fibrations with non rational generic fiber, which therefore cannot contain any  $\mathbb{A}^1$ -cylinder [8], and on the other hand by examples of one parameter families  $f : X \rightarrow S$  of smooth  $\mathbb{A}^1$ -cylindrical affine cubic surfaces whose total spaces do not contain any  $\mathbb{A}^1$ -cylinder at all, relative to  $f : X \rightarrow S$  or not [5]. Intuitively, a global relative cylinder should exist as soon as the fiberwise  $\mathbb{A}^1$ -cylinders are “unique”, in the sense that the intersection of any two of them is again an  $\mathbb{A}^1$ -cylinder. This holds for instance for  $\mathbb{A}^1$ -cylinders inside non rational smooth affine surfaces, and for families  $f : X \rightarrow S$  of such surfaces, it was indeed confirmed in [6, Theorem 10] that  $X$  contains a relative  $\mathbb{A}^1$ -cylinder  $U \simeq Z \times \mathbb{A}^1$  over  $S$ , for which the rational projection  $X \dashrightarrow Z$  coincides, up to birational equivalence, with the Maximally Rationally Connected quotient of a relative smooth projective model  $\bar{f} : \bar{X} \rightarrow S$  of  $X$  over  $S$ .

The natural generalization in higher dimension would be to consider normal varieties  $Y$  which contain  $\mathbb{A}^1$ -cylinders  $U \simeq Z \times \mathbb{A}^1$  over non uniruled bases  $Z$ . But there is a second type of obstruction for uniqueness, which does not appear in the affine case: the fact that a given  $\mathbb{A}^1$ -cylinder  $U \simeq Z \times \mathbb{A}^1$  in a variety  $Y$  can actually be the restriction of a  $\mathbb{P}^1$ -cylinder  $\bar{U} \simeq Z \times \mathbb{P}^1$  inside  $Y$ , with the effect that  $Y$  then contains infinitely many distinct  $\mathbb{A}^1$ -cylinders of the form  $Z \times (\mathbb{P}^1 \setminus \{p\})$ ,  $p \in \mathbb{P}^1$ , all over the same base  $Z$ . This possibility is eliminated by restricting the attention to varieties  $Y$  containing  $\mathbb{A}^1$ -cylinders  $U \simeq Z \times \mathbb{A}^1$  for which the open immersion  $Z \times \mathbb{A}^1 \hookrightarrow Y$  cannot be extended to a birational map  $Z \times \mathbb{P}^1 \dashrightarrow Y$  defined over the generic point of  $Z$ . An  $\mathbb{A}^1$ -cylinder with this property is called *vertically maximal* in  $Y$  (see Definition 10), and our second main result consists of the following characterization:

**Theorem 2.** *Let  $k$  be a field of characteristic zero and let  $f : X \rightarrow S$  be a dominant morphism between normal  $k$ -varieties such that for general closed points  $s \in S$ , the fiber  $X_s$  contains a vertically maximal  $\mathbb{A}^1$ -cylinder  $U_s \simeq Z_s \times \mathbb{A}^1$  over a non uniruled  $\kappa(s)$ -variety  $Z_s$ . Then  $X$  contains an  $\mathbb{A}^1$ -cylinder  $U \simeq Z \times \mathbb{A}^1$  for some  $S$ -variety  $Z$ .*

The article is organized as follows. The first section contains a quick review of rationally connected and uniruled varieties and some explanation concerning the minimal model program for varieties defined over arbitrary fields of characteristic zero which plays a central role in the proof of Theorem 2. In section two, we establish basic properties of  $\mathbb{A}^1$ -cylindrical varieties. Theorem 1 is then derived in section three from quite standard “general-to-generic” Lefschetz principle and specialization arguments. Finally, section four is devoted to the proof of Theorem 2, which proceeds through a careful study of the output of a relative minimal model program applied to a suitably constructed smooth projective model  $\bar{f} : Y \rightarrow S$  of  $X$  over  $S$ .

## 1. PRELIMINARIES

In what follows, unless otherwise stated,  $k$  is a field of characteristic zero, and all objects considered will be assumed to be defined over  $k$ . A  $k$ -variety is a reduced scheme of finite type over  $k$ . For a morphism  $f : X \rightarrow S$  and another morphism  $T \rightarrow S$ , the symbol  $X_T$  will denote the fiber product  $X \times_S T$ . In particular for a point  $s \in S$ , closed or not, we write  $X_s = f^{-1}(s) = X \times_S \text{Spec}(\kappa(s))$  where  $\kappa(s)$  denotes the residue field of  $s$ . In addition, if  $T = \text{Spec}(K)$  for a field  $K$ , then  $X_T$  will also sometimes be denoted by  $X_K$ .

### 1.1. Recollection on rational connectedness and uniruledness.

**Definition 3.** (See [20, IV.3 Definition 3.2 and IV.1 Definition 1.1]) Let  $f : X \rightarrow S$  be an integral scheme over a scheme  $S$ . We say that  $X$  is:

- a) *Rationally connected over  $S$*  if there exists an  $S$ -scheme  $B$  and a morphism  $u : B \times \mathbb{P}^1 \rightarrow X$  of schemes over  $S$  such  $u \times_B u : (B \times \mathbb{P}^1) \times_B (B \times \mathbb{P}^1) \rightarrow X \times_S X$  is dominant.
- b) *Uniruled over  $S$*  if there exists an  $S$ -scheme  $B$  of relative dimension  $\dim(X/S) - 1$  and a dominant rational map  $u : B \times \mathbb{P}^1 \dashrightarrow X$  of schemes over  $S$ .
- c) *Ruled over  $S$*  if there exists an  $S$ -scheme  $B$  of relative dimension  $\dim(X/S) - 1$  and a dominant birational map  $u : B \times \mathbb{P}^1 \dashrightarrow X$  of schemes over  $S$ .

A variety  $X$  defined over a field  $k$  is called rationally connected (resp. uniruled, resp. ruled) if it is rationally connected (resp. uniruled, resp. ruled) over  $\text{Spec}(k)$ . Recall [20, IV.3 3.2.5 and IV.1 Proposition 1.3] that the first two notions are independent of the field over which  $X$  is defined. In particular,  $X$  is rationally connected (resp. uniruled) if and only if  $X_K$  is rationally connected (resp. uniruled) over  $\text{Spec}(K)$  for every field extension  $k \subset K$ . In contrast, it is well-known that the property of being ruled depends on the base field  $k$ : for instance a smooth conic

$C \subset \mathbb{P}_k^2$  without  $k$ -rational point is uniruled but not ruled, but becomes ruled after base extension to a suitable quadratic extension of  $k$ .

The following lemma is probably well-known, but we include a proof because of lack of an appropriate reference.

