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UNRAMIFIED 2-EXTENSIONS OF TOTALLY IMAGINARY NUMBER FIELDS AND 2-ADIC ANALYTIC GROUPS

by

Christian Maire

Abstract. — Let K be a totally imaginary number field. Denote by $G_K^{ur}(2)$ the Galois group of the maximal unramified pro-2 extension of K . By comparing cup-products in étale cohomology of $\text{Spec } \mathcal{O}_K$ and cohomology of uniform pro-2 groups, we obtain situations where $G_K^{ur}(2)$ has no non-trivial uniform analytic quotient, proving some new special cases of the unramified Fontaine-Mazur conjecture. For example, in the family of imaginary quadratic fields K for which the 2-rank of the class group is equal to 5, we obtain that for at least 33.12% of such K , the group $G_K^{ur}(2)$ has no non-trivial uniform analytic quotient.

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Introduction

Given a number field K and a prime number p , the tame version of the conjecture of Fontaine-Mazur (conjecture (5a) of [8]) asserts that every finitely and tamely ramified continuous Galois representation $\rho : \text{Gal}(\overline{K}/K) \rightarrow \text{Gl}_m(\mathbb{Q}_p)$ of the absolute Galois group of K , has finite image. Let us mention briefly two strategies to attack this conjecture.

– The first one is to use the techniques coming from the considerations that inspired the conjecture, *i.e.*, from the Langlands program (geometric Galois representations, modular

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forms, deformation theory, etc.). For more than a decade, many authors have contributed to understanding the foundations of this conjecture with some serious progress having been made. As a partial list of such results, we refer the reader to Buzzard-Taylor [5], Buzzard [4], Kessaei [15], Kisin [16], Pilloni [27], Pilloni-Stroh [28], etc.

– The second one consists in comparing properties of p -adic analytic pro- p groups and arithmetic. Thanks to this strategy, Boston gave in the 90's the first evidence for the tame version of the Fontaine-Mazur conjecture (see [1], [2]). This one has been extended by Wingberg [35]. See also [21] and the recent work of Hajir-Maire [14]. In all of these situations, the key fact is to use a semi-simple action. Typically, this approach gives no information for quadratic number fields when $p = 2$.

In this work, when $p = 2$, we propose to give some families of imaginary quadratic number fields for which the unramified Fontaine-Mazur conjecture is true (conjecture (5b) of [8]). For doing this, we compare the étale cohomology of $\text{Spec } \mathcal{O}_K$ and the cohomology of p -adic analytic pro- p groups. In particular, we exploit the fact that in characteristic 2, cup products in H^2 could be not alternating (meaning $x \cup x \neq 0$), more specifically, a beautiful computation of cup-products in $H_{\text{ét}}^3(\text{Spec } \mathcal{O}_k, \mathbb{F}_2)$ made by Carlson and Schläpke in [6]. And so, surprisingly our strategy works *only* for $p = 2$!

Given a prime number p , denote by $K^{ur}(p)$ the maximal pro- p unramified extension of K ; put $G_K^{ur}(p) := \text{Gal}(K^{ur}(p)/K)$. Here we are interested in uniform quotients of $G_K^{ur}(p)$ (see section 2 for definition) which are related to the unramified Fontaine-Mazur conjecture thanks to the following equivalent version:

Conjecture A. — *Every uniform quotient G of $G_K^{ur}(p)$ is trivial.*

Remark that Conjecture A can be rephrased as follows: the pro- p group $G_K^{ur}(p)$ has no uniform quotient G of dimension d for all $d > 0$. Of course, this is obvious when $d > d_2 \text{Cl}_K$, and when $d \leq 2$, thanks to the fact that $G_K^{ur}(p)$ is FAb (see later the definition).

Now take $p = 2$. Let $(x_i)_{i=1, \dots, n}$ be an \mathbb{F}_2 -basis of $H^1(G_K^{ur}(2), \mathbb{F}_2) \simeq H_{\text{ét}}^1(\text{Spec } \mathcal{O}_k, \mathbb{F}_2)$, and consider the $n \times n$ -square matrix $M_K := (a_{i,j})_{i,j}$ with coefficients in \mathbb{F}_2 , where

$$a_{i,j} = x_i \cup x_i \cup x_j,$$

thanks to the fact that here $H_{\text{ét}}^3(\text{Spec } \mathcal{O}_K, \mathbb{F}_2) \simeq \mathbb{F}_2$. As we will see, this is the Gram matrix of a certain bilinear form defined, via Artin symbol, on the Kummer radical of the 2-elementary abelian maximal unramified extension $K^{ur,2}/K$ of K . We also will see that for imaginary quadratic number fields, this matrix is often of large rank.

First, we prove:

Theorem. — *Let K be a totally imaginary number field. Let $n := d_2 \text{Cl}_K$ be the 2-rank of the class group of K .*

- (i) *Then the pro-2 group $G_K^{ur}(2)$ has no uniform quotient of dimension $d > n - \frac{1}{2} \text{rk}(M_K)$.*
- (ii) *Moreover, Conjecture A holds (for $p = 2$) when:*
 - $n = 3$, and $\text{rk}(M_K) > 0$;
 - $n = 4$, and $\text{rk}(M_K) \geq 3$;
 - $n = 5$, and $\text{rk}(M_K) = 5$.

By relating the matrix M_K to a Rédei-matrix type, and thanks to the work of Gerth [11] and Fouvry-Klüners [9], one can also deduce some density information when K varies in the family \mathcal{F} of imaginary quadratic fields. For $n, d, X \geq 0$, denote by

$$S_X := \{K \in \mathcal{F}, -\text{disc}_K \leq X\}, \quad S_{n,X} := \{K \in S_X, d_2\text{Cl}_K = n\}$$

$$\text{FM}_{n,X}^{(d)} := \{K \in S_{n,X}, G_K^{ur}(2) \text{ has no uniform quotient of dimension } > d\},$$

$$\text{FM}_{n,X} := \{K \in S_{n,X}, \text{Conjecture A holds for } K\},$$

and consider the limits:

$$\text{FM}_n^{(d)} := \liminf_{X \rightarrow +\infty} \frac{\#\text{FM}_{n,X}^{(d)}}{\#S_{n,X}}, \quad \text{FM}_n := \liminf_{X \rightarrow +\infty} \frac{\#\text{FM}_{n,X}}{\#S_{n,X}}.$$

FM_n measures the proportion of imaginary quadratic fields K with $d_2\text{Cl}_K = n$, for which Conjecture A holds (for $p = 2$); and $\text{FM}_n^{(d)}$ measures the proportion of imaginary quadratic fields K with $d_2\text{Cl}_K = n$, for which $G_K^{ur}(2)$ has no uniform quotient of dimension $> d$.

