# An Analytic Calculus for the Intuitionistic Logic of Proofs

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**Abstract**

The goal of this paper is to take a step towards the resolution of the problem of finding an analytic sequent calculus for the logic of proofs. For this, we focus on the system **Ilp**, the intuitionistic version of the logic of proofs. First we present the sequent calculus **Gilp** that is sound and complete with respect to the system **Ilp**; we prove that **Gilp** is cut-free and contraction-free, but it still does not enjoy the subformula property. Then, we enrich the language of the logic of proofs and we formulate in this language a second Gentzen calculus **Gilp**$^*$. We show that **Gilp**$^*$ is a conservative extension of **Gilp**, and that **Gilp**$^*$ satisfies the subformula property.

**Keyword** cut-elimination, logic of proofs, normalisation, proof sequents 2010 MSC: 03F05, 03B60

## 1 Introduction

The old question discussed by Gödel in 1933/38 concerning the intended provability semantics of the classical modal logic **S4** and intuitionistic logic **IPC** was finally settled by the logic of proofs introduced by Artemov [1]. The logic of proofs **Lp** is a natural extension of classical propositional logic by means of proof-carrying formulas. The operations on proofs in the logic of proofs suffice to recover the explicit provability of modal and intuitionistic logic.

Many results have been proved for the logic of proofs **Lp**: e.g. the deduction theorem, the substitution lemma and the internalisation of proofs [1]. Moreover, **Lp** has been shown to be sound and complete with respect to the modal logic **S4**, and with respect to Peano Arithmetic [1].

There also exists a version of **Lp** with an intuitionistic base, namely **Ilp**, introduced in [2]. Unsurprisingly, analogous results can be obtained for the logic of proofs with an intuitionistic base. Indeed, in **Ilp** too, the deduction theorem, the substitution lemma and the internalisation of proofs hold. Moreover, **Ilp** is sound and complete with respect to the modal logic **S4** with an intuitionistic base.

From a Gentzen-style point of view, two similar sequent calculi have been proposed for the two systems **Lp** and **Ilp**, respectively (see [2]). Although simple and cut-free, these sequent calculi fail to satisfy the subformula property; thus they are not analytic calculi. The aim of this paper is to take an important step towards developing a calculus that genuinely enjoys the subformula property.[1] We proceed in the following way and only for the system **Ilp**.

First, we introduce the notion of *proof sequent*, which is a generalisation of the standard notion of sequent. Proof sequents are the result of appending a multiset of natural deduction derivations written in sequent style to a standard sequent. Proof sequents carry the advantage of allowing a distinction between two different levels - the propositional level and the proof polynomial level - which is characteristic of the logic of proofs. By exploiting proof sequents we construct two calculi, **Gilp** and **Gilp**$^*$. **Gilp** is a calculus for the system **Ilp**, which, although it enjoys several interesting properties, does not satisfy the subformula property. It plays the role of an auxiliary calculus useful for obtaining an analytic calculus. After arguing that the lack of analyticity may be traced to the poorness of the language for the logic of proofs, we proceed to develop a second sequent calculus, **Gilp**$^*$, based on an enhanced language. This enhanced language is obtained by using the functional symbols of the lambda calculus instead of the proof constants that are characteristic of the logic of proofs. We prove that **Gilp**$^*$ enjoys the subformula property and that it is a conservative extension of **Gilp**; therefore **Gilp**$^*$ represents the framework where the analyticity of the logic of proofs is reached. Note that calculus **Gilp**$^*$, together with the language on which is based, not only constitute an attempted solution to the problem of the lack of an analytic calculus, but can also be seen as a new natural and interesting development of the logic of proofs where future research can be carried out.

Although the analyticity of his calculi was originally proved by Gentzen to achieve consistency, today this result has become a cornerstone of proof theory, even beyond the concern of consistency. The reasons of its importance are both technical and philosophical and are documented by a wide literature (e.g. see [3, 6, 7, 17]). The importance of having an analytic calculus for the intuitionistic logic of proofs, which is the main goal of this paper, is thus to be understood in this direction: it does not only serve to prove the consistency of the logic (this is also obtainable by embedding **Ilp** into **IS4**), but it serves to provide this logic with a proof theory appealing and significant both from a technical and

[1] It is possible to formulate sequent systems with the subformula property by adding indexes to standard justification logic operators à la Renne [12], or explicitly representing (potentially non cut-free) proofs themselves à la Saveetev [13]. Beyond being somewhat artificial, these approaches offer little insight into the question of the analyticity for the logic of proofs. For this reason, we do not take them to provide a genuine resolution of the problem of the lack of an analytic calculus for the logic of the proofs.

philosophical point of view.[2]

Three works are related to the present one: in the papers [10, 9] the calculus **Gilp** is introduced and a rough idea of how to construct the calculus **Gilp**$^*$ is briefly suggested. The paper [11] mainly concerns a philosophical argument in favor of the analyticity of a sequent calculus, but exploits as a case-study the proof theory for the logic of proofs. The present paper is the only one where both systems **Gilp** and **Gilp**$^*$ are presented and the proof of their relation and of the analyticity of the logic of proof *via* the system **Gilp**$^*$ is fully demonstrated.

The paper is organised as follows. *Section 2.* We will introduce the calculus **Gilp** for the intuitionistic logic of proofs. *Sections 3–4.* We will show that this calculus is contraction-free and cut-free; moreover the rules introducing propositional connectives and proof polynomials are symmetric (see [8] and [16] for a precise description of this property). However, **Gilp** does not satisfy the subformula property. *Section 5.* In the light of this result, we will analyse the logic of proofs in detail and attempt to find the reason for its "resistance" to analyticity. We will show that the reason is linked to the language of the logic of proofs. *Section 6.* We will change the language of the logic of proofs and build the calculus **Gilp**$^*$ in this new language. We will show that **Gilp**$^*$ satisfies the same properties as **Gilp**, namely it is cut-free, contraction-free and the rules introducing propositional connectives and proof polynomials are symmetric. *Section 7.* We will show that **Gilp**$^*$ enjoys the subformula property, and that - *Section 8.* - it can be thought of as a conservative extension of **Gilp**.

## 2 The calculus Gilp

**Definition 2.1.** The language $\mathcal{L}_{lp}$ contains: (i) the usual language of propositional boolean logic, (ii) proof variables $x_0, x_1, x_2, ...$, (iii) proof constants $c_0, c_1, c_2, ...$, (iv) the functional symbols $+$, $!$, and $\cdot$, and (v) the operator symbol of the type "term : formula".

We will use $a, b, c, ...$ for proof constants, and $u, v, w, ...$ for proof variables.

**Definition 2.2.** Terms are defined by the rule

$$t := x_i \mid c_i \mid !t \mid t + s \mid t \cdot s$$

We call these terms *proof polynomials* and denote them by $q, r, s, t, ....$

**Definition 2.3.** Formulas are defined by the rule

$$A := S \mid \bot \mid A \wedge B \mid A \vee B \mid A \rightarrow B \mid t : A$$

where $S$ stands for any sentence variable.

[2]The situation could be thought of as analogous to that of modal logic (e.g. see [8]). For many decades the proof theory of modal logic has encountered a general skepticism because the sequent calculi elaborated for modal logic did not enjoy the subformula property (modal logic was however clearly consistent). As soon as this result was achieved, the proof theory for modal logic started to blossom.

**Definition 2.4.** The set of subformulas of a formula $A$ is the smallest set of formulas containing $A$ such that, if $A_0 \circ A_1$ is in the set, where $\circ \in \{\wedge, \vee, \rightarrow\}$, then so are $A_0$ and $A_1$; and if $t : A_0$ is in the set, then so is $A_0$.

The Hilbert system **Ilp** is composed of:

$A_0$ Axioms of intuitionistic logic formulated in the language $\mathcal{L}_{lp}$

$A_1$ $t{:}(A \rightarrow B) \rightarrow (s{:}A \rightarrow (t \cdot s){:}B)$

$A_2$ $t{:}A \rightarrow A$

$A_3$ $t{:}A \rightarrow !t{:}t{:}A$

$A_4$ $t_i{:}A \rightarrow (t_0 + t_1){:}A$, where $i = 0, 1$

$R_1$ Modus Ponens

$R_2$ If $A$ is one of the axioms $A_0$ - $A_4$, and $c$ is a proof constant, then $c : A$ is a theorem

The principal innovation of the logic of proofs consists in the use of proof polynomials. Where in modal logic we have formulas of the form $\Box A$, in the logic of proofs we have formulas of the form $t{:}A$. Accordingly, from the perspective of the sequent calculus, we want logical rules that introduce this type of formula on the right and on the left side of the sequent, i.e. we want symmetric rules for proof polynomials (for a discussion of the importance of having symmetric rules see [8] and [16]). In order to reach this goal, we look at the semantic interpretation of formulas of the form $t{:}A$, and try to reflect this interpretation in the Gentzen framework. Following Mkrtychev [5]

$t{:}A$ is true if, and only if, $A$ is true and $t$ is a proof of $A$

While it is of course easy to express in the Gentzen framework the fact that the formulas $A$ and $t{:}A$ are true (i.e. it is enough to collocate them on the right side of the sequent), the fact that "$t$ is a proof of $A$" is more difficult to convey. Our solution is to introduce the notion of *typed natural deduction sequent*, or *TND-sequent* for short.

**Definition 2.5.** A *TND-sequent* is an object of the form

$$s_1{:}B_1, ..., s_n{:}B_n \vdash t{:}A$$

where the formulas $s_1{:}B_1, ..., s_n{:}B_n$ form a multiset.

TND-sequents can be seen as natural deduction derivations, written in sequent style, where the only formulas that occur are of the form $t{:}A$. As will become clear below, the idea is to put side by side a standard sequent and a multiset of TND-sequents. This way we can intuitively interpret TND-sequents in the following way: the formula which lies on the right side of the $\vdash$ expresses the fact that $t$ is a proof of $A$, while the formulas that lie on the left side of the $\vdash$ represent the assumptions by means of which we can construct the proof $t$ of $A$. This will become clear once we introduce the rules of the calculus **Gilp**.

**Syntactic Notation**

- $M, N, ...$ stand for multisets of formulas,
- $\mathbf{M}, \mathbf{N}, ...$ stand for multisets of formulas of the form $t\!:\!A$,
- $\mathfrak{M}, \mathfrak{N}, ...$ stand for multisets of formulas that are not of the form $t\!:\!A$,
- $T_1$, $T_2$, ... stand for TND-sequents,
- $\Sigma$, $\Theta$, ... stand for sequents, that is objects of the form $M \Rightarrow C$,
- $G, H, ...$ stand for multisets of TND-sequents.

**Definition 2.6.** The notion of *proof sequent* is defined in the following way:

- if $\Sigma$ is a sequent, then $\Sigma$ is a proof sequent,
- if $\Sigma$ is a sequent and $G \equiv T_1 \mid ... \mid T_n$ is a multiset of TND-sequents, then $G \mid \Sigma$ is a proof sequent.

**Definition 2.7.** The intended interpretation $\tau$ of a proof sequent is:

- $(M \Rightarrow C)^\tau := (\bigwedge M \to C)$,
- $(\mathbf{M}_1 \vdash t_1 : A_1 \mid ... \mid \mathbf{M}_n \vdash t_n : A_n \mid M \Rightarrow C)^\tau := (\bigwedge \mathbf{M}_1 \to t_1 : A_1) \wedge ... \wedge (\bigwedge \mathbf{M}_n \to t_n\!:\!A_n) \wedge (\bigwedge M \to C)$

At first glance, proof sequents might look like hypersequents (e.g. [3]), but in fact they are quite different for the following two reasons. Hypersequents are multisets of sequents, while proof sequents are composed by only one sequent plus a collection of TND-sequents; hypersequents are standardly interpreted disjunctively, while proof sequents are interpreted conjunctively.

By exploiting proof sequents, we formulate the calculus **Gilp**. **Gilp** is composed of:

**Axioms**

$$t_1\!:\!A_1 \vdash t_1\!:\!A_1 \mid ... \mid t_n\!:\!A_n \vdash t_n\!:\!A_n \mid S, M \Rightarrow S$$

$$t_1\!:\!A_1 \vdash t_1\!:\!A_1 \mid ... \mid t_n\!:\!A_n \vdash t_n\!:\!A_n \mid M, \perp \Rightarrow C$$

**Propositional Rules**

$$\frac{G \mid A, B, M \Rightarrow C}{G \mid A \wedge B, M \Rightarrow C} \wedge A \qquad \frac{G \mid M \Rightarrow A \quad G \mid M \Rightarrow B}{G \mid M \Rightarrow A \wedge B} \wedge K$$

$$\frac{G \mid A, M \Rightarrow C \quad G \mid B, M \Rightarrow C}{G \mid A \vee B, M \Rightarrow C} \vee A \qquad \frac{G \mid M \Rightarrow A_i}{G \mid M \Rightarrow A_0 \vee A_1} \vee K$$

$$\frac{G \mid A \rightarrow B, M \Rightarrow A \quad G \mid B, M \Rightarrow C}{G \mid A \rightarrow B, M \Rightarrow C} \rightarrow A \qquad \frac{G \mid A, M \Rightarrow B}{G \mid M \Rightarrow A \rightarrow B} \rightarrow K$$

**Proof Rules**

$$\frac{G \mid t{:}A, A, M \Rightarrow C}{G \mid t{:}A, M \Rightarrow C} PA \qquad \frac{G \mid \mathbf{N}, \mathbf{P} \vdash t{:}A \mid \mathbf{N}, \mathbf{Q}, \mathfrak{M} \Rightarrow A}{G \mid \mathbf{N}, \mathbf{P}, \mathbf{Q}, \mathfrak{M} \Rightarrow t{:}A} PK$$

**Polynomial Rules**

$$\frac{G \mid \mathbf{M} \vdash t_i{:}A \mid \Sigma}{G \mid \mathbf{M} \vdash (t_0 + t_1){:}A \mid \Sigma} + \qquad \frac{G \mid \mathbf{M} \vdash t{:}A \mid \Sigma}{G \mid \mathbf{M} \vdash !t{:}t{:}A \mid \Sigma} \,!$$

$$\frac{G \ \mid \mathbf{M}, \mathbf{P} \vdash t{:}(A \rightarrow F) \mid \mathbf{M}, \mathbf{Q} \vdash t'{:}A \mid \Sigma}{G \mid \mathbf{M}, \mathbf{P}, \mathbf{Q} \vdash (t \cdot t'){:}F \mid \Sigma} \odot$$

$$\frac{G \mid \Sigma}{G \mid \ \vdash c{:}A \mid \Sigma} ci$$

where in the rules $\vee K$ and $+$, $i = 0, 1$, and in the rule $ci$, $A$ is one of the axioms $A_0$ - $A_4$ and $c$ is a constant. We adopt the convention that whenever we write $\mathbf{M}, \mathbf{P} \vdash ... \mid \mathbf{M}, \mathbf{Q} \vdash ...$ then $\mathbf{P}$ and $\mathbf{Q}$ are disjoint, so that $\mathbf{M}$ contains all the formulas of the form $s{:}B$ which are common to these TND-sequents (and similarly for more than two TND-sequents, or a TND-sequent and a sequent). In the calculus **Gilp**, this convention applies to the rule $PK$ and the rule $\odot$.

The rules $PA$ and $PK$ reflect the semantic interpretation of formulas of the form $t{:}A$, when read top-down. Indeed the rule $PA$ tells us that if $A$ is false then $t{:}A$ is false. As for the rule $PK$, it tells us that if $A$ is true and $t$ is a proof of $A$, then $t{:}A$ is true. Note that thanks to $PA$ and $PK$ we have rules that introduce the proof polynomial $t$ on the left and right side of the sequent; that is, we have symmetric rules.

As for the polynomial rules, we observe that they only operate on TND-sequents. Basically, these rules tell us when we can apply the functional symbols $!$, $\cdot$ and $+$ on proofs. The rule $ci$, on the other hand, tells us that we can introduce the fact that $c$ is a proof of one of the axioms $A_0 - A_4$. The fact that $c$ is a proof of one of the axioms $A_0 - A_4$ does not depend on any assumption, since the left side of the $\vdash$ is empty.

*Remark* 2.8. Note that there is no rule which operates on formulas that occur on the left side of the $\vdash$, and so, while the formulas that occur on the right side of the $\vdash$ can be modified, those on the left side remain untouched throughout a derivation. Moreover, in the axioms of **Gilp**, TND-sequents can only have the form $s:B \vdash s:B$. Therefore, for any **Gilp**-derivation of the proof sequent $G \mid s_1 : B_1, ..., s_n : B_n \vdash t : A \mid \Sigma$, the axioms must contain the TND-sequents $s_1:B_1 \vdash s_1:B_1 \mid ... \mid s_n:B_n \vdash s_n:B_n$. From now on we call the TND-sequents of the form $s_1:B_1 \vdash s_1:B_1 \mid ... \mid s_n:B_n \vdash s_n:B_n$ *TND-axioms*. We use $\underline{\mathbf{M}}$, $\underline{\mathbf{N}}$, ... to denote the TND-axioms from which the TND-sequents $\mathbf{M} \vdash t:A$, $\mathbf{N} \vdash t':A'$, ... have been derived, respectively; we use $\underline{G}, \underline{H}, ...$ to denote the TND-axioms from which the proof-sequents $G$ and $H$ have been derived.

**Definition 2.9.** As is standard in the literature, we say that a formula occurring in the premise of a rule $\mathcal{R}$ is an *auxiliary formula* whenever the rule $\mathcal{R}$ operates on that formula. Analogously, we say that a TND-sequent is an *auxiliary TND-sequent* of a rule $\mathcal{R}$ whenever the rule $\mathcal{R}$ operates on the formula $t:A$ occurring on the right side of the $\vdash$ of the TND-sequent. (Since no rule $\mathcal{R}$ operates on formulas occurring on the left side of the $\vdash$, this terminology is not misleading.) Just as a rule $\mathcal{R}$ may have several auxiliary formulas, it may have several auxiliary TND-sequents.

# 3 Admissiblity of Structural Rules

In this section we will show which structural rules are admissible in the calculus **Gilp**. Moreover, in order to show that the rule of contraction is height-preserving admissible, we will show that the propositional left rules and the proof rule $PA$ are height-preserving invertible. In Section 4 it will be proved that the cut-rules, *cut** and *cut*, are admissible.

We will bring this section to a close with the proof that the calculus **Gilp** is sound and complete with respect to the calculus **Ilp**.

**Definition 3.1.** We associate to each derivation $d$ in **Gilp** a natural number $h(d)$, the *height* of $d$. Intuitively, the height corresponds to the length of the longest branch in a tree-derivation $d$, where the length of a branch is the number of nodes in the branch minus one. We omit the standard inductive definition.

**Definition 3.2.** $d \vdash^n G \mid \Sigma$ means that $d$ is a derivation of $G \mid \Sigma$ in **Gilp**, with $h(d) \leq n$. We write ${}^{\langle n \rangle}G \mid \Sigma$ for: $G \mid \Sigma$ is the conclusion of a derivation $d$ with height $\leq n$.

**Lemma 3.3.** *Any proof sequent of the form $G \mid M, C \Rightarrow C$ is derivable in **Gilp**, for any formula $C$, and any multiset of TND-sequents $G$ derivable from the TND-axioms and the polynomial rules.*

*Proof.* By straightforward induction on $C$. $\square$

Figure 1: Structural Rules of **Gilp**

$$\frac{G \mid M \Rightarrow C}{G \mid A, M \Rightarrow C}\, W \qquad \frac{G \mid \Sigma}{G \mid s\!:\!B \vdash s\!:\!B \mid \Sigma}\, EW$$

$$\frac{G \mid A, A, M \Rightarrow C}{G \mid A, M \Rightarrow C}\, C \qquad \frac{G \mid \mathbf{M} \vdash t\!:\!A \mid \Sigma}{G \mid \Sigma}\, El$$

$$\frac{G \mid \mathbf{M}, \mathbf{P} \vdash t\!:\!A \mid t\!:\!A, \mathbf{M}, \mathbf{Q} \vdash s\!:\!B \mid \Sigma}{G \mid \mathbf{M}, \mathbf{P}, \mathbf{Q} \vdash s\!:\!B \mid \Sigma}\, cut^*_{t:A} \qquad \frac{G \mid M \Rightarrow A \quad H \mid A, P \Rightarrow C}{G \mid H \mid M, P \Rightarrow C}\, cut_A$$

**Lemma 3.4.** *In **Gilp** the following holds:*

*(i) the weakening rules $W$ and $EW$ (see Figure 1) together with the rule $El$ are height-preserving admissible.*

*(ii) The propositional left-rules and the proof rule $PA$ are height-preserving invertible.*

*(iii) The contraction rule $C$ (see Figure 1) is height-preserving admissible.*

*Proof.* (i) follows from a straightforward induction on the height of the derivation of the premise. Similarly for the $\wedge A$, $\vee A$ rules in (ii). The inverse of the rule $PA$ is just weakening. In case of the rule $\rightarrow$A, the invertibility is only shown with respect to the premise $G \mid B, M \Rightarrow C$.[3] (iii) is proved by induction on the height of the derivation of the premise $G \mid A, A, M \Rightarrow C$. If $G \mid A, A, M \Rightarrow C$ is an axiom, so is $G \mid A, M \Rightarrow C$. If $G \mid A, A, M \Rightarrow C$ is preceded by a rule $\mathcal{R}$ which does not have any of the two occurrences of the formula $A$ as auxiliary, we proceed by exploiting the inductive hypothesis. Finally let us consider the case where $G \mid A, A, M \Rightarrow C$ is preceded by a propositional or proof rule and one of the two occurrences of the formula $A$ is auxiliary. Then the rule which concludes $G \mid A, A, M \Rightarrow C$ is an $A$-rule and we have to analyze the following four cases: $\wedge A$, $\vee A$, $\rightarrow A$, $PA$. The first three cases are dealt with in the standard way. We just consider the case of the rule $PA$, which is:

$$\frac{{}^{\langle n-1\rangle} G \mid t\!:\!A, t\!:\!A, A, M \Rightarrow C}{{}^{\langle n\rangle} G \mid t\!:\!A, t\!:\!A, M' \Rightarrow C}\, PA \quad \rightsquigarrow^{4} \quad \frac{{}^{\langle n-1\rangle} G \mid t\!:\!A, A, M \Rightarrow C}{{}^{\langle n\rangle} G \mid t\!:\!A, M' \Rightarrow C}\, PA$$

□

[3] For further details on this rule see [15, Ch. 3]

[4] We use the symbol $\rightsquigarrow$ to denote that the premise on the right has been obtained from the premise on the left by applying the inductive hypothesis.

