



HAL
open science

Hyperbolicity of singular spaces

Benoit Cadorel, Erwan Rousseau, Behrouz Taji

► **To cite this version:**

Benoit Cadorel, Erwan Rousseau, Behrouz Taji. Hyperbolicity of singular spaces. 2017. hal-01622037v1

HAL Id: hal-01622037

<https://hal.science/hal-01622037v1>

Preprint submitted on 24 Oct 2017 (v1), last revised 26 Sep 2018 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

HYPERBOLICITY OF SINGULAR SPACES

BENOÎT CADOREL, ERWAN ROUSSEAU, AND BEHROUZ TAJI

ABSTRACT. We study the hyperbolicity of singular quotients of bounded symmetric domains. We give effective criteria for such quotients to satisfy Green-Griffiths-Lang’s conjectures in both analytic and algebraic settings. As an application, we show that Hilbert modular varieties, except for a few possible exceptions, satisfy all expected conjectures.

CONTENTS

1. Introduction	1
2. Cyclic quotient singularities	3
3. Entire curves and singular ball quotients	4
4. Orbifold differential forms	6
5. Quotients of polydiscs	10
6. Hilbert modular varieties	11
References	13

1. INTRODUCTION

As central objects in algebraic geometry, the geometry of quotients of bounded symmetric domains Ω/Γ has been the object of many works. Classical results state that there exist sufficiently small subgroups $\Gamma' \subset \Gamma$ such that Ω/Γ' has remarkable properties: it is of general type [Mum77], it is hyperbolic modulo the boundary ([Nad89], [Rou16]), all its subvarieties are of general type [Bru16].

Nevertheless, it turns out that these properties should be true in most cases without having to take small subgroups $\Gamma' \subset \Gamma$. As an example, Tsuyumine [Tsu85] has shown in the Hilbert modular case ($\Omega = \mathcal{H}^n$ and Γ the Hilbert modular group) that, except finitely many cases, Hilbert modular varieties are of general type. The main difficulty, if one wants to avoid the step of taking small subgroups, is to deal with singularities. Indeed, it is well known that hyperbolicity properties may be completely lost in singular quotients (see for example Keum’s singular ball quotient [Keu08]).

The above mentioned results can be seen as illustrations of the expected following conjectures of Lang and Green-Griffiths (see [Lan86] and [GG80]).

Let $\text{Exc}(X) \subset X$ denote the Zariski closure of the union of the images of all non-constant holomorphic maps $\mathbb{C} \rightarrow X$.

Conjecture 1.1. *Let X be a complex projective manifold. Then X is of general type if and only if $\text{Exc}(X) \neq X$.*

Date: October 24, 2017.

E. R. was partially supported by the ANR project “FOLIAGE”, ANR-16-CE40-0008. E.R. and B.T. thank the University of Sydney for the invitation at the School of Mathematics where part of this work was done.

The setting we consider in this article is the following: a quotient $X = \Omega/\Gamma$ of a bounded symmetric domain by an arithmetic lattice such that

(1.1.1) X has only cyclic quotient singularities p_1, \dots, p_r , with isotropy groups G_i ;

(1.1.2) X admits a toroidal compactification $\bar{X} = X \sqcup D$, where $D \subset \bar{X}$ is a divisor with simple normal crossings.

Denote by $p : \Omega \rightarrow X$ the canonical projection. Let $\tilde{X} \xrightarrow{\pi} \bar{X}$ be a resolution with exceptional divisor E_i above each p_i . Let $\tilde{\Delta} = \sum_i \left(1 - \frac{1}{|G_i|}\right) E_i$, $E = \sum_i E_i$ and let $\tilde{D} = \pi^* D$.

Let h_{Berg} be the Bergman metric on Ω , which we normalize to have $\text{Ric}(h_{\text{Berg}}) = -h_{\text{Berg}}$. Let $\gamma \in \mathbb{Q}_+^*$ such that the holomorphic sectional curvature of h_{Berg} is bounded from above by $-\gamma$.

The first result is a generalization of Nadel's theorem [Nad89] to this singular setting.

Theorem A. *Consider the \mathbb{Q} -line bundle*

$$L = \pi^* K_{\bar{X}} + \tilde{D} - \frac{1}{\gamma} (\tilde{D} + \tilde{\Delta}).$$

Then $\text{Exc}(\tilde{X}) \subset \mathbb{B}^+(L) \cup \tilde{D} \cup E$, where $\mathbb{B}^+(L)$ denotes the augmented base locus. In particular if L is big then $\text{Exc}(\tilde{X}) \neq \tilde{X}$.

Then, we turn to algebraic versions of hyperbolicity in the case of quotients of the ball and polydiscs.

Theorem B. *Let $\Omega = \mathbb{B}^n$ be the unit ball. Then if the \mathbb{Q} -line bundle*

$$L = \pi^* K_{\bar{X}} + \tilde{D} - (n+1) [\tilde{\Delta} + \tilde{D}]$$

is big, any subvariety $V \subset \tilde{X}$ such that $V \not\subset \mathbb{B}^+(L) \cup \tilde{D} \cup E$ is of general type. More precisely, any resolution of singularities \tilde{V} of such a variety has big cotangent bundle.

The proof of these two theorems is given in section 3. They are obtained using negativity properties of the Bergman metric and the methods of [Cad16]. The key idea in both statements is to use sections of the line bundle L to twist the Bergman metric and obtain in this way a singular metric on the compactification with the required negative curvature properties.

Theorem C. *Let $\Omega = \mathbb{D}^n$ be the polydisc and $\Gamma \subset \text{Aut}(\mathbb{D})^n$ a discrete irreducible subgroup. Then if the \mathbb{Q} -line bundle*

$$L = \pi^* K_{\bar{X}} + \tilde{D} - n [\tilde{\Delta} + \tilde{D}]$$

is big, any subvariety $V \subset \tilde{X}$ such that $V \not\subset \mathbb{B}^+(L) \cup \tilde{D} \cup E$ is of general type. More precisely, any resolution of singularities \tilde{V} of such a variety has big cotangent bundle.

This result is proved in section 5. We use extension properties of orbifold symmetric differential forms explained in section 4. The idea is that in this setting, sections of the line bundle L naturally induce symmetric differentials on the regular part of the quotient. Then, extension properties of these differential forms as orbifold symmetric differentials are used to construct global holomorphic symmetric differentials on the compactification.

As an application of the above results, we study in section 6 the case of Hilbert modular varieties and obtain the following version of conjecture 1.1 in this case.

Theorem D. *Let $n \geq 2$. Then, except finitely many possible exceptions, Hilbert modular varieties satisfy the following properties:*

(1.1.3) $\text{Exc}(X) \neq X$.