Then [11] allows us to obtain the following densities for uniform groups of small dimension:

Corollary 1. — *One has:*

- (i) $\text{FM}_3 \geq .992187$,
- (ii) $\text{FM}_4 \geq .874268$, $\text{FM}_4^{(4)} \geq .999695$,
- (iii) $\text{FM}_5 \geq .331299$, $\text{FM}_5^{(4)} \geq .990624$, $\text{FM}_5^{(5)} \geq .9999943$,
- (iv) for all $d \geq 3$, $\text{FM}_d^{(1+d/2)} \geq 0.866364$, and $\text{FM}_d^{(2+d/2)} \geq .999953$.

Remark. — *At this level, one should make two observations.*

1) Perhaps for many $K \in S_{3,X}$ and $S_{4,X}$, the pro-2 group $G_K^{ur}(2)$ is finite but, by the Theorem of Golod-Shafarevich (see for example [17]), for every $K \in S_{n,X}$, $n \geq 5$, the pro-2 group $G_K^{ur}(2)$ is infinite.

2) In our work, it will appear that we have no information about the Conjecture A for number fields K for which the 4-rank of the class group is large. Typically, in FM_i one keeps out all the number fields having maximal 4-rank.

To conclude, let us mention a general asymptotic estimate thanks to the work of Fouvry-Klüners [9]. Put

$$\text{FM}_X^{[i]} := \{K \in S_X, G_K^{ur}(2) \text{ has no uniform quotient of dimension } > i + \frac{1}{2}d_2\text{Cl}_K\}$$

and

$$\text{FM}^{[i]} := \liminf_{X \rightarrow +\infty} \frac{\#\text{FM}_X^{[i]}}{\#S_X}.$$

Our work allows us to obtain:

Corollary 2. — *One has:*

$$\text{FM}^{[1]} \geq .0288788, \quad \text{FM}^{[2]} \geq 0.994714, \quad \text{and} \quad \text{FM}^{[3]} \geq 1 - 9.7 \cdot 10^{-8}.$$

This paper has three sections. In Section 1 and Section 2, we give the basic tools concerning the étale cohomology of number fields and the p -adic analytic groups. Section 3 is devoted to arithmetic considerations. After the presentation of our strategy, we develop some basic facts about bilinear forms over \mathbb{F}_2 , specially for the form introduced in our study (which is defined on a certain Kummer radical). In particular, we insist on the role played by totally isotropic subspaces. To finish, we consider a relation with a Rédei matrix that allows us to obtain density information.

Notations.

Let p be a prime number and let K be a number field. Denote by

- $p^* = (-1)^{(p-1)/2}p$, when p is odd;
- \mathcal{O}_K the ring of integers of K ;
- Cl_K the p -Sylow of the Class group of \mathcal{O}_K ;
- K^{ur} the maximal profinite extension of K unramified everywhere. Put $G_K^{ur} = \text{Gal}(K^{ur}/K)$;
- $K^{ur}(p)$ the maximal pro- p extension of K unramified everywhere. Put $G_K^{ur}(p) := \text{Gal}(K^{ur}(p)/K)$;
- $K^{ur,p}$ the elementary abelian maximal unramified p -extension of K .

Recall that the group $G_K^{ur}(p)$ is a finitely presented pro- p group. See [17]. See also [25] or [12]. Moreover by class field theory, Cl_K is isomorphic to the abelianization of $G_K^{ur}(p)$. In particular it implies that every open subgroup \mathcal{H} of $G_K^{ur}(p)$ has finite abelianization: this property is known as "FAB".

1. Etale cohomology: what we need

1.1. — For what follows, the references are plentiful: [22], [23], [24], [31], [32], etc.

Assume that K is totally imaginary when $p = 2$, and put $X_K = \text{Spec}\mathcal{O}_K$. The Hochschild-Serre spectral sequence (see [23]) gives for every $i \geq 1$ a map

$$\alpha_i : H^i(G_K^{ur}(p)) \longrightarrow H_{et}^i(X_K),$$

where the coefficients are in \mathbb{F}_p (meaning the constant sheaf for the étale site X_K). As α_1 is an isomorphism, one obtains the long exact sequence:

$$H^2(G_K^{ur}(p)) \hookrightarrow H_{et}^2(X_K) \longrightarrow H_{et}^2(X_{K^{ur}(p)}) \longrightarrow H^3(G_K^{ur}(p)) \longrightarrow H_{et}^3(X_K)$$

where $H_{et}^3(X_K) \simeq (\mu_{K,p})^\vee$, here $(\mu_{K,p})^\vee$ is the Pontryagin dual of the group of p th-roots of unity in K .

1.2. — Take now $p = 2$. Let us give $x, y, z \in H_{et}^1(X_K)$. In [6], Carlson and Schrank give a formula in order to determine the cup-product $x \cup y \cup z \in H_{et}^3(X_K)$. In particular, they show how to produce some arithmetical situations for which such cup-products $x \cup x \cup y$ are not zero. Now, one has the commutative diagram:

$$\begin{array}{ccc} H^3(G_K^{ur}(p)) & \xrightarrow{\alpha_3} & H_{et}^3(X_K) \\ \beta \uparrow & & \beta_{et} \uparrow \\ H^1(G_K^{ur}(p))^{\otimes 3} & \xrightarrow[\alpha_1]{\simeq} & H_{et}^1(X_K)^{\otimes 3} \end{array}$$

Hence $(\alpha_3 \circ \beta)(a \otimes b \otimes a) = \alpha_1(a) \cup \alpha_1(b) \cup \alpha_1(c)$. By taking $x = \alpha_1(a) = \alpha_1(b)$ and $y = \alpha_1(c)$, one gets $a \cup a \cup b \neq 0 \in H^3(G_K^{ur}(p))$ when $x \cup x \cup y \neq 0 \in H_{et}^3(X_K)$.

1.3. The computation of Carlson and Schrank. — Take x and y two non-trivial characters of $H^1(G^{ur}(p)) \simeq H_{et}^1(X)$. Put $K_x = K^{ker(x)}$ and $K_y = K^{ker(y)}$. By Kummer theory, there exist $a_x, a_y \in K^\times / (K^\times)^2$ such that $K_x = K(\sqrt{a_x})$ and $K_y = K(\sqrt{a_y})$. As the extension K_y/K is unramified, for every prime ideal \mathfrak{p} of \mathcal{O}_K , the \mathfrak{p} -valuation $v_{\mathfrak{p}}(a_y)$ is even, and then $\sqrt{(a_y)}$ has a sense (as an ideal of \mathcal{O}_K). Let us write

$$\sqrt{(a_y)} := \prod_i \mathfrak{p}_{y,i}^{e_{y,i}}.$$

Denote by I_x the set of prime ideals \mathfrak{p} of \mathcal{O}_K such that \mathfrak{p} is inert in K_x/K (or equivalently, I_x is the set of primes of K such that the Frobenius at \mathfrak{p} generates $\text{Gal}(K_x/K)$).

Proposition 1.1 (Carlson and Schrank). — *The cup-product $x \cup x \cup y \in H_{et}^3(X)$ is non-zero if and only if, $\sum_{\mathfrak{p}_{y,i} \in I_x} e_{y,i}$ is odd.*

Remark 1.2. — The condition of Proposition 1.1 is equivalent to the triviality of the Artin symbol $\left(\frac{K_x/K}{\sqrt{(a_y)}} \right)$. Hence if one takes $b_y = a_y \alpha^2$ with $\alpha \in K$ instead of a_y then, as $\left(\frac{K_x/K}{(\alpha)} \right)$ is trivial, the condition is well-defined.