**Lemma 4.** *Let  $Z$  be a non uniruled  $k$ -variety and let  $h : Y \rightarrow T$  be a surjective proper morphism between normal  $k$ -varieties, with rationally connected general fibers. Then every dominant rational map  $p : Y \dashrightarrow Z$  factors through a rational map  $q : T \dashrightarrow Z$ .*

*Proof.* Since the property of being non uniruled is invariant under birational equivalence, we can assume without loss of generality that  $Z$  is projective. Since  $h : Y \rightarrow T$  is proper, for every blow-up  $\sigma : \tilde{Y} \rightarrow Y$  of  $Y$ , the composition  $h \circ \sigma : \tilde{Y} \rightarrow T$  is again proper with rationally connected general fibers [20, IV.3.3]. So blowing-up  $Y$  to resolve the indeterminacy of  $p$ , we can further assume that  $p$  is a morphism, and by shrinking  $T$  that  $h : Y \rightarrow T$  is faithfully flat and proper, with rationally connected fibers. Then  $Y$  is rationally chain connected over  $T$ . Let  $u : \mathcal{C} = B \times \mathbb{P}^1 \rightarrow Y$  be a morphism of algebraic varieties over  $T$  witnessing this property. The inverse image by  $p \times p : Y \times_T Y \rightarrow Z \times Z$  of the diagonal is a closed subset  $X$  of  $Y \times_T Y$ . If  $X \neq Y \times_T Y$  then since  $h \circ \text{pr}_1 : Y \times_T Y \rightarrow T$  is flat hence open, the image of  $Y \times_T Y \setminus X$  is a dense open subset  $T_0$  of  $T$ . Replacing  $T$  by  $T_0$ , this would imply that the image of  $(p \circ u) \times_B (p \circ u) : \mathcal{C} \times_B \mathcal{C} \rightarrow Z \times Z$  is not contained in the diagonal, in contradiction with the non-uniruledness of  $Z$ . Thus  $X = Y \times_T Y$  and so,  $p$  is constant on the fibers of  $h : Y \rightarrow T$ . By faithfully flat descent, there exists a unique morphism  $q : T \rightarrow Z$  such that  $p = q \circ h$ .  $\square$

*Remark 5.* The conclusion of Lemma 4 does not hold under the weaker assumption that the general fibers of  $h : Y \rightarrow T$  are rationally chain connected. For instance, let  $Y$  be the projective cone over a smooth elliptic curve  $Z \subset \mathbb{P}_k^2$  and let  $h : Y \rightarrow T = \text{Spec}(k)$  be the canonical structure morphism. Then  $Y$  is rationally chain connected over  $\text{Spec}(k)$  and the projection  $p : Y \dashrightarrow Z$  from the vertex of the cone is a dominant rational map, which therefore does not factor through  $\text{Spec}(k)$ .

**1.2. Minimal Model Program over non closed fields.** In the proof of Theorem 2 given in section four below, we will make use of minimal model program over arbitrary fields of characteristic zero. We freely use the standard terminology and conventions in this context, and just recall the mild adaptations needed to run the minimal model program over a non closed field  $k$  in a form appropriate to our needs. It is well known (see e.g. [21, § 2.2] and [15, § 3.9]) that the minimal model program over an algebraically closed field has natural equivariant generalizations to the case of varieties with finite group actions, actually groups whose actions on the Neron-Severi groups of the varieties at hand factor through those of finite groups. This applies in particular to the situation of a smooth projective morphism  $\bar{f} : Y \rightarrow S$  between normal quasi-projective varieties defined over a field  $k$ : after the base change  $\bar{f}_{\bar{k}} : Y_{\bar{k}} \rightarrow S_{\bar{k}}$  to an algebraic closure  $\bar{k}$  of  $k$ , one can perform all the basic steps of  $K_{Y_{\bar{k}}}$ -mmp with scalings relative to  $\bar{f}_{\bar{k}} : Y_{\bar{k}} \rightarrow S_{\bar{k}}$  as in [4] in an equivariant way with respect to the natural action of the Galois group  $G = \text{Gal}(\bar{k}/k)$ . Compared to the genuine relative  $K_{Y_{\bar{k}}}$ -mmp with scalings, this program runs in the category of varieties which are projective over  $S_{\bar{k}}$ , with only terminal  $G$ - $\mathbb{Q}$ -factorial singularities, i.e. varieties with terminal singularities on which every  $G$ -invariant Weil divisor is  $\mathbb{Q}$ -Cartier.

The termination of arbitrary sequences of  $G$ -equivariant flips is not yet verified in a full generality, but as far as  $K_{Y_{\bar{k}}}$  is not pseudo-effective over  $S_{\bar{k}}$ , it follows from [4, Corollary 1.3.3] that there exists a  $G$ -equivariant  $K_{Y_{\bar{k}}}$ -mmp  $\Theta : Y_{\bar{k}} \dashrightarrow \tilde{Y}$  over  $S_{\bar{k}}$  with scalings by an  $\bar{f}_{\bar{k}}$ -ample  $G$ -invariant divisor which ends with a  $G$ -Mori fiber space  $\tilde{\rho} : \tilde{Y} \rightarrow \tilde{T}$  over a normal  $S_{\bar{k}}$ -variety  $\tilde{T}$ . That is,  $\tilde{\rho} : \tilde{Y} \rightarrow \tilde{T}$  is a projective  $G$ -equivariant morphism between quasi-projective  $\bar{k}$ -varieties with the following properties:  $\tilde{\rho}_* \mathcal{O}_{\tilde{Y}} = \mathcal{O}_{\tilde{T}}$ ,  $\dim \tilde{T} < \dim \tilde{Y}$ ,  $\tilde{Y}$  has only terminal  $G$ - $\mathbb{Q}$ -factorial singularities, the anti-canonical divisor  $-K_{\tilde{Y}}$  is  $\tilde{\rho}$ -ample and the relative  $G$ -invariant Picard number of  $\tilde{Y}$  over  $\tilde{T}$  is equal to one.

The birational map  $\Theta : Y_{\bar{k}} \dashrightarrow \tilde{Y}$  is a composition of either divisorial contractions associated to successive  $G$ -invariant extremal faces in the cone  $\overline{\text{NE}}(Y_{\bar{k}}/S_{\bar{k}})$  or flips which are all defined over  $k$ . The last morphism  $\tilde{\rho} : \tilde{Y} \rightarrow \tilde{T}$  corresponds to a  $G$ -equivariant extremal contraction of fiber type and is defined over  $k$  as well. It follows that  $\Theta : Y_{\bar{k}} \dashrightarrow \tilde{Y}$  and  $\tilde{\rho} : \tilde{Y} \rightarrow \tilde{T}$  can be equivalently seen as the base change to  $\bar{k}$  of a sequence  $\theta : Y \dashrightarrow Y'$  of  $K_Y$ -negative divisorial extremal contractions and flips between  $k$ -varieties which are  $\mathbb{Q}$ -factorial over  $k$  and projective over  $S$ , and an extremal contraction of fiber type  $\rho' : Y' \rightarrow T$  between normal  $k$ -varieties, such that  $-K_{Y'}$  is  $\rho'$ -ample and the relative Picard number of  $Y'$  over  $T$  is equal to one.

