Stable and unstable time quasi periodic solutions for a system of coupled NLS equations

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STABLE AND UNSTABLE TIME QUASI PERIODIC SOLUTIONS FOR A SYSTEM OF COUPLED NLS EQUATIONS

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Abstract. We prove that a system of coupled nonlinear Schrödinger equations on the torus exhibits both stable and unstable small KAM tori. In particular the unstable tori are related to a beating phenomena which has been proved recently in [6]. This is the first example of unstable tori for a 1d PDE.

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1. Introduction

We consider the system of coupled nonlinear Schrödinger equations on the torus

NLSC

$$\begin{cases} i\partial_t u + \partial_{xx} u = |v|^2 u, & (t,x) \in \mathbb{R} \times \mathbb{T}, \\ i\partial_t v + \partial_{xx} v = |u|^2 v. \end{cases} \tag{1.1}$$

Key words and phrases. Small amplitude solutions, modulational instabilities, Hamiltonian systems, nonlinear PDE, KAM theory.

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This system is Hamiltonian when considered on the phase space $(u, \bar{u}, v, \bar{v}) \in (L^2(\mathbb{T}))^4$ endowed with the symplectic form $-idu \wedge d\bar{u} - idv \wedge d\bar{v}$. The Hamiltonian of the system is given by

$$H := \int_{\mathbb{T}} \left(|u_x|^2 + |v_x|^2 \right) dx + \int_{\mathbb{T}} |u|^2 |v|^2 dx.$$

In section 4 we will consider a slightly more general case where we add a higher order perturbation R_5 (see (4.1)). In the introduction we prefer fo focus on the simplest case. We also remark that all our results concern small amplitude solutions and thus the sign in front of the linearity doesn't affect our results (but we need the same sign in both line of (1.1) to conserve the Hamiltonian structure).

In order to take profit of the geometry of the torus, we write the Fourier series expansion of u, $\overline{u}$, v and $\overline{v}$:

$$u(t,x) = \sum_{j\in\mathbb{Z}} a_j(t) e^{ijx}, \qquad \overline{u}(t,x) = \sum_{j\in\mathbb{Z}} \bar{a}_j(t) e^{-ijx},$$
$$v(t,x) = \sum_{j\in\mathbb{Z}} b_j(t) e^{ijx}, \qquad \overline{v}(t,x) = \sum_{j\in\mathbb{Z}} \bar{b}_j(t) e^{-ijx}.$$

In this variables, the symplectic structure becomes

$$-i \sum_j da_j \wedge d\bar{a}_j - i \sum_j db_j \wedge d\bar{b}_j,$$

and the Hamiltonian H of the system reads

defNP
$$H(a, \bar{a}, b, \bar{b}) = \sum_{j\in\mathbb{Z}} j^2 (a_j \bar{a}_j + b_j \bar{b}_j) + \sum_{\substack{i,j,k,l\in\mathbb{Z} \\ i+j=k+l}} a_k b_l \bar{a}_i \bar{b}_j = P_2 + P_4. \tag{1.2}$$

In this article we are interested in the persistence of two dimensional linear invariant tori: given $p, q \in \mathbb{Z}$ and $a_p, b_q \in \mathbb{C}$,

sol
$$\begin{cases} u(x,t) &= a_p e^{ipx} e^{-ip^2 t}, \\ v(x,t) &= b_q e^{iqx} e^{-iq^2 t}, \end{cases} \tag{1.3}$$

is a solution to the linear system associated to the quadratic Hamiltonian P_2. Equivalently we can say that for any $c \in \mathbb{R}^2$ and any $p, q \in \mathbb{Z}$, the torus $\mathcal{T}_c(p,q) = \{|a_p|^2 = c_1,\ |b_q|^2 = c_2\}$ is invariant under the flow of P_2. We prove (see Theorem 4.3) that for $p \neq q$, for ρ in a Cantor set of full measure in $[1,2]$ and for ν small enough, the non linear Hamiltonian $P_2 + P_4$ admits an invariant torus close to $\mathcal{T}_{\nu\rho}(p,q)$. Furthermore we prove that this tori are *linearly unstable*: the system (1.1) linearized around $\mathcal{T}_{\nu\rho}(p,q)$ admits one hyperbolic direction. In other words, for $|a_p|^2 = \nu\rho_1$, $|b_q|^2 = \nu\rho_2$ with ρ in a Cantor set and ν small enough, (1.1) admits an unstable small amplitude quasi periodic solution close to (1.3). Precisely we prove in section 4.1:

coro1
Theorem 1.1. *Fix $p \neq q$ and $s > 1/2$. There exists $\nu_0 > 0$ and for $0 < \nu < \nu_0$ there exists $\mathcal{C}_\nu \subset [1,2]^2$ asymptotically of full measure (i.e.*

$\lim_{\nu\to 0}\operatorname{meas}([1,2]^2\setminus\mathcal{C}_\nu)=0$*) such that for* $\rho\in\mathcal{C}_\nu$ *there exists a quasi periodic solution* (u,v) *of* (4.1) *of the form*

$$\begin{cases} u(x,t) &= \sum_{j\in\mathbb{Z}} u_j(t\omega)e^{ijx},\\ v(x,t) &= \sum_{j\in\mathbb{Z}} v_j(t\omega)e^{ijx}, \end{cases}$$

where $U(\cdot)=(u_j(\cdot))_{j\in\mathbb{Z}}$ *and* $V(\cdot)=(v_j(\cdot))_{j\in\mathbb{Z}}$ *are analytic functions from* $\mathbb{T}^2$ *into* ℓ^2_s *satisfying uniformly in* $\theta\in\mathbb{T}^2$

$$\begin{cases} \big||u_p(\theta)|-\sqrt{\nu\rho_1}\big|^2+\sum_{j\neq p}(1+j^2)^s|u_j(\theta)|^2=\mathcal{O}(\nu^3),\\ \big||v_q(\theta)|-\sqrt{\nu\rho_2}\big|^2+\sum_{j\neq q}(1+j^2)^s|v_j(\theta)|^2=\mathcal{O}(\nu^3) \end{cases}$$

and where $\omega\equiv\omega(\rho)\in\mathbb{R}^2$ *is a nonresonant frequency vector that satisfies*

$$\omega=(p^2,q^2)+\mathcal{O}(\nu^{\frac{3}{2}}).$$

Furthermore this solution is linearly unstable.

It is not the first time that one exhibits unstable KAM tori (see for instance [3, 14]) but it turns out that it is the first example in a one dimensional context. This unstable behavior is to be compared with to the *modulational instabilities* extensively studied by physicists since fifty years (see [1, 16] and [10] for a coupled case different from ours).
We also prove (see Theorem 4.5 and Corollary 4.6) the persistence of the invariant torus $\mathcal{T}_\rho(p,p)$ but, in this case, the torus in linearly stable.
We stress out that, although the existence of invariant tori requires a lot of assumptions (and in particular we have to assume that ρ is in a Cantor set), when an invariant torus exists, its stability or instability is only related to the choice of the modes.

The result is obtained by putting H in a normal form $h+f$ suitable to apply a singular KAM theorem (see section 2.6) which is essentially contains in [3]. Notice that P_2 is totally resonant and thus is not adapted to a KAM procedure. The general idea, coming from [12], consists in using P_4 to break the resonances. First we apply a Birkhoff procedure (see Section 3) to eliminate the non resonant part of P_4

$$(P_2+P_4)\circ\tau=P_2+Z_4+\text{higher order term}.$$

Then in sections 4.1 and 4.2 we calculate the effective part of Z_4 in two different cases. This step is highly related to the choice of the torus that we want to perturb.
We notice that we could consider more general tori of any finite dimension (i.e. quasi periodic solutions constructed on finitely many linear modes). The instability of the corresponding torus will appear when one excites initially two modes a_p and b_q with $p\neq q$. To simplify the presentation we prefer to focus on two dimensional tori.
The strategy and the proofs are inspired by [3]. The aim of this paper is to

present these recent technics in a simpler case leading to a surprising result: instability seems typical even in 1d context.

We end this introduction with a remark linking instability of KAM tori and existence of a beating effect. Taking advantage of the resonances between the linear frequencies and of the coupling by the quartic term P_4, Grébert-Paturel-Thomann proved in [6] (see also [15]) that (1.1) exhibits a beating phenomena: roughly speaking when you consider initial data of the form

descr
$$\begin{cases} u(0,x) &= a_p(0)e^{ipx} + a_q(0)e^{iqx}, \\ v(0,x) &= b_p(0)e^{ipx} + b_q(0)e^{iqx}\,, \end{cases} \tag{1.4}$$

with $p \neq q$ and $|a_p(0)| = |b_q(0)| = \gamma\varepsilon$, $|a_q(0)| = |b_p(0)| = (1-\gamma)\varepsilon$ for $0 < \gamma < 1/2$ and ε small enough, then the four modes exchange energy periodically, i.e. they are close to

$$\begin{cases} |a_q(t)|^2 &= |b_p(t)|^2 = K_\gamma(\varepsilon^2 t), \\ |a_p(t)|^2 &= |b_q(t)|^2 = 1 - K_\gamma(\varepsilon^2 t), \end{cases}$$

where K_γ a $2T-$periodic function ($T \sim |\ln\gamma|$) which satisfies $K_\gamma(0) = \gamma$ and $K_\gamma(T) = 1-\gamma$. In [6] the result is proved only for a finite but very long time but in view of [9], we can expect that such beating solution exists for all time.
In this work we consider the case $\gamma = 0$ which corresponds to a two dimensional invariant torus, $\mathcal{T}_{(\sqrt{|a_p(0)|},\sqrt{|b_q(0)|})}$ for the linear system and we prove that the KAM theory applies, i.e. that the non linear Hamiltonian $P_2 + P_4$ admits invariant tori close to $\mathcal{T}_{(\sqrt{|a_p(0)|},\sqrt{|b_q(0)|})}$. Nevertheless, as we have seen, the tori are linearly unstable: when linearized around the torus, the system presents two hyperbolic directions which corresponds to the two other modes of the beating picture above. This means that the beating effect is related to the instability of the two tori: the one construct on the modes a_p, b_q and the one constructed on the modes a_q, b_p. Actually the monomial in P_4 which makes possible the beating effect, namely $a_p\bar{b}_p\bar{a}_q b_q$, is also responsible for the instability of the tori.
The beating phenomena has also be exhibited for the quintic NLS (see [7]) and for cubic NLS with some special nonlinearities (see [8]). It turns out that following the same line we could prove the existence of unstable KAM Tori in these two other cases. The main problem in both cases will be to verify that the hypotheses of the KAM theorem are satisfied, which will lead to computations similar but different from those of Appendix A.

2. An abstract KAM theorem

In this section we state a KAM theorem adapted to our problem. We consider a Hamiltonian $H = h_0 + f$, where h_0 is a quadratic Hamiltonian in normal form

NN
$$h_0 = \Omega(\rho)\cdot r + \sum_{\alpha\in\mathcal{Z}} \Lambda_\alpha(\rho)|\zeta_\alpha|^2. \tag{2.1}$$

Here

- ρ is a parameter in $\mathcal{D}$, which is a compact in the space $\mathbb{R}^n$;
- $r \in \mathbb{R}^n$ are the actions corresponding to the internal modes $(r, \theta) \in (\mathbb{R}^n \times \mathbb{T}^n, dr \wedge d\theta)$;
- $\mathcal{L}$ and $\mathcal{F}$ are respectively infinite and finite sets, $\mathcal{Z}$ is the disjoint union $\mathcal{L} \cup \mathcal{F}$;
- $\zeta = (\zeta_s)_{s\in\mathcal{Z}} \in \mathbb{C}^{\mathcal{Z}}$ are the external modes endowed with the standard complex symplectic structure $-\mathrm{i}\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}$. The external modes decomposes in an infinite part $\zeta_{\mathcal{L}} = (\zeta_s)_{s\in\mathcal{L}}$, corresponding to elliptic directions, which means that $\Lambda_\alpha \in \mathbb{R}$ for $\alpha \in \mathcal{L}$, and a finite part $\zeta_{\mathcal{F}} = (\zeta_s)_{s\in\mathcal{F}}$ corresponding to hyperbolic directions, which means that $\Im\Lambda_s \neq 0$ for $s \in \mathcal{F}$;
- the mappings

properties
$$\begin{cases} \Omega : \mathcal{D} \to \mathbb{R}^n, \\ \Lambda_\alpha : \mathcal{D} \to \mathbb{C}, \qquad \alpha \in \mathcal{Z}, \end{cases} \tag{2.2}$$

  are smooth.
- $f = f(r, \theta, \zeta; \rho)$ is the perturbation, small compare to the integrable part h_0.

2.1. **Setting.** We define precisely the spaces and norms:

Clustering structure on $\mathcal{L}$. We assume that $\mathcal{L}$ has a clustering structure:

$$\mathcal{L} = \cup_{j\in\mathbb{N}}\mathcal{L}_j$$

where $\mathcal{L}_j$, $j \in \mathbb{N}$, are finite sets of cardinality $d_j \leq d < +\infty$. If $\alpha \in \mathcal{L}_j$ we denote $[\alpha] = \mathcal{L}_j$ and $w_\alpha = j$. We consider $\mathcal{F}$ as an extra cluster of $\mathcal{Z} = \mathcal{L}\cup\mathcal{F}$ and for $\alpha \in \mathcal{F}$ we set $w_\alpha = 1$.

ex21 *Example* 2.1. In the second case of NLS systems (see Subsection 4.2), we will set $\mathcal{L} = \mathcal{Z} \subset \mathbb{Z} \times \{\pm\}$, $\mathcal{F} = \emptyset$ and ζ_α, $\alpha \in \mathcal{Z}$ will denote all the external modes: $\zeta_{j_+} = a_j$, $\zeta_{j_-} = b_j$ for $j \in \mathbb{Z} \setminus \{p\}$ and a_p, b_p are the internal modes. The clusters are given by $[j\pm] = \{j_+, j_-, -j_+, -j_-\}$ and $d_j = 4$ for $j \neq |p|$, and an extra cluster is given by $[-p] = \{-p_+, -p_-\}$ and $d_{-p} = 2$. Then we set $w_{j\pm} = |j|$.

ex22 *Example* 2.2. In the first case of NLS systems (see Subsection 4.1), we will set $\mathcal{Z} \subset \mathbb{Z} \times \{\pm\}$. If a_p, b_q $(p \neq q)$ are the two internal modes then $\mathcal{L} = \mathbb{Z} \setminus \{p, q\} \times \{\pm\}$ and ζ_α, $\alpha \in \mathcal{L}$ will denote all the elliptic external modes: $\zeta_{j_+} = a_j$, $\zeta_{j_-} = b_j$ for $j \in \mathbb{Z} \setminus \{p, q\}$. As we will see we have two hyperbolic external modes, b_p and a_q then $\mathcal{F} = \{p_-, q_+\}$. The clusters of $\mathcal{L}$ are given by $[j\pm] = \{j_+, j_-, -j_+, -j_-\}$ with $d_j = 4$ for $j \neq |p|, |q|$. If $p \neq -q$, we have to add two extra clusters given by $[-p] = \{-p_+, -p_-\}$ and $[-q] = \{-q_+, -q_-\}$ with $d_{-p} = d_{-q} = 2$. Then we set $w_{j\pm} = |j|$.

Linear space. Let $s \geq 0$, we consider the complex weighted ℓ_2-space

$$Z_s = \{\zeta = (\zeta_\alpha \in \mathbb{C},\ \alpha \in \mathcal{Z}) \mid \|\zeta\|_s < \infty\},$$

where

$$\|\zeta\|_s^2 = \sum_{\alpha\in\mathcal{Z}} |\zeta_\alpha|^2 w_\alpha^{2s}.$$

We provide the spaces $Z_s \times Z_s$, $s \geq 0$, with the symplectic structure $-\mathrm{i}\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}$. Similarly we define

$$Y_s = \{\zeta_{\mathcal{L}} = (\zeta_\alpha \in \mathbb{C},\ \alpha \in \mathcal{L}) \mid \|\zeta\|_s < \infty\},$$

endowed with the same norm and symplectic structure restricted to indices in $\mathcal{L}$.

Infinite matrices. For the elliptic variables, we denote by $\mathcal{M}_s$ the set of infinite matrices $A : \mathcal{L} \times \mathcal{L} \to \mathbb{C}$ such that A maps linearly Y_s into Y_s. We provide $\mathcal{M}_s$ with the operator norm

$$|A|_s = \|A\|_{\mathcal{L}(Y_s, Y_s)}.$$

We say that a matrix $A \in \mathcal{M}_s$ is in normal form if it is block diagonal and Hermitian, i.e.