**Theorem 3.5.** *For all formulas $A$, and for all proof sequents $G \mid \Sigma$,*

*[i] if $G \mid \Sigma$ is derivable in **Gilp**, then $(G \mid \Sigma)^\tau$ is a theorem of **Ilp**.*

*[ii] If $A$ is a theorem of **Ilp**, then $\Rightarrow A$ is derivable in **Gilp**.*

*Proof.* By induction on the height of derivations in **Gilp** and **Ilp**, respectively. As concerns [i], we omit the proof which is easy but quite tedious. As for [ii], the intuitionistic axioms and the modus ponens rule are proved as usual; the derivations of the axioms $A_2 - A_4$ are straightforward. In order to familiarise the reader with the calculus **Gilp**, we present the derivation of axiom $A_1$.

**Gilp** $\vdash\Rightarrow s\!:\!(A \to B) \to (t\!:\!A \to (s \cdot t)\!:\!B)$[5]

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{t\!:\!A \vdash t\!:\!A \mid t\!:\!A, A \to B, A \Rightarrow A \qquad s\!:\!(A \to B) \vdash s\!:\!(A \to B) \mid s\!:\!(A \to B), B \Rightarrow B}{s\!:\!(A \to B) \vdash s\!:\!(A \to B) \mid t\!:\!A \vdash t\!:\!A \mid s\!:\!(A \to B), A \to B, t\!:\!A, A \Rightarrow B}\,{\scriptstyle \to A}}{s\!:\!(A \to B) \vdash s\!:\!(A \to B) \mid t\!:\!A \vdash t\!:\!A \mid s\!:\!(A \to B), t\!:\!A, A \Rightarrow B}\,{\scriptstyle PA}}{s\!:\!(A \to B) \vdash s\!:\!(A \to B) \mid t\!:\!A \vdash t\!:\!A \mid s\!:\!(A \to B), t\!:\!A \Rightarrow B}\,{\scriptstyle PA}}{s\!:\!(A \to B), t\!:\!A \vdash (s \cdot t)\!:\!B \mid s\!:\!(A \to B), t\!:\!A \Rightarrow B}\,{\scriptstyle \odot}}{s\!:\!(A \to B), t\!:\!A \Rightarrow (s \cdot t)\!:\!B}\,{\scriptstyle PK}}{s\!:\!(A \to B) \Rightarrow (t\!:\!A \to (s \cdot t)\!:\!B)}\,{\scriptstyle \to K}}{\Rightarrow s\!:\!(A \to B) \to (t\!:\!A \to (s \cdot t)\!:\!B)}\,{\scriptstyle \to K}
$$

□

# 4 Cut-admissibility

In this section we will prove that the two cut-rules, *cut*$^*$ and *cut*, are admissible in the calculus **Gilp**. For this, we introduce the following two lemmas.

**Lemma 4.1.** *In **Gilp** each of the polynomial rules $+$, $!$ and $\odot$ permutes up with any rule whose conclusion is not one of its auxiliary TND-sequents.*

*Proof.* The proof is straightforward given that the polynomial rules and any other rule under consideration operate in different parts of a proof sequent. To illustrate this, we consider the case of the rule ! and the rule $PK$:

$$
\dfrac{\dfrac{G \mid \mathbf{M} \vdash t\!:\!A \mid \mathbf{N}, \mathbf{P} \vdash r\!:\!C \mid \mathbf{N}, \mathbf{Q}, \mathfrak{M} \Rightarrow C}{G \mid \mathbf{M} \vdash t\!:\!A \mid \mathbf{N}, \mathbf{P}, \mathbf{Q}, \mathfrak{M} \Rightarrow r \ : \ C}\,{\scriptstyle PK}}{G \mid \mathbf{M} \vdash !t : t\!:\!A \mid \mathbf{N}, \mathbf{P}, \mathbf{Q}, \mathfrak{M} \Rightarrow r \ : \ C}\,{\scriptstyle !}
$$

[5] To aid readability, we use the multiplicative version of the rule $\to A$. This rule can be easily derived by means of the structural rules of **Gilp**.

$$\downarrow$$

$$\dfrac{\dfrac{G \mid \mathbf{M} \vdash t{:}A \mid \mathbf{N}, \mathbf{P} \vdash r{:}C \mid \mathbf{N}, \mathbf{Q}, \mathfrak{M} \Rightarrow C}{G \mid \mathbf{M} \vdash !t : t{:}A \mid \mathbf{N}, \mathbf{P} \vdash r{:}C \mid \mathbf{N}, \mathbf{Q}, \mathfrak{M} \Rightarrow C}\,!}{G \mid \mathbf{M} \vdash !t : t{:}A \mid \mathbf{N}, \mathbf{P}, \mathbf{Q}, \mathfrak{M} \Rightarrow r \ : \ C}\,PK$$

$\square$

**Lemma 4.2.** *The rule*

$$\frac{G \mid \mathbf{M}, \mathbf{Q} \vdash t{:}A \mid t{:}A, \mathbf{P}, \mathbf{Q} \vdash s{:}B \mid \Sigma}{G \mid \mathbf{M}, \mathbf{P}, \mathbf{Q} \vdash s{:}B \mid \Sigma}\,cut^*_{t:A}$$

*is admissible in the calculus* ***Gilp****.*

*Proof.* We consider the derivation $d$ of the premise $G \mid \mathbf{M}, \mathbf{Q} \vdash t : A \mid t : A, \mathbf{P}, \mathbf{Q} \vdash s : B \mid \Sigma$. By Lemma 4.1, we can assume without loss of generality that in $d$, the rules involved in the derivation of the TND-sequent $\mathbf{M}, \mathbf{Q} \vdash t{:}A$ have been applied first, then the rules involved in the derivation of the TND-sequent $t : A, \mathbf{P}, \mathbf{Q} \vdash s : B$, and then all the other rules. Furthermore, by the observation in Remark 2.8, the TND-sequent $t{:}A \vdash t{:}A$ occurs in the axioms of $d$ and it is an auxiliary TND-sequent in the derivation of the TND-sequent $t{:}A, \mathbf{P}, \mathbf{Q} \vdash s{:}B$.

We modify $d$ to obtain a derivation $d'$ as follows. First, we remove the TND-sequent $t{:}A \vdash t{:}A$ from the axioms. Then we apply the first set of rules in $d$ to derive the TND-sequent $\mathbf{M}, \mathbf{Q} \vdash t{:}A$. Then we apply the second set of rules in $d$, with the sole difference that, for any rule for which $t{:}A \vdash t{:}A$ is an auxiliary TND-sequent in $d$, $\mathbf{M}, \mathbf{Q} \vdash t{:}A$ is the corresponding auxiliary TND-sequent in $d'$. (This can be done since what counts in polynomial rules is the formula that occurs on the right side of the $\vdash$, and not the formulas that are on the left side of the $\vdash$.) At this point the TND-sequent $\mathbf{M}, \mathbf{P}, \mathbf{Q} \vdash s{:}B$ has been derived. The rest of the rules applied in $d'$ are identical to the remaining rules in $d$. Hence $d'$ is the required derivation of $G \mid \mathbf{M}, \mathbf{P}, \mathbf{Q} \vdash s{:}B \mid \Sigma$. $\square$

**Lemma 4.3.** *If*

$$\frac{\begin{matrix} \vdots\, {\scriptstyle d_1} \\ G \mid M \Rightarrow A \end{matrix} \quad \begin{matrix} \vdots\, {\scriptstyle d_2} \\ H \mid A, P \Rightarrow C \end{matrix}}{G \mid H \mid M, P \Rightarrow C}\,cut_A$$

*and $d_1$ and $d_2$ do not contain any other application of the cut rule, then we can construct a derivation of $G \mid H \mid M, P \Rightarrow C$ without any application of the cut rule.*

*Proof.* The proof is developed by induction on the complexity of the cut formula, which is the number ($\geq 0$) of occurrences of logical symbols (a proof polynomial $s$ counting as such) in the cut formula $A$, with subinduction on the sum of the heights of the derivations of the premises of the cut-rule. We distinguish cases by the last rule applied on the left premise. There are three general cases to consider. `Case 1.` $G \mid M \Rightarrow A$ is an axiom. `Case 2.` $G \mid M \Rightarrow A$ is inferred by a rule $\mathcal{R}$ in which $A$ is not the auxiliary formula. `Case 3.` $G \mid M \Rightarrow A$ is inferred by a rule $\mathcal{R}$ in which $A$ is the auxiliary formula. The first two cases, as well as `Case 3` where the rule $\mathcal{R}$ inferring the formula $A$ is a propositional rule, are treated in the standard way. We show in detail the subcase where $\mathcal{R}$ is $PK$ and $A = s{:}B$. We have the following situation:

$$\dfrac{\dfrac{G \mid \mathbf{N}, \mathbf{Q} \vdash s{:}B \mid \mathbf{P}, \mathbf{Q}, \mathfrak{M} \Rightarrow B}{G \mid \mathbf{N}, \mathbf{P}, \mathbf{Q}, \mathfrak{M} \Rightarrow s{:}B}\,PK \qquad \begin{array}{c}\vdots\\ H \mid s{:}B, P \Rightarrow C\end{array}}{G \mid H \mid \mathbf{N}, \mathbf{P}, \mathbf{Q}, \mathfrak{M}, P \Rightarrow C}\,cut_{s:B}$$

We have to consider the last rule $\mathcal{R}'$ of $d_2$. If there is no rule $\mathcal{R}'$ which introduces $H \mid s : B, P \Rightarrow C$ because $H \mid s : B, P \Rightarrow C$ is an axiom, then the conclusion is also an axiom. If $\mathcal{R}'$ is a rule in which $s : B$ is not auxiliary, then we can proceed in the standard way by induction on the height (i.e. analogously to `Case 2` above) except for the following situation:

$$\dfrac{\dfrac{G \mid \mathbf{N}, \mathbf{Q} \vdash s{:}B \mid \mathbf{P}, \mathbf{Q}, \mathfrak{M} \Rightarrow B}{G \mid \mathbf{N}, \mathbf{P}, \mathbf{Q}, \mathfrak{M} \Rightarrow s{:}B}\,PK \qquad \dfrac{H \mid s{:}B, \mathbf{N}', \mathbf{Q}' \vdash r{:}C \mid \mathbf{P}', \mathbf{Q}', \mathfrak{P} \Rightarrow C}{H \mid s{:}B, \mathbf{N}', \mathbf{P}', \mathbf{Q}', \mathfrak{P} \Rightarrow r{:}C}\,PK}{G \mid H \mid \mathbf{N}, \mathbf{N}', \mathbf{P}, \mathbf{P}', \mathbf{Q}, \mathbf{Q}', \mathfrak{M}, \mathfrak{P} \Rightarrow r{:}C}\,cut_{s:B}$$

We replace this derivation with the following one:

$$\dfrac{\dfrac{\dfrac{\dfrac{H \mid s{:}B, \mathbf{N}', \mathbf{Q}' \vdash r{:}C \mid \mathbf{P}', \mathbf{Q}', \mathfrak{P} \Rightarrow C}{\underline{G} \mid H \mid \underline{N} \mid \underline{Q} \mid s{:}B, \mathbf{N}', \mathbf{Q}' \vdash r{:}C \mid \mathbf{P}, \mathbf{P}', \mathbf{Q}, \mathbf{Q}', \mathfrak{M}, \mathfrak{P} \Rightarrow C}\,W^*}{G \mid H \mid \mathbf{N}, \mathbf{Q} \vdash s{:}B \mid s{:}B, \mathbf{N}', \mathbf{Q}' \vdash r{:}C \mid \mathbf{P}, \mathbf{P}', \mathbf{Q}, \mathbf{Q}', \mathfrak{M}, \mathfrak{P} \Rightarrow C}\,\star}{G \mid H \mid \mathbf{N}, \mathbf{Q}, \mathbf{N}', \mathbf{Q}' \vdash r{:}C \mid \mathbf{P}, \mathbf{P}', \mathbf{Q}, \mathbf{Q}', \mathfrak{M}, \mathfrak{P} \Rightarrow C}\,Bcut}{G \mid H \mid \mathbf{N}, \mathbf{N}', \mathbf{P}, \mathbf{P}', \mathbf{Q}, \mathbf{Q}', \mathfrak{M}, \mathfrak{P} \Rightarrow r{:}C}\,PK$$

where $W^*$ stands for: repeated application of the weakening rules $W$ and $EW$, while with the $\star$ we refer to those polynomial rules used in the derivation $d_1$ to obtain the TND-sequent $\mathbf{N}, \mathbf{Q} \vdash s{:}B$ and the TND-sequents of $G$.

Note that a case analogous to the previous one is

$$\dfrac{\dfrac{G \mid \mathbf{N},\mathbf{Q} \vdash s{:}B \mid \mathbf{P},\mathbf{Q},\mathfrak{M} \Rightarrow B}{G \mid \mathbf{N},\mathbf{P},\mathbf{Q},\mathfrak{M} \Rightarrow s{:}B}\,{\scriptstyle PK} \qquad \dfrac{H \mid s{:}B,\mathbf{N}',\mathbf{Q}' \vdash r{:}C \mid s{:}B,\mathbf{P}',\mathbf{Q}',\mathfrak{P} \Rightarrow C}{H \mid s{:}B,\mathbf{N}',\mathbf{P}',\mathbf{Q}',\mathfrak{P} \Rightarrow r{:}C}\,{\scriptstyle PK}}{G \mid H \mid \mathbf{N},\mathbf{N}',\mathbf{P},\mathbf{P}',\mathbf{Q},\mathbf{Q}',\mathfrak{M},\mathfrak{P} \Rightarrow r{:}C}\,{\scriptstyle cut_{s:B}}$$

that we replace with the following cut, which is eliminable by induction on the sum of the heights of the derivations of the premises of cut

$$\dfrac{G \mid \mathbf{N},\mathbf{P},\mathbf{Q},\mathfrak{M} \Rightarrow s{:}B \qquad H \mid s{:}B,\mathbf{N}',\mathbf{Q}' \vdash r{:}C \mid s{:}B,\mathbf{P}',\mathbf{Q}',\mathfrak{P} \Rightarrow C}{G \mid H \mid s{:}B,\mathbf{N}',\mathbf{Q}' \vdash r{:}C \mid \mathbf{N},\mathbf{P},\mathbf{P}',\mathbf{Q},\mathbf{Q}',\mathfrak{M},\mathfrak{P} \Rightarrow C}\,{\scriptstyle cut_{s:B}}$$

and then we obtain the conclusion by developing the proof in a way similar to the one shown above.

Let us finally analyse the case where $\mathcal{R}'$ is the rule $PA$. We have the following situation:

$$\dfrac{\dfrac{G \mid \mathbf{N},\mathbf{Q} \vdash s{:}B \mid \mathbf{P},\mathbf{Q},\mathfrak{M} \Rightarrow B}{G \mid \mathbf{N},\mathbf{P},\mathbf{Q},\mathfrak{M} \Rightarrow s{:}B}\,{\scriptstyle PK} \qquad \dfrac{H \mid s{:}B,B,P \Rightarrow C}{H \mid s{:}B,P \Rightarrow C}\,{\scriptstyle PA}}{G \mid H \mid \mathbf{N},\mathbf{P},\mathbf{Q},\mathfrak{M},P \Rightarrow C}\,{\scriptstyle cut_{s:B}}$$

We reduce to

$$\dfrac{\dfrac{\dfrac{G \mid \mathbf{N},\mathbf{Q} \vdash s{:}B \mid \mathbf{P},\mathbf{Q},\mathfrak{M} \Rightarrow B \qquad \dfrac{G \mid \mathbf{N},\mathbf{P},\mathbf{Q},\mathfrak{M} \Rightarrow s{:}B \quad H \mid s{:}B,B,P \Rightarrow C}{G \mid H \mid B,\mathbf{N},\mathbf{P},\mathbf{Q},\mathfrak{M},P \Rightarrow C}\,{\scriptstyle cut_{sB}}}{G \mid G \mid H \mid \mathbf{N},\mathbf{Q} \vdash s{:}B \mid \mathbf{N},\mathbf{P},\mathbf{P},\mathbf{Q},\mathbf{Q},\mathfrak{M},\mathfrak{M},P \Rightarrow C}\,{\scriptstyle cut_B}}{G \mid G \mid H \mid \mathbf{N},\mathbf{P},\mathbf{P},\mathbf{Q},\mathbf{Q},\mathfrak{M},\mathfrak{M},P \Rightarrow C}\,{\scriptstyle E}}{G \mid H \mid \mathbf{N},\mathbf{P},\mathbf{Q},\mathfrak{M},P \Rightarrow C}\,{\scriptstyle C^*+E^*}$$

where the first cut is eliminable by induction on the sum of the heights of the derivations of the premises of cut and the second cut is eliminable by induction on the complexity of the cut formula. □

**Theorem 4.4.** *Every derivation $d$ of a proof sequent $G \mid \Sigma$ in* **Gilp** *+ cut can be effectively transformed into a cut-free derivation $d'$ of $G \mid \Sigma$.*

*Proof.* It follows from the previous lemma by induction on the number of cuts. □

# 5 In between

We have thus introduced a new sequent calculus **Gilp** for the intuitionistic logic of proofs. Despite the fact that this calculus possesses several desirable properties, such as the admissibility of the contraction rules or the invertibility of the left logical rules, it does not solve the problem of the lack of the subformula property for the logic of proofs.

Consider, for example, the following theorem of **Ilp**

$$t{:}A \wedge s{:}B \rightarrow (c \cdot t \cdot s){:}(A \wedge B)$$

Informally speaking this theorem says that, if we have a proof $t$ of the formula $A$, and a proof $s$ of the formula $B$, then we can construct the proof $(c \cdot t \cdot s)$ of the formula $A \wedge B$. The proof (polynomial) $(c \cdot t \cdot s)$ is constructed by means of the rule $ci$, which introduces the formula $c:(A \rightarrow (B \rightarrow A \wedge B))$, and two applications of the rule $\odot$, in the following way:

$$\dfrac{\dfrac{\dfrac{s{:}B \vdash s{:}B \mid t{:}A \vdash t{:}A}{s{:}B \vdash s{:}B \mid t{:}A \vdash t{:}A \mid\ \vdash c{:}A \rightarrow (B \rightarrow (A \wedge B))}\ ci}{s{:}B \vdash s{:}B \mid t{:}A \vdash (c \cdot t){:}B \rightarrow (A \wedge B)}\ \odot}{s{:}B, t{:}A \vdash (c \cdot t \cdot s){:}(A \wedge B)}\ \odot$$

The construction here above involves violation of the subformula property: the formulas $c{:}A \rightarrow (B \rightarrow (A \wedge B))$ and $(c \cdot t){:}B \rightarrow (A \wedge B)$ are not subformulas (according to Definition 2.4) of any of the formulas that occur in the conclusion. This is due to the fact that the formula $c{:}A \rightarrow (B \rightarrow (A \wedge B))$ is introduced into the construction by an application of the $ci$ rule, and the two formulas $A$ and $B$ are subsequently eliminated in two applications of the $\odot$ rule. The constants followed by the two dots, in the proof polynomial $c \cdot t \cdot s$, reflect this double violation of the subformula property.