(1.1.4) *there is a proper subvariety Z such that all subvarieties not contained in Z are of general type.*

The first part of this theorem was already obtained by a different method in [RT17] while the second part is a generalization of results of Tsuyumine [Tsu86] who treated the case of codimension one subvarieties.

2. CYCLIC QUOTIENT SINGULARITIES

In this section, we collect some basic facts about cyclic quotient singularities.

2.1. Resolution. First, we recall how one can resolve this type of singularities. This problem has been addressed by Fujiki ([Fuj75]).

Let G a finite cyclic group with a fixed generator acting on a complex affine m -space $\mathbb{C}^m = \mathbb{C}^m(z_1, \dots, z_m)$ by the formula:

$$g(z_1, \dots, z_m) = (e_1 z_1, \dots, e_m z_m)$$

where, for each i , e_i is a n^{th} root of the unity, $n > 1$. Then the quotient $X = \mathbb{C}^m / G$ has the structure of a normal affine algebraic variety whose singular locus is reduced to the single point $p = \pi(0)$ where $\pi : \mathbb{C}^m \rightarrow X$ denotes the canonical projection.

Following [Fuj75] (theorem 1 p.303) there exists a resolution of the singularity of X , that is a pair (\tilde{X}, f) consisting of a smooth variety \tilde{X} , a proper birational morphism $f : \tilde{X} \rightarrow X$ isomorphic outside $f^{-1}(p)$ with the additional following properties

(2.0.1) $E = f^{-1}(p)$ is a simple normal crossing hypersurface, each component of which being rational.

(2.0.2) There exists an affine open covering $\mathcal{U} = (U_1, \dots, U_l)$ of \tilde{X} with $U_i \simeq \mathbb{C}^m$.

(2.0.3) $f^{-1}(p) \cap U_i$ is defined by $u_{k_1}^i \dots u_{k_t}^i = 0$ in $U_i \simeq \mathbb{C}^m(u^i)$ for some k_1, \dots, k_t (depending on i) if it is non empty.

(2.0.4) For each i the multivaluate map $f \circ \pi : \mathbb{C}^m(u^i) \rightarrow \mathbb{C}^m(z)$ takes the following form:

$$(2.0.1) \quad z_k = \prod_{l=1}^m u_l^{b_{lk}}$$

where b_{lk} are nonnegative rational numbers (depending on i and strictly less than 1).

Remark 2.1. Remark that z_k^n descends to X therefore all denominators of the rational numbers b_{lk} must divide n .

2.2. Discrepancies. Now, we describe some basic estimates on discrepancies associated to cyclic quotient singularities.

Lemma 2.2. *Let $X = \Omega / \Gamma$ be a quotient with only cyclic quotient singularities p_1, \dots, p_r , with isotropy groups G_i such that $|G_i| = m_i$. Let $K_{\tilde{X}} = \pi^* K_X + \sum_i a_i E_i$, where the a_i are the discrepancies. Then, for any i , we have $a_i \leq n(1 - \frac{1}{m_i}) - 1$.*

Proof. On each open chart U_j , we have, for any $k \in [1, n]$,

$$(2.2.1) \quad \pi^*(dz_k) = \sum_l \left[\prod_{s \neq l} (u_s^j)^{b_{sk}} \right] (u_l^j)^{b_{lk}-1} du_l^j.$$

Consequently, $\pi^*(dz_1 \wedge \dots \wedge dz_n)$ can be written as $P(u_1^j, \dots, u_k^j) du_1^j \wedge \dots \wedge du_l^j$ for some $P \in \mathbb{C}[u_1^j, \dots, u_k^j]$. Since for each s , we have $b_{sj} \leq 1 - \frac{1}{m_i}$, we see from (2.2.1) that in each monomial of P , each u_i^j appears with degree at most $n(1 - \frac{1}{m_i}) - 1$.

Consequently, the pluricanonical form $\pi^*(dz_1 \wedge \dots \wedge dz_n)^{\otimes m_i}$ vanishes at order at most $m_i \left[n(1 - \frac{1}{m_i}) - 1 \right]$ along E_i , if we see it as a local section of $K_{\tilde{X}}^{\otimes m_i}$. This means exactly that $a_i \leq n(1 - \frac{1}{m_i}) - 1$. \square

3. ENTIRE CURVES AND SINGULAR BALL QUOTIENTS

This section will be devoted to the proof of Theorems A and B. The main ingredient in both proofs is the use of singular metrics built from the Bergman metric.

3.1. Singular metrics. As explained in the introduction, we consider a quotient $X = \Omega/\Gamma$ of a bounded symmetric domain by an arithmetic lattice such that

(3.0.1) X has only cyclic quotient singularities p_1, \dots, p_r , with isotropy groups G_i such that $|G_i| = m_i$;

(3.0.2) Γ has only unipotent parabolic elements, so that X admits a toroidal compactification $\bar{X} = X \sqcup D$, where $D \subset \bar{X}$ is a divisor with simple normal crossings.

Denote by $p : \Omega \rightarrow X$ the canonical projection. Let $\tilde{X} \xrightarrow{\pi} \bar{X}$ be a resolution with exceptional divisor E_i above each p_i . Let $\tilde{\Delta} = \sum_i \left(1 - \frac{1}{m_i}\right) E_i$, $E = \sum_i E_i$ and finally let $\tilde{D} = \pi^*D$.

We will also denote by h_{Berg} the Bergman metric on the regular part $X \setminus \{p_1, \dots, p_r\}$. It induces a natural singular metric \tilde{h} on $\tilde{X} \setminus (D \cup E_1 \cup \dots \cup E_r)$.

We will use the notations of section 2, concerning resolutions of cyclic singularities.

Now, we state a lemma which controls the singularities of \tilde{h} near exceptional divisors.

Lemma 3.1. *On each one of the open charts U_j^i , there is a constant $C > 0$ such that, for each tangent vector field v ,*

$$(3.1.1) \quad \|v\|_{\tilde{h}}^2 \leq C \max_j |u_j^i|^{2\left(\frac{1}{m_i}-1\right)}.$$

Proof. It suffices to give a similar bound in the case $v = \frac{\partial}{\partial u_j^i}$, for any j . Because of (2.0.1), we have

$$\begin{aligned} \frac{\partial}{\partial u_j^i} &= \sum_{l=1}^n \frac{\partial z_l}{\partial u_j^i} \frac{\partial}{\partial z_l} \\ &= \sum_{l=1}^n \left[1_{\{b_{ij} \neq 0\}} (u_j^i)^{b_{ij}-1} \prod_{s \neq j} (u_s^i)^{b_{is}} \right] \frac{\partial}{\partial z_l}. \end{aligned}$$

But $b_{ij} - 1 \geq \frac{1}{m_i} - 1$ when $b_{ij} \neq 0$, so we get, after taking the norm

$$\left\| \frac{\partial}{\partial u_j^i} \right\|_{\tilde{h}} \leq C |u_j^i|^{\frac{1}{m_i}-1} \max_l \left\| \frac{\partial}{\partial z_l} \right\|_{h_{\text{Berg}}}.$$

Since the $\left\| \frac{\partial}{\partial z_l} \right\|_{h_{\text{Berg}}}$ are bounded, we get the result. \square

In the next lemma, we explain how to extend the metric on \tilde{X} using sections of an appropriate adjoint line bundle.