ANF (2.3) $$A^\alpha_\beta = 0 \quad \text{for} \quad [\alpha] \neq [\beta] \quad \text{and} \quad A^\alpha_\beta = \overline{A^\beta_\alpha} \text{ for } \alpha, \beta \in \mathcal{L}.$$

In particular, we use that if $A \in \mathcal{M}_s$ is in normal form, its eigenvalues are real.

A class of Hamiltonian functions. Let us fix any $n \in \mathbb{N}$. On the space

$$\mathbb{C}^n \times \mathbb{C}^n \times (Z_s \times Z_s)$$

we define the norm

$$\|(r, \theta, z)\|_s = \max(|r|, |\theta|, \|z\|_s).$$

For $\sigma > 0$ we denote

$$\mathbb{T}^n_\sigma = \{\theta \in \mathbb{C}^n : |\Im\theta| < \sigma\}/2\pi\mathbb{Z}^n.$$

For $\sigma, \mu \in (0, 1]$ and $s \geq 0$ we set

$$\mathcal{O}^s(\sigma, \mu) = \{r \in \mathbb{C}^n : |r| < \mu^2\} \times \mathbb{T}^n_\sigma \times \{z \in Z_s \times Z_s : \|z\|_s < \mu\}.$$

We will denote points in $\mathcal{O}^s(\sigma, \mu)$ as $x = (r, \theta, z)$. A function defined on a domain $\mathcal{O}^s(\sigma, \mu)$, is called *real* if it gives real values to real arguments $x = (r, \theta, z)$ with r, θ reals and $z = (\zeta, \bar{\zeta})$.
Let

$$\mathcal{D} = \{\rho\} \subset \mathbb{R}^p$$

be a compact set of positive Lebesgue measure. This is the set of parameters upon which will depend our objects. Differentiability of functions on $\mathcal{D}$ is understood in the sense of Whitney. So $f \in C^1(\mathcal{D})$ if it may be extended to a C^1-smooth function $\tilde{f}$ on $\mathbb{R}^p$, and $|f|_{C^1(\mathcal{D})}$ is the infimum of $|\tilde{f}|_{C^1(\mathbb{R}^p)}$, taken over all C^1-extensions $\tilde{f}$ of f.
Let $f : \mathcal{O}^0(\sigma, \mu) \times \mathcal{D} \to \mathbb{C}$ be a C^1-function, real holomorphic in the first variable x, such that for all $\rho \in \mathcal{D}$

regularize (2.4) $$\mathcal{O}^s(\sigma, \mu) \ni x \mapsto \nabla_z f(x, \rho) \in Z_s \times Z_s$$

and

$$\mathcal{O}^s(\sigma, \mu) \ni x \mapsto \nabla^2_{\zeta_{\mathcal{L}} \bar{\zeta}_{\mathcal{L}}} f(x, \rho) \in \mathcal{M}_s$$

are real holomorphic functions[1]. We denote by $\mathcal{T}^s(\sigma,\mu,\mathcal{D})$ this set of functions. For a function $f\in\mathcal{T}^s(\sigma,\mu,\mathcal{D})$ we define the norm

$$[f]^s_{\sigma,\mu,\mathcal{D}}$$

through

$$\sup \max(|\partial^j_\rho f(x,\rho)|,\mu\|\partial^j_\rho\nabla_z f(x,\rho)\|_s,\mu^2|\partial^j_\rho\nabla^2_{\zeta_\mathcal{L}\bar\zeta_\mathcal{L}}f(x,\rho)|_s),$$

where the supremum is taken over all

$$j=0,1,\ x\in\mathcal{O}^s(\sigma,\mu),\ \rho\in\mathcal{D}.$$

When the function f does not depend on (r,θ) neither on ρ we denote $f\in\mathcal{T}^s(\mu)$.

exf *Example* 2.3. Let $\mathcal{Z}=\mathbb{Z}\setminus\{k_1,\cdots,k_n\}$ and g an analytic function from a neighborhood of the origin in $\mathbb{C}^2$ into $\mathbb{C}$. We define a Hamiltonian f by

$$\mathcal{O}^s(\sigma,\mu)\ni x\mapsto f(x):=\int_{\mathbb{T}}g(\hat u(t),\hat{\bar u}(t))dt$$

with

$$\hat u(t)=\sum_{\ell=1,\cdots,n}r_\ell e^{i\theta_\ell}e^{ik_\ell t}+\sum_{\alpha\in\mathcal{Z}}\zeta_\alpha e^{i\alpha t},$$
$$\hat{\bar u}(t)=\sum_{\ell=1,\cdots,n}r_\ell e^{-i\theta_\ell}e^{-ik_\ell t}+\sum_{\alpha\in\mathcal{Z}}\bar\zeta_\alpha e^{-i\alpha t}.$$

We verify that for $s>1/2$ and $\sigma>0$, $\mu>0$ small enough $f\in\mathcal{T}^s(\sigma,\mu,\mathcal{D})$ (here f does not depend on ρ). A precise proof is given in the Appendix A of [3]. We can recall here the basis of the proof: we have

$$\frac{\partial f}{\partial\zeta_\alpha}=\int_{\mathbb{T}}\partial_1 g(\hat u(t),\hat{\bar u}(t))e^{i\alpha t}dt$$

and since $\hat u$ and $\hat{\bar u}$ have their Fourier coefficients in ℓ^2_s, they are both functions in the Sobolev space H^s which in turns implies that $t\mapsto\partial_1 g(\hat u(t),\hat{\bar u}(t))$ is an H^s function and thus its Fourier coefficients are in ℓ^2_s.

Jet-functions. For any function $f\in\mathcal{T}^s(\sigma,\mu,\mathcal{D})$ we define its jet $f^T(x)$, $x=(r,\theta,z)$, as the following Taylor polynomial of f at $r=0$ and $z=0$

jet (2.5) $$f(0,\theta,0)+d_rf(0,\theta,0)[r]+d_zf(0,\theta,0)[z]+\frac{1}{2}d^2_zf(0,\theta,0)[z,z].$$

Functions of the form f^T will be called *jet-functions.*

Poisson brackets. The Poisson brackets of two Hamiltonian functions is defined by

$$\{f,g\}=\nabla_\theta f\cdot\nabla_r g-\nabla_r f\cdot\nabla_\theta g-i\langle\nabla_z f,J\nabla_z g\rangle\,.$$

[1]In fact by Cauchy's theorem the analyticity of $\nabla_z f$ on $\mathcal{O}^s(\sigma,\mu)$ with value in $Z_s\times Z_s$ yields the analyticity of $\nabla^2_{zz}f$ on $\mathcal{O}^s(\sigma',\mu)$ with values in $\mathcal{L}(Z_s\times Z_s,Z_s\times Z_s)$, and thus the analyticity of $\nabla^2_{\zeta_\mathcal{L}\bar\zeta_\mathcal{L}}f$ on $\mathcal{O}^s(\sigma',\mu)$ with values in $\mathcal{M}_s$, for any $\sigma'<\sigma$. We conserve the two hypothesis to mimic [3] and [5] where an additional property on the Hessian of f was required.

lemma-poisson

Lemma 2.4. *Let $s > 1/2$. Let $f \in \mathcal{T}^s(\sigma, \mu, \mathcal{D})$ and $g \in \mathcal{T}^s(\sigma, \mu, \mathcal{D})$ be two jet functions then for any $0 < \sigma' < \sigma$ we have $\{f, g\} \in \mathcal{T}^s(\sigma', \mu, \mathcal{D})$ and*

$$[\{f, g\}]^s_{\sigma',\mu,\mathcal{D}} \leq C(\sigma - \sigma')^{-1}\mu^{-2}[f]^s_{\sigma,\mu,\mathcal{D}}[g]^s_{\sigma,\mu,\mathcal{D}}.$$

The proof follows as in [5] Lemma 4.3. This stability result is fundamental to apply the KAM scheme.

Normal form. A quadratic Hamiltonian function is on normal form if it reads

HamilNF

$$h = V(\rho) \cdot r + \langle \zeta_{\mathcal{L}}, A(\rho)\bar{\zeta}_{\mathcal{L}} \rangle + \frac{1}{2}\langle z_{\mathcal{F}}, K(\rho) z_{\mathcal{F}} \rangle \tag{2.6}$$

for some vector function $V(\rho) \in \mathbb{R}^n$, some matrix functions $A(\rho) \in \mathcal{M}_s$ on normal form (see (2.3)) and $K(\rho)$ is a matrix $\mathcal{F} \times \mathcal{F} \to gl(2, \mathbb{C})$ symmetric in the following sense[2]: $K^\alpha_\beta = {}^t K^\beta_\alpha$.

subsechypo

2.2. **Hypothesis.** The following three hypotheses concerned only the quadratic Hamiltonian h_0. The first one is related to the asymptotic of Λ_α, the two other are non resonances conditions.

Hypothesis A0 (spectral asymptotic.) There exists $C > 0$ such that

$$|\Lambda_\alpha - |w_\alpha|^2| \leq C, \ \forall\, \alpha \in \mathcal{L}\,.$$

Hypothesis A1 (Conditions on external frequencies.)
There exists $\delta > 0$ such that for all $\rho \in \mathcal{D}$ we have

(a) The elliptic frequencies don't vanish:

$$|\Lambda_\alpha| \geq \delta \ \ \forall\, \alpha \in \mathcal{L}\,;$$

and the hyperbolic frequencies have a non vanishing imaginary part

$$|\Im \Lambda_\alpha| \geq \delta \ \ \forall\, \alpha \in \mathcal{F}\,;$$

(b) The difference between two external frequencies doesn't vanish except if they are in the same cluster:

$$|\Lambda_\alpha(\rho) - \Lambda_\beta(\rho)| \geq \delta \quad \forall \alpha, \beta \in \mathcal{Z} \text{ with } [\alpha] \neq [\beta]\,;$$

(c) The sum of two elliptic frequencies doesn't vanish:

$$|\Lambda_\alpha(\rho) + \Lambda_\beta(\rho)| \geq \delta \text{ for all } \alpha, \beta \in \mathcal{L}.$$

Hypothesis A2 (Transversality conditions.)
These conditions express that the small divisors cannot stay in a resonant position:
There exists $\delta > 0$ such that for all $\tilde{\Omega}(\cdot)$ $\delta-$close in C^1 norm from $\Omega(\cdot)$ and for all $k \in \mathbb{Z}^n \setminus \{0\}$:
(i) either

$$|\tilde{\Omega}(\rho) \cdot k| \geq \delta \quad \forall \rho \in \mathcal{D},$$

or there exits a unit vector[3] $\mathfrak{z} = \mathfrak{z}(k) \in \mathbb{R}^n$ such that

$$(\nabla_\rho \cdot \mathfrak{z})\big(\tilde{\Omega}(\rho) \cdot k\big) \geq \delta \quad \forall \rho \in \mathcal{D}.$$

[2]This symmetry comes from the matrix representation that we chose for Hessian functions and the Schwarz rule.

[3]The notation $(\nabla_\rho \cdot \mathfrak{z}) f(\rho)$ means that we take the gradient of f at the point ρ in the direction $\mathfrak{z}$.

(ii) for all $\alpha \in \mathcal{Z}$ either

$$|\tilde{\Omega}(\rho)\cdot k + \Lambda_\alpha(\rho)| \geq \delta \quad \forall \rho \in \mathcal{D},$$

or there exits a unit vector $\mathfrak{z} = \mathfrak{z}(k) \in \mathbb{R}^n$ such that

$$(\nabla_\rho \cdot \mathfrak{z})\big(\tilde{\Omega}(\rho)\cdot k + \Lambda_\alpha(\rho)\big) \geq \delta \quad \forall \rho \in \mathcal{D},$$

(iii) for all $\alpha, \beta \in \mathcal{Z}$ either

$$|\tilde{\Omega}(\rho)\cdot k + \Lambda_\alpha(\rho) \pm \Lambda_\beta(\rho)| \geq \delta \quad \forall \rho \in \mathcal{D},$$

or there exits a unit vector $\mathfrak{z} = \mathfrak{z}(k) \in \mathbb{R}^n$ such that

$$(\nabla_\rho \cdot \mathfrak{z})\big(\tilde{\Omega}(\rho)\cdot k + \Lambda_\alpha(\rho) \pm \Lambda_\beta(\rho)\big) \geq \delta \quad \forall \rho \in \mathcal{D}\,.$$

Hypotheses A1 and A2 are used (see Proposition 2.7) to control small denominators of the form

$$\begin{array}{ll}
\Omega\cdot k & \forall k \neq 0,\\
\Omega\cdot k + \Lambda_\alpha & \forall \alpha \in \mathcal{Z},\ k \in \mathbb{Z}^n,\\
\Omega\cdot k + \Lambda_\alpha + \Lambda_\beta & \forall \alpha,\beta \in \mathcal{Z},\ k \neq 0,\\
\Lambda_\alpha + \Lambda_\beta & \forall \alpha,\beta \in \mathcal{L},\\
\Omega\cdot k + \Lambda_\alpha - \Lambda_\beta & \forall \alpha,\beta \in \mathcal{Z},\ k \neq 0,\\
\Lambda_\alpha - \Lambda_\beta & \forall \alpha,\beta \in \mathcal{Z} \text{ with } [\alpha] \neq [\beta]
\end{array}$$

which in turns are used to kill the following monomials of the jet of the perturbation f

$$\begin{array}{ll}
e^{ik\cdot\theta} & \forall k \neq 0,\\
e^{ik\cdot\theta}\zeta_\alpha,\ e^{ik\cdot\theta}\bar{\zeta}_\alpha & \forall \alpha \in \mathcal{Z},\ k \in \mathbb{Z}^n,\\
e^{ik\cdot\theta}\zeta_\alpha\zeta_\beta,\ e^{ik\cdot\theta}\bar{\zeta}_\alpha\bar{\zeta}_\beta & \forall \alpha,\beta \in \mathcal{Z},\ k \neq 0,\\
\zeta_\alpha\zeta_\beta,\ \bar{\zeta}_\alpha\bar{\zeta}_\beta & \forall \alpha,\beta \in \mathcal{L},\\
e^{ik\cdot\theta}\zeta_\alpha\bar{\zeta}_\beta & \forall \alpha,\beta \in \mathcal{Z},\ k \neq 0,\\
\zeta_\alpha\bar{\zeta}_\beta & \forall \alpha,\beta \in \mathcal{Z} \text{ with } [\alpha] \neq [\beta].
\end{array}$$

momentum *Remark* 2.5. If f preserves some symmetries, and if these symmetries are preserved by the KAM procedure, then some monomials will never appear in the perturbation terms and the corresponding small denominator has not to be control. This can be used to relax Hypotheses A1 and A2. For instance if f commutes with the momentum $\mathbb{M} = \sum_{j\in\mathbb{Z}} j(|a_j|^2 + |b_j|^2)$ then Hypothesis A1 (c) has not to be satisfied for $\alpha = (p, \pm)$ and $\beta = (-p, \pm)$ (see example 2.1) since $\{a_p b_{-p}, \mathbb{M}\} = \{a_p a_{-p}, \mathbb{M}\} = \{b_p b_{-p}, \mathbb{M}\} = 0$. We can also use the conservation of the mass $\mathbb{L} = \sum_{j\in\mathbb{Z}}(|a_j|^2 + |b_j|^2)$. See also the Appendix A where this remark will be crucial.

2.3. **Statement and comments on the proof.** We recall that we consider a Hamiltonian $H = h_0 + f$, where h_0 is the quadratic Hamiltonian in normal form given by (2.1).