Suppose that you want an analytic proof for the formula $A \wedge B$. This involves two parallel things: an alternative proof polynomial $r$ that labels the formula $A \wedge B$ and an analytic construction for this proof polynomial $r$. A quick reflection on the proof polynomial symbols (other than the dot) of $\mathcal{L}_{lp}$ and their corresponding rules in **Gilp** is enough to realize that there is no way to obtain these two things in **Ilp**: indeed, the only other symbols, and corresponding rules, are ! and +, which are evidently inadequate for this purpose. Thus it seems that the language of the logic of proofs, together with the associated rules of the calculus **Gilp**, are too poor to formulate analytic proofs at the proof polynomial level.

In the light of this diagnosis, the remedy to this situation and the route towards the desired analyticity passes through a modification of the language of the logic of proofs together with an enhancement of the rules of the calculus **Gilp**. This is exactly what we shall do. More precisely, we extend the language of the logic of proofs by means of the functional symbols of the lambda calculus and we add to **Gilp** the corresponding rules.

In order to motivate the use of the functional symbols of the lambda calculus, let us focus on the constants of the logic of proofs. We can think of each constant introduced by the rule *ci* as being labelled by one and only one axiom of the logic of proofs (see Section 8 for further details). The constant $c$ in the example above is labelled by the axiom $A \to (B \to A \wedge B)$. In the typed lambda calculus, we know that each intuitionistic axiom types a (closed) lambda term in normal form. For example, the lambda term $\lambda x.\lambda y.\mathtt{p}(x, y)$ is associated with the axiom $A \to (B \to A \wedge B)$. Therefore there seems to be the following correspondence: constants are associated with axioms, and axioms type lambda terms in normal form.

Suppose that we replace constants by the appropriate lambda terms in normal form. Then the formula

$$t{:}A \wedge s{:}B \to (c \cdot t \cdot s){:}(A \wedge B)$$

becomes

$$t{:}A \wedge s{:}B \to ((\lambda x.\lambda y.\mathtt{p}(x, y)) \cdot t \cdot s){:}(A \wedge B)$$

An important difference between the constants of the logic of proofs and lambda terms is that whilst the former have no internal structure, and so can only be introduced in the *ci* rule, the latter do have internal structure, and can themselves be constructed. Consider, for example, the proof polynomial $(\lambda x.\lambda y.\mathtt{p}(x, y)) \cdot t \cdot s$, which can be constructed in the following way:[6]

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{t{:}A \vdash t{:}A \mid s{:}B \vdash s{:}B \mid x : A \vdash x : A \mid y : B \vdash y : B}{t{:}A \vdash t{:}A \mid s{:}B \vdash s{:}B \mid x : A, y : B \vdash (\mathtt{p}(x, y)){:}(A \wedge B)}}{t{:}A \vdash t{:}A \mid s{:}B \vdash s{:}B \mid x : A \vdash (\lambda y.\mathtt{p}(x, y)){:}(B \to (A \wedge B))}}{t{:}A \vdash t{:}A \mid s{:}B \vdash s{:}B \mid \ \vdash ((\lambda x.\lambda y.\mathtt{p}(x, y))){:}(A \to (B \to (A \wedge B)))}}{s{:}B \vdash s{:}B \mid t{:}A \vdash ((\lambda x.\lambda y.\mathtt{p}(x, y)) \cdot t){:}B \to (A \wedge B)}}{t{:}A, s{:}B \vdash ((\lambda x.\lambda y.\mathtt{p}(x, y)) \cdot t \cdot s){:}(A \wedge B)}$$

The construction above involves the analogue of what, in natural deduction, is an introduction rule followed by an elimination rule – or a 'cut' –, and it is known that such combinations of rules yield violations of the subformula property. Given that the construction is reflected at the proof polynomial level, a similar point holds for the term $((\lambda x.\lambda y.\mathtt{p}(x, y)) \cdot t \cdot s)$, which contains two redexes. Thus the use of lambda terms at the place of constants makes our problems with the loss of analyticity much clearer.

Moreover, the introduction of lambda terms allows us to rely on standard methods for eliminating such "cuts" or redexes: by normalization. Indeed the proof above can be reduced to this one,

[6] We freely use the standard rules of the lamda-calculus (e.g. [15]) to construct the following derivation. We allow ourselves such a freedom for the sake of an explanation in intuitive terms. The rules for constructing appropriate lambda terms will be given in the next section.

$$\frac{t\!:\!A \vdash t\!:\!A \mid s\!:\!B \vdash s\!:\!B}{t\!:\!A, s\!:\!B \vdash (\mathtt{p}(x,y))\!:\!(A \wedge B)}$$

and correspondingly the formula $t : A \wedge s : B \to \mathtt{p}(t,s) : (A \wedge B)$ is a reduct of $t : A \wedge s : B \to ((\lambda x.\lambda y.\mathtt{p}(x,y)) \cdot t \cdot s) : (A \wedge B)$. Hence, given a proof $t$ of $A$ and a proof $s$ of $B$, we can construct a proof, which respects the subformula property, of a formula of the form $r : A \wedge B$. The appropriate $r$ is $\mathtt{p}(t,s)$, which, as suggested, does not belong to the standard language of the logic of proofs, but to a language extended by the addition of lambda terms.

In what follows we exploit these intuitions. We firstly define a new language $\mathcal{L}^*_{lp}$, obtained from the standard language for the logic of proofs $\mathcal{L}_{lp}$ by replacing constants with the functional symbols of the lambda calculus. Beyond the functional symbols of the standard lambda calculus, which type the axioms of intuitionistic logic, we also add the functional symbols of the lambda calculus for the logic of proofs [2], which serve to type the specific axioms of the logic of proofs, i.e. the axioms $A_1 - A_4$. We then construct a sequent calculus **Gilp**$^*$ that extends the calculus **Gilp** thanks to the addition of appropriate rules for the new symbols of $\mathcal{L}^*_{lp}$. We show that **Gilp**$^*$ enjoys the subformula property. Finally we prove that **Gilp**$^*$ can be thought of as the framework where the analyticity of the logic of proofs is reached, since it is a conservative extension of **Gilp**.

# 6 The calculus Gilp$^*$

In this section we will introduce the language $\mathcal{L}^*_{lp}$ and we will formulate in this new language the sequent calculus **Gilp**$^*$. Moreover we will show that **Gilp**$^*$ is cut-free and contraction-free and that its left-side logical rules are invertible.

**Definition 6.1.** The language $\mathcal{L}^*_{lp}$ contains: (i) the usual language of propositional boolean logic, (ii) proof variables $x_0, x_1, x_2, ...$, (iii) the functional symbols $+$, $!$, $\mathtt{p}$, $\mathtt{p}_i$, $\mathtt{k}_i$, $\mathbb{E}^{\vee}_{x,y}$, $\mathbb{E}^{\perp}_{S}$ (for any atomic formula $S$[7]), $\lambda x$, $\cdot$, $\mathbb{P}_{t,t'}$, $\mathbb{U}_t$, $\mathbb{B}_t$, $\mathbb{S}_t$ (for any proof polynomials $t$, $t'$) (iv) the operator symbol of the type "term : formula." Terms, which we call proof polynomials as before, are built from the proof-variables by the functional symbols, and formulas are built as in Definition 2.3. The arities of the functional symbols will be made clear in the lambda and polynomial lambda rules in Figures 2 and 3. $u, v, w, ...$ will denote proof variables, while $p, q, r, s, t, ...$ will denote proof polynomials.

The language $\mathcal{L}^*_{lp}$ is obtained from the language $\mathcal{L}_{lp}$ by dropping the proof constants and adding the functional symbols of the typed lambda calculus for the logic of proofs [2].[8] We recall that this lambda calculus is made up of the

[7]We follow [15, p178] in restricting the function symbol $\mathbb{E}^{\perp}_{S}$ (and consequently applications of the $\perp E$ rule - see Figure 2) to atomic conclusions. As for these authors (see eg. [15, §6.1.8B]), this restriction is adopted in the interest of simplicity.

[8]Let us note that we adopt the notation used in [2].

Figure 2: Lambda Rules

$$\frac{G \;|\; \mathbf{M}, \mathbf{P} \vdash t_0 : A_0 \;|\; \mathbf{M}, \mathbf{Q} \vdash t_1 : A_1 \;|\; \Sigma}{G \;|\; \mathbf{M}, \mathbf{P}, \mathbf{Q} \vdash \mathtt{p}(t_0, t_1) : (A_0 \wedge A_1) \;|\; \Sigma} \wedge I \qquad \frac{G \;|\; \mathbf{M} \vdash t : (A_0 \wedge A_1) \;|\; \Sigma}{G \;|\; \mathbf{M} \vdash \mathtt{p_i}(t) : A_i \;|\; \Sigma} \wedge E$$

$$\frac{G \;|\; \mathbf{M} \vdash t : A_i \;|\; \Sigma}{G \;|\; \mathbf{M} \vdash \mathtt{k_i}(t) : (A_0 \vee A_1) \;|\; \Sigma} \vee I$$

$$\frac{G \;|\; \mathbf{M}, \mathbf{P}, \mathbf{R} \vdash t : (A_0 \vee A_1) \;\;|\; \mathbf{M}, \mathbf{Q}, \mathbf{R}', x : A_0 \vdash q : E \;\;|\; \mathbf{P}, \mathbf{Q}, \mathbf{R}'', y : A_1 \vdash q' : E \;\;|\; \Sigma}{G \;|\; \mathbf{M}, \mathbf{P}, \mathbf{Q}, \mathbf{R}, \mathbf{R}', \mathbf{R}'', \vdash \mathbb{E}^{\vee}_{x,y}(t, q, q') : E \;\;|\; \Sigma} \vee E$$

$$\frac{G \;|\; \mathbf{M}, x : A \vdash t(x) : F \;\;|\; \Sigma}{G \;|\; \mathbf{M} \vdash \lambda x.t(x) : (A \to F) \;\;|\; \Sigma} \lambda \qquad \frac{G \,|\, \mathbf{M} \vdash t : \bot \,|\, \Sigma}{G \,|\, \mathbf{M} \vdash \mathbb{E}^{\bot}_{S}(t) : S \,|\, \Sigma} \bot E$$

where in the rules $\wedge E$ and $\vee I$, $i = 0, 1$, while in the rule $\vee E$, $x$ does not occur in $\mathbf{M}$, $\mathbf{Q}$, $\mathbf{R}^{'}$ and any occurrences in $E$ are shifted down; similarly for $y$ with respect to $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$ and $E$. Finally, in the rule $\lambda$, $x$ does not occur in $\mathbf{M}$ and any occurrences in $F$ are shifted down.

functional symbols of the typed lambda calculus[9] ($\mathtt{p}$, $\mathtt{p}_i$, $\mathtt{k}_i$, $\mathbb{E}^{\vee}_{x,y}$, $\mathbb{E}^{\bot}_{S}$, $\lambda x$) and four other functional symbols ($\mathbb{P}_{t,t'}$, $\mathbb{U}_t$, $\mathbb{B}_t$, $\mathbb{S}_t$) that are used for constructing proofs of the axioms $A_1 - A_4$. As will be clear from the associated rules (Figure 3), the symbols $\mathbb{P}_{t,t'}$, $\mathbb{U}_t$, $\mathbb{B}_t$, $\mathbb{S}_t$ operate on formulas with at least two levels of proof polynomials; unlike [2], indices are used to indicate the inner proof polynomials. Moreover, note that, just as ! 'shifts up' a proof polynomial (see the rule ! in Section 2), $\mathbb{U}_t$ 'shifts down' a proof polynomial. This remark motivates the following definition.

**Definition 6.2.** The occurrence of a proof polynomial $t$ in a formula $A$ is *shifted down* if it appears in the scope of more symbols $\mathbb{U}_t$ than symbols !.

In order to introduce the calculus **Gilp**$^*$, it is useful to introduce the following definition.

**Definition 6.3.** Define the sets $\mathbb{PT}_0, \mathbb{PT}_1, \dots$ of proof polynomials in $\mathcal{L}^*_{lp}$ inductively as follows:

- $\mathbb{PT}_0 = \{x_0, x_1, x_2, \dots\}$
- $t \in \mathbb{PT}_{i+1}$ if, and only if there exists a proof sequent $\mathbf{M} \vdash t : A \mid S \Rightarrow S$ such that: (i) all proof polynomials occurring in $\mathbf{M}$ are variables, (ii) it has

[9] E.g. see [14].

Figure 3: Polynomial Lambda Rules

$$\frac{G \mid \mathbf{M} \vdash r{:}t{:}A \mid \Sigma}{G \mid \mathbf{M} \vdash \mathbb{U}_t(r){:}A \mid \Sigma}\, tE \qquad \frac{G \mid \mathbf{M} \vdash r{:}t{:}A \mid \Sigma}{G \mid \mathbf{M} \vdash \mathbb{B}_t(r){:}!t:t:A \mid \Sigma}\, !I$$

$$\frac{G \mid \mathbf{M} \vdash r{:}t_i{:}A \mid \Sigma}{G \mid \mathbf{M} \vdash \mathbb{S}_{t_i}(r){:}(t_0 + t_1):A \mid \Sigma}\, +I$$

$$\frac{G \mid \mathbf{M}, \mathbf{P} \vdash r{:}t{:}(A \to F) \mid \mathbf{M}, \mathbf{Q} \vdash r'{:}t'{:}A \mid \Sigma}{G \mid \mathbf{M}, \mathbf{P}, \mathbf{Q} \vdash \mathbb{P}_{t,t'}(r, r'){:}(t \cdot t'){:}F \mid \Sigma}\, \odot I$$

where in the rule $+I$, $i$ = 0,1.

been obtained from the proof sequent $\underline{\mathbf{M}} \mid S \Rightarrow S$ by applications of the rules +, !, or $\odot$ or of the rules in Figures 2 and 3, where all applications of the + rule introduce proof polynomials in $\mathbb{PT}_i$ .

$\mathbb{PT}$ is defined to be $\bigcup \mathbb{PT}_i$. $\mathbb{FT}$ is the set of formulas $A$ such that all the proof polynomials occurring in $A$ are in $\mathbb{PT}$.

So, for example, terms such as $(\lambda x.xx)(\lambda x.xx)$, $\lambda x.x + (\lambda x.xx)(\lambda x.xx)$ and $\mathbb{B}_s(\mathbb{U}_t(\lambda x.x + \lambda y.y) + \lambda x \lambda y.x) + (\lambda x.xx)(\lambda x.xx)$ are not in $\mathbb{PT}$. By contrast, terms such as $\lambda x.x$, $\lambda x.x + \lambda y.y$ and $\mathbb{B}_s(\mathbb{U}_t(\lambda x.x + \lambda y.y) + !(\lambda x \lambda y.x + \lambda x.x))$ are in $\mathbb{PT}$ (in fact, they are in $\mathbb{PT}_1$, $\mathbb{PT}_2$ and $\mathbb{PT}_3$ respectively).

It is straightforward to see that, now that proof polynomials are built using new functional symbols, there is a strict correlation between proof polynomials and lambda terms. More precisely, there is a natural mapping between a subclass of the proof polynomials of $\mathcal{L}^*_{lp}$, namely those where there is no ! and +, and the terms of the typed lambda calculus for the logic of proofs. With slight abuse of notation, we speak invariably of proof polynomials and their associated lambda terms; for example, we say that a proof polynomial is in normal form if the associated lambda term is in normal form (e.g. see Section 7).

However, the reader should be aware of important dissimilarities between the role of proof polynomials and that of lambda terms in the typed lambda calculus. Firstly, whilst there is a strict distinction between lambda terms and typing formulas, this is not the case for proof polynomials, which may occur inside, as well as in front of, formulas. Secondly, as will be clear from the specification of the calculus **Gilp**$^*$, nowhere is it demanded that formulas of the form $t : A$ appearing in proofs are "properly typed": no specific relation is required between $t$ and $A$ when such formulas appear in the axioms, for example.

We can now introduce the calculus **Gilp**$^*$.

**Definition 6.4.** The calculus **Gilp**$^*$ is composed of: (i) axioms of the following form

$$t_1 \!:\! A_1 \vdash t_1 \!:\! A_1 \mid ... \mid t_n \!:\! A_n \vdash t_n \!:\! A_n \mid M, S \Rightarrow S$$

$$t_1 \!:\! A_1 \vdash t_1 \!:\! A_1 \mid ... \mid t_n \!:\! A_n \vdash t_n \!:\! A_n \mid M, \bot \Rightarrow C$$

(ii) the propositional, proof rules and polynomial rules, except the rule *ci*, of the calculus **Gilp** (iii) the lambda rules and the polynomial lambda rules in Figures 2 and 3. Items (i)–(iii) satisfy the following restrictions: any formula occurring in the axioms or introduced in the $\vee$, $\lambda$ or $\vee I$ rules belongs to the set $\mathbb{FT}$, and any proof polynomial introduced in the $+$ or $+I$ rules belongs to the set $\mathbb{PT}$.

Let us make the following remarks about the rules of **Gilp**$^*$. While the propositional and proof rules can be divided into left and right introduction rules, the lambda rules together with the rule $\odot$ can be divided into introduction and elimination rules, in the standard way. The rule $\odot$ that *visually* betrays the analyticity property is still present in the calculus **Gilp**$^*$, nevertheless the context in which this rule is to be evaluated has completely changed. Indeed now $\odot$ is only one of the several eliminations rules of the lambda part of the calculus **Gilp**$^*$; all these elimination rules behave as in the natural deduction framework: although visually they seem to betray analyticity, they only do so when combined with certain introduction rules. This crucial point, which has already been illustrated in the previous section, will become further evident in what follows.

Let us note that in the elimination rules we adopt the standard distinction between minor and major premises (e.g. see [15, p. 37]). Finally, the structural rules of **Gilp**$^*$ are the same as those of **Gilp** (see Figure 1) with the sole difference that the rules $W$ and $EW$ carry the proviso that A and s:B are in $\mathbb{FT}$.

Given the resemblance between the calculi **Gilp** and **Gilp**$^*$, it is not surprising that **Gilp**$^*$ inherits many of the important properties of **Gilp**.

**Lemma 6.5.** *In* **Gilp**$^*$*, each of the polynomial, lambda and polynomial lambda rules permutes up with any rule whose conclusion is not one of its auxiliary TND-sequents.*

*Proof.* Analogously to the proof of Lemma 4.1. □

**Lemma 6.6.** *In **Gilp**$^*$ the following holds:*

*(i) the weakening rules $W$ and $EW$ together with the rule El and the contraction rule $C$ are height-preserving admissible.*

*(ii) The propositional left-rules and the proof rule $PA$ are height-preserving invertible.*

*(iii) The rule cut$^*$ is admissible.*

*Proof.* As for (i) and (ii), the proof is developed analogously to the proof of Lemma 3.4. As for (iii), the proof is developed analogously to the proof of Lemma 4.2, exploiting Lemma 6.5. □

**Theorem 6.7.** *Every derivation $d$ in **Gilp**$^*$ + cut of a proof sequent $G \mid \Sigma$ can be effectively transformed into a cut-free derivation $d'$ of $G \mid \Sigma$.*

*Proof.* The proof is developed analogously to the proof of Theorem 4.4. □

# 7 Normalisation for Gilp$^*$

In this section, we will prove that **Gilp**$^*$ satisfies the subformula property, i.e. it is an analytic calculus.[10] In the next section, we will show that **Gilp**$^*$ can be thought of as an analytic calculus for the logic of proofs.

## 7.1 General strategy

Let us start by clarifying what it takes to prove the analyticity of **Gilp**$^*$. **Gilp**$^*$, like **Gilp**, is not a standard sequent calculus: these calculi are composed by two calculi, the sequent one (indicated by the sequent part in Figure 4) and a TND-sequent one (indicated by the TND-sequent part in Figure 4); to get the analyticity of the whole calculus, we will have to prove the analyticity of each of its respective parts. Theorem 6.7 establishes the desired property for the sequent part; so let us focus on the TND-sequent part.

In the TND-sequent part we are basically dealing with a natural deduction calculus. Standardly, in order to prove the analyticity of a natural deduction calculus, we prove a normalization theorem. This is exactly what we are going to do; on the other hand, given the peculiarity of the situation, there are three points that call for careful consideration.

The first is related to the identification of the cuts that we want to get rid of in the normalization theorem for **Gilp**$^*$. In the intuitionistic natural deduction calculus, certain combinations of rules - usually called cuts - give rise to the loss of the subformula property. Thus, via the normalization theorem, one demonstrates that anything which can be proved with these combinations of rules can also be proved without. Such cuts also appear in **Gilp**$^*$ (we shall call them *TND-cuts* to distinguish them from the cuts in the sequent part) and can be treated by techniques akin to the standard ones. But given that **Gilp**$^*$ is composed of many rules other than the standard ones, the question arises of whether there are new *TND-cuts* to be identified and treated. Let us call this point **Point1**.