Lemma 3.2. *Let $-B \leq 0$ be an upper bound for the bisectional curvature of the Bergman metric on Ω . Assume that the line bundle*

$$L = \pi^*(K_{\tilde{X}} + D) - \frac{1}{A} [\tilde{D} + \tilde{\Delta}]$$

is big on \tilde{X} for some constant $A > 0$. Then for any $x \in \tilde{X} \setminus (\mathbb{B}^+(L) \cup \tilde{D} \cup E)$, there exists a singular hermitian metric \tilde{h} on $T_{\tilde{X}}$, such that

(3.2.1) \tilde{h} is smooth and non-degenerate near x ;

(3.2.2) \tilde{h} has bisectional curvature bounded from above by $-B + A$, and holomorphic sectional curvature bounded from above by $-\gamma + A$;

(3.2.3) \tilde{h} is locally bounded everywhere on \tilde{X} .

Proof. Since bigness is an open property, and because $x \notin \mathbb{B}^+(L)$, there exists rational numbers $\beta_i > 1 - \frac{1}{m_i}$ and $\lambda > 1$ such that $L' = \pi^*(K_{\tilde{X}} + D) - \frac{1}{A} [\lambda \tilde{D} + \sum_i \beta_i E_i]$ is big, and $x \notin \text{Bs}(L')$. Consequently, we can find $m \in \mathbb{N}$, and $s \in H^0(\tilde{X}, L'^{\otimes m})$, with $s(x) \neq 0$.

Consider the metric $\tilde{h} = \|s\|_{\frac{2A}{m}}^2 \pi^*(\det h_{\text{Berg}}^*)^{\otimes m} h_{\text{Berg}}$. We will show that it has the required properties. First of all, the metric $\det h_{\text{Berg}}^*$ has at most logarithmic growth near \tilde{D} if we see it as a metric on $\pi^*(K_{\tilde{X}} + D)$ (see [Mum77]). Consequently, $\|s\|_{\frac{2A}{m}}^2 \pi^*(\det h_{\text{Berg}}^*)^{\otimes m}$ vanishes on \tilde{D} at any order strictly smaller than $\frac{2A}{m} \cdot \frac{m\lambda}{A} = 2\lambda$. Since $\lambda > 1$, and since h_{Berg} , seen as a metric on $T_{\tilde{X}}$, diverges at order at most 2 near \tilde{D} (by [Mum77] again), this proves that \tilde{h} is locally bounded near \tilde{D} .

By (3.1.1), h_{Berg} diverges at order at most $2 \left(1 - \frac{1}{m_i}\right)$ near any exceptional divisor E_i . Besides, $\|s\|_{\frac{2A}{m}}^2 \pi^*(\det h_{\text{Berg}}^*)^{\otimes m}$ vanishes at order $\frac{2A}{m} \cdot \frac{m\beta_i}{A} = 2\beta_i > 2 \left(1 - \frac{1}{m_i}\right)$ on any E_i . Consequently, \tilde{h} is locally bounded near E .

Finally, we compute the curvature of \tilde{h} at the points where it is smooth. Because of our normalization assumptions, we find

$$\begin{aligned} i\Theta(\tilde{h}) &= i\frac{A}{m}\Theta\left(\det h_{\text{Berg}}^*{}^{\otimes m}\right) \otimes \mathbb{I} + i\Theta(h_{\text{Berg}}) \\ &= -A\pi^*\text{Ric}(h_{\text{Berg}}) \otimes \mathbb{I} + i\Theta(h_{\text{Berg}}) \\ &= A\pi^*\omega_{h_{\text{Berg}}} \otimes \mathbb{I} + i\Theta(h_{\text{Berg}}). \end{aligned}$$

and the required bounds on the curvatures are then given by an simple computation. \square

3.2. Proofs. Now, we can prove our two criteria for complex hyperbolicity.

Proof of Theorem A. By contradiction, assume there exists an entire curve $\mathbb{C} \xrightarrow{j} \tilde{X}$ not included in $\mathbb{B}^+(L) \cup \tilde{D} \cup E$. By our hypothesis, and since bigness is an open condition, we can pick $A < \gamma$ in Lemma 3.2. This implies that there exists a singular metric \tilde{h} on $T_{\tilde{X}}$, non-degenerate on $j(\mathbb{C})$, locally bounded everywhere, and with holomorphic sectional curvature bounded from above by $-\gamma + A < 0$.

Thus, $j^*\tilde{h}$ defines a smooth metric on \mathbb{C} outside a discrete set of points. This metric has negative curvature, bounded away from zero, and is locally bounded

everywhere. We can apply the usual extension theorem for plurisubharmonic functions, to obtain a singular metric on \mathbb{C} , with negative curvature in the sense of currents, bounded away from zero. This is absurd by Ahlfors-Schwarz lemma. \square

Proof of Theorem B. Let $V \rightarrow \tilde{X}$, with $V \not\subset \mathbb{B}^+(L) \cup D \cup E$, and let \tilde{V} be a resolution of the singularities of V . Then the induced holomorphic map $\tilde{V} \xrightarrow{j} \tilde{X}$ is generically immersive.

In Lemma 3.2, we can take $B = \frac{1}{n+1}$, $\gamma = \frac{2}{n+1}$. Then, choosing any positive constant $A < \frac{1}{n+1}$ will give a metric \tilde{h} such that $j^*\tilde{h}$ is locally bounded everywhere on $T_{\tilde{V}}$, has negative bisectional curvature and negative holomorphic sectional curvature, and is bounded from above by $-\frac{1}{n+1}$. By [Cad16], this implies that $\Omega_{\tilde{V}}$ is big. \square

Remark 3.3. Because of Lemma 2.2, the line bundle $\pi^*K_{\tilde{X}} - K_{\tilde{X}} + (n\tilde{D} - E)$ is always effective. Consequently, Theorem A (resp. Theorem B) can be restated with $L = K_{\tilde{X}} + \left(1 - \frac{1}{\gamma}\right)\tilde{D} + \left(E - (n + \frac{1}{\gamma})\tilde{\Delta}\right)$ (resp. $L = K_{\tilde{X}} - n\tilde{D} + \left(E - (2n + 1)\tilde{\Delta}\right)$).

