Multi-bumps analysis for Trudinger-Moser nonlinearities I-Quantification and location of concentration points

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MULTI-BUMPS ANALYSIS FOR TRUDINGER-MOSER NONLINEARITIES I - QUANTIFICATION AND LOCATION OF CONCENTRATION POINTS

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Abstract. In this paper, we investigate carefully the blow-up behaviour of sequences of solutions of some elliptic PDE in dimension two containing a nonlinearity with Trudinger-Moser growth. A quantification result had been obtained by the first author in [15] but many questions were left open. Similar questions were also explicitly asked in subsequent papers, see Del Pino-Musso-Ruf [12], Malchiodi-Martinazzi [30] or Martinazzi [34]. We answer all of them, proving in particular that blow up phenomenon is very restrictive because of the strong interaction between bubbles in this equation. This work will have a sequel, giving existence results of critical points of the associated functional at all energy levels via degree theory arguments, in the spirit of what had been done for the Liouville equation in the beautiful work of Chen-Lin [8].

1. Introduction

We let Ω be a smooth bounded domain of $\mathbb{R}^2$ and we consider the equation

$$\Delta u = \lambda f u e^{u^2} \text{ in } \Omega,\ u > 0 \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega\ . \tag{1.1}$$

where $\Delta = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$, $\lambda > 0$ and f is a smooth positive function in $\overline{\Omega}$.

This equation is critical with respect to Trudinger-Moser inequality. Indeed, the nonlinearity in e^{u^2} is the best one can hope to control in dimension 2 by the L^2-norm of the gradient. More precisely, we let $H_0^1(\Omega)$ be the standard Sobolev space (with zero boundary condition) endowed with the norm $\|\nabla u\|_2^2 = \int_\Omega |\nabla u|^2\ dx$. Trudinger proved in [40] that $\int_\Omega e^{u^2}\ dx$ is finite for any function u in $H_0^1(\Omega)$. Moser was then a little bit more precise in [35], proving that

$$\sup_{u \in H_0^1(\Omega),\ \|\nabla u\|_2 = 1} \int_\Omega e^{\gamma u^2}\ dx < +\infty \text{ if and only if } \gamma \leq 4\pi\ . \tag{1.2}$$

Solutions of equation (1.1) are in fact critical points of the functional

$$J(u) = \int_\Omega f e^{u^2}\ dx \tag{1.3}$$

under the constraint $\int_\Omega |\nabla u|^2\ dx = \beta$ for some $\beta > 0$. The λ appearing in (1.1) is then the Euler-Lagrange coefficient. This functional is well-defined on $H_0^1(\Omega)$ thanks to Trudinger [40]. It is also easy to find a critical point of J if $\beta < 4\pi$ in the constraint thanks to Moser's inequality (1.2) : these critical points may be found as maxima of J under the constraint $\int_\Omega |\nabla u|^2\ dx = \beta < 4\pi$. However, as studied by Adimurth-Prashanth [2], for $\beta = 4\pi$, finding critical points is more tricky since a lack of compactness appears in Palais-Smale sequences at this level of energy. Nevertheless, it has been proved by Carleson-Chang [6] for the unit disk, by Struwe [37] for Ω

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close to the disk and by Flucher [18] for a general Ω that there are extremals in Moser's inequality (1.2) for $\gamma = 4\pi$, meaning in particular that there are always critical points of J for the critical value $\beta = 4\pi$. Note that existence of critical points for β slightly larger than 4π has also been proved by Struwe [37] and Lamm-Robert-Struwe [20]. Struwe [38] also found critical points of higher energy (for some values of β between 4π and 8π) when the domain contains an annulus (in the spirit of Coron [10]). We refer also to the recent Mancini-Martinazzi [31] for an interesting new proof of the existence of extremal functions for Moser's inequality in the disk without using test-functions computations.

In the last decade, tools have been developed to study sequences of solutions of equation (1.1) and in particular to understand precisely their potential blow-up behaviour. This serie of works started in the minimal energy situation (β close to 4π) with Adimurthi-Struwe [3]. Then Adimurthi-Druet [1] used this blow-up analysis to obtain an improvement of Moser's inequality (completing the result of Lions [28]). In the radial case (that is in the unit disk with $f \equiv 1$), such a blow-up analysis in the minimal energy case was recently used by Malchiodi-Martinazzi [30] to prove that there is a $\beta_0 > 4\pi$ for which there are solutions of (1.1) of energy less than or equal to β_0 but no solutions of energy greater than β_0.

In order to get solutions of higher energies and to describe precisely the set of solutions for all β, one needs a fine analysis of blowing-up solutions. The first result in this direction is the quantification result of the first author [15] that we recall here since the questions we adress in the present work come from it :

Theorem 1.1 (Druet [15])**.** *Let Ω be a smooth bounded domain in $\mathbb{R}^2$ and let $(f_\varepsilon)_{\varepsilon>0}$ be a sequence of functions of uniform critical growth in Ω. Also let $(u_\varepsilon)_{\varepsilon>0}$ be a sequence of solutions of*

$$\Delta u_\varepsilon = f_\varepsilon\big(x, u_\varepsilon(x)\big)$$

verifying that $\|\nabla u_\varepsilon\|_2^2 \to \beta$ as $\varepsilon \to 0$ for some $\beta \in \mathbb{R}$. Then there exists a solution $u_0 \in C^0\left(\bar{\Omega}\right)$ of

$$\Delta u_0 = f_0\big(x, u_0(x)\big) \text{ in } \Omega,\ u_0 = 0 \text{ on } \partial\Omega\ ,$$

and there exists $\tilde{N} \in \mathbb{N}$ such that

$$\|\nabla u_\varepsilon\|_2^2 = \|\nabla u_0\|_2^2 + 4\pi\tilde{N} + o\left(1\right)\ .$$

If $\tilde{N} = 0$, the convergence of u_ε to u_0 is strong in $H_0^1\left(\Omega\right)$ and actually holds in $C^0\left(\bar{\Omega}\right)$.

We do not define here sequences of functions of uniform critical growth in Ω. The only thing we need to know is that they include sequences of the form $f_\varepsilon(x,u) = h_\varepsilon(x)ue^{u^2}$ as soon as $h_\varepsilon > 0$ and $h_\varepsilon \to h_0$ in $\mathcal{C}^1\left(\overline{\Omega}\right)$. But they include much more general nonlinearities behaving like e^{u^2} at infinity. Note also that, in the litterature, the nonlinearity is sometimes written as $e^{4\pi u^2}$ (this is for instance the case in [15]), hence the discrepancy of 4π in some results.

This result describes precisely the lack of compactness in the energy space. Note that this result is not true for Palais-Smale sequences, as proved by Costa-Tintarev [11] : there are Palais-Smale sequences for the above equation which converge to 0 weakly in $H_0^1\left(\Omega\right)$ and which present a lack of compactness at any level above 4π. This shows that the quantification result of Theorem 1.1 is specific to sequences of solutions of the equation and require a pointwise analysis as carried out in [15] and below (not only an analysis in energy space).

Note also that the above result is not empty since Del Pino-Musso-Ruf [13] constructed, via a Liapunov-Schmidt procedure, multi-pikes sequences of solutions of equation (1.1) (with $f \equiv 1$) in annuli. These sequences satisfy the hypothesis of the above theorem, converge weakly to 0 in $H_0^1\left(\Omega\right)$ (that is $u_0 \equiv 0$ in the above result) and have an energy converging to $4\pi\tilde{N}$. They can

construct such solutions for all $\tilde{N} \geq 1$. This suggests that the topology of the domain plays a crucial role in the existence of solutions of arbitrary energies.

However, if one wants to push further the existence results, we need to be more precise than Druet [15]. In particular, we need to answer the following natural questions, left open in this work of the first author (see also Del Pino-Musso-Ruf [12], Malchiodi-Martinazzi [30] or Martinazzi [34] where one can find these, or similar, questions) :

1. Is it possible to have both $u_0 \not\equiv 0$ and $\tilde{N} \geq 1$ in the above theorem ?

2. Are the concentration points appearing when $\tilde{N} \geq 1$ isolated[1] or not ? If yes, where are they ?

These questions are natural and can be motivated by analogy with Liouville type equations (see among others [7, 8, 23, 27, 29, 39]) or Yamabe type equations (see for instance [4, 5, 14, 17, 19, 22, 24, 25, 26, 32, 33, 36]). We refer in particular to [16] for a survey on this kind of questions.

We attack in this paper the questions 1 and 2 above. Our result holds for more general nonlinearities but we restrict, for sake of clearness, to the simplest one. We consider a sequence (u_ε) of smooth positive solutions of

$$\Delta u_\varepsilon = \lambda_\varepsilon f_\varepsilon u_\varepsilon e^{u_\varepsilon^2} \text{ in } \Omega,\ u_\varepsilon = 0 \text{ on } \partial\Omega\ , \tag{1.4}$$

for some sequence (λ_ε) of positive real numbers and some sequence (f_ε) of smooth functions in $\overline{\Omega}$ which satisfies

$$f_\varepsilon \to f_0 \text{ in } \mathcal{C}^1\left(\overline{\Omega}\right) \text{ as } \varepsilon \to 0 \text{ and } \left\|\nabla^2 f_\varepsilon\right\|_{L^\infty(\Omega)} = O(1) \tag{1.5}$$

where $f_0 > 0$ in $\overline{\Omega}$. And we prove the following result :

Theorem 1.2. *Let Ω be a smooth bounded domain of $\mathbb{R}^2$ and let (u_ε) be a sequence of smooth solutions of (1.4) which is bounded in $H_0^1(\Omega)$. Assume that (1.5) holds. Then, after passing to a subsequence, $\lambda_\varepsilon \to \lambda_0$ as $\varepsilon \to 0$ for some $\lambda_0 \in \mathbb{R}$. .*

If $\lambda_0 \neq 0$, then there exists $u_0 \in \mathcal{C}^2\left(\overline{\Omega}\right)$ solution of

$$\Delta u_0 = \lambda_0 f_0 u_0 e^{u_0^2} \text{ in } \Omega,\ u_0 = 0 \text{ on } \partial\Omega$$

such that $u_\varepsilon \to u_0$ in $\mathcal{C}^2\left(\overline{\Omega}\right)$ as $\varepsilon \to 0$.

If $\lambda_0 = 0$, then $u_\varepsilon \rightharpoonup 0$ weakly in $H_0^1(\Omega)$. Moreover there exist $N \geq 1$ such that

$$\int_\Omega |\nabla u_\varepsilon|^2\, dx \to 4\pi N \text{ as } \varepsilon \to 0$$

and N sequences of points $(x_{i,\varepsilon})$ in Ω such that

a) $x_{i,\varepsilon} \to x_i$ as $\varepsilon \to 0$ with $x_i \in \Omega$ (not on the boundary), all the x_i's being distinct.

b) $u_\varepsilon \to 0$ in $\mathcal{C}^2_{loc}\left(\overline{\Omega} \setminus \mathcal{S}\right)$ where $\mathcal{S} = \{x_i\}_{i=1,\dots,N}$.

c) for all $i = 1, \dots, N$, we have that $\gamma_{i,\varepsilon} = u_\varepsilon(x_{i,\varepsilon}) \to +\infty$ as $\varepsilon \to 0$ and that

$$\gamma_{i,\varepsilon}\left(u_\varepsilon\left(x_{i,\varepsilon} + \mu_{i,\varepsilon} x\right) - \gamma_{i,\varepsilon}\right) \to U(x) = -\ln\left(1 + \frac{1}{4}|x|^2\right)$$

in $C^2_{loc}\left(\mathbb{R}^2\right)$ as $\varepsilon \to 0$ where

$$\mu_{i,\varepsilon}^{-2} = \lambda_\varepsilon f_\varepsilon(x_{i,\varepsilon})\, \gamma_{i,\varepsilon}^2 e^{\gamma_{i,\varepsilon}^2} \to +\infty \text{ as } \varepsilon \to 0\ .$$

[1] By isolated, we mean here that the energy at any concentration point is exactly 4π. In other words, we mean that there are no bubble accumulations and we do not wish to rule out only bubbles towers.

d) for all $i = 1, \dots, N$, there exists $m_i > 0$ such that

$$\sqrt{\lambda_\varepsilon}\gamma_{i,\varepsilon} \to \frac{2}{m_i\sqrt{f_0(x_i)}} \text{ as } \varepsilon \to 0 .$$

e) The points x_i are such that

$$2m_i\nabla_y\mathcal{H}(x_i,x_i) + 4\pi\sum_{j\neq i} m_j\nabla_y\mathcal{G}(x_j,x_i) + \frac{1}{2}m_i\frac{\nabla f_0(x_i)}{f_0(x_i)} = 0$$

and that

$$4\pi\sum_{j\neq i} m_j\mathcal{G}(x_j,x_i) + 2m_i\mathcal{H}(x_i,x_i) + m_i\ln\frac{f_0(x_i)}{m_i^2} + m_i = 0$$

for all $i = 1, \dots, N$ where

$$\mathcal{G}(x,y) = \frac{1}{2\pi}\left(\ln\frac{1}{|x-y|} + \mathcal{H}(x,y)\right)$$

is the Green function of the Laplacian with Dirichlet boundary condition.

Note that this theorem proves that, if blow-up occurs, then the weak limit has to be zero so that lack of compactness can occur only at the levels $\beta = 4\pi N$ for $N \geq 1$. This is a key information to get general existence result via degree theory from this theorem; this will be the subject of a subsequent paper. We also obtain a precise characterisation of the location of concentration points. This answers in particular by the affirmative to the conjecture of Del Pino-Musso-Ruf [12] (p. 425) since, in case $f \equiv 1$, the (x_i, m_i) of Theorem 1.2 are critical points of the function

$$\Phi(y_i,\alpha_i) = 2\pi\sum_{i\neq j}\alpha_i\alpha_j\mathcal{G}(y_i,y_j) + \sum_{i=1}^N \alpha_i^2\mathcal{H}(y_i,y_i) + \sum_{i=1}^N\left(\alpha_i^2 - \alpha_i^2\ln\alpha_i\right) .$$

The paper is organized as follows. In Section 2, we recall the main results of Druet [15] and set up the proof of the theorem. Section 3 is devoted to a fine asymptotic analysis in the neighbourhood of a given concentration point while the theorem is proved in Section 4 which deals with the multi-spikes analysis. At last, we collect some useful estimates concerning the standard bubble and the Green function respectively in appendices A and B.

2. Previous results and sketch of the proof

We set up the proof of Theorem 1.2 and we recall some results obtained in Druet [15]. We let Ω be a smooth bounded domain of $\mathbb{R}^2$ and we consider a sequence (u_ε) of smooth positive solutions of

$$\Delta u_\varepsilon = \lambda_\varepsilon f_\varepsilon u_\varepsilon e^{u_\varepsilon^2} \text{ in } \Omega,\ u_\varepsilon = 0 \text{ on } \partial\Omega \tag{2.1}$$

for some sequence (λ_ε) of positive real numbers and some sequence (f_ε) of smooth functions which satisfies (1.5). Note that we necessarily have that

$$\limsup_{\varepsilon\to 0}\lambda_\varepsilon \leq \frac{\lambda_1}{\min_{\overline{\Omega}} f_0} \tag{2.2}$$

where $\lambda_1 > 0$ is the first eigenvalue of the Laplacian with Dirichlet boundary condition in Ω. Indeed, let $\varphi_1 \in \mathcal{C}^\infty\left(\overline{\Omega}\right)$ be a positive (in Ω) eigenfunction associated to λ_1 and multiply equation (2.1) by φ_1. After integration by parts, we get that

$$\lambda_1\int_\Omega u_\varepsilon\varphi_1\,dx = \lambda_\varepsilon\int_\Omega f_\varepsilon u_\varepsilon e^{u_\varepsilon^2}\varphi_1\,dx .$$

Since f_ε becomes positive for ε small thanks to (1.5) and since u_ε and φ_1 are positive, we can write that

$$\lambda_1 \int_\Omega u_\varepsilon \varphi_1 \, dx \geq \lambda_\varepsilon \left(\min_{\overline{\Omega}} f_\varepsilon \right) \int_\Omega u_\varepsilon \varphi_1 \, dx \ ,$$

which leads to (2.2).

We assume in the following that there exists $C > 0$ such that

$$\int_\Omega |\nabla u_\varepsilon|^2 \, dx \leq C \text{ for all } \varepsilon > 0 \ . \tag{2.3}$$

Then we have the following :

Proposition 2.1 (Druet [15]). *After passing to a subsequence, $\lambda_\varepsilon \to \lambda_0$ as $\varepsilon \to 0$, there exists a smooth solution u_0 of the limit equation*

$$\Delta u_0 = \lambda_0 f_0 u_0 e^{u_0^2} \text{ in } \Omega, \ u_0 = 0 \text{ on } \partial\Omega \tag{2.4}$$

and there exist $N \geq 0$ and N sequences $(x_{i,\varepsilon})$ of points in Ω such that the following assertions[2] hold :

a) $u_\varepsilon \rightharpoonup u_0$ weakly in $H_0^1(\Omega)$. If $N = 0$, the convergence of u_ε to u_0 holds in $C^2\left(\overline{\Omega}\right)$.

b) for any $i \in \{1, \dots, N\}$, $u_\varepsilon(x_{i,\varepsilon}) \to +\infty$ as $\varepsilon \to 0$ and $\nabla u_\varepsilon(x_{i,\varepsilon}) = 0$.

c) for any $i, j \in \{1, \dots, N\}$, $i \neq j$,

$$\frac{|x_{i,\varepsilon} - x_{j,\varepsilon}|}{\mu_{i,\varepsilon}} \to +\infty \text{ as } \varepsilon \to 0$$

where

$$\mu_{i,\varepsilon}^{-2} = \lambda_\varepsilon f_\varepsilon(x_{i,\varepsilon}) \, u_\varepsilon(x_{i,\varepsilon})^2 \, e^{u_\varepsilon(x_{i,\varepsilon})^2} \to +\infty \text{ as } \varepsilon \to 0 \ .$$

d) for any $i \in \{1, \dots, N\}$, we have that

$$u_\varepsilon(x_{i,\varepsilon}) \left(u_\varepsilon(x_{i,\varepsilon} + \mu_{i,\varepsilon} x) - u_\varepsilon(x_{i,\varepsilon}) \right) \to U(x) = -\ln\left(1 + \frac{1}{4}|x|^2\right)$$

in $C^2_{loc}\left(\mathbb{R}^2\right)$.

e) there exists $C_1 > 0$ such that

$$\lambda_\varepsilon \left(\min_{i=1,\dots,N} |x_{i,\varepsilon} - x| \right)^2 u_\varepsilon(x)^2 e^{u_\varepsilon(x)^2} \leq C_1$$

for all $x \in \Omega$.

f) there exists $C_2 > 0$ such that

$$\left(\min_{i=1,\dots,N} |x_{i,\varepsilon} - x| \right) u_\varepsilon(x) \, |\nabla u_\varepsilon(x)| \leq C_2 \ .$$

Proof - Even if this result is already contained in [15], we shall give part of the proof here. The first reason is that it is not exactly stated in this way in [15]. The second reason is that it is proved in greater generality in [15] and we thus give a proof which is in some sense more readable here.

First, it is clear thanks to (2.3) that, up to a subsequence, $u_\varepsilon \rightharpoonup u_0$ weakly in $H_0^1(\Omega)$ where u_0 is a solution of (2.4). If $\|u_\varepsilon\|_\infty = O(1)$, then, by standard elliptic theory, this convergence holds in $C^2\left(\overline{\Omega}\right)$ and the proposition is true with $N = 0$. Let us assume from now on that

$$\sup_\Omega u_\varepsilon \to +\infty \text{ as } \varepsilon \to 0 \ . \tag{2.5}$$

[2]We assume for assertions b) to f) that $N \geq 1$.

Given $N \geq 1$ and N sequences $(x_{i,\varepsilon})$ of points in Ω which verify that

$$\gamma_{i,\varepsilon} = u_\varepsilon(x_{i,\varepsilon}) \to +\infty \text{ as } \varepsilon \to 0 \text{ and } \mu_{i,\varepsilon}^{-2} = \lambda_\varepsilon f_\varepsilon(x_{i,\varepsilon})\gamma_{i,\varepsilon}^2 e^{\gamma_{i,\varepsilon}^2} \to +\infty \text{ as } \varepsilon \to 0\,, \tag{2.6}$$

we consider the following assertions :

$\left(P_1^N\right)$ For any $i, j \in \{1, \dots, N\}$, $i \neq j$, $\dfrac{|x_{i,\varepsilon} - x_{j,\varepsilon}|}{\mu_{i,\varepsilon}} \to +\infty$ as $\varepsilon \to 0$.

$\left(P_2^N\right)$ For any $i \in \{1, \dots, N\}$, $\nabla u_\varepsilon(x_{i,\varepsilon}) = 0$ and

$$\gamma_{i,\varepsilon}\left(u_\varepsilon(x_{i,\varepsilon} + \mu_{i,\varepsilon}x) - \gamma_{i,\varepsilon}\right) \to U(x)$$

in $\mathcal{C}^2_{loc}\left(\mathbb{R}^2\right)$ as $\varepsilon \to 0$ where

$$U(x) = -\ln\left(1 + \frac{1}{4}|x|^2\right)$$

is a solution of $\Delta U = e^{2U}$ in $\mathbb{R}^2$.

$\left(P_3^N\right)$ There exists $C > 0$ such that

$$\lambda_\varepsilon\left(\min_{i=1,\dots,N}|x_{i,\varepsilon} - x|\right)^2 u_\varepsilon(x)^2 e^{u_\varepsilon(x)^2} \leq C$$

for all $x \in \Omega$.

A first obvious remark is that

$$\left(P_1^N\right) \text{ and } \left(P_2^N\right) \Longrightarrow \int_\Omega |\nabla u_\varepsilon|^2\, dx \geq 4\pi N + o(1)\,. \tag{2.7}$$

Indeed, one has just to notice that

$$\int_\Omega |\nabla u_\varepsilon|^2\, dx = \lambda_\varepsilon \int_\Omega f_\varepsilon u_\varepsilon^2 e^{u_\varepsilon^2}\, dx\,,$$

that $\mathbb{D}_{x_{i,\varepsilon}}(R\mu_{i,\varepsilon}) \cap \mathbb{D}_{x_{j,\varepsilon}}(R\mu_{j,\varepsilon}) = \emptyset$ for $\varepsilon > 0$ small enough thanks to $\left(P_1^N\right)$ and that

$$\lim_{\varepsilon\to 0} \lambda_\varepsilon \int_{\mathbb{D}_{x_{i,\varepsilon}}(R\mu_{i,\varepsilon})} f_\varepsilon u_\varepsilon^2 e^{u_\varepsilon^2}\, dx = \int_{\mathbb{D}_0(R)} e^{2U}\, dx \to \int_{\mathbb{R}^2} e^{2U}\, dx = 4\pi \text{ as } R \to +\infty$$

thanks to $\left(P_2^N\right)$.

In the following, we shall say that property $\mathcal{P}_N$ holds if there are N sequences $(x_{i,\varepsilon})$ of points in Ω which verify (2.6) such that assertions $\left(P_1^N\right)$ and $\left(P_2^N\right)$ hold.

Step 1 - *Property $\mathcal{P}_1$ holds.*

Proof of Step 1 - Let $x_\varepsilon \in \Omega$ be such that

$$u_\varepsilon(x_\varepsilon) = \max_\Omega u_\varepsilon\,.$$

By (2.5), we have that

$$\gamma_\varepsilon = u_\varepsilon(x_\varepsilon) \to +\infty \text{ as } \varepsilon \to 0\,. \tag{2.8}$$

We just have to check $\left(P_2^1\right)$ since $\left(P_1^1\right)$ is empty. We clearly have that $\nabla u_\varepsilon(x_\varepsilon) = 0$. We set[3]

$$\tilde{u}_\varepsilon(x) = \gamma_\varepsilon\left(u_\varepsilon(x_\varepsilon + \mu_\varepsilon x) - \gamma_\varepsilon\right) \tag{2.9}$$

for $x \in \Omega_\varepsilon$ where

$$\Omega_\varepsilon = \left\{x \in \mathbb{R}^2 \text{ s.t. } x_\varepsilon + \mu_\varepsilon x \in \Omega\right\}$$

and

$$\mu_\varepsilon^{-2} = \lambda_\varepsilon f_\varepsilon(x_\varepsilon)\gamma_\varepsilon^2 e^{\gamma_\varepsilon^2}\,. \tag{2.10}$$

[3]The fact that this rescaling is appropriate to understand the blow up behaviour of solutions of equation (1.1) was first discovered by Adimurthi-Struwe [3].

It is clear that

$$\mu_\varepsilon \to 0 \text{ as } \varepsilon \to 0 \, . \tag{2.11}$$

Indeed, we can write that

$$\lambda_\varepsilon f_\varepsilon u_\varepsilon e^{u_\varepsilon^2} \leq \lambda_\varepsilon \left(\sup_\Omega f_\varepsilon \right) \gamma_\varepsilon e^{\gamma_\varepsilon^2} = \gamma_\varepsilon^{-1} \frac{\sup_\Omega f_\varepsilon}{f_\varepsilon (x_\varepsilon)} \mu_\varepsilon^{-2} = o\left(\mu_\varepsilon^{-2}\right)$$

thanks to (1.5) and (2.8). If ever (2.11) was false, we would have that $\|\Delta u_\varepsilon\|_\infty \to 0$ as $\varepsilon \to 0$ which, together with the fact that $u_\varepsilon = 0$ on $\partial\Omega$, would contradict (2.8). Thus (2.11) holds. Thanks to (2.11), we know that, up to a subsequence and up to a harmless rotation,

$$\Omega_\varepsilon \to \mathbb{R}^2 \text{ or } \Omega_\varepsilon \to \mathbb{R} \times (-\infty, d) \text{ as } \varepsilon \to 0 \tag{2.12}$$

where $d = \lim_{\varepsilon \to 0} \dfrac{d(x_\varepsilon, \partial\Omega)}{\mu_\varepsilon}$. We also have that

$$\Delta \tilde{u}_\varepsilon = \frac{f_\varepsilon (x_\varepsilon + \mu_\varepsilon x)}{f_\varepsilon (x_\varepsilon)} \frac{u_\varepsilon (x_\varepsilon + \mu_\varepsilon x)}{\gamma_\varepsilon} e^{u_\varepsilon (x_\varepsilon + \mu_\varepsilon x)^2 - \gamma_\varepsilon^2} \tag{2.13}$$

in Ω_ε thanks to (2.1) and (2.10). Since $0 \leq u_\varepsilon \leq \gamma_\varepsilon$ in Ω and thanks to (1.5), this leads to $\|\Delta \tilde{u}_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} = O(1)$. Together with the fact that $\tilde{u}_\varepsilon \leq 0 = \tilde{u}_\varepsilon(0)$ and $\tilde{u}_\varepsilon = -\gamma_\varepsilon^2 \to -\infty$ as $\varepsilon \to 0$ on $\partial\Omega_\varepsilon$, one can check that this implies that

$$\Omega_\varepsilon \to \mathbb{R}^2 \text{ as } \varepsilon \to 0$$

and that

$$\tilde{u}_\varepsilon \to U \text{ in } C^1_{loc}\left(\mathbb{R}^2\right) \text{ as } \varepsilon \to 0$$

after passing to a subsequence. We refer here the reader to [3] or [15] for the details of such an assertion. Moreover, we clearly have that $U \leq U(0) = 0$ in $\mathbb{R}^2$. Noting that, as a consequence of the above convergence of $\tilde{u}_\varepsilon$, we have that

$$u_\varepsilon (x_\varepsilon + \mu_\varepsilon x)^2 - \gamma_\varepsilon^2 \to 2U \text{ in } C^0_{loc}\left(\mathbb{R}^2\right) \, ,$$

one can easily pass to the limit in equation (2.13) to obtain that

$$\Delta U = e^{2U} \text{ in } \mathbb{R}^2 \, .$$

Moreover, by standard elliptic theory, one has that

$$\tilde{u}_\varepsilon \to U \text{ in } \mathcal{C}^2_{loc}\left(\mathbb{R}^2\right) \text{ as } \varepsilon \to 0 \, . \tag{2.14}$$

In order to apply the classification result of Chen-Li [9], we need to check that $e^{2U} \in L^1\left(\mathbb{R}^2\right)$. Using (2.9) together with (1.5), (2.10) and (2.14), we can write that

$$\lim_{\varepsilon \to 0} \lambda_\varepsilon \int_{\mathbb{D}_{x_\varepsilon}(R\mu_\varepsilon)} f_\varepsilon u_\varepsilon^2 e^{u_\varepsilon^2} \, dx = \int_{\mathbb{D}_0(R)} e^{2U} \, dx$$

for all $R > 0$. Thanks to (2.1) and (2.3), we know that

$$\lambda_\varepsilon \int_{\mathbb{D}_{x_\varepsilon}(R\mu_\varepsilon)} f_\varepsilon u_\varepsilon^2 e^{u_\varepsilon^2} \, dx \leq \lambda_\varepsilon \int_\Omega f_\varepsilon u_\varepsilon^2 e^{u_\varepsilon^2} \, dx = \int_\Omega |\nabla u_\varepsilon|^2 \, dx \leq M$$

so that $e^{2U} \in L^1\left(\mathbb{R}^2\right)$. Remembering that $U \leq U(0) = 0$, we thus get by [9] that

$$U(x) = -\ln\left(1 + \frac{1}{4}|x|^2\right) \, .$$

This clearly ends the proof of Step 1. ♠

Step 2 - *Assume that property $\mathcal{P}_N$ holds for some $N \geq 1$. Then either $\left(P_3^N\right)$ holds or $\mathcal{P}_{N+1}$ holds.*

Proof of Step 2 - Assume that $\mathcal{P}_N$ holds for some $N \geq 1$ (with associated sequences $(x_{i,\varepsilon})$) and that $\left(P_3^N\right)$ does not hold, meaning that

$$\lambda_\varepsilon \sup_{x\in\Omega} \left(\min_{i=1,\dots,N} |x_{i,\varepsilon} - x| \right)^2 u_\varepsilon(x)^2 e^{u_\varepsilon(x)^2} \to +\infty \text{ as } \varepsilon \to 0 \, . \tag{2.15}$$

We let then $y_\varepsilon \in \Omega$ be such that

$$\left(\min_{i=1,\dots,N} |x_{i,\varepsilon} - y_\varepsilon| \right)^2 u_\varepsilon \left(y_\varepsilon\right)^2 e^{u_\varepsilon(y_\varepsilon)^2} = \sup_{x\in\Omega} \left(\min_{i=1,\dots,N} |x_{i,\varepsilon} - x| \right)^2 u_\varepsilon(x)^2 e^{u_\varepsilon(x)^2} \tag{2.16}$$

and we set

$$u_\varepsilon \left(y_\varepsilon\right) = \hat{\gamma}_\varepsilon \, .$$

Since Ω is bounded and (λ_ε) is bounded, see (2.2), we know that

$$\hat{\gamma}_\varepsilon \to +\infty \text{ as } \varepsilon \to 0$$

thanks to (2.15) and (2.16). Thanks to $\left(P_N^2\right)$, (2.15) and (2.16), we also know that

$$\frac{|x_{i,\varepsilon} - y_\varepsilon|}{\mu_{i,\varepsilon}} \to +\infty \text{ as } \varepsilon \to 0 \text{ for all } 1 \leq i \leq N \, . \tag{2.17}$$

We set

$$\hat{\mu}_\varepsilon^{-2} = \lambda_\varepsilon f_\varepsilon \left(y_\varepsilon\right) \hat{\gamma}_\varepsilon^2 e^{\hat{\gamma}_\varepsilon^2}$$

so that, with (1.5), (2.15) and (2.16),

$$\hat{\mu}_\varepsilon \to 0 \text{ as } \varepsilon \to 0$$

and

$$\frac{|x_{i,\varepsilon} - y_\varepsilon|}{\hat{\mu}_\varepsilon} \to +\infty \text{ as } \varepsilon \to 0 \text{ for all } 1 \leq i \leq N \, . \tag{2.18}$$

We set now

$$\hat{u}_\varepsilon(x) = \hat{\gamma}_\varepsilon \left(u_\varepsilon \left(y_\varepsilon + \hat{\mu}_\varepsilon x \right) - \hat{\gamma}_\varepsilon \right)$$

for $x \in \hat{\Omega}_\varepsilon$ where

$$\hat{\Omega}_\varepsilon = \left\{ x \in \mathbb{R}^2 \text{ s.t. } y_\varepsilon + \hat{\mu}_\varepsilon x \in \Omega \right\} \, .$$

We are exactly in the situation of Step 1 except for one thing : we can not say that $\hat{u}_\varepsilon \leq 0$ in $\hat{\Omega}_\varepsilon$. However, combining (2.16) and (2.18), we can say that

$$\hat{u}_\varepsilon \leq o(1) \text{ in } K \cap \hat{\Omega}_\varepsilon$$

for all compact subset K of $\mathbb{R}^2$. This permits to repeat the arguments of Step 1, see [15] for the details, to obtain that

$$\hat{u}_\varepsilon \to U \text{ in } \mathcal{C}^2_{loc}\left(\mathbb{R}^2\right) \text{ as } \varepsilon \to 0 \, . \tag{2.19}$$

Since U has a strict local maximum at 0, u_ε must possess, for $\varepsilon > 0$ small, a local maximum $x_{N+1,\varepsilon}$ in Ω such that $|x_{N+1,\varepsilon} - y_\varepsilon| = o\left(\hat{\mu}_\varepsilon\right)$. Then $\nabla u_\varepsilon \left(x_{N+1,\varepsilon}\right) = 0$ and defining $\gamma_{N+1,\varepsilon}$, $\mu_{N+1,\varepsilon}$ with respect to this point $x_{N+1,\varepsilon}$, it is easily checked that $\left(P_2^{N+1}\right)$ and $\left(P_1^{N+1}\right)$ hold with the sequences $(x_{i,\varepsilon})_{i=1,\dots,N+1}$ thanks to (2.17), (2.18) and (2.19). This proves that property $\mathcal{P}_{N+1}$ holds and ends the proof of Step 2. ♠

Starting from Step 1, and applying by induction Step 2, using (2.3) and (2.7) to stop the process, we can easily prove the proposition except for point (f). But this point was the subject of Proposition 2 of [15] and we refer the reader to this paper for the proof. ♢

The main result of Druet [15] may be phrased as follows :

Theorem 2.1 (Druet [15]). *In the framework of Proposition 2.1, there exist moreover $M \geq 0$ and M sequences of points $(y_{i,\varepsilon})$ in Ω such that the following assertions hold after passing to a subsequence :*

a) For any $i \in \{1, \dots, M\}$ and any $j \in \{1, \dots, N\}$,

$$\frac{|y_{i,\varepsilon} - x_{j,\varepsilon}|}{\mu_{j,\varepsilon}} \to +\infty \text{ as } \varepsilon \to 0 \, .$$

b) For any $i \in \{1, \dots, M\}$,

$$u_\varepsilon (y_{i,\varepsilon}) \left(u_\varepsilon (y_{i,\varepsilon} + \nu_{i,\varepsilon} x) - u_\varepsilon (y_{i,\varepsilon}) \right) \to U(x) = -\ln \left(1 + \frac{1}{4} |x|^2 \right)$$

in $C^2_{loc} \left(\mathbb{R}^2 \setminus \mathcal{S}_i \right)$ where

$$\nu_{i,\varepsilon}^{-2} = \lambda_\varepsilon f_\varepsilon (y_{i,\varepsilon}) \, u_\varepsilon (y_{i,\varepsilon})^2 \, e^{u_\varepsilon (y_{i,\varepsilon})^2} \to +\infty \text{ as } \varepsilon \to 0$$

and

$$\mathcal{S}_i = \left\{ \lim_{\varepsilon \to 0} \frac{x_{j,\varepsilon} - y_{i,\varepsilon}}{\nu_{i,\varepsilon}}, \, j = 1, \dots, N \right\} \bigcup \left\{ \lim_{\varepsilon \to 0} \frac{y_{k,\varepsilon} - y_{i,\varepsilon}}{\nu_{i,\varepsilon}}, \, k = 1, \dots, M, \, k \neq i \right\} \, .$$

c) The Dirichlet norm of u_ε is quantified by

$$\int_\Omega |\nabla u_\varepsilon|^2 \, dx = \int_\Omega |\nabla u_0|^2 \, dx + 4\pi (N + M) + o(1) \, .$$

It is the way that the main quantification result of Druet [15] is proved. Proposition 1 in Section 3 of [15] corresponds to Proposition 2.1 above (at the exception of f)). Then concentration points are added at the end of Section 3 of [15], point f) of the above proposition is proved in Section 4 of [15] and it is proved during Sections 5 and 6 of [15] that the quantification holds with these concentration points added.

Let us comment on this result. First, it is clear that $u_0 \not\equiv 0 \Rightarrow \lambda_0 > 0$. Second, if $N = 0$, then the convergence of u_ε to u_0 is strong in $H^1_0 (\Omega)$ and in fact even holds in $\mathcal{C}^2 \left(\overline{\Omega} \right)$. The two questions left open in this work of the first author were :

1. Is it possible to have $u_0 \not\equiv 0$ and $N \geq 1$ together ?

2. Are the concentration points $(x_{i,\varepsilon})$ isolated or can there be bubbles accumulation ?

These two questions can be motivated, as explained in the introduction, by the situation in low dimensions for Yamabe type equations, as studied in [14] (see also [16]). But they are also crucial in order to understand precisely the number of solutions of equation (1.1), a question we shall address in a subsequent paper.

Let us briefly sketch the proof of Theorem 1.2. We start from the above results of [15]. We shall first give some fine pointwise estimates on the sequence (u_ε) in small (but not so small) neighbourhoods of the concentration points. This will be the subject of section 3. Then we prove Theorem 1.2 in section 4 through a serie of claims proving successively that : $M = 0$ in Theorem 2.1 above, $\lambda_0 = 0$ so that $u_0 = 0$ and, at last, the concentration points are isolated and of comparable size. All Theorem 1.2 then follows easily.

3. Local blow up analysis

In this section, we get some fine estimates on sequences of solutions of equations (2.1) in the neighbourhood of one of the concentration points $(x_{i,\varepsilon})$ of Theorem 2.1. During all this section, C denotes a constant which is independant of ε or variables x, y, ...