KAM **Theorem 2.6.** *Assume that hypothesis A0, A1, A2 are satisfied*[4] *and that* $f \in \mathcal{T}^s(\sigma,\mu,\mathcal{D})$ *with* $s > 1/2$. *Let* $\gamma > 0$, *there exists a constant* $\varepsilon_0 > 0$ *such that if*

eps (2.7) $$[f]^s_{\sigma,\mu,\mathcal{D}} \leq \varepsilon_0\delta \quad \textit{and} \quad \varepsilon := [f^T]^s_{\sigma,\mu,\mathcal{D}} \leq \varepsilon_0\delta^{1+\gamma},$$

[4]Hypotheses A1 and A2 can be partially relaxed according to Remark 2.5.

then there exists a Cantor set $\mathcal{D}' \subset \mathcal{D}$ asymptotically of full measure (i.e. $\operatorname{meas} \mathcal{D} \setminus \mathcal{D}' \to 0$ *when $\varepsilon \to 0$) and for all $\rho \in \mathcal{D}'$ there exists a symplectic change of variables $\Phi : \mathcal{O}^s(\sigma/2, \mu/2) \to \mathcal{O}^s(\sigma, \mu)$ such that for $\rho \in \mathcal{D}'$*

$$(h_0 + f) \circ \Phi = h + g$$

with $h = \langle \omega(\rho), r\rangle + \langle \zeta_{\mathcal{L}}, A(\rho)\bar{\zeta}_{\mathcal{L}}\rangle + \frac{1}{2}\langle z_{\mathcal{F}}, K(\rho) z_{\mathcal{F}}\rangle$ on normal form (see (2.6)*) and $g \in \mathcal{T}^s(\sigma/2, \mu/2, \mathcal{D}')$ with $g^T \equiv 0$. Furthermore there exists $C > 0$ such that for all $\rho \in \mathcal{D}'$*

$$|\omega - \Omega| \leq C\varepsilon,\ |A - \operatorname{diag}(\Lambda_\alpha,\ \alpha \in \mathcal{L})| \leq C\varepsilon \text{ and } |JK - \operatorname{diag}(\Lambda_\alpha,\ \alpha \in \mathcal{F})| \leq C\varepsilon.$$

As a dynamical consequences $\Phi(\{0\} \times \mathbb{T}^n \times \{0\})$ is an invariant torus for $h_0 + f$ and this torus is linearly stable if and only if $\mathcal{F} = \emptyset$.

Theorem 2.6 is a normal form result, we can explain its dynamical consequences. First, for $\rho \in \mathcal{D}'$, the torus $\{0\} \times \mathbb{T}^n \times \{0\}$ is invariant by the flow of $h + g$ and thus the torus $\Phi(\{0\} \times \mathbb{T}^n \times \{0\})$ is invariant by the flow of $h_0 + f$ and the dynamics on it is the same as that of h.
Moreover, the linearized equation on this torus reads

$$\begin{cases} \dot{\zeta}_{\mathcal{L}} = -iA\zeta_{\mathcal{L}} - i\partial^2_{r\bar{\zeta}} g(0, \theta_0 + \omega t, 0) \cdot r, \\ \dot{z}_{\mathcal{F}} = -iJK z_{\mathcal{F}} - iJ\partial^2_{rz} g(0, \theta_0 + \omega t, 0) \cdot r, \\ \dot{\theta} = \partial^2_{rz} g(0, \theta_0 + \omega t, 0) \cdot z + \partial^2_{rr} g(0, \theta_0 + \omega t, 0) \cdot r, \\ \dot{r} = 0. \end{cases}$$

Since A is on normal form the eigenvalues of the $\zeta_{\mathcal{L}}$-linear part in the first line are purely imaginary (see (2.3)). Since furthermore JK is sufficiently close to the diagonal matrix $\operatorname{diag}(\Lambda_\alpha,\ \alpha \in \mathcal{F})$, the eigenvalues of the $\zeta_{\mathcal{F}}$-linear part in the second line have a non vanishing real part. Finally, the last term in the two first lines is a bounded term, independant on ζ (and on $z = (\zeta, \bar{\zeta})$), and thus doesn't play a role in the linear stability. Therefore the invariant torus is linearly stable if and only if $\mathcal{F} = \emptyset$.

The proof is standard and we don't include it in this article. Nevertheless it is not a direct consequence of an existing KAM theorem. Essentially Theorem 2.6 is a mix between the KAM theorem proved in [12] (see also [13]) and the one proved in [3] (see also [5] for a proof in Sobolev regularity or [2] for a 1d version). In Theorem 2.6 and in the KAM theorem proved in [12] or [13], we have the same asymptotics of the frequencies (Hypothesis A0) which simplifies the proof and doesn't require regularizing perturbation as in [3]. Nevertheless in [12] or [13] the frequencies are non resonant and thus there is no clustering. So we need [3] and the clustering structure to prove Theorem 2.6.

Let us explain why Hypothesis A0, A1, A2 allow to control the so called small divisors. The KAM proof is based on an iterative procedure that requires to solve a homological equation at each step. Roughly speaking, it consists in inverting an infinite dimensional matrix whose eigenvalues are the so-called small divisors:

$$\begin{aligned} &\omega \cdot k \quad k \in \mathbb{Z}^{\mathcal{A}} \setminus \{0\}, \\ &\omega \cdot k + \lambda_\alpha \quad k \in \mathbb{Z}^{\mathcal{A}},\ \alpha \in \mathcal{Z}, \\ &\omega \cdot k + \lambda_\alpha \pm \lambda_\beta \quad k \in \mathbb{Z}^{\mathcal{A}},\ \alpha, \beta \in \mathcal{Z} \end{aligned}$$

where $\omega = \omega(\rho)$ and $\lambda_a = \lambda_a(\rho)$ are small perturbations (changing at each KAM step) of the original frequencies $\Omega(\rho)$ and $\Lambda_a(\rho), a \in \mathcal{L}$.

The transversality condition (Hypothesis A2) ensures that for most values of ρ, all these eigenvalues are far away from zero (at least at the first step):

Melni **Proposition 2.7.** *Let $N \geq 1$ and $0 < \kappa \leq \delta$. Assume Hypothesis A0, A1, A2. Then there exists a closed subset $\mathcal{D}' \equiv \mathcal{D}'(\kappa, N) \subset \mathcal{D}$ satisfying*

$$\text{meas } \mathcal{D} \setminus \mathcal{D}' \leq C\delta^{-1}\kappa N^{n+2},$$

such that for all $\rho \in \mathcal{D}'$, for all $|k| \leq N$ and for all $\alpha, \beta \in \mathcal{Z}$

D00 (2.8) $$|\Omega(\rho) \cdot k| \geq \kappa, \text{ except if } k = 0,$$

D11 (2.9) $$|\Omega(\rho) \cdot k + \Lambda_\alpha(\rho)| \geq \kappa,$$

D22 (2.10) $$|\Omega(\rho) \cdot k + \Lambda_\alpha(\rho) + \Lambda_\beta(\rho)| \geq \kappa,$$

D33 (2.11) $$|\Omega(\rho) \cdot k + \Lambda_\alpha(\rho) - \Lambda_\beta(\rho)| \geq \kappa.$$

Let us recall the following classical result

estim **Lemma 2.8.** *(see for instance [4] appendice A) Let I be an open interval and let $f : I \to \mathbb{R}$ be a $\mathcal{C}^1$-function satisfying*

$$|f'(x)| \geq \delta, \quad \forall x \in I.$$

Then,

$$\text{meas}\{x \in I : |f(x)| < \varepsilon\} \leq C\frac{\varepsilon}{\delta}.$$

Proof of Proposition 2.7. Let us prove the estimates (2.10) and (2.11), the other two being similar but easier. Let us begin with (2.10). Let us fix $k \neq 0,\ \alpha, \beta \in \mathcal{Z}$, by Hypothesis A2 we have, either

$$|\Omega(\rho) \cdot k + \Lambda_\alpha(\rho) + \Lambda_\beta(\rho)| \geq \kappa \quad \forall \rho \in \mathcal{D},$$

or

$$(\nabla_\rho \cdot \jmath)\big(\Omega(\rho) \cdot k + \Lambda_\alpha(\rho) + \Lambda_\beta(\rho)\big) \geq \delta \quad \forall \rho \in \mathcal{D}\,.$$

Then we have using Lemma 2.8

$$\text{meas}\{\rho \mid |\Omega(\rho) \cdot k + \Lambda_\alpha(\rho) + \Lambda_\beta(\rho)| < \kappa\} \leq C\kappa\delta^{-1}$$

where C does not depend on k, α, β. On the other hand, in view of Hypothesis A0, we remark that $\Omega(\rho) \cdot k + \Lambda_\alpha(\rho) + \Lambda_\beta(\rho)$ can be small only if $|\alpha|, |\beta| \leq C|k|^{1/2}$. Therefore

$$\text{meas}\{\rho \mid |\Omega(\rho) \cdot k + \Lambda_\alpha(\rho) + \Lambda_\beta(\rho)| < \kappa, \text{ for some } |k| \leq N \text{ and } \alpha, \beta \in \mathcal{Z}\} \leq C\frac{\kappa}{\delta}N^n N^{\frac{1}{2}} N^{\frac{1}{2}}.$$

The proof of (2.11) is similar except that, since $|j^2 - \ell^2| \geq 2|j| - 1$ for any integers j, ℓ, we deduce that $\Omega(\rho) \cdot k + \Lambda_\alpha(\rho) - \Lambda_\beta(\rho)$ can be small only if $|\alpha|, |\beta| \leq C|k|$. □

Remark 2.9. In the multidimensional case, the transversality condition is not enough to ensure (2.11) and we have to add the so called second Melnikov condition in the list of hypothesis (see for instance [3]). The problem comes from the fact that in $\mathbb{Z}^d$ with $d \geq 2$, $||j|^2 - |\ell|^2|$ can be small even for large j and ℓ.

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3. Birkhoff normal form

In this section we apply a Birkhoff procedure to the Hamiltonian $P_2 + P_4$ (see (1.2)) in order to eliminate the non resonant monomials.

3.1. The framework of the study.

For $s \geq 0$ we define

$$\ell^2_s = \{x \equiv (x_k)_{k\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}} \mid ||x||_s < +\infty\},$$

where

$$||x||^2_s = \sum_{k\in\mathbb{Z}} (1+|k|^2)^s |x_k|^2.$$

We denote $x \star y$ the convolution of sequences in ℓ^2: $(x\star y)_\ell = \sum_{i+j=\ell} x_i y_j$, and recall that for $s > 1/2$, ℓ^2_s is a Hilbert algebra with respect to the convolution product and

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$$||x\star y||_s \leq c_s ||x||_s ||y||_s. \tag{3.1}$$

For a proof of this classic property, see [12], Appendix A for $a = 0$.
We consider the phase space $\mathcal{P}_s = \ell^2_s \times \ell^2_s \times \ell^2_s \times \ell^2_s \ni (a, \bar a, b, \bar b)$ endowed with the canonical symplectic structure

$$-i\sum \mathrm{d}a_k \wedge \mathrm{d}\bar a_k - i\sum \mathrm{d}b_k \wedge \mathrm{d}\bar b_k.$$

Finally, we introduce $B_s(r)$, the ball of radius r centered at the origin in $\mathcal{P}_s$. An interesting feature of the space $\mathcal{P}_s$ is the behavior of the Hamiltonian vectorfields of homogeneous bounded polynomials of $\mathcal{P}_s$, as we can see through the following lemma:

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Lemma 3.1. *Let $s > 1/2$. Let P be a homogeneous polynomial of order 4 on $\mathcal{P}_s$ of the form*

P

$$P(a,b,\bar a,\bar b) = \sum_{\substack{i,j,k,l\in\mathbb{Z}\\ i+j=k+l}} c_{i,j,k,\ell}\ a_i b_j \bar a_k \bar b_\ell \tag{3.2}$$

with $|c_{i,j,k,\ell}| \leq M$ for all $(i,j,k,\ell) \in \mathcal{J} := \{i,j,k,l \in \mathbb{Z} \mid i+j = k+l\}$. Then the Hamiltonian vectorfield X_P is analytic from $B_s(1)$, the unit ball in $\mathcal{P}_s$, into $\mathcal{P}_s$, with

$$||X_P(a,b,\bar a,\bar b)||_s \leq 4M ||(a,b,\bar a,\bar b)||^3_s.$$

In particular $P \in \mathcal{T}^s(1)$. Furthermore P commutes with $\mathbb{L}$ and $\mathbb{M}$.

Of course this lemma extends to polynomials of any order with bounded coefficients and zero momentum (the generalization to any order of the condition $i+j = k+\ell$). In particular this lemma shows that X_{P_4}, X_{Z_4} and X_{χ_4} introduced in lemma below are vectorfields on $\mathcal{P}_s$. Notice that the linear vectorfield X_{P_2} is unbounded on $\mathcal{P}_s$, since it takes values in $\mathcal{P}_{s-2}$.

Proof. As the coefficients of P are bounded by M we get

$$\left|\frac{\partial P}{\partial \bar b_\ell}\right| \leq M \sum_{(i,j,k,\ell)\in\mathcal{J}} |a_i b_j \bar a_k|.$$

Therefore writing

$$\sum_{(i,j,k,\ell)\in\mathcal{J}} |a_i b_j \bar a_k| = (a \star b \star \tilde{\bar a})_\ell,$$

where $\tilde{x}$ denotes the sequence defined by $\tilde{x}_j = x_{-j}$, we deduce using (3.1)

$$\Big|\Big|\frac{\partial P}{\partial \overline{b_\ell}}\Big|\Big|_s \leq M||(a,b,\bar{a},\bar{b})||_s^3.$$

We control the other partial derivatives of P in the same way and we conclude that

$$||X_P(a,b,\bar{a},\bar{b})||_s \leq 4M||(a,b,\bar{a},\bar{b})||_s^3.$$

□

3.2. **The Birkhoff normal form result.** We recall that

$$H = P_2 + P_4$$

is defined by (1.2). We also introduce the mass and momentum Hamiltonians:

$$\mathbb{L} = \sum_{j\in\mathbb{Z}}(|a_j|^2 + |b_j|^2), \quad \text{and} \quad \mathbb{M} = \sum_{j\in\mathbb{Z}} j(|a_j|^2 + |b_j|^2)\,.$$

We notice that H commutes with both $\mathbb{L}$ and $\mathbb{M}$:

$$\{H,\mathbb{L}\} = \{H,\mathbb{M}\} = 0$$

which means that the Hamiltonian flow generated by H preserves the mass and the momentum.

BNF **Proposition 3.2.** *For ε small enough, there exists a symplectic change of variables τ from the ball $B_s(\varepsilon)$ of $\mathcal{P}_s$ into $\mathcal{U}_\varepsilon$ a neighborhood of the origin in $\mathcal{P}_s$ included in $B_s(2\varepsilon)$ such that:*

Hbirk (3.3) $$H^B := H\circ\tau = P_2 + Z_4 + R_6,$$

where :

(i) P_2 depends only on the actions:

$$P_2(a,\overline{a},b,\overline{b}) = \sum_{j\in\mathbb{Z}} j^2(|a_j|^2 + |b_j|^2).$$

(ii) Z_4 is the 4^{th} order homogeneous polynomial:

$$Z_4(a,\overline{a},b,\overline{b}) = \sum_{\substack{k+l=i+j\\ k^2+l^2=i^2+j^2}} a_k b_l \overline{a_i}\overline{b_j} = \sum_{\{k,l\}=\{i,j\}} a_k b_l \overline{a_i}\overline{b_j}.$$

In particular, Z_4 is resonant in the following sense: $\{P_2, Z_4\} = 0$.

(iii) R_6 is a Hamiltonian function in $\mathcal{T}^s(1)$ which satisfies

$$\big\|X_{R_6}(a,\overline{a},b,\overline{b})\big\|_s \leq C\,\big\|(a,\overline{a},b,\overline{b})\big\|_s^5 \quad \textit{for all} \quad (a,\overline{a},b,\overline{b}) \in B_s(\varepsilon).$$

Furthermore R_6 commutes with $\mathbb{L}$ and $\mathbb{M}$.