4. ORBIFOLD DIFFERENTIAL FORMS

Our aim in the current section is to establish an extension result for orbifold differential forms (see Proposition 4.13) which plays a key role in our strategy to prove Theorem C.

4.1. Preliminaries on orbifold differential forms. The notions we are about to introduce in the current section originated in the works of Campana, cf. [Cam04]. For a more thorough account of these preliminary constructions, including background and applications, the reader could also consult [JK11], [Taj16], [CKT16] and [GT16].

Definition 4.1. We define an orbifold pair (X, D) by a normal, algebraic variety X and a divisor $D = \sum_{i=1}^l d_i \cdot D_i$, with $d_i = (1 - \frac{1}{a_i})$, where $a_i \in \mathbb{N}^+ \cup \{\infty\}$ and each D_i is a prime divisor.

We follow the usual convention that, for $a_i = \infty$, we have $(1 - \frac{1}{a_i}) = 1$.

The pair (X, D) in Definition 4.1 is sometimes referred to as a pair with standard coefficients (or a classical or integral orbifold pair).

With Definition 4.1 at hand, given a pair (X, D) , one can naturally associate a notion of multiplicity to each irreducible component D_i of D .

Definition 4.2. Let (X, D) be an orbifold pair. we define the orbifold multiplicity $m_D(D_i)$ of each prime divisor D_i as follows.

$$m_D(D_i) := \begin{cases} a_i & , \text{ if } d_i \neq 1 \\ \infty & , \text{ if } d_i = 1. \end{cases}$$

Our aim is now to introduce morphisms sensitive to the orbifold structure of (X, D) , but to do so we first need to define a notion for pullbacks of Weil divisors over normal varieties.

Definition 4.3. Let $f : Y \rightarrow X$ be a finite morphism of normal, algebraic varieties X and Y . Let $D \subset X$ be a Weil divisor. We define $f^*(D)$ by the Zariski closure of the Weil divisor defined by $f^*(D|_{X_{\text{reg}}})$.

We now turn to morphisms adapted to (X, D) . Such morphisms are guaranteed to exist whenever X is smooth, thanks to Kawamata's covering constructions, cf. [Laz04, Prop. 4.1.12].

Definition 4.4. Let (X, D) be an orbifold pair in Definition 4.1. We call a finite morphism $f : Y \rightarrow X$ of algebraic varieties X and Y , strictly adapted (to D), if Y is normal and $f^*(D_i) = a_i \cdot D'$, for some reduced divisor D' in Y .

We can also consider collections of charts that are adapted to the orbifold structure of a given pair (X, D) .

Definition 4.5 (Orbifold structures). Given an orbifold pair (X, D) , let $\{U_\alpha\}_\alpha$ be a Zariski open covering of X . We call a collection $\mathcal{C}_\alpha = \{(U_\alpha, f_\alpha, X_\alpha)\}_\alpha$ of triples $(U_\alpha, f_\alpha, X_\alpha)$ consisting of strictly adapted morphisms $f_\alpha : X_\alpha \rightarrow (U_\alpha, D|_{U_\alpha})$, an orbifold structure (or strictly adapted orbifold structure) associated to (X, D) .

Notation 4.1. Given a coherent sheaf \mathcal{F} on a normal algebraic variety X , by $S^{[\bullet]} \mathcal{F}$ we denote the reflexive hull $(S^\bullet \mathcal{F})^{\vee\vee}$ of symmetric powers $S^\bullet \mathcal{F}$ of \mathcal{F} . Furthermore, for a morphism of normal algebraic varieties $f : Y \rightarrow X$, we set $f^{[*]} \mathcal{F}$ to denote $(f^* \mathcal{F})^{\vee\vee}$.

We can now define a notion of sheaves of differential forms adapted to the orbifold structure of (X, D) . First, we need to fix some notations.

Notation 4.2. Let \mathcal{C}_α be an orbifold structure for the orbifold pair (X, D) . Let $\{D_{X_\alpha}^{ij}\}_{i,j}$ be the collection of prime divisors in X_α verifying the equality $(f^*(D_i))_{\text{red}} = \sum_j D_{X_\alpha}^{ij}$. Let X_α° denote the smooth locus of $(X_\alpha, f_\alpha^*(D))$ and U_α° an open subset of the smooth locus of (U_α, D) such that $f_\alpha : X_\alpha^\circ \rightarrow U_\alpha^\circ$ is surjective.

Definition 4.6. In the setting of Notation 4.2, we define the orbifold cotangent sheaf $\Omega_{\mathcal{C}_\alpha}^{[1]}$ by the collection of reflexive, \mathcal{O}_{X_α} -module, coherent sheaves $\Omega_{(X_\alpha, f_\alpha, D)}^{[1]}$ uniquely determined as the coherent extension of the kernel of the sheaf morphism

$$f_\alpha^* \Omega_{U_\alpha^\circ}^1(\log(\Gamma D|_{U_\alpha^\circ})) \longrightarrow \bigoplus_{D_{X_\alpha}^{ij}} \mathcal{O}_{D_{X_\alpha}^{ij}},$$

naturally defined by the residue map.

Remark 4.7. This is an instance of the notion of orbifolds sheaves adapted to a pair (X, D) . These are G_α -linearized, coherent sheaves on each X_α satisfying natural compatibility conditions, cf. [GT16, Sect. 2.6].

Example 4.8. Let $(X, D = \sum_{i=1}^l (1 - \frac{1}{a_i}) \cdot D_i)$ be an orbifold pair and $f_\alpha : X_\alpha \rightarrow U_\alpha$ a single, strictly adapted chart in Definition 4.5. Assume that the branch locus of f_α is divisorial and has simple normal crossing support. With this assumption, it follows that X_α is smooth. Furthermore, let us assume that U_α admits a coordinate system (z_1, \dots, z_n) such that $U_\alpha \cap D = \{(z_1, \dots, z_n) \mid \prod_{i=1}^l z_i = 0\}$. Assume that w_1, \dots, w_n is a coordinate system on X_α such that

$$f_\alpha : (w_1, \dots, w_n) \mapsto (w_1^{a_1}, \dots, w_l^{a_l}, w_{l+1}^{b_1}, \dots, w_{l+k}^{b_k}, w_{l+k+1}, \dots, w_n),$$

for some $b_1, \dots, b_k \in \mathbb{N}$. Then, by definition, the sheaf $\Omega_{(X_\alpha, f_\alpha, D)}^1$ is the \mathcal{O}_{X_α} -module generated by

$$\langle dw_1, \dots, dw_l, (w_{l+1}^{b_1-1}) \cdot dw_{l+1}, \dots, (w_{l+k}^{b_k-1}) \cdot dw_{l+k}, dw_{l+k+1}, \dots, dw_n \rangle.$$

Definition 4.9. Give any orbifold pair (X, D) together with an orbifold structure $\mathcal{C}_\alpha = \{(U_\alpha, f_\alpha, X_\alpha)\}_\alpha$, we define the sheaf of symmetric orbifold differential forms $S^{[\bullet]} \Omega_{\mathcal{C}_\alpha}^{[1]}$ by the collection of reflexive sheaves $\{S^{[\bullet]} \Omega_{(X_\alpha, f_\alpha, D)}^{[1]}\}_\alpha$ on each X_α .

