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Testing for long memory in panel random-coefficient AR(1) data

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Abstract

It is well-known that random-coefficient AR(1) process can have long memory depending on the index β of the tail distribution function of the random coefficient, if it is a regularly varying function at unity. We discuss semiparametric estimation of β from panel data comprising N random-coefficient AR(1) series, each of length T . The estimator of β is constructed as a version of the tail index estimator of Goldie and Smith (1987) applied to sample lag 1 autocorrelations of individual time series. Its asymptotic normality is derived under certain conditions on N , T and some parameters of our statistical model. Based on this result, we construct a statistical procedure to test if the panel random-coefficient AR(1) data exhibit long memory. A simulation study illustrates finite-sample performance of the introduced testing procedure.

Keywords: random-coefficient autoregression; panel data; tail index estimator; long memory process

2010 MSC: 62G32, 62M10.

1 Introduction

Dynamic panels (or longitudinal data) comprising observations taken at regular time intervals for the same individuals such as households, firms, etc. in a large heterogeneous population, are often described by time series models with random parameters (for reviews on dynamic panel data analysis, see [2, 3]). One of the simplest models for individual evolution is the random-coefficient AR(1) (RCAR(1)) process

$$X(t) = aX(t-1) + \zeta(t), \quad t \in \mathbb{Z}, \quad (1)$$

where the innovations $\{\zeta(t), t \in \mathbb{Z}\}$ are independent identically distributed (i.i.d.) random variables (r.v.s) with $\mathbb{E}\zeta(0) = 0$, $\mathbb{E}\zeta^2(0) < \infty$ and the autoregressive coefficient $a \in (0, 1)$ is a r.v., independent of $\{\zeta(t), t \in \mathbb{Z}\}$. If the distribution of a is sufficiently dense near unity, then statistical properties of the individual evolution in (1) and the corresponding panel can differ greatly from those in the case of fixed $a \in (0, 1)$. To be more specific, assume that the AR coefficient a has a density function $g(x)$, $x \in (0, 1)$, satisfying

$$g(x) \sim g_1(1-x)^{\beta-1}, \quad x \rightarrow 1-, \quad (2)$$

for some $\beta > 1$ and $g_1 > 0$. Then a stationary solution of RCAR(1) equation (1) has the following autocovariance function

$$\mathbb{E}X(0)X(t) = \mathbb{E}\zeta^2(0)\mathbb{E}\frac{a^t}{1-a^2} \sim \frac{g_1}{2}\Gamma(\beta-1)\mathbb{E}\zeta^2(0)t^{-(\beta-1)}, \quad t \rightarrow \infty, \quad (3)$$

and exhibits long memory in the sense that $\sum_{t \in \mathbb{Z}} |\text{Cov}(X(0), X(t))| = \infty$ for $\beta \in (1, 2]$. The same long memory property applies to the contemporaneous aggregate

$$\bar{X}_N(t) := N^{-1/2} \sum_{i=1}^N X_i(t), \quad t \in \mathbb{Z}, \quad (4)$$

of N independent individual evolutions $\{X_i(t), t \in \mathbb{Z}\}$, $i = 1, \dots, N$, of (1) and its Gaussian limit arising as $N \rightarrow \infty$. For the beta distributed squared AR coefficient a^2 , these facts were first uncovered by Granger [14] and later extended to more general distributions and/or RCAR equations in Gonçalves and Gouriéroux [13], Zaffaroni [30], Celov et al. [5], Oppenheim and Viano [24], Puplinskaitė and Surgailis [28], Philippe et al. [25] and other works, see [18] for review. Assumption (2) and the parameter β play a crucial role for statistical (dependence) properties of the panel $\{X_i(t), t = 1, \dots, T, i = 1, \dots, N\}$ as N and T increase, possibly at different rates. Particularly, Pilipauskaitė and Surgailis [26] proved that for $\beta \in (1, 2)$ the distribution of the normalized sample mean $\sum_{i=1}^N \sum_{t=1}^T X_i(t)$ is asymptotically normal if $N/T^\beta \rightarrow \infty$ and β -stable if $N/T^\beta \rightarrow 0$ (in the ‘intermediate’ case $N/T^\beta \rightarrow c \in (0, \infty)$ this limit distribution is more complicated and given by an integral with respect to a certain Poisson random measure). A similar but non-identical trichotomy of the limit distribution of the sample mean for a panel comprising RCAR(1) series driven by common innovations is proved in Pilipauskaitė and Surgailis [27].

In the above context, a natural statistical problem concerns inference about the distribution of the random AR coefficient a , e.g., its cumulative distribution function (c.d.f.) G or the parameter β in (2). Leipus et al. [17], Celov et al. [6] estimated the density g using sample autocovariances of the limit aggregated process. For estimating parameters of G , Robinson [29] used the method of moments. He proved asymptotic normality of the estimators for moments of G based on the panel RCAR(1) data as $N \rightarrow \infty$ for fixed T , under the condition $\mathbb{E}(1 - a^2)^{-2} < \infty$ which does not allow for long memory in $\{X(t), t \in \mathbb{Z}\}$. For parameters of the beta distribution, Beran et al. [4] discussed maximum likelihood estimation based on (truncated) sample lag 1 autocorrelations computed from $\{X_i(1), \dots, X_i(T)\}$, $i = 1, \dots, N$, and proved consistency and asymptotic normality of the introduced estimator as $N, T \rightarrow \infty$. In nonparametric context, Leipus et al. [19] studied the empirical c.d.f. of a based on (truncated) sample lag 1 autocorrelations similarly to [4], and derived its asymptotic properties as $N, T \rightarrow \infty$, including those of a kernel density estimator. Moreover, [19] proposed another estimator of moments of G and proved its asymptotic normality as $N, T \rightarrow \infty$. Except for parametric situations, the afore mentioned results do not allow for inferences about the tail parameter β in (2) and testing for the presence or absence of long memory in panel RCAR(1) data.