## 2. $\mathbb{A}^1$ -CYLINDRICAL VARIETIES

In this section, we introduce and establish basic properties of a special class of ruled varieties called  $\mathbb{A}^1$ -cylindrical varieties.

**Definition 6.** Let  $f : X \rightarrow S$  be a morphism of schemes. An  $\mathbb{A}^1$ -cylinder in  $X$  over  $S$  is a pair  $(Z, \varphi)$  consisting of an  $S$ -scheme  $Z \rightarrow S$  and an open embedding  $\varphi : Z \times \mathbb{A}^1 \hookrightarrow X$  of  $S$ -schemes. We say that  $X$  is  $\mathbb{A}^1$ -cylindrical over  $S$  if there exists an  $\mathbb{A}^1$ -cylinder  $(Z, \varphi)$  in  $X$  over  $S$ .

A variety  $X$  defined over a field  $k$  is called  $\mathbb{A}^1$ -cylindrical over  $k$  if it is  $\mathbb{A}^1$ -cylindrical over  $\text{Spec}(k)$ . Similarly as for ruledness, the property of being  $\mathbb{A}^1$ -cylindrical depends on the base field  $k$ : a smooth conic  $C \subset \mathbb{P}_k^2$  without  $k$ -rational point is not  $\mathbb{A}^1$ -cylindrical over  $k$  but becomes  $\mathbb{A}^1$ -cylindrical after base extension to a suitable quadratic extension of  $k$ .

**Definition 7.** A sub- $\mathbb{A}^1$ -cylinder of an  $\mathbb{A}^1$ -cylinder  $(Z, \varphi)$  in  $X$  over  $S$  is an  $\mathbb{A}^1$ -cylinder  $(Z', \varphi')$  in  $X$  over  $S$  for which there exists a commutative diagram

$$\begin{array}{ccccc} & & \varphi' & & \\ & & \curvearrowright & & \\ Z' \times \mathbb{A}^1 & \xrightarrow{j} & Z \times \mathbb{A}^1 & \xrightarrow{\varphi} & X \\ \downarrow \text{pr}_{Z'} & & \downarrow \text{pr}_Z & & \\ Z' & \xrightarrow{i} & Z & & \end{array}$$

for some open embeddings of  $S$ -schemes  $i : Z' \hookrightarrow Z$  and  $j : Z' \times \mathbb{A}^1 \hookrightarrow Z \times \mathbb{A}^1$ . Two  $\mathbb{A}^1$ -cylinders in  $X$  over  $S$  are called *equivalent* if they have a common sub- $\mathbb{A}^1$ -cylinder over  $S$ .

**2.1.  $\mathbb{A}^1$ -cylinders and  $\mathbb{P}^1$ -fibrations.** Recall that a  $\mathbb{P}^1$ -fibration is a proper morphism  $h : Y \rightarrow T$  between integral schemes whose fiber over the generic point of  $T$  is a form of  $\mathbb{P}^1$  over the function field  $K$  of  $T$ . Given an  $\mathbb{A}^1$ -cylinder  $(Z, \varphi)$  in an algebraic variety  $X$  over  $k$ , the composition of  $\varphi^{-1} : X \dashrightarrow Z \times \mathbb{A}^1$  with the projection  $\text{pr}_Z : Z \times \mathbb{A}^1 \rightarrow Z$  extends on a suitable complete model  $Y$  of  $X$  to a  $\mathbb{P}^1$ -fibration  $h : Y \rightarrow T$  over a complete model  $T$  of  $Z$ , restricting to a trivial  $\mathbb{P}^1$ -bundle over a non empty open subset  $Z_0$  of  $Z \subset T$ . Conversely, the total space  $Y$  of a  $\mathbb{P}^1$ -fibration  $h : Y \rightarrow T$  is  $\mathbb{A}^1$ -cylindrical over  $T$  provided that  $h$  admits a rational section  $H \subset Y$ . Indeed, if so, there exists a dense open subset  $Z$  of  $T$  such that  $h^{-1}(Z) \simeq Z \times \mathbb{P}^1$  and  $H \cap h^{-1}(Z) \simeq Z \times \{\infty\}$  for some fixed  $k$ -rational point  $\infty \in \mathbb{P}^1$ , which implies in turn that the open subset  $h^{-1}(Z) \cap (Y \setminus H)$  of  $Y$  is isomorphic to  $Z \times \mathbb{A}^1$ .

The following characterization will be useful for the proof of Theorem 2:

**Lemma 8.** Let  $h : Y \rightarrow T$  be a surjective proper morphism between normal varieties over a field  $k$  of characteristic zero, with irreducible and rationally connected general fibers. Suppose that  $Y$  contains an  $\mathbb{A}^1$ -cylinder  $(Z, \varphi)$  for some non-uniruled  $k$ -variety  $Z$ . Then  $h : Y \rightarrow T$  is a  $\mathbb{P}^1$ -fibration and there exists a sub- $\mathbb{A}^1$ -cylinder  $(Z', \varphi')$  of  $(Z, \varphi)$  and commutative diagram

$$\begin{array}{ccc} Z' \times \mathbb{A}^1 & \xrightarrow{\varphi'} & Y \\ \downarrow \text{pr}_{Z'} & & \downarrow h \\ Z' & \xrightarrow{i} & T \end{array}$$

for some open embedding  $i : Z' \hookrightarrow T$ . In particular,  $T$  is not uniruled.

*Proof.* By shrinking  $Z$ , we can assume that it is smooth and affine. Letting  $\overline{Z}$  be a smooth projective model of  $Z$ , the composition  $\text{pr}_Z \circ \varphi^{-1}$  defines a dominant rational map  $\text{pr}_Z \circ \varphi^{-1} : Y \dashrightarrow \overline{Z}$  which lifts to a  $\mathbb{P}^1$ -fibration  $p : \tilde{Y} \rightarrow \overline{Z}$  on some blow-up  $\sigma : \tilde{Y} \rightarrow Y$  of  $Y$ . Since  $\overline{Z}$  is not uniruled, and  $h \circ \sigma : \tilde{Y} \rightarrow T$  is proper with rationally connected general fibers, it follows from Lemma 4 that  $p$  factors through a dominant rational map  $q : T \dashrightarrow \overline{Z}$ . So  $\dim T \geq \dim \overline{Z}$  and since  $\dim \overline{Z} = \dim \tilde{Y} - 1 \geq \dim T$ , we conclude that  $\dim T = \dim Y - 1$ . This implies that  $h \circ \sigma : \tilde{Y} \rightarrow T$  is a  $\mathbb{P}^1$ -fibration, hence that  $h$  is a  $\mathbb{P}^1$ -fibration. Since the general fiber of  $p : \tilde{Y} \rightarrow \overline{Z}$  are irreducible,  $q$  has degree 1, hence is birational. Now it suffices to choose for  $Z'$  an open subset of  $Z \subset \overline{Z}$  on which  $q^{-1}$  restricts to an isomorphism onto its image.  $\square$

