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# Disagreement percolation for marked Gibbs point processes

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## Abstract

We generalise disagreement percolation to Gibbs point processes of balls with varying radii. This allows to establish the uniqueness of the Gibbs measure and exponential decay of correlations in the high temperature regime by comparison with a sub-critical Boolean model. Applications to the continuum random cluster model and the Quermass-interaction model are presented. At the core of our proof lies an explicit dependent thinning from a Poisson point process to a dominated Gibbs point process.

*Keywords.* continuum random cluster model, Quermass-interaction model, Widom-Rowlinson model, disagreement percolation, dependent thinning, Boolean model, stochastic domination, phase transition, unique Gibbs state, exponential decay of correlations.

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## 1 Introduction

The class of Gibbs models is a rich class of point processes, where a model is defined through its microscopic properties. The modern formalism is due to Dobrushin [11], Lanford and Ruelle [23, 27], who gave their names to the DLR equations defining the Gibbs states through their conditional probabilities. A classical question is the question of uniqueness of Gibbs states having the same conditional probabilities. One expects uniqueness at low activities and non uniqueness, usually referred to as *phase transition*, at large activities. This was proven for example for the well-known Widom-Rowlinson model [4].

The aim of this paper is to show uniqueness of the Gibbs state for a large class of Gibbs interactions. The method used is a continuum extension of the classical disagreement percolation technique introduced by van den Berg and Maes [32]. This technique has been recently used to prove uniqueness in the case of the hard-sphere model [18]. The present paper generalises this construction to the case of Gibbs point processes of balls. Two natural restrictions are a stochastic domination of the Gibbs point process by a Poisson point process and a locality assumption about the interaction respecting the geometric structure imposed by the balls. The interactions are not assumed to contain only pair-interactions.

The idea behind disagreement percolation is to construct a coupling, named disagreement coupling, between three point processes on a bounded domain. Two marginals are the studied Gibbs point process with different boundary conditions. The third marginal is a dominating Poisson point process. The key property of this coupling is a control of the disagreement points of the two Gibbs instances by the dominating Poisson point process. Therefore, the Poisson point process seen as a Boolean percolation model controls the influence of the boundary conditions. In the sub-critical percolation regime, this influence is small. Hence, we derive the uniqueness of the Gibbs phase for activities lower than the critical percolation threshold of the dominating Poisson point process. In some cases, we show an exponential decay of correlation, proved as a direct consequence of the exponential decay of connectivity in the sub-critical Boolean model and the existence of a disagreement cluster. Our results apply to several Gibbs model such as the continuum random cluster model or a simplified Quermass-interaction model, as well as every Gibbs model with finite range interaction.

The construction of the disagreement coupling is done by recursion in Section 5 and relies strongly on the measurability of a coupling between the Gibbs point process and a dominating Poisson point process. The classic constructions of dominating couplings [26, 14] are implicit and do not yield measurability directly. In Section 4 we derive a new coupling, namely a dependent thinning from the dominating Poisson point process, with explicitly given thinning probabilities. The thinning probabilities are expressed in terms of the derivative of the free energy of the Gibbs point process.

This paper is organised as follows. Section 2 introduces the set-up of the paper. Section 3 presents the results: uniqueness of the Gibbs state, existence of the disagreement coupling and exponential decay of correlation. Section 3.2 discusses applications to different Gibbs models, showing that they satisfy the assumptions of our theorems. A discussion of possible extension, generalisations, and connections

to related methods is in Section 3.3. We give an explicit expression for the thinning probabilities in Section 4 and construct the disagreement coupling in Section 5. The remaining sections contain proofs of the other statements.

## 2 Preliminaries

This section introduces the notation and models needed to state the results in Section 3.

### 2.1 Space

Consider the state space  $S := \mathbb{R}^d \times \mathbb{R}_+$ . Let  $\mathcal{S}$  be the Borel sets of  $S$ . Let  $\mathcal{S}_{bp}$  be those Borel sets of  $S$  whose projection onto  $\mathbb{R}^d$  is a bounded Borel set. Let  $\Omega$  be the set of locally finite points configurations on  $S$ , meaning that for each configuration  $\omega \in \Omega$  and each bounded subset  $\Lambda$  of  $\mathbb{R}^d$ ,  $|\omega \cap (\Lambda \times \mathbb{R}_+)| < \infty$ . For  $\Delta \in \mathcal{S}$ , denote by  $\Omega_\Delta$  the set of configurations contained in  $\Delta$ . For a configuration  $\omega$ , write  $\omega_\Delta$  for  $\omega \cap \Delta$ . Let  $\mathcal{F}$  be the  $\sigma$ -algebra on  $\Omega$  generated by the counting variables, which is compatible with the *Fell topology*. For  $\Delta \in \mathcal{S}$ , consider the sub  $\sigma$ -algebra  $\mathcal{F}_\Delta$  generated by the events

$$\{\omega \in \Omega \mid \omega_\Delta \in E\}, \quad E \in \mathcal{F}.$$

Let  $\mathcal{B}_b$  be the bounded Borel sets of  $\mathbb{R}^d$ . In the case of  $\Delta = \Lambda \times \mathbb{R}_+$  with  $\Lambda \in \mathcal{B}_b$ , we abbreviate  $\omega_{\Lambda \times \mathbb{R}_+}$ ,  $\Omega_{\Lambda \times \mathbb{R}_+}$  and  $\mathcal{F}_{\Lambda \times \mathbb{R}_+}$  as  $\omega_\Lambda$ ,  $\Omega_\Lambda$  and  $\mathcal{F}_\Lambda$  respectively.

We write  $X := (x, r) \in S$ . The *closed ball* of radius  $r$  around  $x$  is  $B(x, r)$  or  $B(X)$ . We write  $B(\omega) := \cup_{X \in \omega} B(X)$ . We abbreviate  $\omega \cup \{X\}$  to  $\omega \cup X$ . A configuration  $\omega \in \Omega$  has an associated *Gilbert graph*  $\mathcal{G}(\omega)$  with vertex set  $\omega$  and an edge between  $X, Y \in \omega$  whenever  $B(X) \cap B(Y) \neq \emptyset$ . We say that  $X, Y \in S$  are *connected by*  $\omega$ , written  $X \overset{\text{in } \omega}{\longleftrightarrow} Y$ , whenever there is a path in  $\mathcal{G}(\omega \cup \{X, Y\})$  between  $X$  and  $Y$ . For a Borel set  $\Lambda \subseteq \mathbb{R}^d$  and a configuration  $\omega'$ , we write  $\Lambda \overset{\text{in } \omega'}{\longleftrightarrow} \omega'$ , if there exists  $x \in \Lambda$  and  $Y \in \omega'$ , such that  $(x, 0) \overset{\text{in } \omega'}{\longleftrightarrow} Y$ . This extends to the other three combinations of a Borel set and a configuration.

### 2.2 Point processes

This work only considers *simple point process* (short: PP). It treats a PP as a *locally finite random subset of points* of  $S$  instead of as a random measure. Hence, the law of a PP  $\mathcal{P}$  is a probability measure on  $\Omega$  with the canonical variable  $\xi$ . Unless there is ambiguity, we refer to a PP by its law and vice-versa.

Let  $\mathcal{L}^d$  be the Lebesgue measure on  $\mathbb{R}^d$ . Because of the fact that the intensity of the unmarked support PP of our marked PPs are absolutely continuous with respect to  $\mathcal{L}^d$ , we can treat our marked PPs on  $\mathbb{R}^d$  with marks in  $\mathbb{R}_+$  as simple PPs on  $S$ .

The most classical PP is the *Poisson point process*. In this paper we consider only the special case of the Poisson PP with intensity measure  $\alpha \mathcal{L}^d \otimes Q$ , where  $\alpha$  is a positive real number called (*spatial*) *intensity* and  $Q$  is a probability measure on  $\mathbb{R}_+$ . The law of this PP is denoted by  $\mathcal{P}_{\alpha, Q}^{\text{poi}}$ , and the projection on  $\Delta \in \mathcal{S}$  is denoted by  $\mathcal{P}_{\Delta, \alpha, Q}^{\text{poi}}$ . An extensive study of the Poisson PP can be found in [8].