Notation 4.3. Let (X, D) be an orbifold pair. We shall denote by $S^{[\bullet]} \Omega_X^{[1]}(*D_{\text{red}})$ the sheaf of symmetric rational differential forms with poles of arbitrary order along D_{red} , which is defined by:

$$\varinjlim_m \left(S^{[\bullet]} \Omega_X^{[1]} \otimes \mathcal{O}_X(m \cdot D_{\text{red}}) \right)^{\vee\vee}.$$

Sometimes It is more convenient to work with a notion of symmetric differential forms, adapted to (X, D) , as a coherent sheaf on X instead of X_α . This is the purpose of the following definition.

Definition 4.10. Let (X, D) be an orbifold pair equipped with an orbifold structure $\mathcal{C}_\alpha = \{(U_\alpha, f_\alpha, X_\alpha)\}_\alpha$. We define the sheaf of \mathcal{C} -differential forms $S_{\mathcal{C}}^{[\bullet]} \Omega_X^{[1]} \log(D)$ to be the reflexive, coherent subsheaf of $S^{[\bullet]} \Omega_X^{[1]}(*D_{\text{red}})$, defined, at the level of presheaves, by the following property:

$$\sigma \in \Gamma(U_\alpha, S_{\mathcal{C}}^{[\bullet]} \Omega_X^{[1]} \log(D)) \iff \sigma \in \Gamma(U_\alpha, S^{[\bullet]} \Omega_X^{[1]}(*D_{\text{red}})) \text{ and } f_\alpha^{[*]}(\sigma) \in \Gamma(X_\alpha, S^{[\bullet]} \Omega_{(X_\alpha, f_\alpha, D)}^{[1]}).$$

Remark 4.11. One can easily check that the notion of \mathcal{C} -differential forms in Definition 4.10 is independent of the choice of the orbifold structure \mathcal{C}_α .

Remark 4.12. Local calculations show that over X_{reg} the sheaf $S_{\mathcal{C}}^{[\bullet]} \Omega_X^{[1]} \log(D)$ is locally free. More precisely, for every $x \in X_{\text{reg}}$, there exists an open neighbourhood $W_x \subset X_{\text{reg}}$ with $W_x \cap D = \{(z_1, \dots, z_n) \mid \prod_{i=1}^l z_i = 0\}$ such that $S_{\mathcal{C}}^N \Omega_{X_{\text{reg}}}^1 \log(D|_{\text{reg}})$ is the \mathcal{O}_W -module freely generated by

$$\left\langle \frac{dz_1^{m_1}}{z_1^{[m_1 \cdot d_1]}}, \dots, \frac{dz_l^{m_l}}{z_l^{[m_l \cdot d_l]}}, dz_{l+1}^{m_{l+1}}, \dots, dz_n^{m_n} \right\rangle, \quad \sum_{i=1}^n m_i = N,$$

where $d_i = (1 - \frac{1}{a_i})$.

4.2. An extension theorem for symmetric orbifold differential forms. We are now ready to prove an extension result for orbifold differential forms, which can be interpreted as an orbifold version of [GKK10, Prop. 3.1].

Proposition 4.13. *Let $U \subseteq \mathbb{C}^n$ be a normal, algebraic subset. Assume that there is a smooth algebraic subset $V \subseteq \mathbb{C}^n$ and a finite group G acting freely in codimension one such that $U \cong V/G$. Let $\pi : \tilde{U} \rightarrow U$ be a strong log-resolution of U with $E \subseteq \text{Exc}(\pi)$ being the maximal reduced exceptional divisor. Then, for every $m \in \mathbb{N}$, the coherent sheaf*

$$(4.13.1) \quad \mathcal{G} := \pi_* \left(S_{\mathcal{C}}^m \Omega_{\tilde{U}}^1 \log \left(\left(1 - \frac{1}{|G|}\right) \cdot E \right) \right)$$

is reflexive.

Proof. Let $f : V \rightarrow U$ denote the finite map encoding the isomorphism $U \cong V/G$ and set \tilde{V} to be the normalization of the fibre product $V \times_U \tilde{U}$ with the resulting commutative diagram:

$$\begin{array}{ccc}
 \tilde{V} & \xrightarrow{\tilde{f}, \text{ finite}} & \tilde{U} \\
 \tilde{\pi} \downarrow & & \downarrow \pi \\
 V & \xrightarrow{f = \cdot / G} & U
 \end{array}$$

Note that \tilde{f} is a composition of the finite map $V \times_U \tilde{U} \rightarrow \tilde{U}$, which is of degree $|G|$, and its normalization. Therefore $\deg(\tilde{f}) = |G|$.

Now, as the problem is local, to prove the reflexivity of the sheaf \mathcal{G} (4.13.1), it suffices to show that the naturally defined map

$$(4.13.2) \quad \Gamma(\tilde{U}, S_C^m \Omega_{\tilde{U}}^1 \log \left((1 - \frac{1}{|G|}) \cdot E \right)) \longrightarrow \Gamma(\tilde{U} \setminus \text{Exc}(\pi), S^m \Omega_{\tilde{U}}^1)$$

is surjective.

To this end, let $\sigma \in \Gamma(U, S^{[m]} \Omega_U^{[1]})$ be the section defined by $\tilde{\sigma} \in \Gamma(\tilde{U} \setminus \text{Exc}(\pi), S^m \Omega_{\tilde{U}}^1)$. Let $\sigma_V = f^{[*]} \sigma \in \Gamma(V, S^m \Omega_V^1)$. As σ_V is regular, we have

$$\tilde{\pi}^* f^{[*]}(\sigma) = \tilde{\pi}^*(\sigma_V) \in \Gamma(\tilde{V}, S^m \Omega_{\tilde{V}}^1).$$

Viewing $\tilde{\sigma}$ as an element of $\Gamma(\tilde{U}, S^m \Omega_{\tilde{U}}^1(*E))$, and thanks to the commutativity of the diagram above, we have

$$(4.13.3) \quad \tilde{f}^*(\sigma) \in \Gamma(\tilde{V}, S^m \Omega_{\tilde{V}}^1).$$

Now, set $\{E_i\}$ to be the set of irreducible components of E and let

$$E' = \sum (1 - \frac{1}{a_i}) \cdot E_i$$

be the divisor with respect to which $\tilde{f} : \tilde{V} \rightarrow \tilde{U}$ is strictly adapted to the orbifold pair (\tilde{U}, E') (cf. Definition 4.4). Note that, by construction, we have:

(4.13.4) For each i , the inequality $a_i \leq |G|$ holds.