Let us give an easy example inspired by a computation of [6].

Proposition 1.3. — *Let K/\mathbb{Q} be an imaginary quadratic field. Suppose that there exist p and q two different odd prime numbers ramified in K/\mathbb{Q} , and such that: $\left(\frac{p^*}{q} \right) = -1$. Then there exist $x \neq y \in H_{et}^1(X_K)$ such that $x \cup x \cup y \neq 0$.*

Proof. — Take $K_x = K(\sqrt{p^*})$ and $K_y = K(\sqrt{q^*})$, and apply Proposition 1.1. \square

2. Uniform pro- p -groups and arithmetic: what we need

2.1. — Let us start with the definition of a uniform pro- p group (see for example [7]).

Definition 2.1. — Let G be a finitely generated pro- p group. We say that G is uniform if:

- G is torsion free, and

– $[G, G] \subset G^{2p}$.

Remark 2.2. — For a uniform group G , the p -rank of G coincides with the dimension of G .

The uniform pro- p groups play a central role in the study of analytic pro- p group, indeed:

Theorem 2.3 (Lazard [18]). — *Let G be a profinite group. Then G is p -adic analytic i.e. $G \hookrightarrow_c \mathrm{Gl}_m(\mathbb{Z}_p)$ for a certain positive integer m , if and only if, G contains an open uniform subgroup \mathcal{H} .*

Remark 2.4. — For different equivalent definitions of p -adic analytic groups, see [7]. See also [20].

Example 2.5. — The correspondence between p -adic analytic pro- p groups and \mathbb{Z}_p -Lie algebra via the log/exp maps, allows to give examples of uniform pro- p groups (see [7], see also [14]). Typically, let $\mathfrak{sl}_n(\mathbb{Q}_p)$ be the \mathbb{Q}_p -Lie algebra of the square matrices $n \times n$ with coefficients in \mathbb{Q}_p and of zero trace. It is a simple algebra of dimension $n^2 - 1$. Take the natural basis:

- (a) for $i \neq j$, $E_{i,j} = (e_{k,l})_{k,l}$ for which all the coefficient are zero excepted $e_{i,j}$ that takes value $2p$;
- (b) for $i > 1$, $D_i = (d_{k,l})_{k,l}$ which is the diagonal matrix $D_i = (2p, 0, \dots, 0, -2p, 0, \dots, 0)$, where $d_{i,i} = -2p$.

Let \mathfrak{sl}_n be the \mathbb{Z}_p -Lie algebra generated by the $E_{i,j}$ and the D_i . Put $X_{i,j} = \exp E_{i,j}$ and $Y_i = \exp D_i$. Denote by $\mathrm{Sl}_n^1(\mathbb{Z}_p)$ the subgroup of $\mathrm{Sl}_n(\mathbb{Z}_p)$ generated by the matrices $X_{i,j}$ and Y_i . The group $\mathrm{Sl}_n^1(\mathbb{Z}_p)$ is uniform and of dimension $n^2 - 1$. It is also the kernel of the reduction map of $\mathrm{Sl}_n(\mathbb{Z}_p)$ modulo $2p$. Moreover, $\mathrm{Sl}_n^1(\mathbb{Z}_p)$ is also FAb, meaning that every open subgroup \mathcal{H} has finite abelianization.

Recall by Lazard [18] (see also [34] for an alternative proof):

Theorem 2.6 (Lazard [18]). — *Let G be a uniform pro- p group (of dimension $d > 0$). Then for all $i \geq 1$, one has:*

$$H^i(G) \simeq \bigwedge^i (H^1(G)),$$

where here the exterior product is induced by the cup-product.

As consequence, one has immediately:

Corollary 2.7. — *Let G be a uniform pro- p group. Then for all $x, y \in H^1(G)$, one has $x \cup x \cup y = 0 \in H^3(G)$.*

Remark 2.8. — For $p > 2$, Theorem 2.6 is an equivalence: a pro- p group G is uniform if and only if, for $i \geq 1$, $H^i(G) \simeq \bigwedge^i (H^1(G))$. (See [34].)

Let us mention another consequence useful in our context:

Corollary 2.9. — *Let G be a FAb uniform prop- p group of dimension $d > 0$. Then $d \geq 3$.*

Proof. — Indeed, if $\dim G = 1$, then $G \simeq \mathbb{Z}_p$ (G is pro- p free) and, if $\dim G = 2$, then by Theorem 2.3, $H^2(G) \simeq \mathbb{F}_p$ and $G^{ab} \twoheadrightarrow \mathbb{Z}_p$. Hence, $\dim G$ should be at least 3. \square

2.2. — Let us recall the Fontaine-Mazur conjecture (5b) of [8].

Conjecture. — *Let K be a number field. Then every continuous Galois representation $\rho : G_K^{ur} \rightarrow \mathrm{GL}_m(\mathbb{Z}_p)$ has finite image.*

Following the result of Theorem 2.3 of Lazard, we see that proving Conjecture (5b) of [8] for K , is equivalent to proving Conjecture A for every finite extension L/K in K^{ur}/K .

3. Arithmetic consequences

3.1. The strategy. — Usually, when p is odd, cup-products factor through the exterior product. But, for $p = 2$, it is not the case! This is the obvious observation that we will combine with étale cohomology and with the cohomology of uniform pro- p groups.

From now on we assume that $p = 2$.

Suppose given G a non-trivial uniform quotient of $G_K^{ur}(p)$. Then by the inflation map one has:

$$H^1(G) \hookrightarrow H^1(G_K^{ur}(p)).$$

Now take $a, b \in H^1(G_K^{ur}(p))$ coming from $H^1(G)$. Then, the cup-product $a \cup a \cup b \in H^3(G_K^{ur}(p))$ comes from $H^3(G)$ by the inflation map. In other words, one has the following commutative diagram:

$$\begin{array}{ccccc} H^3(G) & \xrightarrow{\text{inf}} & H^3(G_K^{ur}(p)) & \xrightarrow{\alpha_3} & H_{et}^3(X_K) \\ \beta_0 \uparrow & & \beta \uparrow & & \beta_{et} \uparrow \\ H^1(G)^{\otimes 3} & \hookrightarrow & H^1(G_K^{ur}(p))^{\otimes 3} & \xrightarrow{\cong} & H_{et}^1(X_K)^{\otimes 3} \end{array}$$

But by Lazard's result (Theorem 2.6), $\beta_0(a \otimes a \otimes b) = 0$, and then one gets a contradiction if $\alpha_1(a) \cup \alpha_1(a) \cup \alpha_1(b)$ is non-zero in $H_{et}^3(X_K)$: it is at this level that one may use the computation of Carlson-Schläpke.