The percolation properties of  $\mathcal{P}_{\alpha, Q}^{\text{poi}}$  play an important role in this work. A configuration  $\omega \in \Omega$  *percolates* if its Gilbert graph  $\mathcal{G}(\omega)$  contains at least one unbounded (or infinite) connected component. For a radius measure  $Q$ , let  $\alpha_c(d, Q) \in [0, \infty]$  be the *threshold intensity* separating the sub-critical ( $\mathcal{P}_{\alpha, Q}^{\text{poi}}$ -almost-never percolating) and super-critical ( $\mathcal{P}_{\alpha, Q}^{\text{poi}}$ -almost-surely percolating) phases. One of these phases

may not exist, if and only if  $\alpha_c(d, Q) \in \{0, \infty\}$ . This is always the case in dimension one [25, Thm 3.1]. The average volume of a ball under  $Q$  is a dimension-dependent multiple of

$$\rho(Q) := \int_{\mathbb{R}_+} r^d Q(dr). \quad (2.1)$$

**Theorem 2.1** ([25, 16]). *For  $d \geq 2$ , if  $Q$  satisfies  $\rho(Q) < \infty$ , then there exists a percolation threshold  $\alpha_c(d, Q) \in ]0, \infty[$ . Moreover, for each  $\alpha < \alpha_c(d, Q)$  and  $\Lambda \in \mathcal{B}_b$ ,*

$$\mathcal{P}_{\alpha, Q}^{\text{poi}} \left( \Lambda \overset{\text{in } \xi}{\longleftrightarrow} B(0, n)^c \right) \xrightarrow{n \rightarrow \infty} 0. \quad (2.2)$$

*Furthermore, if the radii are bounded, i.e.,  $Q([0, r_0]) = 1$  for some finite  $r_0$ , then the previous quantity decays exponentially fast. There exist  $\kappa, K$  positive such that, for all Borel sets  $\Lambda_1, \Lambda_2$  of  $\mathbb{R}^d$ ,*

$$\mathcal{P}_{\alpha, Q}^{\text{poi}} \left( \Lambda_1 \overset{\text{in } \xi}{\longleftrightarrow} \Lambda_2 \right) \leq K \exp(-\kappa \theta(\Lambda_1, \Lambda_2)), \quad (2.3)$$

where  $\theta(\Lambda_1, \Lambda_2)$  is the Euclidean distance between the sets  $\Lambda_1$  and  $\Lambda_2$ .

In the case of unbounded radii, the exponential decay of connection in the Poisson Boolean model is not proved. In a recent paper [1], a polynomial decay is proved for the Poisson Boolean model in  $\mathbb{R}^2$  with unbounded radii satisfying some integrability assumption.

### 2.3 Gibbs point processes

In this section, we present a general class of *Gibbs point processes*. For every  $\Delta \in \mathcal{S}_{bp}$ , there is a measurable *Hamiltonian*  $H_\Delta : \Omega_\Delta \times \Omega_{\Delta^c} \rightarrow ]-\infty, \infty]$ . The Hamiltonians are additive in the sense that, for all disjoint  $\Delta_1, \Delta_2 \in \mathcal{S}_{bp}$  and  $\omega^1 \in \Omega_{\Delta_1}$ ,  $\omega^2 \in \Omega_{\Delta_2}$  and  $\gamma \in \Omega_{(\Delta_1 \cup \Delta_2)^c}$ ,

$$H_{\Delta_1 \cup \Delta_2}(\omega^1 \cup \omega^2 \mid \gamma) = H_{\Delta_1}(\omega^1 \mid \gamma \cup \omega^2) + H_{\Delta_2}(\omega^2 \mid \gamma). \quad (2.4)$$

Furthermore we assume  $H_\Delta(\emptyset \mid \gamma) = 0$ , which implies together with (2.4), that if  $\tilde{\Delta} \subseteq \Delta \in \mathcal{S}_{bp}$ ,  $\omega \in \Omega_{\tilde{\Delta}}$  and  $\gamma \in \Omega_{\Delta^c}$

$$H_\Delta(\omega \mid \gamma) = H_{\tilde{\Delta}}(\omega \mid \gamma). \quad (2.5)$$

The *partition function* of the Gibbs PP on  $\Delta$  with boundary condition  $\gamma$  is

$$\mathbf{Z}(\Delta, \gamma) := \int_{\Omega_\Delta} e^{-H_\Delta(\omega \mid \gamma)} \mathcal{P}_{\Delta, \lambda, Q}^{\text{poi}}(d\omega). \quad (2.6)$$

The *specification* of the Gibbs PP on  $\Delta$  with boundary condition  $\gamma$  is the PP law

$$\mathcal{P}_{\Delta, \gamma}^{gb}(d\omega) := \frac{e^{-H_\Delta(\omega \mid \gamma)}}{\mathbf{Z}(\Delta, \gamma)} \mathcal{P}_{\Delta, \lambda, Q}^{\text{poi}}(d\omega). \quad (2.7)$$

A PP  $\mathcal{P}$  is a *Gibbs state* of the specification (2.7), if it fulfils the *DLR equations*. These demand that, for every  $\Delta \in \mathcal{S}_{bp}$  and  $\mathcal{P}(\xi_{\Delta^c} = \cdot)$ -a.s.,

$$\mathcal{P}(\xi_\Delta = d\omega \mid \xi_{\Delta^c} = \gamma) = \mathcal{P}_{\Delta, \gamma}^{gb}(d\omega). \quad (2.8)$$

Write  $\mathcal{G}^{gb}$  for the Gibbs states of (2.7). We assume that  $\mathcal{G}^{gb}$  is non-empty.

## 2.4 Stochastic domination

Let  $\Delta \in \mathcal{B}_b$ . On  $\Omega_\Delta^n$ , the standard product  $\sigma$ -algebra is  $\mathcal{F}_\Delta^{\otimes n}$ . The canonical variables on  $\Omega_\Delta^n$  are  $\xi := (\xi^1, \dots, \xi^n)$ . A *coupling*  $\mathcal{P}$  of  $n$  PP laws  $\mathcal{P}^1, \dots, \mathcal{P}^n$  on  $\Delta$  is a probability measure on  $(\Omega_\Delta^n, \mathcal{F}_\Delta^{\otimes n})$  such that, for all  $1 \leq i \leq n$  and  $E \in \mathcal{F}_\Delta$ ,  $\mathcal{P}(\xi^i \in E) = \mathcal{P}^i(\xi \in E)$ .

An event  $E \in \mathcal{F}$  is called *increasing*, if  $\omega \in E$  implies that  $\omega \cup X \in E$ , for every  $X \in S$ . If  $\mathcal{P}^1$  and  $\mathcal{P}^2$  are two probability measures, then we say that  $\mathcal{P}^2$  *stochastically dominates*  $\mathcal{P}^1$  (short: *dominates*), if  $\mathcal{P}^1(E) \leq \mathcal{P}^2(E)$  for all increasing events  $E$ . By Strassen's theorem [24], this is equivalent to the existence of a coupling  $\mathcal{P}$  of  $\mathcal{P}^1$  and  $\mathcal{P}^2$  such that  $\mathcal{P}(\xi^1 \subseteq \xi^2) = 1$ .

The *Papangelou intensity* is the conditional intensity of adding a point at  $X$  to  $\omega \in \Omega$  [8, Section 15.5]. A classic sufficient condition for stochastic domination of a Gibbs PP by a Poisson PP is the uniform boundedness of the Papangelou intensity [26, 14]. That is,

$$\phi_\omega^{gb}(X) := \lambda \exp(-H_{\{X\}}(X | \omega)) \leq \alpha. \quad (\text{Dom})$$

This is trivially equivalent to the existence of an uniform lower bound for the local energy  $H_{\{X\}}(X | \omega)$ .

## 3 Results and discussion

### 3.1 Disagreement percolation

The idea behind disagreement percolation is to couple two instances of a Gibbs PP on the same  $\Delta \in \mathcal{S}_{bp}$  with arbitrary boundary conditions, such that the set of points differing between the two instances (the *disagreement cluster*) is dominated by a Poisson PP.