The resolution of **Point1** presupposes a precise definition of the subformula property for the language $\mathcal{L}^*_{lp}$. For if we have to isolate those combinations of rules that violate this property, then we need to know what this property amounts to in the framework of **Gilp**$^*$. A quick reflection is enough to see that

[10] Henceforth, we use “analytic” and “subformula property” interchangeably.

Figure 4: Summary of the axioms and rules of $\mathbf{Gilp}^*$ (omitting side conditions)

### Axioms

$t_1 : A_1 \vdash t_1 : A_1 \mid ... \mid t_n : A_n \vdash t_n : A_n \mid S, M \Rightarrow S$

$t_1 : A_1 \vdash t_1 : A_1 \mid ... \mid t_n : A_n \vdash t_n : A_n \mid M, \bot \Rightarrow C$

### Sequent Part

$$\frac{G \mid A, B, M \Rightarrow C}{G \mid A \wedge B, M \Rightarrow C}\ {\scriptstyle \wedge A} \qquad \frac{G \mid A, M \Rightarrow C \quad G \mid B, M \Rightarrow C}{G \mid A \vee B, M \Rightarrow C}\ {\scriptstyle \vee A} \qquad \frac{G \mid A \rightarrow B, M \Rightarrow A \quad G \mid B, M \Rightarrow C}{G \mid A \rightarrow B, M \Rightarrow C}\ {\scriptstyle \rightarrow A}$$

$$\frac{G \mid M \Rightarrow A \quad G \mid M \Rightarrow B}{G \mid M \Rightarrow A \wedge B}\ {\scriptstyle \wedge K} \qquad \frac{G \mid M \Rightarrow A_i}{G \mid M \Rightarrow A_0 \vee A_1}\ {\scriptstyle \vee K} \qquad \frac{G \mid A, M \Rightarrow B}{G \mid M \Rightarrow A \rightarrow B}\ {\scriptstyle \rightarrow K}$$

$$\frac{G \mid t : A, A, M \Rightarrow C}{G \mid t : A, M \Rightarrow C}\ {\scriptstyle PA} \qquad \frac{G \mid \mathbf{N}, \mathbf{P} \vdash t : A \mid \mathbf{N}, \mathbf{Q}, \mathfrak{M} \Rightarrow A}{G \mid \mathbf{N}, \mathbf{P}, \mathbf{Q}, \mathfrak{M} \Rightarrow t : A}\ {\scriptstyle PK}$$

### TND – Sequent Part

$$\frac{G \ \mid \mathbf{M}, \mathbf{P} \vdash t_0 : A_0 \mid \mathbf{M}, \mathbf{Q} \vdash t_1 : A_1 \mid \Sigma}{G \ \mid \mathbf{M}, \mathbf{P}, \mathbf{Q} \vdash \mathtt{p}(t_0, t_1) : (A_0 \wedge A_1) \mid \Sigma}\ {\scriptstyle \wedge I} \qquad \frac{G \ \mid \mathbf{M} \vdash t : (A_0 \wedge A_1) \mid \Sigma}{G \ \mid \mathbf{M} \vdash \mathtt{p_i}(t) : A_i \mid \Sigma}\ {\scriptstyle \wedge E}$$

$$\frac{G \ \mid \mathbf{M} \vdash t : A_i \mid \Sigma}{G \ \mid \mathbf{M} \vdash \mathtt{k_i}(t) : (A_0 \vee A_1) \mid \Sigma}\ {\scriptstyle \vee I} \qquad \frac{G \ \mid \mathbf{M}, \mathbf{P}, \mathbf{R} \vdash t : (A_0 \vee A_1) \ \mid \mathbf{M}, \mathbf{Q}, \mathbf{R}', x : A_0 \vdash q : E \ \mid \mathbf{P}, \mathbf{Q}, \mathbf{R}'', y : A_1 \vdash q' : E}{G \ \mid \mathbf{M}, \mathbf{P}, \mathbf{Q}, \mathbf{R}, \mathbf{R}', \mathbf{R}'', \vdash \mathbb{E}^{\vee}_{x,y}(t, q, q') : E \ \mid \Sigma}$$

$$\frac{G \ \mid \mathbf{M}, x : A \vdash t(x) : F \ \mid \Sigma}{G \ \mid \mathbf{M} \vdash \lambda x.t(x) : (A \rightarrow F) \ \mid \Sigma}\ {\scriptstyle \lambda} \qquad \frac{G \ \mid \mathbf{M}, \mathbf{P} \vdash t : (A \rightarrow F) \mid \mathbf{M}, \mathbf{Q} \vdash t' : A \mid \Sigma}{G \mid \mathbf{M}, \mathbf{P}, \mathbf{Q} \vdash (t \cdot t') : F \mid \Sigma}\ {\scriptstyle \odot}$$

$$\frac{G \mid \mathbf{M} \vdash t : \bot \mid \Sigma}{G \mid \mathbf{M} \vdash \mathbb{E}^{\bot}_{S}(t) : S \mid \Sigma}\ {\scriptstyle \bot E}$$

$$\frac{G \mid \mathbf{M} \vdash t_i : A \mid \Sigma}{G \mid \mathbf{M} \vdash (t_0 + t_1) : A \mid \Sigma}\ {\scriptstyle +} \qquad \frac{G \mid \mathbf{M} \vdash t : A \mid \Sigma}{G \mid \mathbf{M} \vdash !t : t : A \mid \Sigma}\ {\scriptstyle !}$$

$$\frac{G \mid \mathbf{M} \vdash r : t : A \mid \Sigma}{G \mid \mathbf{M} \vdash \mathbb{U}(r) : A \mid \Sigma}\ {\scriptstyle tE} \qquad \frac{G \mid \mathbf{M} \vdash r : t : A \mid \Sigma}{G \mid \mathbf{M} \vdash \mathbb{B}(r) : !t : t : A \mid \Sigma}\ {\scriptstyle !I}$$

$$\frac{G \mid \mathbf{M} \vdash r : t_i : A \mid \Sigma}{G \mid \mathbf{M} \vdash \mathbb{S}_i(r) : (t_0 + t_1) : A \mid \Sigma}\ {\scriptstyle +I} \qquad \frac{G \mid \mathbf{M}, \mathbf{P} \vdash r : t : (A \rightarrow F) \mid \mathbf{M}, \mathbf{Q} \vdash r' : t' : A \mid \Sigma}{G \mid \mathbf{M}, \mathbf{P}, \mathbf{Q} \vdash \mathbb{P}(r, r') : (t \cdot t') : F \mid \Sigma}\ {\scriptstyle \odot I}$$

the standard notion of subformula property is no longer adequate. Consider, for example, the following derivation $d$

$$\dfrac{\dfrac{t\!:\!A \vdash t\!:\!A \mid y\!:\!B \vdash y\!:\!B \mid A, B \Rightarrow A}{t\!:\!A, y\!:\!B \vdash \mathtt{p}(t,y) : (A \wedge B) \mid A, B \Rightarrow A}\ {\scriptstyle \wedge I}}{t\!:\!A \vdash \lambda y.\mathtt{p}(t,y) : B \rightarrow (A \wedge B) \mid A, B \Rightarrow A}\ {\scriptstyle \lambda}$$

In $d$ we only use two rules, for the introduction of conjunction and the introduction of implication, both in the TND-sequent part. The rules for the introduction of conjunction and implication are typically inoffensive and thus there should be no loss of analyticity. Nevertheless, if we consider the formulas $y : B$ and $\mathtt{p}(t,y) : (A \wedge B)$ (in the second proof sequent counting from the top) and follow the standard notion of subformula for the logic of proofs (see Definition 2.4), we see that none of them is a subformula of one of the formulas that compose the conclusion. Such a situation is unsatisfactory; an appropriate notion of subformula needs to be defined for $\mathcal{L}^*_{lp}$ according to which, at least, the formulas $y\!:\!B$ and $\mathtt{p}(t,y) : (A \wedge B)$ are counted as subformulas of the formula $\lambda y.\mathtt{p}(t,y) : B \rightarrow (A \wedge B)$. Let us call this point **Point2**.

Let us now turn to the third and final point. It is related to the fact that in **Gilp**$^*$ any TND-cut is "reflected" at the proof polynomial level and thus may have to be taken into account in the reduction of TND-cuts. In order to clarify the point, consider the following derivation $d$

$$\dfrac{\dfrac{\dfrac{t\!:\!A \vdash t\!:\!A \mid y\!:\!B \vdash y\!:\!B \mid t\!:\!A \vdash t\!:\!A \mid y\!:\!B \vdash y\!:\!B \mid A, B \Rightarrow A}{t\!:\!A \vdash t\!:\!A \mid y\!:\!B \vdash y\!:\!B \mid t\!:\!A, y\!:\!B \vdash \mathtt{p}(t,y) : (A \wedge B) \mid A, B \Rightarrow A}\ {\scriptstyle \wedge I}}{t\!:\!A \vdash t\!:\!A \mid y\!:\!B \vdash y\!:\!B \mid t\!:\!A, y\!:\!B \vdash \mathtt{p}(t,y) : (A \wedge B) \mid A \wedge B \Rightarrow A}\ {\scriptstyle \wedge A}}{t\!:\!A \vdash t\!:\!A \mid y\!:\!B \vdash y\!:\!B \mid t\!:\!A, y\!:\!B \vdash \mathtt{p}_0(\mathtt{p}(t,y)) : A \mid A \wedge B \Rightarrow A}\ {\scriptstyle \wedge E}$$

If one concentrates on the TND-sequent part, one easily sees that it contains a quite standard (TND-)cut, namely the conjunction introduction rule immediately followed by the conjunction elimination rule. Such a TND-cut is "reflected" by the proof polynomial $\mathtt{p}_0(\mathtt{p}(t,y))$, which corresponds to a lambda term that contains a contraction.

Suppose that we want to reduce the derivation $d$ to a normal one, which is to say a derivation that does not contain any TND-cut. This will involve not only a new TND-cut-free derivation $d'$, but also a modified conclusion. Indeed, following what is usually done in natural deduction calculi (and in the lambda calculus), we have the following derivation $d'$:

$$\dfrac{t\!:\!A \vdash t\!:\!A \mid y\!:\!B \vdash y\!:\!B \mid t\!:\!A \vdash t\!:\!A \mid A, B \Rightarrow A}{t\!:\!A \vdash t\!:\!A \mid y\!:\!B \vdash y\!:\!B \mid t\!:\!A \vdash t\!:\!A \mid A \wedge B \Rightarrow A}\ {\scriptstyle \wedge A}$$

The conclusion of $d'$ is different from that of $d$: in the same way as the presence of the TND-cut in $d$ is reflected at the proof polynomial level by means of the proof polynomial $\mathtt{p}_0(\mathtt{p}(t,y))$, its absence in the derivation $d'$ is reflected by means of the proof polynomial $t$.[11] Just as a reduction of a derivation in the natural deduction calculus corresponds to a reduction of the corresponding lambda term, in the calculus **Gilp**$^*$ the reduction of a derivation goes hand in hand with a corresponding modification at the proof polynomial level. Hence the analyticity of **Gilp**$^*$ in the TND-sequent part is proved by showing that given a derivation $d$ of $t : A$, we can construct TND-cut-free derivation $d'$ of $s : A$, where $s$ is a "redex" of t.

The example discussed above is a particularly simple case. Since, as noted above, there is no watertight division in **Gilp**$^*$ between proof polynomials and formulas, proof polynomials may occur at several points in a formula. In the reduction of the derivation of such formulas, several of the proof polynomials occurring in it may be replaced by "redexes". Consider for example the following development of the derivation $d$.

$$
\begin{array}{cl}
\vdots & \\
t{:}A \vdash t{:}A \mid y{:}B \vdash y{:}B \mid t{:}A, y{:}B \vdash \mathtt{p}_0(\mathtt{p}(t,y)) : A \mid A \wedge B \Rightarrow A & \\
\hline
t{:}A, y{:}B \vdash \mathtt{p}(t,y) : (A \wedge B) \mid t{:}A, y{:}B \vdash \mathtt{p}_0(\mathtt{p}(t,y)) : A \mid A \wedge B \Rightarrow A & \wedge I \\
\hline
t{:}A, y{:}B \vdash \mathtt{p}_0(\mathtt{p}(t,y)) : A \mid t{:}A, y{:}B \vdash \mathtt{p}_0(\mathtt{p}(t,y)) : A \mid A \wedge B \Rightarrow A & \wedge E \\
\hline
t{:}A, y{:}B \vdash !\mathtt{p}_0(\mathtt{p}(t,y)) : \mathtt{p}_0(\mathtt{p}(t,y)) : A \mid t{:}A, y{:}B \vdash \mathtt{p}_0(\mathtt{p}(t,y)) : A \mid A \wedge B \Rightarrow A & ! \\
\hline
t{:}A, y{:}B \vdash !\mathtt{p}_0(\mathtt{p}(t,y)) : \mathtt{p}_0(\mathtt{p}(t,y)) : A \mid t{:}A, y{:}B, A \wedge B \Rightarrow \mathtt{p}_0(\mathtt{p}(t,y)) : A & PK \\
\hline
t{:}A, y{:}B, A \wedge B \Rightarrow !\mathtt{p}_0(\mathtt{p}(t,y)) : \mathtt{p}_0(\mathtt{p}(t,y)) : A & PK
\end{array}
$$

To reduce the derivation $d$ to a normal one, one would construct the derivation $d''$ as follows

$$
\begin{array}{cl}
t{:}A \vdash t{:}A \mid t{:}A \vdash t{:}A \mid A, B \Rightarrow A & \\
\hline
t{:}A \vdash t{:}A \mid t{:}A \vdash t{:}A \mid A \wedge B \Rightarrow A & \wedge A \\
\hline
t{:}A \vdash !t : t{:}A \mid t{:}A \vdash t{:}A \mid A \wedge B \Rightarrow A & ! \\
\hline
t{:}A \vdash !t : t{:}A \mid t{:}A, A \wedge B \Rightarrow t{:}A & PK \\
\hline
t{:}A, A \wedge B \Rightarrow !t : t{:}A & PK \\
\hline
t{:}A, y{:}B, A \wedge B \Rightarrow !t : t{:}A & WA
\end{array}
$$

[11] Moreover in $d'$ the undischarged assumption $y : B$ is no longer available i.e. where we had $t : A, y : B \vdash \mathtt{p}_0(\mathtt{p}(t,y)) : A$, we now have $t : A \vdash t : A$. This is what normally happens in the lambda calculus and in natural deduction calculi, and so was to be expected. Note that the loss of undischarged assumptions does not jeopardise any continuation of the derivation $d$ where the undischarged assumption might play a role, for there are basically two uses that can be made of such assumptions in **Gilp**$^*$. On the one hand, they could be discharged with a lambda rule; in this case, one may apply the lambda rule with no discharge of assumptions. On the other hand, undischarged assumptions can become part of the sequent, by the application of a $PK$ rule; in this case, the weakening rule can be used to add the appropriate formula to the sequent.

As the reader can easily see, in this case it is not only the outermost proof-polynomial $!\mathtt{p}_0(\mathtt{p}(t,y))$ that reduces to the proof-polynomial $!t$, but both $!\mathtt{p}_0(\mathtt{p}(t,y))$ and $\mathtt{p}_0(\mathtt{p}(t,y))$. Accordingly, we shall say that the whole formula $!\mathtt{p}_0(\mathtt{p}(t,y)) : \mathtt{p}_0(\mathtt{p}(t,y)) : A$ reduces to the formula $!t : t : A$. Hence, when considering reductions of TND-cuts in the calculus **Gilp**$^*$, one must take into account not only *contractions of proof polynomials* – for example, the contraction from $\mathtt{p}_0(\mathtt{p}(t,y))$ to $t$ – just as in the standard lambda calculus, but also *contractions of formulas*, i.e. contractions of the form illustrated above. Indeed, as reductions of TND-cuts are reflected in contractions of formulas, and proof-sequents are composed of formulas, we will require a notion of contraction for proof sequents; in the example above, we will say that the proof-sequent $t : A, y : B, A \wedge B \Rightarrow !t : t : A$ is a *PS-reduct* of the proof-sequent $t : A, y : B, A \wedge B \Rightarrow !\mathtt{p}_0(\mathtt{p}(t,y)) : \mathtt{p}_0(\mathtt{p}(t,y)) : A$. The third point that we will have to settle amounts to precisely identifying and enumerating all the reductions on proof polynomials, on formulas, and on proof-sequents. Let us call this point **Point3**.

In what follows, we will first treat **Point1**–**Point3**. More precisely, Definition 7.7 is the answer to **Point1**, Definition 7.1 the answer to **Point2** and Figure 5 together with Definitions 7.4–7.6 provide an answer to **Point3**. The main results of this section are Lemma 7.8 and Theorem 7.10; the former states that any derivation in the TND-sequent part of **Gilp**$^*$ containing a TND-cut can be converted into a derivation that does not contain any TND-cut; and the latter shows that such a conversion does not affect the sequent part of **Gilp**$^*$.

## 7.2 Preliminary definitions

**Definition 7.1.** The set of *subformulas* of a formula $A$ of $\mathcal{L}^*_{lp}$ is defined in the following way:

- The set of *m-subformulas* of $A$ is the set of Definition 2.4.
- The set of *lp-subformulas* of $t : A$ is defined as the smallest set of formulas containing $t : A$ and satisfying the following versions of the conditions in the standard definition:
  - if $\mathtt{p}(t_0, t_1) : (A_0 \wedge A_1)$ is in the set, then $t_0 : A_0$ and $t_1 : A_1$ are in the set;
  - if $\mathtt{k_i}(t) : (A_0 \vee A_1)$ is in the set, then $t_i : A_i$ is in the set, for $i = 0, 1$;
  - if $\lambda x.t : (A_0 \rightarrow A_1)$ is in the set, then $x : A_0$ and $t : A_1$ are in the set;
  - if $\mathbb{B}_t(r) :!t : t : A$ is in the set, then $r : t : A$ is in the set;
  - if $s + r : A$ is in the set, then $s : A$ and $r : A$ are in the set;
  - if $\mathbb{S}_{t_i}(r) : (t_0 + t_1) : A$ is in the set, then $r : t_i : A$ is in the set, for $i = 0, 1$.
- The set of *subformulas* of $A$ is defined as the smallest set of formulas containing $A$ such that, for any $A'$ in the set, all m-subformulas and lp-subformulas of $A'$ are in the set.

We will show that **Gilp**$^*$ has the subformula property in the sense of Definition 7.1.

We now introduce the notion of *IN-deduction* which will prove useful later. At the intuitive level, the notion of *IN-deduction* can be explained as follows. Any derivation $d$ in **Gilp**$^*$ is composed of rules belonging to the sequent part and rules belonging to the TND-sequent part. The IN-deductions $d'$ gathers together all and only those rules belonging to the TND-sequent part that have been used in $d$.

**Definition 7.2.** An *IN-deduction* $\hat{d}$ is a finite sequence of multisets of TND-sequents, whose first element is a single TND-sequent, whose last element is a TND-axiom, and such that each multiset in the sequence is related with its immediate successor in accordance with one of the polynomial, lambda or polynomial lambda rules.

Since a sequence is a degenerate tree (where each node has at most one successor), we shall continue to use the vocabulary for trees; in particular, we use the term "node" to refer to the places in the sequence, and the multisets occupying them.

**Definition 7.3.** Given a **Gilp**$^*$-derivation $d$ of the proof sequent $G \mid \Sigma$, an *IN-deduction* $\hat{d}$ is said to be *contained* in $d$ if there exists a function $\phi$ assigning to each node in $\hat{d}$ a node in $d$ such that:

- for every node $n$ in $\hat{d}$, $n$ is a multiset of TND-sequents in $\phi(n)$
- for any non-terminal node $n$ in $\hat{d}$, between $\phi(n)$ and $\phi(n+1)$, there is only one application of a polynomial, lambda or polynomial lambda rule on TND-sequents in $n+1$, and the same rule is applied between $n$ and $n+1$.
- if $n$ is a top-node, then $\phi(n)$ is a top-node.

## 7.3 Contractions

We now address **Point3**, by defining contraction of proof polynomials, formulas and ultimately proof sequents.

The basic contraction cases for proof polynomials are given in Figure 5, where the symbol $f$ in 10. and 11. is an eliminating operator or one of the operators $\mathbb{U}$ or $\mathbb{P}$.[12] The set of contraction cases 1.–13. can be thought of as an extension of the standard cases for the simple typed lambda calculus to the logic of proofs, and are divided into detour and permutation contractions. Moreover, there is a rewriting convention (14.) allowing substitution of $t$ for $\mathbb{U}_t(r)$ and vice versa. As standard, we call the terms on left (of cases 1.–13.) redexes and those on the right contracta. The relation *cont* of contraction on proof polynomials is defined from cases 1.–14. in the standard way (see for example [4, 14, 15]).[13]

[12] Note that here and elsewhere we refer to [2, 4] and use the notation of the former.