Before developing this observation in the context of analytic pro-2 group, let us give two immediate consequences:

Corollary 3.1. — *Let K/\mathbb{Q} be a quadratic imaginary number field satisfying the condition of Proposition 1.3. Then $G_K^{ur}(2)$ is of cohomological dimension at least 3.*

Proof. — Indeed, there exists a non-trivial cup-product $x \cup x \cup y \in H_{et}^3(X)$ and then non-trivial in $H^3(G_K(2))$. \square

Corollary 3.2. — *Let p_1, p_2, p_3, p_4 be four prime numbers such that $p_1 p_2 p_3 p_4 \equiv 3 \pmod{4}$. Take $K = \mathbb{Q}(\sqrt{-p_1 p_2 p_3 p_4})$. Suppose that there exist $i \neq j$ such that $\left(\frac{p_i^*}{p_j}\right) = -1$. Then $G_K^{ur}(2)$ has non-trivial uniform quotient.*

Remark 3.3. — Here, one may replace p_1 by 2. But we are not guaranteed in all cases of the infiniteness of $G_K^{ur}(2)$, as we are outside the conditions of the result of Hajir [13]. We will see later the reason.

Proof. — Let us start with a non-trivial uniform quotient G of $G_K^{ur}(2)$. As by class field theory, the pro-2 group G should be FAb, it is of dimension 3, i.e. $H^1(G) \simeq H^1(G_K^{ur}(2))$. By Proposition 1.3, there exist $x, y \in H^1(G)$ such that $x \cup y \neq 0 \in H_{et}^3(X_K)$, and the "strategy" applies. \square

Now, we would like to extend this last construction.

3.2. Bilinear forms over \mathbb{F}_2 and conjecture A. —

3.2.1. Totally isotropic subspaces. — Let \mathcal{B} be a bilinear form over an \mathbb{F}_2 -vector space V of finite dimension. Denote by n the dimension of V and by $\text{rk}(\mathcal{B})$ the rank of \mathcal{B} .

Definition 3.4. — Given a bilinear form \mathcal{B} , one define the index $\nu(\mathcal{B})$ of \mathcal{B} by

$$\nu(\mathcal{B}) := \max\{\dim W, \mathcal{B}(W, W) = 0\}.$$

The index $\nu(K)$ is then an upper bound of the dimension of totally isotropic subspaces W of V . As we will see, the index $\nu(\mathcal{B})$ is well-known when \mathcal{B} is symmetric. For the general case, one has:

Proposition 3.5. — *The index $\nu(\mathcal{B})$ of a bilinear form \mathcal{B} is at most than $n - \frac{1}{2}\text{rk}(\mathcal{B})$.*

Proof. — Let W be a totally isotropic subspace of V of dimension i . Let us complete a basis of W to a basis B of V . It is then easy to see that the Gram matrix of \mathcal{B} in B is of rank at most $2n - 2i$. \square

This bound is in a certain sense optimal as we can achieve it in the symmetric case.

Definition 3.6. — (i) Given $a \in \mathbb{F}_2$. The bilinear form $b(a)$ with matrix $\begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}$ is called a metabolic plan. A metabolic form is an orthogonal sum of metabolic plans (up to isometry).

(ii) A symmetric bilinear form (V, \mathcal{B}) is called alternating if $\mathcal{B}(x, x) = 0$ for all $x \in V$. Otherwise \mathcal{B} is called nonalternating.

Recall now a well-known result on symmetric bilinear forms over \mathbb{F}_2 .

Proposition 3.7. — *Let (V, \mathcal{B}) be a symmetric bilinear form of dimension n over \mathbb{F}_2 . Denote by r the rank of \mathcal{B} . Write $r = 2r_0 + \delta$, with $\delta = 0$ or 1 , and $r_0 \in \mathbb{N}$.*

(i) *If \mathcal{B} is nonalternating, then (V, \mathcal{B}) is isometric to*

$$\overbrace{b(1) \perp \cdots \perp b(1)}^{r_0} \perp \overbrace{\langle 1 \rangle}^{\delta} \perp \overbrace{\langle 0 \rangle \perp \cdots \perp \langle 0 \rangle}^{n-r} \simeq_{iso} \overbrace{\langle 1 \rangle \perp \cdots \perp \langle 1 \rangle}^r \perp \overbrace{\langle 0 \rangle \perp \cdots \perp \langle 0 \rangle}^{n-r};$$

(ii) *If \mathcal{B} is alternating, then \mathcal{B} is isometric to*

$$\overbrace{b(0) \perp \cdots \perp b(0)}^{r_0} \perp \overbrace{\langle 0 \rangle \perp \cdots \perp \langle 0 \rangle}^{n-r}.$$

Moreover, $\nu(\mathcal{B}) = n - r + r_0 = n - r_0 - \delta$.

When (V, \mathcal{B}) is not necessary symmetric, let us introduce the symmetrization \mathcal{B}^{sym} of \mathcal{B} by

$$\mathcal{B}^{sym}(x, y) = \mathcal{B}(x, y) + \mathcal{B}(y, x), \quad \forall x, y \in V.$$

One has:

Proposition 3.8. — Let (V, \mathcal{B}) be a bilinear form of dimension n over \mathbb{F}_2 . Then

$$\nu(\mathcal{B}) \geq n - \lfloor \frac{1}{2} \text{rk}(\mathcal{B}^{\text{sym}}) \rfloor - \lfloor \frac{1}{2} \text{rk}(\mathcal{B}) \rfloor.$$

In particular, $\nu(\mathcal{B}) \geq n - \frac{3}{2} \text{rk}(\mathcal{B})$.

Proof. — It is easy. Let us start with a maximal totally isotropic subspace W of $(V, \mathcal{B}^{\text{sym}})$. Then $\mathcal{B}|_W$ is symmetric: indeed, for any two $x, y \in W$, we get $0 = \mathcal{B}^{\text{sym}}(x, y) = \mathcal{B}(x, y) + \mathcal{B}(y, x)$, and then $\mathcal{B}(x, y) = \mathcal{B}(y, x)$ (recall that V is defined over \mathbb{F}_2). Hence by Proposition 3.7, $\mathcal{B}|_W$ has a totally isotropic subspace of dimension $\nu(\mathcal{B}|_W) = \dim W - \lfloor \frac{1}{2} \text{rk}(\mathcal{B}|_W) \rfloor$. As $\dim W = n - \lfloor \frac{1}{2} \text{rk}(\mathcal{B}^{\text{sym}}) \rfloor$ (by Proposition 3.7), one obtains the first assertion. For the second one, it is enough to note that $\text{rk}(\mathcal{B}^{\text{sym}}) \leq 2 \text{rk}(\mathcal{B})$. \square

3.2.2. Bilinear form over the Kummer radical of the 2-elementary abelian maximal unramified extension. — Let us start with a totally imaginary number field K . Denote by n the 2-rank of $G_K^{\text{ur}}(2)$, in other words, $n = d_2 \text{Cl}_K$.