**2.2. Birational modifications preserving  $\mathbb{A}^1$ -cylinders.** In contrast with ruledness, the property of containing an  $\mathbb{A}^1$ -cylinder is obviously not invariant under birational equivalence. Nevertheless, it is stable under certain particular birational modifications which we record here for later use:

**Lemma 9.** Let  $(Y, \Delta)$  be a pair consisting of a normal  $k$ -variety and a reduced divisor on it, let  $\theta : Y \dashrightarrow Y'$  be a birational map to a normal  $k$ -variety  $Y'$  and let  $\Delta' = \theta_*(\Delta)$  be the proper transform of  $\Delta$  on  $Y'$ . Then the following hold:

- a) If  $\theta$  is an isomorphism in codimension one then  $Y \setminus \Delta$  is  $\mathbb{A}^1$ -cylindrical over  $k$  if and only if so is  $Y' \setminus \Delta'$ .
- b) If  $\theta$  is a proper morphism and  $Y' \setminus \Delta'$  contains an  $\mathbb{A}^1$ -cylinder  $(Z', \varphi')$  over  $k$  then there exists a sub-cylinder  $(Z, \varphi)$  of  $(Z', \varphi')$  such that  $(Z, \theta^{-1} \circ \varphi)$  is an  $\mathbb{A}^1$ -cylinder in  $Y \setminus \Delta$  over  $k$ .



c) If  $\theta$  is a proper morphism, each irreducible component of pure codimension one of the exceptional locus  $\text{Exc}(\theta)$  of  $\theta$  is uniruled and  $Y \setminus \Delta$  contains an  $\mathbb{A}^1$ -cylinder  $(Z, \varphi)$  for some non-uniruled  $k$ -variety  $Z$  then there exists a sub-cylinder  $(Z', \varphi')$  of  $(Z, \varphi)$  such that  $(Z', \theta \circ \varphi')$  is a  $\mathbb{A}^1$ -cylinder in  $Y' \setminus \Delta'$ .

*Proof.* If  $\theta$  is an isomorphism in codimension one, then it restricts to an isomorphism  $\theta : U \rightarrow U'$  between open subsets  $U \subset Y$  and  $U' \subset Y'$  whose complements  $X$  and  $X'$  have codimension at least 2 in  $Y$  and  $Y'$  respectively. Given a cylinder  $(Z, \varphi)$  in  $Y \setminus \Delta$ , the inverse image by  $\varphi$  of  $X \cap (Y \setminus \Delta)$  has codimension at least two in  $Z \times \mathbb{A}^1$ , hence does not dominate  $Z$ . Consequently, there exists a dense open subset  $Z' \subset Z$  such that  $(Z', \varphi' = \varphi|_{Z \times \mathbb{A}^1})$  is a sub- $\mathbb{A}^1$ -cylinder of  $(Z, \varphi)$  whose image is contained in  $U \cap (Y \setminus \Delta)$ , and the composition  $(Z', \theta \circ \varphi')$  is then an  $\mathbb{A}^1$ -cylinder in  $U' \cap Y' \setminus \Delta' \subset Y' \setminus \Delta'$ . Reversing the roles of  $Y$  and  $Y'$ , this yields a).

If  $\theta$  is a proper morphism, then  $C = \theta(\text{Exc}(\theta))$  has codimension at least 2 in  $Y'$  because  $Y'$  is normal. So the restriction of  $\text{pr}_{Z'}$  to  $\varphi'^{-1}(C)$  cannot be dominant. This guarantees the existence of a dense open subset  $Z$  of  $Z'$  such that  $(Z, \varphi = \varphi'|_{Z \times \mathbb{A}^1})$  is a sub- $\mathbb{A}^1$ -cylinder of  $(Z', \varphi')$  whose image is contained  $Y' \setminus \Delta' \cup C$ . Then  $(Z, \theta^{-1} \circ \varphi)$  is an  $\mathbb{A}^1$ -cylinder in  $Y \setminus \Delta \cup \text{Exc}(\theta) \subset Y \setminus \Delta$ , which proves b).

Finally to prove c), we observe that since  $Z$  is not uniruled, the restriction of  $\text{pr}_Z$  to the inverse image by  $\varphi$  of a uniruled irreducible component of pure codimension one of  $\text{Exc}(\theta)$  cannot be dominant. This implies that the restriction of  $\text{pr}_Z$  to  $\varphi^{-1}(\text{Exc}(\theta))$  is not dominant hence that there exists a dense open subset  $Z' \subset Z$  such that  $\varphi(Z' \times \mathbb{A}^1) \subset Y \setminus \Delta \cup \text{Exc}(\theta)$ . Then  $(Z', \varphi' = \varphi|_{Z' \times \mathbb{A}^1})$  is a sub- $\mathbb{A}^1$ -cylinder of  $(Z, \varphi)$  with the desired property.  $\square$

**2.3. Uniqueness properties of  $\mathbb{A}^1$ -cylinders.** The fact that  $\mathbb{P}_k^1$  contains infinitely many non-equivalent cylinders  $\mathbb{P}_k^1 \setminus \{p\}$  over  $k$ , parametrized by the set of its  $k$ -rational points  $p \in \mathbb{P}_k^1(k)$ , shows that in general a given  $k$ -variety  $X$  can contain many non equivalent cylinders even when their respective base spaces are non-uniruled. To ensure some uniqueness property of  $\mathbb{A}^1$ -cylinders, we are led to introduce the following notion:

**Definition 10.** Let  $f : X \rightarrow S$  be a morphism of schemes. An  $\mathbb{A}^1$ -cylinder  $(Z, \varphi)$  in  $X$  over  $S$  is said to be *vertically maximal* in  $X$  over  $S$  if for every generic point  $\xi$  of  $Z$ , the open embedding  $\xi \times \mathbb{A}^1 \hookrightarrow X$  induced by  $\varphi$  cannot be extended to a morphism  $\xi \times \mathbb{P}^1 \rightarrow X$ .

The next result can be thought as another geometric variant of Iitaka and Fujita strong cancellation theorems [13].

**Proposition 11.** Let  $X$  be  $k$ -variety containing a vertically maximal  $\mathbb{A}^1$ -cylinder  $(Z, \varphi)$  over a non uniruled  $k$ -variety  $Z$ . Then every  $\mathbb{A}^1$ -cylinder in  $X$  over  $k$  is equivalent to  $(Z, \varphi)$ .