The present paper discusses in semiparametric context, the estimation of β in (2) from RCAR(1) panel $\{X_i(t), t = 1, \dots, T, i = 1, \dots, N\}$ with finite variance $\mathbb{E}X_i^2(t) < \infty$. We use the fact that (2) implies $\mathbb{P}(1/(1-a) > y) \sim (g_1/\beta)y^{-\beta}$, $y \rightarrow \infty$, i.e. r.v. $1/(1-a)$ follows a heavy-tailed distribution with index $\beta > 1$. Thus, if a_1, \dots, a_N were observed, β could be estimated by a number of tail index estimators. Given panel data, we estimate unobservable a_i by (truncated) sample lag 1 autocorrelation \tilde{a}_i computed from $\{X_i(1), \dots, X_i(T)\}$ similarly to [4, 19], for each $i = 1, \dots, N$. Then we apply the tail-index estimator introduced by Goldie and Smith [10] (also studied by Novak and Utev [23]) to observations $1/(1 - \tilde{a}_1), \dots, 1/(1 - \tilde{a}_N)$. The main result of our paper is Theorem 2 giving sufficient conditions for asymptotic normality of the constructed estimator $\tilde{\beta}_N$. These conditions involve β , rates of growth of N, T and a threshold parameter $\delta = \delta_N \rightarrow 0$ whose choice depends on the second-order regularity parameter ν of G , see (5) below, and the $2p$ -moment of innovations. Based on the above asymptotic result, we construct a statistical procedure to test the presence

of long memory in the panel, more precisely, the null hypothesis $H_0 : \beta \geq 2$ vs. the long memory alternative $H_1 : \beta \in (1, 2)$.

The paper is organized as follows. Section 2 contains the assumptions about the statistical (panel) model and the definition of the estimator $\tilde{\beta}_N$ based on the panel data. We also state the main Theorem 2 together with some consequences. In Section 3 a simulation study illustrates finite-sample properties of the testing procedure for long memory. We also compare our test to the V/S test for long memory based on the aggregated series in (4). Appendix A discusses the choice of the threshold δ_N in terms of other parameters of our RCAR(1) model. Proofs can be found in Appendix B.

In what follows, C stands for a positive constant whose precise value is unimportant and which may change from line to line. We write \rightarrow_p , \rightarrow_d for the convergence in probability and distribution respectively, whereas $\rightarrow_{D[0,1]}$ denotes the weak convergence in the space $D[0, 1]$ with the uniform metric. Notation $\mathcal{N}(\mu, \sigma^2)$ is used for the normal distribution with mean μ and variance σ^2 .

2 Assumptions and the main results

To derive asymptotic results about estimation of β in the RCAR(1) panel model, condition (2) is strengthened as follows.

(G) $a \in (0, 1)$ is a r.v. with c.d.f. $G(x) := \mathbb{P}(a \leq x)$, $x \in [0, 1]$. There exists $\epsilon \in (0, 1)$ such that G is continuously differentiable on $(1 - \epsilon, 1)$ with derivative satisfying

$$g(x) = \kappa\beta(1 - x)^{\beta-1}(1 + O((1 - x)^\nu)), \quad x \rightarrow 1-, \quad (5)$$

for some $\beta > 1$, $\nu > 0$ and $\kappa > 0$.

Let $Y := 1/(1 - a)$. Assumption (G) implies that

$$\mathbb{P}(Y > y) = \kappa y^{-\beta}(1 + O(y^{-\nu})), \quad y \rightarrow \infty. \quad (6)$$

Let Y_1, \dots, Y_N be i.i.d. r.v.s with a c.d.f. satisfying (6). To estimate the tail index β in (6), Goldie and Smith [10] introduced the estimator

$$\beta_N := \frac{\sum_{i=1}^N \mathbf{1}(Y_i \geq v)}{\sum_{i=1}^N \mathbf{1}(Y_i \geq v) \ln(Y_i/v)}, \quad (7)$$

where $v = v_N \rightarrow \infty$ is a threshold level, and proved asymptotic normality and other properties of this estimator.

For independent realizations a_1, \dots, a_N of a under assumption (G), we rewrite the tail-index estimator in (7) as

$$\beta_N = \frac{\sum_{i=1}^N \mathbf{1}(a_i > 1 - \delta)}{\sum_{i=1}^N \mathbf{1}(a_i > 1 - \delta) \ln(\delta/(1 - a_i))}, \quad (8)$$

where $\delta := 1/v$ is a threshold close to 0.

Theorem 1. *Assume that a, a_1, \dots, a_N are i.i.d. r.v.s and (G) holds. If $\delta = \delta_N \rightarrow 0$ and $N\delta^\beta \rightarrow \infty$ and $N\delta^{\beta+2\nu} \rightarrow 0$ as $N \rightarrow \infty$, then*

$$\sqrt{N\delta^\beta}(\beta_N - \beta) \rightarrow_d \mathcal{N}(0, \beta^2/\kappa).$$

Theorem 1 is due to Goldie and Smith [10, Theorem 4.3.2]. The proof in [10] uses Lyapunov's CLT conditionally on the number of exceedances over a threshold. Further sufficient conditions for asymptotic normality of β_N were obtained in Novak and Utev [23]. In Appendix B we give an alternative proof of Theorem 1 based on the tail empirical process. Our proof has the advantage that it can be more easily adapted to prove asymptotic normality of the estimator $\tilde{\beta}_N$ in (13) of parameter β in the panel RCAR(1) model.

Let $X_i := \{X_i(t), t \in \mathbb{Z}\}$, $i = 1, 2, \dots$, be stationary random-coefficient AR(1) processes

$$X_i(t) = a_i X_i(t-1) + \zeta_i(t), \quad t \in \mathbb{Z}, \quad (9)$$

where innovations $\{\zeta_i(t), t \in \mathbb{Z}\}$ admit the following decomposition:

$$\zeta_i(t) = b_i \eta(t) + c_i \xi_i(t), \quad t \in \mathbb{Z}. \quad (10)$$

Let the following assumptions hold:

(A1) $\eta, \eta(t), t \in \mathbb{Z}$, are i.i.d. with $\mathbb{E}\eta = 0$, $\mathbb{E}\eta^2 = 1$, $\mathbb{E}|\eta|^{2p} < \infty$ for some $p > 1$.

(A2) $\xi, \xi_i(t), t \in \mathbb{Z}, i = 1, 2, \dots$, are i.i.d. with $\mathbb{E}\xi = 0$, $\mathbb{E}\xi^2 = 1$, $\mathbb{E}|\xi|^{2p} < \infty$ for the same $p > 1$ as in (A1).

(A3) $(b, c), (b_1, c_1), (b_2, c_2), \dots$, are i.i.d. random vectors with possibly dependent components $b \geq 0, c \geq 0$ satisfying $\mathbb{P}(b + c = 0) = 0$ and $\mathbb{E}(b^2 + c^2) < \infty$.

(A4) a, a_1, a_2, \dots are i.i.d. satisfying assumption (G).