Let $V = \langle a_1, \dots, a_n \rangle (\mathbb{K}^\times)^2 \in \mathbb{K}^\times / (\mathbb{K}^\times)^2$ be the Kummer radical of the 2-elementary abelian maximal unramified extension $\mathbb{K}^{\text{ur},2}/K$. Then V is an \mathbb{F}_2 -vector space of dimension n .

As we have seen in section 1.3, for every prime ideal \mathfrak{p} of \mathcal{O}_K , the \mathfrak{p} -valuation of a_i is even, and then $\sqrt{(a_i)}$ as ideal of \mathcal{O}_K has a sense.

For $x \in V$, denote $K_x := K(\sqrt{x})$, and $\mathfrak{a}(x) := \sqrt{(x)} \in \mathcal{O}_K$. We can now introduce the bilinear form \mathcal{B}_K that plays a central role in our work.

Definition 3.9. — For $a, b \in V$, put:

$$\mathcal{B}_K(a, b) = \left(\frac{K_a/K}{\mathfrak{a}(b)} \right) \cdot \sqrt{a} / \sqrt{a} \in \mathbb{F}_2,$$

where here we use the additive notation.

Remark 3.10. — The Hilbert symbol between a and b is trivial due to the parity of $v_{\mathfrak{q}}(a)$.

Of course, we have:

Lemma 3.11. — The application $\mathcal{B}_K : V \times V \rightarrow \mathbb{F}_2$ is a bilinear form on V .

Proof. — The linearity on the right comes from the linearity of the Artin symbol and the linearity on the left is an easy observation. \square

Remark 3.12. — If we denote by χ_i a generator of $H^1(\text{Gal}(K(\sqrt{a_i})/K))$, then the Gram matrix of the bilinear form \mathcal{B}_K in the basis $\{a_1(\mathbb{K}^\times)^2, \dots, a_n(\mathbb{K}^\times)^2\}$ is exactly the matrix $(\chi_i \cup \chi_j)_{i,j}$ of the cup-products in $H_{\text{et}}^3(\text{Spec } \mathcal{O}_K)$. See Proposition 1.1 and Remark 1.2. Hence the bilinear form \mathcal{B}_K coincides with the bilinear form $\mathcal{B}_K^{\text{et}}$ on $H_{\text{et}}^1(\text{Spec } \mathcal{O}_K)$ defined by $\mathcal{B}_K^{\text{et}}(x, y) = x \cup y \in H_{\text{et}}^3(\text{Spec } \mathcal{O}_K)$.

The bilinear form \mathcal{B}_K is not necessarily symmetric, but we will give later some situations where \mathcal{B}_K is symmetric. Let us give now two types of totally isotropic subspaces W that may appear.

Definition 3.13. — The right-radical $\mathcal{R}ad_r$ of a bilinear form \mathcal{B} on V is the subspace of V defined by: $\mathcal{R}ad_r := \{x \in V, \mathcal{B}(V, x) = 0\}$.

Of course one always has $\dim \mathcal{B} = \text{rk} \mathcal{B} + \dim \mathcal{R}ad_r$. And, remark moreover that the restriction of \mathcal{B} at $\mathcal{R}ad_r$ produces a totally isotropic subspace of V .

Let us come back to the bilinear form \mathcal{B}_K on the Kummer radical of $K^{ur,2}/K$.

Proposition 3.14. — Let $W := \langle \varepsilon_1, \dots, \varepsilon_r \rangle (K^\times)^2 \subset V$ be an \mathbb{F}_2 -subspace of dimension r , generated by some units $\varepsilon_i \in \mathcal{O}_K^\times$. Then $W \subset \mathcal{R}ad_r$, and thus (V, \mathcal{B}_K) contains W as a totally isotropic subspace of dimension r .

Proof. — Indeed, here $\mathfrak{a}(\varepsilon_i) = \mathcal{O}_K$ for $i = 1, \dots, r$. □

Proposition 3.15. — Let $K = k(\sqrt{b})$ be a quadratic extension. Suppose that there exist $a_1, \dots, a_r \in k$ such that the extensions $k(\sqrt{a_i})/k$ are independent and unramified everywhere. Suppose moreover that $b \notin \langle a_1, \dots, a_r \rangle (k^\times)^2$. Then $W := \langle a_1, \dots, a_r \rangle (K^\times)^2$ is a totally isotropic subspace of dimension r .

Proof. — Let $\mathfrak{p} \subset \mathcal{O}_k$ be a prime ideal of \mathcal{O}_k . It is sufficient to prove that $\left(\frac{K_{a_i}/K}{\mathfrak{p}}\right)$ is trivial. Let us study all the possibilities.

- If \mathfrak{p} is inert in K/k , then as $K(\sqrt{a_i})/K$ is unramified at \mathfrak{p} , necessary \mathfrak{p} splits in $K(\sqrt{a_i})/K$ and then $\left(\frac{K_{a_i}/K}{\mathfrak{p}}\right)$ is trivial.

- If $\mathfrak{p} = \mathfrak{P}^2$ is ramified in K/k , then $\left(\frac{K_{a_i}/K}{\mathfrak{p}}\right) = \left(\frac{K_{a_i}/K}{\mathfrak{P}}\right)^2$ is trivial.

- If $\mathfrak{p} = \mathfrak{P}_1 \mathfrak{P}_2$ splits, then obviously $\left(\frac{K_{a_i}/K}{\mathfrak{P}_1}\right) = \left(\frac{K_{a_i}/K}{\mathfrak{P}_2}\right)$, and then $\left(\frac{K_{a_i}/K}{\mathfrak{p}}\right)$ is trivial. □

It is then natural to define the index of K as follows:

Definition 3.16. — The index $\nu(K)$ of K is the index of the bilinear form \mathcal{B}_K .

Of course, if the form \mathcal{B}_K is non-degenerate, one has: $\nu(K) \leq \frac{1}{2}d_2 \text{Cl}_K$. Thus one says that Cl_K is non-degenerate if the form \mathcal{B}_K is non-degenerate.

One can now present the main result of our work:

Theorem 3.17. — Let K/\mathbb{Q} be a totally imaginary number field. Then $G_K^{ur}(2)$ has no uniform quotient of dimension $d > \nu(K)$. In particular:

- (i) if $\nu(K) < 3$, then the Conjecture A holds for K (and $p = 2$);
- (ii) if Cl_K is non-degenerate, then $G_K^{ur}(2)$ has no uniform quotient of dimension $d > \frac{1}{2}d_2 \text{Cl}_K$.

Proof. — Let G be a non-trivial uniform quotient of $G_K^{ur}(p)$ of dimension $d > 0$. Let W be the Kummer radical of $H^1(G)^\vee$; here W is a subspace of the Kummer radical V of $K^{ur,2}/K$. As $d > \nu(K)$, the space W is not totally isotropic. Then, one can find $x, y \in H_{et}^1(G) \subset H^1(X_K)$ such that $x \cup x \cup y \in H_{et}^3(X_K)$ is not zero (by Proposition 1.1). See also Remark 3.12. And thanks to the strategy developed in Section 3.1, we are done for the first part of the theorem.

(i): as $G^{ur}(2)$ is FAb, every non-trivial uniform quotient G of $G^{ur}(2)$ should be of dimension at least $d \geq 3$.