*Proof.* Let  $U = \varphi(Z \times \mathbb{A}^1)$  be the open image of  $Z \times \mathbb{A}^1$  in  $X$ . By shrinking  $Z$  if necessary, we can assume that  $Z$  is affine and that all fibers of the projection  $\text{pr}_Z \circ \varphi^{-1} : U \rightarrow Z$  are closed in  $X$ . Let  $(T, \psi)$  be another cylinder in  $X$  with image  $V = \psi(T \times \mathbb{A}^1)$ , let  $W = U \cap V$  and let  $Z_0$  and  $T_0$  be the open images of  $W$  in  $Z$  and  $T$  by the morphisms  $\text{pr}_Z \circ \varphi^{-1}$  and  $\text{pr}_T \circ \psi^{-1}$  respectively. Since  $\text{pr}_T \circ \psi^{-1} : W \rightarrow T_0$  is a surjective morphism with uniruled fibers and  $Z$ , whence  $Z_0$ , is not uniruled, there exists a unique surjective morphism  $\alpha : T_0 \rightarrow Z_0$  such that  $\text{pr}_Z \circ \varphi^{-1} : W \rightarrow Z_0$  factors as  $\text{pr}_Z \circ \varphi^{-1} = \alpha \circ (\text{pr}_T \circ \psi^{-1}) : W \rightarrow T_0 \rightarrow Z_0$ . So for every point  $t \in T_0$ , there exists a unique  $z = \alpha(t) \in Z_0$  such that  $\psi(\text{pr}_T^{-1}(t)) \cap U$  is equal to  $\varphi(\text{pr}_Z^{-1}(z)) \cap V$ . Since by hypothesis  $\varphi(\text{pr}_Z^{-1}(z)) \simeq \mathbb{A}_{\kappa(z)}^1$  is closed in  $X$ , it follows that  $\psi(\text{pr}_T^{-1}(t)) = \varphi(\text{pr}_Z^{-1}(z))$ . This implies in turn that  $\psi(T_0 \times \mathbb{A}^1) \subset \varphi(Z_0 \times \mathbb{A}^1)$  and that we have a commutative diagram:

$$\begin{array}{ccccc} \psi(T_0 \times \mathbb{A}^1) & \hookrightarrow & \varphi(Z_0 \times \mathbb{A}^1) & \hookrightarrow & U \\ \text{pr}_T \circ \psi^{-1} \downarrow & & \downarrow \text{pr}_Z \circ \varphi^{-1} & & \downarrow \\ T_0 & \xrightarrow{\alpha} & Z_0 & \hookrightarrow & Z. \end{array}$$

It follows in particular that  $\alpha$  is also injective, hence an isomorphism. Thus  $(T_0, \psi|_{T_0 \times \mathbb{A}^1})$  is a sub- $\mathbb{A}^1$ -cylinder of  $(Z, \varphi)$ , which shows that  $(Z, \varphi)$  and  $(T, \psi)$  are equivalent.  $\square$

### 3. PROOF OF THEOREM 1

We now proceed to the proof of Theorem 1. By hypothesis,  $f : X \rightarrow S$  is a dominant morphism between geometrically normal algebraic varieties defined over an uncountable field  $k$  of characteristic zero, with the property that for general closed points  $s \in S$ , the fiber  $X_s$  contains a cylinder  $(Z_s, \varphi_s)$  over a  $\kappa(s)$ -variety  $Z_s$ . Letting  $X_\eta$  be the fiber of  $f$  over the generic point  $\eta$  of  $S$ , the existence of an étale morphism  $T \rightarrow S$  such that  $X \times_S T$  is  $\mathbb{A}^1$ -cylindrical over  $T$ , is equivalent to that of a finite extension  $L \subset L'$  of the function field  $L$  of  $S$  such that  $X_\eta \times_{\text{Spec}(L)} \text{Spec}(L')$  is  $\mathbb{A}^1$ -cylindrical over  $L'$ . In fact, the following specialization lemma implies that it is enough to find any extension  $L'$  of  $L$  for which  $X_\eta \times_{\text{Spec}(L)} \text{Spec}(L')$  is  $\mathbb{A}^1$ -cylindrical over  $L'$ :

**Lemma 12.** *Let  $X$  be a variety defined over a field  $k$  of characteristic zero and let  $k \subset K$  be any field extension. If  $X_K$  is  $\mathbb{A}^1$ -cylindrical over  $K$  then there exists a finite extension  $k \subset k'$  such that  $X_{k'}$  is  $\mathbb{A}^1$ -cylindrical over  $k'$ .*

*Proof.* By hypothesis, there exists an open embedding  $\varphi : Z \times \mathbb{A}^1 \hookrightarrow X_K$  for some  $K$ -variety  $Z$ . This open embedding is defined over a finitely generated sub-extension  $L$  of  $K$ , i.e. there exists an open embedding  $\varphi_0 : Z_0 \times \mathbb{A}^1 \hookrightarrow X_L$  of  $L$ -varieties such that  $\varphi$  is obtained from  $\varphi_0$  by the base change  $\text{Spec}(K) \rightarrow \text{Spec}(L)$ . Being finitely generated over  $k$ ,  $L$  is the function field of an algebraic variety  $S$  defined over  $k$  and we can therefore view  $X_L$  as the fiber  $\mathfrak{X}_\eta$  of the projection  $\text{pr}_S : \mathfrak{X} = X \times S \rightarrow S$  over the generic point  $\eta$  of  $S$ . Let  $\Delta$  and  $T$  be the respective closures of  $\mathfrak{X}_\eta \setminus \varphi_0(Z_0 \times \mathbb{A}^1)$  and  $\varphi_0(Z_0 \times \{0\})$  in  $\mathfrak{X}$ . The projection  $\text{pr}_{Z_0} : Z_0 \times \mathbb{A}^1 \rightarrow Z_0$  induces a rational map  $\rho : \mathfrak{X} \setminus \Delta \dashrightarrow T$  whose generic fiber is isomorphic to  $\mathbb{A}^1$  over the function field of  $T$ . It follows that there exists an open subset  $Y \subset T$  over which  $\rho$  is regular and whose inverse image  $V = \rho^{-1}(Y)$  is isomorphic to  $Y \times \mathbb{A}^1$ . Now for a general closed point  $s \in S$ , the fiber  $\mathfrak{X}_s$  of  $\text{pr}_S$  over  $s$  is isomorphic to  $X_{\kappa(s)}$ , where  $\kappa(s)$  denotes the residue field of  $s$ , and contains an open subset  $V_s$  isomorphic to  $Y_s \times \mathbb{A}^1$ . The induced open immersion  $Y_s \times \mathbb{A}^1 \hookrightarrow X_{\kappa(s)}$  provides the desired  $\mathbb{A}^1$ -cylinder over the finite extension  $\kappa(s)$  of  $k$ .  $\square$