(A5) $\{\eta(t), t \in \mathbb{Z}\}, \{\xi_i(t), t \in \mathbb{Z}\}, a_i$ and (b_i, c_i) are mutually independent for each $i = 1, 2, \dots$

Assumptions (A1)–(A3) about the innovations are very general and allow a uniform treatment of common shock (case $(b, c) = (1, 0)$) and idiosyncratic shock (case $(b, c) = (0, 1)$) situations. Similar assumptions about the innovations are made in [19]. If b or c are random (nonconstant), the innovations $\{\zeta_i(t)\}$ in (10) form a possibly dependent but otherwise uncorrelated stationary process with $\mathbb{E}\zeta_i(0) = 0$, $\mathbb{E}\zeta_i^2(0) = \mathbb{E}(b^2 + c^2)$, $\mathbb{E}\zeta_i(0)\zeta_i(t) = 0, t \neq 0$. Under assumptions (A1)–(A5), for each $i = 1, 2, \dots$ there exists a unique strictly stationary solution of (9) given by

$$X_i(t) = \sum_{s \leq t} a_i^{t-s} \zeta_i(s), \quad t \in \mathbb{Z},$$

with $\mathbb{E}X_i(0) = 0$ and $\mathbb{E}X_i^2(0) = \mathbb{E}(b^2 + c^2)\mathbb{E}(1 - a^2)^{-1} < \infty$, see [19].

From the panel RCAR(1) data $\{X_i(t), t = 1, \dots, T, i = 1, \dots, N\}$ we compute sample lag 1 autocorrelation coefficients

$$\hat{a}_i := \frac{\sum_{t=1}^{T-1} (X_i(t) - \bar{X}_i)(X_i(t+1) - \bar{X}_i)}{\sum_{t=1}^T (X_i(t) - \bar{X}_i)^2}, \quad (11)$$

where $\bar{X}_i := T^{-1} \sum_{t=1}^T X_i(t)$ is the sample mean, $i = 1, \dots, N$. By the Cauchy-Schwarz inequality, the estimator \hat{a}_i in (11) does not exceed 1 in absolute value a.s. Moreover, \hat{a}_i is invariant under the shift and scale transformations of $\{X_i(t)\}$ in (9), i.e., we can replace $\{X_i(t)\}$ by $\{\sigma_i X_i(t) + \mu_i\}$ with some (unknown) $\mu_i \in \mathbb{R}$ and $\sigma_i > 0$ for every $i = 1, 2, \dots$

Next, we choose a threshold level $\delta > 0$ and introduce a truncated estimator

$$\tilde{a}_i := \min(\hat{a}_i, 1 - \delta^2) \quad (12)$$

for $i = 1, \dots, N$. We then define the ‘RCAR’ version of the Goldie-Smith estimator in (8) as

$$\tilde{\beta}_N := \frac{\sum_{i=1}^N \mathbf{1}(\tilde{a}_i > 1 - \delta)}{\sum_{i=1}^N \mathbf{1}(\tilde{a}_i > 1 - \delta) \ln(\delta/(1 - \tilde{a}_i))}. \quad (13)$$

In what follows, let $T = T_N$ be a positive integer-valued function of N , such that $\lim_{N \rightarrow \infty} T_N = \infty$. Let also $\delta = \delta_N > 0$ be a function of N such that $\lim_{N \rightarrow \infty} \delta_N = 0$. For ease of presentation we suppress the dependence of T and δ on N .

Theorem 2. *Assume (A1)–(A5). Let $N \rightarrow \infty$ so that $N\delta^\beta \rightarrow \infty$ and $N\delta^{\beta+2(\beta \wedge \nu)} \rightarrow 0$ and*

$$\sqrt{N\delta^\beta} \gamma \ln \delta \rightarrow 0 \quad \text{if } 1 < p \leq 2, \quad (14)$$

$$\sqrt{N\delta^\beta} ((T\delta^\beta)^{-1} \vee \gamma) \ln \delta \rightarrow 0 \quad \text{if } p > 2, \quad (15)$$

where $\gamma = \gamma_N := (T^{(p-1) \wedge (p/2)} \delta^{p+\beta})^{-1/(p+1)}$. Then

$$\sqrt{N\delta^\beta} (\tilde{\beta}_N - \beta) \rightarrow_d \mathcal{N}(0, \beta^2/\kappa).$$

Corollary 3. *Set $\tilde{K}_N := \sum_{i=1}^N \mathbf{1}(\tilde{a}_i > 1 - \delta)$. Under the assumptions of Theorem 2,*

$$\sqrt{\tilde{K}_N} (\tilde{\beta}_N - \beta) \rightarrow_d \mathcal{N}(0, \beta^2).$$

Remark 1. The reason for truncating sample lag 1 autocorrelation \hat{a}_i at a level less than 1 as in (12) is explained in Beran et al. [4]. In principle, in the estimator (13) we could use a different truncation level from $1 - \delta^2$ in (12), however this new level would enter and further complicate conditions (14)–(15).

Remark 2. In Appendix A, we show that it is possible to construct δ satisfying the assumptions of Theorem 2.

An immediate consequence of Corollary 3 is to provide asymptotic confidence intervals for the parameter β . A more interesting application is to test if the panel RCAR(1) data exhibit long memory. Recall from Introduction that $\beta \in (1, 2)$ corresponds to this situation if RCAR(1) series are independent. Note that the case $\beta = 2$ appears as the boundary between long and short memory. Indeed, in this case the autocovariance function of RCAR(1) is not absolutely summable, but the iterated limit of the sample mean of the panel data follows a normal distribution as for $\beta > 2$ (see [22, 26]). Since it is more important to control the risk of false acceptance of long memory, we choose the following testing hypotheses: the null hypothesis $H_0 : \beta \geq 2$ against the long memory alternative $H_1 : \beta < 2$. We use the following test statistic

$$\tilde{Z}_N := \sqrt{\tilde{K}_N} (\tilde{\beta}_N - 2) / \tilde{\beta}_N. \quad (16)$$

According to Corollary 3, we have

$$\tilde{Z}_N \rightarrow_d \begin{cases} \mathcal{N}(0, 1) & \text{if } \beta = 2, \\ +\infty & \text{if } \beta > 2, \\ -\infty & \text{if } \beta < 2. \end{cases}$$

Fix $\omega \in (0, 1)$ and denote by $z(\omega)$ the ω -quantile of the standard normal distribution. The rejection region $\{\tilde{Z}_N < z(\omega)\}$ has asymptotic level ω for testing the null hypothesis $H_0 : \beta \geq 2$, and is consistent against the alternative $H_1 : \beta < 2$.