**Definition 3.1.** A disagreement coupling family at level  $(\bar{\alpha}, \bar{Q})$  is a family of couplings  $(\mathcal{P}_{\Delta, \gamma^1, \gamma^2}^{dac})$  indexed by  $\Delta \in \mathcal{S}_{bp}$  and  $\gamma^1, \gamma^2 \in \Omega_{\Delta^c}$ , such that they are measurable in the boundary conditions and fulfil

$$\forall 1 \leq i \leq 2: \mathcal{P}_{\Delta, \gamma^1, \gamma^2}^{dac}(\xi^i = d\omega) = \mathcal{P}_{\Delta, \gamma^i}^{gb}(d\omega), \quad (3.1a)$$

$$\mathcal{P}_{\Delta, \gamma^1, \gamma^2}^{dac}(\xi^3 = d\omega) = \mathcal{P}_{\Delta, \bar{\alpha}, \bar{Q}}^{poi}(d\omega), \quad (3.1b)$$

$$\mathcal{P}_{\Delta, \gamma^1, \gamma^2}^{dac}(\xi^1 \cup \xi^2 \subseteq \xi^3) = 1, \quad (3.1c)$$

$$\mathcal{P}_{\Delta, \gamma^1, \gamma^2}^{dac}(\forall X \in \xi^1 \Delta \xi^2 : X \xrightarrow{\text{in } \xi^3} \gamma^1 \cup \gamma^2) = 1. \quad (3.1d)$$

Therefore, control of the disagreement cluster by the percolation cluster of a Boolean model is possible. Hence, a sub-critical Boolean model implies the uniqueness of the Gibbs state.

**Theorem 3.2.** If there exists a disagreement coupling family at level  $(\bar{\alpha}, \bar{Q})$  such that  $\bar{Q}$  satisfies the integrability assumption  $\rho(\bar{Q}) < \infty$  and is sub-critical with  $\bar{\alpha} < \alpha_c(d, \bar{Q})$ , then there is a unique Gibbs state in  $\mathcal{G}^{gb}$ .

The proof of Theorem 3.2 is in Section 6. Theorem 3.2 transfers the question of uniqueness to question of existence of a suitable disagreement coupling family. The key property of a disagreement coupling family to ensure uniqueness is (3.1d), which places the disagreement points of the two Gibbs instances into a percolation cluster of the Boolean model. To ensure this property, we need that the Hamiltonian  $H_\Delta(\omega | \gamma)$  depends only on those points in  $\gamma$  which are connected to  $\omega$  in  $\mathcal{G}(\omega \cup \gamma)$ .

With (2.4), this is equivalent to the following. If  $\omega \in \Omega_\Delta$  and  $\gamma \in \Omega_{\Delta^c}$  are such that  $\omega$  and  $\gamma$  are not connected in  $\mathcal{G}(\omega \cup \gamma)$ , then

$$H_\Delta(\omega \mid \gamma) = H_\Delta(\omega \mid \emptyset). \quad (\text{Loc})$$

**Theorem 3.3.** *If  $\mathcal{P}_{\Delta, \alpha, Q}^{poi}$  dominates  $\mathcal{P}_{\Delta, \gamma}^{gb}$  for all choices of  $\Delta \in \mathcal{S}_{bp}$  and  $\gamma \in \Omega_{\Delta^c}$ , and condition (Loc) is fulfilled, then there exists a disagreement coupling family at level  $(\alpha, Q)$ .*

The proof of Theorem 3.3 is in Section 5. The easy way is to demand a priori stochastic domination of a single Gibbs PP.

In the case of bounded radii, the connection probabilities decay exponentially in a sub-critical Poisson Boolean model. This translates into an exponential decay in influence of the boundary condition of the Gibbs PP and the reduced pair correlation function of the Gibbs state.

**Theorem 3.4.** *Assume that  $\mathcal{P}^{dac}$  is a disagreement coupling family for  $\mathcal{P}^{gb}$  at level  $(\bar{\alpha}, \bar{Q})$  such that  $\bar{Q}$  has bounded support and such that  $\bar{\alpha} < \alpha_c(\bar{Q})$ . Let  $\kappa$  be the constant from (2.3) for  $\mathcal{P}_{\bar{\alpha}, \bar{Q}}^{poi}$ . There exists a unique Gibbs state  $\mathcal{P}$  in  $\mathcal{G}^{gb}$  and  $K' > 0$ , such that:*

*For all  $\Lambda, \bar{\Lambda} \in \mathcal{B}_b$  with  $\Lambda \subseteq \bar{\Lambda}$ ,  $\gamma \in \Omega_{\bar{\Lambda}^c}$  and  $E \in \mathcal{F}_\Lambda$ ,*

$$|\mathcal{P}_{\Lambda, \gamma}^{gb}(E) - \mathcal{P}(E)| \leq K' \exp(-\kappa\theta(\Lambda, \bar{\Lambda}^c)). \quad (3.2a)$$

*For all  $\Lambda_1, \Lambda_2 \in \mathcal{B}_b$ ,  $E \in \mathcal{F}_{\Lambda_1}$  and  $F \in \mathcal{F}_{\Lambda_2}$ ,*

$$|\mathcal{P}(E \cap F) - \mathcal{P}(E)\mathcal{P}(F)| \leq K' \exp(-\kappa\theta(\Lambda_1, \Lambda_2)). \quad (3.2b)$$

The proof of Theorem 3.4 is in Section 7.

## 3.2 Applications

### 3.2.1 Gibbs models with finite range interaction

Consider Gibbs models on  $\mathbb{R}^d$  with finite range interaction  $R > 0$ . Examples of such models are the hard-sphere model and the area interaction model with deterministic radii, the Strauss model [30], and so on. A general result of Preston [26] establishes the existence of a Gibbs state. By taking  $Q = \delta_R$ , this kind of models fits the setting of the present article and the condition (Loc) is automatically fulfilled. If the model satisfies condition (Dom), then, by applying Theorem 3.2, Theorem 3.3 and Theorem 3.4, we obtain the uniqueness of the Gibbs state and the exponential decay of correlation at low activity. In the case of the hard-sphere model, this result was already proved in [18].

### 3.2.2 Continuum random cluster model

The continuum random cluster model, also known as continuum FK-percolation model, is a Gibbs model of random balls whose interaction depends on the number of connected components of the Gilbert graph. This model was introduced in the 1980s as a continuum analogue of the well-known continuum random cluster model [17]. Recently, existence and percolation properties of this model were investigated in [10, 19]. Formally, for  $X \in \mathcal{S}$  and  $\omega \in \Omega$ , we have

$$H_{\{X\}}^{rcm}(X \mid \omega) := -\log(q) (1 - k(X, \omega)),$$

where  $q > 0$  is the connectivity parameter of the model and  $k(X, \omega)$  denotes the number of connected components of  $\mathcal{G}(\omega)$  connected to  $X$  in  $\mathcal{G}(\omega \cup X)$ . The dependence of  $H_{\{X\}}^{ercm}(X | \omega)$  on  $\omega$  via  $k(X, \omega)$  implies that the model satisfies (Loc). It also satisfies (Dom), because

$$H_{\{X\}}^{ercm}(X | \omega) \geq -\log q.$$

By Theorem 3.3, if  $Q$  satisfies  $\rho(Q) < \infty$ , then there exists a disagreement coupling family at level  $(\lambda q, Q)$ . So by Theorem 3.2, if  $\lambda < \alpha_c(d, Q)/q$ , there is a unique Gibbs state.

### 3.2.3 Widom-Rowlinson model

The Widom-Rowlinson model is a well-known model of statistical mechanics introduced in 1970 [35] to model the interaction between two gases. It is also the first continuum model for which a phase transition was proved, by Ruelle [28] using Peierl's argument. A modern proof of this phase transition was done by Chayes, Chayes and Kotecký [4] using percolation properties of the continuum random cluster model. This idea was extended to the case of unbounded radii in [19].

This model does not follow strictly the setting of the article, because each ball is assigned a colour mark  $i$  belonging to some finite set of cardinality  $q$ . The Hamiltonian is a hard-core constraint on the colouring. Configurations without overlapping balls of different colours are forbidden. In other words, each connected component of the Gilbert graph must be mono-coloured.

If one conditions on the colour mark of a point, then this model satisfies assumptions (Loc) and (Dom) with Papangelou intensity bounded by one. Hence, we can apply disagreement percolation to the color-blind projection of this model. If the radius measure satisfies  $\rho(Q) < \infty$ , then there exists a disagreement coupling family at level  $(\lambda, Q)$ . Therefore, there is a unique Widom-Rowlinson Gibbs state for activities  $\lambda < \alpha_c(d, Q)$ .

### 3.2.4 Quermass-interaction model

The Quermass-interaction model is a Gibbs model of random balls in  $\mathbb{R}^2$  whose interaction depends on the perimeter, area and Euler characteristic of the random structure. It was introduced in [20]. The existence of the infinite volume Gibbs model was proved in [9] and the existence of a supercritical percolation phase was proved in [6].