[13] Specifically, *cont* is the reflexive transitive closure of the relation $cont_1$ defined as follows:

Figure 5: Contractions on proof polynomials

| | | | |
|---|---|---|---|
| *Detour contractions* | | | |
| 1. $\mathtt{p}_i(\mathtt{p}(t_0, t_1))$ | $\rightsquigarrow$ | $t_i$ (i $\in \{0, 1\}$) | |
| 2. $(\lambda x.t) \cdot s$ | $\rightsquigarrow$ | $t[x/s]$ | |
| 3. $\mathbb{E}^{\vee}_{x,y}(\mathtt{k}_i(t), t_0, t_1)$ | $\rightsquigarrow$ | $t_i[x/t]$ | |
| 4. $\mathbb{U}_t(!t)$ | $\rightsquigarrow$ | t | |
| 5. $\mathbb{U}_{t \cdot t'}(\mathbb{P}_{t,t'}(r, r'))$ | $\rightsquigarrow$ | $\mathbb{U}_t(r) \cdot \mathbb{U}_{t'}(r')$ | |
| 6. $\mathbb{U}_{!t}(\mathbb{B}_t(r))$ | $\rightsquigarrow$ | $!\mathbb{U}_t(r)$ | |
| 7. $\mathbb{P}_{t,t'}(!t, r')$ | $\rightsquigarrow$ | $!(t \cdot \mathbb{U}_{t'}(r'))$ | and similarly for $\mathbb{P}_{t,t'}(r, !t')$ |
| 8. $\mathbb{P}_{t,!s}(r, \mathbb{B}_s(r'))$ | $\rightsquigarrow$ | $!(\mathbb{U}_t(r) \cdot !\mathbb{U}_s(r'))$ | |
| 9. $\mathbb{P}_{t \cdot t', t''}(\mathbb{P}_{t,t'}(r, r'), r'')$ | $\rightsquigarrow$ | $!((\mathbb{U}_t(r) \cdot \mathbb{U}_{t'}(r')) \cdot \mathbb{U}_{t''}(r''))$ | and similarly for $\mathbb{P}_{t, t' \cdot t''}(r, \mathbb{P}_{t',t''}(r', r''))$ |
| *Permutation contractions* | | | |
| 10. f$(\mathbb{E}^{\vee}_{x,y}(t, t_0, t_1))$ | $\rightsquigarrow$ | $\mathbb{E}^{\vee}_{x,y}(t, f(t_0), f(t_1))$ | |
| 11. $f(t + r)$ | $\rightsquigarrow$ | $f(t) + f(r)$ | |
| 12. $\mathbb{U}_{t_0+t_1}(\mathbb{S}_{t_i}(r))$ | $\rightsquigarrow$ | $\mathbb{U}_{t_i}(r)$ | |
| 13. $\mathbb{P}_{t_0+t_1, t'}(\mathbb{S}_{t_i}(r), r')$ | $\rightsquigarrow$ | $\mathbb{S}_{t_i \cdot t'}(\mathbb{P}_{t_i,t'}(r, r'))$ | and similarly for $\mathbb{P}_{t, t'_0+t'_1}(r, \mathbb{S}_{t'_i}(r'))$ |
| *Rewriting convention* | | | |
| 14. $\mathbb{U}_t(r)$ | $\leftrightsquigarrow$ | $t$ | |

We now extend the notion of contraction to formulas.

---

$t$ $cont_1$ $s$ if $s$ is obtained from $t$ by the replacement of an occurrence of $r$ in $t$ by $r'$, where $r \rightsquigarrow r'$.

**Definition 7.4.** The relation *cont* of contraction on formulas is the smallest reflexive transitive relation such that:

- $t : A$ *cont* $t' : A$ if $t$ *cont* $t'$
- $B \wedge A$ *cont* $B' \wedge A$ if $B$ *cont* $B'$; and similarly for $A \wedge B$, $B \vee A$, $A \vee B$, $B \rightarrow A$ and $A \rightarrow B$
- $t : B$ *cont* $t : B'$ if $B$ *cont* $B'$
- $B$ *cont* $B' \vee B''$ and $B$ *cont* $B' \wedge B''$ if $B$ *cont* $B'$ and $B$ *cont* $B''$

The example given in Section 7.1 provides an instance of the first clause of this definition: $t : A$ is a contractum of $\mathtt{p}_0(\mathtt{p}(t, y)) : A$. The contraction of $B \wedge \mathtt{p}_0(\mathtt{p}(t, y)) : A$ to $B \wedge t : A$, and of $s : \mathtt{p}_0(\mathtt{p}(t, y)) : A$ to $s : t : A$ are examples of the second and third clauses respectively. The fourth clause mandates considering $\mathtt{p}_0(\mathtt{p}(t, y)) : A \vee t : A$ as a contractum of $\mathtt{p}_0(\mathtt{p}(t, y)) : A$.[14]

Analogous notions of reduction are straightforwardly defined for TND-sequents and proof sequents.

**Definition 7.5.** We say that the TND-sequent $s'_1 : B'_1, \ldots, s'_m : B'_m \vdash t' : A'$ is a *TND-reduct* of $T = s_1 : B_1, \ldots, s_n : B_n \vdash t : A$ if $t : A$ *cont* $t' : A'$, and for every $1 \leq j \leq m$, there exists $1 \leq i \leq n$ such that $s_i : B_i$ *cont* $s'_j : B'_j$.

**Definition 7.6.** Let $G|\Sigma$ and $(G)^+|(\Sigma)^+$ be proof sequents. $(G)^+|(\Sigma)^+$ is a *PS-reduct* of $G|\Sigma$ if the former can be obtained from the latter by (i) replacing TND-sequents in $G$ by TND-reducts; (ii) replacing redexes in $\Sigma$ by corresponding contracta.

As per normal, we say that a TND-sequent (respectively proof sequent) *TND-reduces* (PS-reduces) to another when the latter is a TND-reduct (PS-reduct) of the former.

## 7.4 Conversion of IN-deductions

Finally, we turn to the reduction of derivations, for which we have the following notions, that have been adapted from [15, p. 173].

**Definition 7.7.** A *maximal* segment, or a *TND-cut*, in an IN-deduction $\hat{d}$ of **Gilp***$^*$ is a sequence of consecutive occurrences of formulas of the form $t_1 : A_1, ..., t_l : A_l$ plus a formula $s : B$ such that:

- either each $A_i = A$ for some formula $A$, or each $A_i = r_i : A$ for proof polynomials $r_i$ and some formula $A$,

[14]Readers wishing to appreciate the need for the fourth clause are invited to consider the derivation of $(\mathtt{p}(t, y)) : A \wedge B) \vee (t : A \wedge y : B), A \Rightarrow \mathtt{p}_0(\mathtt{p}(t, y)) : A$, and try to remove the TND-cuts. The fourth clause in Definition 7.4 allows us to treat the formula obtained, $(\mathtt{p}(t, y)) : A \wedge B) \vee (t : A \wedge y : B), A \Rightarrow \mathtt{p}_0(\mathtt{p}(t, y)) : A \vee t : A$ as a contractum of the proof sequent.

- each of the $t_1 : A_1, ..., t_l : A_l, s : B$ occurs on the right side of the $\vdash$ and none of them occurs on the left side of the $\vdash$,
- $t_1 : A_1$ is the conclusion of an introduction rule or of the rules $!$, $!I$ or $\odot I$,
- $t_l : A_l$ is the major premise of an elimination rule, the premise of the rule $tE$ or any premise of the rule $\odot I$,
- the proof polynomial $s$ is one of the redexes 1-13.

The *length* of a TND-cut is the number of rules occurring in it that has as auxiliary formulas one of the $t_1 : A_1, ..., t_l : A_l$. The *TND-cutrank* $cr(X)$ of a TND-cut $X$ with formula $t_l : A_l$ is the number ($\geq 0$) of occurrences of logical symbols (a proof polynomial counting as such) in $A_l$. Let $\hat{d}$ be an IN-deduction and let $\dot{X}$ denote the sequence of TND-cuts occurring in $\hat{d}$. The TND-cutrank $cr(\dot{X})$ of $\dot{X}$ is the maximum of the TND-cutranks of cuts in $\dot{X}$. If there is no TND-cut, the TND-cutrank of $\dot{X}$ is zero. A *critical* TND-cut of $\dot{X}$ is a TND-cut of maximal TND-cutrank among all TND-cuts in $\dot{X}$. An IN-deduction whose corresponding $\dot{X}$ contains no critical TND-cut is said to be *normal.* A derivation $d$ is said to be *normal* if it only contains normal IN-deductions.

With the notion of TND-cut we have isolated all the cases of possible violations of the subformula property at the proof polynomial level. Hence, to show that **Gilp**$^*$ has the subformula property, one must demonstrate that all the TND-cuts are eliminable. In order to do this, we first of all require the appropriate conversions of derivations, which extend the conversions in standard normalisation proofs (see [15]) to the case of the lambda terms of the logic of proofs. We first show these; in the cases below, assume that $i = 0, 1$ and $\underset{.}{\imath} = 0, 1$ such that $i \neq \underset{.}{\imath}$.

*Detour conversions*

We first show how to remove TND-cuts of length 1.

$\wedge -$ *conversion*

Consider a derivation $d$ of the proof sequent $G \mid \mathbf{N} \vdash r : C \mid \Sigma$ such that the IN-deductions $\hat{d}_1$, ..., $\hat{d}_n$ of the TND-sequent $\mathbf{N} \vdash r : C$ contain steps of the following form

$$\dfrac{\dfrac{\begin{matrix} \vdots \hat{d}^0_j & & \vdots \hat{d}^1_j \\ \mathbf{M}, \mathbf{Q}_0 \vdash t_0 : A_0 & \mid & \mathbf{M}, \mathbf{Q}_1 \vdash t_1 : A_1 \end{matrix}}{\mathbf{M}, \mathbf{Q}_0, \mathbf{Q}_1 \vdash \mathtt{p}(t_0, t_1) : (A_0 \wedge A_1)}\ {\scriptstyle \wedge I}}{\mathbf{M}, \mathbf{Q}_0, \mathbf{Q}_1 \vdash \mathtt{p}_i(\mathtt{p}(t_0, t_1)) : A_i}\ {\scriptstyle \wedge E}$$

We consider the $n$ IN-deductions $\hat{d}_1$, ..., $\hat{d}_n$ and the derivation $d$ and we go up each of them to their axioms. Let us denote by $\mathcal{A}_j$ the TND-axioms of

$\hat{d}_j$. For each $\hat{d}_j$, we erase in the axioms of $d$ the multisets $\underline{\mathbf{M}}$ and $\underline{\mathbf{Q}}_1$, which belong to $\phi(\mathcal{A}_j)$ (where $\phi$ is as in Definition 7.3). We continue the derivation $d$ as before except for the fact that we omit to apply those rules that form the IN-deduction $\hat{d}^1_j$. This means that now in each of the IN-deductions $\hat{d}_1$, ..., $\hat{d}_n$ the previous steps have been replaced by:

$$\begin{array}{c} \vdots\, \hat{d}^i_j \\ \mathbf{M}, \mathbf{Q}_i \vdash t_i : A_i \end{array}$$

$\rightarrow -conversion$

Consider a derivation $d$ of the proof sequent $G \mid \mathbf{N} \vdash r : C \mid \Sigma$ such that the IN-deductions $\hat{d}_1$, ..., $\hat{d}_n$ of the TND-sequent $\mathbf{N} \vdash r : C$ contain steps of the following form

$$\dfrac{\dfrac{\begin{array}{cc} \vdots\, \hat{d}^0_j & \vdots\, \hat{d}^1_j \\ \mathbf{M}, \mathbf{P} \vdash s : A \mid & \mathbf{M}, \mathbf{Q}, x : A \vdash t : F \end{array}}{\mathbf{M}, \mathbf{P} \vdash s : A \mid \mathbf{M}, \mathbf{Q} \vdash \lambda x.t : (A \rightarrow F)}\ \lambda}{\mathbf{M}, \mathbf{P}, \mathbf{Q} \vdash ((\lambda x.t) \cdot s) : F}\ \odot$$

We consider the $n$ IN-deductions $\hat{d}_1$, ..., $\hat{d}_n$ and the derivation $d$ and we go up each of them to their axioms. Let us denote by $\mathcal{A}_j$ the TND-axioms of $\hat{d}_j$. For each $\hat{d}_j$, we erase in the axioms of $d$ the TND-sequent(s) $x : A \vdash x : A$ which belongs to $\phi(\mathcal{A}_j)$. We apply on these new axioms the rules that belonged to the IN-deduction $d^0_j$ to yield the TND-sequent $\mathbf{M}, \mathbf{P} \vdash s : A$. Note that there can be a modification in the order of application of the lambda and polynomial lambda rules. Such a change does not pose a problem thanks to Lemma 6.5. We then continue the derivation $d$ as before except for the fact that those rules of $d$ that were applied on the TND-sequent(s) $x : A \vdash x : A$ are now applied on the TND-sequent $\mathbf{M}, \mathbf{P} \vdash s : A$. In other words, now in each of the IN-deductions $\hat{d}_1$, ..., $\hat{d}_n$ the previous steps have been replaced by:

$$\dfrac{\dfrac{\begin{array}{c} \vdots\, \hat{d}^0_j \\ \mathbf{M}, \mathbf{P} \vdash s : A \mid \underline{\mathbf{M}} \mid \underline{\mathbf{Q}} \end{array}}{\vdots\, \hat{d}^1_j}}{\mathbf{M}, \mathbf{P}, \mathbf{Q} \vdash t[x/s] : F}$$

$\vee - conversion$

The $\vee$-conversion can be treated analogously to the $\to$-conversion.

$! - conversion$

Consider a derivation $d$ of the proof sequent $G \mid \mathbf{N} \vdash r : C \mid \Sigma$ such that the IN-deductions $\hat{d}_1$, ..., $\hat{d}_n$ of the TND-sequent $\mathbf{N} \vdash r{:}C$ contain steps of the following form

$$\dfrac{\dfrac{\dfrac{\vdots\, \hat{d}_j}{\mathbf{M} \vdash t : A}}{\mathbf{M} \vdash !t{:}t : A}\ !}{\mathbf{M} \vdash \mathbb{U}_t(!t) : A}\ tE$$

We go up to the axioms of the derivation $d$ and we develop the derivation $d$ as before, except for the fact that we omit to apply, each time that they were giving rise to the TND-sequent $\mathbf{M} \vdash \mathbb{U}_t(!t) : A$, the rules ! and $tE$. In other words, now in each of the IN-deductions $\hat{d}_1$, ..., $\hat{d}_n$ the previous steps have been replaced by:

$$\dfrac{\vdots\, \hat{d}_j}{\mathbf{M} \vdash t : A}$$

$\mathbb{P} - conversion$

Consider a derivation $d$ of the proof sequent $G \mid \mathbf{N} \vdash r : C \mid \Sigma$ such that the IN-deductions $\hat{d}_1$, ..., $\hat{d}_n$ of the TND-sequent $\mathbf{N} \vdash r{:}C$ contain steps of the following form

$$\dfrac{\dfrac{\mathbf{M}, \mathbf{P} \vdash \overset{\vdots\, \hat{d}^0_j}{r_0{:}t_0{:}(A \to F)} \mid \mathbf{M}, \mathbf{Q} \vdash \overset{\vdots\, \hat{d}^1_j}{r_1{:}t_1{:}A}}{\mathbf{M}, \mathbf{P}, \mathbf{Q} \vdash \mathbb{P}_{t_0,t_1}(r_0, r_1){:}(t_0 \cdot t_1){:}F}\ \odot I}{\mathbf{M}, \mathbf{P}, \mathbf{Q} \vdash \mathbb{U}_{t_0 \cdot t_1}(\mathbb{P}_{t_0,t_1}(r_0, r_1)){:}F}\ tE$$

We go up to the axioms of the derivation $d$ and we develop the derivation $d$ as before, except for the fact that instead of applying the rules $\odot I$ and $tE$, each time they were giving rise to the TND-sequent $\mathbf{M}, \mathbf{P}, \mathbf{Q} \vdash \mathbb{U}_{t_0 \cdot t_1}(\mathbb{P}_{t_0,t_1}(r_0, r_1)) :$ $F$, we apply the rule $tE$ twice and then the rule $\odot$. In other words, now in each of the IN-deductions $\hat{d}_1$, ..., $\hat{d}_n$ the previous steps have been replaced by:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots \hat{d}_j^0 & \vdots \hat{d}_j^1\\ \mathbf{M},\mathbf{P}\vdash r_0\!:\!t_0\!:\!(A\to F) \mid \mathbf{M},\mathbf{Q}\vdash r_1\!:\!t_1\!:\!A\end{matrix}}{\mathbf{M},\mathbf{P}\vdash r_0\!:\!t_0\!:\!(A\to F) \mid \mathbf{M},\mathbf{Q}\vdash \mathbb{U}_{t_1}(r_1)\!:\!A}\ tE}{\mathbf{M},\mathbf{P}\vdash \mathbb{U}_{t_0}(r_0)\!:\!(A\to F) \mid \mathbf{M},\mathbf{Q}\vdash \mathbb{U}_{t_1}(r_1)\!:\!A}\ tE}{\mathbf{M},\mathbf{P},\mathbf{Q}\vdash \mathbb{U}_{t_0}(r_0)\cdot\mathbb{U}_{t_1}(r_1)\!:\!F}\ \odot$$

$\mathbb{B} - conversion$,

Consider a derivation $d$ of the proof sequent $G \mid \mathbf{N} \vdash r : C \mid \Sigma$ such that the IN-deductions $\hat{d}_1$, ..., $\hat{d}_n$ of the TND-sequent $\mathbf{N} \vdash r\!:\!C$ contain steps of the following form

$$\dfrac{\dfrac{\begin{matrix}\vdots \hat{d}_j\\ \mathbf{M}\vdash r\!:\!t:A\end{matrix}}{\mathbf{M}\vdash \mathbb{B}_t(r)\!:\,!t\!:\!t:A}\ !I}{\mathbf{M}\vdash \mathbb{U}_{!t}(\mathbb{B}_t(r))\!:\!t:A}\ tE$$

We go up to the axioms of the derivation $d$ and we develop the derivation $d$ as before, except for the fact that instead of applying, each time that they were giving rise to the TND-sequent $\mathbf{M} \vdash \mathbb{U}_{!t}(\mathbb{B}_t(r)) : t : A$, the rules $!I$ and $tE$, we apply the rules $tE$ and !. In other words, now in each of the IN-deductions $\hat{d}_1$, ..., $\hat{d}_n$ the previous steps have been replaced by:

$$\dfrac{\dfrac{\begin{matrix}\vdots \hat{d}_j\\ \mathbf{M}\vdash r\!:\!t:A\end{matrix}}{\mathbf{M}\vdash \mathbb{U}_t(r)\!:\!A}\ tE}{\mathbf{M}\vdash !\mathbb{U}_t(r)\!:\!\mathbb{U}_t(r)\!:\!A}\ tE$$

$\mathbb{P}$-! $- conversion$

Consider a derivation $d$ of the proof sequent $G \mid \mathbf{N} \vdash r : C \mid \Sigma$ such that the IN-deductions $\hat{d}_1$, ..., $\hat{d}_n$ of the TND-sequent $\mathbf{N} \vdash r\!:\!C$ contain steps of the following form

$$\dfrac{\dfrac{\begin{matrix}\vdots \hat{d}_j^0 & \vdots \hat{d}_j^1\\ \mathbf{M},\mathbf{P}\vdash t_0\!:\!(A\to F) \mid \mathbf{M},\mathbf{Q}\vdash r_1\!:\!t_1\!:\!A\end{matrix}}{\mathbf{M},\mathbf{P}\vdash !t_0\!:\!t_0\!:\!(A\to F) \mid \mathbf{M},\mathbf{Q}\vdash r_1\!:\!t_1\!:\!A}\ !}{\mathbf{M},\mathbf{P},\mathbf{Q}\vdash \mathbb{P}_{t_0,t_1}(!t_0,r_1)\!:\!(t_0\cdot t_1)\!:\!F}\ \odot I$$

We go up to the axioms of the derivation $d$ and we develop the derivation $d$ as before, except for the fact that instead of applying the rules ! and $\odot I$, each time they were giving rise to the TND-sequent $\mathbf{M}, \mathbf{P}, \mathbf{Q} \vdash \mathbb{P}_{t_0,t_1}(!t_0, r_1) : (t_0 \cdot t_1) : F$, we first apply $tE$, then $\odot$ and finally !. In other words, now in each of the IN-deductions $\hat{d}_1$, ..., $\hat{d}_n$ the previous steps have been replaced by:

$$\dfrac{\dfrac{\dfrac{\begin{matrix} \vdots \hat{d}_j^0 & & \vdots \hat{d}_j^1 \\ \mathbf{M}, \mathbf{P} \vdash t_0 : (A \to F) & | & \mathbf{M}, \mathbf{Q} \vdash r_1 : t_1 : A \end{matrix}}{\mathbf{M}, \mathbf{P} \vdash t_0 : (A \to F) \mid \mathbf{M}, \mathbf{Q} \vdash \mathbb{U}_{t_1}(r_1) : A}\, tE}{\mathbf{M}, \mathbf{P}, \mathbf{Q} \vdash (t_0 \cdot \mathbb{U}_{t_1}(r_1)) : F}\, \odot}{\mathbf{M}, \mathbf{P}, \mathbf{Q} \vdash !(t_0 \cdot \mathbb{U}_{t_1}(r_1)) : (t_0 \cdot \mathbb{U}_{t_1}(r_1)) : F}\, !$$

The case $\mathbb{P}_{t_0,t_1}(r_0, !t_1)$ can be treated similarly.