(iv) τ preserves the class $\mathcal{T}^s(\mu)$ for any $s > 1/2$ and $\mu > 0$ and it also conserves the mass and the momentum. Furthermore τ is close to the identity: there exists C_s such that

$$\big\|\tau(a,\overline{a},b,\overline{b}) - (a,\overline{a},b,\overline{b})\big\|_s \leq C_s\,\big\|(a,\overline{a},b,\overline{b})\big\|_s^3 \quad \textit{for all} \quad (a,\overline{a},b,\overline{b}) \in B_s(\varepsilon).$$

Proof. The idea is to search τ as the time 1 flow φ_1 of χ_4 where χ_4 is a Hamiltonian polynomial of order 4. We write

$$\chi_4(a,\overline{a},b,\overline{b}) := \sum_{p,q,r,s\in\mathbb{Z}} m(p,q,r,s) a_p \overline{a}_q b_r \overline{b}_s.$$

For F a Hamiltonian, the Taylor expansion of $F\circ\varphi_t$ between the times $t=0$ and $t=1$ gives:

$$F\circ\tau := F\circ\varphi_1 = F + \{F,\chi_4\} + \int_0^1 (1-t)\left\{\{F,\chi_4\},\chi_4\right\}\circ\varphi_t \mathrm{d}t.$$

Applying this formula to $F = H = P_2 + P_4$ we obtain

defZ4R6 (3.4)

$$H\circ\tau = \underbrace{P_2}_{P_2} + \underbrace{P_4 + \{P_2,\chi_4\}}_{Z_4} + \underbrace{\{P_4,\chi_4\} + \int_0^1 (1-t)\left\{\{H,\chi_4\},\chi_4\right\}\circ\varphi_t \mathrm{d}t}_{R_6}.$$

Thus defined, the first point of the proposition is already checked. We have now to choose the polynomial χ_4 such as τ, Z_4 and R_6 satisfy the hypothesis of Proposition 3.2. According to the equation (3.4), we want to solve the so called homological equation

eqhomol (3.5) $$\{\chi_4, P_2\} = P_4 - Z_4,$$

where P_2, P_4 are given by equation (1.2) and Z_4 is given by Proposition 3.2. For the right hand-side polynomial term, we have

$$(P_4 - Z_4)(a,\overline{a},b,\overline{b}) = \sum_{\substack{p-q+r-s=0\\ p^2-q^2+r^2-s^2\neq 0}} a_p\overline{a}_q b_r\overline{b}_s.$$

The left hand-side Poisson bracket term gives

$$\begin{aligned}\{\chi_4,P_2\} &= -i\sum_{j\in\mathbb{Z}}\left(\frac{\partial\chi_4}{\partial a_j}\frac{\partial P_2}{\partial\overline{a}_j} - \frac{\partial\chi_4}{\partial\overline{a}_j}\frac{\partial P_2}{\partial a_j} + \frac{\partial\chi_4}{\partial b_j}\frac{\partial P_2}{\partial\overline{b}_j} - \frac{\partial\chi_4}{\partial\overline{b}_j}\frac{\partial P_2}{\partial b_j}\right)\\ &= -i\sum_{p,q,r,s\in\mathbb{Z}}(p^2-q^2+r^2-s^2)m(p,q,r,s)a_p\overline{a}_q b_r\overline{b}_s.\end{aligned}$$

Therefore, in order to solve the homological equation (3.5), it suffices to choose

$$m(p,q,r,s) = \begin{cases}\dfrac{i}{(p^2-q^2+r^2-s^2)} & \text{if } \begin{cases}p-q+r-s=0,\\ p^2-q^2+r^2-s^2\neq 0,\end{cases}\\ 0 \quad \text{else.}\end{cases}$$

Thanks to the choice of χ_4, we have constructed the polynomial Z_4. It remains to prove that Z_4 is resonant, i.e

$$\{P_2, Z_4\} = 0.$$

With the same computations as in the χ_4 construction, we have:

$$\{P_2,Z_4\} = i\sum_{p,q,r,s\in\mathbb{Z}}(p^2-q^2+r^2-s^2)z(p,q,r,s)a_p\overline{a}_q b_r\overline{b}_s,$$

where

$$z(p,q,r,s) = \begin{cases} 1 & \text{if} \quad p-q+r-s=0 \quad \text{and} \quad p^2-q^2+r^2-s^2=0, \\ 0 & \text{else.} \end{cases}$$

Therefore,

$$\forall (p,q,r,s) \in \mathbb{Z}^4,\ (p^2-q^2+r^2-s^2) z(p,q,r,s) = 0,$$

and we have $\{P_2, Z_4\} = 0$. The proof of the second point of the proposition is completed. By construction, the coefficients of χ_4 are bounded. Thus, by Lemma 3.1, we have

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$$\left\| X_{\chi_4}(a,\overline{a},b,\overline{b}) \right\|_s = \mathcal{O}\left\| (a,\overline{a},b,\overline{b}) \right\|_s^3. \tag{3.6}$$

We now want to prove that τ is well defined, i.e. we want to prove that the flow φ_t is defined at least up to $t=1$. For that purpose, the idea here is to use a bootstrap argument. We introduce $T = T(a,\overline{a},b,\overline{b}) > 0$ the existence time of the flow $\varphi_t(a,\overline{a},b,\overline{b})$, and we consider a smaller time $0 \leq s \leq T$. Writing the fundamental theorem of calculus for φ_t, the flow of χ_4, between the times $t=0$ and $t=s$, we obtain

$$\varphi_s(a,\overline{a},b,\overline{b}) - \varphi_0(a,\overline{a},b,\overline{b}) = \int_0^s \dot{\varphi}_t(a,\overline{a},b,\overline{b}) \mathrm{d}t = \int_0^s X_\chi\left(\varphi_t(a,\overline{a},b,\overline{b})\right) \mathrm{d}t.$$

By definition, we have $\varphi_0 = Id$. Thus, the equation (3.6) implies (with $C>0$ a constant):

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$$\|\varphi_s(a,\overline{a},b,\overline{b}) - (a,\overline{a},b,\overline{b})\|_s \leq C \int_0^s \left\|\varphi_t(a,\overline{a},b,\overline{b})\right\|_s^3 \mathrm{d}t. \tag{3.7}$$

Let us choose $(a,\overline{a},b,\overline{b}) \in B_\rho(\varepsilon)$. As long as

$$\|\varphi_s(a,\overline{a},b,\overline{b}) - (a,\overline{a},b,\overline{b})\|_s \leq 2\|(a,\overline{a},b,\overline{b})\|_s,$$

we have (using that $(a,\overline{a},b,\overline{b}) \in B_\rho(\varepsilon)$),

$$\begin{aligned} \|\varphi_s(a,\overline{a},b,\overline{b}) - (a,\overline{a},b,\overline{b})\|_s &\leq Cs(2\|(a,\overline{a},b,\overline{b})\|_s)^3 \\ &\leq \left(8C\varepsilon^2 s\right) \|(a,\overline{a},b,\overline{b})\|_s \end{aligned}$$

For ε small enough, we have $8C\varepsilon^2 \leq 1$. Thus we obtain

$$\|\varphi_s(a,\overline{a},b,\overline{b})\|_s \leq (1+s)\|(a,\overline{a},b,\overline{b})\|_s.$$

This bound is satisfied as soon as

$$\|\varphi_s(a,\overline{a},b,\overline{b}) - (a,\overline{a},b,\overline{b})\|_s \leq 2\|(a,\overline{a},b,\overline{b})\|_s \quad \text{and} \quad s \leq T(a,\overline{a},b,\overline{b}).$$

Therefore, by continuity, we have $T(a,\overline{a},b,\overline{b}) \geq 1$ and

bootstrap

$$\forall s \in [0,1], \quad \|\varphi_s(a,\overline{a},b,\overline{b})\|_s \leq 2\|(a,\overline{a},b,\overline{b})\|_s. \tag{3.8}$$

The fact that $T(a,\overline{a},b,\overline{b}) \geq 1$ implies that $\tau = \varphi_1$ is well defined. Thus, writing the equation (3.7) for $s=1$ and using the bound (3.8), we obtain

$$\|\tau(a,\overline{a},b,\overline{b}) - (a,\overline{a},b,\overline{b})\|_s \lesssim \left\|(a,\overline{a},b,\overline{b})\right\|_s^3.$$

Moreover τ preserves the class $\mathcal{T}^s(\mu)$ for any $\mu \leq 1$ as a consequence of the formula $\nabla_z(f\circ\tau)(z) = (D\tau(z))^* \nabla f \circ \tau(z)$ and the fact that $D\tau(z)$ maps Z_s into Z_s. On the other hand $\{\chi_4, \mathbb{M}\} = \{\chi_4, \mathbb{L}\} = 0$, therefore the flow $\tau = \varphi_1$ of χ_4 conserves the mass and the momentum: $\mathbb{M}(\tau(a,\overline{a},b,\overline{b})) = \mathbb{M}(a,\overline{a},b,\overline{b})$

and $\mathbb{L}(\tau(a,\bar{a},b,\bar{b})) = \mathbb{L}(a,\bar{a},b,\bar{b})$. We have thus proved the forth point of the proposition.

We recall that by construction, the remainder term R_6 is

$$R_6 := \{P_4, \chi_4\} + \int_0^1 (1-t)\{\{H,\chi_4\},\chi_4\} \circ \varphi_t \mathrm{d}t.$$

The polynomials P_4, χ_4 and Z_4 have bounded coefficients and have the prescribed form (3.2). Therefore, using the homological equation (3.5), the same is true for the polynomials Q_1 and Q_2 defined by

$$\begin{cases} Q_1 := \{P_4, \chi_4\}, \\ Q_2 := \{\{H,\chi_4\},\chi_4\} = \{Z_4,\chi_4\} - \{P_4,\chi_4\} + \{\{P_4,\chi_4\},\chi_4\}. \end{cases}$$

Thus using Lemma 3.1 we conclude that $R_6 \in \mathcal{T}^s(1)$.
Moreover, the polynomials P_4, χ_4 and Z_4 are of order 4. Thus, the polynomials Q_1 and Q_2 are of order at least 6. Therefore, the computations of Lemma 3.1 give

$$\left\|X_{Q_1}(a,\overline{a},b,\overline{b})\right\|_s \lesssim \left\|(a,\overline{a},b,\overline{b})\right\|_s^5 \text{ and } \left\|X_{Q_2}(a,\overline{a},b,\overline{b})\right\|_s \lesssim \left\|(a,\overline{a},b,\overline{b})\right\|_s^5.$$

Finally, for ε small enough and $(a,\overline{a},b,\overline{b}) \in B_s(\varepsilon)$, the estimate (3.8) implies

$$\left\|X_{R_6}(a,\overline{a},b,\overline{b})\right\|_s \lesssim \left\|(a,\overline{a},b,\overline{b})\right\|_s^5.$$

On the other hand since τ conserves the mass and the momentum, $R_6 = H \circ \tau - P_2 - Z_4$ commutes with $\mathbb{L}$ and $\mathbb{M}$. The proof of the point (iii) and thus the proof of the proposition is completed. □

secappli

4. Two applications of KAM for the coupled NLS system

We aim to use Theorem 2.6 in order to study the extended coupled Schrödinger systems:

sysgen

$$\begin{cases} i\partial_t u + \partial_{xx} u = |v|^2 u + \dfrac{\partial g}{\partial \overline{v}}(u,\overline{u},v,\overline{v}), & (t,x) \in \mathbb{R}\times\mathbb{T}, \\ i\partial_t v + \partial_{xx} v = |u|^2 v + \dfrac{\partial g}{\partial \overline{u}}(u,\overline{u},v,\overline{v}), \end{cases} \tag{4.1}$$

where g is a real[5] analytic function on a neighborhood of the origin in $\mathbb{C}^4$ and g is of order at least 5 in $(u,\overline{u},v,\overline{v})$. We set

$$R_5(a,\bar{a},b,\bar{b}) = \int_{\mathbb{T}} g(\sum_{\mathbb{Z}} a_j e^{ijx}, \sum_{\mathbb{Z}} \bar{a}_j e^{-ijx}, \sum_{\mathbb{Z}} b_j e^{ijx}, \sum_{\mathbb{Z}} \bar{b}_j e^{-ijx}) dx$$

and we note that, in view of example 2.3 , $R_5 \in \mathcal{T}^s(\mu)$ for some $\mu > 0$. We also notice that R_5 commutes with $\mathbb{L}$ and $\mathbb{M}$. With the notations introduced in (1.2), the Hamiltonian of the system is thus given by

defH

$$H = P_2 + P_4 + R_5. \tag{4.2}$$

After the application of the Birkhoff normal form of Proposition 3.2, we have

defHB

$$H^B = H \circ \tau = P_2 + Z_4 + R_5 \circ \tau + R_6, \tag{4.3}$$

[5]Here real means that $g(z_1,\overline{z_1},z_2,\overline{z_2}) \in \mathbb{R}$.

where we recall that

$$Z_4(a,\overline{a},b,\overline{b}) = \sum_{\{k,l\}=\{i,j\}} a_k b_l \overline{a}_i \overline{b}_j.$$

In order to apply the KAM result, the goal is to study the resonant term Z_4 to determine which part of Z_4 is an effective part, and which part can be treated as a remainder term. Due to the assumptions of Theorem 2.6, the idea is to put the jetless part of Z_4 in the remainder term and to consider the jet part as the effective part of the Hamiltonian.

The idea is to consider the following solution of the linear system:

sollin

$$u(x,t) = a_p e^{ipx} e^{-ip^2 t}, \quad v(x,t) = b_q e^{iqx} e^{-iq^2 t}. \tag{4.4}$$

We introduce the constants ρ_i, the variables x_i and θ_i, the actions I_i $(i=1,2)$ and the variables ζ_α, $\alpha \in \mathcal{Z}$ defined by

defIthetazeta

$$\begin{cases} a_p(t) = (\nu\rho_1 + x_1(t))^{\frac{1}{2}} e^{i\theta_1(t)} =: \sqrt{I_1} e^{i\theta_1(t)}, \\ b_q(t) = (\nu\rho_2 + x_2(t))^{\frac{1}{2}} e^{i\theta_2(t)} =: \sqrt{I_2} e^{i\theta_1(t)}, \\ a_k(t) = \zeta_{k+}(t), \quad k \neq p, \\ b_k(t) = \zeta_{k-}(t), \quad k \neq q, \end{cases} \tag{4.5}$$

where $(\rho_1,\rho_2) \in [1,2]^2$ and ν is a small parameter which controls the size of the solution. The canonical symplectic structure $-i(\mathrm{d}u \wedge \mathrm{d}\overline{u} + \mathrm{d}v \wedge \mathrm{d}\overline{v})$ becomes

$$-\mathrm{d}I \wedge \mathrm{d}\theta - i\mathrm{d}\zeta \wedge \mathrm{d}\overline{\zeta} = -\mathrm{d}x \wedge \mathrm{d}\theta - i\mathrm{d}\zeta \wedge \mathrm{d}\overline{\zeta}.$$

We want to study the linear stability of the torus $\mathbf{T}_\rho^{lin}$ defined by the solution (4.4) of the linear system with $x=0$, θ real and $z=0$ (where we recall that $z=(\zeta,\bar{\zeta})$). This torus can be written as

$$\mathbf{T}_\rho^{lin} := \left\{ (I,\theta,z) \,|\, I = \nu\rho = (\nu\rho_1, \nu\rho_2), \theta \in \mathbb{R}^2/2\pi\mathbb{Z}^2, z = 0 \right\}.$$

For that purpose, we introduce a toroidal neighborhood of the torus $\mathbf{T}_\rho^{lin}$ by

deftaurho

$$\mathbf{T}_\rho(\nu,\sigma,\mu,s) := \left\{ (I,\theta,z) \,|\, |I-\nu\rho| < \nu\mu^2, |\Im\theta| < \sigma, \|z\|_s < \nu^{\frac{1}{2}}\mu \right\}. \tag{4.6}$$

Thanks to a translation in actions between x and $I = \nu\rho + x$, we make an analogy between the toroidal neighborhood $\mathbf{T}_\rho(\nu,\sigma,\mu,s)$ of the torus $\mathbf{T}_\rho^{lin}$ and the neighborhood $\mathcal{O}^s(\sigma,\nu^{\frac{1}{2}}\mu)$ of the origin. Indeed we have

$$\mathbf{T}_\rho(\nu,\sigma,\mu,s) \approx \mathcal{O}^s(\sigma,\nu^{\frac{1}{2}}\mu) = \left\{ (x,\theta,z) \,|\, |x| < \nu\mu^2, |\Im\theta| < \sigma, \|z\|_s < \nu^{\frac{1}{2}}\mu \right\}.$$

As a remark, we see that $\mathbf{T}_\rho(\nu,\sigma,\mu,s)$ is a small neighborhood of $\mathbf{T}_\rho^{lin}$ in the following sense:

$$\lim_{\sigma,\mu\to 0} (\mathbf{T}_\rho(\nu,\sigma,\mu,s)) = \mathbf{T}_\rho^{lin}.$$

Through the two following subsections, we study two examples of application of the KAM theorem. For that purpose, we highlight in Subsection 4.1 some unstable linear tori (corresponding to the case $p \neq q$), whereas we highlight in Subsection 4.2 some stable ones (corresponding to the case $p = q$).

subsecunstable

4.1. **1st case: emphasizing an unstable linear torus.** We consider the case $p \neq q$ and thus the following solution of the linear system:

sollin1 (4.7) $$u(x,t) = a_p e^{ipx} e^{-ip^2 t}, \quad v(x,t) = b_q e^{iqx} e^{-iq^2 t}, \quad \text{with} \quad p \neq q.$$

We use the clustering defined in Example 2.2. The goal of this section is to prove the following:

thmuns **Theorem 4.1.** *There exist $\nu_0 > 0$, $\sigma_0 > 0$ and $\mu_0 > 0$ such that, for $s > \frac{1}{2}$, $0 < \nu \leq \nu_0$, $0 < \sigma \leq \sigma_0$, $0 < \mu \leq \mu_0$ and $\rho \in \mathcal{D}$*

(i) There exist

$$\Phi_\rho : \begin{pmatrix} \mathcal{O}^s(\frac{1}{2}, \frac{e^{-\frac{1}{2}}}{2}) \to & \mathbf{T}_\rho(\nu, 1, 1, s) \\ (r, \theta, z) \mapsto & (I, \theta, z') \end{pmatrix}$$

real holomorphic transformations, analytically depending on ρ, which transform the symplectic structure $-\mathrm{d}r \wedge \mathrm{d}\theta - i\mathrm{d}\zeta \wedge \mathrm{d}\overline{\zeta}$ on $\mathcal{O}^s(\frac{1}{2}, \frac{e^{-\frac{1}{2}}}{2})$ to the symplectic structure $-\nu \mathrm{d}I \wedge \mathrm{d}\theta - i\nu \mathrm{d}\zeta' \wedge \mathrm{d}\overline{\zeta'}$ on $\mathbf{T}_\rho(\nu, 1, 1, s)$. The change of variables Φ_ρ is close to the scaling by the factor $\nu^{\frac{1}{2}}$ on the $\mathcal{L}$-modes but not on the $\mathcal{F}$-modes, where it is close to a certain affine transformation depending on θ.