Fix  $\theta_1, \theta_2, \theta_3 \in \mathbb{R}$ . Let  $Area(X, \omega)$ ,  $Per(X, \omega)$  and  $Euler(X, \omega)$  be the variation of the area, the perimeter and the Euler characteristic respectively, when the ball  $B(X)$  is added to  $B(\omega)$ . The local energy of the Quermass-interaction model is

$$H_{\{X\}}^{quer}(X | \omega) := \theta_1 Area(X, \omega) + \theta_2 Per(X, \omega) + \theta_3 Euler(X, \omega).$$

The contribution of the Euler characteristic is difficult to control. In particular when  $\theta_3 \neq 0$ , the domination condition (Dom) is not satisfied, even with deterministic radii.

From here on, we only consider the case of  $\theta_3 = 0$  and the radius having support on some positive finite interval, meaning that  $Q([r_0, r_1]) = 1$  for some  $0 < r_0 \leq r_1 < \infty$ . In this setting, the interaction is local and satisfies (Loc). Using standard bounds from [6, Lemma 4.12], we have upper and lower bounds for the Papangelou intensity and (Dom) is satisfied. Therefore, by applying Theorem 3.3 and Theorem 3.4, one gets the uniqueness of the Quermass-interaction Gibbs phase and the exponential decay of correlations for small enough the activity  $z$ , depending on the parameters  $\alpha_c(2, Q), \theta_1, \theta_2, r_0$  and  $r_1$ .



### 3.3 Discussion

The domination condition (Dom) is very restrictive. One way to weaken this condition is to demand the following bound on the local energy.

$$H_{\{(x,r)\}}((x,r) \mid \omega) \geq g(r), \quad (\text{Weak-Dom})$$

where  $g$  is a nice enough measurable function. In that case the dominating Poisson PP has a radius measure with unnormalized density  $e^{-g(r)}$  with respect to  $Q$ . However the construction of the dependant thinning done in Section 4 does not carry out without new difficult conditions. But if one gets the existence of a coupling with measurability with respect to the boundary condition, the construction of the disagreement coupling family done in Section 5 would carry out the same and Theorem 3.3 would still be valid.

For difficult radius measures, the analysis of the dominating Boolean model might be complicated. If a radius measure  $Q'$  dominates  $\bar{Q}$ , then the gluing lemma [34, Chapter 1] couples the disagreement coupling with a dominating coupling between  $\mathcal{P}_{\Delta, \bar{\alpha}, \bar{Q}}^{\text{poi}}$  and  $\mathcal{P}_{\Delta, \bar{\alpha}, Q'}^{\text{poi}}$ . This allows to use the condition  $\bar{\alpha} < \alpha_c(d, Q')$  in Theorem 3.2.

If the geometric objects are more general, but can be dominated by a radius law, such that the objects are a.s. contained in larger balls, then one can rewrite the Gibbs interaction with the help of indicators in terms of spheres and apply the theorems in Section 3.1. An example is the segment process in [2].

If the geometric objects are described by more real parameters and the objects are monotone growing in the parameters, a straightforward extension of the derivation approach could work, too. For more general mark spaces, such as compact sets containing zero, and general laws on them, one approach could be to split the derivation in Section 4.2 into a purely spatial component and work with the joint mark distributions conditional on the locations.

The coupling in the proof of Theorem 3.3, in Definition 3.1 in Section 5, reduces to the coupling family for the hard-sphere model used in [18]. For other finite-range Gibbs models, it improves upon the conjectured general product construction discussed in [18] by a factor of two.

Other classic conditions for uniqueness of the low-activity Gibbs measure are cluster expansion and Dobrushin uniqueness. An explicit comparison with cluster expansion has been done for the hard-sphere model in [18]. It shows that disagreement percolation is better in dimensions one to three and asymptotically and suggests that this holds more generally. The exponential decay of correlations in Theorem 3.4 could be used to derive complete analyticity [12], too. Dobrushin uniqueness [11], generalised to finite-range interaction Gibbs PPs in [21, Thm 2.2], derives uniqueness from the summability of the variation distance between two Gibbs instances with the same boundary condition, except on a finite set of points. In the setting of our paper, the Dobrushin condition can be checked using the disagreement coupling and the exponential bounds from Theorem 3.4.

## 4 A dependent thinning

The main result of this section is an explicit construction of a coupling between a dominating Poisson PP and Gibbs PP  $\mathcal{P}_{\Delta, \gamma}^{gb}$  in Proposition 4.1 in Section 4.2. It is realised as a dependent thinning and is measurable in the boundary condition  $\gamma$ . The thinning probability of a single point is related to the derivative of the free energy, as it sweeps through  $\Delta$ . This sweep and the derivative build on a measurable and bijective linearisation of  $S$ , are the focus of Section 4.1.

## 4.1 Measurable ordering

First, map a non-negative real number to its shortest binary digit expansion, filled up with zeros to a bi-infinite sequence of 0s and 1s. In the case of  $x$  being a multiple of  $2^n$ , for some  $n \in \mathbb{Z}$ , this avoids the representation of  $x$  with only 1s below index  $n$ . For example, with  $\bar{a}$  denoting an infinite sequence of the digit  $a \in \{0, 1\}$  and the decimal point “.” to the left of the power 0 coefficient, 2 maps to  $\bar{0}10.\bar{0}$  instead of  $\bar{0}1.\bar{1}$ . With

$$D := \{\iota \in \{0, 1\}^{\mathbb{Z}} \mid \forall n \in \mathbb{Z} : \exists m \leq n \in \mathbb{Z} : \iota_m = 0 \wedge \exists m \in \mathbb{Z} : \forall n \geq m : \iota_n = 0\}$$

the mapping is

$$b : \quad \mathbb{R}_+ \rightarrow D \quad x := \sum_{n \in \mathbb{Z}} \iota_n 2^n \mapsto (\iota_n)_{n \in \mathbb{Z}}. \quad (4.1)$$

The map  $b$  is bijective and measurable in both directions.

Second, we use  $b$  to linearise  $\mathbb{R}_+^m$ . Consider the map

$$B : \quad \mathbb{R}_+^m \rightarrow \mathbb{R}_+ \quad x \mapsto b^{-1}(n \mapsto b(x_{n \bmod m})_{\lfloor n/m \rfloor}). \quad (4.2)$$

The map  $B$  juxtaposes the digits of the same power of 2 of the coordinates of  $x$  and maps the result back to  $\mathbb{R}_+$ . The map  $B$  is bijective and measurable in both directions. It equips  $\mathbb{R}_+^m$  with a measurable total order  $\preceq$  defined by

$$x \preceq y \Leftrightarrow B(x) \leq B(y). \quad (4.3)$$

For  $n \in \mathbb{Z}$  and  $\bar{a} \in \mathbb{N}^m$ , the dyadic hyperblock  $\prod_{i=1}^m [\frac{a_i}{2^n}, \frac{a_i+1}{2^n}] \subseteq \mathbb{R}_+^m$  and the dyadic interval  $[B(\bar{a}), B(\bar{a}) + \frac{1}{2^n}] \subseteq \mathbb{R}_+$  are in bijection.

Without loss of generality, translation on  $\mathbb{R}^d$  allows us to only consider  $\Delta \in \mathcal{S}_{bp}$  with support in  $\mathbb{R}_+^d$  in the remainder of Section 4. Apply the ordering and bijection from above to  $\mathbb{R}_+^{d+1}$ . Although  $Q$  may contain atoms,  $\mathcal{L}^d \otimes Q$  and  $\mathcal{Q}^* := (\mathcal{L}^d \otimes Q) \circ B^{-1}$  are diffuse. Because  $B$  is a measurable bijection, we may not always write it and switch between  $\mathbb{R}_+^{d+1}$  and its linearisation in a notational lightweight and implicit fashion. This also holds for the measures above.

For  $\mathcal{Q}^*$ -a.e  $X \in \Delta$ , there exists  $\varepsilon > 0$  and  $X_\varepsilon^+ \in \Delta$  with  $X \preceq X_\varepsilon^+$  such that  $\mathcal{Q}^*([X, X_\varepsilon^+]) = \varepsilon$ . For a function  $f : \Delta \rightarrow \mathbb{R}$ , we define the derivative at  $X$  as

$$\frac{\partial f}{\partial X}(X) := \lim_{\varepsilon \rightarrow 0} \frac{f(X_\varepsilon^+) - f(X)}{\varepsilon}, \quad (4.4)$$

whenever this limit exists.