(4.13.5) $\Omega_{\tilde{V}}^1 = \Omega_{(\tilde{V}, \tilde{f}, E')}^1$ (cf. Example 4.8).

Therefore, thanks to (4.13.3) we have, by the definition of symmetric \mathcal{C} -differential forms, that $\sigma \in \Gamma(\tilde{U}, S_C^m \Omega_{\tilde{U}}^1 \log(E'))$. In particular, it follows that the map

$$\Gamma(\tilde{U}, S_C^m \Omega_{\tilde{U}}^1 \log(E')) \longrightarrow \Gamma(\tilde{U} \setminus \text{Exc}(\pi), S^m \Omega_{\tilde{U}}^1)$$

is a surjection.

On the other hand, thanks to the inequality $a_i \leq |G|$ in (4.13.4), we have

$$S_C^m \Omega_{\tilde{U}}^1 \log(E') \subseteq S_C^m \Omega_{\tilde{U}}^1 \log \left((1 - \frac{1}{|G|}) \cdot E \right).$$

The surjectivity of the map (4.13.2) now follows. \square

We conclude this section by pointing out that one can easily verify that the extension result in Proposition 4.13 holds for any resolution of U .

5. QUOTIENTS OF POLYDISCS

In this section, we consider the case where Ω is the polydisk \mathbb{D}^n , and consider the same setting as above. Let $\Gamma \subset \text{Aut}(\mathbb{D})^n$ be a discrete subgroup, and $X = \mathbb{D}^n/\Gamma$ a quotient with cyclic quotient singularities p_1, \dots, p_s , whose isotropy groups have cardinal m_i .

We denote $p : \Omega \rightarrow X$ the canonical projection, $\tilde{X} \xrightarrow{\pi} \bar{X}$ a resolution with exceptional divisor E_i above each p_i . Let $\tilde{\Delta} = \sum_i \left(1 - \frac{1}{m_i}\right) E_i$, $E = \sum_i E_i$ and let $\tilde{D} = \pi^* D$.

We first establish the following weak version of Theorem C.

Theorem 5.1. *If the Q-line bundle $\pi^* K_{\bar{X}} - n(\tilde{\Delta} + \tilde{D})$ is big, then $\Omega_{\tilde{X}}$ is big.*

Proof. Let ω be a global section of $\pi^* K_{\bar{X}}^{\otimes m}$. It gives a section of $K_{\mathbb{D}^n}^{\otimes m}$ invariant under Γ . Now we consider it as a section of $S^{mn} \Omega(\mathbb{D}^n)$ invariant under Γ . It gives a section of $S^{mn} \Omega(X^{reg})$. Therefore from the extension property 4.13, we obtain finally a section of $S^{mn} \Omega(\tilde{X}, \tilde{\Delta} + \tilde{D})$. Starting with a section ω of a multiple of

$$\pi^* K_{\bar{X}} - n(\tilde{\Delta} + \tilde{D}) - A,$$

where A is an ample line bundle, one gets a section of $S^{mn} \Omega(\tilde{X}) \otimes A^{-1}$, which implies that $\Omega_{\tilde{X}}$ is big. \square

Example 5.2. Already in the case of surfaces, this statement provides interesting examples of surfaces with $c_1^2 \leq c_2$ and big cotangent bundle (see [GRVR17]).

One can formulate the criterion involving $K_{\tilde{X}}$ instead of $K_{\bar{X}}$.

Corollary 5.3. *In the same setting as above, if $K_{\tilde{X}} + E - n\tilde{D} - 2n\tilde{\Delta}$ is big then $\Omega_{\tilde{X}}$ is big.*

Proof. By lemma 2.2, one obtains $\pi^* K_{\bar{X}} - n(\tilde{\Delta} + \tilde{D}) \geq K_{\tilde{X}} + E - n\tilde{D} - 2n\tilde{\Delta}$. \square

Now, let us suppose that Γ is *irreducible* in the following sense: the restriction of each of the n projections $p_j : \text{Aut}(\mathbb{D})^n \rightarrow \text{Aut}(\mathbb{D})$ to Γ is injective.

Remark that in this setting, singularities of quotients $X = \mathbb{D}^n/\Gamma$ are automatically cyclic quotient.

Then we obtain the stronger statement, Theorem C.

Theorem 5.4. *Let $\Gamma \subset \text{Aut}(\mathbb{D})^n$ be a discrete irreducible subgroup and denote $L := \pi^* K_{\bar{X}} - n(\tilde{\Delta} + \tilde{D})$. Let $\mathbb{B}^+(L)$ be the augmented base locus of L and $Z := \mathbb{B}^+(L) \cup \tilde{D} \cup E$. Then all subvarieties $W \not\subset Z$ have big cotangent bundle.*

In particular, if L is big there is a proper subvariety $Z \subsetneq \tilde{X}$ such that all subvarieties $W \not\subset Z$ have big cotangent bundle.

Proof. Let $W \subset \tilde{X}$ be a subvariety. Following the same steps and notations as in the proof of theorem 5.1, we see that the pull-backs of sections ω on W may vanish in two cases: either $W \subset \mathbb{B}^+(L) \cup E \cup \tilde{D}$ or W is tangent to one of the codimension one holomorphic foliation \mathcal{F}_i induced by $dz_i = 0$ on \mathbb{D}^n . In [RT17] (see proposition 3.1 and remark 3.3), we have established that, under the hypothesis of irreducibility of Γ , leaves passing through elliptic fixed points are Stein therefore cannot contain algebraic subvarieties. Therefore if W is tangent to one of this foliation, we have $\pi(W) \subset X^{reg}$. But then the Bergman metric of the polydisk \mathbb{D}^n induces a Kähler metric on X^{reg} with nonpositive holomorphic bisectional curvature and negative holomorphic sectional curvature. Therefore, we deduce from [BKT13] that in this case W has big cotangent bundle. \square

Finally, we observe that we obtain in this case another proof of Theorem A.