Let  $\bar{k}$  be an algebraic closure of  $k$  and let  $f_{\bar{k}} : X_{\bar{k}} \rightarrow S_{\bar{k}}$  be the morphism obtained by the base extension  $\text{Spec}(\bar{k}) \rightarrow \text{Spec}(k)$ . Since  $S$  is geometrically integral,  $S_{\bar{k}}$  is integral and its field of functions  $\bar{k}(S_{\bar{k}})$  is an extension of the field of functions  $L$  of  $S$ . If the generic fiber of  $f_{\bar{k}}$  becomes  $\mathbb{A}^1$ -cylindrical after the base change to some extension of  $\bar{k}(S_{\bar{k}})$  then by the previous lemma, the generic fiber  $X_\eta$  of  $f : X \rightarrow S$  becomes  $\mathbb{A}^1$ -cylindrical after the base change to a finite extension of  $L$ . We can therefore assume from the very beginning that  $k = \bar{k}$  is an uncountable algebraically closed field of characteristic zero. Up to shrinking  $S$ , we can further assume without loss of generality that it is affine and that for every closed point  $s$  in  $S$ ,  $X_s$  contains a cylinder  $(Z_s, \varphi_s)$  over a  $k$ -variety  $Z_s$ . Since  $X$  and  $S$  are  $k$ -varieties, there exists a subfield  $k_0 \subset k$  of finite transcendence degree over  $\mathbb{Q}$  such that  $f : X \rightarrow S$  is defined over  $k_0$ , i.e. there exists a morphism of  $k_0$ -varieties  $f_0 : X_0 \rightarrow S_0$  and a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & X_0 \\ f \downarrow & & \downarrow f_0 \\ S & \longrightarrow & S_0 \\ \downarrow & & \downarrow \\ \text{Spec}(k) & \longrightarrow & \text{Spec}(k_0) \end{array}$$

in which each square is cartesian. The field of functions  $L_0 = k_0(S_0)$  of  $S_0$  is an extension of  $k_0$  of finite transcendence degree over  $\mathbb{Q}$ , and since  $k$  is uncountable and algebraically closed, there exists a  $k_0$ -embedding  $i : L_0 \hookrightarrow k$  of  $L_0$  in  $k$ . Letting  $(X_0)_{\eta_0}$  be the fiber of  $f_0$  over the generic point  $\eta_0 : \text{Spec}(L_0) \rightarrow S_0$  of  $S_0$ , the composition  $\Gamma(S_0, \mathcal{O}_{S_0}) \hookrightarrow L_0 \hookrightarrow k$  induces a  $k$ -homomorphism  $\Gamma(S_0, \mathcal{O}_{S_0}) \otimes_{k_0} k \rightarrow k$  defining a closed point  $s : \text{Spec}(k) \rightarrow \text{Spec}(\Gamma(S_0, \mathcal{O}_{S_0}) \otimes_{k_0} k) = S$  of  $S$  for which we obtain the following commutative diagram

$$\begin{array}{ccccccc} & & X_s & \longrightarrow & X & & \\ & \swarrow & \downarrow & & \downarrow & \swarrow & \\ (X_0)_{\eta_0} & \longrightarrow & X_0 & \xrightarrow{f} & S & \longrightarrow & \text{Spec}(k) \\ & \downarrow & \downarrow f_0 & & \downarrow s & & \\ & \downarrow i^* & \downarrow & \xrightarrow{f_0} & \downarrow & & \\ \text{Spec}(L_0) & \xrightarrow{\eta_0} & S_0 & \longrightarrow & \text{Spec}(k_0) & & \end{array}$$

Since the bottom square of the cube above is cartesian by construction, we have

$$(X_0)_{\eta_0} \times_{\text{Spec}(L_0)} \text{Spec}(k) \simeq X_0 \times_{S_0} \text{Spec}(k) \simeq X \times_S \text{Spec}(k) = X_s.$$

Since by hypothesis  $X_s$  is  $\mathbb{A}^1$ -cylindrical over  $k$ , we conclude that  $(X_0)_{\eta_0} \times_{\text{Spec}(L_0)} \text{Spec}(k)$  is  $\mathbb{A}^1$ -cylindrical over  $k$ . Lemma 12 then guarantees that there exists a finite extension  $L_0 \subset L'_0$  such that  $(X_0)_{\eta_0} \times_{\text{Spec}(L_0)} \text{Spec}(L'_0)$  is  $\mathbb{A}^1$ -cylindrical over  $L'_0$ . Finally, the tensor product  $L \otimes_{L_0} L'_0$  decomposes as a direct product of finitely many finite extensions  $L'$  of  $L$  with the property that  $X_\eta \times_{\text{Spec}(L)} \text{Spec}(L')$  is  $\mathbb{A}^1$ -cylindrical over  $L'$ , which completes the proof of Theorem 1.

*Remark 13.* Combined with Proposition 11, Lemma 12 implies that for a  $k$ -variety  $X$ , the property of containing a vertically maximal  $\mathbb{A}^1$ -cylinder over a non uniruled variety is independent of the base field. Indeed, by Lemma 12 if  $X_K$  contains a cylinder for some arbitrary field extension  $k \subset K$ , then  $X_{k'}$  contains a cylinder for a finite extension  $k \subset k'$ . Letting  $k''$  be the Galois closure of the extension  $k \subset k'$  in an algebraic closure of  $k'$ , Proposition 11 implies that the translates of a given cylinder  $(Z, \varphi)$  in  $X_{k''}$  over  $k''$  by the action of the Galois group  $G = \text{Gal}(k''/k)$  are all equivalent. Since  $G$  is a finite group, it follows that there exists a dense affine open subset  $Z_0$  of  $Z$ , an action of  $G$  on  $Z_0$  lifting to a  $G$ -action on  $Z_0 \times \mathbb{A}^1$  such that the induced open embedding  $(Z_0, \varphi|_{Z_0 \times \mathbb{A}^1}) \hookrightarrow X_{k''}$  is  $G$ -equivariant. The quotients  $(Z_0 \times \mathbb{A}^1)/G$  and  $Z_0/G$  are then affine varieties defined over  $k$  while the projection  $Z_0 \times \mathbb{A}^1 \rightarrow Z_0$  and the open embedding  $\varphi|_{Z_0 \times \mathbb{A}^1} : Z_0 \times \mathbb{A}^1 \hookrightarrow X_{k''}$  descend respectively to a locally trivial  $\mathbb{A}^1$ -bundle  $\pi : (Z_0 \times \mathbb{A}^1)/G \rightarrow Z_0/G$  and an open embedding  $\psi : (Z_0 \times \mathbb{A}^1)/G \hookrightarrow X_{k''}/G \simeq X$ . A cylinder in  $X$  over  $k$  is then obtained by restricting  $\psi$  to the inverse image of a dense open subset of  $Z_0/G$  over which  $\pi$  is a trivial  $\mathbb{A}^1$ -bundle.