## 4.2 The thinning approach

For  $\Delta \in \mathcal{S}_{bp}$  with support in  $\mathbb{R}_+^d$  and  $\gamma \in \Omega_{\Delta^c}$ , the partition function as traditionally given in statistical mechanics is

$$\mathcal{Z}(\Delta, \gamma) := \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \int_{\Delta^n} e^{H_\Delta(\cup_{i=1}^n X_i | \gamma)} \prod_{i=1}^n \mathcal{L}^d \otimes Q(dX_i). \quad (4.5)$$

The relation between  $\mathcal{Z}(\Delta, \gamma)$  and  $\mathbf{Z}(\Delta, \gamma)$  is an exponential factor for the void space of the Poisson PP

$$\mathcal{Z}(\Delta, \gamma) = \mathbf{Z}(\Delta, \gamma) e^{\lambda \mathcal{L}^d \otimes Q(\Delta)}. \quad (4.6)$$

The use of  $\mathcal{Z}$  allows for more succinct expressions. In particular,  $\mathcal{Z}(\Delta, \gamma) \geq 1$  always.

Given  $\Delta \in \mathcal{S}_{bp}$  and  $\gamma \in \Omega_{\Delta^c}$ , we want to thin  $\mathcal{P}_{\Delta, \alpha, Q}^{\text{poi}}$  to  $\mathcal{P}_{\Delta, \gamma}^{gb}$ . We order  $\Delta$  by  $\preceq$  and restrict intervals  $]X, Y]$  (and all variants thereof) to  $\Delta$ .

**Proposition 4.1.** *A thinning from  $\mathcal{P}_{\Delta,\alpha,Q}^{\text{poi}}$  to  $\mathcal{P}_{\Delta,\gamma}^{\text{gb}}$  is given by*

$$\mathcal{P}_{\Delta,\gamma,\alpha,Q}^{\text{thin}}(\text{d}(\omega^1, \omega^2)) := p_{\Delta,\gamma}^j(\omega' | \omega) \mathcal{P}_{\Delta,\alpha,Q}^{\text{poi}}(\text{d}\omega), \quad (4.7)$$

with the joint thinning probability being

$$p_{\Delta,\gamma}^j(\omega' | \omega) := [\omega' \subseteq \omega] \left( \prod_{Y \in \omega'} p_{\Delta,\gamma}^s(Y | \omega'_{]-\infty, Y[}) \right) \times \left( \prod_{Z \in \omega \setminus \omega'} (1 - p_{\Delta,\gamma}^s(Z | \omega'_{]-\infty, Z[})) \right), \quad (4.8)$$

and the dependent single point thinning probability being

$$p_{\Delta,\gamma}^s(X | \omega') := -\frac{1}{\alpha} \frac{\partial}{\partial X} \log \mathcal{Z}([X, \infty[, \omega' \cup \gamma) \quad (4.9)$$

$$= \frac{\lambda}{\alpha} e^{-H_{\{X\}}(X, \gamma \cup \omega')} \frac{\mathcal{Z}([X, \infty[, \gamma \cup \omega' \cup X)}{\mathcal{Z}([X, \infty[, \gamma \cup \omega')}. \quad (4.10)$$

The derivative in (4.9) is as in (4.4). The thinning probabilities and the thinning itself are measurable in the boundary condition.

*Proof.* Consider the points of a realisation  $\omega$  of  $\mathcal{P}_{\Delta,\alpha,Q}^{\text{poi}}$  sequentially. The decision of whether to keep or thin a point  $X$  depends only on decisions already taken in  $] -\infty, X[$ . In particular, the only information we admit is the location of the already kept points  $\omega' \subseteq \omega \cap ] -\infty, X[$ . We name the thinning probability  $p_{\Delta,\gamma}^s(X | \omega')$ .

We consider what happens if, starting at some  $X \in \Delta$ , we delete all points in  $\omega_{[X, \infty[}$ , i.e., all not yet considered points in  $\omega$ . On the one side, this is the void probability of a thinned Poisson PP with intensity  $\alpha p_{\Delta,\gamma}^s(\cdot | \omega')$ .

$$\mathcal{P}_{[X, \infty[, \alpha p_{\Delta,\gamma}^s(\cdot | \omega'), Q}^{\text{poi}}(\xi = \emptyset) = \exp \left( -\alpha \int_{[X, \infty[} p_{\Delta,\gamma}^s(Y | \omega') \mathcal{Q}^*(\text{d}Y) \right). \quad (4.11a)$$

On the other side, the resulting empty realisation should follow the  $\mathcal{P}_{\Delta,\gamma}^{\text{gb}}$  law. The DLR equations (2.8) imply that

$$\begin{aligned} \mathcal{P}_{\Delta,\gamma}^{\text{gb}}(\xi_{[X, \infty[} = \emptyset | \xi_{]-\infty, X[} = \omega') \\ = \mathcal{P}_{[X, \infty[, \omega' \cup \gamma}^{\text{gb}}(\xi = \emptyset) = \frac{1}{\mathcal{Z}([X, \infty[, \omega' \cup \gamma)}. \end{aligned} \quad (4.11b)$$

Equating the left hand sides of (4.11a) and (4.11b) leads to

$$\int_{[X, \infty[} \alpha p_{\Delta,\gamma}^s(Y | \omega') \mathcal{Q}^*(\text{d}Y) = \log \mathcal{Z}([X, \infty[, \omega' \cup \gamma). \quad (4.12)$$

Taking the derivative along the ordered space  $(\Delta, \preccurlyeq)$  yields (4.9). On the lhs of (4.12) we apply a one-sided version of the *Lebesgue differentiation theorem* [3, Thm 5.6.2] to extract the integrand as the  $\mathcal{Q}^*$ -a.s derivative. There is an additional minus in (4.9), because differentiation of the above integral proceeds in decreasing direction in  $\preccurlyeq$ , reverse to the usual direction. Section 4.3 shows that (4.12) determines the law of the thinning. Section 4.4 proves (4.10).  $\square$

### 4.3 Conditional void probabilities

Let  $\mathcal{P}$  be a PP law on  $(\Omega_\Delta, \mathcal{F}_\Delta)$ . Assume that  $\mathcal{P}$  is absolutely continuous with respect to  $\mathcal{P}_{\Delta,1,Q}^{\text{poi}}$ . The *interval void function*  $\mathcal{V}$  is, for  $A, B \in \Delta \cup \{\pm\infty\}$  with  $A \preceq B$ ,

$$\mathcal{V}(A, B) := \mathcal{P}(\xi_{]A,B]} = \emptyset). \quad (4.13)$$

Because of the nice properties of  $\preceq$  from Section 4.1, it determines all void probabilities of  $\mathcal{P}$ . The void probabilities determine the law  $\mathcal{P}$  [8, Thm 9.2.XIII]. For  $A, B \in \Delta \cup \{\pm\infty\}$  with  $A \preceq B$  and  $\omega \in \Omega_{]-\infty,A]}$ , let the *conditional void interval probability* be

$$\mathcal{V}(A, B \mid \omega) := \mathcal{P}(\xi_{]A,B]} = \emptyset \mid \xi_{]-\infty,A]} = \omega).$$

Because

$$\mathcal{V}(A, B) = \int_{\Omega_{]-\infty,A]}} \mathcal{V}(A, B \mid \omega) \mathcal{P}(\xi_{]-\infty,A]} = d\omega),$$

the conditional void interval probabilities determine the law  $\mathcal{P}$ . The *one-sided conditional void interval probability* is  $\mathcal{V}(A \mid \omega) := \mathcal{V}(A, \infty \mid \omega)$ . Because of the identity

$$\mathcal{V}(A, B \mid \omega) \mathcal{V}(B \mid \omega) = \mathcal{V}(A \mid \omega),$$

it is possible to define the conditional void interval probabilities by the one-sided versions via

$$\mathcal{V}(A, B \mid \omega) := \begin{cases} \frac{\mathcal{V}(A \mid \omega)}{\mathcal{V}(B \mid \omega)} & \text{if } \mathcal{V}(B \mid \omega) > 0 \\ 0 & \text{else.} \end{cases}$$

Thus, the one-sided conditional void interval probabilities determine the law  $\mathcal{P}$ . Because  $\mathcal{Q}^*$  is diffuse, the negligible change of the left interval border from closed to open from  $\mathcal{V}(X \mid \omega)$  to  $\mathcal{P}(\xi_{]X,\infty[} = \emptyset \mid \xi_{]-\infty,X]} = \omega)$  does not matter.