We let (ρ_ε) be a bounded sequence of positive real numbers (possibly converging to 0 as $\varepsilon \to 0$) and we consider a sequence of smooth positive functions (v_ε) which are solutions of

$$\Delta v_\varepsilon = \lambda_\varepsilon f_\varepsilon v_\varepsilon e^{v_\varepsilon^2} \text{ in } \mathbb{D}_0(\rho_\varepsilon) \tag{3.1}$$

where (λ_ε) is a bounded sequence of positive real numbers, (f_ε) is a sequence of smooth positive functions satisfying that there exists $C_0 > 0$ such that

$$\frac{1}{C_0} \leq f_\varepsilon(0) \leq C_0,\ |\nabla f_\varepsilon| \leq C_0 \text{ and } \left|\nabla^2 f_\varepsilon\right| \leq C_0 \text{ in } \mathbb{D}_0(\rho_\varepsilon)\ . \tag{3.2}$$

Here and in all what follows, $\mathbb{D}_x(r)$ denotes the disk of center x and radius r. We assume moreover that

$$\gamma_\varepsilon = v_\varepsilon(0) \to +\infty \text{ as } \varepsilon \to 0 \text{ and } \nabla v_\varepsilon(0) = 0\ , \tag{3.3}$$

that

$$\mu_\varepsilon^{-2} = \lambda_\varepsilon f_\varepsilon(0)\, \gamma_\varepsilon^2 e^{\gamma_\varepsilon^2} \to +\infty \text{ as } \varepsilon \to 0 \text{ with } \frac{\rho_\varepsilon}{\mu_\varepsilon} \to +\infty \text{ as } \varepsilon \to 0\ , \tag{3.4}$$

that

$$\gamma_\varepsilon\left(v_\varepsilon(\mu_\varepsilon x) - \gamma_\varepsilon\right) \to U(x) = -\ln\left(1 + \frac{1}{4}|x|^2\right) \text{ in } \mathcal{C}^2_{loc}\left(\mathbb{R}^2\right) \text{ as } \varepsilon \to 0\ , \tag{3.5}$$

that there exists $C_1 > 0$ such that

$$\lambda_\varepsilon |x|^2 v_\varepsilon^2 e^{v_\varepsilon^2} \leq C_1 \text{ in } \mathbb{D}_0(\rho_\varepsilon) \tag{3.6}$$

and that there exists $C_2 > 0$ such that

$$|x|\,|\nabla v_\varepsilon| \leq \frac{C_2}{\gamma_\varepsilon} \text{ in } \mathbb{D}_0(\rho_\varepsilon)\ . \tag{3.7}$$

The aim of this section will be to compare in a suitable disk the sequence (v_ε) with the bubble B_ε defined as the radial solution in $\mathbb{R}^2$ of

$$\Delta B_\varepsilon = \lambda_\varepsilon f_\varepsilon(0) B_\varepsilon e^{B_\varepsilon^2} \text{ with } B_\varepsilon(0) = \gamma_\varepsilon\ . \tag{3.8}$$

Thanks to the results of Appendix A, see in particular Claims 5.2 and 5.3, we know that

$$\left|B_\varepsilon(x) - \left(\gamma_\varepsilon - \frac{t_\varepsilon(x)}{\gamma_\varepsilon} - \frac{t_\varepsilon(x)}{\gamma_\varepsilon^3}\right)\right| \leq C_3 \gamma_\varepsilon^{-2} \text{ for } x \text{ s.t. } t_\varepsilon(x) \leq \gamma_\varepsilon^2 \tag{3.9}$$

and that

$$\left|\nabla B_\varepsilon(x) - \gamma_\varepsilon^{-1} \frac{2\vec{x}}{|x|^2 + 4\mu_\varepsilon^2}\right| \leq C_4 \gamma_\varepsilon^{-2} \frac{|x|}{|x|^2 + \mu_\varepsilon^2} \text{ for } x \text{ s.t. } t_\varepsilon(x) \leq \gamma_\varepsilon^2 \tag{3.10}$$

where $C_3 > 0$ and $C_4 > 0$ are some universal constants and

$$t_\varepsilon(x) = \ln\left(1 + \frac{|x|^2}{4\mu_\varepsilon^2}\right)\ . \tag{3.11}$$

We prove the following :

Proposition 3.1. *We have that :*

a) if $\overline{v}_\varepsilon(r) = \dfrac{1}{2\pi r}\displaystyle\int_{\partial\mathbb{D}_0(r)} v_\varepsilon \, d\sigma$,

$$\sup_{0\le r\le\rho_\varepsilon} |\overline{v}_\varepsilon(r) - B_\varepsilon(r)| = o\left(\gamma_\varepsilon^{-1}\right) \ .$$

As a consequence, we have that

$$t_\varepsilon(\rho_\varepsilon) \le \gamma_\varepsilon^2 - 1 + o(1) \ .$$

b) There exists $C > 0$ *such that*

$$|v_\varepsilon - B_\varepsilon| \le C\gamma_\varepsilon^{-1}$$

and

$$|\nabla(v_\varepsilon - B_\varepsilon)| \le C\gamma_\varepsilon^{-1}\rho_\varepsilon^{-1}$$

in $\mathbb{D}_0(\rho_\varepsilon)$.

c) After passing to a subsequence,

$$\gamma_\varepsilon\left(v_\varepsilon(\rho_\varepsilon\,\cdot) - B_\varepsilon(\rho_\varepsilon)\right) \to 2\ln\frac{1}{|x|} + \mathcal{H}$$

as $\varepsilon\to 0$ *in* $\mathcal{C}^1_{loc}(\mathbb{D}_0(1)\setminus\{0\})$ *where* $\mathcal{H}$ *is some harmonic function in the unit disk satisfying*

$$\mathcal{H}(0) = 0 \text{ and } \nabla\mathcal{H}(0) = -\frac{1}{2}\lim_{\varepsilon\to 0}\frac{\rho_\varepsilon\nabla f_\varepsilon(0)}{f_\varepsilon(0)} \ .$$

Proof of Proposition 3.1 - Let us first remark that we may assume without loss of generality that

$$t_\varepsilon(\rho_\varepsilon) \le \gamma_\varepsilon^2 \ . \tag{3.12}$$

Indeed, up to reduce ρ_ε, this is the case and once a) is proved, we know that $t_\varepsilon(\rho_\varepsilon) \le \gamma_\varepsilon^2 - 1 + o(1)$. This will easily permit to prove that, for the original ρ_ε, we had $t_\varepsilon(\rho_\varepsilon) \le \gamma_\varepsilon^2$ since $t_\varepsilon(r) \le \gamma_\varepsilon^2 - \frac{1}{2}$ as long as $t_\varepsilon(r) \le \gamma_\varepsilon^2$.

Fix $0 < \eta < 1$ and let

$$r_\varepsilon = \sup\left\{ r\in(0,\rho_\varepsilon) \text{ s.t. } |\overline{v}_\varepsilon(s) - B_\varepsilon(s)| \le \frac{\eta}{\gamma_\varepsilon} \text{ for all } 0\le s\le r\right\} \tag{3.13}$$

where

$$\overline{v}_\varepsilon(r) = \frac{1}{2\pi r}\int_{\partial\mathbb{D}_0(r)} v_\varepsilon \, d\sigma \ .$$

Note that we know thanks to (3.4) and (3.5) that

$$\frac{r_\varepsilon}{\mu_\varepsilon} \to +\infty \text{ as } \varepsilon\to 0 \ . \tag{3.14}$$

We have that

$$|\overline{v}_\varepsilon(r) - B_\varepsilon(r)| \le \frac{\eta}{\gamma_\varepsilon} \text{ for all } 0\le r\le r_\varepsilon \tag{3.15}$$

and that

$$|\overline{v}_\varepsilon(r_\varepsilon) - B_\varepsilon(r_\varepsilon)| = \frac{\eta}{\gamma_\varepsilon} \text{ if } r_\varepsilon < \rho_\varepsilon \ . \tag{3.16}$$

We set

$$v_\varepsilon = B_\varepsilon + w_\varepsilon \tag{3.17}$$

in $\mathbb{D}_0(\rho_\varepsilon)$. Thanks to (3.7) and (3.15), we know that

$$|w_\varepsilon| \le \frac{\eta + \pi C_2}{\gamma_\varepsilon} \text{ in } \mathbb{D}_0(r_\varepsilon) \ . \tag{3.18}$$

This clearly implies since $|B_\varepsilon| \leq \gamma_\varepsilon$ that

$$\left|v_\varepsilon^2 - B_\varepsilon^2\right| \leq 3\left(\eta + \pi C_2\right) \text{ in } \mathbb{D}_0\left(r_\varepsilon\right) . \tag{3.19}$$

Thanks to (3.1), we can write that

$$\begin{aligned}\Delta w_\varepsilon &= \lambda_\varepsilon f_\varepsilon v_\varepsilon e^{v_\varepsilon^2} - \lambda_\varepsilon f_\varepsilon(0) B_\varepsilon e^{B_\varepsilon^2} \\ &= \lambda_\varepsilon e^{B_\varepsilon^2}\left(f_\varepsilon v_\varepsilon e^{v_\varepsilon^2 - B_\varepsilon^2} - f_\varepsilon(0) B_\varepsilon\right) \\ &= \lambda_\varepsilon e^{B_\varepsilon^2}\left(f_\varepsilon w_\varepsilon e^{v_\varepsilon^2 - B_\varepsilon^2} + f_\varepsilon B_\varepsilon e^{v_\varepsilon^2 - B_\varepsilon^2} - f_\varepsilon(0) B_\varepsilon\right)\end{aligned}$$

in $\mathbb{D}_0\left(r_\varepsilon\right)$ so that, using (3.2), (3.18) and (3.19) but also (3.9), we get the existence of some $C > 0$ such that

$$|\Delta w_\varepsilon| \leq C\lambda_\varepsilon f_\varepsilon(0)\left(1 + B_\varepsilon^2\right) e^{B_\varepsilon^2} |w_\varepsilon| + C\lambda_\varepsilon |x| \left(\frac{2}{\gamma_\varepsilon} + B_\varepsilon\right) e^{B_\varepsilon^2} \text{ in } \mathbb{D}_0\left(r_\varepsilon\right) . \tag{3.20}$$

We let φ_ε be such that

$$\Delta \varphi_\varepsilon = 0 \text{ in } \mathbb{D}_0\left(r_\varepsilon\right) \text{ and } \varphi_\varepsilon = w_\varepsilon \text{ on } \partial\mathbb{D}_0\left(r_\varepsilon\right) . \tag{3.21}$$

Using (3.7) and (3.10), we know that

$$|\nabla w_\varepsilon| \leq C\gamma_\varepsilon^{-1} r_\varepsilon^{-1} \text{ on } \partial\mathbb{D}_0\left(r_\varepsilon\right)$$

for some $C > 0$ so that

$$\left\|\nabla \varphi_\varepsilon\right\|_{L^\infty\left(\mathbb{D}_0\left(r_\varepsilon\right)\right)} = O\left(\frac{1}{\gamma_\varepsilon r_\varepsilon}\right) . \tag{3.22}$$

Note also that, up to a subsequence,

$$\gamma_\varepsilon \varphi_\varepsilon\left(r_\varepsilon \cdot\right) \to \varphi_0 \text{ in } \mathcal{C}^2_{loc}\left(\mathbb{D}_0(1)\right) \text{ as } \varepsilon \to 0 \tag{3.23}$$

since $\left|\overline{\varphi}_\varepsilon\left(r_\varepsilon\right)\right| \leq \eta\gamma_\varepsilon^{-1}$ thanks to (3.15) and (3.17). It follows from standard elliptic theory thanks to (3.21).

Step 1 - *There exists $C > 0$ such that*

$$\left|\nabla\left(w_\varepsilon - \varphi_\varepsilon\right)(y)\right| \leq C\left(\left\|\nabla w_\varepsilon\right\|_{L^\infty\left(\mathbb{D}_0\left(r_\varepsilon\right)\right)} + \gamma_\varepsilon^{-1}\right)\left(\frac{\mu_\varepsilon}{\mu_\varepsilon + |y|} + \gamma_\varepsilon^{-2}\right) + C\gamma_\varepsilon^{-2} + Cr_\varepsilon^{-1}\gamma_\varepsilon^{-3}$$

for all $y \in \mathbb{D}_0\left(r_\varepsilon\right)$.

Proof of Step 1 - Let $y_\varepsilon \in \mathbb{D}_0\left(r_\varepsilon\right)$. Using the Green representation formula and (3.20), we can write that

$$\begin{aligned}\left|\nabla\left(w_\varepsilon - \varphi_\varepsilon\right)\left(y_\varepsilon\right)\right| \leq\ & C\lambda_\varepsilon f_\varepsilon(0) \int_{\mathbb{D}_0\left(r_\varepsilon\right)} \frac{1}{|x - y_\varepsilon|}\left(1 + B_\varepsilon(x)^2\right) e^{B_\varepsilon(x)^2} |w_\varepsilon(x)|\, dx \\ & + C\lambda_\varepsilon \int_{\mathbb{D}_0\left(r_\varepsilon\right)} \frac{1}{|x - y_\varepsilon|}|x| \left(\frac{2}{\gamma_\varepsilon} + B_\varepsilon(x)\right) e^{B_\varepsilon(x)^2}\, dx .\end{aligned} \tag{3.24}$$

We let in the following

$$t_{1,\varepsilon} = \frac{1}{4}\gamma_\varepsilon^2 \text{ and } t_{2,\varepsilon} = \gamma_\varepsilon^2 - \gamma_\varepsilon \tag{3.25}$$

and we let

$$\begin{aligned}\Omega_{0,\varepsilon} &= \mathbb{D}_0\left(r_\varepsilon\right) \cap \left\{t_\varepsilon(x) \leq t_{1,\varepsilon}\right\}, \\ \Omega_{1,\varepsilon} &= \mathbb{D}_0\left(r_\varepsilon\right) \cap \left\{t_{1,\varepsilon} \leq t_\varepsilon(x) \leq t_{2,\varepsilon}\right\} \text{ and} \\ \Omega_{2,\varepsilon} &= \mathbb{D}_0\left(r_\varepsilon\right) \cap \left\{t_\varepsilon(x) \geq t_{2,\varepsilon}\right\} .\end{aligned} \tag{3.26}$$

We also set, for $i = 0, 1, 2$,

$$I_{i,\varepsilon} = \lambda_\varepsilon f_\varepsilon(0) \int_{\Omega_{i,\varepsilon}} \frac{1}{|x - y_\varepsilon|} \left(1 + B_\varepsilon(x)^2\right) e^{B_\varepsilon(x)^2} |w_\varepsilon(x)| \, dx \tag{3.27}$$

and

$$J_{i,\varepsilon} = \lambda_\varepsilon \int_{\Omega_{i,\varepsilon}} \frac{1}{|x - y_\varepsilon|} |x| \left(\frac{2}{\gamma_\varepsilon} + B_\varepsilon(x)\right) e^{B_\varepsilon(x)^2} \, dx \, . \tag{3.28}$$

Case 1 - We assume first that $|y_\varepsilon| = O(\mu_\varepsilon)$. Since $w_\varepsilon(0) = 0$ and using (3.9), we can write that

$$I_{0,\varepsilon} \leq C\lambda_\varepsilon f_\varepsilon(0) \gamma_\varepsilon^2 e^{\gamma_\varepsilon^2} \|\nabla w_\varepsilon\|_{L^\infty(\Omega_{0,\varepsilon})} \int_{\Omega_{0,\varepsilon}} \frac{|x|}{|x - y_\varepsilon|} e^{\frac{t_\varepsilon(x)^2}{\gamma_\varepsilon^2} - 2t_\varepsilon(x)} \, dx \, .$$

Thanks to (3.4), we can rewrite this as

$$I_{0,\varepsilon} \leq C\mu_\varepsilon^{-2} \|\nabla w_\varepsilon\|_{L^\infty(\Omega_{0,\varepsilon})} \int_{\Omega_{0,\varepsilon}} \frac{|x|}{|x - y_\varepsilon|} e^{\frac{t_\varepsilon(x)^2}{\gamma_\varepsilon^2} - 2t_\varepsilon(x)} \, dx \, .$$

Since

$$\frac{t_\varepsilon(x)^2}{\gamma_\varepsilon^2} - 2t_\varepsilon(x) \leq -\frac{7}{4} t_\varepsilon(x) \text{ in } \Omega_{0,\varepsilon} \, ,$$

this leads to

$$\begin{aligned} I_{0,\varepsilon} &\leq C\mu_\varepsilon^{-2} \|\nabla w_\varepsilon\|_{L^\infty(\Omega_{0,\varepsilon})} \int_{\Omega_{0,\varepsilon}} \frac{|x|}{|x - y_\varepsilon|} \left(1 + \frac{|x|^2}{4\mu_\varepsilon^2}\right)^{-\frac{7}{4}} dx \\ &\leq C \|\nabla w_\varepsilon\|_{L^\infty(\Omega_{0,\varepsilon})} \int_{\mathbb{R}^2} \frac{|x|}{\left|x - \frac{y_\varepsilon}{\mu_\varepsilon}\right|} \left(1 + \frac{|x|^2}{4}\right)^{-\frac{7}{4}} dx \, . \end{aligned}$$

Since $|y_\varepsilon| = O(\mu_\varepsilon)$, we obtain by the dominated convergence theorem that

$$I_{0,\varepsilon} = O\left(\|\nabla w_\varepsilon\|_{L^\infty(\Omega_{0,\varepsilon})}\right) \, . \tag{3.29}$$

In $\Omega_{1,\varepsilon}$, we have that $|x| \leq (1 + o(1)) |x - y_\varepsilon|$ since $|y_\varepsilon| = O(\mu_\varepsilon)$ so that, we can write, as above

$$\begin{aligned} I_{1,\varepsilon} &\leq C\mu_\varepsilon^{-2} \|\nabla w_\varepsilon\|_{L^\infty(\Omega_{1,\varepsilon})} \int_{\Omega_{1,\varepsilon}} e^{\frac{t_\varepsilon(x)^2}{\gamma_\varepsilon^2} - 2t_\varepsilon(x)} dx \\ &\leq C \|\nabla w_\varepsilon\|_{L^\infty(\Omega_{1,\varepsilon})} \int_{t_{1,\varepsilon}}^{t_{2,\varepsilon}} e^{\frac{t^2}{\gamma_\varepsilon^2} - t} dt \end{aligned}$$

by the change of variables $t = \ln\left(1 + \frac{|x|^2}{4\mu_\varepsilon^2}\right)$. Since

$$\frac{t^2}{\gamma_\varepsilon^2} - t \leq -\frac{t}{\gamma_\varepsilon} \leq -\frac{1}{4}\gamma_\varepsilon$$

for $\frac{1}{4}\gamma_\varepsilon^2 = t_{1,\varepsilon} \leq t \leq t_{2,\varepsilon} = \gamma_\varepsilon^2 - \gamma_\varepsilon$, we immediately get that

$$I_{1,\varepsilon} \leq C \|\nabla w_\varepsilon\|_{L^\infty(\Omega_{1,\varepsilon})} \gamma_\varepsilon^2 e^{-\frac{1}{4}\gamma_\varepsilon} \, . \tag{3.30}$$

In $\Omega_{2,\varepsilon}$, we have that $B_\varepsilon = O(1)$ thanks to (3.9) so that, using (3.2) and (3.18) , we can write that

$$I_{2,\varepsilon} \leq C\lambda_\varepsilon \gamma_\varepsilon^{-1} \int_{\mathbb{D}_0(r_\varepsilon)} \frac{1}{|x - y_\varepsilon|} \, dx$$

so that

$$I_{2,\varepsilon} \leq C\lambda_\varepsilon r_\varepsilon \gamma_\varepsilon^{-1} \, . \tag{3.31}$$

Now we notice that $t_\varepsilon(r_\varepsilon) \leq \gamma_\varepsilon^2$ implies that

$$\frac{r_\varepsilon^2}{\mu_\varepsilon^2} \leq 4e^{\gamma_\varepsilon^2} .$$

Using (3.2) and (3.4), this gives that

$$\lambda_\varepsilon r_\varepsilon^2 \leq \frac{C}{\gamma_\varepsilon^2} . \tag{3.32}$$

Thus we get that

$$I_{2,\varepsilon} \leq C r_\varepsilon^{-1} \gamma_\varepsilon^{-3} . \tag{3.33}$$

For the second set of integrals, things are similar and easier. We write that

$$J_{0,\varepsilon} \leq C\mu_\varepsilon^{-2}\gamma_\varepsilon^{-1} \int_{\Omega_{0,\varepsilon}} \frac{|x|}{|x-y_\varepsilon|} e^{\frac{t_\varepsilon(x)^2}{\gamma_\varepsilon^2} - 2t_\varepsilon(x)}\, dx$$

so that, see above,

$$J_{0,\varepsilon} \leq C\gamma_\varepsilon^{-1} . \tag{3.34}$$

We also have that

$$J_{1,\varepsilon} \leq C\gamma_\varepsilon e^{-\frac{1}{4}\gamma_\varepsilon} \tag{3.35}$$

in the same way than above. At last, for $J_{2,\varepsilon}$, we write that

$$J_{2,\varepsilon} \leq C\lambda_\varepsilon r_\varepsilon \int_{\mathbb{D}_0(r_\varepsilon)} \frac{1}{|x-y_\varepsilon|}\, dx \leq C\lambda_\varepsilon r_\varepsilon^2 .$$

Thus we have thanks to (3.32) that

$$J_{2,\varepsilon} \leq C\gamma_\varepsilon^{-2} . \tag{3.36}$$

Summarizing, we obtain in this first case, coming back to (3.24) with (3.29), (3.30), (3.33), (3.34), (3.35) and (3.36), that

$$|\nabla(w_\varepsilon - \varphi_\varepsilon)(y_\varepsilon)| \leq C\|\nabla w_\varepsilon\|_{L^\infty(\mathbb{D}_0(r_\varepsilon))} + C\gamma_\varepsilon^{-1} + Cr_\varepsilon^{-1}\gamma_\varepsilon^{-3} . \tag{3.37}$$

Case 2 - We assume now that $\frac{|y_\varepsilon|}{\mu_\varepsilon} \to +\infty$ as $\varepsilon \to 0$.

We follow the lines of the first case to estimate most of the integrals. Thus we only emphasize on the changes. First, we write that

$$\begin{aligned} I_{0,\varepsilon} &\leq C\mu_\varepsilon^{-2}\|\nabla w_\varepsilon\|_{L^\infty(\Omega_{0,\varepsilon})} \int_{\Omega_{0,\varepsilon}} \frac{|x|}{|x-y_\varepsilon|}\left(1+\frac{|x|^2}{4\mu_\varepsilon^2}\right)^{-\frac{7}{4}} dx \\ &\leq C\|\nabla w_\varepsilon\|_{L^\infty(\Omega_{0,\varepsilon})} \int_{\mathbb{R}^2} \frac{|x|}{|x-\mu_\varepsilon^{-1}y_\varepsilon|}\left(1+\frac{|x|^2}{4}\right)^{-\frac{7}{4}} dx . \end{aligned}$$

Now we can write that

$$\begin{aligned} &\int_{\mathbb{R}^2} \frac{|x|}{|x-\mu_\varepsilon^{-1}y_\varepsilon|}\left(1+\frac{|x|^2}{4}\right)^{-\frac{7}{4}} dx \\ &= \left(\frac{|y_\varepsilon|}{\mu_\varepsilon}\right)^{-\frac{3}{2}} \int_{\mathbb{R}^2} \frac{|x|}{|x-|y_\varepsilon|^{-1}y_\varepsilon|}\left(\frac{\mu_\varepsilon^2}{|y_\varepsilon|^2}+\frac{|x|^2}{4}\right)^{-\frac{7}{4}} dx \\ &\leq C\left(\frac{|y_\varepsilon|}{\mu_\varepsilon}\right)^{-\frac{3}{2}} + 2\left(\frac{|y_\varepsilon|}{\mu_\varepsilon}\right)^{-\frac{3}{2}} \int_{\mathbb{D}_0\left(\frac{1}{2}\right)} |x|\left(\frac{\mu_\varepsilon^2}{|y_\varepsilon|^2}+\frac{|x|^2}{4}\right)^{-\frac{7}{4}} dx \\ &\leq C\left(\frac{|y_\varepsilon|}{\mu_\varepsilon}\right)^{-\frac{3}{2}} + 2\frac{\mu_\varepsilon}{|y_\varepsilon|}\int_{\mathbb{R}^2} |x|\left(1+\frac{|x|^2}{4}\right)^{-\frac{7}{4}} dx \end{aligned}$$

so that

$$I_{0,\varepsilon} \leq C \left\| \nabla w_\varepsilon \right\|_{L^\infty(\Omega_{0,\varepsilon})} \frac{\mu_\varepsilon}{|y_\varepsilon|} \, . \tag{3.38}$$

Let us write once again that

$$I_{1,\varepsilon} \leq C\mu_\varepsilon^{-2} \left\| \nabla w_\varepsilon \right\|_{L^\infty(\Omega_{1,\varepsilon})} \int_{\Omega_{1,\varepsilon}} \frac{|x|}{|x-y_\varepsilon|} e^{\frac{t_\varepsilon(x)^2}{\gamma_\varepsilon^2} - 2t_\varepsilon(x)} \, dx \, .$$

Let us split this integral into two parts. First,

$$\begin{aligned} \mu_\varepsilon^{-2} \int_{\Omega_{1,\varepsilon} \setminus \mathbb{D}_{y_\varepsilon}\left(\frac{1}{2}|y_\varepsilon|\right)} \frac{|x|}{|x-y_\varepsilon|} e^{\frac{t_\varepsilon(x)^2}{\gamma_\varepsilon^2} - 2t_\varepsilon(x)} \, dx &\leq 3\mu_\varepsilon^{-2} \int_{\Omega_{1,\varepsilon}} e^{\frac{t_\varepsilon(x)^2}{\gamma_\varepsilon^2} - 2t_\varepsilon(x)} \, dx \\ &\leq C \int_{t_{1,\varepsilon}}^{t_{2,\varepsilon}} e^{\frac{t^2}{\gamma_\varepsilon^2} - t} \, dt \\ &\leq C\gamma_\varepsilon^2 e^{-\frac{1}{4}\gamma_\varepsilon} \end{aligned}$$

as in Case 1. Second,

$$\mu_\varepsilon^{-2} \int_{\Omega_{1,\varepsilon} \cap \mathbb{D}_{y_\varepsilon}\left(\frac{1}{2}|y_\varepsilon|\right)} \frac{|x|}{|x-y_\varepsilon|} e^{\frac{t_\varepsilon(x)^2}{\gamma_\varepsilon^2} - 2t_\varepsilon(x)} \, dx \leq \frac{3}{2}\mu_\varepsilon^{-2} \left|y_\varepsilon\right| e^{\frac{s_\varepsilon^2}{\gamma_\varepsilon^2} - 2s_\varepsilon} \int_{\mathbb{D}_{y_\varepsilon}\left(\frac{1}{2}|y_\varepsilon|\right)} \frac{1}{|x-y_\varepsilon|} \, dx$$

where

$$s_\varepsilon = t_\varepsilon\left(\frac{y_\varepsilon}{2}\right) \, .$$

Thus we have that

$$\mu_\varepsilon^{-2} \int_{\Omega_{1,\varepsilon} \cap \mathbb{D}_{y_\varepsilon}\left(\frac{1}{2}|y_\varepsilon|\right)} \frac{|x|}{|x-y_\varepsilon|} e^{\frac{t_\varepsilon(x)^2}{\gamma_\varepsilon^2} - 2t_\varepsilon(x)} \, dx \leq C \frac{\left|y_\varepsilon\right|^2}{\mu_\varepsilon^2} e^{\frac{s_\varepsilon^2}{\gamma_\varepsilon^2} - 2s_\varepsilon}$$

Note that $\Omega_{1,\varepsilon} \cap \mathbb{D}_{y_\varepsilon}\left(\dfrac{1}{2}\left|y_\varepsilon\right|\right) = \emptyset$ if

$$t_\varepsilon\left(\frac{3}{2}\left|y_\varepsilon\right|\right) = \ln\left(1 + \frac{9\left|y_\varepsilon\right|^2}{16\mu_\varepsilon^2}\right) \leq t_{1,\varepsilon} = \frac{1}{4}\gamma_\varepsilon^2$$

so that we may assume that

$$\ln\left(1 + \frac{9\left|y_\varepsilon\right|^2}{16\mu_\varepsilon^2}\right) > \frac{1}{4}\gamma_\varepsilon^2 \, .$$

Thus

$$s_\varepsilon = \ln\left(1 + \frac{\left|y_\varepsilon\right|^2}{16\mu_\varepsilon^2}\right) \geq \frac{1}{4}\gamma_\varepsilon^2 - \ln 9 \, .$$

It is also clear that if $\Omega_{1,\varepsilon} \cap \mathbb{D}_{y_\varepsilon}\left(\dfrac{1}{2}\left|y_\varepsilon\right|\right) \neq \emptyset$, $s_\varepsilon \leq t_{2,\varepsilon} = \gamma_\varepsilon^2 - \gamma_\varepsilon$. Thus we have that

$$\frac{s_\varepsilon^2}{\gamma_\varepsilon^2} - s_\varepsilon \leq -\frac{1}{4}\gamma_\varepsilon + O(1) \, .$$

We deduce that, if not zero,

$$\begin{aligned}\mu_\varepsilon^{-2}\int_{\Omega_{1,\varepsilon}\cap\mathbb{D}_{y_\varepsilon}\left(\frac{1}{2}|y_\varepsilon|\right)}\frac{|x|}{|x-y_\varepsilon|}e^{\frac{t_\varepsilon(x)^2}{\gamma_\varepsilon^2}-2t_\varepsilon(x)}\,dx &\leq C\frac{|y_\varepsilon|^2}{\mu_\varepsilon^2}e^{-\frac{1}{4}\gamma_\varepsilon}e^{-s_\varepsilon}\\ &\leq C\frac{|y_\varepsilon|^2}{\mu_\varepsilon^2}e^{-\frac{1}{4}\gamma_\varepsilon}\left(1+\frac{|y_\varepsilon|^2}{16\mu_\varepsilon^2}\right)^{-1}\\ &\leq Ce^{-\frac{1}{4}\gamma_\varepsilon}\ .\end{aligned}$$

Thus we arrive to

$$I_{1,\varepsilon}\leq C\gamma_\varepsilon^2e^{-\frac{1}{4}\gamma_\varepsilon}\left\|\nabla w_\varepsilon\right\|_{L^\infty(\Omega_{1,\varepsilon})}\ . \tag{3.39}$$

At last, for $I_{2,\varepsilon}$, we have nothing to change to get that

$$I_{2,\varepsilon}\leq Cr_\varepsilon^{-1}\gamma_\varepsilon^{-3}\ . \tag{3.40}$$

For $J_{0,\varepsilon}$, $J_{1,\varepsilon}$ and $J_{2,\varepsilon}$, we proceed as above or as in Case 1 to get that

$$J_{0,\varepsilon}\leq C\gamma_\varepsilon^{-1}\frac{\mu_\varepsilon}{|y_\varepsilon|},\ J_{1,\varepsilon}\leq C\gamma_\varepsilon e^{-\frac{1}{4}\gamma_\varepsilon}\text{ and }J_{2,\varepsilon}\leq\frac{C}{\gamma_\varepsilon^2}\ .$$

Thus, in this second case, we obtain coming back to (3.24) with (3.38), (3.39), (3.40) and these last estimates that

$$\left|\nabla\left(w_\varepsilon-\varphi_\varepsilon\right)(y_\varepsilon)\right|\leq C\left(\left\|\nabla w_\varepsilon\right\|_{L^\infty(\mathbb{D}_0(r_\varepsilon))}+\gamma_\varepsilon^{-1}\right)\left(\frac{\mu_\varepsilon}{|y_\varepsilon|}+\gamma_\varepsilon^2e^{-\frac{1}{4}\gamma_\varepsilon}\right)+C\gamma_\varepsilon^{-2}+Cr_\varepsilon^{-1}\gamma_\varepsilon^{-3}\ . \tag{3.41}$$

The study of these two cases clearly permits to conclude Step 1. ♠

Step 2 - *We have that*

$$\left\|\nabla\left(w_\varepsilon-\varphi_\varepsilon\right)\right\|_{L^\infty(\mathbb{D}_0(r_\varepsilon))}=o\left(\gamma_\varepsilon^{-1}r_\varepsilon^{-1}\right)+O\left(\gamma_\varepsilon^{-1}\right)$$

and that

$$\left\|w_\varepsilon-\varphi_\varepsilon\right\|_{L^\infty(\mathbb{D}_0(r_\varepsilon))}=o\left(\gamma_\varepsilon^{-1}\right)\ .$$

Moreover, if $r_\varepsilon\not\to 0$ *as* $\varepsilon\to 0$, *we have that*

$$\lim_{\varepsilon\to 0}\frac{\nabla f_\varepsilon(0)}{f_\varepsilon(0)}=-2\left(\lim_{\varepsilon\to 0}\frac{1}{r_\varepsilon}\right)\nabla\varphi_0(0)\ .$$

Proof of Step 2 - Let $y_\varepsilon\in\mathbb{D}_0\left(r_\varepsilon\right)$ be such that

$$\left|\nabla\left(w_\varepsilon-\varphi_\varepsilon\right)(y_\varepsilon)\right|=\left\|\nabla\left(w_\varepsilon-\varphi_\varepsilon\right)\right\|_{L^\infty(\mathbb{D}_0(r_\varepsilon))} \tag{3.42}$$

and let us assume that

$$\alpha_\varepsilon=\left|\nabla\left(w_\varepsilon-\varphi_\varepsilon\right)(y_\varepsilon)\right|\geq\frac{\delta}{r_\varepsilon\gamma_\varepsilon}+\frac{1}{\delta\gamma_\varepsilon} \tag{3.43}$$

for some $\delta>0$. Thanks to (3.22), we have that

$$\left\|\nabla w_\varepsilon\right\|_{L^\infty(\mathbb{D}_0(r_\varepsilon))}\leq\alpha_\varepsilon+Cr_\varepsilon^{-1}\gamma_\varepsilon^{-1}\leq\alpha_\varepsilon\left(1+\frac{C}{\delta}\right)\ . \tag{3.44}$$

Applying Step 1 to this sequence (y_ε), we get thanks to (3.42), (3.43) and (3.44) that

$$\left(\frac{1}{\delta}+\frac{\delta}{r_\varepsilon}\right)\gamma_\varepsilon^{-1}\leq\alpha_\varepsilon=\left|\nabla\left(w_\varepsilon-\varphi_\varepsilon\right)(y_\varepsilon)\right|\leq C_\delta\alpha_\varepsilon\left(\frac{\mu_\varepsilon}{\mu_\varepsilon+|y_\varepsilon|}+\gamma_\varepsilon^{-2}\right)+C\gamma_\varepsilon^{-2}+Cr_\varepsilon^{-1}\gamma_\varepsilon^{-3}\ .$$

This proves that

$$\frac{y_\varepsilon}{\mu_\varepsilon}\to y_0\in\mathbb{R}^2\text{ as }\varepsilon\to 0 \tag{3.45}$$

after passing to a subsequence and, thanks to Step 1 and (3.43), that

$$|\nabla(\tilde{w}_\varepsilon - \tilde{\varphi}_\varepsilon)(x)| \leq \frac{C_\delta}{1+|x|} + o(1) \text{ for all } x \in \mathbb{R}^2 \tag{3.46}$$

where C_δ depends only on δ and

$$\tilde{w}_\varepsilon(x) = \frac{1}{\mu_\varepsilon \alpha_\varepsilon} w_\varepsilon(\mu_\varepsilon x)\,,\ \tilde{\varphi}_\varepsilon(x) = \frac{1}{\mu_\varepsilon \alpha_\varepsilon}\varphi_\varepsilon(\mu_\varepsilon x)\ . \tag{3.47}$$

We know that

$$\tilde{w}_\varepsilon(0) = 0,\ \nabla\tilde{w}_\varepsilon(0) = 0 \text{ and } \left|\nabla(\tilde{w}_\varepsilon - \tilde{\varphi}_\varepsilon)\left(\frac{y_\varepsilon}{\mu_\varepsilon}\right)\right| = 1\ . \tag{3.48}$$

We also know thanks to (3.23) and (3.43) that, after passing to a subsequence,

$$\nabla\tilde{\varphi}_\varepsilon(x) \to \left(\lim_{\varepsilon\to 0}\frac{1}{\gamma_\varepsilon r_\varepsilon \alpha_\varepsilon}\right)\nabla\varphi_0(0) = \vec{A} \text{ in } \mathcal{C}^1_{loc}\left(\mathbb{R}^2\right) \text{ as } \varepsilon \to 0\ . \tag{3.49}$$

Using (3.20), we can write that

$$|\Delta\tilde{w}_\varepsilon| \leq C\lambda_\varepsilon\mu_\varepsilon^2 f_\varepsilon(0)\left(1 + B_\varepsilon(\mu_\varepsilon x)^2\right)e^{B_\varepsilon(\mu_\varepsilon x)^2}|\tilde{w}_\varepsilon| + C\lambda_\varepsilon\alpha_\varepsilon^{-1}\mu_\varepsilon^2|x|\left(\frac{2}{\gamma_\varepsilon} + B_\varepsilon(\mu_\varepsilon x)\right)e^{B_\varepsilon(\mu_\varepsilon x)^2}\ .$$

Noting thanks to (3.46), (3.48) and (3.49) that

$$|\tilde{w}_\varepsilon(x)| \leq C_\delta \ln(1+|x|) + \left|\vec{A}\right||x| + o(|x|)$$

and is thus uniformly bounded on any compact subset of $\mathbb{R}^2$, we easily deduce from the above estimate together with the definition (3.4) of μ_ε and (3.43) that $(\Delta\tilde{w}_\varepsilon)$ is uniformly bounded in any compact subset of $\mathbb{R}^2$. Thus, by standard elliptic theory, we have that, after passing to a subsequence,

$$\tilde{w}_\varepsilon \to w_0 \text{ in } C^{1,\eta}_{loc}\left(\mathbb{R}^2\right) \text{ as } \varepsilon \to 0\ . \tag{3.50}$$

Moreover, we have thanks to (3.45), (3.46), (3.48) and (3.50) that

$$w_0(0) = 0,\ \nabla w_0(0) = 0,\ \left|\nabla w_0(y_0) - \vec{A}\right| = 1 \text{ and } \left|\nabla w_0(x) - \vec{A}\right| \leq \frac{C_\delta}{1+|x|} \text{ in } \mathbb{R}^2\ . \tag{3.51}$$

Thus $w_0 \not\equiv 0$. Since we know that $\gamma_\varepsilon w_\varepsilon(\mu_\varepsilon x) \to 0$ in $\mathcal{C}^1_{loc}\left(\mathbb{R}^2\right)$ as $\varepsilon \to 0$ thanks to (3.5), we deduce that

$$\gamma_\varepsilon\mu_\varepsilon\alpha_\varepsilon \to 0 \text{ as } \varepsilon \to 0\ . \tag{3.52}$$

Thanks to (3.1), (3.4), (3.8), (3.17) and (3.47), we can write that

$$\begin{aligned}
\Delta\tilde{w}_\varepsilon(x) &= \frac{1}{\alpha_\varepsilon}\mu_\varepsilon\lambda_\varepsilon\Big(f_\varepsilon(\mu_\varepsilon x)\left(B_\varepsilon(\mu_\varepsilon x) + w_\varepsilon(\mu_\varepsilon x)\right)e^{(B_\varepsilon(\mu_\varepsilon x)+w_\varepsilon(\mu_\varepsilon x))^2} \\
&\quad - f_\varepsilon(0)B_\varepsilon(\mu_\varepsilon x)\,e^{B_\varepsilon(\mu_\varepsilon x)^2}\Big)\ . \\
&= \frac{B_\varepsilon(\mu_\varepsilon x)}{\gamma_\varepsilon}e^{B_\varepsilon(\mu_\varepsilon x)^2-\gamma_\varepsilon^2}\frac{1}{\alpha_\varepsilon\mu_\varepsilon\gamma_\varepsilon}\left(\frac{f_\varepsilon(\mu_\varepsilon x)}{f_\varepsilon(0)}e^{2B_\varepsilon(\mu_\varepsilon x)w_\varepsilon(\mu_\varepsilon x)+w_\varepsilon(\mu_\varepsilon x)^2} - 1\right) \\
&\quad + \gamma_\varepsilon^{-2}\frac{f_\varepsilon(\mu_\varepsilon x)}{f_\varepsilon(0)}\tilde{w}_\varepsilon(x)e^{B_\varepsilon(\mu_\varepsilon x)^2-\gamma_\varepsilon^2+2B_\varepsilon(\mu_\varepsilon x)w_\varepsilon(\mu_\varepsilon x)+w_\varepsilon(\mu_\varepsilon x)^2}\ .
\end{aligned}$$

Let us write now that

$$\gamma_\varepsilon^{-1}B_\varepsilon(\mu_\varepsilon x) \to 1 \text{ in } \mathcal{C}^0_{loc}\left(\mathbb{R}^2\right) \text{ as } \varepsilon \to 0\ ,$$

that

$$B_\varepsilon(\mu_\varepsilon x)^2 - \gamma_\varepsilon^2 \to 2U(x) \text{ in } \mathcal{C}^0_{loc}\left(\mathbb{R}^2\right) \text{ as } \varepsilon \to 0$$

where

$$U(x) = -\ln\left(1+\frac{|x|^2}{4}\right)$$

thanks to (3.9) and (3.11). We can also write that

$$\frac{f_\varepsilon(\mu_\varepsilon x)}{f_\varepsilon(0)} = 1 + f_\varepsilon(0)^{-1}\mu_\varepsilon x^i \partial_i f_\varepsilon(0) + O\left(\mu_\varepsilon^2|x|^2\right)$$

thanks to (3.2) and that

$$2B_\varepsilon(\mu_\varepsilon x)\, w_\varepsilon(\mu_\varepsilon x) + w_\varepsilon(\mu_\varepsilon x)^2 = 2\mu_\varepsilon\alpha_\varepsilon\gamma_\varepsilon\left(w_0 + o(1)\right) = o(1)$$

thanks to (3.50) and (3.52). Thus we can write that

$$\begin{aligned}&\frac{f_\varepsilon(\mu_\varepsilon x)}{f_\varepsilon(0)} e^{2B_\varepsilon(\mu_\varepsilon x)w_\varepsilon(\mu_\varepsilon x)+w_\varepsilon(\mu_\varepsilon x)^2} - 1\\ &\quad = 2\mu_\varepsilon\alpha_\varepsilon\gamma_\varepsilon w_0 + \mu_\varepsilon f_\varepsilon(0)^{-1}x^i\partial_i f_\varepsilon(0) + o\left(\mu_\varepsilon\alpha_\varepsilon\gamma_\varepsilon\right)\end{aligned}$$

Thus we obtain that

$$\Delta\tilde{w}_\varepsilon(x) = e^{2U(x)}\left(2w_0(x) + \frac{1}{\alpha_\varepsilon\gamma_\varepsilon} f_\varepsilon(0)^{-1}x^i\partial_i f_\varepsilon(0)\right) + o(1)\ .$$

Thanks to (3.43), we know that, after passing to a subsequence,

$$\frac{1}{\alpha_\varepsilon\gamma_\varepsilon}\frac{\partial_i f_\varepsilon(0)}{f_\varepsilon(0)} \to X_i \text{ as } \varepsilon\to 0\ . \tag{3.53}$$

Note that we have, again thanks to (3.43), that

$$\vec{X} = 0 \text{ if } r_\varepsilon \to 0 \text{ as } \varepsilon\to 0\ . \tag{3.54}$$

Then we can write that

$$\Delta\tilde{w}_\varepsilon(x) = e^{2U(x)}\left(2w_0(x) + X_i x^i\right) + o(1)$$

so that

$$\Delta w_0 = e^{2U}\left(2w_0 + X_i x^i\right) \text{ in } \mathbb{R}^2\ . \tag{3.55}$$

Now, thanks to [7], lemma 2.3 or [21], lemma C.1, we know that the only solution of this equation satisfying (3.51) is

$$w_0(x) = \frac{|x|^2}{4+|x|^2}A^i x_i \tag{3.56}$$

and, moreover, we must have

$$\vec{A} = -\frac{1}{2}\vec{X}\ . \tag{3.57}$$

Since $w_0 \not\equiv 0$, we must have $\vec{A}\neq 0$ and thus $\vec{X}\neq 0$.