Theorem 5.5. *In the same setting as above, let $f : \mathbb{C} \rightarrow X$ be an entire curve. Then $f(\mathbb{C}) \subset Z := \mathbb{B}^+(L) \cup \tilde{D} \cup E$. In particular, if L is big then $\text{Exc}(X) \neq X$.*

Proof. Let $f : \mathbb{C} \rightarrow X$ be an entire curve. From the construction above, consider global symmetric differentials vanishing on an ample divisor induced by sections of $\pi^* K_{\tilde{X}} - n(\tilde{\Delta} + \tilde{D}) - A$. The classical vanishing theorem (see for example corollary 7.9 in [Dem97]) implies that any entire curve $f : \mathbb{C} \rightarrow X$ satisfies $f^* \omega \equiv 0$. Let us suppose that $f(\mathbb{C})$ does not lie in Z then f has to be tangent to one of the holomorphic codimension-one foliations induced by $dz_i = 0$. But this is impossible because proposition 3.1 of [RT17] says that these leaves are hyperbolic. \square

6. HILBERT MODULAR VARIETIES

Let K be a totally real algebraic number field of degree $n > 1$ over the rational number field \mathbb{Q} , and let \mathcal{O}_K be the ring of integers in K . Then $\Gamma = \Gamma_K = SL_2(\mathcal{O}_K)$ acts on the product \mathfrak{H}^n of n copies of the upper half plane $\mathfrak{H} = \{z \in \mathbb{C} \mid \Im z > 0\}$:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot (z_1, \dots, z_n) = \left(\frac{\alpha^{(1)}z_1 + \beta^{(1)}}{\gamma^{(1)}z_1 + \delta^{(1)}}, \dots, \frac{\alpha^{(n)}z_n + \beta^{(n)}}{\gamma^{(n)}z_n + \delta^{(n)}} \right),$$

where $\alpha = \alpha^{(1)}, \alpha^{(2)}, \dots, \alpha^{(n)}$ denote the conjugates of $\alpha \in K$.

A holomorphic function f on \mathfrak{H}^n is called a Hilbert modular form of weight k if

$$f(Mz) = N(\gamma z + \delta)^k f(z)$$

for all $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma$, where $N(z) = \prod_{i=1}^n z_i$.

Hilbert modular forms are classically interpreted in terms of differential forms: if $\omega = dz_1 \wedge \dots \wedge dz_n$ and f is a Hilbert modular form of weight k then $f\omega^{\otimes k}$ gives an invariant holomorphic top-differential forms which descends on (the smooth part of) \mathfrak{H}^n/Γ .

The observation already used in the previous sections is that one can also look at Hilbert modular forms as symmetric differential forms. Indeed, in the above notations, $f(dz_1 \dots dz_n)^k$ is also invariant under Γ and therefore provides a symmetric differential on (the smooth part of) \mathfrak{H}^n/Γ .

Recall that there is a natural compactification $Y := \overline{\mathfrak{H}^n/\Gamma}$ adding finitely many cusps. Then one can take a projective resolution $X \rightarrow Y$.

Now we can apply the results of the previous section.

First, a corollary of Theorem 5.1 gives the following result.

Theorem 6.1. *Let $n \geq 2$. Then except finitely many possible exceptions Hilbert modular varieties have a big cotangent bundle.*

Proof. Let $E = E_e + E_c$ be the exceptional divisor where $E_e = \sum_i E_e^i$ corresponds to the resolution of elliptic points and E_c corresponds to the resolution of cusps. Theorem 5.1 tells us that there are constants α_i depending only on the order of the stabilizer of elliptic fixed points such that if $K_X + E - \sum_i \alpha_i E_e^i - nE_c$ is big then Ω_X is big.

Therefore we are reduced to prove that $K_X + E - \sum_i \alpha_i E_e^i - nE_c$ is big except finitely many possible exceptions. Let S_k^m denote the space of Hilbert modular forms of weight k and vanishing order at least m over cusps. Sections of

$$k(K_X + E - \sum_i \alpha_i E_e^i - nE_c)$$

corresponds to modular forms. We have to show the maximal growth of the space of corresponding modular forms. So we have to prove that one can produce more sections than the number of conditions imposed by the vanishing along the exceptional components. We shall use the following result of [Tsu85] (Sect. 4)

$$(6.1.1) \quad \dim S_k^{v_k}(\Gamma_K) \geq (2^{-2n+1}\pi^{-2n}d_K^{3/2}\zeta_K(2) - 2^{n-1}v^n n^{-n}d_K^{1/2}hR)k^n + O(k^{n-1})$$

for even $k \geq 0$, where h, d_K, R, ζ_K denote the class number of K , the absolute value of the discriminant, the positive regulator and the zeta function of K . In particular, there is a modular form F with $\text{ord}(f) / \text{weight}(f) \geq v$, if

$$(6.1.2) \quad v < 2^{-3}\pi^{-2}n \left(\frac{4d_K\zeta_K(2)}{hR} \right)^{1/n}.$$

If we fix n , then $\zeta_K(2)$ has a positive lower bound independent of K . Since $hR \sim d_K^{1/2}$ by the Brauer-Siegel Theorem, for any constant C there are only a finite number of K such that the right hand side of (6.1.2) is smaller than C . Therefore we obtain sections with the requested order of vanishing along E_c .

Let us now deal with the elliptic points. Prestel [Pre68] has obtained precise formula on the number of elliptic points of the Hilbert modular group. In particular, one can deduce (see section 6.5 of [RT17] for details) that for fixed n , there are only finitely many different type of elliptic points and the number of equivalence classes of elliptic fixed points is $O(d_K^{\frac{1}{2}+\epsilon})$ for every $\epsilon > 0$. This immediately gives the maximal growth of the space of modular forms satisfying the vanishing conditions with finitely many possible exceptions. \square

As a consequence, we recover the main result of [Tsu85]

Corollary 6.2. *Let $n \geq 2$. Then except finitely many possible exceptions Hilbert modular varieties are of general type.*

Proof. This is an immediate application of [CP15] who prove that if the cotangent bundle is big then the canonical bundle is big. \square

Now, we can give the proof of the two statements announced in Theorem D of the introduction as corollaries of Theorems 5.4 and 5.5.

Theorem 6.3. *Let $n \geq 2$. Then, except finitely many possible exceptions, Hilbert modular varieties satisfy $\text{Exc}(X) \neq X$.*

Proof. Let X be a Hilbert modular variety. The proof of Theorem 6.1 tells that $L := K_X + E - \sum_i \alpha_i E_e^i - nE_c$ is big except finitely many possible exceptions. Then Theorem 5.5 tells us that $\text{Exc}(X) \subset \mathbb{B}^+(L) \cup E$. \square

Finally, we obtain the second statement.