### 4.4 Derivative

This section shows that the thinning probability (4.9) is a fraction of partition functions (4.10). In short,

$$-\frac{\partial}{\partial X} \log \mathcal{Z}([X, \infty[, \omega' \cup \gamma) = \lambda e^{-H_{\{X\}}(X, \gamma \cup \omega')} \frac{\mathcal{Z}([X, \infty[, \gamma \cup \omega' \cup X)}{\mathcal{Z}([X, \infty[, \gamma \cup \omega')}. \quad (4.14)$$

With  $z(X) := \mathcal{Z}([X, \infty[, \gamma \cup \omega')$ , this amounts to

$$\frac{\partial}{\partial X} \log z(X) = \frac{z'(X)}{z(X)}.$$

Thus, to show (4.14), it remains to verify that

$$z'(X) = -\lambda e^{-H_{\{X\}}(X, \gamma \cup \omega')} \mathcal{Z}([X, \infty[, \gamma \cup \omega' \cup X).$$

We use the notation from (4.4). Let  $\gamma' := \gamma \cup \omega'$ ,  $\Delta' := [X, \infty[$ ,  $\Delta'_\varepsilon := [X_\varepsilon^+, \infty[$  and  $\Delta_\varepsilon := [X, X_\varepsilon^+]$ . With  $b := \mathcal{Q}^*(\Delta')$ , we have  $b - \varepsilon = \mathcal{Q}^*(\Delta'_\varepsilon)$  and  $\varepsilon = \mathcal{Q}^*(\Delta_\varepsilon)$ . We

have

$$\begin{aligned}
& z'(X) \\
&= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (z(X_\varepsilon^+) - z(X)) \\
&= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left( \int_{\Omega_{\Delta'_\varepsilon}} e^{-H_{\Delta'_\varepsilon}(\omega|\gamma')} \mathcal{P}_{\Delta'_\varepsilon, \lambda, Q}^{\text{poi}}(d\omega) \exp(\lambda \mathcal{Q}^*(\Delta'_\varepsilon)) \right. \\
&\quad \left. - \int_{\Omega_{\Delta'}} e^{-H_{\Delta'}(\omega|\gamma')} \mathcal{P}_{\Delta', \lambda, Q}^{\text{poi}}(d\omega) \exp(\lambda \mathcal{Q}^*(\Delta')) \right) \\
&= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left( \int_{\Omega_{\Delta'_\varepsilon}} e^{-H_{\Delta'_\varepsilon}(\omega|\gamma')} \mathcal{P}_{\Delta'_\varepsilon, \lambda, Q}^{\text{poi}}(d\omega) e^{\lambda(b-\varepsilon)} - \int_{\Omega_{\Delta'}} e^{-H_{\Delta'}(\omega|\gamma')} \mathcal{P}_{\Delta', \lambda, Q}^{\text{poi}}(d\omega) e^{\lambda b} \right) \\
&= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{\Omega_{\Delta_\varepsilon}} \int_{\Omega_{\Delta'_\varepsilon}} e^{\lambda b} \left( e^{-H_{\Delta'_\varepsilon}(\omega|\gamma') - \lambda\varepsilon} - e^{-H_{\Delta'}(\omega \cup \gamma''|\gamma')} \right) \mathcal{P}_{\Delta'_\varepsilon, \lambda, Q}^{\text{poi}}(d\omega) \mathcal{P}_{\Delta_\varepsilon, \lambda, Q}^{\text{poi}}(d\gamma'') \\
&\tag{*} \\
&= \underbrace{-\lambda z(X)}_{\text{if } |\gamma''|=0} + \underbrace{\lambda z(X) - \lambda e^{-H_{\{X\}}(X|\gamma')} \mathcal{Z}(\downarrow X, \infty, \gamma' \cup X)}_{\text{if } |\gamma''|=1} + \underbrace{0}_{\text{if } |\gamma''| \geq 2} \\
&= -\lambda e^{-H_{\{X\}}(X|\gamma')} \mathcal{Z}(\downarrow X, \infty, \gamma' \cup X).
\end{aligned}$$

Case  $\gamma'' = \emptyset$  in (\*):

$$\begin{aligned}
& \tag{*} \stackrel{(2.5)}{=} \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} e^{-\lambda\varepsilon} \int_{\Omega_{\Delta'_\varepsilon}} e^{\lambda b} e^{-H_{\Delta'_\varepsilon}(\omega|\gamma')} (e^{-\lambda\varepsilon} - 1) \mathcal{P}_{\Delta'_\varepsilon, \lambda, Q}^{\text{poi}}(d\omega) \\
&= \lim_{\varepsilon \rightarrow 0} \frac{e^{-\lambda\varepsilon} - 1}{\varepsilon} \lim_{\varepsilon \rightarrow 0} e^{\lambda(b-\varepsilon)} \int_{\Omega_{\Delta'_\varepsilon}} e^{-H_{\Delta'_\varepsilon}(\omega|\gamma')} \mathcal{P}_{\Delta'_\varepsilon, \lambda, Q}^{\text{poi}}(d\omega) \\
&= -\lambda \lim_{\varepsilon \rightarrow 0} \mathcal{Z}(\Delta'_\varepsilon, \gamma') \\
&= -\lambda \mathcal{Z}(\Delta', \gamma') \quad \text{using the continuity of } \mathcal{Z} \text{ in the domain} \\
&= -\lambda z(X).
\end{aligned}$$

Case  $\gamma'' =: \{Y\}$  in (\*):

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{\Delta_\varepsilon} \int_{\Omega_{\Delta'_\varepsilon}} e^{\lambda b} \left( e^{-H_{\Delta'_\varepsilon}(\omega|\gamma') - \lambda\varepsilon} - e^{-H_{\Delta'}(\omega \cup Y|\gamma')} \right) \mathcal{P}_{\Delta'_\varepsilon, \lambda, Q}^{\text{poi}}(d\omega) \lambda e^{-\lambda\varepsilon} \mathcal{Q}^*(dY) \\
&= \lambda \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} e^{-\lambda\varepsilon} \int_{\Delta_\varepsilon} e^{\lambda(b-\varepsilon)} \int_{\Omega_{\Delta'_\varepsilon}} e^{-H_{\Delta'_\varepsilon}(\omega|\gamma')} \mathcal{P}_{\Delta'_\varepsilon, \lambda, Q}^{\text{poi}}(d\omega) \mathcal{Q}^*(dY) \\
&\quad - \lambda \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{\Delta_\varepsilon} e^{-H_{\{Y\}}(Y|\gamma')} e^{\lambda(b-\varepsilon)} \int_{\Omega_{\Delta'_\varepsilon}} e^{-H_{\Delta'_\varepsilon}(\omega|\gamma' \cup Y)} \mathcal{P}_{\Delta'_\varepsilon, \lambda, Q}^{\text{poi}}(d\omega) \mathcal{Q}^*(dY) \\
&= \lambda \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} e^{-\lambda\varepsilon} \varepsilon \mathcal{Z}(\Delta'_\varepsilon, \gamma') - \lambda \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{\Omega_{\Delta_\varepsilon}} e^{-H_{\{Y\}}(Y|\gamma')} \mathcal{Z}(\Delta'_\varepsilon, \gamma' \cup Y) \mathcal{Q}^*(dY) \\
&= \lambda \mathcal{Z}(\Delta', \gamma') - \lambda e^{-H_{\{X\}}(X, \gamma \cup \omega')} \mathcal{Z}(\Delta', \gamma' \cup X).
\end{aligned}$$

Case  $|\gamma''| \geq 2$  in (\*): Straightforward computation gives

$$\int_{\Omega_{\Delta'_\varepsilon}} \left( \frac{\alpha}{\lambda} \right)^{|\omega|} \mathcal{P}_{\Delta'_\varepsilon, \lambda, Q}^{\text{poi}}(d\omega) = e^{\lambda(b-\varepsilon)} \left( \frac{\alpha}{\lambda} - 1 \right) \leq e^{b\alpha + \lambda\varepsilon}.$$