This permits to prove the step. Indeed, if $r_\varepsilon\to 0$, then we have that $\vec{X}=0$ by (3.54), which is a contradiction. Thus, if $r_\varepsilon\to 0$, we get that (3.43) is impossible so that $\alpha_\varepsilon = o\left(\gamma_\varepsilon^{-1}r_\varepsilon^{-1}\right)$ in this case. This proves the first estimate of the step in the case $r_\varepsilon\to 0$ as $\varepsilon\to 0$. If $r_\varepsilon \not\to 0$ as $\varepsilon\to 0$, we know thanks to the fact that $\vec{X}\neq 0$ and to (3.53) that $\alpha_\varepsilon = O\left(\gamma_\varepsilon^{-1}\right)$ if (3.43) holds and if it does not hold, we again have that $\alpha_\varepsilon = O\left(\gamma_\varepsilon^{-1}\right)$. Thus we also have that the first estimate of the step holds if $r_\varepsilon \not\to 0$ as $\varepsilon\to 0$. Moreover, in this second case, we know that

$$\lim_{\varepsilon\to 0}\frac{\partial_i f_\varepsilon(0)}{f_\varepsilon(0)} = -2\left(\lim_{\varepsilon\to 0}\frac{1}{r_\varepsilon}\right)\nabla\varphi_0(0)$$

thanks to (3.49), (3.53) and (3.57). This proves the last part of the step.

It remains to notice that the second estimate of the step is a simple consequence of the first. Indeed, coming back to the estimate of Step 1 with the estimate on the gradient just proved, we have that

$$|\nabla (w_\varepsilon - \varphi_\varepsilon)(y)| \leq C\gamma_\varepsilon^{-1}\left(1+O\left(r_\varepsilon^{-1}\right)\right)\left(\frac{\mu_\varepsilon}{\mu_\varepsilon+|y|}+\gamma_\varepsilon^{-2}\right)+C\gamma_\varepsilon^{-2}+Cr_\varepsilon^{-1}\gamma_\varepsilon^{-3}$$

Since $w_\varepsilon - \varphi_\varepsilon = 0$ on $\partial\mathbb{D}_0(r_\varepsilon)$, this leads after integration to

$$|w_\varepsilon(y)-\varphi_\varepsilon(y)| \leq C\gamma_\varepsilon^{-1}\left(1+O\left(r_\varepsilon^{-1}\right)\right)\mu_\varepsilon \ln\frac{\mu_\varepsilon+r_\varepsilon}{\mu_\varepsilon+|y|}+O\left(\gamma_\varepsilon^{-2}\right)$$

for all $y \in \mathbb{D}_0(r_\varepsilon)$. This leads to

$$\|w_\varepsilon-\varphi_\varepsilon\|_{L^\infty(\mathbb{D}_0(r_\varepsilon))} = O\left(\gamma_\varepsilon^{-1}\mu_\varepsilon\ln\left(1+\frac{r_\varepsilon}{\mu_\varepsilon}\right)\right)+O\left(\gamma_\varepsilon^{-1}\frac{\mu_\varepsilon}{r_\varepsilon}\ln\left(1+\frac{r_\varepsilon}{\mu_\varepsilon}\right)\right)+O\left(\gamma_\varepsilon^{-2}\right)=o\left(\gamma_\varepsilon^{-1}\right)$$

thanks to (3.14).

This ends the proof of Step 2. ♠

We are now in position to prove Proposition 3.1. First, since $w_\varepsilon(0)=0$ and $\nabla w_\varepsilon(0)=0$, we get with Step 2 that

$$\varphi_\varepsilon(0) = o\left(\gamma_\varepsilon^{-1}\right) \tag{3.58}$$

and that

$$|\nabla\varphi_\varepsilon(0)| = o\left(\gamma_\varepsilon^{-1}r_\varepsilon^{-1}\right)+O\left(\gamma_\varepsilon^{-1}\right)\ . \tag{3.59}$$

Since φ_ε is harmonic, (3.58) gives that

$$\gamma_\varepsilon\varphi_\varepsilon(0) = \frac{\gamma_\varepsilon}{2\pi r_\varepsilon}\int_{\partial\mathbb{D}_0(r_\varepsilon)}\varphi_\varepsilon\, d\sigma \to 0 \text{ as } \varepsilon\to 0\ .$$

Since $v_\varepsilon - B_\varepsilon = w_\varepsilon = \varphi_\varepsilon$ on $\partial\mathbb{D}_0(r_\varepsilon)$, this leads to

$$\gamma_\varepsilon\left|\overline{v}_\varepsilon(r_\varepsilon)-B_\varepsilon(r_\varepsilon)\right| \to 0 \text{ as } \varepsilon\to 0\ ,$$

which is impossible if $r_\varepsilon < \rho_\varepsilon$ thanks to (3.16). Thus we have proved that

$$r_\varepsilon = \rho_\varepsilon\ , \tag{3.60}$$

for any choice of $\eta\in(0,1)$. This proves the first part of a). The second part of a) is then just a consequence of (3.9). Indeed, $\gamma_\varepsilon\left|\overline{v}_\varepsilon(\rho_\varepsilon)-B_\varepsilon(\rho_\varepsilon)\right| = o(1)$ implies that $\gamma_\varepsilon B_\varepsilon(\rho_\varepsilon) \geq \gamma_\varepsilon\overline{v}_\varepsilon(\rho_\varepsilon)+o(1)$. And (3.9) gives that

$$\gamma_\varepsilon^2 - t_\varepsilon(\rho_\varepsilon) - \gamma_\varepsilon^{-2}t_\varepsilon(\rho_\varepsilon) \geq \gamma_\varepsilon\overline{v}_\varepsilon(\rho_\varepsilon)+o(1)\ .$$

which leads to $t_\varepsilon(\rho_\varepsilon)\leq\gamma_\varepsilon^2-1+o(1)$ since $\overline{v}_\varepsilon(\rho_\varepsilon)\geq 0$. Point b) of the proposition is a consequence of Step 2 together with (3.22). It remains to prove c). Let us write that

$$\gamma_\varepsilon\left(v_\varepsilon(\rho_\varepsilon x)-B_\varepsilon(\rho_\varepsilon)\right) = \gamma_\varepsilon w_\varepsilon(\rho_\varepsilon x)+\gamma_\varepsilon\left(B_\varepsilon(\rho_\varepsilon x)-B_\varepsilon(\rho_\varepsilon)\right)\ .$$

We write that

$$\gamma_\varepsilon\left(B_\varepsilon(\rho_\varepsilon x)-B_\varepsilon(\rho_\varepsilon)\right) \to 2\ln\frac{1}{|x|} \text{ in } \mathcal{C}^1_{loc}\left(\mathbb{D}_0(1)\setminus\{0\}\right) \text{ as } \varepsilon\to 0$$

thanks to (3.9) and (3.10). Moreover, thanks to Step 2, we know that

$$\gamma_\varepsilon\left\|w_\varepsilon(\rho_\varepsilon x)-\varphi_\varepsilon(\rho_\varepsilon x)\right\|_{L^\infty(\mathbb{D}_0(1))} = o(1)$$

and, combining Steps 1 and 2, that

$$\begin{aligned}&\gamma_\varepsilon\rho_\varepsilon\left|\nabla w_\varepsilon(\rho_\varepsilon x)-\nabla\varphi_\varepsilon(\rho_\varepsilon x)\right|\\ &\quad\leq C\left(\frac{\mu_\varepsilon}{\mu_\varepsilon+\rho_\varepsilon|x|}+\gamma_\varepsilon^{-2}\right)+C\rho_\varepsilon\gamma_\varepsilon^{-1}+C\gamma_\varepsilon^{-2}\end{aligned}$$

in $\mathbb{D}_0(1)$. Thus we have that

$$\gamma_\varepsilon w_\varepsilon\left(\rho_\varepsilon x\right) \to \varphi_0 \text{ in } \mathcal{C}^1_{loc}\left(\mathbb{D}_0(1)\setminus\{0\}\right) \text{ as } \varepsilon \to 0$$

thanks to (3.23). We thus have obtained that

$$\gamma_\varepsilon\left(v_\varepsilon\left(\rho_\varepsilon\,\cdot\right) - B_\varepsilon\left(\rho_\varepsilon\right)\right) \to 2\ln\frac{1}{|x|} + \varphi_0$$

in $\mathcal{C}^1_{loc}\left(\mathbb{D}_0(1)\setminus\{0\}\right)$ as $\varepsilon \to 0$. Moreover, we have thanks to Step 2 that

$$\varphi_0(0) = 0 \text{ and } \nabla\varphi_0(0) = -\frac{1}{2}\lim_{\varepsilon\to 0}\frac{\rho_\varepsilon \nabla f_\varepsilon(0)}{f_\varepsilon(0)}\ .$$

This ends the proof of the proposition. ♢

4. Proof of Theorem 1.2

Let (u_ε) be a sequence of smooth positive solutions of

$$\Delta u_\varepsilon = \lambda_\varepsilon f_\varepsilon u_\varepsilon e^{u_\varepsilon^2} \text{ in } \Omega,\ u_\varepsilon = 0 \text{ on } \partial\Omega \tag{4.1}$$

for some sequence (λ_ε) of positive real numbers and some sequence (f_ε) of functions in $\mathcal{C}^1\left(\overline{\Omega}\right)$ which satisfies (1.5). We assume that there exists $C > 0$ such that

$$\int_\Omega |\nabla u_\varepsilon|^2\, dx \le C\ . \tag{4.2}$$

We consider the concentration points $(x_{i,\varepsilon})_{i=1,\dots,N}$ given by Proposition 2.1 together with the $\gamma_{i,\varepsilon}$'s and $\mu_{i,\varepsilon}$'s. For any $i \in \{1,\dots,N\}$, we let

$$r_{i,\varepsilon} = \frac{1}{2}\min\left\{\min_{j\in\{1,\dots,N\}, j\neq i}|x_{i,\varepsilon} - x_{j,\varepsilon}|\,,\, d\left(x_{i,\varepsilon},\partial\Omega\right)\right\}\ . \tag{4.3}$$

Note that we have

$$\lambda_\varepsilon |x - x_{i,\varepsilon}|^2 u_\varepsilon(x)^2 e^{u_\varepsilon(x)^2} \le C_1 \text{ in } \mathbb{D}_{x_{i,\varepsilon}}\left(r_{i,\varepsilon}\right) \tag{4.4}$$

and

$$|x - x_{i,\varepsilon}|\, u_\varepsilon(x)\, |\nabla u_\varepsilon(x)| \le C_2 \text{ in } \mathbb{D}_{x_{i,\varepsilon}}\left(r_{i,\varepsilon}\right) \tag{4.5}$$

thanks to assertions e) and f) of Proposition 2.1.

We let, for $i \in \{1,\dots,N\}$, $B_{i,\varepsilon}$ be the radial solution, studied in Appendix A, of

$$\Delta B_{i,\varepsilon} = \lambda_\varepsilon f_\varepsilon\left(x_{i,\varepsilon}\right) B_{i,\varepsilon} e^{B_{i,\varepsilon}^2} \text{ and } B_{i,\varepsilon}(0) = \gamma_{i,\varepsilon}$$

and we shall write, by an obvious and not misleading abuse of notation,

$$B_{i,\varepsilon}(x) = B_{i,\varepsilon}\left(|x - x_{i,\varepsilon}|\right)\ . \tag{4.6}$$

We let also

$$t_{i,\varepsilon}(r) = \ln\left(1 + \frac{r^2}{4\mu_{i,\varepsilon}^2}\right) \text{ and } t_{i,\varepsilon}(x) = t_{i,\varepsilon}\left(|x - x_{i,\varepsilon}|\right)\ . \tag{4.7}$$

At last, we define for $i = 1,\dots,N$

$$d_{i,\varepsilon} = d\left(x_{i,\varepsilon},\partial\Omega\right)\ . \tag{4.8}$$

Let us first state a claim which explains how we shall use the results of Section 3 for the multi-bumps analysis :

Claim 4.1. *Assume that* (u_ε) *satisfies equation* (4.1) *with* (f_ε) *satisfying* (1.5). *Assume also that* (4.2) *holds so that we have concentration points* $(x_{i,\varepsilon})$ *satisfying* (4.4) *and* (4.5). *Let* $0 \leq r_\varepsilon \leq r_{i,\varepsilon}$ *be such that there exists* $C_3 > 0$ *such that*

$$|x - x_{i,\varepsilon}| \, |\nabla u_\varepsilon(x)| \leq C_3 \gamma_{i,\varepsilon}^{-1} \text{ in } \mathbb{D}_{x_{i,\varepsilon}}(r_\varepsilon) \ .$$

Then we have that :

a) $t_{i,\varepsilon}(r_\varepsilon) \leq \gamma_{i,\varepsilon}^2 - 1 + o(1)$ *and*

$$\frac{1}{2\pi r_\varepsilon} \int_{\partial \mathbb{D}_{x_{i,\varepsilon}}(r_\varepsilon)} u_\varepsilon \, d\sigma = B_{i,\varepsilon}(r_\varepsilon) + o\left(\gamma_{i,\varepsilon}^{-1}\right) \ .$$

b) There exists $C > 0$ *such that*

$$|u_\varepsilon - B_{i,\varepsilon}| \leq C \gamma_{i,\varepsilon}^{-1}$$

and

$$|\nabla (u_\varepsilon - B_{i,\varepsilon})| \leq C \gamma_{i,\varepsilon}^{-1} r_\varepsilon^{-1}$$

in $\mathbb{D}_{x_{i,\varepsilon}}(r_\varepsilon)$.

c) If $r_\varepsilon = r_{i,\varepsilon}$, *after passing to a subsequence,*

$$\gamma_{i,\varepsilon} \left(u_\varepsilon \left(x_{i,\varepsilon} + r_{i,\varepsilon} \cdot \right) - B_{i,\varepsilon}(r_{i,\varepsilon}) \right) \to 2 \ln \frac{1}{|x|} + \mathcal{H}_i$$

as $\varepsilon \to 0$ *in* $\mathcal{C}^1_{loc}(\mathbb{D}_0(1) \setminus \{0\})$ *where* $\mathcal{H}_i$ *is some harmonic function in the unit disk satisfying*

$$\mathcal{H}_i(0) = 0 \text{ and } \nabla \mathcal{H}_i(0) = -\frac{1}{2} \lim_{\varepsilon \to 0} \frac{r_{i,\varepsilon} \nabla f_\varepsilon(x_{i,\varepsilon})}{f_\varepsilon(x_{i,\varepsilon})} \ .$$

Let us start with a simple consequence of this claim :

Claim 4.2. *For any* $i \in \{1, \dots, N\}$ *and any sequence* (r_ε) *of positive real numbers such that* $\mathbb{D}_{x_{i,\varepsilon}}(r_\varepsilon) \subset \Omega$, *we have that :*

a) If $r_\varepsilon \leq r_{i,\varepsilon}$ *and* $B_{i,\varepsilon}(r_\varepsilon) \geq \delta \gamma_{i,\varepsilon}$ *for some* $\delta > 0$, *there exists* $C > 0$ *such that*

$$|u_\varepsilon - B_{i,\varepsilon}| \leq \frac{C}{\gamma_{i,\varepsilon}} \text{ in } \mathbb{D}_{x_{i,\varepsilon}}(r_\varepsilon) \ .$$

Moreover, we have that

$$\frac{1}{2\pi r_\varepsilon} \int_{\partial \mathbb{D}_{x_{i,\varepsilon}}(r_\varepsilon)} u_\varepsilon \, d\sigma = B_{i,\varepsilon}(r_\varepsilon) + o\left(\gamma_{i,\varepsilon}^{-1}\right) \ .$$

b) If $\limsup_{\varepsilon \to 0} \gamma_{i,\varepsilon}^{-1} B_{i,\varepsilon}(r_\varepsilon) \leq 0$ *and* $\limsup_{\varepsilon \to 0} \gamma_{i,\varepsilon}^{-1} B_{i,\varepsilon}(r_{i,\varepsilon}) \leq 0$, *then we have that*

$$\inf_{\partial \mathbb{D}_{x_{i,\varepsilon}}(r_\varepsilon)} u_\varepsilon \leq B_{i,\varepsilon}(r_\varepsilon) + o\left(\gamma_{i,\varepsilon}^{-1}\right) \ .$$

c) If $\limsup_{\varepsilon \to 0} \gamma_{i,\varepsilon}^{-1} B_{i,\varepsilon}(r_{i,\varepsilon}) \leq 0$, *we have that* $t_{i,\varepsilon}(d_{i,\varepsilon}) \leq \gamma_{i,\varepsilon}^2$ *for* $\varepsilon > 0$ *small enough. In other words, we have that*

$$\lambda_\varepsilon f_\varepsilon(x_{i,\varepsilon}) \gamma_{i,\varepsilon}^2 d_{i,\varepsilon}^2 \leq 4$$

for ε *small enough. Here,* $d_{i,\varepsilon}$ *is as in* (4.8).

Proof of Claim 4.2 - We first prove a). We assume that $B_{i,\varepsilon}(r_\varepsilon) \geq \delta\gamma_{i,\varepsilon}$ for some $\delta > 0$. Define $0 \leq s_\varepsilon \leq r_\varepsilon$ as

$$s_\varepsilon = \max\left\{0 \leq s \leq r_\varepsilon \text{ s.t. } u_\varepsilon \geq \frac{1}{2}\delta\gamma_{i,\varepsilon} \text{ in } \mathbb{D}_{x_{i,\varepsilon}}(s)\right\} .$$

Thanks to (4.5), we have that

$$|x - x_{i,\varepsilon}|\,|\nabla u_\varepsilon| \leq C\gamma_{i,\varepsilon}^{-1} \text{ in } \mathbb{D}_{x_{i,\varepsilon}}(s_\varepsilon)$$

for some $C > 0$ so that we can apply Claim 4.1. Assertion b) of this claim gives that

$$|u_\varepsilon - B_{i,\varepsilon}| \leq C\gamma_{i,\varepsilon}^{-1} \text{ in } \mathbb{D}_{x_{i,\varepsilon}}(s_\varepsilon)$$

for some $C > 0$. Since $B_{i,\varepsilon}(s_\varepsilon) \geq B_{i,\varepsilon}(r_\varepsilon) \geq \delta\gamma_{i,\varepsilon}$, we obtain in particular that $s_\varepsilon = r_\varepsilon$. Indeed, if $s_\varepsilon < r_\varepsilon$, there would exist some $x_\varepsilon \in \partial\mathbb{D}_{x_{i,\varepsilon}}(s_\varepsilon)$ such that $u_\varepsilon(x_\varepsilon) = \frac{\delta}{2}\gamma_{i,\varepsilon}$, which is impossible by what we just proved. Thus a) is clearly proved, applying again Claim 4.1 this time with r_ε.

Let us now prove b). Let us assume first that $1 + \frac{r_\varepsilon^2}{4\mu_{i,\varepsilon}^2} \leq e^{\gamma_{i,\varepsilon}^2}$, that $\limsup_{\varepsilon\to 0} \gamma_{i,\varepsilon}^{-1} B_{i,\varepsilon}(r_\varepsilon) \leq 0$ and that $\limsup_{\varepsilon\to 0} \gamma_{i,\varepsilon}^{-1} B_{i,\varepsilon}(r_{i,\varepsilon}) \leq 0$ and assume by contradiction that there exists $0 < \eta < 1$ such that

$$\inf_{\mathbb{D}_{x_{i,\varepsilon}}(r_\varepsilon)} u_\varepsilon \geq B_{i,\varepsilon}(r_\varepsilon) + \eta\gamma_{i,\varepsilon}^{-1} . \tag{4.9}$$

We claim that

$$u_\varepsilon \geq \gamma_{i,\varepsilon} + \frac{1}{\gamma_{i,\varepsilon}} \ln\frac{4\mu_{i,\varepsilon}^2}{|x_{i,\varepsilon} - x|^2} - \frac{1-\eta}{\gamma_{i,\varepsilon}} + o\left(\gamma_{i,\varepsilon}^{-1}\right) \text{ in } \mathbb{D}_{x_{i,\varepsilon}}(r_\varepsilon) \setminus \mathbb{D}_{x_{i,\varepsilon}}(R_0\mu_{i,\varepsilon}) \tag{4.10}$$

where

$$R_0 = \frac{4}{\sqrt{e^{1-\eta} - 1}} .$$

The right-hand side of (4.10) being harmonic and u_ε being super-harmonic, it is sufficient to check the inequality on $\partial\mathbb{D}_{x_{i,\varepsilon}}(r_\varepsilon)$ and on $\partial\mathbb{D}_{x_{i,\varepsilon}}(R_0\mu_{i,\varepsilon})$. For that purpose, let us write that

$$B_{i,\varepsilon}(r_\varepsilon) = \gamma_{i,\varepsilon} - \gamma_{i,\varepsilon}^{-1} t_{i,\varepsilon}(r_\varepsilon) - \gamma_{i,\varepsilon}^{-3} t_{i,\varepsilon}(r_\varepsilon) + O\left(\gamma_{i,\varepsilon}^{-2}\right)$$

as proved in Appendix A, Claim 5.2, since we assumed for the moment that $t_{i,\varepsilon}(r_\varepsilon) \leq \gamma_{i,\varepsilon}^2$. Since we assumed that $\limsup_{\varepsilon\to 0} \gamma_{i,\varepsilon}^{-1} B_{i,\varepsilon}(r_\varepsilon) \leq 0$, this gives that $\frac{t_{i,\varepsilon}(r_\varepsilon)}{\gamma_{i,\varepsilon}^2} \to 1$ as $\varepsilon \to 0$ so that

$$\begin{aligned} B_{i,\varepsilon}(r_\varepsilon) &= \gamma_{i,\varepsilon} - \gamma_{i,\varepsilon}^{-1} \ln\left(1 + \frac{r_\varepsilon^2}{4\mu_{i,\varepsilon}^2}\right) - \gamma_{i,\varepsilon}^{-1} + o\left(\gamma_{i,\varepsilon}^{-1}\right) \\ &= \gamma_{i,\varepsilon} + \frac{1}{\gamma_{i,\varepsilon}} \ln\frac{4\mu_{i,\varepsilon}^2}{r_\varepsilon^2} - \gamma_{i,\varepsilon}^{-1} + o\left(\gamma_{i,\varepsilon}^{-1}\right) . \end{aligned}$$

This implies with (4.9) that

$$u_\varepsilon \geq \gamma_{i,\varepsilon} + \frac{1}{\gamma_{i,\varepsilon}} \ln\frac{4\mu_{i,\varepsilon}^2}{|x_{i,\varepsilon} - x|^2} - \frac{1-\eta}{\gamma_{i,\varepsilon}} + o\left(\gamma_{i,\varepsilon}^{-1}\right) \text{ on } \partial\mathbb{D}_{x_{i,\varepsilon}}(r_\varepsilon) . \tag{4.11}$$

Let us write now that

$$u_\varepsilon - B_{i,\varepsilon} = o\left(\gamma_{i,\varepsilon}^{-1}\right) \text{ on } \partial\mathbb{D}_{x_{i,\varepsilon}}(R_0\mu_{i,\varepsilon})$$

thanks to d) of Proposition 2.1. Since

$$B_{i,\varepsilon}(R_0\mu_{i,\varepsilon}) = \gamma_{i,\varepsilon} - \gamma_{i,\varepsilon}^{-1} \ln\left(1 + \frac{R_0^2}{4}\right) + o\left(\gamma_{i,\varepsilon}^{-1}\right) ,$$

we obtain that

$$u_\varepsilon \geq \gamma_{i,\varepsilon} + \frac{1}{\gamma_{i,\varepsilon}} \ln \frac{4\mu_{i,\varepsilon}^2}{|x_{i,\varepsilon} - x|^2} - \frac{1-\eta}{\gamma_{i,\varepsilon}} + o\left(\gamma_{i,\varepsilon}^{-1}\right) \text{ on } \partial \mathbb{D}_{x_{i,\varepsilon}}(R_0 \mu_{i,\varepsilon}) \tag{4.12}$$

provided that

$$\ln\left(1 + \frac{4}{R_0^2}\right) < 1 - \eta \,,$$

which is the case with our choice of R_0. Thus (4.10) is proved.

Now there exists $R_0\mu_{i,\varepsilon} \leq s_\varepsilon \leq \min\{r_\varepsilon, r_{i,\varepsilon}\}$ such that $B_{i,\varepsilon}(s_\varepsilon) = \frac{\eta}{2}\gamma_{i,\varepsilon}$ since $\limsup_{\varepsilon\to 0} \gamma_{i,\varepsilon}^{-1} B_{i,\varepsilon}(r_\varepsilon) \leq 0$ and $\limsup_{\varepsilon\to 0} \gamma_{i,\varepsilon}^{-1} B_{i,\varepsilon}(r_{i,\varepsilon}) \leq 0$. We can apply a) of the claim to get that

$$\frac{1}{2\pi s_\varepsilon} \int_{\partial \mathbb{D}_{x_{i,\varepsilon}}(s_\varepsilon)} u_\varepsilon \, d\sigma = \frac{\eta}{2}\gamma_{i,\varepsilon} + o\left(\gamma_{i,\varepsilon}^{-1}\right) \,.$$

Applying (4.10), this leads to

$$\gamma_{i,\varepsilon} + \frac{1}{\gamma_{i,\varepsilon}} \ln \frac{4\mu_{i,\varepsilon}^2}{s_\varepsilon^2} - \frac{1-\eta}{\gamma_{i,\varepsilon}} + o\left(\gamma_{i,\varepsilon}^{-1}\right) \leq \frac{\eta}{2}\gamma_{i,\varepsilon} + o\left(\gamma_{i,\varepsilon}^{-1}\right) \,. \tag{4.13}$$

Since $B_{i,\varepsilon}(s_\varepsilon) = \frac{\eta}{2}\gamma_{i,\varepsilon}$, it is not difficult to check thanks to Claim 5.2 of Appendix A that

$$t_{i,\varepsilon}(s_\varepsilon) = \left(1 - \frac{\eta}{2}\right)\left(\gamma_{i,\varepsilon}^2 - 1\right) + O\left(\gamma_{i,\varepsilon}^{-1}\right)$$

so that, since $\frac{s_\varepsilon}{\mu_{i,\varepsilon}} \to +\infty$ as $\varepsilon \to 0$,

$$\ln \frac{4\mu_{i,\varepsilon}^2}{s_\varepsilon^2} = -\left(\gamma_{i,\varepsilon}^2 - 1\right)\left(1 - \frac{\eta}{2}\right) + o(1) \,.$$

Coming back to (4.13) with this leads to a contradiction. This proves that (4.9) is absurd for any $0 < \eta < 1$. Thus we have proved assertion b) as long as $t_{i,\varepsilon}(r_\varepsilon) \leq \gamma_{i,\varepsilon}^2$.

We shall now prove c), which will by the way prove that b) holds since the condition $t_{i,\varepsilon}(r_\varepsilon) \leq \gamma_{i,\varepsilon}^2$ will always be satisfied. Let us assume by contradiction that $t_{i,\varepsilon}(d_{i,\varepsilon}) \geq \gamma_{i,\varepsilon}^2$. Then $\mathbb{D}_{x_{i,\varepsilon}}(r_\varepsilon) \subset \Omega$ for $\varepsilon > 0$ small where

$$1 + \frac{r_\varepsilon^2}{4\mu_\varepsilon^2} = e^{\gamma_\varepsilon^2 - \frac{1}{2}} \,.$$

We can apply b) in this case since $t_{i,\varepsilon}(r_\varepsilon) \leq \gamma_{i,\varepsilon}^2$ and

$$B_{i,\varepsilon}(r_\varepsilon) = -\frac{1}{2}\gamma_{i,\varepsilon}^{-1} + O\left(\gamma_{i,\varepsilon}^{-2}\right)$$

by Claim 5.2 of Appendix A. This leads to a contradiction since $u_\varepsilon \geq 0$ in Ω. Thus c) is proved thanks to the definition of $\mu_{i,\varepsilon}$ and b) is also proved. This ends the proof of this claim. ♢

Claim 4.3. *For any $i \in \{1, \dots, N\}$, we have that*

$$\limsup_{\varepsilon\to 0} \gamma_{i,\varepsilon}^{-1} B_{i,\varepsilon}(r_{i,\varepsilon}) \leq 0 \,.$$

Proof of Claim 4.3 - Let us reorder for this proof the concentration points in such a way that

$$r_{1,\varepsilon} \leq r_{2,\varepsilon} \leq \cdots \leq r_{N,\varepsilon} \,. \tag{4.14}$$

We prove the assertion by induction on i. Let $i \in \{1, \dots, N\}$ and let us assume that

$$\limsup_{\varepsilon\to 0} \gamma_{j,\varepsilon}^{-1} B_{j,\varepsilon}(r_{j,\varepsilon}) \leq 0 \text{ for all } 1 \leq j \leq i-1 \,. \tag{4.15}$$

Note that we do not assume anything if $i = 1$. We proceed by contradiction, assuming that, after passing to a subsequence,

$$\gamma_{i,\varepsilon}^{-1} B_{i,\varepsilon}(r_{i,\varepsilon}) \geq 2\varepsilon_0 \tag{4.16}$$

for some $\varepsilon_0 > 0$.

Step 1 - *If* (4.16) *holds, then* $\frac{d(x_{i,\varepsilon},\partial\Omega)}{r_{i,\varepsilon}} \to +\infty$ *as* $\varepsilon \to 0$. *In particular, this implies that* $r_{i,\varepsilon} \to 0$ *as* $\varepsilon \to 0$.

Proof of Step 1 - For any $\eta > 0$ small enough, there exists a path of length less than or equal to $Cd(x_{i,\varepsilon},\partial\Omega)$ joining the boundary of Ω and the boundary of the disk $\mathbb{D}_{x_{i,\varepsilon}}(\eta d(x_{i,\varepsilon},\partial\Omega))$, and avoiding all the disks $\mathbb{D}_{x_{j,\varepsilon}}(\eta d(x_{i,\varepsilon},\partial\Omega))$ for $j = 1,\dots,N$. Using f) of Proposition 2.1, we deduce that, for any $\eta > 0$, there exists $C > 0$ such that

$$u_\varepsilon \leq C \text{ on } \partial\mathbb{D}_{x_{i,\varepsilon}}(\eta d(x_{i,\varepsilon},\partial\Omega)) \ .$$

If $d(x_{i,\varepsilon},\partial\Omega) = O(r_{i,\varepsilon})$, we can find $\eta > 0$ small enough such that $\eta d(x_{i,\varepsilon},\partial\Omega) \leq r_{i,\varepsilon}$. Then the above estimate would clearly contradict a) of Claim 4.2 together with (4.16). Thus Step 1 is proved. ♠

Thanks to Step 1, we know that, if (4.16) holds, then

$$\mathcal{D}_i = \{j \in \{1,\dots,N\}, j \neq i \text{ s.t. } |x_{j,\varepsilon} - x_{i,\varepsilon}| = O(r_{i,\varepsilon})\} \neq \emptyset \ . \tag{4.17}$$

There exists $0 < \delta < 1$ such that, for any $j \in \mathcal{D}_i$, any point of $\partial\mathbb{D}_{x_{j,\varepsilon}}(\delta r_{i,\varepsilon})$ can be joined to a point of $\partial\mathbb{D}_{x_{i,\varepsilon}}(\delta r_{i,\varepsilon})$ by a path $\gamma_\varepsilon : [0,1] \to \Omega$ such that $|\gamma_\varepsilon(t) - x_{k,\varepsilon}| \geq \delta r_{i,\varepsilon}$ for all $k = 1,\dots,N$ and all $0 \leq t \leq 1$ and such that $|\gamma_\varepsilon'(t)| \leq \delta^{-1} r_{i,\varepsilon}$. Thanks to assertion f) of Proposition 2.1, the existence of such paths give that

$$\inf_{\partial\mathbb{D}_{x_{j,\varepsilon}}(\delta r_{i,\varepsilon})} u_\varepsilon^2 \geq \inf_{\partial\mathbb{D}_{x_{i,\varepsilon}}(\delta r_{i,\varepsilon})} u_\varepsilon^2 - 2C_2\delta^{-2} \text{ for all } j \in \mathcal{D}_i \ .$$

Thanks to (4.16), we can apply a) of Claim 4.2 to obtain also that

$$u_\varepsilon \geq B_{i,\varepsilon}(\delta r_{i,\varepsilon}) - C\gamma_{i,\varepsilon}^{-1} \text{ on } \partial\mathbb{D}_{x_{i,\varepsilon}}(\delta r_{i,\varepsilon})$$

for some $C > 0$. Since

$$B_{i,\varepsilon}(\delta r_{i,\varepsilon}) = B_{i,\varepsilon}(r_{i,\varepsilon}) + O\left(\gamma_{i,\varepsilon}^{-1}\right) \ ,$$

the two previous estimates , together with (4.16), lead to the existence of some $C > 0$ such that

$$u_\varepsilon \geq B_{i,\varepsilon}(r_{i,\varepsilon}) - C\gamma_{i,\varepsilon}^{-1} \text{ on } \partial\mathbb{D}_{x_{j,\varepsilon}}(\delta r_{i,\varepsilon}) \text{ for all } j \in \mathcal{D}_i \ . \tag{4.18}$$

Step 2 - *If* (4.16) *holds, then for any* $j \in \mathcal{D}_i$, *we have that*

$$\liminf_{\varepsilon\to 0} \gamma_{j,\varepsilon}^{-1} B_{j,\varepsilon}(r_{j,\varepsilon}) > 0 \ .$$

In particular, we have that $j \geq i + 1$.