Theorem 6.4. *Let $n \geq 2$. Then except finitely many possible exceptions Hilbert modular varieties contain a proper subvariety Z such that all subvarieties not contained in Z have big cotangent bundle and are of general type.*

Proof. Let X be a Hilbert modular variety such that $L := K_X + E - \sum_i \alpha_i E_e^i - nE_c$ is big. Then define $Z := \mathbb{B}^+(L) \cup E$. Let $Y \subset X$ be a subvariety not contained in Z . Theorem 5.4 gives that all subvarieties not contained in Z have big cotangent bundle and are of general type. \square

REFERENCES

- [BKT13] Yohan Brunebarbe, Bruno Klingler, and Burt Totaro. Symmetric differentials and the fundamental group. *Duke Math. J.*, 162(14):2797–2813, 2013. [↑ 10](#)
- [Bru16] Yohan Brunebarbe. A strong hyperbolicity property of locally symmetric varieties. *preprint*, 2016. [↑ 1](#)
- [Cad16] Benoît Cadorel. Symmetric differentials on complex hyperbolic manifolds with cusps. *preprint arXiv:1606.05470*, 2016. [↑ 2, 6](#)
- [Cam04] Frédéric Campana. Orbifolds, special varieties and classification theory. *Ann. Inst. Fourier (Grenoble)*, 54(3):499–630, 2004. <http://eudml.org/doc/116120>. [↑ 6](#)
- [CKT16] Benoît Claudon, Stefan Kebekus, and Behrouz Taji. Generic positivity and applications to hyperbolicity of moduli spaces. *Preprint arXiv:1610.09832*, October 2016. [↑ 6](#)
- [CP15] Frédéric Campana and Mihai Păun. Orbifold generic semi-positivity: an application to families of canonically polarized manifolds. *Ann. Inst. Fourier (Grenoble)*, 65(2):835–861, 2015. [↑ 12](#)
- [Dem97] Jean-Pierre Demailly. Algebraic criteria for Kobayashi hyperbolic projective varieties and jet differentials. In *Algebraic geometry—Santa Cruz 1995*, volume 62 of *Proc. Sympos. Pure Math.*, pages 285–360. Amer. Math. Soc., Providence, RI, 1997. [↑ 11](#)
- [Fuj75] Akira Fujiki. On resolutions of cyclic quotient singularities. *Publ. Res. Inst. Math. Sci.*, 10(1):293–328, 1974/75. [↑ 3](#)
- [GG80] Mark Green and Phillip Griffiths. Two applications of algebraic geometry to entire holomorphic mappings. In *The Chern Symposium 1979 (Proc. Internat. Sympos., Berkeley, Calif., 1979)*, pages 41–74. Springer, New York, 1980. [↑ 1](#)
- [GKK10] Daniel Greb, Stefan Kebekus, and Sándor J. Kovács. Extension theorems for differential forms, and Bogomolov-Sommese vanishing on log canonical varieties. *Compositio Math.*, 146:193–219, January 2010. DOI:10.1112/S0010437X09004321. A slightly extended version is available as [arXiv:0808.3647](#). [↑ 8](#)
- [GRVR17] Julien Grivaux, Juliana Restrepo Velasquez, and Erwan Rousseau. On Lang’s conjecture for some product-quotient surfaces. *to appear in Ann. Sc. Norm. Super. Pisa.*, 2017. [↑ 10](#)
- [GT16] Henri Guenancia and Behrouz Taji. Orbifold stability and Miyaoka-Yau inequality for minimal pairs. *Preprint arXiv:1611.05981*, November 2016. [↑ 6, 7](#)
- [JK11] Kelly Jabbusch and Stefan Kebekus. Families over special base manifolds and a conjecture of Campana. *Math. Z.*, 269(3-4):847–878, 2011. DOI: 10.1007/s00209-010-0758-6. [↑ 6](#)
- [Keu08] Jonghae Keum. Quotients of fake projective planes. *Geom. Topol.*, 12(4):2497–2515, 2008. [↑ 1](#)
- [Lan86] Serge Lang. Hyperbolic and Diophantine analysis. *Bull. Amer. Math. Soc. (N.S.)*, 14(2):159–205, 1986. [↑ 1](#)
- [Laz04] Robert Lazarsfeld. *Positivity in algebraic geometry. I*, volume 48 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2004. Classical setting: line bundles and linear series. [↑ 7](#)
- [Mum77] D. Mumford. Hirzebruch’s proportionality theorem in the noncompact case. *Invent. Math.*, 42:239–272, 1977. [↑ 1, 5](#)
- [Nad89] Alan Michael Nadel. The nonexistence of certain level structures on abelian varieties over complex function fields. *Ann. of Math. (2)*, 129(1):161–178, 1989. [↑ 1, 2](#)
- [Pre68] Alexander Prestel. Die elliptischen Fixpunkte der Hilbertschen Modulgruppen. *Math. Ann.*, 177:181–209, 1968. [↑ 12](#)
- [Rou16] Erwan Rousseau. Hyperbolicity, automorphic forms and Siegel modular varieties. *Ann. Sci. Éc. Norm. Supér. (4)*, 49(1):249–255, 2016. [↑ 1](#)
- [RT17] Erwan Rousseau and Frédéric Touzet. Curves in Hilbert modular varieties. *to appear in Asian J. Math.*, 2017. [↑ 3, 10, 11, 12](#)
- [Taj16] Behrouz Taji. The isotriviality of smooth families of canonically polarized manifolds over a special quasi-projective base. *Compositio Math.*, 152:1421–1434, 7 2016. DOI:10.1112/S0010437X1600734X. [↑ 6](#)
- [Tsu85] Shigeaki Tsuyumine. On the Kodaira dimensions of Hilbert modular varieties. *Invent. Math.*, 80(2):269–281, 1985. [↑ 1, 12](#)
- [Tsu86] Shigeaki Tsuyumine. Multitensors of differential forms on the Hilbert modular variety and on its subvarieties. *Math. Ann.*, 274(4):659–670, 1986. [↑ 3](#)

BENOÎT CADOREL, AIX MARSEILLE UNIV, CNRS, CENTRALE MARSEILLE, I2M, MARSEILLE, FRANCE

E-mail address: benoit.cadorel@univ-amu.fr

ERWAN ROUSSEAU, INSTITUT UNIVERSITAIRE DE FRANCE & AIX MARSEILLE UNIV, CNRS, CENTRALE MARSEILLE, I2M, MARSEILLE, FRANCE

E-mail address: erwan.rousseau@univ-amu.fr

BEHROUZ TAJI, UNIVERSITY OF NOTRE DAME, DEPARTMENT OF MATHEMATICS, 278 HURLEY, NOTRE DAME, IN 46556, USA

E-mail address: btaji@nd.edu

URL: <http://sites.nd.edu/b-taji/>