(ii) Φ_ρ puts the Hamiltonian $H = P_2 + P_4 + R_5$ in normal form in the following sense:

H1 (4.8) $$\tilde{H} = \frac{1}{\nu} \left(H \circ \Phi_\rho - C \right)(r, \theta, z) = h_0(r, z) + f(r, \theta, z),$$

where $C = \nu^2 \rho_1 \rho_2 + \nu p^2 \rho_1 + \nu q^2 \rho_2$ is a constant and the effective part h_0 of the Hamiltonian reads

h01 (4.9) $$h_0 = \Omega(\rho) \cdot r + \sum_{j \neq p,q} \left(\Lambda_j^a(\rho) |a_j|^2 + \Lambda_j^b(\rho) |b_j|^2 \right) + \frac{1}{2} \langle z_f, K(\rho) z_f \rangle.$$

The frequencies Ω_ρ are given by

$$\Omega(\rho) = \begin{pmatrix} p^2 + \nu\rho_2 \\ q^2 + \nu\rho_1 \end{pmatrix},$$

the eigenvalues Λ_j^a and Λ_j^b are defined by

$$\Lambda_j^a(\rho) = j^2 + \nu\rho_2 \quad \text{and} \quad \Lambda_j^b(\rho) = j^2 + \nu\rho_1,$$

and the symmetric real matrix $K(\rho)$, acting on the four external modes $z_f := \Phi_\rho \begin{pmatrix} b_p \\ a_q \\ \overline{b}_p \\ \overline{a}_q \end{pmatrix}$, is given by

$$K(\rho) = \begin{pmatrix} 0 & \nu\sqrt{\rho_1\rho_2} & p^2 - q^2 & 0 \\ \nu\sqrt{\rho_1\rho_2} & 0 & 0 & q^2 - p^2 \\ p^2 - q^2 & 0 & 0 & \nu\sqrt{\rho_1\rho_2} \\ 0 & q^2 - p^2 & \nu\sqrt{\rho_1\rho_2} & 0 \end{pmatrix}.$$

(iii) The remainder term f belongs to $\mathcal{T}^s(\sigma, \mu, \mathcal{D})$ and satisfies

$$[f]^s_{\sigma,\mu,\mathcal{D}} \lesssim \nu \quad \text{and} \quad [f^T]^s_{\sigma,\mu,\mathcal{D}} \lesssim \nu^{\frac{3}{2}}.$$

4.1.1. *Set up of the change of variables Φ_ρ.* The construction of the change of variables Φ_ρ is decomposed in two steps. First we eliminate the angles and then we rescale the variables.

4.1.1.1. *Structure of the Hamiltonian and elimination of the angles.* We study the Hamiltonian H^B, defined in equation (4.3), obtained after the Birkhoff normal form iteration:

$$H^B = H \circ \tau = P_2 + Z_4 + R_5 \circ \tau + R_6.$$

Here, the term P_2 already contributes to the desired effective Hamiltonian h_0 and the constant term C of Theorem 4.1, whereas the terms $R_5 \circ \tau$ and R_6 contribute to the remainder term f. Therefore, we just have to deal with the resonant term Z_4, separating the constant, the effective and the remainder parts. We split the polynomial Z_4 according to the number of inner modes a_p and b_q. The term of order 4 in a_p, b_q from Z_4 is given by

$$\begin{aligned} Z_{4,4} := |a_p|^2|b_q|^2 &= (\nu\rho_1 + x_1)(\nu\rho_2 + x_2) \\ &= \underbrace{\nu^2\rho_1\rho_2}_{\text{constant}} + \underbrace{\nu\rho_2 x_1 + \nu\rho_1 x_2}_{\text{effective part}} + \underbrace{x_1 x_2}_{\text{remainder}} . \end{aligned}$$

Due to the structure of the resonant set, there is no term of order 3 in a_p, b_q in Z_4. The terms of order 2 in a_p, b_q from Z_4 are

$$Z_{4,2} := |a_p|^2 \sum_{k\neq q} |b_k|^2 + |b_q|^2 \sum_{k\neq p} |a_k|^2 + a_p b_q \overline{a}_q \overline{b}_p + \overline{a}_p \overline{b}_q a_q b_p.$$

For the first term (the same goes for the second one), we write

$$|a_p|^2 \sum_{k\neq q} |b_k|^2 = \underbrace{\nu\rho_1 \sum_{k\neq q} |b_k|^2}_{\text{effective part}} + \underbrace{x_1 \sum_{k\neq q} |b_k|^2}_{\text{remainder}} .$$

The study of the two last terms is trickier because in each term there is four different modes and thus there are angles. In order to split these terms between effective and remainder terms, we recall that we want in Theorem 4.1 the remainder term f to be small and to have a jet smaller. Therefore, the idea is to keep the jet part of these terms in the effective part and to put the other part (without jet) in the remainder part. We obtain for example for the term $a_p b_q \overline{a}_q \overline{b}_p$:

$$\underbrace{\nu\sqrt{\rho_1\rho_2} e^{i(\theta_1+\theta_2)} \overline{a}_q \overline{b}_p}_{\text{effective part}} + \underbrace{\left(\sqrt{(\nu\rho_1 + x_1)(\nu\rho_2 + x_2)} - \nu\sqrt{\rho_1\rho_2} \right) e^{i(\theta_1+\theta_2)} \overline{a}_q \overline{b}_p}_{\text{remainder (jetless part)}} .$$

The effective part of $Z_{4,2}$ is thus given by

$$Z^e_{4,2} := \nu\big(\rho_1 \sum_{k\neq q} |b_k|^2 + \rho_2 \sum_{k\neq p} |a_k|^2 + \sqrt{\rho_1\rho_2}(e^{i(\theta_1+\theta_2)} \overline{a}_q \overline{b}_p + e^{-i(\theta_1+\theta_2)} a_q b_p)\big).$$

Then, in order to kill the angles, we introduce the symplectic change of variables

$$\Psi_{ang}\left(x, \theta, z = (a, b)\right) = \left(y, \theta, z' = (c, d)\right),$$

where the new variables c, d and y are defined by

defpsiang
$$\begin{cases} c_q = a_q e^{-i\theta_1}, & c_k = a_k,\ k \neq p, q, \\ d_p = b_p e^{-i\theta_2}, & d_k = b_k,\ k \neq p, q, \\ y_1 = x_1 + |a_q|^2, & y_2 = x_2 + |b_p|^2. \end{cases} \tag{4.10}$$

This change of variables Ψ_{ang} is the reason why the change of variable Φ_ρ of Theorem 4.1 is not close to a scaling for the $\mathcal{F}$-modes a_q and b_p. Finally, the terms $Z_{4,1}$ and $Z_{4,0}$ of order 1 and 0 in a_p, b_q from Z_4 are remainder terms. We can thus write the Hamiltonian $\tilde{H} = H^B \circ \Psi_{ang}$ as

defhtilde
$$\tilde{H} = H^B \circ \Psi_{ang} = C + H^e + R, \tag{4.11}$$

where the constant part C is given by

$$C = \nu^2 \rho_1 \rho_2 + \nu p^2 \rho_1 + \nu q^2 \rho_2,$$

the remainder term R is defined by

$$\begin{aligned} R =& Z_{4,0} \circ \Psi_{ang} + Z_{4,1} \circ \Psi_{ang} + (y_1 - |c_q|^2) \sum_{k \neq q} |d_k|^2 + (y_2 - |d_p|^2) \sum_{k \neq p} |c_k|^2 \\ &+ \left(\sqrt{(\nu\rho_1 + y_1 - |c_q|^2)(\nu\rho_2 + y_2 - |d_p|^2)} - \nu\sqrt{\rho_1\rho_2} \right) (\overline{c}_q \overline{d}_p + c_q d_p) \\ &+ (y_1 - |c_q|^2)(y_2 - |d_p|^2) + R_5 \circ \tau \circ \Psi_{ang} + R_6 \circ \Psi_{ang}, \end{aligned}$$

and the effective Hamiltonian H^e reads

$$\begin{aligned} H^e =& (p^2 + \nu\rho_2) y_1 + (q^2 + \nu\rho_1) y_2 + (p^2 - q^2)|d_p|^2 + (q^2 - p^2)|c_q|^2 \\ &+ \nu\sqrt{\rho_1\rho_2}(\overline{c}_q \overline{d}_p + c_q d_p) + \sum_{k \neq p,q} (k^2 + \nu\rho_2)|c_k|^2 + (k^2 + \nu\rho_1)|d_k|^2. \end{aligned}$$

The new frequencies are thus given by

$$\Omega(\rho) = \begin{pmatrix} p^2 + \nu\rho_2 \\ q^2 + \nu\rho_1 \end{pmatrix}.$$

This expression of the Hamiltonian $\tilde{H}$ is really close to the desired one of Theorem 4.1. In order to control the size of the remainder term, the idea is now to rescale the variables c, d and y.

4.1.1.2. *Rescaling of the variables and introduction of* Φ_ρ. In order to study the initial Hamiltonian H on the torus $\mathbf{T}_\rho(\nu, \frac{1}{2}, \frac{e^{-\frac{1}{2}}}{2}, s)$, we introduce the rescaling of the variables y and z by the change of variables

$$\chi_\rho\left(y, \theta, z'\right) = (r, \theta, z),$$

where

$$(r, \theta, z) := (\nu y, \theta, \nu^{\frac{1}{2}} z').$$

The symplectic structure becomes

$$-\nu \mathrm{d}r \wedge \mathrm{d}\theta - i\nu \mathrm{d}\zeta \wedge \mathrm{d}\overline{\zeta}.$$

By definition, the change of variables χ_ρ sends the neighborhood $\mathcal{O}^s(\sigma, \mu)$ of the origin into a toroidal neighborhood of $\mathbf{T}^{lin}_\rho$:

$$\chi_\rho\left(\mathcal{O}^s(\sigma, \mu)\right) = \mathbf{T}_\rho(\nu, \sigma, \mu, s), \quad \forall \sigma, \mu, \nu, s > 0.$$

We can now define the change of variable Φ_ρ of Theorem 4.1 by:

defphirho
$$\Phi_\rho := \tau \circ \Psi_{ang} \circ \chi_\rho. \tag{4.12}$$

Thanks to this definition, we are now able to prove Theorem 4.1.

4.1.2. *Proof of Theorem 4.1.* First, we can check the first point of the theorem by proving that $\Phi_\rho\left(\mathcal{O}^s(\frac{1}{2},\frac{e^{-\frac{1}{2}}}{2})\right) \subset \mathbf{T}_\rho(\nu,1,1,s)$. By definition of the rescaling χ_ρ, we have

$$\chi_\rho\left(\mathcal{O}^s(\frac{1}{2},\frac{e^{-\frac{1}{2}}}{2})\right) = \mathbf{T}_\rho(\nu,\frac{1}{2},\frac{e^{-\frac{1}{2}}}{2},s).$$

Applying the change of variable Ψ_{ang} (see (4.10)), we obtain

$$\Psi_{ang}\circ\chi_\rho\left(\mathcal{O}^s(\frac{1}{2},\frac{e^{-\frac{1}{2}}}{2})\right) \subset \mathbf{T}_\rho(\nu,\frac{1}{2},\frac{1}{2},s).$$

Thus, applying the Birkhoff change of variables τ which is close to the identity (see Theorem 3.2), we obtain

$$\tau\circ\Psi_{ang}\circ\chi_\rho\left(\mathcal{O}^s(\frac{1}{2},\frac{e^{-\frac{1}{2}}}{2})\right) = \Phi_\rho\left(\mathcal{O}^s(\frac{1}{2},\frac{e^{-\frac{1}{2}}}{2})\right) \subset \mathbf{T}_\rho(\nu,1,1,s).$$

Then, we have to check the structure of the Hamiltonian $H\circ\Phi_\rho$. For that purpose, we start from the Hamiltonian $\tilde{H} = H\circ\tau\circ\Psi_{ang}$ defined in (4.11), and we write

$$H\circ\Phi_\rho = \tilde{H}\circ\chi_\rho = C + \nu h_0 + \nu f,$$

where h_0 and f are defined by

$$h_0 := \frac{1}{\nu}H^e\circ\chi_\rho \quad \text{and} \quad f := \frac{1}{\nu}R\circ\chi_\rho.$$

By construction, h_0 satisfies the properties of the point (ii) of the theorem.

For the study of f, we first need to show that for σ and μ small enough, we have $f \in \mathcal{T}^s(\sigma,\mu,\mathcal{D})$. We recall that R is defined in the previous subsubsection. From the definition of R, we write

decompo f
$$f = f_Z + f_e + f_5 + f_6, \tag{4.13}$$

where f_Z is the part of f that contains the terms $Z_{4,0}$ and $Z_{4,1}$, f_e is the explicit part of f and f_5 (respectively f_6) is the part with the term R_5 (respectively R_6). We remark that for all these terms, the explicit changes of variables Ψ_{ang} and $\frac{1}{\nu}\chi_\rho$ don't play a role here. Applying Lemma 3.1, we first have $f_Z \in \mathcal{T}^s(1,1,\mathcal{D})$. For the explicit part f_e, it is straightforward to check that $\nabla_z f(r,\theta,z,\rho) \in Z_s\times Z_s$ as soon as $z \in Z_s\times Z_s$. Therefore, we have $f_e \in \mathcal{T}^s(1,1,\mathcal{D})$ too. Now, by the third point of the Birkhoff normal form Propositon 3.2, we also have $f_6 \in \mathcal{T}^s(1,1,\mathcal{D})$. Using once again Propositon 3.2 for the behavior of the change of variable τ (forth point) and Example 2.3, there exists $\sigma_0 > 0$ and $\mu_0 > 0$ such that $f_5 \in \mathcal{T}^s(\sigma,\mu,\mathcal{D})$ for $0 < \sigma \le \sigma_0$ and $0 < \mu \le \mu_0$. Finally, we obtain

$$f \in \mathcal{T}^s(\sigma,\mu,\mathcal{D}) \quad \text{for} \quad 0<\sigma\le\sigma_0 \quad \text{and} \quad 0<\mu\le\mu_0.$$

We fix now $0 < \sigma \le \sigma_0$ and $0 < \mu \le \mu_0$.