(ii): in this case $\nu(K) \leq \frac{1}{2}\text{rk}(\mathcal{B}_K)$. □

We finish this section with the proof of the theorem presented in the introduction.

• As $\nu(K) \leq n - \frac{1}{2}\text{rk}(\mathcal{B}_K)$, see Proposition 3.5 and Remark 3.12, the assertion (i) can be deduced by Theorem 3.17.

• This is an obvious observation for the small dimensions. In the three cases, $\nu(K) \leq n - \frac{1}{2}\text{rk}(\mathcal{B}_K) < 3$.

3.3. The imaginary quadratic case. —

3.3.1. The context. — Let us consider an imaginary quadratic field $K = \mathbb{Q}(\sqrt{D})$, $D \in \mathbb{Z}_{<0}$ square-free. Let p_1, \dots, p_{k+1} be the *odd* prime numbers dividing D . Let us write the discriminant disc_K of K by: $\text{disc}_K = p_0^* \cdot p_1^* \cdots p_{k+1}^*$, where $p_0^* \in \{1, -4, \pm 8\}$.

The 2-rank n of Cl_K depends on the ramification of 2 in K/\mathbb{Q} . Put $K^{ur,2}$ the 2-elementary abelian maximal unramified extension of K :

- if 2 is unramified in K/\mathbb{Q} , i.e. $p_0^* = 1$, then $n = k$ and $V = \langle p_1^*, \dots, p_k^* \rangle (\mathbb{K}^\times)^2 \subset \mathbb{K}^\times$ is the Kummer radical of $K^{ur,2}/K$;
- if 2 is ramified in K/\mathbb{Q} , i.e. $p_0^* = -4$ or ± 8 , then $n = k+1$ and $V = \langle p_1^*, \dots, p_{k+1}^* \rangle (\mathbb{K}^\times)^2 \subset \mathbb{K}^\times$ is the Kummer radical of $K^{ur,2}/K$.

We denote by $\mathcal{S} = \{p_1^*, \dots, p_n^*\}$ the \mathbb{F}_2 -basis of V , where here $n = d_2\text{Cl}_K (= k \text{ or } k+1)$.

Lemma 3.18. — (i) For $p^* \neq q^* \in \mathcal{S}$, one has: $\mathcal{B}_K(p^*, q^*) = 0$ if and only if, $\left(\frac{p^*}{q}\right) = 1$.

(ii) For $p|D$, put $D_p := D/p^*$. Then for $p^* \in \mathcal{S}$, one has: $\mathcal{B}_K(p^*, p^*) := \left(\frac{D_p}{p}\right)$.

Proof. — Obvious. □

Hence the matrix of the bilinear form \mathcal{B}_K in the basis \mathcal{S} is a square $n \times n$ Rédei-matrix type $M_K = (m_{i,j})_{i,j}$, where

$$m_{i,j} = \begin{cases} \left(\frac{p_i^*}{p_j}\right) & \text{if } i \neq j, \\ \left(\frac{D_{p_i}}{p_i}\right) & \text{if } i = j. \end{cases}$$

Here as usual, one uses the additive notation (the 1's are replaced by 0's and the -1 's by 1).

Example 3.19. — Take $K = \mathbb{Q}(\sqrt{-4 \cdot 3 \cdot 5 \cdot 7 \cdot 13})$. This quadratic field has a root discriminant $|\text{disc}_K|^{1/2} = 73.89 \dots$, but we don't know actually if $G_K^{ur}(2)$ is finite or not; see the recent works of Boston and Wang [3]. Take $\mathcal{S} = \{-3, -5, -7, -13\}$. Then the Gram matrix of \mathcal{B}_K in \mathcal{S} is:

$$M_K = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

Hence $\text{rk}(\mathcal{B}_K) = 3$ and $\nu(K) \leq 4 - \frac{3}{2} = 2.5$. By Theorem 3.17, one concludes that $G_K^{ur}(2)$ has no non-trivial uniform quotient. By Proposition 3.8, remark that here one has: $\nu(K) = 2$.

Let us recall at this level a part of the theorem of the introduction:

Corollary 3.20. — *The Conjecture A holds when:*

- (i) $d_2\text{Cl}_K = 5$ and \mathcal{B}_K is non-degenerate;
- (ii) $d_2\text{Cl}_K = 4$ and $\text{rk}(\mathcal{B}_K) \geq 3$;
- (iii) $d_2\text{Cl}_K = 3$ and $\text{rk}(\mathcal{B}_K) > 0$.

Remark that (iii) is an extension of corollary 3.2.

3.3.2. Symmetric bilinear forms. Examples. — Let us conserve the context of the previous section 3.3.1. Then, thanks to the quadratic reciprocity law, one gets:

Proposition 3.21. — *The bilinear form $\mathcal{B}_K : V \times V \rightarrow \mathbb{F}_2$ is symmetric, if and only if, there is at most one prime $p \equiv 3 \pmod{4}$ dividing D .*

Proof. — Obvious. □

Let us give some examples (the computations have been done by using PARI/GP [26]).

Example 3.22. — Take $k + 1$ prime numbers p_1, \dots, p_{k+1} , such that

- $p_1 \equiv \dots \equiv p_k \equiv 1 \pmod{4}$ and $p_{k+1} \equiv 3 \pmod{4}$;
- for $1 \leq i < j \leq k$, $\left(\frac{p_i}{p_j}\right) = 1$;
- for $i = 1, \dots, k$, $\left(\frac{p_i}{p_{k+1}}\right) = -1$

Put $K = \mathbb{Q}(\sqrt{-p_1 \dots p_{k+1}})$. In this case the matrix of the bilinear form \mathcal{B}_K in the basis $(p_i)_{1 \leq i \leq k}$ is the identity matrix of dimension $k \times k$ and, $\nu(K) = \lfloor \frac{k}{2} \rfloor$. Hence, $G_K^{ur}(p)$ has no uniform quotient of dimension at least $\lfloor \frac{k}{2} \rfloor + 1$.

In particular, if we choose an integer $t > 0$ such that $\sqrt{k+1} \geq t \geq \sqrt{\lfloor \frac{k}{2} \rfloor + 2}$, then there is no quotient of $G_K^{ur}(2)$ onto $\text{Sl}_t^1(\mathbb{Z}_2)$. (If $t > \sqrt{k+1}$, it is obvious.)

Here are some more examples. For $K_1 = \mathbb{Q}(\sqrt{-5 \cdot 29 \cdot 109 \cdot 281 \cdot 349 \cdot 47})$, $G_{K_1}^{ur}(2)$ has no uniform quotient; here $\text{Cl}(K_1) \simeq (\mathbb{Z}/2\mathbb{Z})^5$.

Take $K_2 = \mathbb{Q}(\sqrt{-5 \cdot 29 \cdot 109 \cdot 281 \cdot 349 \cdot 1601 \cdot 1889 \cdot 5581 \cdot 3847})$; here $\text{Cl}(K_2) \simeq (\mathbb{Z}/2\mathbb{Z})^8$. Then $G_{K_2}^{ur}(2)$ has no uniform quotient of dimension at least 5. In particular, there is no unramified extension of K_2 with Galois group isomorphic to $\text{Sl}_3^1(\mathbb{Z}_2)$.