Thus,

$$\begin{aligned}
|(\star)| &\leq e^{\lambda b} \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{\Omega_{\Delta_\varepsilon}} [|\gamma''| \geq 2] \\
&\quad \times \int_{\Omega_{\Delta'_\varepsilon}} \underbrace{\left| e^{-H_{\Delta'_\varepsilon}(\omega|\gamma')} - \lambda \varepsilon - e^{-H_{\Delta'_\varepsilon}(\omega \cup \gamma''|\gamma')} \right|}_{\leq \left(\frac{\alpha}{\lambda}\right)^{|\gamma''|+|\omega|} + \left(\frac{\alpha}{\lambda}\right)^{|\omega|}} \mathcal{P}_{\Delta'_\varepsilon, \lambda, Q}^{\text{poi}}(d\omega) \mathcal{P}_{\Delta_\varepsilon, \lambda, Q}^{\text{poi}}(d\gamma'') \\
&\leq e^{\lambda b} \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} e^{b\alpha + \lambda \varepsilon} \left( \int_{\Omega_{\Delta_\varepsilon}} [|\gamma''| \geq 2] \left( 1 + \left(\frac{\alpha}{\lambda}\right)^{|\gamma''|} \right) \mathcal{P}_{\Delta_\varepsilon, \lambda, Q}^{\text{poi}}(d\gamma'') \right) \\
&= 0.
\end{aligned}$$

## 5 Proof of Theorem 3.3

We construct a disagreement coupling family with slightly stronger properties than needed. See Proposition 5.1 for the full statement. Let  $\Delta \in \mathcal{S}_{bp}$  and  $\gamma^1, \gamma^2 \in \Omega_{\Delta^c}$ . We may thin a Poisson PP to two conditionally independent copies of the Gibbs PP.

$$\mathcal{P}_{\Delta, \gamma^1, \gamma^2}^{\text{thin2}}(d\omega) := p_{\Delta, \gamma^1}^j(\omega^1 | \omega^3) p_{\Delta, \gamma^2}^j(\omega^2 | \omega^3) \mathcal{P}_{\Delta, \alpha, Q}^{\text{poi}}(d\omega^3). \quad (5.1)$$

The *influence zone* is

$$\Gamma := \{X \in \Delta \mid \exists Y \in \gamma^1 \cup \gamma^2 : B(X) \cap B(Y) \neq \emptyset\}. \quad (5.2)$$

Define the joint Janossy intensity of the law  $\mathcal{P}_{\Delta, \gamma^1, \gamma^2}^{\text{da-zone}}$  on  $(\Omega_\Gamma^3, \mathcal{F}_\Gamma^{\otimes 3})$  by

$$\mathcal{P}_{\Delta, \gamma^1, \gamma^2}^{\text{da-zone}}(d\omega) := \mathcal{P}_{\Delta, \gamma^1, \gamma^2}^{\text{thin2}}(d\omega_\Gamma). \quad (5.3a)$$

Define the joint Janossy intensity of law  $\mathcal{P}_{\Delta, \gamma^1, \gamma^2}^{\text{da-rec}}$  on  $(\Omega_\Delta^3, \mathcal{F}_\Delta^{\otimes 3})$  recursively by

$$\begin{aligned}
\mathcal{P}_{\Delta, \gamma^1, \gamma^2}^{\text{da-rec}}(d\omega) &:= [\Gamma = \emptyset] [\omega^1 = \omega^2] \mathcal{P}_{\Delta, \emptyset}^{\text{thin}}(d(\omega^1, \omega^3)) \\
&\quad + [\Gamma \neq \emptyset] \mathcal{P}_{\Delta, \gamma^1, \gamma^2}^{\text{da-zone}}(d\omega_\Gamma) \mathcal{P}_{\Delta \setminus \Gamma, \gamma^1 \cup \omega_\Gamma^1, \gamma^2 \cup \omega_\Gamma^2}^{\text{da-rec}}(d\omega_{\Delta \setminus \Gamma}). \quad (5.3b)
\end{aligned}$$

**Proposition 5.1.** *The coupling  $\mathcal{P}_{\Delta, \gamma^1, \gamma^2}^{\text{da-rec}}$  has the following properties:*

$$\forall 1 \leq i \leq 2, \omega \in \Omega_B : \quad \mathcal{P}_{\Delta, \gamma^1, \gamma^2}^{\text{da-rec}}(\xi^i = d\omega) = \mathcal{P}_{\Delta, \gamma^i}^{gb}(d\omega), \quad (5.4a)$$

$$\mathcal{P}_{\Delta, \gamma^1, \gamma^2}^{\text{da-rec}}(\xi^3 = d\omega) = \mathcal{P}_{\Delta, \alpha, Q}^{\text{poi}}(d\omega), \quad (5.4b)$$

$$\mathcal{P}_{\Delta, \gamma^1, \gamma^2}^{\text{da-rec}}(\xi^1 \cup \xi^2 \subseteq \xi^3) = 1, \quad (5.4c)$$

$$\mathcal{P}_{\Delta, \gamma^1, \gamma^2}^{\text{da-rec}}(\forall X \in \xi_1 \Delta \xi_2 : X \xrightarrow{\text{in } \xi^3} \gamma^1 \cup \gamma^2). \quad (5.4d)$$

Further, it is jointly measurable in the boundary conditions  $(\gamma^1, \gamma^2)$ .

*Proof.* The first step is to check the termination of the recursion in (5.3b). The recursion is made with respect to the influence zone  $\Gamma$ , which is decreasing and whose volume is bounded by  $\mathcal{L}^d \otimes Q(\Delta)$ . The recursion stops when no Gibbs point (of  $\xi^1$  and  $\xi^2$ ) is placed in  $\Gamma$ . This happens in particular when there is no Poisson point (of  $\xi^3$ ) in the influence zone  $\Gamma$ . At each step of the recursion, this happens independently with probability bounded from below by  $e^{-\alpha \mathcal{L}^d \otimes Q(\Delta)}$ . Therefore, the recursion stops after an almost-surely finite number of steps.

The next step is to show the measurability in the boundary conditions. Proposition 4.1 asserts that  $p_{\Delta, \gamma^1}^j(\omega^1 | \omega^3)$  and  $p_{\Delta, \gamma^2}^j(\omega^2 | \omega^3)$  are measurable in  $\gamma^1$  and

$\gamma^2$  respectively. Hence, the coupling  $\mathcal{P}_{\Delta, \gamma^1, \gamma^2}^{\text{da-rec}}$  is jointly measurable in the boundary conditions  $(\gamma^1, \gamma^2)$ . The measurability is needed for the well-definedness of the recursive definition (5.3b) and the proof of (5.4)

Finally, we show equations (5.4). Equation (5.4a) is a straightforward consequence of the DLR equations (2.8) and the assumption (Loc). Equations (5.4b) and (5.4c) are also a straightforward consequence of the construction. Concerning (5.4d), the only points of the Poisson configuration  $\xi^3$  which are not connected (in  $\mathcal{G}(\xi^3)$ ) to the boundary conditions  $\gamma^1 \cup \gamma^2$  are the ones sampled at the end of the recursion, when  $\Gamma = \emptyset$ . These points thin to both Gibbs PPs  $\xi^1$  and  $\xi^2$  identically, as outlined in the  $\Gamma = \emptyset$  case of (5.3b). By construction (5.3a), from those points the ones belonging to the first Gibbs configuration  $\xi^1$  also belong to the second Gibbs configuration  $\xi^2$ . Therefore, the only points where the two Gibbs configurations may differ are the one sample when the influence zone is not empty. By (5.3a), those points are connected to the boundary conditions  $\gamma^1 \cup \gamma^2$ .  $\square$

## 6 Proof of Theorem 3.2

Let  $\mathcal{P}^1, \mathcal{P}^2 \in \mathcal{G}^{gb}$ . We are going to prove that  $\mathcal{P}^1 = \mathcal{P}^2$ . Let  $\Lambda \in \mathcal{B}_b$  and  $E \in \mathcal{F}_\Lambda$ . For  $n \in \mathbb{N}$ , consider the closed ball  $\Lambda_n := B(0, n)$  in  $\mathbb{R}^d$ . Let  $\mathcal{P}_n^{1 \otimes 2} := \mathcal{P}_{\Lambda_n}^1 \otimes \mathcal{P}_{\Lambda_n}^2$ . For  $n$  large enough,  $\Lambda \subseteq \Lambda_n$ . The DLR equation (2.8) for  $\Lambda_n$  implies that

$$|\mathcal{P}^1(E) - \mathcal{P}^2(E)| \leq \int_{\Omega_{\Lambda_n}^2} |\mathcal{P}_{\Lambda_n, \gamma^1}^{gb}(E) - \mathcal{P}_{\Lambda_n, \gamma^2}^{gb}(E)| \mathcal{P}_n^{1 \otimes 2}(d\gamma). \quad (6.1)$$