3 Simulation study

We examine the finite sample performance of the test $\tilde{Z}_N < z(\omega)$ for $H_0 : \beta \geq 2$ at level ω . We compare our test with its analogue $Z_N := \sqrt{K_N}(\beta_N - 2)/\beta_N < z(\omega)$, where β_N defined in (8) and $K_N := \sum_{i=1}^N \mathbf{1}(a_i > 1 - \delta)$ are based on unobservable AR coefficients a_1, \dots, a_N . We also compare our test with the V/S test of Giraitis et al. [9] which allows to detect long memory in time series, applying the latter method to the aggregated series in (4).

We consider a panel which comprises N i.i.d. RCAR(1) series $\{X_i(t)\}$, $i = 1, \dots, N$ of length T , each of them driven by i.i.d. standard normal innovations $\{\zeta_i(t)\} \equiv \{\xi_i(t)\}$ in (10), with AR coefficients a_i drawn from the following beta-type density

$$g(x) = \frac{2}{B(\alpha, \beta)} x^{2\alpha-1} (1-x^2)^{\beta-1}, \quad x \in (0, 1), \quad (17)$$

with parameters $\alpha > 0$, $\beta > 1$ and $B(\alpha, \beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha + \beta)$ denotes the beta function. In this case, the squared coefficient a_i^2 is beta distributed with parameters (α, β) . We simulate 5000 panels for each configuration of N , T , α and β , where

- $(N, T) = (1000, 5000), (5000, 10000)$,
- $\beta = 1.5, 2, 2.5$,
- $\alpha = 0.75, 1.5, 2.5$.

Given a panel, the test statistics \tilde{Z}_N and Z_N require us to choose the threshold δ . Let us explain our data-driven choice in more detail. The distribution in (17) satisfies $1 - G(1 - \delta) = \mathbb{P}(a > 1 - \delta) = \kappa \delta^\beta (1 + \lambda \delta^\nu + o(\delta^\nu))$, $\delta \rightarrow 0$ for the same $\beta > 1$ and some parameters $\nu > 0$, $\kappa > 0$, $\lambda \neq 0$. According to [10, (4.3.8)], as $N \rightarrow \infty$, the sequence

$$\delta^* := \left(\frac{\beta(\beta + \nu)^2}{2\lambda^2 \nu^3 \kappa N} \right)^{1/(\beta+2\nu)} \quad (18)$$

yields asymptotic normality of β_N in (8) with non-zero mean, β^2 -variance and asymptotically minimizes the mean square error of β_N . In that case on average β_N uses

$$N(1 - G(1 - \delta^*)) = \mathbb{E} \sum_{i=1}^N \mathbf{1}(a_i > 1 - \delta^*) \sim \left(\frac{(1 - \rho)N^{-\rho}}{B\sqrt{-2\rho}} \right)^{2/(1-2\rho)} =: k^* \quad (19)$$

upper order statistics $a_{(1)} < \dots < a_{(N)}$ of a_1, \dots, a_N , where $\rho := -\nu/\beta < 0$, $B := (\nu/\beta)\kappa^{-\nu/\beta}\lambda \neq 0$. If we predetermine a smaller sample fraction $(k^*)^\epsilon < k^*$ with $\epsilon \in (0, 1)$ and choose an albeit random $\delta = 1 - a_{(N - [(k^*)^\epsilon])}$, then asymptotic normality of β_N holds with zero mean as in Theorem 1. Therefore, in our simulations, given a panel and the order statistics $\hat{a}_{(1)} \leq \dots \leq \hat{a}_{(N)}$ calculated on a panel, we choose δ as follows

$$\delta = 1 - \hat{a}_{(N - [(\hat{k}^*)^\epsilon])} \quad (20)$$

for several values of $\epsilon \in (0, 1)$. We obtain \hat{k}^* by replacing ρ , B in (19) by their semiparametric estimates proposed in [8, 11]. We calculate the latter estimates from $\hat{a}_1, \dots, \hat{a}_N$ using the algorithm in [12].

Table 1 reports empirical rejection frequencies for \tilde{Z}_N and Z_N computed with δ in (20) when testing $H_0 : \beta \geq 2$ at significance level $\omega = 5\%$. For $(N, T) = (5000, 10000)$ and different values of α, β, ϵ , the empirical c.d.f.s of computed p -values of \tilde{Z}_N are plotted in Figure 1. Recall that when the significance level of

		$\beta = 1.5$			$\beta = 2$			$\beta = 2.5$		
ϵ	$\alpha =$	0.75	1.5	2.5	0.75	1.5	2.5	0.75	1.5	2.5
(N, T)		\tilde{Z}_N statistic								
(1000, 5000)	0.5	34.2	27.4	22.1	5.5	5.9	5.3	0.4	0.5	0.6
	0.6	50.0	42.2	36.4	5.5	5.8	6.0	0.1	0.2	0.4
	0.7	69.9	64.7	58.5	3.8	7.4	8.4	0.0	0.1	0.2
(5000, 10000)	0.5	47.0	36.4	27.7	5.3	4.7	4.8	0.1	0.3	0.3
	0.6	72.9	61.5	52.8	5.6	5.6	5.2	0.0	0.0	0.1
	0.7	93.9	88.2	82.9	5.4	7.9	7.5	0.0	0.0	0.1
N		Z_N statistic								
1000	0.5	41.9	39.7	38.9	7.0	7.7	7.6	0.4	0.8	0.9
	0.6	59.2	54.6	53.5	8.1	9.1	9.0	0.2	0.4	0.7
	0.7	79.8	75.0	72.5	10.0	10.8	12.0	0.1	0.2	0.4
5000	0.5	57.0	53.2	51.4	6.4	7.3	8.0	0.1	0.2	0.2
	0.6	79.7	75.5	72.2	7.6	8.3	8.8	0.0	0.1	0.1
	0.7	96.0	93.0	91.2	9.6	10.7	11.9	0.0	0.0	0.0

Table 1: Empirical rejection rates (in %) for \tilde{Z}_N and Z_N when testing $H_0 : \beta \geq 2$ at significance level $\omega = 5\%$ based on 5000 replications. Each panel comprises N i.i.d. RCAR(1) series of length T . The AR coefficients are drawn from (17) with parameters (α, β) . Computations of \tilde{Z}_N and Z_N use δ in (20) with estimated parameters B, ρ , for several values of ϵ .

the test is correct, the (asymptotic) distribution of the p -value is uniform on $[0, 1]$. Our comments on Table 1 and Figure 1 are summarized at the end of the section.