Proof of Step 2 - Assume on the contrary that there exists $j \in \mathcal{D}_i$ such that, after passing to a subsequence,

$$\limsup_{\varepsilon\to 0} \gamma_{j,\varepsilon}^{-1} B_{j,\varepsilon}(r_{j,\varepsilon}) \leq 0 \ . \tag{4.19}$$

Since $j \in \mathcal{D}_i$, we also know that

$$r_{j,\varepsilon} \leq \frac{1}{2}|x_{i,\varepsilon} - x_{j,\varepsilon}| \leq Cr_{i,\varepsilon} \ . \tag{4.20}$$

Thus we also have that

$$\limsup_{\varepsilon\to 0} \gamma_{j,\varepsilon}^{-1} B_{j,\varepsilon}(\delta r_{i,\varepsilon}) \leq 0 \ . \tag{4.21}$$

We can apply b) of Claim 4.2 with $r_\varepsilon = \delta r_{i,\varepsilon}$ to obtain that

$$B_{i,\varepsilon}(r_{i,\varepsilon}) - C\gamma_{i,\varepsilon}^{-1} \leq B_{j,\varepsilon}(\delta r_{i,\varepsilon}) + o\left(\gamma_{j,\varepsilon}^{-1}\right) \tag{4.22}$$

thanks to (4.18). Combining (4.16) and (4.21), we get that

$$\gamma_{i,\varepsilon} = o(\gamma_{j,\varepsilon}) \ . \tag{4.23}$$

Thus we also have that $\mu_{j,\varepsilon} \leq \mu_{i,\varepsilon}$. Let us write now thanks to Claim 5.2 of Appendix A that

$$B_{j,\varepsilon}(\delta r_{i,\varepsilon}) = -\gamma_{j,\varepsilon}^{-1} \ln\left(\frac{r_{i,\varepsilon}^2}{4}\right) - \gamma_{j,\varepsilon}^{-1} \ln\left(\lambda_\varepsilon \gamma_{j,\varepsilon}^2\right) + O\left(\gamma_{j,\varepsilon}^{-1}\right)$$

and that

$$B_{i,\varepsilon}(r_{i,\varepsilon}) = -\gamma_{i,\varepsilon}^{-1} \ln\left(\frac{r_{i,\varepsilon}^2}{4}\right) - \gamma_{i,\varepsilon}^{-1} \ln\left(\lambda_\varepsilon \gamma_{i,\varepsilon}^2\right) + O\left(\gamma_{i,\varepsilon}^{-1}\right)$$

to obtain that

$$B_{j,\varepsilon}(\delta r_{i,\varepsilon}) = \frac{\gamma_{i,\varepsilon}}{\gamma_{j,\varepsilon}} B_{i,\varepsilon}(r_{i,\varepsilon}) + \gamma_{j,\varepsilon}^{-1} \ln\left(\frac{\gamma_{i,\varepsilon}^2}{\gamma_{j,\varepsilon}^2}\right) + O\left(\gamma_{j,\varepsilon}^{-1}\right) \ .$$

Coming back to (4.22) with this, (4.16) and (4.23), we obtain that

$$(2\varepsilon_0 + o(1))\,\gamma_{i,\varepsilon} \leq \gamma_{j,\varepsilon}^{-1} \ln\left(\frac{\gamma_{i,\varepsilon}^2}{\gamma_{j,\varepsilon}^2}\right) + O\left(\gamma_{i,\varepsilon}^{-1}\right) \leq O\left(\gamma_{i,\varepsilon}^{-1}\right) \ ,$$

which is a clear contradiction. Step 2 is proved. ♠

We can now conclude the proof of the claim by proving that (4.16) is absurd if (4.15) holds. Continue to assume that (4.16) holds. Then we know thanks to Step 2 that for any $j \in \mathcal{D}_i$, $j \geq i+1$ so that $r_{j,\varepsilon} \geq r_{i,\varepsilon}$. We set, for $j \in \mathcal{D}_i$, and up to a subsequence,

$$\hat{x}_j = \lim_{\varepsilon \to 0} \frac{x_{j,\varepsilon} - x_{i,\varepsilon}}{r_{i,\varepsilon}} \tag{4.24}$$

and we let

$$\hat{\mathcal{S}} = \{\hat{x}_j,\ j \in \mathcal{D}_i\} \ . \tag{4.25}$$

We know thanks to Step 1 that there exists $j \in \mathcal{D}_i$ such that

$$|\hat{x}_j| = 2 \tag{4.26}$$

and that

$$|\hat{x}_k - \hat{x}_l| \geq 2 \text{ for all } k, l \in \mathcal{D}_i,\ k \neq l \ . \tag{4.27}$$

Since $r_{j,\varepsilon}$ and $r_{i,\varepsilon}$ are comparable, we also have thanks to Step 2 that

$$\liminf_{\varepsilon \to 0} \gamma_{j,\varepsilon}^{-1} B_{j,\varepsilon}(r_{j,\varepsilon}) > 0 \ . \tag{4.28}$$

Let K be a compact subset of $\mathbb{R}^2 \setminus \hat{\mathcal{S}}$. We can use assertion f) of Proposition 2.1 to write[4] that

$$\gamma_{i,\varepsilon} \left| u_\varepsilon(x_{i,\varepsilon} + r_{i,\varepsilon} x) - B_{i,\varepsilon}(r_{i,\varepsilon}) \right| \leq C_K \text{ in } K \ . \tag{4.29}$$

Thanks to (4.1), we can write that

$$\Delta \hat{u}_\varepsilon = \lambda_\varepsilon r_{i,\varepsilon}^2 \gamma_{i,\varepsilon} f_\varepsilon(x_{i,\varepsilon} + r_{i,\varepsilon} x)\, u_\varepsilon(x_{i,\varepsilon} + r_{i,\varepsilon} x)^2\, e^{u_\varepsilon(x_{i,\varepsilon} + r_{i,\varepsilon} x)^2}$$

where

$$\hat{u}_\varepsilon = \gamma_{i,\varepsilon}\left(u_\varepsilon(x_{i,\varepsilon} + r_{i,\varepsilon} x) - B_{i,\varepsilon}(r_{i,\varepsilon})\right) \ .$$

[4] see the argument between Steps 1 and 2.

Using (4.29), we can write that

$$|\Delta \hat{u}_\varepsilon| \leq C_K \mu_{i,\varepsilon}^{-2} r_{i,\varepsilon}^2 e^{B_{i,\varepsilon}(r_{i,\varepsilon})^2 - \gamma_{i,\varepsilon}^2} \text{ in } K$$

for any compact subset K of $\mathbb{R}^2 \setminus \hat{\mathcal{S}}$. Thanks to (4.16), we have that

$$e^{B_{i,\varepsilon}(r_{i,\varepsilon})^2 - \gamma_{i,\varepsilon}^2} \leq C \left(1 + \frac{r_{i,\varepsilon}^2}{4\mu_{i,\varepsilon}^2}\right)^{-1-2\varepsilon_0}$$

so that

$$|\Delta \hat{u}_\varepsilon| \leq C_K \left(\frac{\mu_{i,\varepsilon}}{r_{i,\varepsilon}}\right)^{4\varepsilon_0} \to 0 \text{ uniformly in } K \ .$$

By standard elliptic theory, we thus have that

$$\hat{u}_\varepsilon = \gamma_{i,\varepsilon} \left(u_\varepsilon \left(x_{i,\varepsilon} + r_{i,\varepsilon} x\right) - B_{i,\varepsilon}\left(r_{i,\varepsilon}\right)\right) \to \hat{u}_0 \text{ in } \mathcal{C}^1_{loc}\left(\mathbb{R}^2 \setminus \hat{\mathcal{S}}\right) \text{ as } \varepsilon \to 0 \tag{4.30}$$

where

$$\Delta \hat{u}_0 = 0 \text{ in } \mathbb{R}^2 \setminus \hat{\mathcal{S}} \ . \tag{4.31}$$

Since $r_{j,\varepsilon} \geq r_{i,\varepsilon}$ for $j \in \mathcal{D}_i$, (4.28) permits to apply a) of Claim 4.2, which in turn implies thanks to (4.5) that we can apply Claim 4.1 for all $j \in \mathcal{D}_i$ with $r_\varepsilon = r_{i,\varepsilon}$. Assertion c) of this claim gives that

$$\gamma_{j,\varepsilon} \left(u_\varepsilon \left(x_{j,\varepsilon} + r_{i,\varepsilon} x\right) - B_{j,\varepsilon}\left(r_{i,\varepsilon}\right)\right) \to 2 \ln \frac{1}{|x|} + \mathcal{H}_j \tag{4.32}$$

in $\mathcal{C}^1_{loc}\left(\mathbb{D}_0(1) \setminus \{0\}\right)$ as $\varepsilon \to 0$ where $\mathcal{H}_j$ is harmonic in the unit disk and satifies $\mathcal{H}_j(0) = 0$ and $\nabla \mathcal{H}_j(0) = 0$ (note here that we know thanks to Step 1 that $r_{i,\varepsilon} \to 0$ as $\varepsilon \to 0$). This gives that

$$\frac{\gamma_{j,\varepsilon}}{\gamma_{i,\varepsilon}} \hat{u}_\varepsilon + \gamma_{j,\varepsilon} \left(B_{i,\varepsilon}\left(r_{i,\varepsilon}\right) - B_{j,\varepsilon}\left(r_{i,\varepsilon}\right)\right) \to 2 \ln \frac{1}{|x - \hat{x}_j|} + \hat{\mathcal{H}}_j \tag{4.33}$$

in $\mathcal{C}^1_{loc}\left(\mathbb{D}_{\hat{x}_j}(1) \setminus \{\hat{x}_j\}\right)$ as $\varepsilon \to 0$ for all $j \in \mathcal{D}_i$ (and also for $j = i$ if we set $\hat{x}_i = 0$). It remains to write thanks to Claim 5.2 of Appendix A and to (1.5) that

$$B_{i,\varepsilon}\left(r_{i,\varepsilon}\right) - B_{j,\varepsilon}\left(r_{i,\varepsilon}\right) = \left(1 - \frac{\gamma_{i,\varepsilon}}{\gamma_{j,\varepsilon}}\right) B_{i,\varepsilon}\left(r_{i,\varepsilon}\right) + \gamma_{j,\varepsilon}^{-1} \ln \frac{\gamma_{j,\varepsilon}^2}{\gamma_{i,\varepsilon}^2} + O\left(\gamma_{j,\varepsilon}^{-1}\right)$$

to deduce from (4.16), (4.30) and (4.33) that

$$\frac{\gamma_{j,\varepsilon}}{\gamma_{i,\varepsilon}} \to 1 \text{ as } \varepsilon \to 0 \text{ and } \gamma_{i,\varepsilon} \left|\gamma_{i,\varepsilon} - \gamma_{j,\varepsilon}\right| = O\left(1\right) \ . \tag{4.34}$$

Then (4.30) and (4.33) just lead to

$$\hat{u}_0 = 2 \ln \frac{1}{|x - \hat{x}_j|} + \varphi_j \tag{4.35}$$

in $\mathbb{D}_{\hat{x}_j}\left(1\right) \setminus \{\hat{x}_j\}$ where φ_j is smooth and harmonic and satisfies $\nabla \varphi_j\left(\hat{x}_j\right) = 0$. Thus we can write that

$$\hat{u}_0 = 2 \ln \frac{1}{|x|} + 2 \sum_{j \in \mathcal{D}_i} \ln \frac{1}{|x - \hat{x}_j|} + \varphi \tag{4.36}$$

where φ is a smooth harmonic function in $\mathbb{R}^2$. Thanks to assertion f) of Proposition 2.1, we also know that

$$|\nabla \varphi(x)| \leq \frac{C}{1 + |x|} \text{ in } \mathbb{R}^2$$

for some $C > 0$ so that $\varphi \equiv Cst$. Now this gives that for any $k \in \mathcal{D}_i$,

$$\nabla \left(\ln \frac{1}{|x|} + \sum_{j \in \mathcal{D}_i, j \neq k} \ln \frac{1}{|x - \hat{x}_j|} \right) (\hat{x}_k) = 0 \, .$$

Let $k \in \mathcal{D}_i$ be such that $|\hat{x}_k| \geq |\hat{x}_j|$ for all $j \in \mathcal{D}_i$. Then

$$\begin{aligned} \left\langle \nabla \left(\ln \frac{1}{|x|} + \sum_{j \in \mathcal{D}_i, \, j \neq k} \ln \frac{1}{|x - \hat{x}_j|} \right) (\hat{x}_k) \, , \hat{x}_k \right\rangle &= -|\hat{x}_k| - \sum_{j \in \mathcal{D}_i, \, j \neq k} \frac{|\hat{x}_k|^2 - \langle \hat{x}_k, \hat{x}_j \rangle}{|\hat{x}_k - \hat{x}_j|} \\ &\leq -|\hat{x}_k| < 0 \, , \end{aligned}$$

which gives the desired contradiction. This proves that (4.16) is absurd as soon as (4.15) holds. And this ends the proof of the claim by an induction argument. ♢

Claim 4.4. *For any $i = 1, \ldots, N$, we have that*

$$\lambda_\varepsilon f_\varepsilon \left(x_{i,\varepsilon} \right) \gamma_{i,\varepsilon}^2 d_{i,\varepsilon}^2 \leq 4$$

for ε small enough.

Proof of Claim 4.4 - It is a direct consequence of c) of Claim 4.2 together with Claim 4.3. ♢

Claim 4.5. *We have that*

$$\int_\Omega |\nabla u_\varepsilon|^2 \, dx = \int_\Omega |\nabla u_0|^2 \, dx + 4\pi N + o(1) \, .$$

In other words, $M = 0$ in Theorem 2.1.

Proof of Claim 4.5 - We prove that $M = 0$ in Theorem 2.1. Assume on the contrary that there exists some sequence $(y_{1,\varepsilon})$ such that the assertion b) of Theorem 2.1 holds. This means that

$$\nu_{1,\varepsilon}^{-2} = \lambda_\varepsilon f_\varepsilon \left(y_{1,\varepsilon} \right) u_\varepsilon \left(y_{1,\varepsilon} \right)^2 e^{u_\varepsilon (y_{1,\varepsilon})^2} \to +\infty \text{ as } \varepsilon \to 0 \, .$$

By e) of Proposition 2.1, we know that

$$\begin{aligned} \nu_{1,\varepsilon}^{-2} \left(\min_{i=1,\ldots,N} |x_{i,\varepsilon} - y_{1,\varepsilon}| \right)^2 &= \left(\min_{i=1,\ldots,N} |x_{i,\varepsilon} - y_{1,\varepsilon}| \right)^2 \lambda_\varepsilon f_\varepsilon \left(y_{1,\varepsilon} \right) u_\varepsilon \left(y_{1,\varepsilon} \right)^2 e^{u_\varepsilon (y_{1,\varepsilon})^2} \\ &\leq C_1 f_\varepsilon \left(y_{1,\varepsilon} \right) \, . \end{aligned}$$

This proves that there exists $i \in \{1, \ldots, N\}$ such that

$$|x_{i,\varepsilon} - y_{1,\varepsilon}| = O \left(\nu_{1,\varepsilon} \right) \, .$$

Since

$$\frac{|x_{i,\varepsilon} - y_{1,\varepsilon}|}{\mu_{i,\varepsilon}} \to +\infty \text{ as } \varepsilon \to 0$$

by a) of Theorem 2.1, we have that

$$\frac{\nu_{1,\varepsilon}}{\mu_{i,\varepsilon}} \to +\infty \text{ as } \varepsilon \to 0 \, .$$

Thanks to the definition of $\nu_{1,\varepsilon}$ and $\mu_{i,\varepsilon}$, this leads to

$$e^{\gamma_{i,\varepsilon}^2 - u_\varepsilon (y_{1,\varepsilon})^2} \frac{\gamma_{i,\varepsilon}^2}{u_\varepsilon \left(y_{1,\varepsilon} \right)^2} \to +\infty \text{ as } \varepsilon \to 0 \, ,$$

which implies that

$$\gamma_{i,\varepsilon}^2 - u_\varepsilon \left(y_{1,\varepsilon} \right)^2 \to +\infty \text{ as } \varepsilon \to 0 \, . \tag{4.37}$$

Now, by the convergence of b) of Theorem 2.1, we know that

$$u_\varepsilon \geq u_\varepsilon\left(y_{1,\varepsilon}\right) - Cu_\varepsilon\left(y_{1,\varepsilon}\right)^{-1} \text{ on } \partial\mathbb{D}_{x_{i,\varepsilon}}\left(R\nu_{1,\varepsilon}\right)$$

for some $R > 0$ and $C > 0$. Thanks to Claim 4.3, we can use assertion b) of Claim 4.2 to deduce that

$$u_\varepsilon\left(y_{1,\varepsilon}\right) - Cu_\varepsilon\left(y_{1,\varepsilon}\right)^{-1} \leq B_{i,\varepsilon}\left(R\nu_{1,\varepsilon}\right) + o\left(\gamma_{i,\varepsilon}^{-1}\right) .$$

This leads after some simple computations, using Claim 5.2 of Appendix A, to

$$u_\varepsilon\left(y_{1,\varepsilon}\right) - Cu_\varepsilon\left(y_{1,\varepsilon}\right)^{-1} \leq \gamma_{i,\varepsilon}^{-1} \ln\left(\frac{u_\varepsilon\left(y_{1,\varepsilon}\right)^2}{\gamma_{i,\varepsilon}^2}\right) + \frac{u_\varepsilon\left(y_{1,\varepsilon}\right)^2}{\gamma_{i,\varepsilon}} + C\gamma_{i,\varepsilon}^{-1}$$

so that, thanks to (4.37),

$$u_\varepsilon\left(y_{1,\varepsilon}\right)^2\left(1 - \frac{u_\varepsilon\left(y_{1,\varepsilon}\right)}{\gamma_{i,\varepsilon}}\right) \leq C .$$

This clearly implies that

$$\frac{u_\varepsilon\left(y_{1,\varepsilon}\right)}{\gamma_{i,\varepsilon}} \to 1 \text{ as } \varepsilon \to 0$$

and then that

$$u_\varepsilon\left(y_{1,\varepsilon}\right) \geq \gamma_{i,\varepsilon} - C\gamma_{i,\varepsilon}^{-1}$$

for some $C > 0$. This contradicts (4.37). Thus we have proved that $M = 0$ in Theorem 2.1 and the claim follows. ♢

For any $i \in \{1, \dots, N\}$, thanks to Claim 4.3 and a) of Claim (4.2), there exists $0 \leq s_{i,\varepsilon} \leq r_{i,\varepsilon}$ such that

$$\limsup_{\varepsilon\to 0} \gamma_{i,\varepsilon}^{-1} B_{i,\varepsilon}\left(s_{i,\varepsilon}\right) \leq 0 \text{ and } \left|u_\varepsilon - B_{i,\varepsilon}\right| \leq \frac{D_i}{\gamma_{i,\varepsilon}} \text{ in } \mathbb{D}_{x_{i,\varepsilon}}\left(s_{i,\varepsilon}\right) \tag{4.38}$$

for some $D_i > 0$.

Claim 4.6. *We have that*

$$\liminf_{\varepsilon\to 0} \int_{\mathbb{D}_{x_{i,\varepsilon}}(s_{i,\varepsilon})} \left|\nabla u_\varepsilon\right|^2 dx \geq 4\pi .$$

Proof of Claim 4.6 - Let $\delta > 0$. Let us write thanks to (4.38) that

$$\int_{\mathbb{D}_{x_{i,\varepsilon}}(s_{i,\varepsilon})} \left|\nabla u_\varepsilon\right|^2 dx \geq \int_{\mathbb{D}_{x_{i,\varepsilon}}(s_{i,\varepsilon})} \left|\nabla\left(u_\varepsilon - \delta\gamma_{i,\varepsilon}\right)^+\right|^2 dx = \int_{\mathbb{D}_{x_{i,\varepsilon}}(s_{i,\varepsilon})} \left(u_\varepsilon - \delta\gamma_{i,\varepsilon}\right)^+ \Delta u_\varepsilon \, dx .$$

Thanks to (4.1), this leads to

$$\begin{aligned}\int_{\mathbb{D}_{x_{i,\varepsilon}}(s_{i,\varepsilon})} \left|\nabla u_\varepsilon\right|^2 dx &\geq \lambda_\varepsilon \int_{\mathbb{D}_{x_{i,\varepsilon}}(s_{i,\varepsilon})} f_\varepsilon\left(u_\varepsilon - \delta\gamma_{i,\varepsilon}\right)^+ u_\varepsilon e^{u_\varepsilon^2} dx \\ &\geq \lambda_\varepsilon \int_{\mathbb{D}_{x_{i,\varepsilon}}(R\mu_{i,\varepsilon})} f_\varepsilon\left(u_\varepsilon - \delta\gamma_{i,\varepsilon}\right)^+ u_\varepsilon e^{u_\varepsilon^2} dx\end{aligned}$$

for all $R > 0$. Now we have that

$$\lim_{\varepsilon\to 0} \lambda_\varepsilon \int_{\mathbb{D}_{x_{i,\varepsilon}}(R\mu_{i,\varepsilon})} f_\varepsilon\left(u_\varepsilon - \delta\gamma_{i,\varepsilon}\right)^+ u_\varepsilon e^{u_\varepsilon^2} dx = (1-\delta) \int_{\mathbb{D}_0(R)} e^{2U} dx$$

thanks to d) of Proposition 2.1. Since

$$\int_{\mathbb{R}^2} e^{2U} dx = 4\pi ,$$

the result follows by letting R go to $+\infty$ and δ go to 0. ♢

Let us set now

$$\Omega_\varepsilon = \Omega \setminus \bigcup_{i=1}^{N} \mathbb{D}_{x_{i,\varepsilon}}(s_{i,\varepsilon}) \tag{4.39}$$

where $s_{i,\varepsilon}$ is as in (4.38) and

$$w_\varepsilon = \begin{cases} u_\varepsilon & \text{in } \Omega_\varepsilon \\ \min\left\{u_\varepsilon, B_{i,\varepsilon}(s_{i,\varepsilon}) + 2\dfrac{D_i}{\gamma_{i,\varepsilon}}\right\} & \text{in } \mathbb{D}_{x_{i,\varepsilon}}(s_{i,\varepsilon}) \text{ for } i = 1,\dots,N \end{cases} \tag{4.40}$$

Claim 4.7. *We have that*

$$\int_\Omega |\nabla(w_\varepsilon - u_0)|^2 \, dx \to 0 \text{ as } \varepsilon \to 0 \, .$$

Proof of Claim 4.7 - Let us write that

$$\begin{aligned} \int_\Omega |\nabla(w_\varepsilon - u_0)|^2 \, dx &= \int_\Omega |\nabla w_\varepsilon|^2 \, dx - 2\int_\Omega \langle \nabla w_\varepsilon, \nabla u_0\rangle \, dx + \int_\Omega |\nabla u_0|^2 \, dx \\ &= \int_\Omega |\nabla u_\varepsilon|^2 \, dx - 2\int_\Omega \langle \nabla u_\varepsilon, \nabla u_0\rangle \, dx + \int_\Omega |\nabla u_0|^2 \, dx \\ &\quad + \int_\Omega \langle \nabla(w_\varepsilon - u_\varepsilon), \nabla u_\varepsilon + \nabla w_\varepsilon - 2\nabla u_0\rangle \, dx \\ &= 4\pi N + o(1) + \int_\Omega \langle \nabla(w_\varepsilon - u_\varepsilon), \nabla u_\varepsilon + \nabla w_\varepsilon - 2\nabla u_0\rangle \, dx \end{aligned}$$

thanks to the weak convergence of u_ε to u_0 in H^1 and to Claim 4.5. Let us remark now that $\nabla(w_\varepsilon - u_\varepsilon) \equiv 0$ in Ω_ε and that $\langle \nabla(w_\varepsilon - u_\varepsilon), \nabla w_\varepsilon\rangle = 0$ a.e. Thus we can write that

$$\int_\Omega |\nabla(w_\varepsilon - u_0)|^2 \, dx = 4\pi N + o(1) + \sum_{i=1}^{N} \int_{\mathbb{D}_{x_{i,\varepsilon}}(s_{i,\varepsilon})} \langle \nabla(w_\varepsilon - u_\varepsilon), \nabla u_\varepsilon - 2\nabla u_0\rangle \, dx \, . \tag{4.41}$$

Since $w_\varepsilon - u_\varepsilon$ is null on the boundary of $\mathbb{D}_{x_{i,\varepsilon}}(s_{i,\varepsilon})$, we can proceed as in the proof of Claim 4.6 to get that

$$\begin{aligned} \int_{\mathbb{D}_{x_{i,\varepsilon}}(s_{i,\varepsilon})} \langle \nabla(w_\varepsilon - u_\varepsilon), \nabla u_\varepsilon - 2\nabla u_0\rangle \, dx &= \int_{\mathbb{D}_{x_{i,\varepsilon}}(s_{i,\varepsilon})} (w_\varepsilon - u_\varepsilon)(\Delta u_\varepsilon - 2\Delta u_0) \, dx \\ &\le -4\pi + o(1) + O\left(\gamma_{i,\varepsilon} \int_{\mathbb{D}_{x_{i,\varepsilon}}(s_{i,\varepsilon})} |\Delta u_0| \, dx\right) \, . \end{aligned}$$

Here we used the fact that $w_\varepsilon \le u_\varepsilon$ and $|w_\varepsilon| = o(\gamma_{i,\varepsilon})$ in $\mathbb{D}_{x_{i,\varepsilon}}(s_{i,\varepsilon})$. If $u_0 \equiv 0$, the last term disappears. If $u_0 \not\equiv 0$, then $\lambda_\varepsilon \to \lambda_0$ with $\lambda_0 > 0$ and Claim 4.4 gives that $\gamma_{i,\varepsilon} s_{i,\varepsilon}^2 = o(1)$. Thus, in any case, we have that

$$\int_{\mathbb{D}_{x_{i,\varepsilon}}(s_{i,\varepsilon})} \langle \nabla(w_\varepsilon - u_\varepsilon), \nabla u_\varepsilon - 2\nabla u_0\rangle \, dx \le -4\pi + o(1) \, .$$

Coming back to (4.41) with this proves the claim. ♢

The next two claims are devoted to obtaining good pointwise estimates on u_ε and ∇u_ε.

Claim 4.8. *For any sequence (x_ε) of points in Ω such that*

$$\frac{|x_\varepsilon - x_{i,\varepsilon}|}{\mu_{i,\varepsilon}} \to +\infty \text{ as } \varepsilon \to 0 \text{ for } i = 1,\dots,N \, ,$$

we have that

a) if $d_\varepsilon = d(x_\varepsilon, \partial\Omega) \not\to 0$ *as* $\varepsilon \to 0$*, then*

$$\begin{aligned} u_\varepsilon(x_\varepsilon) &= \psi_\varepsilon(x_\varepsilon) + \sum_{i=1}^N \frac{4\pi + o(1)}{\gamma_{i,\varepsilon}} \mathcal{G}(x_{i,\varepsilon}, x_\varepsilon) \\ &\quad + O\left(\sum_{i=1}^N \gamma_{i,\varepsilon}^{-1} \left(\frac{\mu_{i,\varepsilon}}{|x_{i,\varepsilon} - x_\varepsilon|} + \gamma_{i,\varepsilon}^{-1} \ln\left(\frac{s_{i,\varepsilon}}{|x_{i,\varepsilon} - x_\varepsilon|} + 2 \right) \right) \right) , \end{aligned}$$

b) if $d_\varepsilon \to 0$ *as* $\varepsilon \to 0$*, then*

$$\begin{aligned} u_\varepsilon(x_\varepsilon) &= \psi_\varepsilon(x_\varepsilon) + \sum_{i=1}^N \frac{4\pi + o(1)}{\gamma_{i,\varepsilon}} \mathcal{G}(x_{i,\varepsilon}, x_\varepsilon) \\ &\quad + O\left(\sum_{i\in\mathcal{A}} \gamma_{i,\varepsilon}^{-1} \left(\frac{\mu_{i,\varepsilon}}{|x_{i,\varepsilon} - x_\varepsilon|} + \gamma_{i,\varepsilon}^{-1} \ln\left(\frac{s_{i,\varepsilon}}{|x_{i,\varepsilon} - x_\varepsilon|} + 2 \right) \right) \right) \\ &\quad + O\left(\sum_{i\in\mathcal{B}} \frac{d_\varepsilon}{d_\varepsilon + d_{i,\varepsilon}} \gamma_{i,\varepsilon}^{-1} \left(\mu_{i,\varepsilon} + \gamma_{i,\varepsilon}^{-1} s_{i,\varepsilon} \right) \right) \end{aligned}$$

where $\mathcal{G}$ *is the Green function of the Laplacian with Dirichlet boundary condition in* Ω *and* ψ_ε *is a solution of*

$$\Delta\psi_\varepsilon = \lambda_\varepsilon f_\varepsilon w_\varepsilon e^{w_\varepsilon^2} \text{ in } \Omega \text{ and } \psi_\varepsilon = 0 \text{ on } \partial\Omega .$$

In b), $\mathcal{A}$ *is defined as the set of* $i \in \{1, \dots, N\}$ *such that* $|x_{i,\varepsilon} - x_\varepsilon| \leq s_{i,\varepsilon} + o(d_\varepsilon)$ *and* $\mathcal{B}$ *as its complementary.*

Proof of Claim 4.8 - We let $\mathcal{G}$ be the Green function of the Laplacian with Dirichlet boundary condition in Ω. We let (x_ε) be a sequence of points in Ω such that

$$\frac{|x_\varepsilon - x_{i,\varepsilon}|}{\mu_{i,\varepsilon}} \to +\infty \text{ as } \varepsilon \to 0 \text{ for } i = 1, \dots, N . \tag{4.42}$$

Then we have thanks to (4.1) and to the definition of ψ_ε that

$$u_\varepsilon(x_\varepsilon) - \psi_\varepsilon(x_\varepsilon) = \lambda_\varepsilon \int_\Omega \mathcal{G}(x_\varepsilon, x) f_\varepsilon(x) \left(u_\varepsilon(x) e^{u_\varepsilon(x)^2} - w_\varepsilon(x) e^{w_\varepsilon(x)^2} \right) dx .$$

Using the definition (4.40) of w_ε, this gives that

$$\begin{aligned} &u_\varepsilon(x_\varepsilon) - \psi_\varepsilon(x_\varepsilon) \\ &= \sum_{i=1}^N \lambda_\varepsilon \mathcal{G}(x_\varepsilon, x_{i,\varepsilon}) \int_{\mathbb{D}_{x_{i,\varepsilon}}(s_{i,\varepsilon})} f_\varepsilon(x) \left(u_\varepsilon(x) e^{u_\varepsilon(x)^2} - w_\varepsilon(x) e^{w_\varepsilon(x)^2} \right) dx + \sum_{i=1}^N A_{i,\varepsilon} \end{aligned} \tag{4.43}$$

where

$$A_{i,\varepsilon} = \lambda_\varepsilon \int_{\mathbb{D}_{x_{i,\varepsilon}}(s_{i,\varepsilon})} \left(\mathcal{G}(x_\varepsilon, x) - \mathcal{G}(x_\varepsilon, x_{i,\varepsilon}) \right) f_\varepsilon(x) \left(u_\varepsilon(x) e^{u_\varepsilon(x)^2} - w_\varepsilon(x) e^{w_\varepsilon(x)^2} \right) dx . \tag{4.44}$$

We fix $i \in \{1, \dots, N\}$ in the following and we let

$$\begin{aligned} \Omega_{0,\varepsilon} &= \mathbb{D}_{x_{i,\varepsilon}}(s_{i,\varepsilon}) \cap \{ t_\varepsilon(x) \leq t_{1,\varepsilon} \} , \\ \Omega_{1,\varepsilon} &= \mathbb{D}_{x_{i,\varepsilon}}(s_{i,\varepsilon}) \cap \{ t_{1,\varepsilon} \leq t_\varepsilon(x) \leq t_{2,\varepsilon} \} \text{ and} \\ \Omega_{2,\varepsilon} &= \mathbb{D}_{x_{i,\varepsilon}}(s_{i,\varepsilon}) \cap \{ t_\varepsilon(x) \geq t_{2,\varepsilon} \} . \end{aligned} \tag{4.45}$$

where $t_\varepsilon(x) = \ln\left(1 + \frac{|x - x_{i,\varepsilon}|^2}{4\mu_{i,\varepsilon}^2}\right)$, $t_{1,\varepsilon} = \frac{1}{4}\gamma_{i,\varepsilon}^2$ and $t_{2,\varepsilon} = \gamma_{i,\varepsilon}^2 - \gamma_{i,\varepsilon}$.

STEP 1 - *We have that*

$$\lambda_\varepsilon \int_{\mathbb{D}_{x_{i,\varepsilon}}(s_{i,\varepsilon})} f_\varepsilon(x) \left(u_\varepsilon(x) e^{u_\varepsilon(x)^2} - w_\varepsilon(x) e^{w_\varepsilon(x)^2} \right) dx = 4\pi \gamma_{i,\varepsilon}^{-1} + o\left(\gamma_{i,\varepsilon}^{-1}\right) .$$

PROOF OF STEP 1 - We write that

$$\begin{aligned}
&\lambda_\varepsilon \int_{\mathbb{D}_{x_{i,\varepsilon}}(s_{i,\varepsilon})} f_\varepsilon(x) \left(u_\varepsilon(x) e^{u_\varepsilon(x)^2} - w_\varepsilon(x) e^{w_\varepsilon(x)^2} \right) dx \\
&= \lambda_\varepsilon \int_{\mathbb{D}_{x_{i,\varepsilon}}(R_\varepsilon \mu_{i,\varepsilon})} f_\varepsilon(x) \left(u_\varepsilon(x) e^{u_\varepsilon(x)^2} - w_\varepsilon(x) e^{w_\varepsilon(x)^2} \right) dx \\
&\quad + \lambda_\varepsilon \int_{\Omega_{0,\varepsilon} \setminus \mathbb{D}_{x_{i,\varepsilon}}(R_\varepsilon \mu_{i,\varepsilon})} f_\varepsilon(x) \left(u_\varepsilon(x) e^{u_\varepsilon(x)^2} - w_\varepsilon(x) e^{w_\varepsilon(x)^2} \right) dx \\
&\quad + \lambda_\varepsilon \int_{\Omega_{1,\varepsilon}} f_\varepsilon(x) \left(u_\varepsilon(x) e^{u_\varepsilon(x)^2} - w_\varepsilon(x) e^{w_\varepsilon(x)^2} \right) dx \\
&\quad + \lambda_\varepsilon \int_{\Omega_{2,\varepsilon}} f_\varepsilon(x) \left(u_\varepsilon(x) e^{u_\varepsilon(x)^2} - w_\varepsilon(x) e^{w_\varepsilon(x)^2} \right) dx
\end{aligned}$$

where $R_\varepsilon \to +\infty$ is such that $|u_\varepsilon - B_{i,\varepsilon}| = o\left(\gamma_{i,\varepsilon}^{-1}\right)$ and $\gamma_{i,\varepsilon}^{-1} B_{i,\varepsilon}(x) = 1 + o(1)$ in $\mathbb{D}_{x_{i,\varepsilon}}(R_\varepsilon \mu_{i,\varepsilon})$. Such a R_ε does exist thanks to d) of Proposition 2.1. Then we have, using also (4.38), that

$$\begin{aligned}
&\lambda_\varepsilon \int_{\mathbb{D}_{x_{i,\varepsilon}}(R_\varepsilon \mu_{i,\varepsilon})} f_\varepsilon(x) \left(u_\varepsilon(x) e^{u_\varepsilon(x)^2} - w_\varepsilon(x) e^{w_\varepsilon(x)^2} \right) dx \\
&= \lambda_\varepsilon \left(f_\varepsilon\left(x_{i,\varepsilon}\right) + o(1) \right) \int_{\mathbb{D}_{x_{i,\varepsilon}}(R_\varepsilon \mu_{i,\varepsilon})} B_{i,\varepsilon}(x) e^{B_{i,\varepsilon}(x)^2} dx \\
&\quad + O\left(\lambda_\varepsilon B_{i,\varepsilon}\left(s_{i,\varepsilon}\right) e^{B_{i,\varepsilon}(s_{i,\varepsilon})^2} R_\varepsilon^2 \mu_{i,\varepsilon}^2 \right) \\
&= \gamma_{i,\varepsilon}^{-1} \left(\int_{\mathbb{D}_0(R_\varepsilon)} e^{2U} dx + o(1) \right) + o\left(\gamma_{i,\varepsilon}^{-1}\right) \\
&= 4\pi \gamma_{i,\varepsilon}^{-1} + o\left(\gamma_{i,\varepsilon}^{-1}\right) .
\end{aligned}$$

In $\Omega_{0,\varepsilon}$, we write that

$$B_{i,\varepsilon}\left(x\right)^2 = \gamma_{i,\varepsilon}^2 - 2t_\varepsilon\left(x\right) + \frac{t_\varepsilon(x)^2}{\gamma_{i,\varepsilon}^2} + O(1) \leq \gamma_{i,\varepsilon}^2 - \frac{7}{4} t_\varepsilon(x) + O(1)$$

so that

$$e^{B_{i,\varepsilon}(x)^2} \leq e^{\gamma_{i,\varepsilon}^2} \left(1 + \frac{|x - x_{i,\varepsilon}|^2}{4\mu_{i,\varepsilon}^2} \right)^{-\frac{7}{4}} .$$

Thus we can write that

$$\begin{aligned}
0 \leq \quad & \lambda_\varepsilon \int_{\Omega_{0,\varepsilon} \setminus \mathbb{D}_{x_{i,\varepsilon}}(R_\varepsilon \mu_{i,\varepsilon})} f_\varepsilon(x) \left(u_\varepsilon(x) e^{u_\varepsilon(x)^2} - w_\varepsilon(x) e^{w_\varepsilon(x)^2} \right) dx \\
& \leq C \gamma_{i,\varepsilon}^{-1} \mu_{i,\varepsilon}^{-2} \int_{\Omega \setminus \mathbb{D}_{x_{i,\varepsilon}}(R_\varepsilon \mu_{i,\varepsilon})} \left(1 + \frac{|x - x_{i,\varepsilon}|^2}{4\mu_{i,\varepsilon}^2} \right)^{-\frac{7}{4}} dx = o\left(\gamma_{i,\varepsilon}^{-1}\right) .
\end{aligned}$$