#### 4. PROOF OF THEOREM 2

We first consider the case where  $f : X \rightarrow S$  is a smooth projective morphism whose general closed fibers contain vertically maximal  $\mathbb{A}^1$ -cylinders over non uniruled varieties. The case of an arbitrary morphism  $f : X \rightarrow S$  between normal algebraic varieties is then deduced by considering a suitably constructed smooth relative projective model of  $X$  over  $S$ .

##### 4.1. Case of a smooth projective morphism.

**Proposition 14.** *Let  $\bar{f} : Y \rightarrow S$  be a smooth projective morphism between normal  $k$ -varieties and let  $\Delta \subset Y$  be a divisor on  $Y$  such that for a general closed point  $s \in S$ ,  $Y_s \setminus \Delta_s$  contains an  $\mathbb{A}^1$ -cylinder  $(Z_s, \varphi_s)$  over a non uniruled  $\kappa(s)$ -variety  $Z_s$ . Then there exists a  $K_Y$ -MMP  $\theta : Y \dashrightarrow Y'$  relative to  $\bar{f} : Y \rightarrow S$  whose output  $\bar{f}' : Y' \rightarrow S$  has the structure of a Mori conic bundle  $\rho' : Y' \rightarrow T$  over a non uniruled normal  $S$ -variety  $h : T \rightarrow S$ . Furthermore, for a general closed point  $s \in S$ , there exists a sub-cylinder  $(Z'_s, \varphi'_s)$  of  $(Z_s, \varphi_s)$  and a commutative diagram*

$$\begin{array}{ccc} Z'_s \times \mathbb{A}^1 & \xrightarrow{\theta_s \circ \varphi'_s} & Y'_s \\ \text{pr}_{Z'_s} \downarrow & & \downarrow \rho'_s \\ Z'_s & \xrightarrow{\alpha_s} & T_s \end{array}$$

where the top and bottom arrows are open embeddings.

*Proof.* Since the general fibers of  $\bar{f} : Y \rightarrow S$  are in particular uniruled, it follows that  $K_Y$  is not  $\bar{f}$ -pseudo-effective. By virtue of [4, Corollary 1.3.3] (see § 1.2), there exists a  $K_Y$ -mmp  $\theta : Y \dashrightarrow Y'$  relative to  $\bar{f} : Y \rightarrow S$  whose output  $\bar{f}' : Y' \rightarrow S$  has the structure of a Mori fiber space  $\rho' : Y' \rightarrow T$  over some normal  $S$ -variety  $h : T \rightarrow S$ . Since for a general closed point  $s \in S$  the restriction  $\theta_s : Y_s \dashrightarrow Y'_s$  of  $\theta$  is a part of a  $K_{Y_s}$ -mmp ran from the smooth projective variety  $Y_s$ , it follows from [11, Corollary 1.7] that every irreducible component of pure codimension one of the exceptional locus of  $\theta_s$  is uniruled. Since  $\theta_s$  is a composition of divisorial contractions and isomorphisms in codimension one, we deduce from Lemma 9 a) and c) that there exists a sub-cylinder  $(Z'_s, \varphi'_s)$  of  $(Z_s, \varphi_s)$  such that  $(Z'_s, \theta_s \circ \varphi'_s)$  is an  $\mathbb{A}^1$ -cylinder in  $Y'_s$ . Since  $Y'$  has terminal singularities and  $-K_{Y'}$  is  $\rho'$ -ample, we deduce from [11, Corollary 1.4] that every fiber of  $\rho'$  is rationally chain connected. Since a general closed fiber of  $\rho'$  has again terminal singularities, we deduce in turn from [11, Corollary 1.8] that it is in fact rationally connected. The assertion then follows from Lemma 8.  $\square$

**Lemma 15.** *In the setting of Proposition 14, suppose further that for a general closed point  $s \in S$ , the  $\mathbb{A}^1$ -cylinder  $(Z_s, \varphi_s)$  in  $Y_s \setminus \Delta_s$  is maximally vertical. Then  $Y \setminus \Delta$  is  $\mathbb{A}^1$ -cylindrical over  $S$ .*

*Proof.* Since  $Y_s$  is projective, the hypothesis that  $(Z_s, \varphi_s)$  is maximally vertical in  $Y_s \setminus \Delta_s$  implies that the subset  $\Delta_0$  of irreducible components of  $\Delta$  which are horizontal for  $\bar{f} : Y \rightarrow S$  is not empty. Furthermore, for a general closed point  $s \in S$ ,  $\Delta_{0,s}$  intersects the closures in  $Y_s$  of the general fibers of  $\text{pr}_{Z_s} \circ \varphi_s^{-1} : \varphi_s(Z_s \times \mathbb{A}^1) \rightarrow Z_s$  in a unique place. Let  $(Z'_s, \varphi'_s)$  be a sub- $\mathbb{A}^1$ -cylinder of  $(Z_s, \varphi_s)$  with the property that  $(Z'_s, \theta_s \circ \varphi'_s)$  is an  $\mathbb{A}^1$ -cylinder in  $Y'_s$  and  $\alpha_s : Z'_s \hookrightarrow T_s$  is an open embedding. Since the only divisors that could be contracted by  $\theta_s : Y_s \dashrightarrow Y'_s$  are uniruled hence do not dominate  $Z'_s$ , we can assume up to shrinking  $Z'_s$  further if necessary that the restriction of  $\theta_s^{-1}$  to  $\rho'^{-1}(Z'_s)$  is an isomorphism onto its image  $V_s$  in  $Y_s$ . Consequently,  $\rho'_s \circ \theta_s|_{V_s} : V_s \rightarrow Z'_s$  is a  $\mathbb{P}^1$ -fibration extending  $\text{pr}_{Z'_s} \circ \varphi'^{-1}_s : \varphi'_s(Z'_s \times \mathbb{A}^1) \rightarrow Z'_s$ . Since  $(Z_s, \varphi_s)$  is vertically maximal in  $Y_s \setminus \Delta_s$ , so is  $(Z'_s, \varphi'_s)$ , and it follows that  $\Delta_{0,s} \cap V_s$  is a section of  $\rho'_s \circ \theta_s|_{V_s} : V_s \rightarrow Z'_s$ . This implies in turn that  $\Delta_0$  is irreducible and that there exists an open subset  $T_0$  of  $T$  such that  $(\rho' \circ \theta)^{-1}(T_0) \simeq T_0 \times \mathbb{P}^1$  and  $(\rho' \circ \theta)^{-1}(T_0) \setminus \Delta \simeq T_0 \times \mathbb{A}^1$ . So  $Y \setminus \Delta$  is  $\mathbb{A}^1$ -cylindrical over  $T_0$  whence over  $S$ .  $\square$