In time series theory, several tests for long memory were developed, see [9, 15, 20]. Clearly, these tests cannot be applied to individual RCAR(1) series, the latter being always short memory a.s., independent of the value of β and the distribution of the random coefficient a . However, in practical situations one can apply the above-mentioned tests to the aggregated RCAR(1) series $\{\tilde{X}_N(t), 1 \leq t \leq T\}$ in (4) with covariance decaying as $t^{-(\beta-1)}, t \rightarrow \infty$, see (3). Below, we report a Monte Carlo analysis of the finite sample performance of the V/S test (see [9]) applied to the aggregated RCAR(1) data with short memory ($\beta = 2.5$). The panel RCAR(1) data is simulated from the same model as before for $(N, T) = (5000, 10000)$ and different values of α . Following the same strategy as in Lavancier et al. [16], we derive a data-driven choice of bandwidth ensuring a good size. We use the expansion of the HAC estimator proved by Abadir et al. [1] and minimize its L^2 error under the null hypothesis. We get a bandwidth of the form

$$q^* = \left(\frac{3B^2T}{8\pi^2} \right)^{1/3},$$

where

$$B := \int_0^\infty \left(\frac{f(x)}{f(0) \sin^2(x/2)} \mathbf{1}(0 < x < \pi) - \frac{4}{x^2} \right) dx.$$

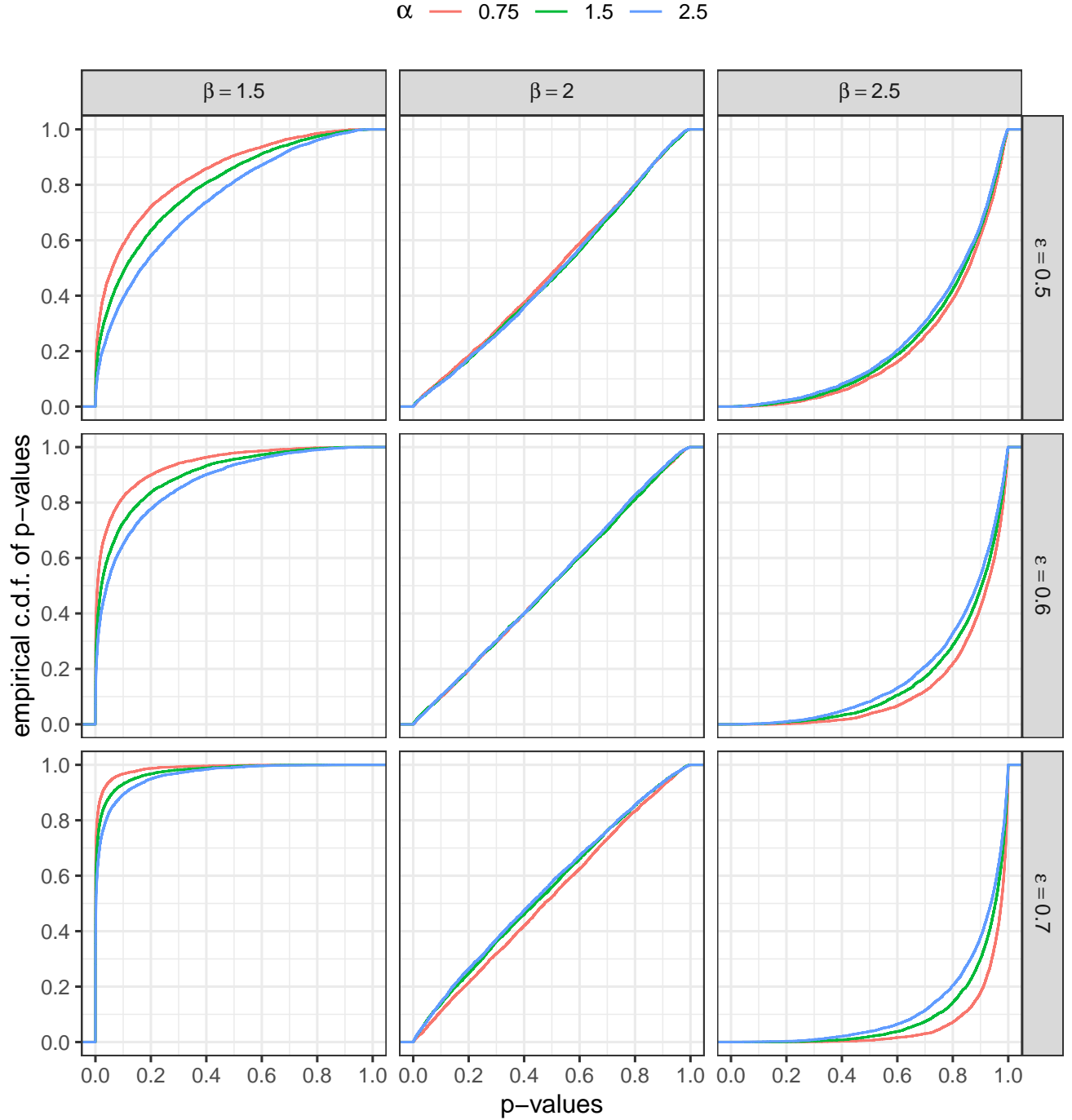


Figure 1: Empirical c.d.f. of p -values of \tilde{Z}_N for testing $H_0 : \beta \geq 2$. Each panel comprises $N = 5000$ i.i.d. RCAR(1) series of length $T = 10000$. The AR coefficients are drawn from (17) with parameters (α, β) . Computations of \tilde{Z}_N use δ in (20) with estimated parameters B, ρ , for several values of ϵ . The number of replications of each experiment is 5000.

depends on the spectral density $f(x) = (1/2\pi)\mathbb{E}|1 - ae^{ix}|^{-2}$ of $\{\tilde{X}_N(t)\}$. To evaluate q^* , we replace $f(x)/f(0)$ by its estimate $\sum_{j=1}^N |1 - \hat{a}_j e^{ix}|^{-2} / \sum_{j=1}^N (1 - \hat{a}_j)^{-2}$, calculated from the estimated AR coefficients.

Figure 2 represents the empirical c.d.f. of computed p -values of the V/S test. Figure 2 shows that the test

is not valid, in the sense that its empirical size is not close to the nominal level. Therefore, its power is not comparable to the one of our test and we do not provide its results. The reason why the V/S test fails for our panel model may be due to the form of the spectral density (see Figure 3) which exhibits a sharp peak at the origin indicating a considerable amount of spurious long memory under short memory null $\beta = 2.5$ for different values of α .