Then, for the estimates on the norms of f, we remark that $R\circ\chi_\rho$ contains only terms of order at least 2 in ν (for example, $Z_{4,0}\circ\Psi_{ang}\circ\chi_\rho$ and all the other terms, except those with R_5 or R_6 which are smaller, are of order ν^2), thus we do have

$$[f]^s_{\sigma,\mu,\mathcal{D}} \lesssim \nu.$$

Finally, for the estimate on the jet part of f, we first remark that by construction [6],

$$f^T = \frac{1}{\nu}(R_5\circ\tau\circ\Psi_{ang}\circ\chi_\rho)^T + \frac{1}{\nu}(R_6\circ\Psi_{ang}\circ\chi_\rho)^T.$$

Therefore, using that R_5 and R_6 are of order 5 and 6, we have

$$[f^T]^s_{\sigma,\mu,\mathcal{D}} \lesssim \nu^{\frac{3}{2}}.$$

□

4.1.3. *Study of the $\mathcal{F}$-modes.* Assume that the hypothesis A0, A1 and A2 are satisfied (see Appendix A), we can apply Theorem 2.6. To study the linear stability of the torus $\Phi(\mathbf{T}^{lin})$, with Φ defined in Theorem 2.6, we thus have to check whether there exists hyperbolic directions or not. In the effective Hamiltonian h_0 defined in Theorem 4.1, the modes a_j and b_j ($j\neq p,q$) are elliptic modes and thus don't have influence on this linear stability. Therefore, in order to study the linear stability of the torus $\Phi(\mathbf{T}^{lin})$, we have to study the semi-external modes (d_p,c_q) to determine if they are elliptic or hyperbolic modes. The variables $(d_p,c_q,\overline{d}_p,\overline{c}_q)$ satisfy

$$\begin{pmatrix}\dot{d}_p\\ \dot{c}_q\\ \dot{\overline{d}}_p\\ \dot{\overline{c}}_q\end{pmatrix} = -i\begin{pmatrix}\frac{\partial h_0}{\partial \overline{d}_p}\\ \frac{\partial h_0}{\partial \overline{c}_q}\\ -\frac{\partial h_0}{\partial d_p}\\ -\frac{\partial h_0}{\partial c_q}\end{pmatrix} = M\begin{pmatrix}d_p\\ c_q\\ \overline{d}_p\\ \overline{c}_q\end{pmatrix},$$

where we denote by $M=-iJK(\rho)$ the matrix

$$M := -i\begin{pmatrix}p^2-q^2 & 0 & 0 & \nu\sqrt{\rho_1\rho_2}\\ 0 & q^2-p^2 & \nu\sqrt{\rho_1\rho_2} & 0\\ 0 & -\nu\sqrt{\rho_1\rho_2} & q^2-p^2 & 0\\ -\nu\sqrt{\rho_1\rho_2} & 0 & 0 & p^2-q^2\end{pmatrix}.$$

A straightforward computation shows that the characteristic polynomial χ_M is

$$\chi_M(\lambda) = \left((\lambda-i(p^2-q^2))^2-\nu^2\rho_1\rho_2\right)\left((\lambda+i(p^2-q^2))^2-\nu^2\rho_1\rho_2\right).$$

Thus, the eigenvalues of M are

112 (4.14) $$\lambda_{1,2} = i(p^2-q^2)\pm\nu\sqrt{\rho_1\rho_2},\qquad \lambda_{3,4} = -i(p^2-q^2)\pm\nu\sqrt{\rho_1\rho_2}.$$

Therefore, we have four eigenvalues λ_i, $i=1..4$ which satisfy

$$\Re(\lambda_i)\neq 0,\quad \text{for}\quad i=1\dots 4.$$

For $p\neq -q$, we have four different eigenvalues and we can diagonalize the matrix M. For $p=-q$, we have two double eigenvalues: $\pm\nu\sqrt{\rho_1\rho_2}$ but

[6] Here we use that the Hamiltonian $G = (\sqrt{(\rho_1+y_1-|c_q|^2)(\rho_2+y_2-|d_p|^2)}-\sqrt{\rho_1\rho_2})c_qd_p$ satisfies $G^T=0$.

the matrix M is still diagonalizable (we can check by example that the two vectors ${}^t(1,0,0,i)$ and ${}^t(0,1,-i,0)$ are eigenvectors associated to the eigenvalue $\nu\sqrt{\rho_1\rho_2}$, another way to prove this is to use the fact that all real skew matrix is diagonalizable with purely imaginary eigenvalues). Therefore, in both cases, we have two hyperbolic directions, this implies the linear instability of the torus $\mathcal{T} := \{y = 0\} \times \mathbb{T}^2 \times \{z = 0\}$.

Remark 4.2. The real part of the eigenvalues of M does not depend on the choice of the modes $p \neq q$. Moreover, the case $p = -q$ gives real eigenvalues.

4.1.4. *Structure of h_0.* In order to apply the KAM theorem, let us see that we can write the Hamiltonian h_0 with the normal structure (2.1). By equation (4.9), we have

$$h_0 = \Omega(\rho)\cdot r + \sum_{j\neq p,q}\Big(\Lambda_j^a(\rho)|a_j|^2 + \Lambda_j^b(\rho)|b_j|^2\Big) + \frac{1}{2}\langle z_f, K(\rho)z_f\rangle.$$

We use the notations introduced in Example 2.2. First we remark for the elliptic part that

$$\sum_{j\neq p,q}\Big(\Lambda_j^a(\rho)|a_j|^2 + \Lambda_j^b(\rho)|b_j|^2\Big) = \sum_{\alpha\in\mathcal{L}}\Lambda_\alpha(\rho)|\zeta_\alpha|^2,$$

where

$$\Lambda_{j\pm}(\rho) = j^2 + \nu\rho_\pm \text{ with } \rho_+ = \rho_2 \text{ and } \rho_- = \rho_1.$$

For the part related to the matrix $K(\rho)$, the study of the matrix M in the previous subsection and the eigenvectors associated to the eigenvalues computed in (4.14) suggest the introduction of the following symplectic change of variables

def_ef

$$\begin{cases} \zeta_e = \dfrac{1}{\sqrt{2}}(c_q + i\bar{d}_p), & \bar{\zeta}_e = \dfrac{1}{\sqrt{2}}(\bar{c}_q + id_p), \\ \zeta_f = \dfrac{1}{\sqrt{2}}(d_p + i\bar{c}_q), & \bar{\zeta}_f = \dfrac{1}{\sqrt{2}}(\bar{d}_p + ic_q), \end{cases} \tag{4.15}$$

where the variables (d_p, c_q) are defined in (4.10). We remark here that $\bar{\zeta}_e$ and $\bar{\zeta}_f$ are not the complex conjugates of ζ_e and ζ_f, but the Hamiltonian dual variables of ζ_e and ζ_f in the following sense (for example for ζ_e):

$$\partial_t\zeta_e = -i\frac{\partial h_0}{\bar{\zeta}_e}, \qquad \partial_t\bar{\zeta}_e = i\frac{\partial h_0}{\zeta_e}.$$

In the new variables, we have

$$\begin{aligned}\frac{1}{2}\langle z_f, K(\rho)z_f\rangle :&= (p^2-q^2)(|d_p|^2 - |c_q|^2) + \nu\sqrt{\rho_1\rho_2}(\bar{c}_q\bar{d}_p + c_qd_p)\\ &= (q^2-p^2-i\nu\sqrt{\rho_1\rho_2})|\zeta_e|^2 + (p^2-q^2-i\nu\sqrt{\rho_1\rho_2})|\zeta_f|^2.\end{aligned}$$

Finally, we can write

h01nf

$$h_0 = \Omega(\rho)\cdot r + \sum_{\alpha\in\mathcal{Z}=\mathcal{L}\cup\mathcal{F}}\Lambda_\alpha(\rho)|\zeta_\alpha|^2, \tag{4.16}$$

where $\Omega(\rho) = \begin{pmatrix} p^2 + \nu\rho_2 \\ q^2 + \nu\rho_1 \end{pmatrix}$, ζ_α is defined by (4.5) for $\alpha \in \mathcal{L} = \mathbb{Z}\backslash\{p,q\}\times\{\pm\}$, ζ_α is defined by (4.15) for $\alpha \in \mathcal{F} = \{e, f\}$, and

$$\begin{cases} \Lambda_\alpha(\rho) & = j^2 + \nu\rho_\pm \text{ for } \alpha = (j,\pm) \in \mathcal{L} \text{ with } \rho_+ = \rho_2 \text{ and } \rho_- = \rho_1, \\ \Lambda_\alpha(\rho) & = \begin{cases} q^2 - p^2 - i\nu\sqrt{\rho_1\rho_2} \text{ for } \alpha = e \in \mathcal{F}, \\ p^2 - q^2 - i\nu\sqrt{\rho_1\rho_2} \text{ for } \alpha = f \in \mathcal{F}. \end{cases} \end{cases}$$

subsubAppli

4.1.5. *Application of our KAM result.* To apply Theorem 2.6 to the Hamiltonian $\tilde{H}$ given by (4.8) it remains to verify Hypothesis A0, A1 and A2 for h_0 given by (4.16). This is done in Appendix A where we prove that Hypothesis A1 and A2 are satisfied for $\delta = (\sqrt{2}-1)\nu$. Therefore, we obtain

KAMunstable **Theorem 4.3.** *Fix $p \neq q$. There exists $\nu_0 > 0$ and for $0 < \nu < \nu_0$ there exists $\mathcal{C}_\nu \subset [1,2]^2$ asymptotically of full measure (i.e. $\lim_{\nu\to 0} \mathrm{meas}([1,2]^2 \setminus \mathcal{C}_\nu) = 0$) such that for $\rho \in \mathcal{C}_\nu$ the torus $\mathcal{T}_{\nu\rho} := \{|a_p|^2 = \nu\rho_1,\ |b_q|^2 = \nu\rho_2, \text{all other modes vanishing}\}$, which is invariant for the Hamiltonian flow associated to P_2, persists in slightly deformed way under the perturbation $P_4 + R_5$. Furthermore this invariant torus is linearly unstable.*

We can formulate our result in terms of small amplitude quasi periodic solutions. We notice that, in view of Theorem 4.1 we have $\varepsilon = [f^T]^s_{\frac{1}{2},\frac{e^{-\frac{1}{2}}}{2},\mathcal{D}} = \mathcal{O}(\nu^{3/2})$ therefore Theorem 1.1 is a corollary of Theorem 4.3.

subsecstable

4.2. **2nd case: emphasizing a stable linear torus.** Here, we study the missing case of the previous subsection: the case $p = q$. Therefore, we start with the following solution of the linear system:

sollin2 (4.17) $$u(x,t) = a_p e^{ipx} e^{-ip^2 t}, \quad v(x,t) = b_p e^{ipx} e^{-ip^2 t}.$$

This time, we use the clustering defined in Example 2.1. The goal of this section is to prove the following:

thmsta **Theorem 4.4.** *There exist $\nu_0 > 0$, $\sigma_0 > 0$ and $\mu_0 > 0$ such that, for $s > \frac{1}{2}$, $0 < \nu \leq \nu_0$, $0 < \sigma \leq \sigma_0$, $0 < \mu \leq \mu_0$ and $\rho \in \mathcal{D}$*

(i) There exist

$$\Phi_\rho : \begin{pmatrix} \mathcal{O}^s(\frac{1}{2}, \frac{e^{-\frac{1}{2}}}{2}) \to & \mathbf{T}_\rho(\nu,1,1,s) \\ (r,\theta,z) \mapsto & (I,\theta,z') \end{pmatrix}$$

real holomorphic transformations, analytically depending on ρ, which transform the symplectic structure $-\mathrm{d}r \wedge \mathrm{d}\theta - i\mathrm{d}\zeta \wedge \mathrm{d}\overline{\zeta}$ on $\mathcal{O}^s(\frac{1}{2}, \frac{e^{-\frac{1}{2}}}{2})$ to the symplectic structure $-\nu\mathrm{d}I \wedge \mathrm{d}\theta - i\nu\mathrm{d}\zeta' \wedge \mathrm{d}\overline{\zeta'}$ on $\mathbf{T}_\rho(\nu,1,1,s)$. The change of variables Φ_ρ is close to a certain affine transformation depending on θ.

(ii) Φ_ρ puts the Hamiltonian $H = P_2 + P_4 + R_5$ in normal form in the following sense:

H2 (4.18) $$\frac{1}{\nu}\left(H \circ \Phi_\rho - C\right)(r,\theta,z) = h_0(r,z) + f(r,\theta,z),$$

where $C = \nu^2\rho_1\rho_2 + \nu p^2(\rho_1+\rho_2)$ is a constant and the effective part h_0 of the Hamiltonian reads

h02 (4.19) $$h_0 = \Omega(\rho)\cdot r + \sum_{j\neq p}\left(\Lambda_j^a(\rho)|a_j|^2 + \Lambda_j^b(\rho)|b_j|^2\right).$$

The frequencies Ω_ρ are given by

$$\Omega(\rho) = \begin{pmatrix} p^2+\nu\rho_2 \\ p^2+\nu\rho_1 \end{pmatrix},$$

the eigenvalues Λ_j^a and Λ_j^b are defined by

$$\Lambda_j^a(\rho) = j^2 - p^2 + \nu\sqrt{\rho_1\rho_2} \quad \text{and} \quad \Lambda_j^b(\rho) = j^2 - p^2 - \nu\sqrt{\rho_1\rho_2}.$$

(iii) The remainder term f belongs to $\mathcal{T}^s(\sigma,\mu,\mathcal{D})$ and satisfies

$$[f]^s_{\sigma,\mu,\mathcal{D}} \lesssim \nu \quad \text{and} \quad [f^T]^s_{\sigma,\mu,\mathcal{D}} \lesssim \nu^{\frac{3}{2}}.$$

4.2.1. *Set up of the change of variables Φ_ρ.* As in the previous subsection, we have to eliminate the angles, and then to perform a rescaling of the variables.

4.2.1.1. *Structure of the Hamiltonian and elimination of the angles.* As the linear term P_2 contributes already to the constant part C and to the effective Hamiltonian h_0, the main difference with the study of two different modes is the behavior of the resonant term Z_4. We split the polynomial Z_4 according to the number of inner modes a_p and b_p. The term of order 4 in a_p, b_p from Z_4 is given by

$$Z_{4,4} := |a_p|^2|b_p|^2 = (\rho_1+x_1)(\rho_2+x_2) = \underbrace{\nu^2\rho_1\rho_2}_{\text{constant}} + \underbrace{\nu\rho_2 x_1 + \nu\rho_1 x_2}_{\text{effective part}} + \underbrace{x_1x_2}_{\text{remainder}}.$$

The terms of order 2 in a_p, b_p from Z_4 are

$$Z_{4,2} := |a_p|^2\sum_{k\neq p}|b_k|^2 + |b_p|^2\sum_{k\neq p}|a_k|^2 + a_p\bar{b}_p\sum_{k\neq p}\overline{a}_k b_k + \overline{a}_p b_p\sum_{k\neq p}a_k\bar{b}_k.$$

Separating the effective part (with jet) from the remainder part (without jet), we show that the effective part of $Z_{4,2}$ is given by

$$\begin{aligned} Z^e_{4,2} :=& \nu\rho_1\sum_{k\neq p}|b_k|^2 + \nu\rho_2\sum_{k\neq p}|a_k|^2 \\ &+ \nu\sqrt{\rho_1\rho_2}\left(e^{i(\theta_1+\theta_2)}\sum_{k\neq p}\overline{a}_k b_k + e^{-i(\theta_1+\theta_2)}\sum_{k\neq p}\overline{a}_p b_p\right). \end{aligned}$$

Then, in order to kill the angles, we introduce the symplectic change of variables

$$\Psi_{ang}(x,\theta,z=(a,b)) = \left(y,\theta,z'=(c,d)\right),$$

where the new variables c, d and y are defined by

defpsiang2 (4.20) $$\begin{cases} c_k &= a_k e^{-i\theta_1}, \qquad d_k = b_k e^{-i\theta_2}, \quad k\neq p, \\ y_1 &= x_1 + \sum_{k\neq p}|a_k|^2, \qquad y_2 = x_2 + \sum_{k\neq p}|b_k|^2. \end{cases}$$