**4.2. General case.** The case of a general morphism  $f : X \rightarrow S$  between normal algebraic varieties is now obtained as follows. By desingularization theorems [12], we can find a desingularization  $\sigma : \tilde{X} \rightarrow X$  which restricts to an isomorphism over the regular locus  $X_{\text{reg}}$ . Since  $X$  is normal, it follows in particular that the image of the exceptional locus of  $\sigma$  is a closed subset of  $X$  of codimension at least two. By Nagata completion theorems [23] and desingularization theorems again, there exists an open embedding  $j : \tilde{X} \hookrightarrow \tilde{Y}$  into a smooth algebraic variety  $\tilde{Y}$  proper over  $S$ . Then by Chow lemma [9, 5.6.1] there exists a smooth algebraic variety  $Y$  projective over  $S$ , say  $\bar{f} : Y \rightarrow S$ , and a birational morphism  $\tau : Y \rightarrow \tilde{Y}$ . Applying desingularization again, we can further assume that the reduced total transform of  $\tilde{Y} \setminus j(\tilde{X})$  in  $Y$  is an SNC divisor  $\Delta$ . Since  $\tilde{Y}$  is smooth, the image of the exceptional locus of  $\tau$  has codimension at least two in  $\tilde{Y}$ , and so the image of the exceptional locus of  $\beta = \sigma \circ \tau|_{\tau^{-1}(\tilde{X})} : \tau^{-1}(\tilde{X}) \rightarrow X$  is a closed subset of codimension at least two in  $X$ . Summing up, we get a sequence of birational maps of  $S$ -varieties

$$\begin{array}{ccccc}
 & & \delta & & \\
 & \nearrow & & \nwarrow & \\
 X & \xrightarrow{\sigma^{-1}} & \tilde{X} & \xrightarrow{j} & \tilde{Y} & \xrightarrow{\tau^{-1}} & Y \\
 f \downarrow & & & & & & \downarrow \bar{f} \\
 S & \xlongequal{\quad} & & & & & S
 \end{array}$$

which we refer to as a *good relative smooth projective completion* of  $f : X \rightarrow S$ .

**Lemma 16.** *Let  $f : X \rightarrow S$  be a morphism between normal  $k$ -varieties and let  $\delta : X \dashrightarrow Y$  be a good relative smooth projective completion of  $f : X \rightarrow S$ . Suppose that for a general closed point  $s \in S$ ,  $X_s$  contains an  $\mathbb{A}^1$ -cylinder  $(Z_s, \varphi_s)$  over a  $\kappa(s)$ -variety  $Z_s$ . Then for a general closed point  $s$ , there exists a dense open subset  $Z'_s$  of  $Z_s$  such that  $(Z'_s, \delta_s \circ \varphi_s)$  is an  $\mathbb{A}^1$ -cylinder in  $Y_s \setminus \Delta_s$ . Furthermore, if  $(Z_s, \varphi_s)$  is vertically maximal in  $X_s$  then  $(Z'_s, \delta_s \circ \varphi_s)$  is vertically maximal in  $Y_s \setminus \Delta_s$ .*

*Proof.* Since  $X$  is normal, for a general closed point  $s \in S$ ,  $X_s$  is a normal variety. The morphism  $(\sigma \circ \tau)_s : \tau^{-1}(j(\tilde{X}))_s \rightarrow X_s$  being proper and birational by construction, the first assertion follows from Lemma 9 b). The second one is clear from the definition of  $\Delta$ .  $\square$

The following proposition combined with Proposition 14, Lemma 15 Lemma 16 completes the proof of Theorem 2.

**Proposition 17.** *Let  $f : X \rightarrow S$  be a morphism between normal  $k$ -varieties and let  $\delta : X \dashrightarrow Y$  be a good relative smooth projective completion of  $f : X \rightarrow S$ . Suppose that for a general closed point  $s \in S$ ,  $X_s$  contains a vertically maximal  $\mathbb{A}^1$ -cylinder  $(Z_s, \varphi_s)$  over a non uniruled  $\kappa(s)$ -variety  $Z_s$ . If  $Y \setminus \Delta$  is  $\mathbb{A}^1$ -cylindrical over  $S$  then so is  $X$ .*

*Proof.* Let  $\psi : T \times \mathbb{A}^1 \hookrightarrow Y \setminus \Delta$  be an  $\mathbb{A}^1$ -cylinder in  $Y$  over  $S$ . It is enough to show that the restriction of  $\text{pr}_T$  to the inverse image by  $\psi$  of the exceptional locus  $\text{Exc}(\beta)$  of  $\beta = \sigma \circ \tau|_{\tau^{-1}(\tilde{X})} : \tau^{-1}(\tilde{X}) \rightarrow X$  is not dominant. Indeed, if so, there exists an open subset  $T_0$  of  $T$  such that  $\psi(T_0 \times \mathbb{A}^1)$  is contained in  $Y \setminus \text{Exc}(\beta) \cup \Delta \simeq \delta(X \setminus \beta(\text{Exc}(\beta)))$ . So suppose on the contrary that there exists an irreducible component  $E$  of  $\text{Exc}(\beta)$  such that  $\text{pr}_T|_{\psi^{-1}(E)}$  is dominant. For a general closed point  $s \in S$ , the fiber  $Y_s$  is smooth and the restriction  $\beta_s : \tau_s^{-1}(\tilde{X}_s) \rightarrow X_s$  is an isomorphism outside a closed subset of codimension at least two in  $X_s$ . So there exists a dense open subset  $Z'_s$  of  $Z_s$  such that  $(Z'_s, \varphi'_s = \beta_s^{-1} \circ \varphi_s|_{Z'_s \times \mathbb{A}^1})$  is  $\mathbb{A}^1$ -cylinder in  $\tau_s^{-1}(\tilde{X}_s)$ . Since  $(Z_s, \varphi_s)$  is a vertically maximal  $\mathbb{A}^1$ -cylinder in  $X_s$ ,  $(Z'_s, \varphi'_s)$  is vertically maximal in  $\tau_s^{-1}(\tilde{X}_s)$ . On the other hand, for a general closed point  $s \in S$ , the restriction  $\psi_s : T_s \times \mathbb{A}^1 \rightarrow \tau_s^{-1}(\tilde{X}_s)$  is also an embedding. Since  $Z_s$  whence  $Z'_s$  is not uniruled, it follows from Proposition 11 that  $(Z'_s, \varphi'_s)$  and  $(T_s, \psi_s)$  are equivalent  $\mathbb{A}^1$ -cylinders in  $\tau_s^{-1}(\tilde{X}_s)$ . But then the restriction of  $\text{pr}_{Z'_s}$  to  $\varphi'^{-1}_s(E)$  would be dominant, implying in turn that  $(Z'_s, \varphi'_s)$  is not a cylinder, a contradiction.  $\square$

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