Example 3.23. — Take $2m + 1$ prime numbers p_1, \dots, p_{2m+1} , such that

- $p_1 \equiv \dots \equiv p_{2m} \equiv 1 \pmod{4}$ and $p_{2m+1} \equiv 3 \pmod{4}$;
- $\left(\frac{p_1}{p_2}\right) = \left(\frac{p_3}{p_4}\right) = \dots = \left(\frac{p_{2m-1}}{p_{2m}}\right) = -1$;
- for the other indices $1 \leq i < j \leq 2m$, $\left(\frac{p_i}{p_j}\right) = 1$;
- for $i = 1, \dots, 2m$, $\left(\frac{p_i}{p_{2m+1}}\right) = -1$

Put $K = \mathbb{Q}(\sqrt{-p_1 \cdots p_{2m+1}})$. In this case the bilinear form \mathcal{B}_K is nondegenerate and alternating, then isometric to $\overbrace{b(0) \perp \cdots \perp b(0)}^m$. Hence, $\nu(K) = m$, and $G_K^{ur}(p)$ has no uniform quotient of dimension at least $m + 1$.

For example, for $K = \mathbb{Q}(\sqrt{-5 \cdot 13 \cdot 29 \cdot 61 \cdot 1049 \cdot 1301 \cdot 743})$, $G_K^{ur}(2)$ has no uniform quotient of dimension at least 4.

3.3.3. Relation with the 4-rank of the Class group - Density estimations. — The study of the 4-rank of the class group of quadratic number fields started with the work of Rédei [29] (see also [30]). Since, many authors have contributed to its extensions, generalizations and applications. Let us cite an article of Lemmermeyer [19] where one can find a large litterature about the question. See also a nice paper of Stevenhagen [33], and the work of Gerth [11] and Fouvry-Klüners [9] concerning the density question.

Definition 3.24. — Let K be a number field, define by $R_{K,4}$ the 4-rank of K as follows:

$$R_{K,4} := \dim_{\mathbb{F}_2} \text{Cl}_K[4] / \text{Cl}_K[2],$$

where $\text{Cl}_K[m] = \{c \in \text{Cl}_K, c^m = 1\}$.

Let us conserve the context and the notations of the section 3.3.1: here $K = \mathbb{Q}(\sqrt{D})$ is an imaginary quadratic field of discriminant disc_K , $D \in \mathbb{Z}_{<0}$ square-free. Denote by $\{q_1, \cdots, q_{n+1}\}$ the set of prime numbers that ramify in K/\mathbb{Q} ; $d_2 \text{Cl}_K = n$. Here we can take $q_i = p_i$ for $1 \leq i \leq n$, and $q_n = p_{k+1}$ or $q_n = 2$ following the ramification at 2. Then, consider the Rédei matrix $M'_K = (m_{i,j})_{i,j}$ of size $(n+1) \times (n+1)$ with coefficients in \mathbb{F}_2 , where

$$m_{i,j} = \begin{cases} \begin{pmatrix} q_i^* \\ q_j \end{pmatrix} & \text{if } i \neq j, \\ \begin{pmatrix} D_{q_i} \\ q_i \end{pmatrix} & \text{if } i = j. \end{cases}$$

It is not difficult to see that the sum of the rows is zero, hence the rank of M'_K is smaller than n .

Theorem 3.25 (Rédei). — *Let K be an imaginary quadratic number field. Then $R_{K,4} = d_2 \text{Cl}_K - \text{rk}(M'_K)$.*

Remark 3.26. — The strategy of Rédei is to construct for every couple (D_1, D_2) "of second kind", a degree 4 cyclic unramified extension of K . Here to be of second kind means that $\text{disc}_K = D_1 D_2$, where D_i are fundamental discriminants such that $\left(\frac{D_1}{p_2}\right) = \left(\frac{D_2}{p_1}\right) = 1$, for every prime $p_i | D_i$, $i = 1, 2$. And clearly, this condition corresponds exactly to the existence of orthogonal subspaces W_i of the Kummer radical V , $i = 1, 2$, generated by the p_i^* , for all $p_i | D_i$: $\mathcal{B}_K(W_1, W_2) = \mathcal{B}_K(W_2, W_1) = \{0\}$. Such orthogonal subspaces allow us to construct totally isotropic subspaces. And then, the larger the 4-rank of Cl_K , the larger $\nu(K)$ must be.

Consider now the matrix M''_K obtained from M'_K after missing the last row. Remark here that the matrix M_K is a submatrix of the Rédei matrix M''_K :

$$M''_K = \begin{pmatrix} \boxed{M_K} & * \\ & \vdots \\ & * \end{pmatrix}$$

Hence, $\text{rk}(M_K) + 1 \geq \text{rk}(M'_K) \geq \text{rk}(M_K)$. Remark that in example 3.19, $\text{rk}(M_K) = 3$ and $\text{rk}(M'_K) = 4$. But sometimes one has $\text{rk}(M'_K) = \text{rk}(M_K)$, as for example:

(A): when: $p_0 = 1$ (the set of primes $p_i \equiv 3 \pmod{4}$ is odd);

(B): or, when \mathcal{B}_K is non-degenerate.

For situation (A), it suffices to note that the sum of the columns is zero (thanks to the properties of the Legendre symbol).

From now on we follow the work of Gerth [11]. Denote by \mathcal{F} the set of imaginary quadratic number fields. For $0 \leq r \leq n$ and $X \geq 0$, put

$$S_X = \{K \in \mathcal{F}, |\text{disc}_K| \leq X\},$$

$$S_{n,X} = \{K \in S_X, d_2\text{Cl}_K = n\}, \quad S_{n,r,X} = \{K \in S_{n,X}, R_{K,4} = r\}.$$

Denote also

$$A_X = \{K \in S_X, \text{satisfying (A)}\}$$

$$A_{n,X} = \{K \in A_X, d_2\text{Cl}_K = n\}, \quad A_{n,r,X} = \{K \in A_{n,X}, R_{K,4} = r\}.$$

One has the following density theorem due to Gerth:

Theorem 3.27 (Gerth [11]). — *The limits $\lim_{X \rightarrow \infty} \frac{|A_{n,r,X}|}{|A_{n,X}|}$ and $\lim_{X \rightarrow \infty} \frac{|S_{n,r,X}|}{|S_{n,X}|}$ exist and are equal. Denote by $d_{n,r}$ this quantity. Then $d_{n,r}$ is explicit and,*

$$d_{\infty,r} := \lim_{n \rightarrow \infty} d_{n,r} = \frac{2^{-r^2} \prod_{k=1}^{\infty} (1 - 2^{-k})}{\prod_{k=1}^r (1 - 2^{-k})}.$$