In $\Omega_{1,\varepsilon}$, we write that

$$e^{B_{i,\varepsilon}(x)^2} \leq e^{\gamma_{i,\varepsilon}^2 - \frac{1}{4}\gamma_{i,\varepsilon}} e^{-t_\varepsilon(x)}$$

so that

$$\begin{aligned} 0 \quad &\leq \quad \lambda_\varepsilon \int_{\Omega_{1,\varepsilon}} f_\varepsilon(x) \left(u_\varepsilon(x) e^{u_\varepsilon(x)^2} - w_\varepsilon(x) e^{w_\varepsilon(x)^2} \right) dx \\ &\leq C \mu_{i,\varepsilon}^{-2} e^{-\frac{1}{4}\gamma_{i,\varepsilon}} \gamma_{i,\varepsilon}^{-1} \int_{\Omega_{1,\varepsilon}} \left(1 + \frac{|x - x_{i,\varepsilon}|^2}{4\mu_{i,\varepsilon}^2} \right)^{-1} dx \\ &\leq C e^{-\frac{1}{4}\gamma_{i,\varepsilon}} \gamma_{i,\varepsilon}^{-1} \ln \frac{s_{i,\varepsilon}}{\mu_{i,\varepsilon}} = o\left(\gamma_{i,\varepsilon}^{-2} \right) \end{aligned}$$

since $2 \ln \frac{s_{i,\varepsilon}}{\mu_{i,\varepsilon}} = \gamma_{i,\varepsilon}^2 + O(1)$ thanks to Claim 4.4. At last, in $\Omega_{2,\varepsilon}$, we have that $B_{i,\varepsilon} = O(1)$ so that

$$0 \leq \lambda_\varepsilon \int_{\Omega_{2,\varepsilon}} f_\varepsilon(x) \left(u_\varepsilon(x) e^{u_\varepsilon(x)^2} - w_\varepsilon(x) e^{w_\varepsilon(x)^2} \right) dx \leq C \lambda_\varepsilon s_{i,\varepsilon}^2 = O\left(\gamma_{i,\varepsilon}^{-2} \right)$$

thanks to Claim 4.4. Combining all these estimates clearly proves Step 1. ♠

We shall now estimate the A_i's involved in (4.43) and defined in (4.44). We write since $u_\varepsilon \geq w_\varepsilon$ and thanks to (4.38) that

$$|A_{i,\varepsilon}| \leq C \lambda_\varepsilon \int_{\mathbb{D}_{x_{i,\varepsilon}}(s_{i,\varepsilon})} |\mathcal{G}(x_\varepsilon, x) - \mathcal{G}(x_\varepsilon, x_{i,\varepsilon})| \left(B_{i,\varepsilon} + C\gamma_{i,\varepsilon}^{-1} \right) e^{B_{i,\varepsilon}(x)^2} dx \, . \tag{4.46}$$

Step 2 - *Assume that $d_\varepsilon = d(x_\varepsilon, \partial\Omega) \geq d$ for some $d > 0$. Then we have that*

$$|A_{i,\varepsilon}| \leq C \gamma_{i,\varepsilon}^{-1} \left(\frac{\mu_{i,\varepsilon}}{|x_{i,\varepsilon} - x_\varepsilon|} + \gamma_{i,\varepsilon}^{-1} \ln \left(\frac{s_{i,\varepsilon}}{|x_{i,\varepsilon} - x_\varepsilon|} + 2 \right) \right) \, .$$

Proof of Step 2 - We use (6.1) to write that

$$|\mathcal{G}(x_\varepsilon, x) - \mathcal{G}(x_\varepsilon, x_{i,\varepsilon})| \leq \frac{1}{2\pi} \left| \ln \frac{|x_{i,\varepsilon} - x_\varepsilon|}{|x_\varepsilon - x|} \right| + C |x - x_{i,\varepsilon}| \, .$$

Thus we have thanks to (4.46) that

$$|A_{i,\varepsilon}| \leq C \lambda_\varepsilon \int_{\mathbb{D}_{x_{i,\varepsilon}}(s_{i,\varepsilon})} \left(\left| \ln \frac{|x_{i,\varepsilon} - x_\varepsilon|}{|x_\varepsilon - x|} \right| + |x_{i,\varepsilon} - x| \right) \left(B_{i,\varepsilon} + C\gamma_{i,\varepsilon}^{-1} \right) e^{B_{i,\varepsilon}(x)^2} dx \, .$$

In $\Omega_{0,\varepsilon}$, we have that

$$B_{i,\varepsilon}(x) \leq \gamma_{i,\varepsilon}^2 - \frac{7}{4} \ln \left(1 + \frac{|x_{i,\varepsilon} - x|^2}{4\mu_{i,\varepsilon}^2} \right)$$

so that

$$\begin{aligned} &\lambda_\varepsilon \int_{\Omega_{0,\varepsilon}} \left(\left| \ln \frac{|x_{i,\varepsilon} - x_\varepsilon|}{|x_\varepsilon - x|} \right| + |x_{i,\varepsilon} - x| \right) \left(B_{i,\varepsilon} + C\gamma_{i,\varepsilon}^{-1} \right) e^{B_{i,\varepsilon}(x)^2} dx \\ &\leq C \mu_{i,\varepsilon}^{-2} \gamma_{i,\varepsilon}^{-1} \int_{\Omega_{0,\varepsilon}} \left(\left| \ln \frac{|x_{i,\varepsilon} - x_\varepsilon|}{|x_\varepsilon - x|} \right| + |x_{i,\varepsilon} - x| \right) \left(1 + \frac{|x_{i,\varepsilon} - x|^2}{4\mu_{i,\varepsilon}^2} \right)^{-\frac{7}{4}} dx \, . \end{aligned}$$

This leads after simple computations, since $\frac{|x_{i,\varepsilon} - x_\varepsilon|}{\mu_{i,\varepsilon}} \to +\infty$, as $\varepsilon \to 0$ to

$$\lambda_\varepsilon \int_{\Omega_{0,\varepsilon}} \left(\left| \ln \frac{|x_{i,\varepsilon} - x_\varepsilon|}{|x_\varepsilon - x|} \right| + |x_{i,\varepsilon} - x| \right) \left(B_{i,\varepsilon} + C\gamma_{i,\varepsilon}^{-1} \right) e^{B_{i,\varepsilon}(x)^2} dx \leq C \gamma_{i,\varepsilon}^{-1} \frac{\mu_{i,\varepsilon}}{|x_{i,\varepsilon} - x_\varepsilon|} \, .$$

In $\Omega_{1,\varepsilon}$, we can write that

$$e^{B_{i,\varepsilon}(x)^2} \leq e^{\gamma_{i,\varepsilon}^2 - \frac{1}{4}\gamma_{i,\varepsilon}} \left(1 + \frac{|x - x_{i,\varepsilon}|^2}{4\mu_{i,\varepsilon}^2}\right)^{-1}$$

so that

$$\begin{aligned}
&\lambda_\varepsilon \int_{\Omega_{1,\varepsilon}} \left(\left|\ln \frac{|x_{i,\varepsilon} - x_\varepsilon|}{|x_\varepsilon - x|}\right| + |x_{i,\varepsilon} - x|\right) \left(B_{i,\varepsilon} + C\gamma_{i,\varepsilon}^{-1}\right) e^{B_{i,\varepsilon}(x)^2}\, dx \\
&\leq C\gamma_{i,\varepsilon}^{-1} e^{-\frac{1}{4}\gamma_{i,\varepsilon}} \int_{\Omega_{1,\varepsilon}} \left(\left|\ln \frac{|x_{i,\varepsilon} - x_\varepsilon|}{|x_\varepsilon - x|}\right| + |x_{i,\varepsilon} - x|\right) |x - x_{i,\varepsilon}|^{-2}\, dx \\
&\leq C\gamma_{i,\varepsilon}^{-1} e^{-\frac{1}{4}\gamma_{i,\varepsilon}} \left(\ln \frac{r_{2,\varepsilon}}{\mu_{i,\varepsilon}}\right)^2
\end{aligned}$$

where $t_{i,\varepsilon}(r_{2,\varepsilon}) = t_{2,\varepsilon}$. We have that

$$\ln \frac{r_{2,\varepsilon}}{\mu_{i,\varepsilon}} \leq C\gamma_\varepsilon^2$$

so that

$$\lambda_\varepsilon \int_{\Omega_{1,\varepsilon}} \left(\left|\ln \frac{|x_{i,\varepsilon} - x_\varepsilon|}{|x_\varepsilon - x|}\right| + |x_{i,\varepsilon} - x|\right) \left(B_{i,\varepsilon} + C\gamma_{i,\varepsilon}^{-1}\right) e^{B_{i,\varepsilon}(x)^2}\, dx = O\left(\gamma_{i,\varepsilon}^3 e^{-\frac{1}{4}\gamma_{i,\varepsilon}}\right) = o\left(\gamma_{i,\varepsilon}^{-2}\right)\ .$$

At last, in $\Omega_{2,\varepsilon}$, we have that $B_{i,\varepsilon} = O(1)$ so that

$$\begin{aligned}
&\lambda_\varepsilon \int_{\Omega_{2,\varepsilon}} \left(\left|\ln \frac{|x_{i,\varepsilon} - x_\varepsilon|}{|x_\varepsilon - x|}\right| + |x_{i,\varepsilon} - x|\right) \left(B_{i,\varepsilon} + C\gamma_{i,\varepsilon}^{-1}\right) e^{B_{i,\varepsilon}(x)^2}\, dx \\
&\leq \lambda_\varepsilon \int_{\Omega_{2,\varepsilon}} \left(\left|\ln \frac{|x_{i,\varepsilon} - x_\varepsilon|}{|x_\varepsilon - x|}\right| + |x_{i,\varepsilon} - x|\right) dx \\
&\leq \lambda_\varepsilon s_{i,\varepsilon}^2 \ln\left(\frac{s_{i,\varepsilon}}{|x_{i,\varepsilon} - x_\varepsilon|} + 2\right) \leq C\gamma_{i,\varepsilon}^{-2} \ln\left(\frac{s_{i,\varepsilon}}{|x_{i,\varepsilon} - x_\varepsilon|} + 2\right)
\end{aligned}$$

by direct computations and Claim 4.4. Combining the above estimates gives Step 2. ♠

Step 3 - *Assume now that $d_\varepsilon = d(x_\varepsilon, \partial\Omega) \to 0$ as $\varepsilon \to 0$ and that $|x_{i,\varepsilon} - x_\varepsilon| \geq s_{i,\varepsilon} + \delta d_\varepsilon$ for some $\delta > 0$. Then we have that*

$$|A_{i,\varepsilon}| \leq C \frac{d_\varepsilon}{d_\varepsilon + d_{i,\varepsilon}} \left(\gamma_{i,\varepsilon}^{-1}\mu_{i,\varepsilon} + \gamma_{i,\varepsilon}^{-2} s_{i,\varepsilon}\right)\ .$$

Proof of Step 3 - In this setting, we can apply (6.12) to write that

$$|\mathcal{G}(x_\varepsilon, x) - \mathcal{G}(x_\varepsilon, x_{i,\varepsilon})| \leq C \frac{d_\varepsilon}{d_\varepsilon + d_{i,\varepsilon}} |x - x_{i,\varepsilon}|$$

so that

$$|A_{i,\varepsilon}| \leq C\lambda_\varepsilon \frac{d_\varepsilon}{d_\varepsilon + d_{i,\varepsilon}} \int_{\mathbb{D}_{x_{i,\varepsilon}}(s_{i,\varepsilon})} |x - x_{i,\varepsilon}| \left(B_{i,\varepsilon} + C\gamma_{i,\varepsilon}^{-1}\right) e^{B_{i,\varepsilon}^2}\, dx$$

thanks to (4.44). In $\Omega_{0,\varepsilon}$, we have that

$$\left(B_{i,\varepsilon} + C\gamma_{i,\varepsilon}^{-1}\right) e^{B_{i,\varepsilon}^2} \leq C\gamma_{i,\varepsilon} e^{\gamma_{i,\varepsilon}^2} \left(1 + \frac{|x - x_{i,\varepsilon}|^2}{4\mu_{i,\varepsilon}^2}\right)^{-\frac{7}{4}}$$

so that

$$\lambda_\varepsilon \int_{\Omega_{0,\varepsilon}} |x-x_{i,\varepsilon}| \left(B_{i,\varepsilon}+C\gamma_{i,\varepsilon}^{-1}\right) e^{B_{i,\varepsilon}^2}\, dx$$
$$\leq C\mu_{i,\varepsilon}^{-2}\gamma_{i,\varepsilon}^{-1} \int_{\mathbb{D}_{x_{i,\varepsilon}}(s_{i,\varepsilon})} |x-x_{i,\varepsilon}| \left(1+\frac{|x-x_{i,\varepsilon}|^2}{4\mu_{i,\varepsilon}^2}\right)^{-\frac{7}{4}} dx$$
$$\leq C\mu_{i,\varepsilon}\gamma_{i,\varepsilon}^{-1}\ .$$

In $\Omega_{1,\varepsilon}$, we have that

$$\left(B_{i,\varepsilon}+C\gamma_{i,\varepsilon}^{-1}\right) e^{B_{i,\varepsilon}(x)^2} \leq \gamma_{i,\varepsilon} e^{\gamma_{i,\varepsilon}^2-\frac{1}{4}\gamma_{i,\varepsilon}} \left(1+\frac{|x-x_{i,\varepsilon}|^2}{4\mu_{i,\varepsilon}^2}\right)^{-1}$$

so that

$$\lambda_\varepsilon \int_{\Omega_{1,\varepsilon}} |x-x_{i,\varepsilon}| \left(B_{i,\varepsilon}+C\gamma_{i,\varepsilon}^{-1}\right) e^{B_{i,\varepsilon}^2}\, dx$$
$$\leq C\mu_{i,\varepsilon}^{-2}\gamma_{i,\varepsilon}^{-1} e^{-\frac{1}{4}\gamma_{i,\varepsilon}} \int_{\mathbb{D}_{x_{i,\varepsilon}}(s_{i,\varepsilon})} |x-x_{i,\varepsilon}| \left(1+\frac{|x-x_{i,\varepsilon}|^2}{4\mu_{i,\varepsilon}^2}\right)^{-1} dx$$
$$\leq \gamma_{i,\varepsilon}^{-1} e^{-\frac{1}{4}\gamma_{i,\varepsilon}} s_{i,\varepsilon}\ .$$

At last, in $\Omega_{2,\varepsilon}$, we have that $B_{i,\varepsilon}=O(1)$ so that

$$\lambda_\varepsilon \int_{\Omega_{2,\varepsilon}} |x-x_{i,\varepsilon}| \left(B_{i,\varepsilon}+C\gamma_{i,\varepsilon}^{-1}\right) e^{B_{i,\varepsilon}^2}\, dx \leq \lambda_\varepsilon s_{i,\varepsilon}^3\ .$$

Combining the above estimates with Claim 4.4, we get the estimate of Step 3. ♠

Step 4 - *Assume now that* $d_\varepsilon = d(x_\varepsilon, \partial\Omega) \to 0$ *as* $\varepsilon \to 0$ *and that* $|x_{i,\varepsilon}-x_\varepsilon| \leq s_{i,\varepsilon} + o(d_\varepsilon)$. *Then we have that*

$$|A_{i,\varepsilon}| \leq C\gamma_{i,\varepsilon}^{-1} \left(\frac{\mu_{i,\varepsilon}}{|x_{i,\varepsilon}-x_\varepsilon|} + \gamma_{i,\varepsilon}^{-1} \ln\left(\frac{s_{i,\varepsilon}}{|x_{i,\varepsilon}-x_\varepsilon|}+2\right)\right)\ .$$

Proof of Step 4 - Let us remark that in this case, we necessarily have that

$$d_\varepsilon \leq |x_{i,\varepsilon}-x_\varepsilon| + d_{i,\varepsilon} \leq s_{i,\varepsilon} + d_{i,\varepsilon} + o(d_\varepsilon) \leq \frac{3}{2} d_{i,\varepsilon} + o(d_\varepsilon)$$

so that $d_\varepsilon = O(d_{i,\varepsilon})$. This leads in turn to $|x_{i,\varepsilon}-x_\varepsilon| \leq s_{i,\varepsilon} + o(d_{i,\varepsilon})$. And then we can write that

$$d_{i,\varepsilon} \leq d_\varepsilon + |x_{i,\varepsilon}-x_\varepsilon| \leq s_{i,\varepsilon} + o(d_{i,\varepsilon}) + d_\varepsilon \leq \frac{1}{2} d_{i,\varepsilon} + d_\varepsilon + o(d_{i,\varepsilon})$$

so that $d_{i,\varepsilon} = O(d_\varepsilon)$. Thanks to (6.12), we can write that

$$|\mathcal{G}(x_\varepsilon, x) - \mathcal{G}(x_\varepsilon, x_{i,\varepsilon})| \leq C\frac{|x-x_{i,\varepsilon}|}{d_{i,\varepsilon}} + C\left|\ln\frac{|x_{i,\varepsilon}-x_\varepsilon|}{|x_\varepsilon-x|}\right|$$

so that the computations of Step 2 lead to the result of Step 4. ♠

Of course, the combination of Steps 1 to 4 gives the estimate of the claim. ♢

Claim 4.9. *There exists* $C>0$ *such that*

$$|\nabla(u_\varepsilon-\psi_\varepsilon)(x)| \leq C\sum_{i=1}^N \gamma_{i,\varepsilon}^{-1} \left(\mu_{i,\varepsilon}+|x-x_{i,\varepsilon}|\right)^{-1}$$

where ψ_ε *is as in Claim 4.8.*

Proof of Claim 4.9 - We use again the Green representation formula with equation (4.1) (together with the equation satisfied by ψ_ε, see Claim 4.8) to write that

$$|\nabla (u_\varepsilon - \psi_\varepsilon)(x)| \leq \lambda_\varepsilon \int_\Omega |\nabla \mathcal{G}(x,y)| f_\varepsilon(y) \left(u_\varepsilon(y) e^{u_\varepsilon(y)^2} - w_\varepsilon(y) e^{w_\varepsilon(y)^2} \right) dy \, .$$

Thanks to standard estimates on the Green function and to the definition (4.40), this leads to

$$|\nabla (u_\varepsilon - \psi_\varepsilon)(x)| \leq C \sum_{i=1}^N \lambda_\varepsilon \int_{\mathbb{D}_{x_{i,\varepsilon}}(s_{i,\varepsilon})} |x-y|^{-1} u_\varepsilon(y) e^{u_\varepsilon(y)^2} dy \, . \tag{4.47}$$

Thanks to (4.38), we have that

$$\begin{aligned} &\lambda_\varepsilon \int_{\mathbb{D}_{x_{i,\varepsilon}}(s_{i,\varepsilon})} |x-y|^{-1} u_\varepsilon(y) e^{u_\varepsilon(y)^2} dy \\ &\leq C\lambda_\varepsilon \sum_{k=0,1,2} \int_{\Omega_{k,\varepsilon}} |x-y|^{-1} \left(B_{i,\varepsilon}(y) + C_i \gamma_{i,\varepsilon}^{-1} \right) e^{B_{i,\varepsilon}(y)^2} dy \end{aligned} \tag{4.48}$$

where the $\Omega_{\alpha,\varepsilon}$'s are as in (4.45). In $\Omega_{0,\varepsilon}$, we write that

$$\left(B_{i,\varepsilon} + C_i \gamma_{i,\varepsilon}^{-1} \right)(y) e^{B_{i,\varepsilon}(y)^2} \leq C\gamma_{i,\varepsilon} e^{\gamma_{i,\varepsilon}^2} \left(1 + \frac{|y - x_{i,\varepsilon}|^2}{4\mu_{i,\varepsilon}^2} \right)^{-\frac{7}{4}}$$

so that

$$\begin{aligned} &\lambda_\varepsilon \int_{\Omega_{0,\varepsilon}} |x-y|^{-1} \left(B_{i,\varepsilon}(y) + C_i \gamma_{i,\varepsilon}^{-1} \right) e^{B_{i,\varepsilon}(y)^2} dy \\ &\leq C\mu_{i,\varepsilon}^{-2} \gamma_{i,\varepsilon}^{-1} \int_{\Omega_{0,\varepsilon}} |x-y|^{-1} \left(1 + \frac{|y - x_{i,\varepsilon}|^2}{4\mu_{i,\varepsilon}^2} \right)^{-\frac{7}{4}} dy \, . \end{aligned}$$

Direct computations give that

$$\lambda_\varepsilon \int_{\Omega_{0,\varepsilon}} |x-y|^{-1} \left(B_{i,\varepsilon}(y) + C_i \gamma_{i,\varepsilon}^{-1} \right) e^{B_{i,\varepsilon}(y)^2} dy \leq C\gamma_{i,\varepsilon}^{-1} \left(\mu_{i,\varepsilon} + |x - x_{i,\varepsilon}| \right)^{-1} \, . \tag{4.49}$$

In $\Omega_{1,\varepsilon}$, we write that

$$\left(B_{i,\varepsilon} + C_i \gamma_{i,\varepsilon}^{-1} \right)(y) e^{B_{i,\varepsilon}(y)^2} \leq C\gamma_{i,\varepsilon} e^{\gamma_{i,\varepsilon}^2} e^{\frac{t_\varepsilon(y)^2}{\gamma_{i,\varepsilon}^2} - 2t_\varepsilon(y)}$$

so that

$$\lambda_\varepsilon \int_{\Omega_{1,\varepsilon}} |x-y|^{-1} \left(B_{i,\varepsilon}(y) + C_i \gamma_{i,\varepsilon}^{-1} \right) e^{B_{i,\varepsilon}(y)^2} dy \leq C\gamma_{i,\varepsilon}^{-1} \mu_{i,\varepsilon}^{-2} \int_{\Omega_{1,\varepsilon}} |x-y|^{-1} e^{\frac{t_\varepsilon(y)^2}{\gamma_{i,\varepsilon}^2} - 2t_\varepsilon(y)} \, .$$

In $\Omega_{1,\varepsilon}$, we have that

$$\frac{t_\varepsilon(y)^2}{\gamma_{i,\varepsilon}^2} - 2t_\varepsilon(y) \leq -t_\varepsilon(y) - \frac{1}{4}\gamma_{i,\varepsilon}$$

so that

$$\begin{aligned} &\lambda_\varepsilon \int_{\Omega_{1,\varepsilon}} |x-y|^{-1} \left(B_{i,\varepsilon}(y) + C_i \gamma_{i,\varepsilon}^{-1} \right) e^{B_{i,\varepsilon}(y)^2} dy \\ &\leq C\gamma_{i,\varepsilon}^{-1} e^{-\frac{1}{4}\gamma_{i,\varepsilon}} \int_{\Omega_{1,\varepsilon}} |x-y|^{-1} \left(\mu_{i,\varepsilon}^2 + |y - x_{i,\varepsilon}|^2 \right)^{-1} dy \, . \end{aligned}$$

In $\Omega_{1,\varepsilon}$ we have that $|y-x_{i,\varepsilon}| \geq \mu_{i,\varepsilon}$ so that

$$\lambda_\varepsilon \int_{\Omega_{1,\varepsilon}} |x-y|^{-1} \left(B_{i,\varepsilon}(y) + C_i\gamma_{i,\varepsilon}^{-1}\right) e^{B_{i,\varepsilon}(y)^2}\, dy \leq C\gamma_{i,\varepsilon}^{-1} e^{-\frac{1}{4}\gamma_{i,\varepsilon}} \int_{\Omega_{1,\varepsilon}} |x-y|^{-1} |y-x_{i,\varepsilon}|^{-2}\, dy\,.$$

Noting that $\mathbb{D}_{x_{i,\varepsilon}}(r_{1,\varepsilon}) \cap \Omega_{1,\varepsilon} = \emptyset$ for ε small where

$$r_{1,\varepsilon} = \mu_{i,\varepsilon} e^{\frac{1}{8}\gamma_{i,\varepsilon}^2}\,,$$

we get by direct computations that

$$\begin{aligned}\lambda_\varepsilon &\int_{\Omega_{1,\varepsilon}} |x-y|^{-1} \left(B_{i,\varepsilon}(y) + C_i\gamma_{i,\varepsilon}^{-1}\right) e^{B_{i,\varepsilon}(y)^2}\, dy \\ &\leq C\gamma_{i,\varepsilon}^{-1} e^{-\frac{1}{4}\gamma_{i,\varepsilon}} \left(|x-x_{i,\varepsilon}| + r_{1,\varepsilon}\right)^{-1} \ln\left(2 + \frac{|x-x_{i,\varepsilon}|}{r_{1,\varepsilon}}\right)\,.\end{aligned}$$

Thanks to the value of $r_{1,\varepsilon}$, this leads to

$$\lambda_\varepsilon \int_{\Omega_{1,\varepsilon}} |x-y|^{-1} \left(B_{i,\varepsilon}(y) + C_i\gamma_{i,\varepsilon}^{-1}\right) e^{B_{i,\varepsilon}(y)^2}\, dy = o\left(\gamma_{i,\varepsilon}^{-1} \left(\mu_{i,\varepsilon} + |x-x_{i,\varepsilon}|\right)^{-1}\right)\,. \tag{4.50}$$

At last, in $\Omega_{2,\varepsilon}$, we have that $B_\varepsilon(y) = O(1)$ so that

$$\lambda_\varepsilon \int_{\Omega_{2,\varepsilon}} |x-y|^{-1} \left(B_{i,\varepsilon}(y) + C_i\gamma_{i,\varepsilon}^{-1}\right) e^{B_{i,\varepsilon}(y)^2}\, dy \leq C\lambda_\varepsilon \int_{\Omega_{2,\varepsilon}} |x-y|^{-1}\, dy \leq C\lambda_\varepsilon \frac{s_{i,\varepsilon}^2}{s_{i,\varepsilon} + |x-x_{i,\varepsilon}|}\,.$$

Thanks to Claim 4.4, this leads to

$$\lambda_\varepsilon \int_{\Omega_{2,\varepsilon}} |x-y|^{-1} \left(B_{i,\varepsilon}(y) + C_i\gamma_{i,\varepsilon}^{-1}\right) e^{B_{i,\varepsilon}(y)^2}\, dy \leq C\gamma_{i,\varepsilon}^{-2} \left(s_{i,\varepsilon} + |x-x_{i,\varepsilon}|\right)^{-1}\,. \tag{4.51}$$

Coming back to (4.47) with (4.48), (4.49), (4.50) and (4.51), we obtain the claim. ♢

Let us reorder the concentration points in a suitable way. For this purpose, we notice that, up to a subsequence, for any $i,j \in \{1,\dots,N\}$, there exists $C_{i,j}$, possibly 0 or $+\infty$ (but nonnegative) such that

$$\lim_{\varepsilon\to 0} \frac{\gamma_{i,\varepsilon}}{\gamma_{j,\varepsilon}} = C_{i,j}\,. \tag{4.52}$$

Note that $C_{i,j} = C_{j,i}^{-1}$ (with obvious conventions when $C_{i,j} = 0$ or $+\infty$). Then there exists $\tilde{C} \geq 1$ such that

$$\text{for any } i,j \in \{1,\dots,N\}\,,\ \text{either } C_{i,j} = 0 \text{ or } C_{i,j} = +\infty \text{ or } \frac{1}{\tilde{C}} \leq C_{i,j} \leq \tilde{C}\,. \tag{4.53}$$

It is then easily checked that we can order the concentration points in such a way that

$$\text{for any } i,j \in \{1,\dots,N\}\,,\ i<j \Rightarrow C_{i,j} < +\infty \tag{4.54}$$

and

$$\text{for any } i,j \in \{1,\dots,N\}\,,\ i<j \text{ and } C_{i,j} > 0 \Rightarrow r_{i,\varepsilon} \leq r_{j,\varepsilon}\,. \tag{4.55}$$

Let us give some estimates on ψ_ε, involved in Claims 4.8 and 4.9. Using Claim 4.7, we clearly have that $\lambda_\varepsilon^{-1}\Delta\psi_\varepsilon$ is uniformly bounded in any $L^p(\Omega)$ thanks to Trudinger-Moser inequality. Thus we know that there exists $C>0$ such that

$$\|\psi_\varepsilon\|_{\mathcal{C}^{1,\alpha}(\overline{\Omega})} \leq C\lambda_\varepsilon \tag{4.56}$$

for $0<\alpha<1$ by standard elliptic theory. Now, if $\lambda_\varepsilon \to 0$, we know that $u_0 \equiv 0$ and we can be a little bit more precise. Indeed,

$$\|\Delta\psi_\varepsilon\|_{L^p(\Omega)} \leq \lambda_\varepsilon \|f_\varepsilon\|_{L^\infty(\Omega)} \left\|w_\varepsilon e^{w_\varepsilon^2}\right\|_{L^p(\Omega)} \leq \lambda_\varepsilon \|f_\varepsilon\|_{L^\infty(\Omega)} \|u_\varepsilon\|_{L^{2p}(\Omega)} \left\|e^{w_\varepsilon^2}\right\|_{L^{2p}(\Omega)}\,.$$

Since $u_0 \equiv 0$, we know thanks to Claim 4.7 and to Trudinger-Moser inequality that $\left(e^{w_\varepsilon^2}\right)$ is bounded in any L^q. Thus we have that

$$\|\Delta\psi_\varepsilon\|_{L^p(\Omega)} \leq C\lambda_\varepsilon \|u_\varepsilon\|_{L^{2p}(\Omega)}$$

thanks to (1.5). Using Claim (4.9), we get that

$$\|\nabla (u_\varepsilon - \psi_\varepsilon)\|_{L^q(\Omega)} \leq \frac{C_q}{\gamma_{1,\varepsilon}}$$

for some $C_q > 0$ for all $1 \leq q < 2$. Remember that concentration points are ordered such that (4.54) holds. This gives that

$$\|u_\varepsilon\|_{L^{2p}(\Omega)} \leq C_p \left(\gamma_{1,\varepsilon}^{-1} + \|\nabla\psi_\varepsilon\|_{\mathcal{C}^1(\overline{\Omega})}\right)$$

so that

$$\|\Delta\psi_\varepsilon\|_{L^p(\Omega)} \leq C\lambda_\varepsilon \left(\gamma_{1,\varepsilon}^{-1} + \|\nabla\psi_\varepsilon\|_{\mathcal{C}^1(\overline{\Omega})}\right) .$$

By standard elliptic theory and since we assumed that $\lambda_\varepsilon \to 0$, we finally obtain that

$$\text{if } \lambda_\varepsilon \to 0 \text{ as } \varepsilon \to 0, \text{ then } \|\psi_\varepsilon\|_{\mathcal{C}^{1,\alpha}(\overline{\Omega})} \leq C\frac{\lambda_\varepsilon}{\gamma_{1,\varepsilon}} . \tag{4.57}$$

Claim 4.10. *We have that $r_{1,\varepsilon} \geq \delta_0$ for some $\delta_0 > 0$.*

Proof of Claim 4.10 - We assume by contradiction that $r_{1,\varepsilon} \to 0$ as $\varepsilon \to 0$. We let in the following

$$\mathcal{D}_1^\star = \{i \in \{2, \dots, N\} \text{ s.t. } |x_{i,\varepsilon} - x_{1,\varepsilon}| = O(r_{1,\varepsilon})\} \text{ and } \mathcal{D}_1 = \mathcal{D}_1^\star \cup \{1\} . \tag{4.58}$$

After passing to a subsequence, we let

$$\mathcal{S}_1^\star = \left\{\tilde{x}_i = \lim_{\varepsilon\to 0} \frac{x_{i,\varepsilon} - x_{1,\varepsilon}}{r_{1,\varepsilon}},\ i \in \mathcal{D}_1^\star\right\} \text{ and } \mathcal{S}_1 = \mathcal{S}_1^\star \cup \{\tilde{x}_1 = 0\} . \tag{4.59}$$

We also let

$$\Omega_{1,\varepsilon} = \left\{y \in \mathbb{R}^2 \text{ s.t. } x_{1,\varepsilon} + r_{1,\varepsilon} y \in \Omega\right\} . \tag{4.60}$$

Note that, after passing to a subsequence (and up to a harmless rotation if necessary), we have that

$$\Omega_{1,\varepsilon} \to \Omega_0 \text{ as } \varepsilon \to 0 \text{ where } \begin{cases} \Omega_0 = \mathbb{R}^2 & \text{if } \dfrac{d_{1,\varepsilon}}{r_{1,\varepsilon}} \to +\infty \text{ as } \varepsilon \to 0 \\ \Omega_0 = \mathbb{R} \times (-\infty, L) & \text{if } \dfrac{d_{1,\varepsilon}}{r_{1,\varepsilon}} \to L \text{ as } \varepsilon \to 0 \end{cases} \tag{4.61}$$

Here $d_{1,\varepsilon} = d(x_{1,\varepsilon}, \partial\Omega)$, as defined in (4.8). For $R > 0$, we shall also let

$$\Omega_0^R = (\Omega_0 \cap \mathbb{D}_0(R)) \setminus \bigcup_{i\in\mathcal{D}_1} \mathbb{D}_{\tilde{x}_i}\left(\frac{1}{R}\right) . \tag{4.62}$$

We shall distinguish three cases, depending on the behaviour of $d_{1,\varepsilon} = d(x_{1,\varepsilon}, \partial\Omega)$ and $r_{1,\varepsilon}$.

Case 1 - We assume that $d_{1,\varepsilon} \not\to 0$ as $\varepsilon \to 0$, meaning that, after passing to a subsequence, $x_{1,\varepsilon} \to x_1$ as $\varepsilon \to 0$ with $x_1 \in \Omega$.

We let $y \in \Omega_0^R$ for some $R > 0$ and we set $x_\varepsilon = x_{1,\varepsilon} + r_{1,\varepsilon} y$. Since $d_{1,\varepsilon} \not\to 0$ and $r_{1,\varepsilon} \to 0$, we are in situation a) of Claim 4.8. Note indeed that

$$\frac{|x_\varepsilon - x_{i,\varepsilon}|}{\mu_{i,\varepsilon}} \to +\infty \text{ as } \varepsilon \to 0 \text{ for all } i = 1, \dots, N .$$

It is obvious if $i \in \mathcal{D}_1$ since we clearly have in this case

$$\frac{|x_\varepsilon - x_{i,\varepsilon}|}{\mu_{i,\varepsilon}} = \frac{|x_\varepsilon - x_{i,\varepsilon}|}{r_{1,\varepsilon}} \frac{r_{1,\varepsilon}}{r_{i,\varepsilon}} \frac{r_{i,\varepsilon}}{\mu_{i,\varepsilon}}$$

with $\frac{|x_\varepsilon - x_{i,\varepsilon}|}{r_{1,\varepsilon}} \geq R^{-1} + o(1)$, $\frac{r_{1,\varepsilon}}{r_{i,\varepsilon}} \geq 2\,|\tilde{x}_i|^{-1} + o(1)$ for $i \in \mathcal{D}_1^\star$ and equal to 1 if $i = 1$, and $\frac{r_{i,\varepsilon}}{\mu_{i,\varepsilon}} \to +\infty$ as $\varepsilon \to 0$ thanks to assertion c) of Proposition 2.1. While, if $i \notin \mathcal{D}_1$, we can write that

$$\frac{|x_\varepsilon - x_{i,\varepsilon}|}{\mu_{i,\varepsilon}} \geq (1 + o(1)) \frac{|x_{i,\varepsilon} - x_{1,\varepsilon}|}{\mu_{i,\varepsilon}} \geq (2 + o(1)) \frac{r_{i,\varepsilon}}{\mu_{i,\varepsilon}} \to +\infty \text{ as } \varepsilon \to 0\,.$$

Thus, applying a) of Claim 4.8, we can write that

$$\begin{aligned} u_\varepsilon(x_\varepsilon) &= \psi_\varepsilon(x_\varepsilon) + \sum_{i=1}^N (4\pi + o(1))\,\gamma_{i,\varepsilon}^{-1} \mathcal{G}(x_{i,\varepsilon}, x_\varepsilon) \\ &\quad + O\left(\sum_{i=1}^N \gamma_{i,\varepsilon}^{-1}\left(\frac{\mu_{i,\varepsilon}}{|x_{i,\varepsilon} - x_\varepsilon|} + \gamma_{i,\varepsilon}^{-1} \ln\left(\frac{s_{i,\varepsilon}}{|x_{i,\varepsilon} - x_\varepsilon|} + 2\right)\right)\right)\,. \end{aligned}$$

Now, for any $i \in \{1, \dots, N\}$,

$$\gamma_{i,\varepsilon}^{-1} \frac{\mu_{i,\varepsilon}}{|x_{i,\varepsilon} - x_\varepsilon|} = o\left(\gamma_{i,\varepsilon}^{-1}\right) = o\left(\gamma_{1,\varepsilon}^{-1}\right)$$

thanks to (4.54) and

$$\gamma_{i,\varepsilon}^{-2} \ln\left(\frac{s_{i,\varepsilon}}{|x_{i,\varepsilon} - x_\varepsilon|} + 2\right) = o\left(\gamma_{1,\varepsilon}^{-1}\right)$$

thanks to the fact that $s_{i,\varepsilon} \leq r_{i,\varepsilon} = O(|x_{i,\varepsilon} - x_\varepsilon|)$. Note that Claim 4.4 implies that $\lambda_\varepsilon = O\left(\gamma_{1,\varepsilon}^{-2}\right)$ in our case so that (4.57) gives that

$$\psi_\varepsilon(x_\varepsilon) = O\left(\gamma_{1,\varepsilon}^{-3}\right) = o\left(\gamma_{1,\varepsilon}^{-1}\right)\,.$$

Thus we have that

$$u_\varepsilon(x_\varepsilon) = \sum_{i=1}^N (4\pi + o(1))\,\gamma_{i,\varepsilon}^{-1} \mathcal{G}(x_{i,\varepsilon}, x_\varepsilon) + o\left(\gamma_{1,\varepsilon}^{-1}\right)\,. \tag{4.63}$$

We can now use (6.3) to write that

$$\mathcal{G}(x_{i,\varepsilon}, x_\varepsilon) = \frac{1}{2\pi} \ln \frac{1}{r_{1,\varepsilon}} + O(1)$$

if $i \in \mathcal{D}_1$ and that

$$\mathcal{G}(x_{i,\varepsilon}, x_\varepsilon) = \mathcal{G}(x_{i,\varepsilon}, x_{1,\varepsilon}) + O(1)$$

if $i \notin \mathcal{D}_1$. Thus we have that

$$\begin{aligned} u_\varepsilon(x_\varepsilon) &= (2 + o(1))\,\gamma_{1,\varepsilon}^{-1} \ln \frac{1}{r_{1,\varepsilon}} \left(1 + \sum_{i \in \mathcal{D}_1^\star} C_{1,i}\right) \\ &\quad + \sum_{i \notin \mathcal{D}_1} (4\pi + o(1))\,\gamma_{i,\varepsilon}^{-1} \mathcal{G}(x_{i,\varepsilon}, x_{1,\varepsilon}) + O\left(\gamma_{1,\varepsilon}^{-1}\right)\,. \end{aligned} \tag{4.64}$$

Note that $C_{1,i} \leq \tilde{C}$ for all $i > 1$ thanks to (4.53). Thus we have in particular that

$$\gamma_{1,\varepsilon}^{-1} \ln \frac{1}{r_{1,\varepsilon}} \left(1 + \sum_{i \in \mathcal{D}_1^\star} C_{1,i} \right) \leq \left(\frac{1}{2} + o(1) \right) u_\varepsilon (x_\varepsilon) \leq \gamma_{1,\varepsilon}^{-1} \ln \frac{1}{r_{1,\varepsilon}} \left(1 + (N-1)\tilde{C} \right) . \tag{4.65}$$

Note that we also have thanks to Claim 4.9 and to (4.57) that

$$|\nabla u_\varepsilon(x)| \leq C \gamma_{1,\varepsilon}^{-1} |x_{1,\varepsilon} - x|^{-1} \text{ for all } x \in \mathbb{D}_{x_{1,\varepsilon}} (r_{1,\varepsilon}) . \tag{4.66}$$

We are thus in position to apply Claim 4.1 for $i = 1$ to write that, if $|x| = \frac{1}{2}$,

$$u_\varepsilon (x_{1,\varepsilon} + r_{1,\varepsilon} x) = B_{1,\varepsilon} (r_{1,\varepsilon}) + O \left(\gamma_{1,\varepsilon}^{-1} \right) .$$