Our conclusions from the simulation experiment are as follows:

- For the choice of δ in (20), the test with the rejection region $\{\tilde{Z}_N < z(\omega)\}$ at nominal level ω is consistent under the null hypothesis $H_0 : \beta \geq 2$. For $\beta = 2$, the empirical probability for \tilde{Z}_N to reject H_0 is close to the nominal level ω almost uniformly in ω .
- The power of the test constructed on \tilde{Z}_N increases with ϵ (see Table 1, Figure 1), however increasing ϵ may worsen the empirical size. As expected, the power of the test increases with the sample size.
- The V/S test applied to contemporaneously aggregated RCAR(1) series seems to be inconsistent in spite of the adapted choice of the tuning parameter.

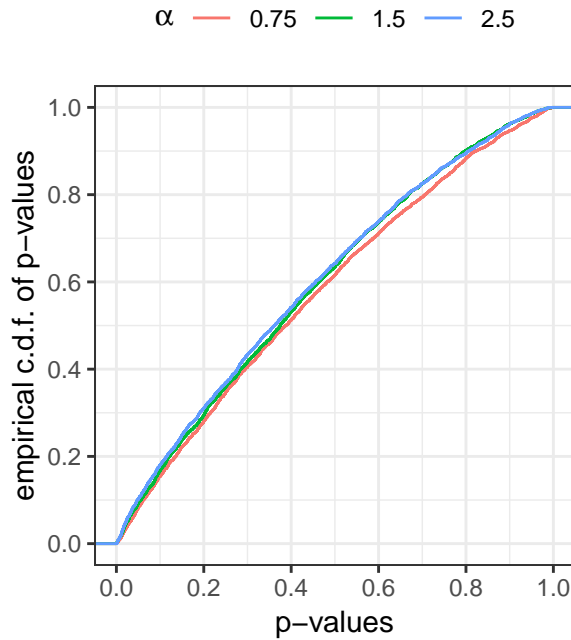


Figure 2: Empirical c.d.f. of p -values of the V/S statistic for testing short memory against long memory. The sample size $T = 10000$. Samples are drawn from the finite sum of $N = 5000$ RCAR(1) with AR coefficient distributed as in (17) with parameters $(\alpha, 2.5)$. The null short memory hypothesis holds. The number of replications of each experiment is 5000.

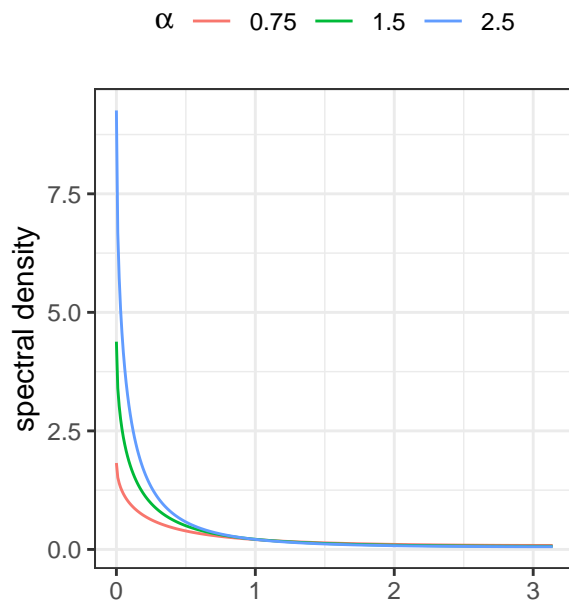


Figure 3: The spectral density $f(x)$, $x \in [0, \pi]$, of an RCAR(1) process $\{X(t)\}$ with AR coefficient distributed as in (17) with parameters $(\alpha, 2.5)$. The aggregated process $\{\bar{X}_N(t)\}$ in (4) of independent copies of $\{X(t)\}$ has the same spectral density.

A The choice of the threshold in Theorem 2

Let us discuss conditions for the choice of the threshold δ in Theorem 2. Note that (A4) restricts this result to the case $\beta > 1$. Assume $p > 2$ in (A1), (A2) and also that $T = T_N$, $\delta = \delta_N$ vary as

$$T \sim C_1 N^a, \quad \delta \sim C_2 N^{-b}$$

for some $a > 0$, $b > 0$ and $C_1 > 0$, $C_2 > 0$. Then $N\delta^\beta \rightarrow \infty$ is equivalent to

$$\beta < \frac{1}{b}. \quad (21)$$

Condition $N\delta^{\beta+2(\beta\wedge\nu)} \rightarrow 0$ is equivalent to

$$\max \left\{ \frac{1}{3b}, \frac{1}{b} - 2\nu \right\} < \beta, \quad (22)$$

whereas (15) is satisfied if and only if

$$\frac{1 + p(1 - a + 2b)}{b(p - 1)} < \beta < \frac{2a - 1}{b}. \quad (23)$$

Inequalities (21)–(23) can be summarized as

$$\max \left\{ 1, \frac{1}{3b}, \frac{1}{b} - 2\nu, \frac{1 + p(1 - a + 2b)}{b(p - 1)} \right\} < \beta < \min \left\{ \frac{1}{b}, \frac{2a - 1}{b} \right\}. \quad (24)$$

In order that the interval for β in (24) is nonempty, we restrict the set of possible values of the parameters a and b . In particular, this is the case and (24) holds if

$$\begin{cases} 1 < \beta < \frac{1}{b}, \\ \max \left\{ \frac{1}{3}, \frac{1}{1 + 2\nu} \right\} \leq b < 1, \quad a \geq \frac{(1 + b)(1 + p)}{p}, \quad \nu > 0. \end{cases} \quad (25)$$

Indeed, the upper bound in (24) is obvious since (25) implies $a > 1$ and the lower bound in (24) holds due to

$$\max \left\{ \frac{1}{3b}, \frac{1}{b} - 2\nu, \frac{1 + p(1 - a + 2b)}{b(p - 1)} \right\} \leq 1, \quad a > 1,$$

which follow from (25).

Albeit being only sufficient for Theorem 2, inequalities (25) provide some limitations and recommendations for estimation of β . Note that (25) restricts the range of β to the interval $(1, 3)$ provided the second-order parameter ν in (5) satisfies $\nu \geq 1$ (which roughly means that the density $g(x)$, $x \in (0, 1)$, is well-approximated by power function $C(1 - x)^{\beta-1}$ in the vicinity of $x = 1$). Condition $\beta < 1/b$ in (25) says that for larger values of β the threshold δ should decrease slower with N , or should be taken larger for fixed N , compared to the choice of δ for smaller β . Finally, the lower bound for a in (25) reflects the fact that the panel length $T \sim C_1 N^a$ should grow much faster than N , with exponent $a > 1 + b > 4/3$ in the limiting case $p = \infty$, in other words, the results of the present paper apply to long panels, similarly to [19].