Recall also the following quantities introduced at the beginning of our work:

$$\text{FM}_{n,X}^{(d)} := \{K \in S_{n,X}, G_K^{ur}(2) \text{ has no uniform quotient of dimension } > d\},$$

$$\text{FM}_{n,X} := \{K \in S_{n,X}, \text{Conjecture A holds for } K\},$$

and the limits:

$$\text{FM}_n := \liminf_{X \rightarrow +\infty} \frac{\#\text{FM}_{n,X}}{\#S_{n,X}}, \quad \text{FM}_n^{(d)} := \liminf_{X \rightarrow +\infty} \frac{\#\text{FM}_{n,X}^{(d)}}{\#S_{n,X}}.$$

After combining all our observations, we obtain (see also Corollary 1):

Corollary 3.28. — *For $d \leq n$, one has*

$$\text{FM}_n^{(d)} \geq d_{n,0} + d_{n,1} + \cdots + d_{n,2d-n-1}.$$

In particular:

- (i) $\text{FM}_3 \geq .992187$;
- (ii) $\text{FM}_4 \geq .874268$, $\text{FM}_4^{(4)} \geq .999695$;
- (iii) $\text{FM}_5 \geq .331299$, $\text{FM}_5^{(4)} \geq .990624$, $\text{FM}_5^{(5)} \geq .9999943$;
- (iv) $\text{FM}_6^{(4)} \geq .867183$, $\text{FM}_6^{(5)} \geq .999255$, $\text{FM}_6^{(6)} \geq 1 - 5.2 \cdot 10^{-8}$;
- (v) for all $d \geq 3$, $\text{FM}_d^{(1+d/2)} \geq .866364$, $\text{FM}_d^{(2+d/2)} \geq .999953$.

Proof. — As noted by Gerth in [11], the dominating set in the density computation is the set $A_{n,X}$ of imaginary quadratic number fields $K = \mathbb{Q}(\sqrt{D})$ satisfying (A). But for K in $A_{n,X}$, one has $\text{rk}(\mathcal{B}_K) = \text{rk}(M_K) = n - R_{K,4}$. Hence for $K \in A_{n,X,r}$, by Proposition 3.5

$$\nu(K) \leq n - \frac{1}{2}(n - R_{K,4}) = \frac{1}{2}(n + R_{K,4}).$$

Now one uses Corollary 3.20. Or equivalently, one sees that Conjecture A holds when $3 > \frac{1}{2}(n + R_{K,4})$, i.e., when $R_{K,4} < 6 - n$. More generally, $G_K^{ur}(2)$ has no uniform quotient of dimension d when $R_{K,4} < 2d - n$. In particular,

$$\text{FM}_n^{(d)} \geq d_{n,0} + d_{n,1} + \cdots + d_{n,2d-n-1}.$$

Now one uses the estimates of Gerth in [11], to obtain:

- (i) $\text{FM}_3 \geq d_{3,0} + d_{3,1} + d_{3,2} \simeq 0.992187 \cdots$
- (ii) $\text{FM}_4 \geq d_{4,0} + d_{4,1} \simeq 0.874268 \cdots$, $\text{FM}_4^{(4)} \geq d_{4,0} + d_{4,1} + d_{4,2} + d_{4,3} \simeq .999695 \cdots$,
- (iii) $\text{FM}_5 \geq d_{5,0} \simeq 0.331299 \cdots$, $\text{FM}_5^{(4)} \geq d_{5,0} + d_{5,1} + d_{5,2} \simeq .990624 \cdots$, $\text{FM}_5^{(5)} \geq d_{5,0} + d_{5,1} + d_{5,2} + d_{5,3} + d_{5,4} \simeq .9999943 \cdots$,
- (iv) $\text{FM}_6^{(4)} \geq d_{6,0} + d_{6,1} \simeq 0.867183 \cdots$, $\text{FM}_6^{(5)} \geq d_{6,0} + d_{6,1} + d_{6,2} + d_{6,3} \simeq .999255 \cdots$,
 $\text{FM}_6^{(6)} \geq 1 - d_{6,6} \simeq 1 - 5.2 \cdot 10^{-8}$,
- (v) $\text{FM}_d^{(1+d/2)} \geq d_{\infty,0} + d_{\infty,1} \simeq .866364 \cdots$, $\text{FM}_d^{(2+d/2)} \geq d_{\infty,0} + d_{\infty,1} + d_{\infty,2} + d_{\infty,3} \simeq .999953 \cdots$.

□

In the spirit of the Cohen-Lenstra heuristics, the work of Gerth has been improved by Fouvry-Klüners [9]. This work allows us to give a more general density estimation as announced in Introduction. Recall

$$\text{FM}_X^{[i]} := \{K \in S_X, G_K^{ur}(2) \text{ has no uniform quotient of dimension } > i + \frac{1}{2}d_2\text{Cl}_K\}$$

and

$$\text{FM}^{[i]} := \liminf_{X \rightarrow +\infty} \frac{\#\text{FM}_X^{[i]}}{\#S_X}.$$

Our work allows us to obtain (see Corollary 2):

Corollary 3.29. — *For $i \geq 1$, one has:*

$$\text{FM}^{[i]} \geq d_{\infty,0} + d_{\infty,1} + \cdots + d_{\infty,2i-2}.$$

In particular,

$$\text{FM}^{[1]} \geq .288788, \quad \text{FM}^{[2]} \geq .994714, \quad \text{and} \quad \text{FM}^{[3]} \geq 1 - 9.7 \cdot 10^{-8}.$$

Proof. — By Fouvry-Klüners [9], the density of imaginary quadratic fields for which $R_{K,4} = r$, is equal to $d_{\infty,r}$. Remind of that for $K \in \mathcal{F}$: $\text{rg}(M_K) \geq \text{rg}(M'_K) - 1$. Then thanks to Proposition 3.5 and Theorem 3.25, we get

$$\nu(K) \leq \frac{1}{2}d_2\text{Cl}_K + \frac{1}{2} + \frac{1}{2}R_{K,4}.$$

Putting this fact together with Theorem 3.17, we obtain that $G_K^{ur}(2)$ has no uniform quotient of dimension $d > \frac{1}{2}d_2\text{Cl}_K + \frac{1}{2} + \frac{1}{2}R_{K,4}$. Then for $i \geq 1$, the proportion of the

fields K in $\text{FM}^{[i]}$ is at least the proportion of $K \in \mathcal{F}$ for which $R_{K,4} < 2i - 1$, hence at least $d_{\infty,0} + d_{\infty,1} + \cdots + d_{\infty,2i-2}$ by [9]. To conclude:

$$\begin{aligned} \text{FM}^{[1]} &\geq d_{\infty,0} \simeq .288788 \cdots \\ \text{FM}^{[2]} &\geq d_{\infty,0} + d_{\infty,1} + d_{\infty,2} \simeq .994714 \cdots \\ \text{FM}^{[3]} &\geq d_{\infty,0} + d_{\infty,1} + d_{\infty,2} + d_{\infty,3} + d_{\infty,4} \simeq 1 - 9.7 \cdot 10^{-8}. \end{aligned}$$

□

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