Combined with (4.65), this gives that

$$(2 + o(1)) \gamma_{1,\varepsilon}^{-1} \ln \frac{1}{r_{1,\varepsilon}} \left(1 + \sum_{i \in \mathcal{D}_1^\star} C_{1,i} \right) \leq B_{1,\varepsilon} (r_{1,\varepsilon}) \leq (2 + o(1)) \gamma_{1,\varepsilon}^{-1} \ln \frac{1}{r_{1,\varepsilon}} \left(1 + (N-1)\tilde{C} \right) . \tag{4.67}$$

We write now thanks to Claim 5.2 of Appendix A that

$$B_{1,\varepsilon} (r_{1,\varepsilon}) = 2\gamma_{1,\varepsilon}^{-1} \ln \frac{1}{r_{1,\varepsilon}} - \gamma_{1,\varepsilon}^{-1} \ln \left(\lambda_\varepsilon \gamma_{1,\varepsilon}^2 \right) + O \left(\gamma_{1,\varepsilon}^{-1} \right) \tag{4.68}$$

to deduce that

$$(2 + o(1)) \gamma_{1,\varepsilon}^{-1} \ln \frac{1}{r_{1,\varepsilon}} \sum_{i \in \mathcal{D}_1^\star} C_{1,i} \leq -\gamma_{1,\varepsilon}^{-1} \ln \left(\lambda_\varepsilon \gamma_{1,\varepsilon}^2 \right) \leq \left(2(N-1)\tilde{C} + o(1) \right) \gamma_{1,\varepsilon}^{-1} \ln \frac{1}{r_{1,\varepsilon}} . \tag{4.69}$$

Fix now $i \in \mathcal{D}_1^\star$. It is clear that there exists $\delta > 0$ such that $\partial \mathbb{D}_{x_{i,\varepsilon}} (\delta r_{1,\varepsilon}) \subset \left\{ x_{1,\varepsilon} + r_{1,\varepsilon} y,\, y \in \Omega_0^R \right\}$ for some $R > 0$. Thus we can write that

$$\inf_{\partial \mathbb{D}_{x_{i,\varepsilon}} (\delta r_{1,\varepsilon})} u_\varepsilon \geq (2 + o(1)) \gamma_{1,\varepsilon}^{-1} \ln \frac{1}{r_{1,\varepsilon}} \left(1 + \sum_{i \in \mathcal{D}_1^\star} C_{1,i} \right)$$

thanks to (4.65). We can also apply b) of Claim 4.2 with $r_\varepsilon = \delta r_{1,\varepsilon}$ thanks to Claim 4.3 and to the fact that $\dfrac{r_{1,\varepsilon}}{r_{i,\varepsilon}} \geq 2 |\tilde{x}_i|^{-1} + o(1)$. This leads to

$$(2 + o(1)) \gamma_{1,\varepsilon}^{-1} \ln \frac{1}{r_{1,\varepsilon}} \left(1 + \sum_{i \in \mathcal{D}_1^\star} C_{1,i} \right) \leq B_{i,\varepsilon} (\delta r_{1,\varepsilon}) + o \left(\gamma_{i,\varepsilon}^{-1} \right) .$$

We have that

$$B_{i,\varepsilon} (\delta r_{1,\varepsilon}) = 2\gamma_{i,\varepsilon}^{-1} \ln \frac{1}{r_{1,\varepsilon}} - \gamma_{i,\varepsilon}^{-1} \ln \left(\lambda_\varepsilon \gamma_{i,\varepsilon}^2 \right) + O \left(\gamma_{i,\varepsilon}^{-1} \right) .$$

This leads together with (4.69) to

$$(2 + o(1)) \gamma_{1,\varepsilon}^{-1} \ln \frac{1}{r_{1,\varepsilon}} \left(1 + \sum_{i \in \mathcal{D}_1^\star} C_{1,i} \right) \leq \left(2(N-1)\tilde{C} + 2 + o(1) \right) \gamma_{i,\varepsilon}^{-1} \ln \frac{1}{r_{1,\varepsilon}} - \gamma_{i,\varepsilon}^{-1} \ln \frac{\gamma_{i,\varepsilon}^2}{\gamma_{1,\varepsilon}^2} .$$

This is clearly impossible if $C_{1,i} = 0$. Thus we have proved that

$$\text{for any } i \in \mathcal{D}_1^\star,\, C_{1,i} > 0 . \tag{4.70}$$

This implies thanks to (4.55) that $r_{i,\varepsilon} \geq r_{1,\varepsilon}$ for all $i \in \mathcal{D}_1^\star$. Then we can apply Claim 4.1 to all $i \in \mathcal{D}_1^\star$ thanks to Claim 4.9 and to what we just said to get that, for any $\tilde{x}_i \in \mathcal{D}_1$,

$$\gamma_{i,\varepsilon}\left(u_\varepsilon\left(x_{1,\varepsilon}+r_{1,\varepsilon}x\right)-B_{i,\varepsilon}\left(r_{1,\varepsilon}\right)\right) \to 2\ln\frac{1}{|x-\tilde{x}_i|}+\mathcal{H}_i \text{ in } \mathcal{C}^1_{loc}\left(\mathbb{D}_{\tilde{x}_i}\left(1\right)\setminus\{\tilde{x}_i\}\right) \text{ as } \varepsilon\to 0 \quad (4.71)$$

where $\mathcal{H}_i$ is some harmonic function in $\mathbb{D}_{\tilde{x}_i}(1)$ satisfying $\mathcal{H}_i(\tilde{x}_i)=0$ and $\nabla\mathcal{H}_i(\tilde{x}_i)=0$ (note here that we assumed that $r_{1,\varepsilon}\to 0$ as $\varepsilon\to 0$). Let us set now

$$v_\varepsilon(x)=\gamma_{1,\varepsilon}\left(u_\varepsilon\left(x_{1,\varepsilon}+r_{1,\varepsilon}x\right)-B_{1,\varepsilon}\left(r_{1,\varepsilon}\right)\right) .$$

Thanks to Claim 4.9, we have that

$$|\nabla v_\varepsilon| \leq C_R \text{ in } \Omega_0^R$$

for all $R>0$. This clearly proves that (v_ε) is uniformly bounded in any Ω_0^R. Since

$$\Delta v_\varepsilon=\lambda_\varepsilon r_{1,\varepsilon}^2\gamma_{1,\varepsilon}f_\varepsilon\left(x_{1,\varepsilon}+r_{1,\varepsilon}x\right)u_\varepsilon\left(x_{1,\varepsilon}+r_{1,\varepsilon}x\right)e^{u_\varepsilon\left(x_{1,\varepsilon}+r_{1,\varepsilon}x\right)^2}$$

in Ω_0^R, we have that

$$|\Delta v_\varepsilon|=O\left(\lambda_\varepsilon r_{1,\varepsilon}^2\gamma_{1,\varepsilon}\left(B_{1,\varepsilon}\left(r_{1,\varepsilon}\right)+\gamma_{1,\varepsilon}^{-1}\right)e^{B_{1,\varepsilon}\left(r_{1,\varepsilon}\right)^2}\right) \text{ in } \Omega_0^R .$$

Thanks to (4.68), we know that

$$\lambda_\varepsilon r_{1,\varepsilon}^2 \leq C\gamma_{1,\varepsilon}^{-2}e^{-\gamma_{1,\varepsilon}B_{1,\varepsilon}\left(r_{1,\varepsilon}\right)}$$

so that

$$|\Delta v_\varepsilon|=O\left(\gamma_{1,\varepsilon}^{-1}\left(B_{1,\varepsilon}\left(r_{1,\varepsilon}\right)+\gamma_{1,\varepsilon}^{-1}\right)e^{B_{1,\varepsilon}\left(r_{1,\varepsilon}\right)^2-\gamma_{1,\varepsilon}B_{1,\varepsilon}\left(r_{1,\varepsilon}\right)}\right)=o(1) \text{ in } \Omega_0^R$$

thanks to Claim 4.3. Thus we have by standard elliptic theory that

$$v_\varepsilon \to v_0 \text{ in } \mathcal{C}^1_{loc}\left(\mathbb{R}^2\setminus\mathcal{S}_1\right) \text{ as } \varepsilon\to 0 \quad (4.72)$$

where v_0 is some harmonic funtion in $\mathbb{R}^2\setminus\mathcal{S}_1$ which satisfies, thanks to Claim 4.9,

$$|\nabla v_0| \leq \frac{C}{|x|} \text{ for } |x| \text{ large.} \quad (4.73)$$

Thanks to (4.71), we know that

$$v_0(x)=2C_{1,i}\ln\frac{1}{|x-\tilde{x}_i|}+C_{1,i}\mathcal{H}_i+B_i \text{ in } \mathbb{D}_{\tilde{x}_i}(1)$$

for all $i\in\mathcal{D}_1$ where B_i is a constant given by

$$B_i=(1-C_{1,i})\left(1+\ln\frac{f_0\left(x_1\right)}{4}\right)+2C_{1,i}\ln C_{1,i}+\lim_{\varepsilon\to 0}\left(\left(1-\frac{\gamma_{1,\varepsilon}}{\gamma_{i,\varepsilon}}\right)\ln\left(\lambda_\varepsilon\gamma_{1,\varepsilon}^2r_{1,\varepsilon}^2\right)\right) .$$

Thus we have that

$$v_0(x)=2\ln\frac{1}{|x|}+2\sum_{i\in\mathcal{D}_1^\star}C_{1,i}\ln\frac{1}{|x-\tilde{x}_i|}+w_0$$

where w_0 is harmonic in $\mathbb{R}^2$ and satisfies thanks to (4.73) that $|\nabla w_0|\leq C|x|^{-1}$ for $|x|$ large. This implies that $w_0\equiv A_0$ for some constant A_0. Thus we have that

$$v_0(x)=2\ln\frac{1}{|x|}+2\sum_{i\in\mathcal{D}_1^\star}C_{1,i}\ln\frac{1}{|x-\tilde{x}_i|}+A_0 . \quad (4.74)$$

Moreover, the $\mathcal{H}_i$'s of (4.71) are given by

$$\mathcal{H}_i(x)=2\ln\frac{1}{|x|}+2\sum_{j\in\mathcal{D}_1^\star,\,j\neq i}C_{1,j}\ln\frac{1}{|x-\tilde{x}_j|}+A_0$$

and they satisfy $\nabla \mathcal{H}_i(\tilde{x}_i) = 0$ for all $i \in \mathcal{D}_1$. Note that, by the definition of $r_{1,\varepsilon}$ and since we assumed that $\frac{r_{1,\varepsilon}}{d_{1,\varepsilon}} \to +\infty$ as $\varepsilon \to 0$, we know that $\mathcal{D}_1^\star \neq \emptyset$. Let us pick up $i \in \mathcal{D}_1^\star$ such that $|\tilde{x}_i| \geq |\tilde{x}_j|$ for all $j \in \mathcal{D}_1^\star$. It is then clear that

$$\langle \nabla \mathcal{H}_i(\tilde{x}_i), \tilde{x}_i \rangle = -2 - 2 \sum_{j \in \mathcal{D}_1^\star, j \neq i} C_{1,i} \frac{\langle \tilde{x}_i - \tilde{x}_j, \tilde{x}_i \rangle}{|\tilde{x}_i - \tilde{x}_j|^2} \leq -2 \, ,$$

which contradicts the fact that $\nabla \mathcal{H}_i(\tilde{x}_i) = 0$. This is the contradiction we were looking for and this proves that, if $r_{1,\varepsilon} \to 0$ as $\varepsilon \to 0$, this first case can not happen, that is we must have $d_{1,\varepsilon} \to 0$ as $\varepsilon \to 0$. ♠

Case 2 - We assume that $d_{1,\varepsilon} \to 0$ and that $\frac{r_{1,\varepsilon}}{d_\varepsilon} \to 0$ as $\varepsilon \to 0$.

We let $y \in \Omega_0^R$ for some $R > 0$ and we set $x_\varepsilon = x_{1,\varepsilon} + r_{1,\varepsilon} x$. Since $d_{1,\varepsilon} \to 0$ and $r_{1,\varepsilon} \to 0$, we are in situation b) of Claim 4.8. Indeed, as in Case 1, we have that

$$\frac{|x_\varepsilon - x_{i,\varepsilon}|}{\mu_{i,\varepsilon}} \to +\infty \text{ as } \varepsilon \to 0 \text{ for all } i = 1, \dots, N \, .$$

$$\begin{aligned} u_\varepsilon(x_\varepsilon) &= \psi_\varepsilon(x_\varepsilon) + \sum_{i=1}^N \frac{4\pi + o(1)}{\gamma_{i,\varepsilon}} \mathcal{G}(x_{i,\varepsilon}, x_\varepsilon) \\ &\quad + O\left(\sum_{i \in \mathcal{A}} \gamma_{i,\varepsilon}^{-1} \left(\frac{\mu_{i,\varepsilon}}{|x_{i,\varepsilon} - x_\varepsilon|} + \gamma_{i,\varepsilon}^{-1} \ln\left(\frac{s_{i,\varepsilon}}{|x_{i,\varepsilon} - x_\varepsilon|} + 2 \right) \right) \right) \\ &\quad + O\left(\sum_{i \in \mathcal{B}} \frac{d_\varepsilon}{d_\varepsilon + d_{i,\varepsilon}} \left(\gamma_{i,\varepsilon}^{-1} \mu_{i,\varepsilon} + \gamma_{i,\varepsilon}^{-2} s_{i,\varepsilon} \right) \right) \end{aligned}$$

where $\mathcal{A}$ is defined as the set of $i \in \{1, \dots, N\}$ such that $|x_{i,\varepsilon} - x_\varepsilon| \leq s_{i,\varepsilon} + o(d_\varepsilon)$ and $\mathcal{B}$ as its complementary. Noting that $|x_{i,\varepsilon} - x_\varepsilon| \geq C r_{i,\varepsilon}$ for all $i \in \{1, \dots, N\}$, we have that for any $i \in \mathcal{A}$,

$$\gamma_{i,\varepsilon}^{-1} \frac{\mu_{i,\varepsilon}}{|x_{i,\varepsilon} - x_\varepsilon|} + \gamma_{i,\varepsilon}^{-2} \ln\left(\frac{s_{i,\varepsilon}}{|x_{i,\varepsilon} - x_\varepsilon|} + 2 \right) = o\left(\gamma_{i,\varepsilon}^{-1} \right)$$

and, for any $i \in \mathcal{B}$,

$$\frac{d_\varepsilon}{d_\varepsilon + d_{i,\varepsilon}} \left(\gamma_{i,\varepsilon}^{-1} \mu_{i,\varepsilon} + \gamma_{i,\varepsilon}^{-2} s_{i,\varepsilon} \right) = o\left(\gamma_{i,\varepsilon}^{-1} \right) \, .$$

Thus we have that

$$u_\varepsilon(x_\varepsilon) = \psi_\varepsilon(x_\varepsilon) + \sum_{i=1}^N \frac{4\pi + o(1)}{\gamma_{i,\varepsilon}} \mathcal{G}(x_{i,\varepsilon}, x_\varepsilon) + o\left(\gamma_{1,\varepsilon}^{-1} \right)$$

thanks to (4.54). For $i \in \mathcal{D}_1$, we have that $|x_{i,\varepsilon} - x_\varepsilon| = o(d_{1,\varepsilon})$ so that, thanks to (6.12),

$$\mathcal{G}(x_{i,\varepsilon}, x_\varepsilon) = \frac{1}{2\pi} \left(\ln \frac{2 d_{1,\varepsilon}}{r_{1,\varepsilon}} \right) + O(1) \, .$$

For any $i \notin \mathcal{D}_1$, we know that

$$\mathcal{G}(x_{i,\varepsilon}, x_\varepsilon) = \mathcal{G}(x_{i,\varepsilon}, x_{1,\varepsilon}) + o(1)$$

thanks to (6.12). Thus we can write that

$$u_\varepsilon(x_\varepsilon) = \psi_\varepsilon(x_\varepsilon) + \sum_{i \in \mathcal{D}_1} \frac{2 + o(1)}{\gamma_{i,\varepsilon}} \left(\ln \frac{2 d_{1,\varepsilon}}{r_{1,\varepsilon}} \right) + \sum_{i \notin \mathcal{D}_1} \gamma_{i,\varepsilon}^{-1} \mathcal{G}(x_{i,\varepsilon}, x_{1,\varepsilon}) + O\left(\gamma_{1,\varepsilon}^{-1} \right) \, .$$

If $\lambda_0 \neq 0$, then we can write thanks to the fact that $\psi_\varepsilon = 0$ on $\partial\Omega$ and to (4.56) that $\psi_\varepsilon(x_\varepsilon) = O(d_{1,\varepsilon})$. This leads with Claim 4.4 to $\psi_\varepsilon(x_\varepsilon) = O\left(\gamma_{1,\varepsilon}^{-1}\right)$. If $\lambda_0 = 0$, then we can use (4.57) to arrive to the same result. Thus we finally get that

$$u_\varepsilon(x_\varepsilon) = \frac{2}{\gamma_{1,\varepsilon}}\left(1 + \sum_{i\in\mathcal{D}_1^\star} C_{1,i}\right)\left(\ln\frac{d_{1,\varepsilon}}{r_{1,\varepsilon}}\right) + \sum_{i\notin\mathcal{D}_1}\gamma_{i,\varepsilon}^{-1}\mathcal{G}(x_{i,\varepsilon}, x_{1,\varepsilon}) + o\left(\gamma_{1,\varepsilon}^{-1}\left(\ln\frac{d_{1,\varepsilon}}{r_{1,\varepsilon}}\right)\right) . \quad (4.75)$$

Note that $C_{1,i} \leq \tilde{C}$ for all $i > 1$ thanks to (4.53). Thus we have in particular that

$$2\gamma_{1,\varepsilon}^{-1}\ln\frac{d_{1,\varepsilon}}{r_{1,\varepsilon}}\left(1 + \sum_{i\in\mathcal{D}_1^\star} C_{1,i}\right) \leq u_\varepsilon(x_\varepsilon) + o\left(\gamma_{1,\varepsilon}^{-1}\ln\frac{d_{1,\varepsilon}}{r_{1,\varepsilon}}\right) \leq 2\gamma_{1,\varepsilon}^{-1}\ln\frac{d_{1,\varepsilon}}{r_{1,\varepsilon}}\left(1 + (N-1)\tilde{C}\right) . \quad (4.76)$$

Here we used (6.12) to estimate $\mathcal{G}(x_{i,\varepsilon}, x_{1,\varepsilon})$ for $i \notin \mathcal{D}_1$. Note that we also have thanks to Claim 4.9 and to (4.57) that

$$|\nabla u_\varepsilon(x)| \leq C\gamma_{1,\varepsilon}^{-1}|x_{1,\varepsilon} - x| \text{ for all } x \in \mathbb{D}_{x_{1,\varepsilon}}(r_{1,\varepsilon}) . \quad (4.77)$$

The proof now follows exactly Case 1, from (4.66) to the end. We will not repeat it here. ♠

Case 3 - We assume that $d_{1,\varepsilon} \to 0$ as $\varepsilon \to 0$ and that $\dfrac{d_{1,\varepsilon}}{r_{1,\varepsilon}} \to L$ as $\varepsilon \to 0$ where $L \geq 2$.

We are thus in the case where, after some harmless rotation,

$$\Omega_0 = \mathbb{R}\times(-\infty, L) .$$

We let $y \in \Omega_0^R$ for some $R > 0$ and we set $x_\varepsilon = x_{1,\varepsilon} + r_{1,\varepsilon}y$. Since $d_{1,\varepsilon} \to 0$ and $r_{1,\varepsilon} \to 0$, we are in situation b) of Claim 4.8. Indeed, as in Case 1, we have that

$$\frac{|x_\varepsilon - x_{i,\varepsilon}|}{\mu_{i,\varepsilon}} \to +\infty \text{ as } \varepsilon \to 0 \text{ for all } i = 1, \dots, N .$$

Thus we can write that

$$\begin{aligned} u_\varepsilon(x_\varepsilon) &= \psi_\varepsilon(x_\varepsilon) + \sum_{i=1}^N \frac{4\pi + o(1)}{\gamma_{i,\varepsilon}}\mathcal{G}(x_{i,\varepsilon}, x_\varepsilon) \\ &\quad + O\left(\sum_{i\in\mathcal{A}}\gamma_{i,\varepsilon}^{-1}\left(\frac{\mu_{i,\varepsilon}}{|x_{i,\varepsilon} - x_\varepsilon|} + \gamma_{i,\varepsilon}^{-1}\ln\left(\frac{s_{i,\varepsilon}}{|x_{i,\varepsilon} - x_\varepsilon|} + 2\right)\right)\right) \\ &\quad + O\left(\sum_{i\in\mathcal{B}}\frac{d_\varepsilon}{d_\varepsilon + d_{i,\varepsilon}}\left(\gamma_{i,\varepsilon}^{-1}\mu_{i,\varepsilon} + \gamma_{i,\varepsilon}^{-2}s_{i,\varepsilon}\right)\right) \end{aligned}$$

where $\mathcal{A}$ is defined as the set of $i \in \{1, \dots, N\}$ such that $|x_{i,\varepsilon} - x_\varepsilon| \leq s_{i,\varepsilon} + o(d_\varepsilon)$ and $\mathcal{B}$ as its complementary. As in Case 2, we have that

$$\gamma_{i,\varepsilon}^{-1}\frac{\mu_{i,\varepsilon}}{|x_{i,\varepsilon} - x_\varepsilon|} + \gamma_{i,\varepsilon}^{-2}\ln\left(\frac{s_{i,\varepsilon}}{|x_{i,\varepsilon} - x_\varepsilon|} + 2\right) = o\left(\gamma_{i,\varepsilon}^{-1}\right)$$

for all $i \in \mathcal{A}$ while

$$\frac{d_\varepsilon}{d_\varepsilon + d_{i,\varepsilon}}\left(\gamma_{i,\varepsilon}^{-1}\mu_{i,\varepsilon} + \gamma_{i,\varepsilon}^{-2}s_{i,\varepsilon}\right) = o\left(\gamma_{i,\varepsilon}^{-1}\right)$$

for all $i \in \mathcal{B}$. Thus we have that

$$u_\varepsilon(x_\varepsilon) = \psi_\varepsilon(x_\varepsilon) + \sum_{i=1}^N \frac{4\pi + o(1)}{\gamma_{i,\varepsilon}}\mathcal{G}(x_{i,\varepsilon}, x_\varepsilon) + o\left(\gamma_{1,\varepsilon}^{-1}\right) .$$

If $i \in \mathcal{D}_1$, we have that

$$\mathcal{G}(x_{i,\varepsilon}, x_\varepsilon) = \frac{1}{2\pi} \ln \frac{|\tilde{y}_i - y|}{|\tilde{x}_i - y|} + o(1)$$

where

$$\tilde{y}_i = \mathcal{R}(\tilde{x}_i) \ ,$$

$\mathcal{R}$ being the reflection with respect to the straight line $\mathbb{R} \times \{L\}$. Here we used (6.12). If $i \notin \mathcal{D}_1$, we have that

$$\mathcal{G}(x_{i,\varepsilon}, x_\varepsilon) = o(1)$$

thanks to (6.12). Thus we can write, remembering (4.54), that

$$u_\varepsilon(x_\varepsilon) = \psi_\varepsilon(x_\varepsilon) + 2\gamma_{1,\varepsilon}^{-1} \sum_{i \in \mathcal{D}_1} C_{1,i} \ln \frac{|\tilde{y}_i - y|}{|\tilde{x}_i - y|} + o\left(\gamma_{1,\varepsilon}^{-1}\right) \ .$$

Using (4.56), we know that

$$\frac{\psi_\varepsilon(x_\varepsilon)}{r_{1,\varepsilon}} \to A(L - y_2)$$

where $y = (y_1, y_2)$ for some A independent of y. Moreover, we have that $A \geq 0$ by the maximum principle since $\Delta\psi_\varepsilon \geq 0$ in Ω and $\psi_\varepsilon = 0$ on $\partial\Omega$. If $\lambda_0 \neq 0$, we can use Claim 4.4 to deduce that

$$\gamma_{1,\varepsilon}\psi_\varepsilon(x_\varepsilon) \to B(L - y_2)$$

for some $B > 0$, independent of y. If $\lambda_0 = 0$, then (4.57) implies that

$$\gamma_{1,\varepsilon}\psi_{1,\varepsilon}(x_\varepsilon) = O(\lambda_\varepsilon r_{1,\varepsilon}) = o(1) \ .$$

Thus, up to change the B above, we can write that

$$\gamma_{1,\varepsilon}u_\varepsilon(x_\varepsilon) \to B(L - y_2) + 2 \sum_{i \in \mathcal{D}_1} C_{1,i} \ln \frac{|\tilde{y}_i - y|}{|\tilde{x}_i - y|} \text{ as } \varepsilon \to 0 \ . \tag{4.78}$$

Then, by the equation satisfied by u_ε, it is clear that

$$v_\varepsilon(x) = \gamma_{1,\varepsilon}u_\varepsilon(x_{1,\varepsilon} + r_{1,\varepsilon}x)$$

has a Laplacian uniformly converging to 0 in any Ω_0^R. Thus, by standard elliptic theory, we can conclude that

$$\gamma_{1,\varepsilon}u_\varepsilon(x_{1,\varepsilon} + r_{1,\varepsilon}y) \to B(L - y_2) + 2 \sum_{i \in \mathcal{D}_1} C_{1,i} \ln \frac{|\tilde{y}_i - y|}{|\tilde{x}_i - y|} \text{ in } \mathcal{C}^1_{loc}(\Omega_0 \setminus \mathcal{S}_1) \text{ as } \varepsilon \to 0 \ . \tag{4.79}$$

Writing that

$$|\nabla\psi_\varepsilon| \leq C\lambda_\varepsilon \text{ in } \mathbb{D}_{x_{1,\varepsilon}}(r_{1,\varepsilon})$$

thanks to (4.56), we get with Claim 4.4 that

$$|\nabla\psi_\varepsilon| \leq C\sqrt{\lambda_\varepsilon} d_{1,\varepsilon}^{-1} \gamma_{1,\varepsilon}^{-1} \text{ in } \mathbb{D}_{x_{1,\varepsilon}}(r_{1,\varepsilon})$$

so that we can use Claim 4.9 and (4.54) to obtain that

$$|\nabla u_\varepsilon| \leq C\gamma_{1,\varepsilon}^{-1} |x_{1,\varepsilon} - x_\varepsilon|^{-1} \text{ in } \mathbb{D}_{x_{1,\varepsilon}}(r_{1,\varepsilon}) \ .$$

We are thus in position to apply Claim 4.1 to $i = 1$. In particular, combined with (4.79), we get that

$$\gamma_{1,\varepsilon}B_{1,\varepsilon}(r_{1,\varepsilon}) = O(1) \ .$$

This leads with Claim 5.2 of Appendix A to

$$\ln\left(\lambda_\varepsilon r_{1,\varepsilon}^2 \gamma_{1,\varepsilon}^2\right) = O(1) \ .$$

Thus we have that, up to a subsequence,

$$\lambda_\varepsilon f_\varepsilon\left(x_{1,\varepsilon}\right) r_{1,\varepsilon}^2 \gamma_{1,\varepsilon}^2 \to \alpha_0 \text{ as } \varepsilon \to 0 \tag{4.80}$$

for some $\alpha_0 > 0$. Let now $i \in \mathcal{D}_1$ be such that the second coordinate of $\tilde{x}_i$ satisfies $\left(\tilde{x}_i\right)_2 < L$ and

$$\left(\tilde{x}_i\right)_2 \geq \left(\tilde{x}_j\right)_2 \text{ or } \left(\tilde{x}_j\right)_2 = L \text{ for all } j \in \mathcal{D}_1 .$$

Note that such a i does exists since $1 \in \mathcal{D}_1$. Moreover, we have that

$$L > \left(\tilde{x}_i\right)_2 \geq \left(\tilde{x}_1\right)_2 = 0 .$$

Note also that $d_{i,\varepsilon} \geq \left(L - \left(\tilde{x}_i\right)_2 + o(1)\right) r_{1,\varepsilon}$ so that Claim 4.4 implies that

$$O(1) = \lambda_\varepsilon f_\varepsilon\left(x_{1,\varepsilon}\right) \gamma_{i,\varepsilon}^2 d_{i,\varepsilon}^2 = \frac{\gamma_{i,\varepsilon}^2}{\gamma_{1,\varepsilon}^2} \frac{d_{i,\varepsilon}^2}{r_{1,\varepsilon}^2} \lambda_\varepsilon f_\varepsilon\left(x_{1,\varepsilon}\right) \gamma_{1,\varepsilon}^2 r_{1,\varepsilon}^2 \geq \left(\left(L - \left(\tilde{x}_i\right)_2\right)^2 \alpha_0 + o(1)\right) \frac{\gamma_{i,\varepsilon}^2}{\gamma_{1,\varepsilon}^2}$$

thanks to (4.80). This implies that $C_{1,i} \neq 0$. Thanks to (4.55), we then have that $r_{i,\varepsilon} \geq r_{1,\varepsilon}$. Once again, thanks to Claim 4.9 and to (4.54), we see now that

$$\left|\nabla u_\varepsilon\right| \leq C \gamma_{i,\varepsilon}^{-1} \left|x_{i,\varepsilon} - x_\varepsilon\right|^{-1} \text{ in } \mathbb{D}_{x_{i,\varepsilon}}\left(r_{1,\varepsilon}\right)$$

and that we can apply Claim 4.1. In particular, using (4.80), we get that

$$\gamma_{i,\varepsilon} u_\varepsilon\left(x_{1,\varepsilon} + r_{1,\varepsilon} x\right) \to 2 \ln \frac{1}{\left|x - \tilde{x}_i\right|} + \mathcal{H}_i - \ln\left(\frac{\alpha_0}{4C_{1,i}^2}\right) \text{ in } \mathcal{C}^1_{loc}\left(\mathbb{D}_{\tilde{x}_i}(1) \setminus \{\tilde{x}_i\}\right) \text{ as } \varepsilon \to 0$$

where $\mathcal{H}_i$ is harmonic in $\mathbb{D}_{\tilde{x}_i}(1)$ and satisfies $\nabla \mathcal{H}_i\left(\tilde{x}_i\right) = 0$ (since $r_{1,\varepsilon} \to 0$ as $\varepsilon \to 0$ by assumption). Now, combining this with (4.79), we know that

$$\mathcal{H}_i = \frac{B}{C_{1,i}}\left(L - x_2\right) + 2 \sum_{j \in \mathcal{D}_1, j \neq i} \frac{C_{1,j}}{C_{1,i}} \ln \frac{\left|\tilde{y}_j - x\right|}{\left|\tilde{x}_j - x\right|} + 2 \ln\left|\tilde{y}_i - x\right| + \ln\left(\frac{\alpha_0}{4C_{1,i}^2}\right) .$$

The derivative of $\mathcal{H}_i$ with respect to the second coordinate at $\tilde{x}_i$ is

$$\frac{\partial \mathcal{H}_i}{\partial x_2}\left(\tilde{x}_i\right) = -\frac{B}{C_{1,i}} + 2 \sum_{j \in \mathcal{D}_1, j \neq i} \frac{C_{1,j}}{C_{1,i}} \left(\frac{\left(\tilde{x}_i\right)_2 - \left(\tilde{y}_j\right)_2}{\left|\tilde{y}_j - \tilde{x}_i\right|^2} - \frac{\left(\tilde{x}_i\right)_2 - \left(\tilde{x}_j\right)_2}{\left|\tilde{x}_j - \tilde{x}_i\right|^2}\right) + 2 \frac{\left(\tilde{x}_i\right)_2 - \left(\tilde{y}_i\right)_2}{\left|\tilde{y}_i - \tilde{x}_i\right|^2} .$$

Note now that

$$\left(\tilde{y}_j\right)_2 = 2L - \left(\tilde{x}_j\right)_2$$

so that

$$\frac{\partial \mathcal{H}_i}{\partial x_2}\left(\tilde{x}_i\right) = -\frac{B}{C_{1,i}} + 2 \sum_{j \in \mathcal{D}_1, j \neq i} \frac{C_{1,j}}{C_{1,i}} \left(\frac{\left(\tilde{x}_i\right)_2 + \left(\tilde{x}_j\right)_2 - 2L}{\left|\tilde{y}_j - \tilde{x}_i\right|^2} - \frac{\left(\tilde{x}_i\right)_2 - \left(\tilde{x}_j\right)_2}{\left|\tilde{x}_j - \tilde{x}_i\right|^2}\right) + 4 \frac{\left(\tilde{x}_i\right)_2 - L}{\left|\tilde{y}_i - \tilde{x}_i\right|^2} .$$

We claim that

$$\frac{\left(\tilde{x}_i\right)_2 + \left(\tilde{x}_j\right)_2 - 2L}{\left|\tilde{y}_j - \tilde{x}_i\right|^2} \leq \frac{\left(\tilde{x}_i\right)_2 - \left(\tilde{x}_j\right)_2}{\left|\tilde{x}_j - \tilde{x}_i\right|^2} \tag{4.81}$$

for all $j \in \mathcal{D}_1$ with $j \neq i$. This will imply that

$$\frac{\partial \mathcal{H}_i}{\partial x_2}\left(\tilde{x}_i\right) < 0 ,$$

all the terms above being nonpositive, the last one being negative. This will give a contradiction with the fact that $\nabla \mathcal{H}_i\left(\tilde{x}_i\right) = 0$, thus proving that this last case is not possible either. In order to prove (4.81), we first notice that

$$\left|\tilde{x}_i - \tilde{y}_j\right|^2 = \left|\tilde{x}_i - \tilde{x}_j\right|^2 + 4\left(L - \left(\tilde{x}_i\right)_2\right)\left(L - \left(\tilde{x}_j\right)_2\right) .$$

Thus we can write that

$$\begin{aligned}
&\left((\tilde{x}_i)_2 + (\tilde{x}_j)_2 - 2L\right)|\tilde{x}_i - \tilde{x}_j|^2 - \left((\tilde{x}_i)_2 - (\tilde{x}_j)_2\right)|\tilde{x}_i - \tilde{y}_j|^2 \\
&\quad = 2|\tilde{x}_i - \tilde{x}_j|^2\left((\tilde{x}_j)_2 - L\right) - 4\left(L - (\tilde{x}_i)_2\right)\left(L - (\tilde{x}_j)_2\right)\left((\tilde{x}_i)_2 - (\tilde{x}_j)_2\right) \\
&\quad = 2\left((\tilde{x}_j)_2 - L\right)\left(|\tilde{x}_i - \tilde{x}_j|^2 + 2\left(L - (\tilde{x}_i)_2\right)\left((\tilde{x}_i)_2 - (\tilde{x}_j)_2\right)\right) \\
&\quad \leq 0
\end{aligned}$$

since $(\tilde{x}_j)_2 - L \leq 0$ and if $(\tilde{x}_j)_2 - L \neq 0$, $(\tilde{x}_i)_2 - (\tilde{x}_j)_2 \geq 0$. This clearly proves (4.81) and, as already said, proves that this last case is not possible. ♠

The study of these three cases proves that the assumption $r_{1,\varepsilon} \to 0$ is absurd and thus proves the claim. ♢

Note that this claim implies that

$$x_{1,\varepsilon} \to x_1 \text{ as } \varepsilon \to 0 \text{ with } x_1 \in \Omega\ . \tag{4.82}$$

We also have thanks to Claims 4.4 and 4.10 that

$$\lambda_\varepsilon = O\left(\gamma_{1,\varepsilon}^{-2}\right)$$

so that $\lambda_0 = 0$ and $u_0 \equiv 0$. Moreover, we can transform (4.57) into

$$\|\nabla\psi_\varepsilon\|_{\mathcal{C}^{1,\alpha}\left(\overline{\Omega}\right)} = O\left(\gamma_{1,\varepsilon}^{-3}\right)\ . \tag{4.83}$$

Let us now give a simple consequence of the previous claim :

Claim 4.11. *After passing to a subsequence,*

$$\lambda_\varepsilon \gamma_{1,\varepsilon}^2 \to \alpha_0 \text{ as } \varepsilon \to 0$$

for some

$$0 < \alpha_0 \leq \frac{4}{f_0(x_1)\, d(x_1, \partial\Omega)^2}\ .$$

Proof of Claim 4.11 - We already said that $\lambda_\varepsilon = O\left(\gamma_{1,\varepsilon}^{-2}\right)$. Claim 4.9 with (4.83) gives that

$$|\nabla u_\varepsilon| \leq C\sum_{i=1}^{N} \gamma_{i,\varepsilon}^{-1}\left(\mu_{i,\varepsilon} + |x - x_{i,\varepsilon}|\right)^{-1} \text{ in } \Omega\ .$$

This gives in particular that

$$|\nabla u_\varepsilon| \leq C\gamma_{1,\varepsilon}^{-1}|x - x_{1,\varepsilon}|^{-1}$$

in $\mathbb{D}_{x_{1,\varepsilon}}(\delta_0)$ where δ_0 is as in Claim 4.10. Thus we are in position to apply Claim 4.1 to $i = 1$. This gives in particular that

$$\gamma_{1,\varepsilon}\left(u_\varepsilon(x) - B_{1,\varepsilon}(\delta_0)\right) = O(1)$$

for all $|x - x_{1,\varepsilon}| = \frac{\delta_0}{2}$. Now Claim 4.8 combined with (4.83) gives that

$$\gamma_{1,\varepsilon} u_\varepsilon(x) = O(1) \text{ on } \partial\mathbb{D}_{x_{1,\varepsilon}}\left(\frac{\delta_0}{2}\right)$$

so that the above leads to

$$\gamma_{1,\varepsilon} B_{1,\varepsilon}(\delta_0) = O(1)\ .$$

Since

$$\gamma_{1,\varepsilon} B_{1,\varepsilon}(\delta_0) = -\ln\left(\lambda_\varepsilon\gamma_{1,\varepsilon}^2\right) + O(1)\ ,$$

we obtain that

$$\ln\left(\lambda_\varepsilon\gamma_{1,\varepsilon}^2\right) = O(1)\ .$$

This clearly permits to prove the claim. ♢

Claim 4.12. *We have that $r_{i,\varepsilon} \geq \delta_1$ for some $\delta_1 > 0$ for all $i = 1, \dots, N$.*

Proof of Claim 4.12 - We shall prove it by induction on i. This is already proved for $i = 1$ in the previous claim. Fix $2 \leq i \leq N$ and assume that

$$r_{j,\varepsilon} \geq \delta_1 > 0 \text{ for all } j < i \, . \tag{4.84}$$

In particular, after passing to a subsequence, we have that

$$x_{j,\varepsilon} \to x_j \text{ as } \varepsilon \to 0 \text{ with } x_j \in \Omega \, . \tag{4.85}$$

Assume by contradiction that

$$r_{i,\varepsilon} \to 0 \text{ as } \varepsilon \to 0 \, . \tag{4.86}$$

By (4.55), this implies that $C_{j,i} = 0$ for all $j < i$ so that

$$\gamma_{j,\varepsilon} = o\left(\gamma_{i,\varepsilon}\right) \text{ for all } j < i \, . \tag{4.87}$$

We shall now proceed as in the proof of Claim 4.10 and distinguish three cases.