B Proofs

Notation. In what follows, let $G_N(x) := N^{-1} \sum_{i=1}^N \mathbf{1}(a_i \leq x)$, where a_1, \dots, a_N are i.i.d. with $G(x) := \mathbb{P}(a_1 \leq x)$, $x \in [0, 1]$. Let $\widehat{G}_N(x) := N^{-1} \sum_{i=1}^N \mathbf{1}(\widehat{a}_i \leq x)$, where $\widehat{a}_1, \dots, \widehat{a}_N$ defined by (11) have a common c.d.f. $\widehat{G}(x) := \mathbb{P}(\widehat{a}_1 \leq x)$, $x \in [-1, 1]$.

Proof of Theorem 1. We rewrite the estimator in (8) as

$$\beta_N = \frac{1 - G_N(1 - \delta)}{\int_{1-\delta}^1 \ln(\delta/(1-x)) dG_N(x)} = \frac{1 - G_N(1 - \delta)}{\int_{1-\delta}^1 (1 - G_N(x)) \frac{dx}{1-x}} = \frac{1 - G_N(1 - \delta)}{\int_0^\delta (1 - G_N(1-x)) \frac{dx}{x}}.$$

Next, we decompose $\beta_N - \beta = D^{-1} \sum_{i=1}^4 I_i$, where

$$\begin{aligned} I_1 &:= \beta \int_0^\delta (G_N(1-x) - G(1-x)) \frac{dx}{x}, & I_2 &:= -(G_N(1-\delta) - G(1-\delta)), \\ I_3 &:= -\beta \int_0^\delta (1 - \kappa x^\beta - G(1-x)) \frac{dx}{x}, & I_4 &:= 1 - \kappa \delta^\beta - G(1-\delta) \end{aligned} \quad (26)$$

and

$$D := \int_0^\delta (1 - G_N(1-x)) \frac{dx}{x} = \frac{1}{\beta} (\kappa \delta^\beta - I_1 - I_3). \quad (27)$$

According to the assumptions $(N\delta^\beta)^{1/2} \delta^\nu \rightarrow 0$ and (G), we get $(N\delta^{-\beta})^{1/2} I_4 \rightarrow 0$ and $(N\delta^{-\beta})^{1/2} I_3 \rightarrow 0$.

From the tail empirical process theory, see e.g. [7, Theorem 1], [21, (1.1)–(1.3)], we have that

$$(N\delta^{-\beta})^{1/2} (G_N(1-x\delta) - G(1-x\delta)) \rightarrow_{D[0,1]} \kappa^{1/2} B(x^\beta), \quad (28)$$

where $\{B(x), x \in [0, 1]\}$ is a standard Brownian motion. Therefore, we can expect that

$$(N\delta^{-\beta})^{1/2} (I_1 + I_2) \rightarrow_d \kappa^{1/2} \left(\beta \int_0^1 B(x^\beta) \frac{dx}{x} - B(1) \right). \quad (29)$$

The main technical point to prove (29) is to justify the application of the invariance principle (28) to the integral $(N\delta^{-\beta})^{1/2} I_1$, which is not a continuous functional in the uniform topology on the whole space $D[0, 1]$.

For $\varepsilon > 0$, we split $I_1 := \beta(I_0^\varepsilon + I_\varepsilon^1)$, where

$$I_0^\varepsilon := \int_0^\varepsilon (G_N(1-\delta x) - G(1-\delta x)) \frac{dx}{x}, \quad I_\varepsilon^1 := \int_\varepsilon^1 (G_N(1-\delta x) - G(1-\delta x)) \frac{dx}{x}.$$

By (28), $(N\delta^{-\beta})^{1/2} I_\varepsilon^1 \rightarrow_d \kappa^{1/2} \int_\varepsilon^1 B(x^\beta) \frac{dx}{x}$, where $\mathbb{E} \left| \int_\varepsilon^1 B(x^\beta) \frac{dx}{x} - \int_0^1 B(x^\beta) \frac{dx}{x} \right|^2 \rightarrow 0$ as $\varepsilon \rightarrow 0$. Hence, (29) follows from

$$\lim_{\varepsilon \rightarrow 0} \limsup_{N \rightarrow \infty} \mathbb{E} |(N\delta^{-\beta})^{1/2} I_0^\varepsilon|^2 = 0. \quad (30)$$

In the i.i.d. case $\mathbb{E} |I_0^\varepsilon|^2 = \int_0^\varepsilon \int_0^\varepsilon \text{Cov}(G_N(1-\delta x), G_N(1-\delta y)) \frac{dx dy}{xy}$, where

$$\text{Cov}(G_N(x), G_N(y)) = N^{-1} G(x \wedge y) (1 - G(x \vee y)) \leq N^{-1} (1 - G(x \vee y)),$$

and

$$\mathbb{E} |I_0^\varepsilon|^2 \leq \frac{C}{N} \int_0^\varepsilon \frac{dx}{x} \int_0^x (1 - G(1-\delta y)) \frac{dy}{y} \leq \frac{C}{N} \int_0^\varepsilon \frac{dx}{x} \int_0^x (\delta y)^\beta \frac{dy}{y} = \frac{C}{N\delta^{-\beta}} \int_0^\varepsilon x^{\beta-1} dx = \frac{C\varepsilon^\beta}{N\delta^{-\beta}}, \quad (31)$$

proving (30) and hence (29) too.

Finally, we obtain $\delta^{-\beta} D \rightarrow_p \kappa/\beta$ in view of $(N\delta^{-\beta})^{1/2}(I_1 + I_3) = O_p(1)$ and $N\delta^\beta \rightarrow \infty$.

We conclude that

$$(N\delta^\beta)^{1/2}(\beta_N - \beta) \rightarrow_d \frac{\beta}{\kappa^{1/2}} \left(\beta \int_0^1 B(x^\beta) \frac{dx}{x} - B(1) \right) =: W. \quad (32)$$

Clearly, W follows a normal distribution with zero mean and variance

$$\mathbb{E}W^2 = \frac{\beta^2}{\kappa} \left(2\beta^2 \int_0^1 \frac{dx}{x} \int_0^x y^{\beta-1} dy - 2\beta \int_0^1 x^{\beta-1} dx + 1 \right) = \frac{\beta^2}{\kappa},$$

which agrees with the one in [10]. The proof is complete. \square

In the sequel we will need the following result of [19].