Finally, the term $Z_{4,0}$ of order 0 in a_p, b_p from Z_4 is still a remainder term. We can thus write the Hamiltonian $\tilde{H} = H^B \circ \Psi_{ang}$ as

defhtilde2 (4.21) $$\tilde{H} = H^B \circ \Psi_{ang} = C + H^e + R,$$

where the constant part C is given by

$$C = \nu^2\rho_1\rho_2 + \nu p^2(\rho_1+\rho_2),$$

the remainder term R is defined by

$$\begin{aligned} R =& Z_{4,0}\circ\Psi_{ang} + (y_1 - \sum_{k\neq p}|c_k|^2)\sum_{k\neq q}|d_k|^2 + (y_2 - \sum_{k\neq p}|d_k|^2)\sum_{k\neq p}|c_k|^2 \\ &+ \alpha(\nu,\rho,z')\sum_{k\neq p}(\overline{c}_k d_k + \overline{d}_k c_k) + (y_1 - \sum_{k\neq p}|c_k|^2)(y_2 - \sum_{k\neq p}|d_k|^2) \\ &+ R_5\circ\tau\circ\Psi_{ang} + R_6\circ\Psi_{ang}, \end{aligned}$$

with

$$\alpha(\nu,\rho,z') = \sqrt{(\nu\rho_1 + y_1 - \sum_{k\neq p}|c_k|^2)(\nu\rho_2 + y_2 - \sum_{k\neq p}|d_k|^2)} - \nu\sqrt{\rho_1\rho_2},$$

and the effective Hamiltonian H^e reads

$$\begin{aligned} H^e =& (p^2+\nu\rho_2)y_1 + (p^2+\nu\rho_1)y_2 + \sum_{k\neq p}(k^2-p^2)(|c_k|^2+|d_k|^2) \\ &+ \nu\sqrt{\rho_1\rho_2}\left(\sum_{k\neq p}\overline{c_k}d_k + \sum_{k\neq p}c_k\overline{d_k}\right). \end{aligned}$$

The new frequencies are thus given by

$$\Omega(\rho) = \begin{pmatrix} p^2+\nu\rho_2 \\ p^2+\nu\rho_1 \end{pmatrix}.$$

The last term in H^e is not a diagonal term (*i.e* in $|c_k|^2$ and $|d_k|^2$). Nevertheless, due to its symmetries, the good idea is to introduce a new symplectic change of variables

$$\Psi_{sym}\left(I', \theta, z' = (c,d)\right) = \left(I', \theta, z'' = (e,f)\right),$$

where

$$e_k = \frac{c_k + d_k}{\sqrt{2}}, \quad f_k = \frac{c_k - d_k}{\sqrt{2}}.$$

A good way to see how this change of variables appears is to look at the equations satisfied by $(c_k, \bar{c}_k, d_k, \bar{d}_k)$ (for $k \neq p$). We have

$$i\begin{pmatrix} \dot{c}_k \\ \dot{\bar{c}}_k \\ \dot{d}_k \\ \dot{\bar{d}}_k \end{pmatrix} = \begin{pmatrix} \frac{\partial H^e}{\partial \bar{c}_k} \\ -\frac{\partial H^e}{\partial c_k} \\ \frac{\partial H^e}{\partial \bar{d}_k} \\ -\frac{\partial H^e}{\partial d_k} \end{pmatrix} = \begin{pmatrix} k^2-p^2 & 0 & \sqrt{\rho_1\rho_2} & 0 \\ 0 & p^2-k^2 & 0 & -\sqrt{\rho_1\rho_2} \\ \sqrt{\rho_1\rho_2} & 0 & k^2-p^2 & 0 \\ 0 & -\sqrt{\rho_1\rho_2} & 0 & p^2-k^2 \end{pmatrix} \begin{pmatrix} c_k \\ \overline{c}_k \\ d_k \\ \overline{d}_k \end{pmatrix}.$$

The diagonalization of the matrix M, defined by

$$M := -i \begin{pmatrix} k^2-p^2 & 0 & \sqrt{\rho_1\rho_2} & 0 \\ 0 & p^2-k^2 & 0 & -\sqrt{\rho_1\rho_2} \\ \sqrt{\rho_1\rho_2} & 0 & k^2-p^2 & 0 \\ 0 & -\sqrt{\rho_1\rho_2} & 0 & p^2-k^2 \end{pmatrix},$$

allows to introduce the variables $e_k = \frac{c_k+d_k}{\sqrt{2}}$ and $f_k = \frac{c_k-d_k}{\sqrt{2}}$. In these new variables, we obtain

$$\begin{aligned} H^e \circ \Psi_{sym} =& (p^2+\nu\rho_2)y_1 + (p^2+\nu\rho_1)y_2 + \sum_{k\neq p}(k^2-p^2+\nu\sqrt{\rho_1\rho_2})|e_k|^2 \\ &+ \sum_{k\neq p}(k^2-p^2-\nu\sqrt{\rho_1\rho_2})|f_k|^2. \end{aligned}$$

To finish the construction of the change of variables Φ_ρ, we just have now to rescale the variables e, f and y.

4.2.1.2. *Rescaling of the variables and introduction of* Φ_ρ. As in the previous case, we introduce the rescaling by the change of variables

$$\chi_\rho\left(r,\theta,z''\right) = (x,\theta,z),$$

where

$$(x,\theta,z) := (\nu r,\theta,\nu^{\frac{1}{2}}z'').$$

The symplectic structure becomes

$$-\nu \mathrm{d}r\wedge\mathrm{d}\theta - i\nu \mathrm{d}\zeta\wedge\mathrm{d}\overline{\zeta}.$$

By definition of χ_ρ we have

$$\chi_\rho\left(\mathcal{O}^s(\sigma,\mu)\right) = \mathbf{T}_\rho(\nu,\sigma,\mu,s).$$

We can now define the change of variable Φ_ρ of Theorem 4.4 by:

defphirho2 (4.22) $$\Phi_\rho := \tau\circ\Psi_{ang}\circ\Psi_{sym}\circ\chi_\rho.$$

Thanks to this definition, we are now able to prove Theorem 4.4.

4.2.2. *Proof of Theorem 4.4.* The only difference with the proof of Theorem 4.1 lies in the construction of the change of variables Φ_ρ. First, we can remark that the change of variables Ψ_{sym} doesn't change the norm. Indeed, writing $\Psi_{sym}\left(I',\theta,z'=(c,d)\right) = (I',\theta,z''=(e,f))$, we remark that

$$|c_k|^2+|d_k|^2 = |e_k|^2+|f_k|^2, \quad \forall k\neq p.$$

Thus, we have

$$\Psi_{sym}\circ\chi_\rho\left(\mathcal{O}^s(\frac{1}{2},\frac{e^{-\frac{1}{2}}}{2})\right) = \mathbf{T}_\rho(\nu,\frac{1}{2},\frac{e^{-\frac{1}{2}}}{2},s).$$

The change of variables Ψ_{ang} (see (4.20)) is constructed as in (4.10), and the change of variables τ from the Birkhoff normal form is the same as in the proof of Theorem 4.1. Therefore, we have

$$\tau\circ\Psi_{ang}\circ\Psi_{sym}\circ\chi_\rho\left(\mathcal{O}^s(\frac{1}{2},\frac{e^{-\frac{1}{2}}}{2})\right) = \Phi_\rho\left(\mathcal{O}^s(\frac{1}{2},\frac{e^{-\frac{1}{2}}}{2})\right) \subset \mathbf{T}_\rho(\nu,1,1,s).$$

Therefore, to conclude the proof of Theorem 4.4, it suffices to write

$$H \circ \Phi_\rho = \tilde{H} \circ \Psi_{sym} \circ \chi_\rho = C + \nu h_0 + \nu f,$$

with h_0 and f defined by

$$h_0 := \frac{1}{\nu} H^e \circ \Psi_{sym} \circ \chi_\rho \quad \text{and} \quad f := \frac{1}{\nu} R \circ \Psi_{sym} \circ \chi_\rho.$$

Thus defined, h_0 satisfies the point (ii) of the theorem. For the third point of the theorem about the term f, the same study as in previous case (thanks to a decomposition as in equation (4.13)) shows that there exists $\sigma_0 > 0$ and $\mu_0 > 0$ such that

$$f \in \mathcal{T}^s(\sigma, \mu, \mathcal{D}) \quad \text{for} \quad 0 < \sigma \leq \sigma_0 \quad \text{and} \quad 0 < \mu \leq \mu_0,$$

and for $0 < \sigma \leq \sigma_0$ and $0 < \mu \leq \mu_0$, we also have

$$[f]^s_{\sigma,\mu,\mathcal{D}} \lesssim \nu \quad \text{and} \quad [f^T]^s_{\sigma,\mu,\mathcal{D}} \lesssim \nu^{\frac{3}{2}}.$$

□

4.2.3. *Structure of h_0.* As $\mathcal{F} = \emptyset$, it's easy to write the Hamiltonian h_0 with the normal structure (2.1). By equation (4.19), we have

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$$h_0 = \Omega(\rho) \cdot r + \sum_{\alpha \in \mathcal{Z} = \mathcal{L}} \Lambda_\alpha(\rho) |\zeta_\alpha|^2, \tag{4.23}$$

where $\Omega(\rho) = \begin{pmatrix} p^2 + \nu\rho_2 \\ p^2 + \nu\rho_1 \end{pmatrix}$, ζ_α is defined by (4.5) (we recall that we have here $p = q$) for $\alpha \in \mathcal{L} = \mathbb{Z} \setminus \{p\} \times \{\pm\}$, and

$$\Lambda_\alpha(\rho) = j^2 - p^2 \pm \nu\sqrt{\rho_1\rho_2} \quad \text{for} \quad \alpha = (j, \pm) \in \mathcal{L}.$$

4.2.4. *Application of our KAM result.* We see in Appendix A that the hypotheses A0, A1 and A2 are satisfied for the Hamiltonian (4.18) for the paramter $\delta = (\sqrt{2} - 1)\nu$. Thus we can apply Theorem 2.6 and we obtain

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Theorem 4.5. *Fix p. There exists $\nu_0 > 0$ and for $0 < \nu < \nu_0$ there exists $\mathcal{C}_\nu \subset [1,2]^2$ asymptotically of full measure (i.e. $\lim_{\nu \to 0} \mathrm{meas}([1,2]^2 \setminus \mathcal{C}_\nu) = 0$) such that for $\rho \in \mathcal{C}_\nu$ the torus $\mathcal{T}_{\nu\rho} := \{|a_p|^2 = \nu\rho_1,\ |b_p|^2 = \nu\rho_2$, all other modes vanishing$\}$, which is invariant for the Hamiltonian flow associated to P_2, persists in slightly deformed way under the perturbation P_4. Furthermore this invariant torus is linearly stable.*

In terms of small amplitude quasi periodic solutions, Theorem 4.5 reads

coro2

Corollary 4.6. *Fix p and integer and $s > 1/2$. There exists $\nu_0 > 0$ and for $0 < \nu < \nu_0$ there exists $\mathcal{C}_\nu \subset [1,2]^2$ asymptotically of full measure (i.e. $\lim_{\nu \to 0} \mathrm{meas}([1,2]^2 \setminus \mathcal{C}_\nu) = 0$) such that for $\rho \in \mathcal{C}_\nu$ there exists a quasi periodic solution (u, v) of (4.1) of the form*

$$\begin{cases} u(x,t) &= \sum_{j \in \mathbb{Z}} u_j(t\omega) e^{ijx}, \\ v(x,t) &= \sum_{j \in \mathbb{Z}} v_j(t\omega) e^{ijx}, \end{cases}$$

where $U(\cdot) = (u_j(\cdot))_{j\in\mathbb{Z}}$ *and* $V(\cdot) = (v_j(\cdot))_{j\in\mathbb{Z}}$ *are analytic functions from* $\mathbb{T}^2$ *into* ℓ^2_s *satisfying uniformly in* $\theta \in \mathbb{T}^2$

$$\begin{cases} \big||u_p(\theta)| - \sqrt{\nu\rho_1}\big|^2 + \sum_{j\neq p}(1+j^2)^s|u_j(\theta)|^2 = \mathcal{O}(\nu^3), \\ \big||v_p(\theta)| - \sqrt{\nu\rho_2}\big|^2 + \sum_{j\neq p}(1+j^2)^s|v_j(\theta)|^2 = \mathcal{O}(\nu^3) \end{cases}$$

and where $\omega \equiv \omega(\rho) \in \mathbb{R}^2$ *is a nonresonant frequency vector that satisfies*

$$\omega = (p^2, p^2) + \mathcal{O}(\nu^{\frac{3}{2}}).$$

Furthermore this solution is linearly stable.

AppA

Appendix A. Verification of the non resonance hypotheses

In this appendix, the goal is to check that the hypotheses A0, A1 and A2 are satisfied for the Hamiltonians h_0 given by (4.16) and (4.23). The first one, Hypothesis A0, is trivially satisfied and we focus on A1 and A2.

In this process we will use the conservation of the mass and of the momentum (see Remark 2.5). The expression of $\mathbb{L}$ and $\mathbb{M}$ depends on the change of variable Φ_ρ so we have to distinguish between the two examples studied in section 4. We recall that we have initially

$$\mathbb{L} = \sum_{j\in\mathbb{Z}}(|a_j|^2 + |b_j|^2), \quad \text{and} \quad \mathbb{M} = \sum_{j\in\mathbb{Z}} j(|a_j|^2 + |b_j|^2)\,.$$

The first case corresponding to (4.16).
In the new variables, we have for the mass and momentum:

$$\begin{aligned} \mathbb{L} &= \sum_{j\neq p,q}(|c_j|^2 + |d_j|^2) + |a_q|^2 + |b_p|^2 + \nu\rho_1 + x_1 + \nu\rho_2 + x_2 \\ &= \sum_{j\neq p,q}(|c_j|^2 + |d_j|^2) + \nu(\rho_1+\rho_2) + y_1 + y_2 \\ &= \nu\sum_{j\neq p,q}(|e_j|^2 + |f_j|^2) + \nu(\rho_1+\rho_2+r_1+r_2) \\ &= \nu\sum_{\alpha\in\mathcal{L}}|\zeta_\alpha|^2 + \nu(\rho_1+\rho_2+r_1+r_2) \end{aligned}$$

and

$$\begin{aligned} \mathbb{M} &= \sum_{j\neq p,q} j(|c_j|^2 + |d_j|^2) + p|a_q|^2 + q|b_p|^2 + p(\nu\rho_1 + x_1 + \nu\rho_2 + x_2) \\ &= \nu\big(\sum_{\alpha\in\mathcal{L}}\alpha_1|\zeta_\alpha|^2 + p(r_1+\rho_1) + q(r_2+\rho_2)\big). \end{aligned}$$

So the perturbation f of (4.8) commutes with

com1 (A.1) $\quad \mathbb{L}_1 = r_1 + r_2 + \sum_{\alpha\in\mathcal{L}}|\zeta_\alpha|^2 \quad$ and $\quad \mathbb{M}_1 = pr_1 + qr_2 + \sum_{\alpha\in\mathcal{L}}\alpha_1|\zeta_\alpha|^2.$

The second case corresponding to (4.23).
In the new variables we have for the mass and momentum:

$$\begin{aligned}\mathbb{L} &= \sum_{j\neq p}(|c_j|^2+|d_j|^2)+\nu\rho_1+x_1+\nu\rho_2+x_2\\ &= \nu(\rho_1+\rho_2)+y_1+y_2=\nu(\rho_1+\rho_2+r_1+r_2),\end{aligned}$$

and

$$\begin{aligned}\mathbb{M} &= \sum_{j\neq p} j(|c_j|^2+|d_j|^2)+p(\nu\rho_1+x_1+\nu\rho_2+x_2)\\ &= \nu\sum_{j\neq p}(j-p)(|e_j|^2+|f_j|^2)+\nu p(r_1+r_2\rho_1+\rho_2)\\ &= \nu\big(\sum_{\alpha\in\mathcal{Z}}(\alpha_1-p)|\zeta_\alpha|^2+p(r_1+r_2+\rho_1+\rho_2)\big).\end{aligned}$$

So the perturbation f of (4.18) commutes with

com2 (A.2) $$\mathbb{L}_2=r_1+r_2 \quad \text{and} \quad \mathbb{M}_2=\sum_{\alpha\in\mathcal{Z}}(\alpha_1-p)|\zeta_\alpha|^2.$$

A.1. **Verification of Hypothesis A1.** Conditions (a) and (b) trivially hold true with $\delta\leq\nu<1/5$ for h_0 given by (4.16) or by (4.23). For (4.16) condition (c) also holds trivially true with $\delta\leq 1/2$ as soon as $\nu<1/4$. Nevertheless condition (c) is not always true for (4.23), for instance we have

$$\Lambda_{-p,+}+\Lambda_{-p,-}=\nu\sqrt{\rho_1\rho_2}-\nu\sqrt{\rho_1\rho_2}=0.$$

More generally $\Lambda_{j,+}+\Lambda_{k,-}=j^2+k^2-2p^2$ can vanish for some values of (j,k,p) and these are the only problems in order to verify the condition (c). The small divisor $\Lambda_{j,+}+\Lambda_{k,-}$ corresponds to the quadratic term $\zeta_{j,+}\zeta_{k,-}$. As we know that the perturbation f commutes with $\mathbb{M}_2$ we have to consider only the case $\{\mathbb{M}_2,\zeta_{j,+}\zeta_{k,-}\}=0$. This yields the two conditions on (j,k,p)

$$j^2+k^2-2p^2=0 \quad \text{and} \quad j+k-2p=0$$

whose only solution is $k=j=p$ which is not possible since $(p,\pm)\notin\mathcal{Z}$.