We let in the following

$$\mathcal{D}_i^\star = \{j > i \text{ s.t. } |x_{i,\varepsilon} - x_{j,\varepsilon}| = O\left(r_{i,\varepsilon}\right)\} \text{ and } \mathcal{D}_i = \mathcal{D}_i^\star \cup \{i\} \, . \tag{4.88}$$

After passing to a subsequence, we let

$$\mathcal{S}_i^\star = \left\{ \tilde{x}_j = \lim_{\varepsilon \to 0} \frac{x_{j,\varepsilon} - x_{i,\varepsilon}}{r_{i,\varepsilon}}, \, j \in \mathcal{D}_i^\star \right\} \text{ and } \mathcal{S}_i = \mathcal{S}_i^\star \cup \{\tilde{x}_i = 0\} \, . \tag{4.89}$$

We also let

$$\Omega_{i,\varepsilon} = \left\{ y \in \mathbb{R}^2 \text{ s.t. } x_{i,\varepsilon} + r_{i,\varepsilon} y \in \Omega \right\} \, . \tag{4.90}$$

Note that, after passing to a subsequence (and up to a harmless rotation if necessary), we have that

$$\Omega_{i,\varepsilon} \to \Omega_0 \text{ as } \varepsilon \to 0 \text{ where } \begin{cases} \Omega_0 = \mathbb{R}^2 & \text{if } \dfrac{d_{i,\varepsilon}}{r_{i,\varepsilon}} \to +\infty \text{ as } \varepsilon \to 0 \\ \Omega_0 = \mathbb{R} \times (-\infty, L) & \text{if } \dfrac{d_{i,\varepsilon}}{r_{i,\varepsilon}} \to L \text{ as } \varepsilon \to 0 \end{cases} \tag{4.91}$$

Here $d_{i,\varepsilon} = d\left(x_{i,\varepsilon}, \partial\Omega\right)$, as defined in (4.8). For $R > 0$, we shall also let

$$\Omega_0^R = \left(\Omega_0 \cap \mathbb{D}_0(R)\right) \setminus \bigcup_{j \in \mathcal{D}_i} \mathbb{D}_{\tilde{x}_j}\left(\frac{1}{R}\right) \, . \tag{4.92}$$

Case 1 - We assume that $d_{i,\varepsilon} \not\to 0$ as $\varepsilon \to 0$, meaning that, after passing to a subsequence, $x_{i,\varepsilon} \to x_i$ as $\varepsilon \to 0$ with $x_i \in \Omega$.

We let $y \in \Omega_0^R$ for some $R > 0$ and we set $x_\varepsilon = x_{i,\varepsilon} + r_{i,\varepsilon} y$. Since $d_{i,\varepsilon} \not\to 0$ and $r_{i,\varepsilon} \to 0$ as $\varepsilon \to 0$, we are in situation a) of Claim 4.8. Note indeed that

$$\frac{|x_\varepsilon - x_{j,\varepsilon}|}{\mu_{j,\varepsilon}} \to +\infty \text{ as } \varepsilon \to 0 \text{ for all } j = 1, \dots, N \, .$$

It is obvious if $j < i$ since $r_{j,\varepsilon} \geq \delta_1 > 0$ and $r_{i,\varepsilon} \to 0$ as $\varepsilon \to 0$. It is also obvious if $j \in \mathcal{D}_i$ since we clearly have in this case

$$\frac{|x_\varepsilon - x_{j,\varepsilon}|}{\mu_{j,\varepsilon}} = \frac{|x_\varepsilon - x_{j,\varepsilon}|}{r_{i,\varepsilon}} \frac{r_{i,\varepsilon}}{r_{j,\varepsilon}} \frac{r_{j,\varepsilon}}{\mu_{j,\varepsilon}}$$

with $\frac{|x_\varepsilon - x_{j,\varepsilon}|}{r_{i,\varepsilon}} \geq R^{-1} + o(1)$, $\frac{r_{i,\varepsilon}}{r_{j,\varepsilon}} \geq 2\,|\tilde{x}_j|^{-1} + o(1)$ for $j \in \mathcal{D}_i^\star$ and equal to 1 if $j = i$, and $\frac{r_{j,\varepsilon}}{\mu_{j,\varepsilon}} \to +\infty$ as $\varepsilon \to 0$ thanks to assertion c) of Proposition 2.1. While, if $j > i$ and $j \notin \mathcal{D}_i$, we can write that

$$\frac{|x_\varepsilon - x_{j,\varepsilon}|}{\mu_{i,\varepsilon}} \geq (1 + o(1))\,\frac{|x_{i,\varepsilon} - x_{j,\varepsilon}|}{\mu_{j,\varepsilon}} \geq (2 + o(1))\,\frac{r_{j,\varepsilon}}{\mu_{j,\varepsilon}} \to +\infty \text{ as } \varepsilon \to 0\,.$$

Thus, applying a) of Claim 4.8, we can write that

$$\begin{aligned} u_\varepsilon(x_\varepsilon) &= \psi_\varepsilon(x_\varepsilon) + \sum_{j=1}^N (4\pi + o(1))\,\gamma_{j,\varepsilon}^{-1}\mathcal{G}(x_{j,\varepsilon}, x_\varepsilon) \\ &\quad + O\left(\sum_{i=1}^N \left(\gamma_{j,\varepsilon}^{-1}\frac{\mu_{j,\varepsilon}}{|x_{j,\varepsilon} - x_\varepsilon|} + \gamma_{j,\varepsilon}^{-2}\ln\left(\frac{s_{j,\varepsilon}}{|x_{j,\varepsilon} - x_\varepsilon|} + 2\right)\right)\right)\,. \end{aligned}$$

For $j < i$, we have that

$$(4\pi + o(1))\,\gamma_{j,\varepsilon}^{-1}\mathcal{G}(x_{j,\varepsilon}, x_\varepsilon) = 4\pi\gamma_{1,\varepsilon}^{-1}C_{1,j}\mathcal{G}(x_j, x_i) + o\left(\gamma_{1,\varepsilon}^{-1}\right)$$

thanks to the assumption that $x_{i,\varepsilon} \to x_i$ with $x_i \in \Omega$, to (4.85) and to (4.52), (4.54). We also obviously have that

$$\gamma_{j,\varepsilon}^{-1}\frac{\mu_{j,\varepsilon}}{|x_{j,\varepsilon} - x_\varepsilon|} + \gamma_{j,\varepsilon}^{-2}\ln\left(\frac{s_{j,\varepsilon}}{|x_{j,\varepsilon} - x_\varepsilon|} + 2\right) = o\left(\gamma_{1,\varepsilon}^{-1}\right)\,.$$

We also know thanks to (4.83) that $\psi_\varepsilon(x_\varepsilon) = o\left(\gamma_{1,\varepsilon}^{-1}\right)$. For $j > i$, we can proceed exactly as in Case 1 of Claim 4.10 to finally obtain that

$$\begin{aligned} u_\varepsilon(x_\varepsilon) &= 4\pi\gamma_{1,\varepsilon}^{-1}\sum_{j=1}^{i-1} C_{1,j}\mathcal{G}(x_j, x_i) + o\left(\gamma_{1,\varepsilon}^{-1}\right) \\ &\quad + (2 + o(1))\,\gamma_{i,\varepsilon}^{-1}\ln\frac{1}{r_{i,\varepsilon}}\left(1 + \sum_{j\in\mathcal{D}_i^\star} C_{i,j}\right) \\ &\quad + \sum_{j>i,\, j\notin\mathcal{D}_i} (4\pi + o(1))\,\gamma_{j,\varepsilon}^{-1}\mathcal{G}(x_{i,\varepsilon}, x_{j,\varepsilon}) + O\left(\gamma_{i,\varepsilon}^{-1}\right) \end{aligned} \tag{4.93}$$

This gives in particular that $u_\varepsilon \geq C\gamma_{1,\varepsilon}^{-1}$ on $\partial\mathbb{D}_{x_{i,\varepsilon}}(r_{i,\varepsilon})$ for some $C > 0$. Using b) of Claim 4.2, we deduce that

$$C\gamma_{1,\varepsilon}^{-1} \leq B_{i,\varepsilon}(r_{i,\varepsilon}) + O\left(\gamma_{i,\varepsilon}^{-1}\right) = -\gamma_{i,\varepsilon}^{-1}\ln\left(\lambda_\varepsilon\gamma_{i,\varepsilon}^2 r_{i,\varepsilon}^2\right) + O\left(\gamma_{i,\varepsilon}^{-1}\right)\,.$$

Since $\gamma_{1,\varepsilon} = o(\gamma_{i,\varepsilon})$, see (4.87), we deduce that

$$\lambda_\varepsilon\gamma_{i,\varepsilon}^2 r_{i,\varepsilon}^2 \to 0 \text{ as } \varepsilon \to 0\,.$$

Thanks to Claim 4.11, this gives that

$$\gamma_{i,\varepsilon} r_{i,\varepsilon} = o(\gamma_{1,\varepsilon})\,. \tag{4.94}$$

We apply now Claim 4.9 combined with (4.83) and this last estimate to write that

$$|\nabla u_\varepsilon(x)| \leq C\gamma_{i,\varepsilon}^{-1}|x_{i,\varepsilon} - x|^{-1} + C\gamma_{1,\varepsilon}^{-1} \leq C'\gamma_{i,\varepsilon}^{-1}|x_{i,\varepsilon} - x|^{-1}$$

in $\mathbb{D}_{x_{i,\varepsilon}}(r_{i,\varepsilon})$. Thus we can apply Claim 4.1 to i : this gives that, if $|x| = \frac{1}{2}$,

$$u_\varepsilon(x_{i,\varepsilon} + r_{i,\varepsilon}x) = B_{i,\varepsilon}(r_{i,\varepsilon}) + O\left(\gamma_{i,\varepsilon}^{-1}\right)\,.$$

Combined with (4.93), this leads to

$$\begin{aligned} B_{i,\varepsilon}\left(r_{i,\varepsilon}\right) &= 4\pi\gamma_{1,\varepsilon}^{-1}\sum_{j=1}^{i-1} C_{1,j}\mathcal{G}\left(x_j,x_i\right)+o\left(\gamma_{1,\varepsilon}^{-1}\right) \\ &\quad +\left(2+o(1)\right)\gamma_{i,\varepsilon}^{-1}\ln\frac{1}{r_{i,\varepsilon}}\left(1+\sum_{j\in\mathcal{D}_i^\star} C_{i,j}\right) \\ &\quad +\sum_{j>i,\, j\notin\mathcal{D}_i}\left(4\pi+o(1)\right)\gamma_{j,\varepsilon}^{-1}\mathcal{G}\left(x_{i,\varepsilon},x_{j,\varepsilon}\right)+O\left(\gamma_{i,\varepsilon}^{-1}\right)\ . \end{aligned}$$

Since

$$B_{i,\varepsilon}\left(r_{i,\varepsilon}\right)=-\gamma_{i,\varepsilon}^{-1}\ln\left(\lambda_\varepsilon\gamma_{i,\varepsilon}^2 r_{i,\varepsilon}^2\right)+O\left(\gamma_{i,\varepsilon}^{-1}\right)\ ,$$

this leads to

$$\begin{aligned} -\ln\left(\lambda_\varepsilon\gamma_{i,\varepsilon}^2 r_{i,\varepsilon}^2\right) &= \gamma_{i,\varepsilon}\gamma_{1,\varepsilon}^{-1}\left(4\pi\sum_{j=1}^{i-1} C_{1,j}\mathcal{G}\left(x_j,x_i\right)+o(1)\right) \\ &\quad +2\ln\frac{1}{r_{i,\varepsilon}}\left(1+\sum_{j\in\mathcal{D}_i^\star} C_{i,j}\right) \\ &\quad +\sum_{j>i,\, j\notin\mathcal{D}_i}\left(4\pi+o(1)\right)C_{i,j}\mathcal{G}\left(x_{i,\varepsilon},x_{j,\varepsilon}\right)+o\left(\ln\frac{1}{r_{i,\varepsilon}}\right)\ . \end{aligned} \tag{4.95}$$

Thanks to Claim 4.11 and (4.87), we deduce that

$$4\pi\sum_{j=1}^{i-1} C_{1,j}\mathcal{G}\left(x_j,x_i\right)+o(1)+\frac{\gamma_{1,\varepsilon}}{\gamma_{i,\varepsilon}}\ln\frac{1}{r_{i,\varepsilon}}\left(2\sum_{j\in\mathcal{D}_1^\star} C_{i,j}+o(1)\right)\le 0\ . \tag{4.96}$$

Let $k\in\mathcal{D}_i^\star$. It is clear that there exists $\delta>0$ such that $\partial\mathbb{D}_{x_{k,\varepsilon}}\left(\delta r_{i,\varepsilon}\right)\subset\left\{x_{i,\varepsilon}+r_{i,\varepsilon}y,\ y\in\Omega_0^R\right\}$ for some $R>0$. Thus we can write that

$$\inf_{\partial\mathbb{D}_{x_{k,\varepsilon}}(\delta r_{i,\varepsilon})} u_\varepsilon\ge C\gamma_{1,\varepsilon}^{-1}+2\gamma_{i,\varepsilon}^{-1}\ln\frac{1}{r_{i,\varepsilon}}\left(1+\sum_{j\in\mathcal{D}_i^\star} C_{i,j}+o(1)\right)$$

thanks to (4.93). We can also apply b) of Claim 4.2 with $r_\varepsilon=\delta r_{i,\varepsilon}$ thanks to Claim 4.3 and to the fact that $\dfrac{r_{i,\varepsilon}}{r_{k,\varepsilon}}\ge 2\left|\tilde{x}_k\right|^{-1}+o(1)$. This leads to

$$\begin{aligned} &C\gamma_{1,\varepsilon}^{-1}+2\gamma_{i,\varepsilon}^{-1}\ln\frac{1}{r_{i,\varepsilon}}\left(1+\sum_{j\in\mathcal{D}_i^\star} C_{i,j}+o(1)\right) \\ &\quad\le B_{k,\varepsilon}\left(\delta r_{i,\varepsilon}\right)+o\left(\gamma_{k,\varepsilon}^{-1}\right)=-\gamma_{k,\varepsilon}^{-1}\ln\left(\lambda_\varepsilon r_{i,\varepsilon}^2\gamma_{k,\varepsilon}^2\right)+O\left(\gamma_{k,\varepsilon}^{-1}\right)\ . \end{aligned}$$

Combined with (4.95), this gives that

$$\begin{aligned} C\gamma_{1,\varepsilon}^{-1}+2\gamma_{i,\varepsilon}^{-1}\ln\frac{1}{r_{i,\varepsilon}}\left(1+\sum_{j\in\mathcal{D}_i^\star} C_{i,j}+o(1)\right) &\le \gamma_{k,\varepsilon}^{-1}\gamma_{i,\varepsilon}\gamma_{1,\varepsilon}^{-1}\left(4\pi\sum_{j=1}^{i-1} C_{1,j}\mathcal{G}\left(x_j,x_i\right)+o(1)\right) \\ &\quad +O\left(\gamma_{k,\varepsilon}^{-1}\ln\frac{1}{r_{i,\varepsilon}}\right)-\gamma_{k,\varepsilon}^{-1}\ln\left(\frac{\gamma_{k,\varepsilon}^2}{\gamma_{i,\varepsilon}^2}\right)\ . \end{aligned}$$

Assume by contradiction that $\gamma_{i,\varepsilon} = o\left(\gamma_{k,\varepsilon}\right)$. We then have that

$$C\gamma_{1,\varepsilon}^{-1} + 2\gamma_{i,\varepsilon}^{-1} \ln \frac{1}{r_{i,\varepsilon}} \left(1 + \sum_{j \in \mathcal{D}_i^\star} C_{i,j} + o(1)\right) \leq o\left(\gamma_{1,\varepsilon}^{-1}\right) + o\left(\gamma_{i,\varepsilon}^{-1} \ln \frac{1}{r_{i,\varepsilon}}\right) ,$$

which is absurd. Thus we have proved that

$$C_{i,j} > 0 \text{ for all } j \in \mathcal{D}_i^\star . \tag{4.97}$$

Since $\frac{r_{i,\varepsilon}}{d_{i,\varepsilon}} \to +\infty$ as $\varepsilon \to 0$ and since $\frac{|x_{i,\varepsilon} - x_{j,\varepsilon}|}{r_{i,\varepsilon}} \to +\infty$ for all $j < i$, we are sure that $\mathcal{D}_i^\star \neq \emptyset$ and, with (4.97), that

$$\sum_{j \in \mathcal{D}_i^\star} C_{i,j} > 0 .$$

Then (4.96) leads to a contradiction. This proves that this first case is absurd. ♠

Case 2 - We assume that $d_{i,\varepsilon} \to 0$ and that $\frac{r_{i,\varepsilon}}{d_{i,\varepsilon}} \to 0$ as $\varepsilon \to 0$.

We let $x_i = \lim_{\varepsilon \to 0} x_{i,\varepsilon}$. Note that $x_i \in \partial\Omega$. We let $y \in \Omega_0^R$ for some $R > 0$ and we set $x_\varepsilon = x_{i,\varepsilon} + r_{i,\varepsilon} y$. Since $d_{i,\varepsilon} \to 0$ and $r_{i,\varepsilon} \to 0$, we are in situation b) of Claim 4.8. Indeed, as in Case 1, we have that

$$\frac{|x_\varepsilon - x_{j,\varepsilon}|}{\mu_{j,\varepsilon}} \to +\infty \text{ as } \varepsilon \to 0 \text{ for all } j = 1, \dots, N .$$

Thus we have that

$$\begin{aligned} u_\varepsilon\left(x_\varepsilon\right) &= \psi_\varepsilon\left(x_\varepsilon\right) + \sum_{j=1}^N \frac{4\pi + o(1)}{\gamma_{j,\varepsilon}} \mathcal{G}\left(x_{j,\varepsilon}, x_\varepsilon\right) \\ &\quad + O\left(\sum_{j \in \mathcal{A}} \left(\gamma_{j,\varepsilon}^{-1} \frac{\mu_{j,\varepsilon}}{|x_{j,\varepsilon} - x_\varepsilon|} + \gamma_{j,\varepsilon}^{-2} \ln\left(\frac{s_{j,\varepsilon}}{|x_{j,\varepsilon} - x_\varepsilon|} + 2\right)\right)\right) \\ &\quad + O\left(\sum_{j \in \mathcal{B}} \frac{d_\varepsilon}{d_\varepsilon + d_{j,\varepsilon}} \left(\gamma_{j,\varepsilon}^{-1} \mu_{j,\varepsilon} + \gamma_{j,\varepsilon}^{-2} s_{j,\varepsilon}\right)\right) \end{aligned}$$

where $\mathcal{A}$ is defined as the set of $j \in \{1, \dots, N\}$ such that $|x_{j,\varepsilon} - x_\varepsilon| \leq s_{j,\varepsilon} + o\left(d_\varepsilon\right)$ and $\mathcal{B}$ as its complementary. Noting that $|x_{j,\varepsilon} - x_\varepsilon| \geq C r_{j,\varepsilon}$ for all $j \in \{1, \dots, N\}$, we have that for any $j \in \mathcal{A}$,

$$\gamma_{j,\varepsilon}^{-1} \frac{\mu_{j,\varepsilon}}{|x_{j,\varepsilon} - x_\varepsilon|} + \gamma_{j,\varepsilon}^{-2} \ln\left(\frac{s_{j,\varepsilon}}{|x_{j,\varepsilon} - x_\varepsilon|} + 2\right) = o\left(\gamma_{j,\varepsilon}^{-1}\right) .$$

And, for any $j \in \mathcal{B}$,

$$\frac{d_\varepsilon}{d_\varepsilon + d_{j,\varepsilon}} \left(\gamma_{j,\varepsilon}^{-1} \mu_{j,\varepsilon} + \gamma_{j,\varepsilon}^{-2} s_{j,\varepsilon}\right) = o\left(\gamma_{j,\varepsilon}^{-1}\right) .$$

Note also that, if $j < i$, we have that $j \in \mathcal{B}$ thanks to (4.84) and that

$$\frac{d_\varepsilon}{d_\varepsilon + d_{j,\varepsilon}} \left(\gamma_{j,\varepsilon}^{-1} \mu_{j,\varepsilon} + \gamma_{j,\varepsilon}^{-2} s_{j,\varepsilon}\right) = o\left(d_\varepsilon \gamma_{j,\varepsilon}^{-1}\right) .$$

Thus we have that

$$u_\varepsilon\left(x_\varepsilon\right) = \psi_\varepsilon\left(x_\varepsilon\right) + \sum_{j=1}^N \frac{4\pi + o(1)}{\gamma_{j,\varepsilon}} \mathcal{G}\left(x_{j,\varepsilon}, x_\varepsilon\right) + o\left(\gamma_{i,\varepsilon}^{-1}\right) + o\left(d_\varepsilon \gamma_{1,\varepsilon}^{-1}\right) .$$

We can write thanks to (4.83) and since $\psi_\varepsilon = 0$ on $\partial\Omega$ that

$$\psi_\varepsilon\left(x_\varepsilon\right) = O\left(d_\varepsilon \gamma_{1,\varepsilon}^{-3}\right) .$$

Then we have that, for any $j < i$,

$$\mathcal{G}(x_{j,\varepsilon}, x_\varepsilon) = -d_\varepsilon \partial_\nu \mathcal{G}(x_j, x_i) + o(d_\varepsilon) \ .$$

And, for $j \geq i$, we have thanks to (6.12) that

$$\mathcal{G}(x_{j,\varepsilon}, x_\varepsilon) = \frac{1}{2\pi} \ln \frac{2d_{i,\varepsilon}}{r_{i,\varepsilon}} + \frac{1}{2\pi} \ln \frac{r_{i,\varepsilon}}{|x_{j,\varepsilon} - x_\varepsilon|} + O\left(\frac{r_{i,\varepsilon}}{d_{i,\varepsilon}}\right) + O(d_{i,\varepsilon})$$

if $j \in \mathcal{D}_i$ and that

$$\mathcal{G}(x_{j,\varepsilon}, x_\varepsilon) = \mathcal{G}(x_{i,\varepsilon}, x_{j,\varepsilon}) + o(1)$$

if $j \notin \mathcal{D}_i$. We thus arrive to

$$\begin{aligned} u_\varepsilon(x_\varepsilon) &= 4\pi \frac{d_{i,\varepsilon}}{\gamma_{1,\varepsilon}} \left(\sum_{j=1}^{i-1} C_{1,j} \left(-\partial_\nu \mathcal{G}(x_j, x_i) \right) \right) \\ &\quad + 2 \left(\sum_{j \in \mathcal{D}_1} C_{i,j} + o(1) \right) \gamma_{i,\varepsilon}^{-1} \left(\ln \frac{2d_{i,\varepsilon}}{r_{i,\varepsilon}} \right) \\ &\quad + \sum_{j>i,\, j \notin \mathcal{D}_i} (4\pi + o(1)) \gamma_{j,\varepsilon}^{-1} \mathcal{G}(x_{i,\varepsilon}, x_{j,\varepsilon}) + o\left(d_{i,\varepsilon} \gamma_{1,\varepsilon}^{-1}\right) \ . \end{aligned} \tag{4.98}$$

This gives in particular that $u_\varepsilon \geq C d_{i,\varepsilon} \gamma_{1,\varepsilon}^{-1}$ on $\partial \mathbb{D}_{x_{i,\varepsilon}}(r_{i,\varepsilon})$ for some $C > 0$. Using b) of Claim 4.2, we deduce that

$$C d_{i,\varepsilon} \gamma_{1,\varepsilon}^{-1} \leq B_{i,\varepsilon}(r_{i,\varepsilon}) + O\left(\gamma_{i,\varepsilon}^{-1}\right) = -\gamma_{i,\varepsilon}^{-1} \ln\left(\lambda_\varepsilon \gamma_{i,\varepsilon}^2 r_{i,\varepsilon}^2\right) + O\left(\gamma_{i,\varepsilon}^{-1}\right) \ .$$

Thanks to Claim 4.11, this gives that

$$C \frac{d_{i,\varepsilon} \gamma_{i,\varepsilon}}{\gamma_{1,\varepsilon}} \leq -\ln \left(\frac{\gamma_{i,\varepsilon}^2 r_{i,\varepsilon}^2}{\gamma_{1,\varepsilon}^2} \right) + O(1) \ .$$

Since $\frac{d_{i,\varepsilon}}{r_{i,\varepsilon}} \to +\infty$ as $\varepsilon \to 0$ in our case, this implies that

$$\gamma_{i,\varepsilon} r_{i,\varepsilon} = o(\gamma_{1,\varepsilon}) \ . \tag{4.99}$$

We apply now Claim 4.9 combined with (4.83), (4.84) and this last estimate to write that

$$|\nabla u_\varepsilon(x)| \leq C \gamma_{i,\varepsilon}^{-1} |x_{i,\varepsilon} - x|^{-1} + C \gamma_{1,\varepsilon}^{-1} \leq C' \gamma_{i,\varepsilon}^{-1} |x_{i,\varepsilon} - x|^{-1}$$

in $\mathbb{D}_{x_{i,\varepsilon}}(r_{i,\varepsilon})$. Thus we can apply Claim 4.1 to i : this gives that, if $|x| = \frac{1}{2}$,

$$u_\varepsilon(x_{i,\varepsilon} + r_{i,\varepsilon} x) = B_{i,\varepsilon}(r_{i,\varepsilon}) + O\left(\gamma_{i,\varepsilon}^{-1}\right) \ .$$

Combined with (4.98) and (4.99), this leads to

$$\begin{aligned} B_{i,\varepsilon}(r_{i,\varepsilon}) &= 4\pi \frac{d_{i,\varepsilon}}{\gamma_{1,\varepsilon}} \left(\sum_{j=1}^{i-1} C_{1,j} \left(-\partial_\nu \mathcal{G}(x_j, x_i) \right) + o(1) \right) \\ &\quad + 2 \left(\sum_{j \in \mathcal{D}_i} C_{i,j} + o(1) \right) \gamma_{i,\varepsilon}^{-1} \left(\ln \frac{2d_{i,\varepsilon}}{r_{i,\varepsilon}} \right) \\ &\quad + \sum_{j>i,\, j \notin \mathcal{D}_i} (4\pi + o(1)) \gamma_{j,\varepsilon}^{-1} \mathcal{G}(x_{i,\varepsilon}, x_{j,\varepsilon}) \ . \end{aligned} \tag{4.100}$$

Since

$$B_{i,\varepsilon}\left(r_{i,\varepsilon}\right)=-\gamma_{i,\varepsilon}^{-1}\ln\left(\lambda_\varepsilon\gamma_{i,\varepsilon}^2r_{i,\varepsilon}^2\right)+O\left(\gamma_{i,\varepsilon}^{-1}\right)=-\gamma_{i,\varepsilon}^{-1}\ln\left(\frac{\gamma_{i,\varepsilon}^2d_{i,\varepsilon}^2}{\gamma_{1,\varepsilon}^2}\right)-2\gamma_{i,\varepsilon}^{-1}\ln\left(\frac{r_{i,\varepsilon}}{d_{i,\varepsilon}}\right)+O\left(\gamma_{i,\varepsilon}^{-1}\right)$$

thanks to Claim 4.11, this leads to

$$\begin{aligned}-\ln\left(\frac{\gamma_{i,\varepsilon}^2d_{i,\varepsilon}^2}{\gamma_{1,\varepsilon}^2}\right) &= 4\pi\frac{d_{i,\varepsilon}\gamma_{i,\varepsilon}}{\gamma_{1,\varepsilon}}\left(\sum_{j=1}^{i-1}C_{1,j}\left(-\partial_\nu\mathcal{G}\left(x_j,x_i\right)\right)+o(1)\right)\\ &\quad+2\left(\sum_{j\in\mathcal{D}_i^\star}C_{i,j}+o(1)\right)\left(\ln\frac{d_{i,\varepsilon}}{r_{i,\varepsilon}}\right)\\ &\quad+\sum_{j>i,\,j\notin\mathcal{D}_i}\left(4\pi+o(1)\right)\frac{\gamma_{i,\varepsilon}}{\gamma_{j,\varepsilon}}\mathcal{G}\left(x_{i,\varepsilon},x_{j,\varepsilon}\right)\ ,\end{aligned}\tag{4.101}$$

from which we can infer that, for ε small,

$$\begin{aligned}2\frac{\gamma_{1,\varepsilon}}{d_{i,\varepsilon}\gamma_{i,\varepsilon}}\ln\left(\frac{\gamma_{1,\varepsilon}}{d_{i,\varepsilon}\gamma_{i,\varepsilon}}\right) &\geq 2\pi\sum_{j=1}^{i-1}C_{1,j}\left(-\partial_\nu\mathcal{G}\left(x_j,x_i\right)\right)\\ &\quad+2\left(\sum_{j\in\mathcal{D}_i^\star}C_{i,j}+o(1)\right)\frac{\gamma_{1,\varepsilon}}{d_{i,\varepsilon}\gamma_{i,\varepsilon}}\left(\ln\frac{d_{i,\varepsilon}}{r_{i,\varepsilon}}\right)\ .\end{aligned}\tag{4.102}$$

Let $j\in\mathcal{D}_i^\star$. Note that, since $\frac{d_{i,\varepsilon}}{r_{i,\varepsilon}}\to+\infty$ as $\varepsilon\to0$, we know that $\mathcal{D}_i^\star\neq\emptyset$. There exists $\delta>0$ such that $\partial\mathbb{D}_{x_{j,\varepsilon}}\left(\delta r_{i,\varepsilon}\right)\subset\Omega_0^R$ for some $R>0$. Thus we can write that

$$\inf_{\partial\mathbb{D}_{x_{j,\varepsilon}}(\delta r_{i,\varepsilon})}u_\varepsilon\geq\big(1+o(1)\big)B_{i,\varepsilon}\left(r_{i,\varepsilon}\right)$$

thanks to (4.98) and (4.100). We can also apply b) of Claim 4.2 with $r_\varepsilon=\delta r_{i,\varepsilon}$ thanks to Claim 4.3 and to the fact that $\dfrac{r_{i,\varepsilon}}{r_{j,\varepsilon}}\geq2\left|\tilde{x}_j\right|^{-1}+o(1)$. This leads to

$$B_{j,\varepsilon}\left(\delta r_{i,\varepsilon}\right)\geq\big(1+o(1)\big)B_{i,\varepsilon}\left(r_{i,\varepsilon}\right)\ .$$

Since

$$B_{j,\varepsilon}\left(\delta r_{i,\varepsilon}\right)=-\gamma_{j,\varepsilon}^{-1}\ln\left(\lambda_\varepsilon\gamma_{j,\varepsilon}^2r_{i,\varepsilon}^2\right)+O\left(\gamma_{j,\varepsilon}^{-1}\right)$$

and

$$B_{i,\varepsilon}\left(r_{i,\varepsilon}\right)=-\gamma_{i,\varepsilon}^{-1}\ln\left(\lambda_\varepsilon\gamma_{i,\varepsilon}^2r_{i,\varepsilon}^2\right)+O\left(\gamma_{i,\varepsilon}^{-1}\right)$$

thanks to Claim 4.11, we obtain that

$$-\gamma_{j,\varepsilon}^{-1}\ln\left(\lambda_\varepsilon\gamma_{j,\varepsilon}^2r_{i,\varepsilon}^2\right)+O\left(\gamma_{j,\varepsilon}^{-1}\right)\geq-\big(1+o(1)\big)\gamma_{i,\varepsilon}^{-1}\ln\left(\lambda_\varepsilon\gamma_{i,\varepsilon}^2r_{i,\varepsilon}^2\right)+O\left(\gamma_{i,\varepsilon}^{-1}\right)\ .$$

This implies since $\gamma_{i,\varepsilon}=O\left(\gamma_{j,\varepsilon}\right)$, see (4.54), that

$$\ln\left(\lambda_\varepsilon\gamma_{i,\varepsilon}^2r_{i,\varepsilon}^2\right)\left(1+o(1)-\frac{\gamma_{i,\varepsilon}}{\gamma_{j,\varepsilon}}\right)\geq-C$$

for some $C>0$. Since $\frac{r_{i,\varepsilon}}{d_{i,\varepsilon}}\to0$ as $\varepsilon\to0$, we get with Claim 4.4 that $\lambda_\varepsilon r_{i,\varepsilon}^2\gamma_{i,\varepsilon}^2\to0$ as $\varepsilon\to0$ and the above implies that $C_{i,j}\geq1$. Thus we have obtained that

$$C_{i,j}\geq1\text{ for all }j\in\mathcal{D}_i^\star\ .\tag{4.103}$$

Thanks to (4.55), we know that $r_{j,\varepsilon} \geq r_{i,\varepsilon}$ for all $i \in \mathcal{D}_i^\star$. Using Claim 4.9, (4.83) and (4.84), we thus obtain that

$$|\nabla u_\varepsilon| \leq C \left(\gamma_{1,\varepsilon}^{-1} + \gamma_{i,\varepsilon}^{-1} \sum_{j \in \mathcal{D}_i} |x_{j,\varepsilon} - x|^{-1} \right)$$

in $\mathbb{D}_{x_{i,\varepsilon}}(Rr_{i,\varepsilon})$ for all $R > 0$. Thanks to (4.99), this leads to

$$|\nabla u_\varepsilon| \leq C\gamma_{i,\varepsilon}^{-1} \sum_{j \in \mathcal{D}_i} |x_{j,\varepsilon} - x|^{-1}$$

in $\mathbb{D}_{x_{i,\varepsilon}}(Rr_{i,\varepsilon})$ for all $R > 0$. We are now in position to follow exactly the end of the proof of Case 2 of Claim 4.10. We can prove that

$$\gamma_{i,\varepsilon}\left(u_\varepsilon(x_{i,\varepsilon} + r_{i,\varepsilon}x) - B_{i,\varepsilon}(r_{i,\varepsilon})\right) \to 2\ln\frac{1}{|x|} + 2\sum_{j \in \mathcal{D}_i^\star} C_{i,j} \ln\frac{1}{|x - \tilde{x}_j|} + A_0$$

in $\mathcal{C}^1_{loc}\left(\mathbb{R}^2 \setminus \mathcal{S}_i\right)$ as $\varepsilon \to 0$ for some constant A_0 and then get a contradiction with Claim 4.1 for $j \in \mathcal{D}_i^\star$ (which is non-empty) such that $|\tilde{x}_j| \geq |\tilde{x}_k|$ for all $k \in \mathcal{D}_i^\star$. Note here that we assumed that $r_{i,\varepsilon} \to 0$ as $\varepsilon \to 0$, see (4.86). This proves that this second case can not happen either. ♠

Case 3 - We assume that $d_{i,\varepsilon} \to 0$ as $\varepsilon \to 0$ and that $\dfrac{d_{i,\varepsilon}}{r_{i,\varepsilon}} \to L$ as $\varepsilon \to 0$ where $L \geq 2$.