The disagreement coupling allows to majorize

$$\begin{aligned} |\mathcal{P}_{\Lambda_n, \gamma^1}^{gb}(E) - \mathcal{P}_{\Lambda_n, \gamma^2}^{gb}(E)| &\stackrel{(3.1a)}{=} |\mathcal{P}_{\Lambda_n, \gamma^1, \gamma^2}^{\text{dac}}(\xi^1 \in E) - \mathcal{P}_{\Lambda_n, \gamma^1, \gamma^2}^{\text{dac}}(\xi^2 \in E)| \\ &\stackrel{(3.1d)}{\leq} \mathcal{P}_{\Lambda_n, \gamma^1, \gamma^2}^{\text{dac}}(\xi_\Lambda^3 \xleftrightarrow{\text{in } \xi^3} \gamma^1 \cup \gamma^2) \\ &\leq \mathcal{P}_{\Lambda_n, \gamma^1, \gamma^2}^{\text{dac}}(\Lambda \xleftrightarrow{\text{in } \xi^3} \gamma^1 \cup \gamma^2). \end{aligned} \quad (6.2)$$

Combining (6.1) and (6.2) yields

$$\begin{aligned} |\mathcal{P}^1(E) - \mathcal{P}^2(E)| &\leq \int_{\Omega_{\Lambda_n}^2} \mathcal{P}_{\Lambda_n, \gamma^1, \gamma^2}^{\text{dac}}(\Lambda \xleftrightarrow{\text{in } \xi^3} \gamma^1 \cup \gamma^2) \mathcal{P}_n^{1 \otimes 2}(d\gamma) \\ &\stackrel{(3.1c)}{=} \int_{\Omega_{\Lambda_n}^2} \mathcal{P}_{\Lambda_n, \bar{\alpha}, \bar{Q}}^{\text{poi}}(\Lambda \xleftrightarrow{\text{in } \xi} \gamma^1 \cup \gamma^2) \mathcal{P}_n^{1 \otimes 2}(d\gamma). \end{aligned} \quad (6.3)$$

As we are in the sub-critical regime of the Boolean model, we expect the integrated probability to converge to 0 as  $n$  grows to infinity. Unfortunately, this convergence depends on the outside configurations  $\gamma^1, \gamma^2$  and we need uniform convergence.

Let  $\varepsilon > 0$ . Since the integrated event is increasing in  $\gamma^1$  and  $\gamma^2$ , the stochastic domination  $\mathcal{P}_{\Lambda_n, \gamma}^{gb} \prec \mathcal{P}_{\Lambda_n, \bar{\alpha}, \bar{Q}}^{\text{poi}}$  implies that

$$|\mathcal{P}^1(E) - \mathcal{P}^2(E)| \leq \int_{\Omega_{\Lambda_n}^2} \mathcal{P}_{\Lambda_n, \bar{\alpha}, \bar{Q}}^{\text{poi}}(\Lambda \xleftrightarrow{\text{in } \xi} \gamma^1 \cup \gamma^2) \mathcal{P}_{\Lambda_n, \bar{\alpha}, \bar{Q}}^{\text{poi}} \otimes \mathcal{P}_{\Lambda_n, \bar{\alpha}, \bar{Q}}^{\text{poi}}(d\gamma).$$

The following lemma gives a control, with high probability, of the radii in a Boolean model.

**Lemma 6.1.** *For a positive integer  $k$ , let*

$$\Upsilon_k := \left\{ \omega \in \Omega \mid \forall (x, r) \in \omega, r \leq \frac{\|x\|}{2} + k \right\}.$$

If  $Q$  satisfies the integrability assumption  $\rho(Q) < \infty$ , then for  $k$  large enough we have

$$\mathcal{P}_{\alpha, Q}^{\text{poi}}(\Upsilon_k) \geq 1 - \varepsilon. \quad (6.4)$$

The proof of Lemma 6.1 can be adapted from [9, Lemma 3.2]. If  $X \in \gamma^1 \cup \gamma^2 \in \Upsilon_k$ , then  $B(X) \cap B(0, n/2 - k - 1) = \emptyset$ . Therefore, for large enough  $k$ , we have

$$|\mathcal{P}^1(E) - \mathcal{P}^2(E)| \leq \varepsilon + \mathcal{P}_{\Lambda_n, \bar{\alpha}, \bar{Q}}^{\text{poi}}(\Lambda \xrightarrow{\text{in } \xi} B(0, n/2 - k - 1)).$$

Using (2.2) from Theorem 2.1, for large enough  $n$ , we have

$$|\mathcal{P}^1(E) - \mathcal{P}^2(E)| \leq 2\varepsilon.$$

Letting  $\varepsilon$  tend to 0 shows that  $\mathcal{P}^1 = \mathcal{P}^2$ .

## 7 Proof of Theorem 3.4

The existence of a unique  $\mathcal{P} \in \mathcal{G}^{gb}$  follows from Theorem 3.2.

First, we prove (3.2a). Recall that we assume that  $\bar{Q}$  has bounded support, i.e.,  $\bar{Q}([0, r_0]) = 1$  for some  $r_0 \in \mathbb{R}_+$ .

$$\begin{aligned} |\mathcal{P}_{\bar{\Lambda}, \gamma}^{gb}(E) - \mathcal{P}(E)| &\stackrel{(2.8)}{=} |\mathcal{P}_{\bar{\Lambda}, \gamma}^{gb}(E) - \int_{\Omega_{\bar{\Lambda}^c}} \mathcal{P}_{\bar{\Lambda}, \gamma'}^{gb}(E) \mathcal{P}(\xi_{\bar{\Lambda}^c} = d\gamma')| \\ &\leq \int_{\Omega_{\bar{\Lambda}^c}} |\mathcal{P}_{\bar{\Lambda}, \gamma}^{gb}(E) - \mathcal{P}_{\bar{\Lambda}, \gamma'}^{gb}(E)| \mathcal{P}(\xi_{\bar{\Lambda}^c} = d\gamma'). \end{aligned} \quad (7.1)$$

Combining (7.1) with (6.2), we get

$$\begin{aligned} |\mathcal{P}_{\bar{\Lambda}, \gamma}^{gb}(E) - \mathcal{P}(E)| &\leq \int_{\Omega_{\bar{\Lambda}^c}} \mathcal{P}_{\bar{\Lambda}, \bar{\alpha}, \bar{Q}}^{\text{poi}}(\Lambda \xrightarrow{\text{in } \xi} \gamma \cup \gamma') \mathcal{P}(\xi_{\bar{\Lambda}^c} = d\gamma') \\ &\leq \mathcal{P}_{\bar{\Lambda}, \bar{\alpha}, \bar{Q}}^{\text{poi}}(\Lambda \xrightarrow{\text{in } \xi} (\bar{\Lambda} + B(0, r_0))^c). \end{aligned}$$

Therefore, using (2.3) from Theorem 2.1, we obtain

$$|\mathcal{P}_{\bar{\Lambda}, \gamma}^{gb}(E) - \mathcal{P}(E)| \leq K \exp(-\kappa[\theta(\Lambda, \bar{\Lambda}^c) - r_0]).$$

Setting  $K' := Ke^{r_0}$ , we obtain (3.2a).

Second, we show (3.2b). Let  $\Lambda$  be a bounded set containing  $\Lambda_1 \cup \Lambda_2$  such that  $\theta(\Lambda_1 \cup \Lambda_2, \Lambda^c) \geq \theta(\Lambda_1, \Lambda_2)$  and let  $\bar{\Lambda} := \Lambda \setminus \Lambda_2$ . Thus, we have  $\theta(\Lambda_1, \bar{\Lambda}^c) \geq \theta(\Lambda_1, \Lambda_2)$  and

$$\begin{aligned} |\mathcal{P}(E \cap F) - \mathcal{P}(E)\mathcal{P}(F)| &\stackrel{(2.8)}{=} \left| \int_{\Omega_{\bar{\Lambda}^c}} [\gamma \in F] (\mathcal{P}_{\bar{\Lambda}, \gamma}^{gb}(E) - \mathcal{P}(E)) \mathcal{P}_{\bar{\Lambda}^c}(d\gamma) \right| \\ &\leq \int_{\Omega_{\bar{\Lambda}^c}} |\mathcal{P}_{\bar{\Lambda}, \gamma}^{gb}(E) - \mathcal{P}(E)| \mathcal{P}_{\bar{\Lambda}^c}(d\gamma) \\ &\leq K' \exp(-\kappa' \theta(\Lambda_1, \Lambda_2)). \end{aligned}$$

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