$\mathbb{P}$-$\mathbb{B}$ – *conversion*

Consider a derivation $d$ of the proof sequent $G \mid \mathbf{N} \vdash r : C \mid \Sigma$ such that the IN-deductions $\hat{d}_1$, ..., $\hat{d}_n$ of the TND-sequent $\mathbf{N} \vdash r : C$ contain steps of the following form

$$\dfrac{\dfrac{\begin{matrix} \vdots \hat{d}_j^0 & & \vdots \hat{d}_j^1 \\ \mathbf{M}, \mathbf{P} \vdash r_0 : t_0 : (s : A \to F) & | & \mathbf{M}, \mathbf{Q} \vdash r_1 : s : A \end{matrix}}{\mathbf{M}, \mathbf{P} \vdash r_0 : t_0 : (s : A \to F) \mid \mathbf{M}, \mathbf{Q} \vdash \mathbb{B}_s(r_1) \, !s : s : A}\, !I}{\mathbf{M}, \mathbf{P}, \mathbf{Q} \vdash \mathbb{P}_{t_0, !s}(r_0, \mathbb{B}_s(r_1)) : (t_0 \cdot !s) : F}\, \odot I$$

We consider the $n$ IN-deductions $\hat{d}_1$, ..., $\hat{d}_n$ and the derivation $d$ and we go up each of them to their axioms. Let us denote by $\mathcal{A}_j$ the TND-axioms of $\hat{d}_j$. We focus on the formulas $s : A$ in $\mathcal{A}_j$ from which the formula $r_0 : t_0 : (s : A \to F)$ has been derived. (In case the formula $s : A$ has been introduced by an application of the rule $\lambda$ without discharge of assumptions, we operate in a similar way.) For each $\hat{d}_j$, we replace in the axioms of $d$ these formulas $s : A$ by $\mathbb{U}_s(r_1) : A$. Then we develop the derivation $d$ as before, except for the fact that instead of applying the rules $!I$ and $\odot I$, each time they were giving rise to the TND-sequent $\mathbf{M}, \mathbf{P}, \mathbf{Q} \vdash \mathbb{P}_{t_0, !s}(r_0, \mathbb{B}_s(r_1)) : (t_0 \cdot !s) : F$, we first apply the $tE$ and ! rules, and then the $\odot$ rule and the ! rule. In other words, now in each of the IN-deductions $\hat{d}_1$, ..., $\hat{d}_n$ the previous steps have been replaced by:

$$
\begin{array}{c}
\vdots\hat{d}_j^{0\prime} \qquad\qquad\qquad\qquad \vdots\hat{d}_j^{1} \\
\mathbf{M},\mathbf{P'}\vdash r_0 : t_0 : (\mathbb{U}_s(r_1) : A \to F') \mid \mathbf{M},\mathbf{Q}\vdash r_1 : s : A \\
\hline
\mathbf{M},\mathbf{P'}\vdash r_0 : t_0 : (\mathbb{U}_s(r_1) : A \to F') \mid \mathbf{M},\mathbf{Q}\vdash \mathbb{U}_s(r_1) : A \quad tE \\
\hline
\mathbf{M},\mathbf{P'}\vdash r_0 : t_0 : (\mathbb{U}_s(r_1) : A \to F') \mid \mathbf{M},\mathbf{Q}\vdash !\mathbb{U}_s(r_1) : \mathbb{U}_s(r_1) : A \quad ! \\
\hline
\mathbf{M},\mathbf{P'}\vdash \mathbb{U}_{t_0}(r_0) : (\mathbb{U}_s(r_1) : A \to F') \mid \mathbf{M},\mathbf{Q}\vdash !\mathbb{U}_s(r_1) : \mathbb{U}_s(r_1) : A \quad tE \\
\hline
\mathbf{M},\mathbf{P'},\mathbf{Q}\vdash (\mathbb{U}_{t_0}(r_0)\cdot !\mathbb{U}_s(r_1)) : F' \quad \odot \\
\hline
\mathbf{M},\mathbf{P'},\mathbf{Q}\vdash !(\mathbb{U}_{t_0}(r_0)\cdot !\mathbb{U}_s(r_1)) : (\mathbb{U}_{t_0}(r_0)\cdot !\mathbb{U}_s(r_1)) : F' \quad !
\end{array}
$$

$\mathbb{P}$-$\mathbb{P}$ $-$ *conversion*

Consider a derivation $d$ of the proof sequent $G \mid \mathbf{N} \vdash r : C \mid \Sigma$ such that the IN-deductions $\hat{d}_1$, ..., $\hat{d}_n$ of the TND-sequent $\mathbf{N} \vdash r : C$ contain steps of the following form

$$
\begin{array}{c}
\vdots\hat{d}_j^{0} \qquad\qquad\qquad\qquad\qquad \vdots\hat{d}_j^{1} \qquad\qquad\qquad\qquad\qquad \vdots\hat{d}_j^{2} \\
\mathbf{M},\mathbf{N_{01}},\mathbf{N_{02}},\mathbf{P_1}\vdash r_0 : t_0 : (B \to (A \to F)) \mid \mathbf{M},\mathbf{N_{01}},\mathbf{N_{12}},\mathbf{P_1}\vdash r_1 : t_1 : B \mid \mathbf{M},\mathbf{N_{02}},\mathbf{N_{12}},\mathbf{P_2}\vdash r_2 : t_2 : A \\
\hline
\mathbf{M},\mathbf{N_{01}},\mathbf{N_{02}},\mathbf{N_{12}},\mathbf{P_0},\mathbf{P_1}\vdash \mathbb{P}_{t_0,t_1}(r_0,r_1) : (t_0\cdot t_1) : (A \to F) \mid \mathbf{M},\mathbf{N_{02}},\mathbf{N_{12}},\mathbf{P_2}\vdash r_2 : t_2 : A \quad \odot I \\
\hline
\mathbf{M},\mathbf{N_{01}},\mathbf{N_{02}},\mathbf{N_{12}},\mathbf{P_0},\mathbf{P_1},\mathbf{P_2}\vdash \mathbb{P}_{t_0\cdot t_1,t_2}(\mathbb{P}_{t_0,t_1}(r_0,r_1),r_2) : ((t_0\cdot t_1)\cdot t_2) : F \quad \odot I
\end{array}
$$

We go up to the axioms of the derivation $d$ and we develop the derivation $d$ as before, except for the fact that instead of applying the rules $\odot I$, each time they were giving rise to the TND-sequent $\mathbf{M},\mathbf{N_{01}},\mathbf{N_{02}},\mathbf{N_{12}},\mathbf{P_0},\mathbf{P_1},\mathbf{P_2}\vdash \mathbb{P}_{t_0\cdot t_1,t_2}(\mathbb{P}_{t_0,t_1}(r_0,r_1),r_2) : ((t_0\cdot t_1)\cdot t_2) : F$, we first apply $tE$, then $\odot$ and finally $!$. In other words, now in each of the IN-deductions $\hat{d}_1$, ..., $\hat{d}_n$ the previous steps have been replaced by:

$$
\begin{array}{c}
\vdots\hat{d}_j^{0} \qquad\qquad\qquad\qquad\qquad \vdots\hat{d}_j^{1} \qquad\qquad\qquad\qquad\qquad \vdots\hat{d}_j^{2} \\
\mathbf{M},\mathbf{N_{01}},\mathbf{N_{02}},\mathbf{P_1}\vdash r_0 : t_0 : (B \to (A \to F)) \mid \mathbf{M},\mathbf{N_{01}},\mathbf{N_{12}},,\mathbf{P_1}\vdash r_1 : t_1 : B \mid \mathbf{M},\mathbf{N_{02}},\mathbf{N_{12}},\mathbf{P_2}\vdash r_2 : t_2 : A \\
\hline
\mathbf{M},\mathbf{N_{01}},\mathbf{N_{02}},\mathbf{P_1}\vdash \mathbb{U}_{t_0}(r_0) : (B \to (A \to F)) \mid \mathbf{M},\mathbf{N_{01}},\mathbf{N_{12}},\mathbf{P_1}\vdash r_1 : t_1 : B \mid \mathbf{M},\mathbf{N_{02}},\mathbf{N_{12}},\mathbf{P_2}\vdash r_2 : t_2 : A \quad tE \\
\hline
\mathbf{M},\mathbf{N_{01}},\mathbf{N_{02}},\mathbf{P_1}\vdash \mathbb{U}_{t_0}(r_0) : (B \to (A \to F)) \mid \mathbf{M},\mathbf{N_{01}},\mathbf{N_{12}},\mathbf{P_1}\vdash \mathbb{U}_{t_1}(r_1) : B \mid \mathbf{M},\mathbf{N_{02}},\mathbf{N_{12}},\mathbf{P_2}\vdash r_2 : t_2 : A \quad tE \\
\hline
\mathbf{M},\mathbf{N_{01}},\mathbf{N_{02}},\mathbf{P_1}\vdash \mathbb{U}_{t_0}(r_0) : (B \to (A \to F)) \mid \mathbf{M},\mathbf{N_{01}},\mathbf{N_{12}},\mathbf{P_1}\vdash \mathbb{U}_{t_1}(r_1) : B \mid \mathbf{M},\mathbf{N_{02}},\mathbf{N_{12}},\mathbf{P_2}\vdash \mathbb{U}_{t_2}(r_2) : A \quad tE \\
\hline
\mathbf{M},\mathbf{N_{01}},\mathbf{N_{02}},\mathbf{N_{12}},\mathbf{P_0},\mathbf{P_1}\vdash (\mathbb{U}_{t_0}(r_0)\cdot \mathbb{U}_{t_1}(r_1)) : (A \to F) \mid \mathbf{M},\mathbf{N_{02}},\mathbf{N_{12}},\mathbf{P_2}\vdash \mathbb{U}_{t_2}(r_2) : A \quad \odot \\
\hline
\mathbf{M},\mathbf{N_{01}},\mathbf{N_{02}},\mathbf{N_{12}},\mathbf{P_0},\mathbf{P_1},\mathbf{P_2}\vdash ((\mathbb{U}_{t_0}(r_0)\cdot \mathbb{U}_{t_1}(r_1))\cdot \mathbb{U}_{t_2}(r_2)) : F \quad \odot \\
\hline
\mathbf{M},\mathbf{N_{01}},\mathbf{N_{02}},\mathbf{N_{12}},\mathbf{P_0},\mathbf{P_1},\mathbf{P_2}\vdash !((\mathbb{U}_{t_0}(r_0)\cdot \mathbb{U}_{t_1}(r_1))\cdot \mathbb{U}_{t_2}(r_2)) : ((\mathbb{U}_{t_0}(r_0)\cdot \mathbb{U}_{t_1}(r_1))\cdot \mathbb{U}_{t_2}(r_2)) : F \quad !
\end{array}
$$

The case $\mathbb{P}_{t_0,t_1\cdot t_2}(r_0,\mathbb{P}_{t_1,t_2}(r_1,r_2))$ can be treated similarly.

*Permutation conversions*

We now show how to remove TND-cuts of length $> 1$,

$\vee -$ *perm conversion* $(\{X\} = f(\mathbb{E}^{\vee}_{x,y}(\mathtt{k}_i(t), t_0, t_1))$

We permute the elimination rules and the rules $tE$ and $\odot I$ upwards over the minor premises of $\vee E$.

$+ -$ *perm conversion* $(\{X\} = f(t + r))$

We only consider the case where $f = \cdot s$, so the redex is $(t+r) \cdot s$. The other cases can be treated analogously.

Consider a derivation $d$ of the proof sequent $G \mid \mathbf{N} \vdash r : C \mid \Sigma$ such that the IN-deductions $\hat{d}_1$, ..., $\hat{d}_n$ of the TND-sequent $\mathbf{N} \vdash r\!:\!C$ contain steps of the following form

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\,\hat{d}^0_j & & \vdots\,\hat{d}^1_j\end{matrix}}{\mathbf{M},\mathbf{P} \vdash s\!:\!A \mid \mathbf{M},\mathbf{Q}, x\!:\!A \vdash t\!:\!F}}{\mathbf{M},\mathbf{P} \vdash s\!:\!A \mid \mathbf{M},\mathbf{Q} \vdash \lambda x.t\!:\!(A \to F)}\,\lambda}{\mathbf{M},\mathbf{P} \vdash s\!:\!A \mid \mathbf{M},\mathbf{Q} \vdash ((\lambda x.t) + r)\!:\!(A \to F)}\,+}{\mathbf{M},\mathbf{P},\mathbf{Q} \vdash (((\lambda x.t) + r) \cdot s) : F}\,\odot$$

We go up to the axioms of the derivation $d$ and we develop the derivation $d$ as before, except for the fact that instead of applying the rules $+$ and $\odot$, each time they were giving rise to the TND-sequent $\mathbf{M},\mathbf{P},\mathbf{Q} \vdash (((\lambda x.t) + r) \cdot s) : F$, we apply first the rule $\odot$ and then the rule $+$. In other words, now in each of the IN-deductions $\hat{d}_1$, ..., $\hat{d}_n$ the previous steps have been replaced by:

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\,\hat{d}^0_j & & \vdots\,\hat{d}^1_j\end{matrix}}{\mathbf{M},\mathbf{P} \vdash s\!:\!A \mid \mathbf{M},\mathbf{Q}, x\!:\!A \vdash t\!:\!F}}{\mathbf{M},\mathbf{P} \vdash s\!:\!A \mid \mathbf{M},\mathbf{Q} \vdash \lambda x.t\!:\!(A \to F)}\,\lambda}{\mathbf{M},\mathbf{P},\mathbf{Q} \vdash ((\lambda x.t) \cdot s) : F}\,\odot}{\mathbf{M},\mathbf{P},\mathbf{Q} \vdash (((\lambda x.t) \cdot s) + (r \cdot s)) : F}\,+$$

$\mathbb{U}$-$\mathbb{S}$ $-$ *conversion* $(\{X\} = \mathbb{U}_{t_0+t_1}(\mathbb{S}_{t_i}(r))$

Consider a derivation $d$ of the proof sequent $G \mid \mathbf{N} \vdash r : C \mid \Sigma$ such that the IN-deductions $\hat{d}_1$, ..., $\hat{d}_n$ of the TND-sequent $\mathbf{N} \vdash r\!:\!C$ contain steps of the following form

$$\dfrac{\dfrac{\begin{array}{c}\vdots\, \hat{d}_j \\ \mathbf{M} \vdash r : t_i : A\end{array}}{\mathbf{M} \vdash \mathbb{S}_{t_i}(r) : (t_0 + t_1) : A}\ {+I}}{\mathbf{M} \vdash \mathbb{U}_{t_0+t_1}(\mathbb{S}_{t_i}(r)) : A}\ {tE}$$

We go up to the axioms of the derivation $d$ and we develop the derivation $d$ as before, except for the fact that instead of applying the rules $+I$ and $tE$, each time that they were giving rise to the TND-sequent $\mathbf{M} \vdash \mathbb{U}_{t_0+t_1}(\mathbb{S}_{t_i}(r)) : A$, we only apply the rule $tE$. In other words, now in each of the IN-deductions $\hat{d}_1$, ..., $\hat{d}_n$ the previous steps have been replaced by:

$$\dfrac{\begin{array}{c}\vdots\, \hat{d}_j \\ \mathbf{M} \vdash r : t_i : A\end{array}}{\mathbf{M} \vdash \mathbb{U}_{t_i}(r) : A}\ {tE}$$

$\mathbb{P}$-$\mathbb{S}$ $-$ *perm conversion* $(\{X\} = \mathbb{P}_{t_0+t_1,t'}(\mathbb{S}_{t_i}(r), r'))$

The $\mathbb{P}$-$\mathbb{S}$$-$perm conversion can be treated analogously to the $+$-perm conversion.

It is straightforward to check that in all of the above conversions, the conclusion of the IN-deduction obtained is a TND-reduct of the conclusion of the original IN-deduction. Moreover, each of the conversions corresponds to one of the contractions of proof polynomials 1.-13. in Figure 5.

Armed with the preceding conversions, we can prove the first important result of this section.

**Lemma 7.8.** *Let $d$ be a derivation of a proof sequent $\mathbf{M}_1 \vdash t_1 : A_1 \mid ... \mid \mathbf{M}_k \vdash t_k : A_k \mid \Sigma$ where the only rules applied are polynomial, lambda and polynomial lambda rules. Then there exists a proof sequent $(\mathbf{M}_1 \vdash t_1 : A_1 \mid ... \mid \mathbf{M}_k \vdash t_k : A_k)^+ \mid \Sigma$ and a TND-cut-free derivation of this proof sequent where $(\mathbf{M}_1 \vdash t_1 : A_1 \mid ... \mid \mathbf{M}_k \vdash t_k : A_k)^+ \mid \Sigma$ is a PS-reduct of $\mathbf{M}_1 \vdash t_1 : A_1 \mid ... \mid \mathbf{M}_k \vdash t_k : A_k \mid \Sigma$.*

*Proof.* Consider the IN-deduction $\hat{d}_1$ of $\mathbf{M}_1 \vdash t_1 : A_1$ and let $\dot{X}_1$ be the sequence of TND-cuts in $\hat{d}_1$. If $\dot{X}_1$ is empty, then no conversions are required; suppose henceforth that this is not the case. We eliminate all the TND-cuts of $\dot{X}_1$ by developing the proof along the lines of the normalisation proof in [15, Ch 6 ]. Following them, we assume that, in the application of the elimination rules and the $\odot I$ rule, the major premise is to the left of the minor premise(s), if there are any minor premises. We proceed by main induction on the cutrank of $\dot{X}_1$, with subinduction on the sum of the lengths of the critical cuts in $\dot{X}_1$. Let us call $Y$ a t.c.c. (top critical cut) in $\dot{X}_1$ if no critical cut occurs above $Y$.

The induction step is as follows. For $Y$ the rightmost t.c.c, apply the conversion corresponding to $Y$. The resulting $\dot{X}'_1$ has a lower cutrank, or has the same cutrank, but a lower value for the sum of lengths of all critical cuts. Continue the IN-deduction $\hat{d}_1$ as before. For the $\odot$, $\vee E$ and $\odot I$ rules, their application requires that certain formulas on the right hand side of different TND-sequents are the same. In all these cases, one of the formulas comes from an occurrence of the formula in a TND-axiom or from an application of the $\lambda$ rule without discharge of assumptions. In the case of the $\vee E$, this is evident, since the formulas in question ($A_0$ and $A_1$) are on the left hand side of the TND-sequent. For the case of the $\odot$ rule, the occurrence of $A$ in the formula $t : (A \to F)$ must have been derived from a TND-axiom – either from an occurrence on the right hand side of the TND-sequent if the $\lambda$ rule was not used in the derivation of $t : (A \to F)$, or on the left hand side of the TND-sequent if the $\lambda$ rule was used – or from an application of the $\lambda$ rule without discharge of assumptions. Similarly for the $\odot I$ case. If, after the application of the conversion, there is a subsequent application of a $\odot$, $\vee E$ and $\odot I$ rule where the formulas required for the application no longer match, one proceeds by changing the appropriate TND-axiom or application of the $\lambda$ rule (substituting the redex by the appropriate contractum) so that the formula matches the corresponding formula in the other TND-sequent. One thus obtains an IN-deduction of a TND-reduct of $\mathbf{M}_1 \vdash t_1 : A_1$, which has a lower cutrank or the same cutrank but a lower value for the sum of the lengths of the critical cuts.

Repeat the induction step until the IN-deduction contains no critical cuts. One thus obtains an IN-deduction $\hat{d}'_1$ without TND-cuts of a TND-sequent $\mathbf{M}'_1 \vdash t'_1 : A'_1$ that is a TND-reduct of $\mathbf{M}_1 \vdash t_1 : A_1$. Repeating this procedure for all other TND-sequents $\mathbf{M}_m \vdash t_m : A_m$, we obtain a TND-cut-free derivation of $(\mathbf{M}_1 \vdash t_1 : A_1 \mid ... \mid \mathbf{M}_k \vdash t_k : A_k)^+ \mid \Sigma$, where this proof sequent is a PS-reduct of $\mathbf{M}_1 \vdash t^i_1 : A_1 \mid ... \mid \mathbf{M}_k \vdash t_k : A_k \mid \Sigma$, as required.