We are thus in the case where, after some harmless rotation,

$$\Omega_0 = \mathbb{R} \times (-\infty, L) \ .$$

We let $y \in \Omega_0^R$ for some $R > 0$ and we set $x_\varepsilon = x_{i,\varepsilon} + r_{i,\varepsilon}x$. Since $d_{i,\varepsilon} \to 0$ and $r_{i,\varepsilon} \to 0$, we are in situation b) of Claim 4.8. Indeed, as in Case 1, we have that

$$\frac{|x_\varepsilon - x_{j,\varepsilon}|}{\mu_{j,\varepsilon}} \to +\infty \text{ as } \varepsilon \to 0 \text{ for all } j = 1, \dots, N \ .$$

Thus we can write that

$$\begin{aligned} u_\varepsilon(x_\varepsilon) &= \psi_\varepsilon(x_\varepsilon) + \sum_{j=1}^N \frac{4\pi + o(1)}{\gamma_{j,\varepsilon}} \mathcal{G}(x_{j,\varepsilon}, x_\varepsilon) \\ &\quad + O\left(\sum_{j \in \mathcal{A}} \left(\gamma_{j,\varepsilon}^{-1} \frac{\mu_{j,\varepsilon}}{|x_{j,\varepsilon} - x_\varepsilon|} + \gamma_{j,\varepsilon}^{-2} \ln\left(\frac{s_{j,\varepsilon}}{|x_{j,\varepsilon} - x_\varepsilon|} + 2 \right) \right) \right) \\ &\quad + O\left(\sum_{j \in \mathcal{B}} \frac{d_\varepsilon}{d_\varepsilon + d_{j,\varepsilon}} \left(\gamma_{j,\varepsilon}^{-1} \mu_{j,\varepsilon} + \gamma_{j,\varepsilon}^{-2} s_{j,\varepsilon} \right) \right) \end{aligned}$$

where $\mathcal{A}$ is defined as the set of $j \in \{1, \dots, N\}$ such that $|x_{j,\varepsilon} - x_\varepsilon| \leq s_{j,\varepsilon} + o(d_\varepsilon)$ and $\mathcal{B}$ as its complementary. As in Case 2, we have that

$$\gamma_{j,\varepsilon}^{-1} \frac{\mu_{j,\varepsilon}}{|x_{j,\varepsilon} - x_\varepsilon|} + \gamma_{j,\varepsilon}^{-2} \ln\left(\frac{s_{j,\varepsilon}}{|x_{j,\varepsilon} - x_\varepsilon|} + 2 \right) = o\left(\gamma_{j,\varepsilon}^{-1}\right)$$

for all $j \in \mathcal{A}$ while

$$\frac{d_\varepsilon}{d_\varepsilon + d_{j,\varepsilon}} \left(\gamma_{j,\varepsilon}^{-1} \mu_{j,\varepsilon} + \gamma_{j,\varepsilon}^{-2} s_{j,\varepsilon} \right) = o\left(\gamma_{j,\varepsilon}^{-1}\right)$$

for all $j \in \mathcal{B}$. Note also that, if $j < i$, we have that $j \in \mathcal{B}$ thanks to (4.84) and that

$$\frac{d_\varepsilon}{d_\varepsilon + d_{j,\varepsilon}} \left(\gamma_{j,\varepsilon}^{-1} \mu_{j,\varepsilon} + \gamma_{j,\varepsilon}^{-2} s_{j,\varepsilon} \right) = o\left(d_\varepsilon \gamma_{j,\varepsilon}^{-1}\right) = o\left(r_{i,\varepsilon} \gamma_{j,\varepsilon}^{-1}\right) \ .$$

Thus we have that

$$u_\varepsilon(x_\varepsilon) = \psi_\varepsilon(x_\varepsilon) + \sum_{i=1}^{N} \frac{4\pi + o(1)}{\gamma_{i,\varepsilon}} \mathcal{G}(x_{i,\varepsilon}, x_\varepsilon) + o\left(\gamma_{i,\varepsilon}^{-1}\right) + o\left(r_{i,\varepsilon}\gamma_{1,\varepsilon}^{-1}\right) .$$

We can write thanks to (4.83) and since $\psi_\varepsilon = 0$ on $\partial\Omega$ that

$$\psi_\varepsilon(x_\varepsilon) = O\left(r_{i,\varepsilon}\gamma_{1,\varepsilon}^{-3}\right) .$$

Then we have that, for any $j < i$,

$$\mathcal{G}(x_{j,\varepsilon}, x_\varepsilon) = -d_\varepsilon \partial_\nu \mathcal{G}(x_j, x_i) + o(r_{i,\varepsilon}) .$$

And, for $j \geq i$, we have that

$$\mathcal{G}(x_{j,\varepsilon}, x_\varepsilon) = \frac{1}{2\pi} \ln \frac{|\tilde{y}_j - y|}{|\tilde{x}_j - y|} + o(1)$$

if $j \in \mathcal{D}_i$ where

$$\tilde{y}_j = \mathcal{R}(\tilde{x}_j) ,$$

$\mathcal{R}$ being the reflection with respect to the straight line $\mathbb{R} \times \{L\}$. Here we used (6.12). At last, for $j \geq i$ and $j \notin \mathcal{D}_i$, we have that

$$\mathcal{G}(x_{j,\varepsilon}, x_\varepsilon) = o(1)$$

thanks to (6.12). This leads to

$$\begin{aligned} u_\varepsilon(x_\varepsilon) &= 4\pi d_\varepsilon \gamma_{1,\varepsilon}^{-1} \sum_{j=1}^{i-1} \left(-C_{1,j} \partial_\nu \mathcal{G}(x_j, x_i)\right) \\ &\quad + \sum_{j \in \mathcal{D}_i} \frac{2 + o(1)}{\gamma_{j,\varepsilon}} \ln \frac{|\tilde{y}_j - y|}{|\tilde{x}_j - y|} + o\left(\gamma_{i,\varepsilon}^{-1}\right) + o\left(r_{i,\varepsilon}\gamma_{1,\varepsilon}^{-1}\right) . \end{aligned}$$

This gives in particular that $u_\varepsilon \geq C r_{i,\varepsilon}\gamma_{1,\varepsilon}^{-1}$ on $\partial\mathbb{D}_{x_{i,\varepsilon}}(r_{i,\varepsilon})$ for some $C > 0$. Using b) of Claim 4.2, we deduce that

$$C r_{i,\varepsilon}\gamma_{1,\varepsilon}^{-1} \leq B_{i,\varepsilon}(r_{i,\varepsilon}) + O\left(\gamma_{i,\varepsilon}^{-1}\right) = -\gamma_{i,\varepsilon}^{-1} \ln\left(\lambda_\varepsilon \gamma_{i,\varepsilon}^2 r_{i,\varepsilon}^2\right) + O\left(\gamma_{i,\varepsilon}^{-1}\right) .$$

Thanks to Claim 4.11, this gives that

$$C \frac{r_{i,\varepsilon}\gamma_{i,\varepsilon}}{\gamma_{1,\varepsilon}} \leq -\ln\left(\frac{\gamma_{i,\varepsilon}^2 r_{i,\varepsilon}^2}{\gamma_{1,\varepsilon}^2}\right) + O(1) .$$

This proves that

$$r_{i,\varepsilon}\gamma_{i,\varepsilon} = O(\gamma_{1,\varepsilon})$$

so that, up to a subsequence,

$$\frac{r_{i,\varepsilon}\gamma_{i,\varepsilon}}{\gamma_{1,\varepsilon}} \to B_0 \text{ as } \varepsilon \to 0 . \tag{4.104}$$

Then, by the equation satified by u_ε, it is clear that

$$v_\varepsilon(x) = \gamma_{i,\varepsilon} u_\varepsilon(x_{1,\varepsilon} + r_{i,\varepsilon}x)$$

has a Laplacian uniformly converging to 0 in any Ω_0^R. Thus, by standard elliptic theory, we can conclude that

$$\gamma_{i,\varepsilon} u_\varepsilon(x_{i,\varepsilon} + r_{i,\varepsilon}x) \to B_1(L - y_2) + 2 \sum_{j \in \mathcal{D}_i} C_{i,j} \ln \frac{|\tilde{y}_j - y|}{|\tilde{x}_j - y|} \text{ in } \mathcal{C}^1_{loc}(\Omega_0 \setminus \mathcal{S}_1) \text{ as } \varepsilon \to 0 . \tag{4.105}$$

Using (4.83), (4.104), (4.105) and Claim 4.9, we have that

$$|\nabla u_\varepsilon| \leq C\gamma_{i,\varepsilon}^{-1} |x_{i,\varepsilon} - x|^{-1} \text{ in } \mathbb{D}_{x_{i,\varepsilon}}(r_{i,\varepsilon}) .$$

We are thus in position to apply the results of Section 3 to $u_\varepsilon(x_{i,\varepsilon}+\cdot)$ in the disk $\mathbb{D}_0(r_{i,\varepsilon})$. In particular, applying c) of Proposition 3.1 and combining it with (4.105), we get that

$$\gamma_{i,\varepsilon}B_{i,\varepsilon}(r_{i,\varepsilon})=O(1)\ .$$

This leads with Claim 5.2 of Appendix A to

$$\ln\left(\lambda_\varepsilon r_{i,\varepsilon}^2\gamma_{i,\varepsilon}^2\right)=O(1)\ .$$

Thanks to Claim 4.11, we thus have that $B_0>0$ in (4.104) and $B_1>0$ in (4.105). We can then proceed exactly as in Case 3 of Claim 4.10 to get a contradiction in this last case. ♠

The study of these three cases, all leading to a contradiction, proves that (4.87) is absurd when we assume (4.84). As already said, this permits to prove the claim by induction on i. ♢

We are now in position to prove Theorem 1.2. We know thanks to Claim 4.12 that

$$x_{i,\varepsilon}\to x_i \text{ as } \varepsilon\to 0 \text{ where } x_i\in\Omega\ . \tag{4.106}$$

Claim 4.4 then gives that $\lambda_\varepsilon\gamma_{i,\varepsilon}^2=O(1)$ for all $i=1,\dots,N$. Thanks to Claim 4.11 and (4.54), this implies that, up to a subsequence

$$\frac{1}{\sqrt{\lambda_\varepsilon}\gamma_{i,\varepsilon}}\to m_i \text{ as } \varepsilon\to 0 \tag{4.107}$$

for all $i=1,\dots,N$ with $m_i>0$. Thanks to Claim 4.8, to (4.83) and to the equation satisfied by u_ε, by standard elliptic theory, we obtain that

$$\frac{u_\varepsilon}{\sqrt{\lambda_\varepsilon}}\to 4\pi\sum_{i=1}^N m_i\mathcal{G}(x_i,x) \text{ in } \mathcal{C}^1_{loc}(\Omega\setminus\mathcal{S}) \tag{4.108}$$

where $\mathcal{S}=\{x_i\}_{i=1,\dots,N}$. Moreover, using again (4.83) this time together with Claim 4.9, we know that

$$|\nabla u_\varepsilon|\le C\sqrt{\lambda_\varepsilon}\sum_{i=1}^N|x_{i,\varepsilon}-x|^{-1}$$

in Ω. We are thus in position to apply Claim 4.1 for all $i=1,\dots,N$. This gives that

$$\gamma_{i,\varepsilon}\left(u_\varepsilon(x_{i,\varepsilon}+\delta x)-B_{i,\varepsilon}(\delta)\right)\to 2\ln\frac{1}{|x|}+\mathcal{H}_i(x) \text{ in } \mathcal{C}^1_{loc}(\mathbb{D}_0(1)\setminus\{0\}) \text{ as } \varepsilon\to 0 \tag{4.109}$$

where $\mathcal{H}_i(0)=0$ and $\nabla\mathcal{H}_i(0)=-\frac{1}{2}\delta\frac{\nabla f_0(x_i)}{f_0(x_i)}$. Let us write thanks to Claim 5.2 that

$$\begin{aligned}
B_{i,\varepsilon}(\delta) &= \gamma_{i,\varepsilon}-\gamma_{i,\varepsilon}^{-1}\left(1+\gamma_{i,\varepsilon}^{-2}\right)\ln\left(1+\frac{\delta^2}{4\mu_{i,\varepsilon}^2}\right)+O\left(\gamma_{i,\varepsilon}^{-2}\right)\\
&= \gamma_{i,\varepsilon}-\gamma_{i,\varepsilon}^{-1}\left(1+\gamma_{i,\varepsilon}^{-2}\right)\ln\frac{1}{\mu_{i,\varepsilon}^2}-\gamma_{i,\varepsilon}^{-1}\ln\frac{\delta^2}{4}+o\left(\gamma_{i,\varepsilon}^{-1}\right)\\
&= -\gamma_{i,\varepsilon}^{-1}-\gamma_{i,\varepsilon}^{-1}\ln\left(f_0(x_i)\lambda_\varepsilon\gamma_{i,\varepsilon}^2\right)-\gamma_{i,\varepsilon}^{-1}\ln\frac{\delta^2}{4}+o\left(\gamma_{i,\varepsilon}^{-1}\right)
\end{aligned}$$

so that, thanks to (4.107),

$$\gamma_{i,\varepsilon}B_{i,\varepsilon}(\delta)\to-\ln\frac{\delta^2 f_0(x_i)}{4m_i^2}-1\ .$$

Coming back to (4.109) with this, we get that

$$\gamma_{i,\varepsilon}u_\varepsilon(x)\to 2\ln\frac{1}{|x-x_i|}+\mathcal{H}_i\left(\frac{x-x_i}{\delta}\right)-\ln\frac{f_0(x_i)}{4m_i^2}-1 \text{ in } \mathcal{C}^1_{loc}(\mathbb{D}_{x_i}(\delta)\setminus\{x_i\}) \text{ as } \varepsilon\to 0\ . \tag{4.110}$$

On the other hand, using (4.107) and (4.108), we also have that

$$\gamma_{i,\varepsilon} u_\varepsilon (x) \to \frac{4\pi}{m_i} \sum_{j=1}^{N} m_j \mathcal{G}(x_j, x) \text{ in } \mathcal{C}^1_{loc}(\mathbb{D}_{x_i}(\delta) \setminus \{x_i\}) \text{ as } \varepsilon \to 0 . \tag{4.111}$$

Combining (4.110) and (4.111), we get that

$$m_i \mathcal{H}_i \left(\frac{x - x_i}{\delta} \right) = 4\pi \sum_{j=1}^{N} m_j \mathcal{G}(x_j, x) - 2m_i \ln \frac{1}{|x - x_i|} + m_i \ln \frac{f_0(x_i)}{4m_i^2} + m_i .$$

Writing

$$\mathcal{G}(x, y) = \frac{1}{2\pi} \left(\ln \frac{1}{|x - y|} + \mathcal{H}(x, y) \right) ,$$

this leads to

$$m_i \mathcal{H}_i \left(\frac{x - x_i}{\delta} \right) = 4\pi \sum_{j \neq i} m_j \mathcal{G}(x_j, x) + 2m_i \mathcal{H}(x_i, x) + m_i \ln \frac{f_0(x_i)}{4m_i^2} + m_i .$$

The conditions that $\mathcal{H}_i(0) = 0$ and $\nabla \mathcal{H}_i(0) = -\frac{1}{2}\delta \frac{\nabla f_0(x_i)}{f_0(x_i)}$ read as

$$4\pi \sum_{j \neq i} m_j \mathcal{G}(x_j, x_i) + 2m_i \mathcal{H}(x_i, x_i) + m_i \ln \frac{f_0(x_i)}{4m_i^2} + m_i = 0 \tag{4.112}$$

and

$$4\pi \sum_{j \neq i} m_j \nabla_y \mathcal{G}(x_j, x_i) + 2m_i \nabla_y \mathcal{H}(x_i, x_i) = -\frac{1}{2} m_i \frac{\nabla f_0(x_i)}{f_0(x_i)} . \tag{4.113}$$

This ends the proof of Theorem 1.2, up to change the m_i's as in the statement of the theorem. ♢

5. Appendix A - The standard bubble

In this appendix, we develop the exact form of the standard bubble B_ε which is defined as the radial solution of

$$\Delta B_\varepsilon = \mu_\varepsilon^{-2} \gamma_\varepsilon^{-2} B_\varepsilon e^{B_\varepsilon^2 - \gamma_\varepsilon^2} \text{ in } \mathbb{R}^2 \text{ with } B_\varepsilon(0) = \gamma_\varepsilon \tag{5.1}$$

where $\gamma_\varepsilon \to +\infty$ and $\mu_\varepsilon \to 0$ as $\varepsilon \to 0$. Note that, by standard ordinary differential equations theory, this function is defined on $[0, +\infty)$ and is decreasing.

We perform the change of variables

$$t = \ln \left(1 + \frac{r^2}{4\mu_\varepsilon^2} \right) \tag{5.2}$$

so that we can rewrite equation (5.1) as

$$e^t \left(\left(1 - e^{-t} \right) B_\varepsilon' \right)' = -\frac{B_\varepsilon}{\gamma_\varepsilon^2} e^{2t + B_\varepsilon^2 - \gamma_\varepsilon^2} . \tag{5.3}$$

We shall need the following lemma which can be proved by direct computations :

Lemma 5.1. *The solution φ of*

$$\mathcal{L}(\varphi) = e^t \left(\left(1 - e^{-t} \right) \varphi' \right)' + 2\varphi = F$$

with $\varphi(0) = 0$ and F smooth is

$$\varphi(t) = \int_0^t e^{-s} F(s) \left(\left(1 - 2e^{-t} \right) \left(1 - 2e^{-s} \right) \ln \frac{e^t - 1}{e^s - 1} + 4 \left(e^{-s} - e^{-t} \right) \right) ds .$$

Proof of Lemma 5.1 - We clearly have that $\varphi(0) = 0$ so that we just have to check that φ satisfies the given differential equation. Let us differentiate to obtain that

$$\varphi'(t) = \int_0^t e^{-s}F(s)\left(2e^{-t}\left(1-2e^{-s}\right)\ln\frac{e^t-1}{e^s-1} + \frac{e^t-2}{e^t-1}\left(1-2e^{-s}\right) + 4e^{-t}\right)ds$$

so that

$$\left(1-e^{-t}\right)\varphi'(t) = \int_0^t e^{-s}F(s)\left(2\frac{e^t-1}{e^{2t}}\left(1-2e^{-s}\right)\ln\frac{e^t-1}{e^s-1} + \frac{e^t-2}{e^t}\left(1-2e^{-s}\right) + 4\frac{e^t-1}{e^{2t}}\right)ds\ .$$

Differentiating again, we get that

$$\begin{aligned}\left(\left(1-e^{-t}\right)\varphi'\right)'(t) &= e^{-t}F(t)\left(\frac{e^t-2}{e^t}\left(1-2e^{-t}\right) + 4\frac{e^t-1}{e^{2t}}\right)\\ &\quad + \int_0^t e^{-s}F(s)\left(-2e^{-t}\left(1-2e^{-t}\right)\left(1-2e^{-s}\right)\ln\frac{e^t-1}{e^s-1} + 8e^{-2t} - 8e^{-t}e^{-s}\right)ds\\ &= e^{-t}F(t) - 2e^{-t}\varphi(t)\ ,\end{aligned}$$

which proves the lemma. ♢

Let us define

$$\varphi_0(t) = \int_0^t e^{-s}\left(s-s^2\right)\left(\left(1-2e^{-t}\right)\left(1-2e^{-s}\right)\ln\frac{e^t-1}{e^s-1} + 4\left(e^{-s}-e^{-t}\right)\right)ds \tag{5.4}$$

so that, by lemma 5.1,

$$\mathcal{L}\left(\varphi_0\right)(t) = t - t^2\ . \tag{5.5}$$

We claim now that

$$|\varphi_0(t)+t| \le C_0 \text{ and } \varphi_0'(t) \to 1 \text{ as } t \to +\infty \tag{5.6}$$

for some $C_0 > 0$. Let us write that

$$\begin{aligned}\varphi_0(t) &= \int_0^t e^{-s}\left(s-s^2\right)\left(2e^{-t}\left(1-2e^{-s}\right)\ln\frac{e^t-1}{e^s-1} + \frac{e^t-2}{e^t-1}\left(1-2e^{-s}\right) + 4e^{-t}\right)ds\\ &= \left(2e^{-t}\ln\left(e^t-1\right) + \frac{e^t-2}{e^t-1}\right)\int_0^t e^{-s}\left(s-s^2\right)\left(1-2e^{-s}\right)ds\\ &\quad + e^{-t}\int_0^t e^{-s}\left(s-s^2\right)\left(2\left(1-2e^{-s}\right)\ln\frac{1}{e^s-1} + 4\right)ds\\ &= \left(2e^{-t}\ln\left(e^t-1\right) + \frac{e^t-2}{e^t-1}\right)\left(\left(1+t+t^2\right)e^{-t} - t^2e^{-2t} - 1\right)\\ &\quad + O\left(e^{-t}\right)\\ &= -1 + O\left(\left(1+t^2\right)e^{-t}\right)\ .\end{aligned}$$

This proves the second part of (5.6) by passing to the limit $t \to +\infty$ and the first part by integration.

We set now

$$B_\varepsilon(t) = \gamma_\varepsilon - \frac{t}{\gamma_\varepsilon} + \gamma_\varepsilon^{-3}\varphi_0 + R_\varepsilon\ . \tag{5.7}$$

Claim 5.1. *There exists $D_0 > 0$ such that*

$$|R_\varepsilon'(t)| \le D_0\gamma_\varepsilon^{-5} \text{ for all } 0 \le t \le \gamma_\varepsilon^2 - T_\varepsilon$$

where T_ε is any sequence such that $T_\varepsilon = o\left(\gamma_\varepsilon\right)$ and $\gamma_\varepsilon^k e^{-T_\varepsilon} \to 0$ as $\varepsilon \to 0$ for all k.

Proof of Claim 5.1 - Fix such a sequence T_ε. Let $D_0 > 0$ that we shall choose later. Since $R'_\varepsilon(0) = 0$, there exists $0 < t_\varepsilon \leq \gamma_\varepsilon^2 - T_\varepsilon$ such that

$$|R'_\varepsilon(t)| \leq D_0 \gamma_\varepsilon^{-5} \text{ for all } 0 \leq t \leq t_\varepsilon \ . \tag{5.8}$$

Note that this implies since $R_\varepsilon(0) = 0$ that

$$|R_\varepsilon(t)| \leq D_0 \gamma_\varepsilon^{-5} t \text{ for all } 0 \leq t \leq t_\varepsilon \ . \tag{5.9}$$

We will prove that, for some choice of D_0, this t_ε may be chosen equal to $\gamma_\varepsilon^2 - T_\varepsilon$, which will prove the claim. Now, assume this is not the case, then, for the maximal t_ε such that (5.8) holds, we have that

$$|R'_\varepsilon(t_\varepsilon)| = D_0 \gamma_\varepsilon^{-5} \ . \tag{5.10}$$

This is the statement we will contradict by an appropriate choice of D_0. Let us use (5.3), (5.5) and (5.7) to write that

$$\mathcal{L}(R_\varepsilon) = F_\varepsilon$$

where

$$F_\varepsilon = \frac{1}{\gamma_\varepsilon} - \frac{B_\varepsilon}{\gamma_\varepsilon^2} e^{2t + B_\varepsilon^2 - \gamma_\varepsilon^2} + 2R_\varepsilon - \gamma_\varepsilon^{-3}\left(t - t^2 - 2\varphi_0\right) \ .$$

For $0 \leq t \leq \min\{t_\varepsilon, T_\varepsilon\}$, we have that

$$2t + B_\varepsilon^2 - \gamma_\varepsilon^2 = \frac{t^2}{\gamma_\varepsilon^2} + 2\gamma_\varepsilon R_\varepsilon + 2\gamma_\varepsilon^{-2}\left(1 - \frac{t}{\gamma_\varepsilon^2}\right)\varphi_0 + o\left(\gamma_\varepsilon^{-4}\right)$$

and that

$$\frac{B_\varepsilon}{\gamma_\varepsilon^2} = \gamma_\varepsilon^{-1} - \gamma_\varepsilon^{-2} t + \gamma_\varepsilon^{-5}\varphi_0 + o\left(\gamma_\varepsilon^{-6}\right)$$

thanks to (5.6) and (5.9). Thus we have in particular that

$$\left|2t + B_\varepsilon^2 - \gamma_\varepsilon^2\right| \leq \frac{2t^2}{\gamma_\varepsilon^2} + 2D_0\gamma_\varepsilon^{-4} t + 2\gamma_\varepsilon^{-2}(C_0 + 1) + o\left(\gamma_\varepsilon^{-4}\right) = o(1)$$

again with (5.6) and (5.9). We can write that

$$\begin{aligned} \left|e^{2t + B_\varepsilon^2 - \gamma_\varepsilon^2} - 1 - \left(2t + B_\varepsilon^2 - \gamma_\varepsilon^2\right)\right| &\leq 2\left(2t + B_\varepsilon^2 - \gamma_\varepsilon^2\right)^2 \\ &\leq 20\gamma_\varepsilon^{-4}\left(t^4 + (C_0 + 1)^2\right) \end{aligned}$$

for all $0 \leq t \leq \min\{t_\varepsilon, T_\varepsilon\}$ for ε small. Coming back to F_ε, this leads to

$$|F_\varepsilon| \leq D_1\left(1 + t^4\right)\gamma_\varepsilon^{-5}$$

for all $0 \leq t \leq \min\{t_\varepsilon, T_\varepsilon\}$ where D_1 depends on C_0 but not on D_0. We can use the representation formula of Lemma 5.1 to deduce that

$$\begin{aligned} |R'_\varepsilon(t)| &\leq D_1\gamma_\varepsilon^{-5}\int_0^t e^{-s}\left(1 + s^4\right)\left|2e^{-t}\left(1 - 2e^{-s}\right)\ln\frac{e^t - 1}{e^s - 1} + \frac{e^t - 2}{e^t - 1}\left(1 - 2e^{-s}\right) + 4e^{-t}\right| ds \\ &\leq D_2\gamma_\varepsilon^{-5} \end{aligned}$$

for all $0 \leq t \leq \min\{T_\varepsilon, t_\varepsilon\}$ where D_2 depends only on C_0, not on D_0. Up to choose $D_0 > 2D_2$, we get that $t_\varepsilon > T_\varepsilon$ thanks to (5.10). Moreover we have that

$$|R'_\varepsilon(T_\varepsilon)| \leq D_2\gamma_\varepsilon^{-5} \ . \tag{5.11}$$

From now on, we assume that $t_\varepsilon \geq T_\varepsilon$. For all $T_\varepsilon \leq t \leq \gamma_\varepsilon^2 - T_\varepsilon$, we can write that

$$|F_\varepsilon(t)| \leq C\gamma_\varepsilon e^{\frac{t^2}{\gamma_\varepsilon^2}}$$

for some $C > 0$, depending on D_0 and C_0. Then we write that

$$\begin{aligned}
|R'_\varepsilon(t) - R'_\varepsilon(T_\varepsilon)| &\leq C\gamma_\varepsilon \int_{T_\varepsilon}^t e^{\frac{s^2}{\gamma_\varepsilon^2}-s} \left| 2e^{-t}\left(1-2e^{-s}\right)\ln\frac{e^t-1}{e^s-1} + \frac{e^t-2}{e^t-1}\left(1-2e^{-s}\right) + 4e^{-t}\right| ds \\
&\leq C\gamma_\varepsilon \int_{T_\varepsilon}^t e^{\frac{s^2}{\gamma_\varepsilon^2}-s}\, ds \\
&= O\left(\gamma_\varepsilon \int_{T_\varepsilon}^{\gamma_\varepsilon^2 - T_\varepsilon} e^{\frac{s^2}{\gamma_\varepsilon^2}-s}\, ds\right) \\
&= O\left(\gamma_\varepsilon \int_{T_\varepsilon}^{\frac{1}{2}\gamma_\varepsilon^2} e^{\frac{s^2}{\gamma_\varepsilon^2}-s}\, ds\right) \\
&= O\left(\gamma_\varepsilon \int_{T_\varepsilon}^{\frac{1}{2}\gamma_\varepsilon^2} e^{-\frac{1}{2}s}\, ds\right) \\
&= O\left(\gamma_\varepsilon e^{-\frac{1}{2}T_\varepsilon}\right) = o\left(\gamma_\varepsilon^{-5}\right) \ .
\end{aligned}$$

Combined with (5.11), this gives that

$$|R'_\varepsilon(t)| \leq D_2\gamma_\varepsilon^{-5} + o\left(\gamma_\varepsilon^{-5}\right) \ .$$

This proves that (5.10) is impossible, up to choose $D_0 \geq 2D_2$. This ends the proof of this claim. $\diamondsuit$.

If we want to push a little bit further the estimates, we can get

Claim 5.2. *There exists $C_0 > 0$ such that*

$$\left| B_\varepsilon - \gamma_\varepsilon + \frac{t}{\gamma_\varepsilon} + \frac{t}{\gamma_\varepsilon^3} \right| \leq C_0\gamma_\varepsilon^{-2}$$

for all $0 \leq t \leq \gamma_\varepsilon^2$.

Proof of Claim 5.2 - It is clear that it holds for any $0 \leq t \leq \gamma_\varepsilon^2 - T_\varepsilon$ for T_ε as in Claim 5.1. This is a consequence of Claim 5.1 and of (5.6). We also know that

$$B_\varepsilon\left(\gamma_\varepsilon^2 - T_\varepsilon\right) = \frac{T_\varepsilon}{\gamma_\varepsilon} - \frac{1}{\gamma_\varepsilon} + \frac{T_\varepsilon}{\gamma_\varepsilon^3} + O\left(\gamma_\varepsilon^{-3}\right) \ . \tag{5.12}$$

and that

$$B'_\varepsilon\left(\gamma_\varepsilon^2 - T_\varepsilon\right) = -\frac{1}{\gamma_\varepsilon} - \frac{1}{\gamma_\varepsilon^3} + O\left(\gamma_\varepsilon^{-5}\right) \ . \tag{5.13}$$

Let us integrate twice the equation (5.3) between $\gamma_\varepsilon^2 - T_\varepsilon$ and $t_\varepsilon = \gamma_\varepsilon^2 - \alpha_\varepsilon$ for $0 \leq \alpha_\varepsilon \leq T_\varepsilon$ to write that

$$\begin{aligned}
B_\varepsilon(t_\varepsilon) &= B_\varepsilon\left(\gamma_\varepsilon^2 - T_\varepsilon\right) + B'_\varepsilon\left(\gamma_\varepsilon^2 - T_\varepsilon\right)\left(1 - e^{T_\varepsilon - \gamma_\varepsilon^2}\right)\ln\left(\frac{e^{\gamma_\varepsilon^2-\alpha_\varepsilon}-1}{e^{\gamma_\varepsilon^2-T_\varepsilon}-1}\right) \\
&\quad -\frac{1}{\gamma_\varepsilon^2}\int_{\gamma_\varepsilon^2-T_\varepsilon}^{\gamma_\varepsilon^2-\alpha_\varepsilon} \ln\left(\frac{e^{t_\varepsilon}-1}{e^t-1}\right) B_\varepsilon(t) e^{t+B_\varepsilon(t)^2-\gamma_\varepsilon^2}\, dt \ .
\end{aligned} \tag{5.14}$$

Using (5.12) and (5.13), and remembering that $\alpha_\varepsilon \leq T_\varepsilon = o(\gamma_\varepsilon)$, we obtain that

$$B_\varepsilon(t_\varepsilon) = \gamma_\varepsilon - \frac{t_\varepsilon}{\gamma_\varepsilon} - \frac{t_\varepsilon}{\gamma_\varepsilon^3} + O\left(\gamma_\varepsilon^{-3}\right) - \frac{1}{\gamma_\varepsilon^2}\int_{\gamma_\varepsilon^2-T_\varepsilon}^{\gamma_\varepsilon^2-\alpha_\varepsilon} \ln\left(\frac{e^{t_\varepsilon}-1}{e^t-1}\right) B_\varepsilon(t) e^{t+B_\varepsilon(t)^2-\gamma_\varepsilon^2}\, dt \ . \tag{5.15}$$

Assume that the statement of the Claim holds up to t_ε. If we are able to prove that, under this condition,

$$\int_{\gamma_\varepsilon^2-T_\varepsilon}^{\gamma_\varepsilon^2-\alpha_\varepsilon} \ln\left(\frac{e^{t_\varepsilon}-1}{e^t-1}\right) B_\varepsilon(t) e^{t+B_\varepsilon(t)^2-\gamma_\varepsilon^2}\, dt = o(1)\ , \tag{5.16}$$

then the argument already used in the previous claim will conclude.

If

$$\left|B_\varepsilon - \gamma_\varepsilon + \frac{t}{\gamma_\varepsilon} + \frac{t}{\gamma_\varepsilon^3}\right| \le C_0\gamma_\varepsilon^{-2}$$

for all $0 \le t \le t_\varepsilon$, then we can write that

$$\ln\left(\frac{e^{t_\varepsilon}-1}{e^t-1}\right) |B_\varepsilon(t)|\, e^{t+B_\varepsilon(t)^2-\gamma_\varepsilon^2} = O\left(\gamma_\varepsilon^{-1}\left(1+s^2\right)e^{-s}\right)$$

in $\left[\gamma_\varepsilon^2 - T_\varepsilon, t_\varepsilon\right]$ with $t = \gamma_\varepsilon^2 - s$ so that it is easily checked that

$$\int_{\gamma_\varepsilon^2-T_\varepsilon}^{\gamma_\varepsilon^2-\alpha_\varepsilon} \ln\left(\frac{e^{t_\varepsilon}-1}{e^t-1}\right) B_\varepsilon(t) e^{t+B_\varepsilon(t)^2-\gamma_\varepsilon^2}\, dt = O\left(\gamma_\varepsilon^{-1}\right)\ ,$$

which ends the proof of this claim. ♢

Claim 5.3. *There exists $C_1 > 0$ such that*

$$\left|B_\varepsilon'(t) + \gamma_\varepsilon^{-1}\right| \le C_1\gamma_\varepsilon^{-2}$$

for all $0 \le t \le \gamma_\varepsilon^2$.

Proof of Claim 5.3 - Let us start from the fact that

$$B_\varepsilon'(t) = -\gamma_\varepsilon^{-2}\frac{e^t}{e^t-1}\int_0^t B_\varepsilon(s) e^{s+B_\varepsilon(s)^2-\gamma_\varepsilon^2}\, ds$$

obtained by integrating (5.3). This leads to

$$\left|B_\varepsilon'(t) + \gamma_\varepsilon^{-1}\right| \le \gamma_\varepsilon^{-2}\frac{e^t}{e^t-1}\int_0^t \left|B_\varepsilon(s) e^{2s+B_\varepsilon(s)^2-\gamma_\varepsilon^2} - \gamma_\varepsilon\right| e^{-s}\, ds\ .$$

Let us use Claim 5.2 to write that

$$\left|B_\varepsilon(s) e^{2s+B_\varepsilon(s)^2-\gamma_\varepsilon^2} - \gamma_\varepsilon\right| \le C\gamma_\varepsilon\left(e^{\frac{s^2}{\gamma_\varepsilon^2}}-1\right)e^{-s} + C\frac{s+\gamma_\varepsilon}{\gamma_\varepsilon}e^{\frac{s^2}{\gamma_\varepsilon^2}-s}$$

for some $C > 0$ independent of ε and of $0 \le s \le \gamma_\varepsilon^2$. Thus we get that

$$\left|B_\varepsilon'(t) + \gamma_\varepsilon^{-1}\right| \le C\gamma_\varepsilon^{-1}\frac{e^t}{e^t-1}\int_0^t \left(e^{\frac{s^2}{\gamma_\varepsilon^2}}-1\right)e^{-s}\, ds + C\gamma_\varepsilon^{-3}\frac{e^t}{e^t-1}\int_0^t (s+\gamma_\varepsilon)\, e^{\frac{s^2}{\gamma_\varepsilon^2}-s}\, ds\ .$$

Arguing as above, one gets that

$$\frac{e^t}{e^t-1}\int_0^t \left(e^{\frac{s^2}{\gamma_\varepsilon^2}}-1\right)e^{-s}\, ds \le C\gamma_\varepsilon^{-2}$$

and that

$$\frac{e^t}{e^t-1}\int_0^t (s+\gamma_\varepsilon)\, e^{\frac{s^2}{\gamma_\varepsilon^2}-s}\, ds \le C\gamma_\varepsilon$$

for all $0 \le t \le \gamma_\varepsilon^2$. This permits to end the proof of the claim. ♢

6. Appendix B - Estimates on the Green function

We list and prove some useful estimates on the Green function of the Laplacian with Dirichlet boundary condition in some smooth domain Ω. We fix such a two-dimensional domain and we let $\mathcal{G}(x,y)$ be such that

$$\Delta_x \mathcal{G}(x,y) = \delta_y \text{ with } \mathcal{G}(x,y) = 0 \text{ if } x \in \partial\Omega \, .$$

It is well known that $\mathcal{G}$ is symmetric and smooth outside of the diagonal. Except on the disk of radius R where $\mathcal{G}$ is explicitly given by

$$\mathcal{G}(x,y) = \frac{1}{4\pi} \ln \frac{\left| \frac{|y|}{R} x - \frac{Ry}{|y|} \right|^2}{|x-y|^2}$$

and so where all the estimates below follow from explicit computations, we need to be a little bit careful to estimate the Green function for various x and y.

We know that

$$\mathcal{G}(x,y) = \frac{1}{2\pi} \ln \frac{1}{|x-y|} + \mathcal{H}_y(x) \tag{6.1}$$

where

$$\Delta_x \mathcal{H}_y(x) = 0 \text{ in } \Omega \text{ and } \mathcal{H}_y(x) = -\frac{1}{2\pi} \ln \frac{1}{|x-y|} \text{ on } \partial\Omega \, .$$

First, if $y \in K$ for some compact subset K of Ω, we clearly have that

$$|\mathcal{H}_y(x)| \leq C_K \text{ and } |\nabla \mathcal{H}_y(x)| \leq C_K \tag{6.2}$$

for some $C_K > 0$ for all $x \in \Omega$ so that

$$\begin{aligned}
&\left| \mathcal{G}(x,y) - \frac{1}{2\pi} \ln \frac{1}{|x-y|} \right| \leq C_K \, , \\
&\left| \nabla_x \mathcal{G}(x,y) + \frac{1}{2\pi} \frac{x-y}{|x-y|^2} \right| \leq C_K \, , \\
&|\nabla_x \mathcal{G}(x,y)| \leq C_K |x-y|^{-1} \, , \\
&\left| \mathcal{G}(x,y) - \mathcal{G}(z,y) - \frac{1}{2\pi} \ln \frac{|z-y|}{|x-y|} \right| \leq C_K |x-z|
\end{aligned} \tag{6.3}$$

for all $x, y, z \in K \subset\subset \Omega$ (distinct points).

We let now (y_ε) be a sequence of points in Ω such that

$$d_\varepsilon = d(y_\varepsilon, \partial\Omega) \to 0 \text{ as } \varepsilon \to 0 \, . \tag{6.4}$$

We let now $\tilde{y}_\varepsilon \in \mathbb{R}^2$ be such that

$$\tilde{y}_\varepsilon = 2\pi(y_\varepsilon) - y_\varepsilon \tag{6.5}$$

where π is the projection on the boundary of Ω. Note that $\pi(y_\varepsilon)$ is unique thanks to (6.4) and to the fact that Ω is smooth. Moreover, we have that

$$\tilde{y}_\varepsilon = y_\varepsilon + 2d_\varepsilon \nu_\varepsilon \tag{6.6}$$

where ν_ε is the unit outer normal of $\partial\Omega$ at $\pi(y_\varepsilon)$. We let now

$$\mathcal{G}(x, y_\varepsilon) = \frac{1}{2\pi} \ln \frac{|x-\tilde{y}_\varepsilon|}{|x-y_\varepsilon|} + \tilde{\mathcal{H}}_\varepsilon(x) \tag{6.7}$$

where $\tilde{\mathcal{H}}_\varepsilon$ is harmonic in Ω and satisfies

$$\tilde{\mathcal{H}}_\varepsilon(x) = -\frac{1}{2\pi} \ln \frac{|x-\tilde{y}_\varepsilon|}{|x-y_\varepsilon|} \text{ on } \partial\Omega \, . \tag{6.8}$$

It is easily checked since $\Omega \in \mathcal{C}^2$ that

$$\left|\tilde{\mathcal{H}}_\varepsilon(x)\right| \leq C_\Omega d_\varepsilon$$

for some $C_\Omega > 0$ independent of ε and for all $x \in \partial\Omega$. Thus we have that

$$\left|\tilde{\mathcal{H}}_\varepsilon(x)\right| \leq C_\Omega d_\varepsilon \text{ in } \Omega\,. \tag{6.9}$$

It is also easily checked that

$$\left|\nabla^T \tilde{\mathcal{H}}_\varepsilon(x)\right| \leq C_\Omega \tag{6.10}$$

for all $x \in \partial\Omega$ where ∇^T denotes the tangential derivative. Thus we have that

$$\left|\nabla \tilde{\mathcal{H}}_\varepsilon(x)\right| \leq C_\Omega \frac{d_\varepsilon}{d_\varepsilon + d(x, \partial\Omega)} \text{ in } \Omega\,. \tag{6.11}$$

Let us give some useful consequences of (6.9) and (6.11). Let y_ε be such that $d_\varepsilon = d(y_\varepsilon, \partial\Omega) \to 0$ as $\varepsilon \to 0$, then we have that for any sequence (x_ε) in Ω

$$\begin{aligned}
&\mathcal{G}(x_\varepsilon, y_\varepsilon) = O\left(\frac{d_\varepsilon}{|x_\varepsilon - y_\varepsilon|}\right) && \text{if } d_\varepsilon = O(|x_\varepsilon - y_\varepsilon|)\\
&\mathcal{G}(x_\varepsilon, y_\varepsilon) = \frac{1}{2\pi}\ln\frac{2d_\varepsilon}{|x_\varepsilon - y_\varepsilon|} + O\left(\frac{|x_\varepsilon - y_\varepsilon|}{d_\varepsilon}\right) + O(d_\varepsilon) && \text{if } |x_\varepsilon - y_\varepsilon| = o(d_\varepsilon)\\
&|\nabla_x \mathcal{G}(x_\varepsilon, y_\varepsilon)| = O\left(\frac{d_\varepsilon}{d_\varepsilon + d(x_\varepsilon, \partial\Omega)}\right) && \text{if } d_\varepsilon = O(|x_\varepsilon - y_\varepsilon|)\\
&|\nabla_x \mathcal{G}(x_\varepsilon, y_\varepsilon)| = \frac{1}{2\pi|x_\varepsilon - y_\varepsilon|} + O\left(\frac{1}{d_\varepsilon}\right) && \text{if } |x_\varepsilon - y_\varepsilon| = o(d_\varepsilon)
\end{aligned} \tag{6.12}$$

These are the only estimates which were used in this paper.

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