A.2. **Verification of Hypothesis A2.** In both cases, (4.16) or (4.23), we have for $k\in\mathbb{Z}^2\setminus 0$ and $\mathfrak{z}=|k|^{-1}(k_2,k_1)$

$$(\nabla_\rho\cdot\mathfrak{z})(\Omega(\rho)\cdot k)=\nu|k|.$$

Therefore, since in both case $|(\nabla_\rho\cdot\mathfrak{z})\Lambda_\alpha(\rho)|\leq\nu$ for all $\alpha\in\mathcal{Z}$, we get choosing $\delta\leq\nu$:

- the second part of alternative A2 (i) is always satisfied for $k\neq 0$,
- the second part of alternative A2 (ii) is always satisfied for $|k|\geq 2$,
- the second part of alternative A2 (iii) is always satisfied for $|k|\geq 3$.

So it remains to verify alternative A2 (ii)-(iii) for a finite number of choice of k: $0<|k|<2$ for (ii) and $0<|k|<3$ for (iii). The condition $0<|k|<2$ implies that

k<2 (A.3) $$k\in\mathcal{K}_2:=\{\pm(0,1),\pm(1,0),(\pm1,\pm1)\},$$

and the condition $0<|k|<3$ implies that

k<3 (A.4) $$k\in\mathcal{K}_3:=\mathcal{K}_2\cup\{\pm(0,2),\pm(2,0),(\pm1,\pm2),(\pm2,\pm1),(\pm2,\pm2)\}.$$

As in the previous section we will use the conservation of the mass and momenta. The strategy is to to show that for each small divisor, either the small divisor corresponds to a term that doesn't exists (we use the conservation of the mass and momenta to show this) and we don't need to estimate it, either the small divisor satisfies one of the two alternatives of the hypotheses A2 (ii) and (iii).

Let us first consider the case of (4.16).

The small divisor $\Omega \cdot k + \Lambda_\alpha$ corresponds to the monomial $e^{ik\cdot\theta}\bar{\zeta}_\alpha$ with $k \in \mathcal{K}_2$ defined in (A.3), in the following sense:

$$\left\{e^{ik\cdot\theta}\bar{\zeta}_\alpha, h_0\right\} = i(\Omega \cdot k + \Lambda_\alpha).$$

The conservation of the mass $\mathbb{L}_1$ defined in (A.1) gives

$$\left\{e^{ik\cdot\theta}\bar{\zeta}_\alpha, \mathbb{L}_1\right\} = ie^{ik\cdot\theta}\bar{\zeta}_\alpha(k_1 + k_2 + 1) = 0.$$

Therefore, we just have to study the cases $k \in \{(0,-1),(-1,0)\}$. Moreover, the conservation of the momentum $\mathbb{M}_1$ defined in (A.1) gives

$$\left\{e^{ik\cdot\theta}\bar{\zeta}_\alpha, \mathbb{M}_1\right\} = ie^{ik\cdot\theta}\bar{\zeta}_\alpha(pk_1 + qk_2 + j) = 0.$$

For $k \in \{(0,-1),(-1,0)\}$, the conservation of the momentum $\mathbb{M}_1$ implies $j \in \{p, q\}$, which implies that $\alpha \in \mathcal{F}$. But for $\alpha \in \mathcal{F}$, thanks to the imaginary part of the eigenvalue Λ_α, we have for all $\rho \in \mathcal{D}$:

$$|\Omega(\rho)\cdot k + \Lambda_\alpha(\rho)| = |k_1p^2 + k_2q^2 \pm (q^2 - p^2) + \nu(k_1\rho_2 + k_2\rho_1 - i\sqrt{\rho_1\rho_2})| \geq \nu.$$

We consider the small divisor $\Omega \cdot k + \Lambda_\alpha + \Lambda_\beta$ in the same way and we notice that it corresponds to the monomial $e^{ik\cdot\theta}\bar{\zeta}_\alpha\bar{\zeta}_\beta$ with $k \in \mathcal{K}_3$ defined in (A.4). The conservation of the mass $\mathbb{L}_1$ implies $k_1 + k_2 = -2$. Thus, we just have to consider the cases $k \in \{(0,-2),(-2,0),(-1,-1)\}$. For these three cases, we show that the second part of alternative A2 (iii) is always satisfied. We have

$$\Omega(\rho)\cdot k + \Lambda_\alpha(\rho) + \Lambda_\beta(\rho) = f(p,q,\alpha,\beta,k) + \nu(k_1\rho_2 + k_2\rho_1 + \eta(\rho)),$$

where $\eta(\rho) \in \{2\rho_1, 2\rho_2, \rho_1 + \rho_2\}$. Using that the coordinates of $\nabla_\rho\eta(\rho)$ and those of k are positives, we have for $\mathfrak{z} = |k|^{-1}(k_2, k_1)$:

$$(\nabla_\rho \cdot \mathfrak{z})(\Omega(\rho)\cdot k + \Lambda_\alpha(\rho) + \Lambda_\beta(\rho)) = \nu|k| + \frac{\nu}{|k|}(k_2,k_1)\cdot\nabla_\rho\eta(\rho) \geq \sqrt{2}\nu.$$

The last small divisor to study here is $\Omega \cdot k + \Lambda_\alpha - \Lambda_\beta$, corresponding to the monomial $e^{ik\cdot\theta}\bar{\zeta}_\alpha\zeta_\beta$, with $k \in \mathcal{K}_3$. This time, the conservation of the mass $\mathbb{L}_1$ gives the equation $k_1 + k_2 = 0$. Therefore we have to study in $\mathcal{K}_3$ the four cases $k \in \{(-1,1),(-1,1),(-2,2),(-2,2)\}$. First, if $\alpha \in \mathcal{F}$ and $\beta \in \mathcal{F}$, we have for $\mathfrak{z} = |k|^{-1}(k_2,k_1)$

$$(\nabla_\rho \cdot \mathfrak{z})(\Omega(\rho)\cdot k + \Lambda_\alpha(\rho) - \Lambda_\beta(\rho)) = \nu|k| \geq \sqrt{2}\nu.$$

Then, if $\alpha \in \mathcal{F}$ and $\beta \in \mathcal{L}$ (or $\alpha \in \mathcal{L}$ and $\beta \in \mathcal{F}$), we have just one eigenvalue with a nonzero imaginary part. We conclude, as in the study of the small divisor $\Omega \cdot k + \Lambda_\alpha$, that we have

$$|\Omega(\rho)\cdot k + \Lambda_\alpha(\rho) - \Lambda_\beta(\rho)| \geq \nu.$$

Finally, for $\alpha = (j, \pm_1) \in \mathcal{L}$ and $\beta = (\ell, \pm_2) \in \mathcal{L}$, we split the study with respect to the values of $k \in \{(-1,1),(-1,1),(-2,2),(-2,2)\}$. On the one hand, if $k \in \{(-2,2),(-2,2)\}$, for $\mathfrak{z} = |k|^{-1}(k_2, k_1)$, we have

$$(\nabla_\rho \cdot \mathfrak{z})(\Omega(\rho)\cdot k + \Lambda_\alpha(\rho) - \Lambda_\beta(\rho)) \geq \nu(|k| - \frac{|k_1 - k_2|}{|k|}) = \sqrt{2}\nu.$$

On the other hand, for $k \in \{(-1,1),(-1,1)\}$, the previous method doesn't work because $(|k| - \frac{|k_1-k_2|}{|k|}) = 0$. The idea is to show that the part without a factor ν of $\Omega(\rho)\cdot k + \Lambda_\alpha(\rho) - \Lambda_\beta(\rho)$ is an nonzero integer. We have

$$\Omega(\rho)\cdot k + \Lambda_\alpha(\rho) - \Lambda_\beta(\rho) = k_1 p^2 + k_2 q^2 + j^2 - \ell^2 + \nu(k_1\rho_2 + k_2\rho_1 + \rho_{\pm_1} - \rho_{\pm_2}).$$

For that purpose, we use the conservation of the moment $\mathbb{M}_1$:

$$\left\{ e^{ik\cdot\theta}\bar{\zeta}_\alpha \zeta_\beta, \mathbb{M}_1 \right\} = 0 \Rightarrow j - l + pk_1 + qk_2 = 0.$$

This leads to the equations

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$$k_1 p^2 + k_2 q^2 + j^2 - \ell^2 = 0 \text{ and } k_1 p + k_2 q + j - \ell = 0. \tag{A.5}$$

For example, for $k = (1,-1)$, this implies $(j-l)(j+l) = (j-l)(p+q)$. But, with the equation (A.5), we have $j = l \Rightarrow p = q$ which is impossible. Thus we can conclude that $j + l = p + q$, and using once again equation (A.5), we obtain $j = q$ and $l = p$, which is excluded as $\alpha = (j, \pm_1) \in \mathcal{L}$ and $\beta = (\ell, \pm_2) \in \mathcal{L}$. With the same computations, the case $k = (-1,1)$ leads to the solution $j = p$ and $l = q$ which is also excluded for the same reason. Therefore, we have $|k_1 p^2 + k_2 q^2 + j^2 - \ell^2| \geq 1$ (it is an nonzero integer). Thus, using that ν is small enough ($\nu \leq \frac{1}{6}$ is enough), we obtain

$$|\Omega(\rho)\cdot k + \Lambda_\alpha(\rho) - \Lambda_\beta(\rho)| \geq \frac{1}{2} \geq \nu.$$

Now we consider the case of (4.23).

First we remark that the mass $\mathbb{L}_2 = r_1 + r_2$ doesn't depend on the external modes $\zeta_\alpha, \bar{\zeta}_\alpha$. Therefore, the conservation of the mass

$$\left\{ e^{ik\cdot\theta}\bar{\zeta}_\alpha, \mathbb{L}_2 \right\} = \left\{ e^{ik\cdot\theta}\bar{\zeta}_\alpha\bar{\zeta}_\beta, \mathbb{L}_2 \right\} = \left\{ e^{ik\cdot\theta}\bar{\zeta}_\alpha\zeta_\beta, \mathbb{L}_2 \right\} = 0$$

implies for the three cases the relation $k_1 + k_2 = 0$. Thus, we can reduce the sets $\mathcal{K}_2$ and $\mathcal{K}_3$ to

$$\tilde{\mathcal{K}}_2 := \{\pm(1,-1)\} \quad \text{and} \quad \tilde{\mathcal{K}}_3 := \tilde{\mathcal{K}}_2 \cup \{\pm(2,-2)\}.$$

For the Hypothesis A2 (ii), we have $k = \pm(1,-1)$ and (using that $k_1 + k_2 = 0$):

$$\Omega(\rho)\cdot k + \Lambda_{j,\pm}(\rho) = j^2 - p^2 + \nu(k_1\rho_2 + k_2\rho_1 \pm \sqrt{\rho_1\rho_2}).$$

Therefore, once again for $\mathfrak{z} = |k|^{-1}(k_2, k_1)$, and using that

$$|\sqrt{\frac{\rho_2}{\rho_1}}k_2 + \sqrt{\frac{\rho_1}{\rho_2}}k_1| = \frac{|\rho_1 - \rho_2|}{\sqrt{\rho_1\rho_2}} \leq 1,$$

we have

$$(\nabla_\rho \cdot \mathfrak{z})(\Omega(\rho)\cdot k + \Lambda_\alpha(\rho)) = \sqrt{2}\nu \pm \frac{\nu}{2\sqrt{2}}(\sqrt{\frac{\rho_2}{\rho_1}}k_2 + \sqrt{\frac{\rho_1}{\rho_2}}k_1) \geq \frac{3}{4}\sqrt{2}\nu.$$

The small divisor $\Omega \cdot k + \Lambda_\alpha + \Lambda_\beta$, for $\alpha = (j, \pm)$ and $\beta = (\ell, \pm)$ in $\mathcal{L}$, corresponding to the monomial $e^{ik\cdot\theta}\bar{\zeta}_\alpha\bar{\zeta}_\beta$ with $k \in \mathcal{K}_3$, is given by (using that $k_1 + k_2 = 0$):

$$\Omega(\rho)\cdot k + \Lambda_{j,\pm}(\rho) + \Lambda_{\ell,\pm}(\rho) = j^2 + \ell^2 - 2p^2 + \nu(k_1\rho_2 + k_2\rho_1 + \eta(\rho)),$$

where $\eta(\rho) \in \{-2\sqrt{\rho_1\rho_2}, 0, 2\sqrt{\rho_1\rho_2}\}$. Here, we use the conservation of the moment $\mathbb{M}_2$ (it implies $j+l = 2p$) to show that the integer part of the small divisor $\Omega \cdot k + \Lambda_\alpha + \Lambda_\beta$ never vanishes. Indeed, we have

$$(j^2 + \ell^2 + -2p^2 = 0 \text{ and } j + l - 2p = 0) \Rightarrow j = l = p,$$

which is impossible because $\alpha = (j, \pm)$ and $\beta = (\ell, \pm)$ are in $\mathcal{L}$.

Finally, we have to study the small divisor $\Omega \cdot k + \Lambda_\alpha - \Lambda_\beta$, for $\alpha = (j, \pm)$ and $\beta = (\ell, \pm)$ in $\mathcal{L}$, corresponding to the monomial $e^{ik\cdot\theta}\bar{\zeta}_\alpha\zeta_\beta$ with $k \in \mathcal{K}_3$. We have

$$\Omega(\rho)\cdot k + \Lambda_{j,\pm}(\rho) - \Lambda_{\ell,\pm}(\rho) = j^2 - \ell^2 + \nu(k_1\rho_2 + k_2\rho_1 + \eta(\rho)),$$

where $\eta(\rho) \in \{-2\sqrt{\rho_1\rho_2}, 0, 2\sqrt{\rho_1\rho_2}\}$. The conservation of the moment $\mathbb{M}_2$ gives $j = l$, thus we can write

$$\Omega(\rho)\cdot k + \Lambda_{j,\pm}(\rho) - \Lambda_{\ell,\pm}(\rho) = \nu(k_1(\rho_2 - \rho_1) + \eta(\rho)).$$

Always with the choice $\mathfrak{z} = |k|^{-1}(k_2, k_1) = (\sqrt{2}|k_1|)^{-1}(-k_1, k_1)$, we obtain here

$$(\nabla_\rho \cdot \mathfrak{z})(\Omega(\rho)\cdot k + \Lambda_{j,\pm}(\rho) - \Lambda_{\ell,\pm}(\rho)) \geq \nu|k_1|(\sqrt{2} - |\sqrt{\frac{\rho_2}{\rho_1}} - \sqrt{\frac{\rho_1}{\rho_2}}|) \geq (\sqrt{2}-1)\nu,$$

where we use in the last inequality that $|\sqrt{\frac{\rho_2}{\rho_1}} - \sqrt{\frac{\rho_1}{\rho_2}}| = \frac{|\rho_1-\rho_2|}{\sqrt{\rho_1\rho_2}} \leq 1$ and $|k_1| \in \{1, 2\}$.

Conclusion. In order to show that the spectrum of h_0 satisfies the hypotheses A1 and A2, it suffices to choose the minimum of the explicit values we obtain for both cases (h_0 defined by (4.16) or (4.23)), which is

$$\delta = (\sqrt{2} - 1)\nu.$$

Therefore, the hypotheses of Theorem 2.6 are satisfied, and we can apply this theorem to obtain Theorem 4.1 for the linear instability of the torus

$$\mathcal{T}^u_{\nu\rho} := \{|a_p|^2 = \nu\rho_1,\ |b_q|^2 = \nu\rho_2, \text{all other modes vanishing}\} \text{ where } p \neq q,$$

and to obtain Theorem 4.4 for the linear stability of

$$\mathcal{T}^s_{\nu\rho} := \{|a_p|^2 = \nu\rho_1,\ |b_p|^2 = \nu\rho_2, \text{all other modes vanishing}\}$$

for the extended cubic coupled Schrödinger systems (4.1).

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