□

## 7.5 Main result

Lemma 7.8 can be thought of as an analogue of the standard natural-deduction normalization theorem for the TND-sequent part of **Gilp**$^*$. It establishes analyticity for the TND-sequent part of **Gilp**$^*$. In this section, we show that the whole of **Gilp**$^*$ is analytic. This does not follow immediately from Lemma 7.8 and the cut-freeness of the sequent part of **Gilp**$^*$ (Theorem 6.7): in particular, it remains to be shown that the procedure for reducing the TND-cuts in a derivation of **Gilp**$^*$ does not jeopardize the sequent part of the derivation, either by blocking the applicability of some of the sequent rules, or by introducing the cut rule. In the demonstration of the main result, the following Lemma shall prove useful.

**Lemma 7.9.** *Given a TND-cut-free IN-deduction $d$ of $M \vdash t : A$, there exists a TND-cut-free derivation $d'$ of $M \Rightarrow A$ in* ***Gilp***$^*$.

*Proof.* The proof is the development of the proof of Theorem 6.3.1 of [15, pp.

190-191], where the Theorem 6.3.1 states that, given a normal natural deduction derivation $d$ of a formula $A$ from some premises $M$, one can construct a cut-free derivation $d'$ of the sequent $M \Rightarrow A$.

We proceed by induction on the height of $d$, which corresponds to the number of rules used in $d$.

$h(d) = 0$. Then $M \vdash t : A$ is the axiom $t : A \vdash t : A$. By Lemma 3.3, we have a derivation $d'$ of $A \Rightarrow A$. Applying the weakening rule to introduce the formula $t : A$ and the $PA$ rule yields the required derivation $d'$ of $t : A \Rightarrow A$.

$h(d) > 0$. We distinguish five subcases according to the last rule applied in $d$. `Case 1.` The final rule $\mathcal{R}$ applied in $d$ is one of the following ones $\wedge I$, $\vee I$ or $\lambda$. By applying the inductive hypothesis on the premise of $\mathcal{R}$ and then the corresponding right rule of the sequent part of **Gilp**$^*$ (namely, $\wedge K$, $\vee K$ and $\rightarrow K$) one obtains the desired derivation $d'$. `Case 2.` Suppose that the final rule $\mathcal{R}$ applied in $d$ is the $+$ rule. By simply applying the inductive hypothesis on the premise of $\mathcal{R}$, we obtain the desired derivation $d'$.

`Case 3.` The final rule $\mathcal{R}$ applied in $d$ is the $!$ rule; let $M \vdash t : A$ be the premise of $\mathcal{R}$. By the inductive hypothesis, there is a derivation $d^*$ of $M \Rightarrow A$. By applying $EW$ together with all rules forming the IN-deduction $d$ to $M \Rightarrow A$, one obtains the proof sequent $M \vdash t : A \mid M \Rightarrow A$. Applying $PK$ yields the required derivation $d'$ of $M \Rightarrow t : A$.

`Case 4.` The final rule $\mathcal{R}$ applied in $d$ is either the rule $+I$ or the rule $!I$; let $M \vdash r : t : A$ be the premise of this rule. By the inductive hypothesis, there is a derivation $d^*$ of $M \Rightarrow t : A$; we distinguish cases according to whether $d^*$ involves an application of the $PK$ rule with $t : A$ as auxiliary formula or not. If there is no application of the $PK$ rule of this sort, then $t : A$ occurs on the right hand side of the sequent in an axiom of $d^*$; given the axioms of **Gilp**$^*$, it follows that $\perp$ occurs on the left hand side of the axiom. Replacing the right hand side of the axiom in $d^*$ appropriately (i.e. by $t + s : A$ if $\mathcal{R}$ is $+I$, and by $!t : t : A$ if $\mathcal{R}$ is $!I$) yields the required derivation $d'$. Now consider the case in which $d^*$ involves an application of the $PK$ rule to a proof sequent of the form $M' \vdash t : A \mid M'' \Rightarrow A$. If $\mathcal{R}$ is the rule $+I$, then applying the rule $+$ before the $PK$ rule and continuing $d^*$ as previously yields the required sequent. If $\mathcal{R}$ is $!I$, then applying the $EW$ rule and the rules that gave the TND-sequent $M' \vdash t : A$ yields the proof sequent $M' \vdash t : A \mid M' \vdash t : A \mid M'' \Rightarrow A$. Applying the $!$ rule and the $PK$ rule twice on this proof sequent, with appropriate applications of the weakening and $PA$ rule if necessary, one gets the required derivation $d'$.

`Case 5.` The final rule $\mathcal{R}$ applied in $d$ is one of the rules $\perp E$, $\wedge E$, $\vee E$, $\odot$, $tE$ or $\odot I$. Let $\tau = (\tau_0, \ldots, \tau_n)$ be a sequence of TND-sequents and $\mathcal{R}_\tau = (\mathcal{R}_{\tau,1}, \ldots, \mathcal{R}_{\tau,n})$ be a sequence of rules jointly defined as follows: $\tau_0$ is the conclusion of $d$, and for every $i$ such that a rule in $d$ has as conclusion the TND-sequent $\tau_i$, $\mathcal{R}_{\tau,i+1}$ is the rule which has been applied, $\tau_{i+1}$ is the major premise of the application of $\mathcal{R}_{\tau,i+1}$, if it is $\vee E$, $\odot$ or $\odot I$, and (any) premise of $\mathcal{R}_{\tau,i+1}$ if not. ($\tau$ corresponds to what [15] call a main branch of the derivation, and $\mathcal{R}_\tau$ to the sequence of rules applied in this branch. Note that, in the numbering used here, $\tau_n$ is the top of the branch and $\mathcal{R}_{\tau,n}$ is the first rule applied.) For essentially the same reasons as those noted in [15, §6.3.1], the TND-cut-freeness of $d$ implies that: (i) the only rules in $\mathcal{R}_\tau$ are $\bot E$, $\wedge E$, $\vee E$, $\odot$, $tE$ or $\odot E$. It follows that there is a unique sequence $\tau$ defined as above. (ii) If $\vee E$ is a rule in $\mathcal{R}_\tau$, then $\mathcal{R}_{\tau,1}$ is $\vee E$ and $\mathcal{R}_{\tau,i}$ is not $\vee E$, for all $i \neq 1$. (iii) If $\odot I$ is a rule in $\mathcal{R}_\tau$, then $\mathcal{R}_{\tau,i}$ is $\odot I$ whenever $\mathcal{R}_{\tau,i+1}$ is $\odot I$, for all $1 \leq i \leq n-1$.

We now essentially apply the same technique as in [15, §6.3.1], which distinguishes cases according to $\mathcal{R}_{\tau,n}$ (the first rule applied in the branch). For example, if $\mathcal{R}_{\tau,n}$ is $\odot$, and $\tau_n$ is $s : C_1 \to C_2$, then, as in [15, §6.3.1], we use the inductive hypothesis to construct derivations of $M \Rightarrow C_1$ and $M, C_2 \Rightarrow A$, and then apply weakening and $\to A$ to obtain a derivation of $M, C_1 \to C_2 \Rightarrow A$. Finally, adding $s : C_1 \to C_2$ by weakening and applying $PA$ yields the desired derivation of $M, s : C_1 \to C_2 \Rightarrow A$. The only new cases are those of the $tE$ and $\odot I$ rules. Suppose that $\mathcal{R}_{\tau,n}$ is $tE$, so we have an IN-deduction of the form:

$$\dfrac{\dfrac{r{:}r'{:}s{:}B \vdash r{:}r'{:}s{:}B}{r{:}r'{:}s{:}B \vdash \mathbb{U}_{r'}(r){:}s{:}B}\, tE}{\dfrac{\vdots\ \hat{d}}{r{:}r'{:}s{:}B, M \vdash t{:}A}}$$

We distinguish two cases. If $\mathbb{U}_{r'}(r)$ does not occur in $A$, then take the IN-deduction obtained by applying all the rules in $\hat{d}$ starting from the TND-axiom $r'{:}s{:}B \vdash r'{:}s{:}B$. By the inductive hypothesis applied to this IN-deduction, there exists a derivation of $r' : s : B, M \Rightarrow A$; applying the weakening rule to add $r : r' : s : B$ and the $PA$ rule yields the required derivation $d'$ of $r : r' : s : B, M \Rightarrow A$. On the other hand, if $\mathbb{U}_{r'}(r)$ does occur in $A$, then apply the inductive hypothesis to the IN-deduction obtained by applying all the rules in $\hat{d}$ starting from the TND-axiom $\mathbb{U}_{r'}(r) : s : B \vdash \mathbb{U}_{r'}(r) : s : B$, to yield an derivation of $\mathbb{U}_{r'}(r) : s : B, M \Rightarrow A$. By the specification of the case, $\mathbb{U}_{r'}(r) : s : B \vdash \mathbb{U}_{r'}(r) : s : B$ occurs among the TND-axioms of this derivation. Replacing each such TND-axiom by the TND-axiom $r{:}r'{:}s{:}B \vdash r{:}r'{:}s{:}B$ and an application of the $tE$ rule (and any appropriate occurrences of $\mathbb{U}_{r'}(r){:}s{:}B$ in the axioms but not in the TND-axioms by $r : r' : s : B$) yields the required derivation $d'$ of $r{:}r'{:}s{:}B, M \Rightarrow A$.

Now suppose that $\mathcal{R}_{\tau,n}$ is $\odot I$; as noted, it follows that $\mathcal{R}_{\tau,i}$ is $\odot I$ for all $1 \leq i \leq n$. Let $\tau_i$ (the major premise of $\mathcal{R}_{\tau,i}$) be $r_i : t_i : A_i \to F_i$, where, for all $i > 1$, $F_i = A_{i-1} \to F_{i-1}$; and let the minor premise of $\mathcal{R}_{\tau,i}$ be $N_i \vdash r'_i : t'_i : A_i$. So $\tau_0$ (the conclusion of $d$) is $N_n, \ldots, N_1, r_n : t_n : A_n \to F_n \vdash \mathbb{P}_{((t_n \cdot t'_n)\cdot \ldots \cdot t'_2, t'_1}(\ldots(\mathbb{P}_{t_n, t'_n}(r_n, r'_n; t_n, t'_n), \ldots, r'_1) : (((t_n \cdot t'_n) \cdot t'_{n-1} \cdot \ldots \cdot t'_1) : F_1$. By the inductive hypothesis, for each $1 \leq i \leq n$, there is a derivation of $N_i \Rightarrow t'_i : A_i$. We distinguish cases according to whether all of these derivations involve an application of the $PK$ rule with $t'_i : A_i$ as auxiliary formula. First consider the case where, for some $1 \leq j \leq n$, there is no application of the $PK$ rule of this sort in the derivation of $N_j \Rightarrow t'_j : A_j$. In this case, $t'_j : A_j$ occurs on the right hand side of the sequent in an axiom; given the axioms of **Gilp**$^*$, it follows that $\bot$ occurs on the left hand side of the axiom. Replacing the right hand side of the axiom in this derivation with $(((t_n \cdot t'_n) \cdot t'_{n-1} \cdot \ldots \cdot t'_1) : F_1$ yields a derivation of $N_j \Rightarrow (((t_n \cdot t'_n) \cdot t'_{n-1} \cdot \ldots \cdot t'_1) : F_1$. Suitable applications of weakening yield the required derivation $d'$ of $N_n, \ldots, N_1, r_n : t_n : A_n \to F_n \vdash (((t_n \cdot t'_n) \cdot t'_{n-1} \cdot \ldots \cdot t'_1) : F_1$. Now consider the case in which each of the derivations of $N_i \Rightarrow t'_i : A_i$ involves the application of a $PK$ rule to a proof sequent of the form $N_i^1 \vdash t'_i : A_i \mid N_i^2 \Rightarrow A_i$. Applying repeatedly the rule $E$, weakening and $PA$, we thus obtain derivations of $N_i \Rightarrow A_i$ for each $i$; moreover, focusing on the TND-sequents, we have IN-deductions of $N_i^1 \vdash t'_i : A_i$, for each $i$. Applying the $\to A$ rule repeatedly to the derivations $N_i \Rightarrow A_i$ and then weakening and $PA$, as in the $\odot$ case considered above, we obtain a derivation of $N_n, \ldots, N_1, r_n : t_n : A_n \to F_n \Rightarrow F_1$. Beginning from the TND-axiom $t_n : A_n \to F_n \vdash t_n : A_n \to F_n$ and the TND-axioms of the IN-deductions of $N_i^1 \vdash t'_i : A_i$ for $1 \leq i \leq n$, applying the rules in each of the IN-deductions of the $N_i^1 \vdash t'_i : A_i$, and then applying the $\odot$ rule repeatedly, one obtains an IN-deduction of $N_n^1, \ldots, N_1^1, t_n : A_n \to F_n \vdash (((t_n \cdot t'_n) \cdot t'_{n-1} \cdot \ldots \cdot t'_1) : F_1$. Applying the $EW$ rule repeatedly and the rules in this IN-deduction to the derivation of $N_n, \ldots, N_1, r_n : t_n : A_n \to F_n \Rightarrow F_l$, one obtains the proof sequent $N_n^1, \ldots, N_1^1, t_n : A_n \to F_n \vdash (((t_n \cdot t'_n) \cdot t'_{n-1} \cdot \ldots \cdot t'_1) : F_1 \mid N_n, \ldots, N_1, r_n : t_n : A_n \to F_n \Rightarrow F_1$. Applying the $PK$ and $PA$ rules yields the required derivation $d'$ of $N_n, \ldots, N_1, r_n : t_n : A_n \to F_n \vdash (((t_n \cdot t'_n) \cdot t'_{n-1} \cdot \ldots \cdot t'_1) : F_1$.

□

We now state and prove the main theorem of this section.

**Theorem 7.10.** *For each derivation $d$ in **Gilp**$^*$ of a proof sequent $G \mid \Sigma$, there exists a proof sequent $G^+ \mid \Sigma^+$ and a TND-cut-free derivation $d'$ of this proof sequent such that $G^+ \mid \Sigma^+$ is a PS-reduct of $G \mid \Sigma$.*

*Proof.* Let $d$ be a derivation of the proof-sequent $G \mid \Sigma$. By Lemma 6.5 we can assume that all polynomial, lambda and polynomial lambda rules have been applied before any other rule in $d$. Furthermore, since each pair of propositional and proof rules with different auxiliary formulas can be permuted, we can assume that each application of the $PK$ rule in $d$ has occurred before the other rules in the following sense: the only propositional or proof rules which have been applied before that application of the $PK$ rule have as conclusion a subformula of the auxiliary formula of that application of the $PK$ rule.

There may be one or several proof sequents obtained in the derivation $d$ after all applications of polynomial, lambda and polynomial lambda rules and before application of any other rules (depending on whether $d$ begins from one or several axioms). Let $\mathbf{M}_1^i \vdash t_1^i : A_1^i \mid \ldots \mid \mathbf{M}_k^i \vdash t_k^i : A_k^i \mid (\Sigma)^i$ be these proof sequents. By Lemma 7.8, for each $i$, we obtain a TND-cut-free derivation, $d_i$, of $(\mathbf{M}_1^i \vdash t_1^i : A_1^i \mid \ldots \mid \mathbf{M}_k^i \vdash t_k^i : A_k^i)^+ \mid (\Sigma)^i$, where this proof sequent is a PS-reduct of $\mathbf{M}_1^i \vdash t_1^i : A_1^i \mid \ldots \mid \mathbf{M}_k^i \vdash t_k^i : A_k^i \mid (\Sigma)^i$. Let $\bar{d}$ be the multiset of derivations $d_i$.

We construct a sequence of multisets of derivations, $\bar{d}_j$, $0 \leq j \leq n$ such that (i) $\bar{d}_0 = \bar{d}$; (ii) the conclusions of each of the derivations in $\bar{d}_j$ are PS-reducts of proof sequents at corresponding points in $d$; (iii) $\bar{d}_n$ contains a single derivation of a PS-reduct of $G \mid \Sigma$. The sequence is defined by induction, following the construction of $d$. The induction step is as follows.

Consider an application of a rule $\mathcal{R}$ in $d$ whose premise(s) correspond to conclusion(s) of derivations in $\bar{d}_j$. We will essentially apply the same rule to form $\bar{d}_{j+1}$; it is necessary however to ensure that the rule can always be applied, and that the conclusion obtained is a PS-reduct of the corresponding proof sequent in $d$. If the rule $\mathcal{R}$ is one of the $\wedge A$, $\vee K$ or $\rightarrow K$ rules, then it can always be applied to the corresponding proof sequent in $\bar{d}_j$; let $\bar{d}_{j+1}$ be the derivation obtained. If the rule $\mathcal{R}$ is one of the $\wedge K$, $\rightarrow A$, $\vee A$, $PA$ or $PK$ rules, and it can be directly applied to the corresponding proof sequent (or sequents) in $\bar{d}_j$ to yield a PS-reduct of the corresponding proof sequent in $d$, then $\bar{d}_{j+1}$ is the derivation obtained when the rule is applied. It may be that the rule $\mathcal{R}$ is one of the $\wedge K$, $\rightarrow A$, $\vee A$, $PA$ or $PK$ rules but it cannot be applied to yield a PS-reduct of the appropriate proof sequent. This happens in any of the following cases. (a) There are two premises of the rule in $d$, corresponding to conclusions of derivations $d'$ and $d''$ in $\bar{d}_j$, and there is a formula $D$ in the sequent of each premise in $d$ such that the application of the rule in $d$ requires these occurrences to be the same, but such that the corresponding formula in the conclusion of $d'$ is $D'$, the corresponding formula in the conclusion of $d''$ is $D''$, and $D' \neq D''$ (this may happen with the $\wedge K$, $\rightarrow A$ and $\vee A$ rules). (b) There are auxiliary formulas of the rule in $d$ that are of the form $A$ and $t : A$, whereas the corresponding formulas in $\bar{d}_j$ are $A'$ and $t'' : A''$ with $A' \neq A''$ (this may occur for the $PA$ and $PK$ rules). (c) There is a formula $r : C$ appearing both in an auxiliary TND-sequent and in the sequent of the premise of the rule in $d$ such that the specific application of the rule relies on the fact that these occurrences are of the same formula, but in $\bar{d}_j$ the corresponding formula in the TND-sequent is $r' : C'$, the corresponding formula in the sequent is $r'' : C''$ and $r' : C' \neq r'' : C''$ (this may occur with the $PK$ rule).

We distinguish cases according to the rule $\mathcal{R}$ applied at the corresponding point in $d$.

$\wedge K$ **or** $\rightarrow A$ Consider the two proof sequents in $\bar{d}_j$ that correspond to the premises of the rule $\mathcal{R}$ in $d$, and suppose that there exists at least one formula $D$ in the premises of $\mathcal{R}$ that has now become, in one case, the formula $D'$ and in the other case the formula $D''$. Apply the weakening rule to each premise to introduce the formulas $D''$ and $D'$ respectively; then apply the rule $\wedge A$ to get formulas of the form $D' \wedge D''$. Repeat the same operation for all such $D$. Apply, on the two proof sequents thus obtained, the rule $\mathcal{R}$ and call the derivation $\bar{d}_{j+1}$. By construction, the conclusion is a PS-reduct of the conclusion of the application of the rule $\mathcal{R}$ in $d$.

$\vee A$ Consider the two proof sequents in $\bar{d}_j$ that correspond to the premises of the rule $\mathcal{R}$ in $d$, and suppose that there exists at least one formula $D$ in the left hand sides of the premises of $\mathcal{R}$ that has now become, in one case, the formula $D'$ and in the other case the formula $D''$. For each such $D$, proceed as in the $\wedge K$- $\rightarrow A$ case. Moreover, suppose that in the right hand sides of these proof sequents, their respective formulas $C'$ and $C''$ differ. Apply to each of them the rule $\vee K$ to get $C' \vee C''$. Then apply the rule $\vee A$ and call the resulting derivation $\bar{d}_{j+1}$. By construction, the conclusion is a PS-reduct of the conclusion of the corresponding application of the rule in $d$. (Note that these two subcases are not exclusive.)

$PA$ Consider the proof sequent in $\bar{d}_j$ that corresponds to the premise of the rule $\mathcal{R}$ in $d$, and suppose that the two auxiliary formulas $A$ and $t : A$ of the rule $\mathcal{R}$ in $d$ have now become the formulas $A'$ and $t'' : A''$, respectively. First, add, by means of the weakening rule, the formula $t'' : A'$; then apply to it and to the formula $A'$ the rule $PA$. Finally apply $\wedge A$ to $t'' : A'$ and $t'' : A''$. Let $\bar{d}_{j+1}$ be the resulting derivation. By construction, the conclusion is a PS-reduct of the conclusion of the application of the $PA$ rule in $d$.