Proposition 4 (Leipus et al. [19]). *Under assumptions (A1)–(A5), for all $\varepsilon \in (0, 1)$ and $T \geq 1$, it holds*

$$\mathbb{P}(|\hat{a}_1 - a_1| > \varepsilon) \leq C(T^{-((p-1) \wedge (p/2))} \varepsilon^{-p} + T^{-1})$$

with $C > 0$ independent of ε, T .

In the proof of Theorem 2 we will use the following proposition.

Proposition 5. *Let assumptions (A1)–(A5) hold. Let $N \rightarrow \infty$ so that $N\delta^\beta \rightarrow \infty$ and (14), (15) hold. Then*

$$(N\delta^{-\beta})^{1/2}(\hat{G}_N(1 - \delta) - G_N(1 - \delta)) = o_p(1), \quad (33)$$

$$(N\delta^{-\beta})^{1/2} \int_{\delta^2}^{\delta} (\hat{G}_N(1 - x) - G_N(1 - x)) \frac{dx}{x} = o_p(1). \quad (34)$$

Proof. For $x \in [1 - \delta, 1]$, write

$$\hat{G}_N(x) - G_N(x) = \frac{1}{N} \sum_{i=1}^N (\mathbf{1}(a_i + \hat{\rho}_i \leq x) - \mathbf{1}(a_i \leq x)) = D'_N(x) - D''_N(x),$$

where $\hat{\rho}_i := \hat{a}_i - a_i$, $i = 1, \dots, N$, and

$$D'_N(x) := \frac{1}{N} \sum_{i=1}^N \mathbf{1}(x < a_i \leq x - \hat{\rho}_i, \hat{\rho}_i \leq 0),$$

$$D''_N(x) := \frac{1}{N} \sum_{i=1}^N \mathbf{1}(x - \hat{\rho}_i < a_i \leq x, \hat{\rho}_i > 0).$$

For all $\gamma > 0$,

$$0 \leq D'_N(x) \leq \frac{1}{N} \sum_{i=1}^N \mathbf{1}(x < a_i \leq x + \gamma\delta) + \frac{1}{N} \sum_{i=1}^N \mathbf{1}(|\hat{\rho}_i| > \gamma\delta) =: I'_N(x) + I''_N,$$

where by Proposition 4,

$$\mathbb{E}I''_N = \mathbb{P}(|\hat{\rho}_1| > \gamma\delta) \leq C(T^{-((p-1) \wedge (p/2))} (\gamma\delta)^{-p} + T^{-1}), \quad (35)$$

and by the relation (5),

$$\mathbb{E}I'_N(x) = \mathbb{P}(x < a_1 \leq x + \gamma\delta) \leq C \int_x^{x+\gamma\delta} (1-u)^{\beta-1} du \leq C\gamma\delta^\beta \quad (36)$$

holds uniformly for all $x \in [1-\delta, 1]$. Choose $\gamma = \gamma_N = o(1)$ so that $T^{-((p-1)\wedge(p/2))}(\gamma\delta)^{-p} \sim \gamma\delta^\beta$ as $N \rightarrow \infty$. Under the conditions (14), (15) and $N\delta^\beta \rightarrow \infty$, from (35), (36) it follows that

$$(N\delta^{-\beta})^{1/2} \int_{\delta^2}^{\delta} \mathbb{E}D'_N(1-x) \frac{dx}{x} \leq C |\ln \delta| (N\delta^{-\beta})^{1/2} (\mathbb{E}I''_N + \sup_{x \in [0, \delta]} \mathbb{E}I'_N(1-x)) = o(1),$$

hence

$$(N\delta^{-\beta})^{1/2} \int_{\delta^2}^{\delta} D'_N(1-x) \frac{dx}{x} = o_p(1)$$

by Markov's inequality. Since

$$(N\delta^{-\beta})^{1/2} \int_{\delta^2}^{\delta} D''_N(1-x) \frac{dx}{x} = o_p(1)$$

is analogous, this proves (34). The same proof works for the relation (33). \square

Proof of Theorem 2. Rewrite

$$\tilde{\beta}_N = \frac{1 - \widehat{G}_N(1-\delta)}{\int_{\delta^2}^{\delta} (1 - \widehat{G}_N(1-x)) \frac{dx}{x}}.$$

Split $\tilde{\beta}_N - \beta = \tilde{D}^{-1}(\sum_{i=1}^4 I_i + \sum_{i=1}^4 R_i)$, where I_i , $i = 1, \dots, 4$, are defined in (26) and

$$\begin{aligned} R_1 &:= \beta \int_{\delta^2}^{\delta} (\widehat{G}_N(1-x) - G_N(1-x)) \frac{dx}{x}, & R_2 &:= G_N(1-\delta) - \widehat{G}_N(1-\delta), \\ R_3 &:= \beta \int_0^{\delta^2} (G(1-x) - G_N(1-x)) \frac{dx}{x}, & R_4 &:= \beta \int_0^{\delta^2} (1 - G(1-x)) \frac{dx}{x} \end{aligned}$$

and

$$\tilde{D} := \int_{\delta^2}^{\delta} (1 - \widehat{G}_N(1-x)) \frac{dx}{x} = D - \frac{1}{\beta}(R_1 + R_3 + R_4)$$

with D given by (27). By Proposition 5, $(N\delta^{-\beta})^{1/2}R_2 = o_p(1)$ and $(N\delta^{-\beta})^{1/2}R_1 = o_p(1)$. In view of (31), we have $\mathbb{E}|(N\delta^{-\beta})^{1/2}R_3|^2 \leq C\delta^{2\beta} = o(1)$ and so $(N\delta^{-\beta})^{1/2}R_3 = o_p(1)$. Finally, $(N\delta^{-\beta})^{1/2}R_4 = o(1)$ as $N\delta^{3\beta} \rightarrow 0$. \square

Proof of Corollary 3. Let $K_N = \sum_{i=1}^N \mathbf{1}(a_i > 1-\delta)$. Since $\text{Var}(K_N) \leq N(1-G(1-\delta))$ and $N(1-G(1-\delta)) \rightarrow \infty$, Markov's inequality yields

$$\frac{K_N}{N(1-G(1-\delta))} \rightarrow_p 1,$$

consequently, $(N\delta^\beta)^{-1}K_N \rightarrow_p \kappa$. By Proposition 5, we have $(N\delta^\beta)^{-1}(\tilde{K}_N - K_N) = o_p(1)$. We conclude that $(N\delta^\beta)^{-1}\tilde{K}_N \rightarrow_p \kappa$. \square

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