$PK$ We distinguish two subcases depending on the two formulas $A$ and $t : A$ that are auxiliary in the application of the rule $\mathcal{R}$ in $d$. `Subcase 1.` In $\bar{d}_j$ the formulas $A$ and $t : A$ do not coincide; more precisely, the formula $A$ has become the formula $A'$, while the formula $t : A$ has become $t'' : A''$. Let $M \vdash t'' : A''$ be the TND-sequent where $t'' : A''$ occurs. By Lemma 7.9, there exists a cut-free derivation of $M \Rightarrow A''$. By appropriate applications of the $EW$ rule, and of the rules involved in the (TND-cut-free) IN-deduction of $M \vdash t'' : A''$, we obtain a TND-cut-free derivation of $M \vdash t'' : A'' \mid M \Rightarrow A''$; then, by an application of the $PK$ rule and several applications of the rule $W$, $EW$ and $PA$, one gets the desired derivation $\bar{d}_{j+1}$. By construction, the conclusion is a PS-reduct of the conclusion of the corresponding application of the rule in $d$. `Subcase 2.` In $\bar{d}_j$ the formulas $A$ and $t : A$ coincide as in $d$. Consider the proof sequent in $\bar{d}_j$ that corresponds to the premise of the rule $\mathcal{R}$ in $d$ and suppose that it contains an occurrence of the formula $r' : C'$ in the auxiliary TND-sequent and an occurrence of the formulas $r'' : C''$ on the left hand side of the sequent, where $r' : C'$ and $r'' : C''$ correspond to the same formula $r : C$ in the premise of the rule in $d$. In this case, the $PK$ rule can be applied, yielding a proof sequent where $r' : C'$ and $r'' : C''$ occur on the left hand side. Apply the $\wedge A$ rule to these two formulas, and let the resulting derivation be $\bar{d}_{j+1}$. By construction, the conclusion is a PS-reduct of the conclusion of the corresponding application of the $PK$ rule in $d$.

At the end of this induction, we obtain a TND-cut-free derivation $d'$ of a proof sequent $G^+ \mid \Sigma^+$. By construction, $G^+ \mid \Sigma^+$ is a PS-reduct of $G \mid \Sigma$, as required.

□

This result establishes the analyticity of **Gilp**$^*$, in the sense set out in Section 7.1. As noted there, TND-cuts cannot be reduced without changes in the proof polynomials (that in a sense "encode" the presence of these cuts in a proof). Thus Theorem 7.10 (in tandem with Theorem 6.7) establishes the only reasonable form of analyticity that can be demanded in **Gilp**$^*$: for any derivation $d$, containing both cuts at the sequent and the TND-sequent level, there exists a cut- and TND-cut-free derivation $d'$ of a proof sequent which is a *reduced version* of the original one; that is, which it can be obtained from it by appropriate contractions.

# 8 Realisation of Gilp

We have thus shown that **Gilp**$^*$ is an analytic sequent calculus on the language $\mathcal{L}^*_{lp}$. However, to what extent can it be thought of as a sequent calculus for the intuitionistic logic of proofs? We will answer this question in this section, showing that **Gilp**$^*$ realises all theorems of **Gilp** and that to any theorem in **Gilp**$^*$ which belongs to the image of $\mathcal{L}_{lp}$, there is an associated theorem in **Gilp**. In this sense, **Gilp**$^*$ can be thought of as a conservative extension of **Gilp**.

Let us start with the following remark made by Artemov [1, p.9],

> A constant specification ($\mathcal{CS}$) is a finite set of formulas $c_1 : A_1, ..., c_n : A_n$ such that $c_i$ is a constant and $A_i$ is an axiom $A_0 - A_4$. ($\mathcal{CS}$) is injective if for each constant $c$ there is at most one formula $c : A \in$ ($\mathcal{CS}$) (each constant denotes a proof of not more than one axiom). Each derivation in **LP** naturally generates the $\mathcal{CS}$ consisting of all formulas introduced in this derivation by the rule $R_2$. [...] *One might restrict* **LP** *to injective constant specifications only* without changing the ability of **LP** to emulate modal logic, or the functional and arithmetical completeness theorems for **LP**. [1, p.9] [Italics ours.]

In the light of this we may assume that in **Gilp** each constant introduced by the rule *ci* is associated with at most one axiom (that is, that all constant specifications generated by derivations in **Gilp** are injective).[15] Accordingly, we have assignments of constants to proof polynomials, defined as follows.

**Definition 8.1.** An *assignment* $\sigma$ is a function from the set of constants of $\mathcal{L}_{lp}$ to the set of proof polynomials in $\mathcal{L}^*_{lp}$ such that, for every constant $c$, $\sigma(c)$ is either a proof polynomial corresponding to a lambda term in normal form whose type is an intuitionistic axiom, or it is one of $\lambda x.\ \mathbb{U}_t(x)$, $\lambda x.\ \mathbb{B}_t(x)$, $\lambda x.\ \mathbb{S}_t(x)$, $\lambda x.\lambda y.\ \mathbb{P}_{t,t'}(x, y)$.

**Definition 8.2.** Given an assignment $\sigma$, the *realizing translation* $\delta_\sigma$ from the language $\mathcal{L}_{lp}$ of **Gilp** to the language $\mathcal{L}^*_{lp}$ of **Gilp**$^*$ is defined in the following way. Proof variables, propositional atoms, boolean connectives and the functional symbols !, +, $\cdot$ are translated by their equivalents in $\mathcal{L}^*_{lp}$; and, for each constant

[15] The restriction to injective constant specifications is helpful but not necessary for the general form of the results presented below: similar results can be obtained, involving an appropriately modified notion of proper translation (see Definitions 8.1 and 8.2), in the absence of injectivity.

$c$ of $\mathcal{L}_{lp}$, $(c)^{\delta_\sigma} = \sigma(c)$. The translation is extended to terms of $\mathcal{L}_{lp}$ by induction on the construction of terms, and similarly for formulas.

Different assignments of terms to constants correspond to different realizing translations. Hence we have in fact defined a family of possible translations from $\mathcal{L}_{lp}$ to $\mathcal{L}^*_{lp}$; in what follows, we will only be concerned with translations in this family. Each realizing translation can be thought of as yielding a "realization" of the formulas of $\mathcal{L}_{lp}$ in $\mathcal{L}^*_{lp}$, in which constants are replaced by appropriate proof polynomials; this is analogous to the realization of formulas of modal logic in the language of the logic of proofs [1]. We drop the subscript indicating the assignment, unless it is specifically required.

**Definition 8.3.** $\mathbb{PR}$ is the smallest set of proof polynomials such that:

- $\mathbb{PR}$ contains
    1. proof polynomials corresponding to lambda terms in normal form typed by intuitionistic axioms (e.g. see [14]),
    2. the proof polynomials $\lambda x.\, \mathbb{U}_t(x)$, $\lambda x.\, \mathbb{B}_t(x)$, $\lambda x.\, \mathbb{S}_t(x)$, $\lambda x.\lambda y.\, \mathbb{P}_{t,t'}(x, y)$,
    3. the proof variables $x_0$, $x_1$, $x_2$, ...
- $\mathbb{PR}$ is closed under the operations of $!$, $+$ and $\cdot$.

Moreover, $\mathbb{FR}$ is the set of formulas $A$ such that all the proof polynomials occurring in $A$ are in $\mathbb{PR}$.

Let $\mathcal{L}^*_{lp}|_{lp}$ be the fragment of the language $\mathcal{L}^*_{lp}$ containing proof polynomials in $\mathbb{PR}$ and formulas in $\mathbb{FR}$. $\mathcal{L}^*_{lp}|_{lp}$ can be thought of as the "image" of $\mathcal{L}_{lp}$ in $\mathcal{L}^*_{lp}$ in the following sense: for any formula $A$ (respectively proof polynomial $t$) in $\mathcal{L}_{lp}$, $(A)^\delta$ (resp. $(t)^\delta$) is in $\mathcal{L}^*_{lp}|_{lp}$ for any realizing translation $\delta$; and for any formula $A$ (respectively proof polynomial $t$) in $\mathcal{L}^*_{lp}|_{lp}$, there is a formula $B$ (resp. proof polynomial $s$) in $\mathcal{L}_{lp}$ and a realizing translation $\delta$ such that $(B)^\delta = A$ (resp. $(s)^\delta = t$). We say that a proof sequent $G \mid \Sigma \in \mathcal{L}^*_{lp}|_{lp}$ if all formulas occurring in it are in $\mathcal{L}^*_{lp}|_{lp}$.

We first show that for any theorem in **Gilp**, there is a realizing translation under which the theorem is derivable in **Gilp**$^*$.

**Theorem 8.4.** *For every derivation $d$ of $G \mid \Sigma$ in* **Gilp**, *there exists a derivation $d'$ of $(G)^\delta \mid (\Sigma)^\delta$ in* **Gilp**$^*$, *for some realizing translation $\delta$.*

*Proof.* Let $\mathcal{CS}$ be the constant specification generated by the derivation $d$ (this is defined in a way similar to that in [1]; see also the citation above), which, as noted above, can be assumed to be injective. Take any assignment $\sigma$ respecting this constant specification in the following sense: if $c$ is assigned to an axiom $A$ of the logic of proofs under $\mathcal{CS}$, then $\sigma(c)$ corresponds to a lambda term typed by $A$. Let $\delta$ be the realizing translation generated by $\sigma$.

We now proceed by induction on the height of $d$.

$h(d)=0$. $G \mid \Sigma$ is an axiom. $(G)^\delta \mid (\Sigma)^\delta$ is an axiom too.

$h(d)>0$. If $G \mid \Sigma$ has been derived by a propositional rule, or by a proof rule, or by one of the rules !, + and $\odot$, then the procedure is straightforward. If $G \mid \Sigma$ has been derived by the rule $ci$, then we consider the following two examples. The cases that are not considered here can be analysed in an analogous way.

$$\frac{G \mid \Sigma}{\vdash a\colon(A \to (B \to A)) \mid G \mid \Sigma}\,{\scriptstyle ci} \quad \rightsquigarrow \quad \cfrac{\cfrac{\cfrac{(G)^\delta \mid (\Sigma)^\delta}{x : (A)^\delta \vdash x\colon(A)^\delta \mid (G)^\delta \mid (\Sigma)^\delta}\,{\scriptstyle EW}}{x : (A)^\delta \vdash \lambda y.x\colon((B)^\delta \to (A)^\delta) \mid (G)^\delta \mid (\Sigma)^\delta}\,{\scriptstyle \lambda}}{\vdash \lambda x.\lambda y.x\colon((A)^\delta \to ((B)^\delta \to (A)^\delta)) \mid (G)^\delta \mid (\Sigma)^\delta}\,{\scriptstyle \lambda}$$

$$\frac{G \mid \Sigma}{\vdash d\colon(t\colon(A \to B) \to (s\colon A \to (t \cdot s)\colon B)) \mid G \mid \Sigma}\,{\scriptstyle ci} \quad \rightsquigarrow$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{(G)^\delta \mid (\Sigma)^\delta}{y\colon(s\colon A)^\delta \vdash y\colon(s\colon A)^\delta \mid (G)^\delta \mid (\Sigma)^\delta}\,{\scriptstyle EW}}{x\colon(t\colon(A \to B))^\delta \vdash x\colon(t\colon(A \to B))^\delta \mid y\colon(s\colon A)^\delta \vdash y\colon(s\colon A)^\delta \mid (G)^\delta \mid (\Sigma)^\delta}\,{\scriptstyle EW}}{x\colon(t\colon(A \to B))^\delta, y\colon(s\colon A)^\delta \vdash \mathbb{P}_{t,s}(x,y)\colon((t \cdot s)\colon B)^\delta \mid (G)^\delta \mid (\Sigma)^\delta}\,{\scriptstyle \odot I}}{x\colon(t\colon(A \to B))^\delta \vdash \lambda y.\mathbb{P}_{t,s}(x,y)\colon((s\colon A)^\delta \to ((t \cdot s)\colon B)^\delta) \mid (G)^\delta \mid (\Sigma)^\delta}\,{\scriptstyle \lambda}}{\vdash \lambda x.\lambda y.\mathbb{P}_{t,s}(x,y)\colon((t\colon(A \to B))^\delta \to ((s\colon A)^\delta \to ((t \cdot s)\colon B)^\delta)) \mid (G)^\delta \mid (\Sigma)^\delta}\,{\scriptstyle \lambda}$$

□

It is clear on inspection of the proof that the result holds for any realizing translation which respects the constant specification generated by the derivation $d$.

We have thus shown that any theorem of **Gilp** is realized in **Gilp**$^*$: that for any derivable sequent in **Gilp**, there exists an associated sequent of **Gilp**$^*$, differing only in that constants are replaced by appropriate proof polynomials, which is derivable. We now prove a converse result: that for any proof sequent in $\mathcal{L}^*_{lp}|_{lp}$ that is derivable in **Gilp**$^*$, there exists a proof sequent derivable in **Gilp** and a realizing translation which maps it to the initial proof sequent (so the latter sequent can be obtained from the former by replacing appropriate proof polynomials by constants).

**Theorem 8.5.** *Given a derivation $d$ in **Gilp**$^*$ of $G \mid \Sigma \in \mathcal{L}^*_{lp}|_{lp}$, there exists a derivation $d'$ in **Gilp** of $G' \mid \Sigma'$ and a realizing translation $\delta$ such that $(G')^\delta \mid (\Sigma')^\delta = G \mid \Sigma$.*

*Proof.* Given Lemma 6.5, it can be assumed that in $d$ all the applications of lambda and polynomial lambda rules come first, in the following sense: there is no application of a propositional, proof, or polynomial rule that is followed by a lambda or polynomial lambda rule except if it applies to the same TND-sequent.

Recall from the discussion at the beginning of this section that any realizing translation is determined by the assignment of proof polynomials to constants (Definitions 8.1 and 8.2). Let a *partial assignment* be a partial function from the set of constants in $\mathcal{L}_{lp}$ to the set of proof polynomials in $\mathcal{L}^*_{lp}$ satisfying the conditions in Definition 8.1 – that is, a function that is only defined for some constants. Likewise, a *partial realizing translation* is a realizing translation generated by a partial assignment of constants – the translation is only defined on a fragment of $\mathcal{L}_{lp}$; in particular, it is not defined for the constants that are not assigned to proof polynomials, and for any proof polynomials containing them. Since $G \mid \Sigma \in \mathcal{L}^*_{lp}|_{lp}$, each proof polynomial in it belongs to $\mathbb{PR}$. Consider any partial assignment such that, for each proof polynomial of the sort specified in Definition 8.1 occurring in $G \mid \Sigma \in \mathcal{L}^*_{lp}|_{lp}$, there is a constant which is assigned to it; and let $\delta_0$ be the partial realizing translation generated by this assignment. Hence there is a $G' \mid \Sigma'$ in **Gilp** such that $(G')^{\delta_0} \mid (\Sigma')^{\delta_0} = G \mid \Sigma$.

We construct sequences $d_1, \dots d_n$ of trees labelled by proof sequents in **Gilp**, $d^1, \dots, d^n$ of derivations of $G \mid \Sigma$ in **Gilp**$^*$, maps $f_1 \dots f_n$ from $d_i$ to $d^i$ and partial realizing translations $\delta_1, \dots, \delta_n$ such that, for all $1 \leq i \leq n$, (i) the root of $d_i$ is mapped to the root of $d^i$ and it is labelled by $G' \mid \Sigma'$; (ii) for every node in $d_i$, if it is labelled by $G'' \mid \Sigma''$, then its image under $f_i$ is labelled by $(G'')^{\delta_i} \mid (\Sigma'')^{\delta_i}$, and (iii) each node in $d_i$ is related to its successor nodes by a rule in **Gilp**.

Let $d_1$ be a single-noded tree labelled $G' \mid \Sigma'$, $d^1 = d$, $f_1$ be the function taking the node in $d_1$ to the root of $d^1$ and $\delta_1 = \delta_0$. The rest of the sequences are defined by induction, where the induction step involves the following two cases:

- $d_i$ contains a leaf $m$ such that $f_i(m)$ has been obtained in $d^i$ by a propositional rule, a proof rule or a polynomial rule except for the $\odot$ rule. In this case, let $d_{i+1}$ be the result of adding the appropriate nodes to $d_i$ above $m$, labelled by the premises of the same rule in **Gilp**; let $d^{i+1} = d^i$; let $f_{i+1}$ extend $f_i$ by mapping these new nodes to the premises of the rule in $d^{i+1}$; and let $\delta_{i+1} = \delta_i$.

- $d_i$ contains a leaf $m$ such that $f_i(m)$ has been obtained in $d^i$ by the $\odot$ rule:

$$\frac{G \;\mid \mathbf{M}, \mathbf{P} \vdash t_0 : (A \to F) \mid \mathbf{M}, \mathbf{Q} \vdash t_1 : A \mid \Sigma}{G \mid \mathbf{M}, \mathbf{P}, \mathbf{Q} \vdash (t_0 \cdot t_1) : F \mid \Sigma} \odot$$

  If $A$ is in $\mathcal{L}^*_{lp}|_{lp}$, then proceed as in the previous case, extending the partial realizing translation $\delta_i$ so that $A$ is in the image of $\delta_{i+1}$ if necessary. Now consider the case where $A$ contains a subformula $r : B$ where $r$ is not in $\mathcal{L}^*_{lp}|_{lp}$. Since $\mathbf{M}, \mathbf{P}, \mathbf{Q} \vdash (t_0 \cdot t_1) : F \in \mathcal{L}^*_{lp}|_{lp}$ by the inductive hypothesis, the only way this can occur is if: (i) $r : B$ appears in the axioms of the IN-deductions of $\mathbf{M}, \mathbf{Q} \vdash t_1 : A$ (either alone or as a subformula of another formula) and is then discharged in a $\lambda$ rule or a $\vee E$ rule; or (ii) it is introduced in an instance of the $\lambda$ rule without discharge of assumptions; or (iii) it is introduced in an instance of the $\vee I$ rule; or (iv) $r$ contains a subterm $\dot{r} + \ddot{r}$, with $\dot{r}$ or $\ddot{r}$ introduced in an instance of the $+$ or $+I$ rule. Similar cases hold for the IN-deductions of $\mathbf{M}, \mathbf{P} \vdash t_0 : (A \rightarrow F)$. By (i) replacing all occurrences of $r : B$ in the axioms, (ii) replacing all occurrences of $r : B$ introduced in instances of the $\lambda$ rule without discharge of assumptions, (iii) replacing all occurrences of $r : B$ introduced in instances of the $\vee I$ rule, and (iv) removing appropriate instances of the $+$ and $+I$ rules or replacing occurrences of $r : B$ introduced in them, we obtain a derivation where, in all IN-deductions of $\mathbf{M}, \mathbf{P}, \mathbf{Q} \vdash (t_0 \cdot t_1) : F$, all occurrences of $r : B$ are replaced by a formula $r' : B'$ belonging to $\mathcal{L}^*_{lp}|_{lp}$. Repeating for all such subformulas of $A$, one obtains a derivation $d^{i+1}$ of $G \mid \Sigma$ where this instance of the $\odot$ rule is replaced by an application of the $\odot$ rule on the proof sequent $G \ \mid \mathbf{M}, \mathbf{P} \vdash t_0 : (C \rightarrow F) \mid \mathbf{M}, \mathbf{Q} \vdash t_1 : C \mid \Sigma$ with $C \in \mathcal{L}^*_{lp}|_{lp}$. Let $\delta_{i+1}$ be the result of extending $\delta_i$ by assigning new constants to the proof polynomials in $C$. Note that there is a canonical mapping from $d^{i+1}$ to $d^i$, obtained by replacing any occurrences of $C$ (or subformulas) by $A$ (or appropriate subformulas). $d_{i+1}$ is the result of adding the appropriate node to $d_i$ above $m$, labelled by the premises of the equivalent application of the $\odot$ rule, and $f_{i+1}$ is a mapping from $d_{i+1}$ to $d^{i+1}$ which, when composed with the canonical mapping from $d^{i+1}$ to $d^i$, coincides with $f_i$ and which maps the new node to the premise of this instance of the $\odot$ rule in $d^{i+1}$.

The construction halts when all leaves $m$ of $d_i$ are such that either $m$ is an axiom of **Gilp** or $f_i(m)$ has been obtained in $d^i$ by a lambda or polynomial lambda rule. It is straightforward to check that $d_1, \ldots, d_n, f_1, \ldots, f_n, d^1, \ldots, d^n$, and $\delta_1, \ldots, \delta_n$ satisfy the desired properties. Moreover, since $d$ is finite, the sequence stops at some $d_n$.

We now construct a derivation $d'$ of $G' \mid \Sigma'$ in **Gilp** as follows:

- if all leaves of $d_n$ are axioms of **Gilp**, then $d_n$ is the desired derivation of $G' \mid \Sigma'$ and any realizing translation extending $\delta_n$ is the required $\delta$.

- if not, then for any leaf $m$ in $d_n$ that is not an axiom, it must contain a TND-sequent which is not of the form $s : B \vdash s : B$; call it $\mathbf{M}'' \vdash t'' : A''$. For each such TND-sequent, consider its image $(\mathbf{M}'')^{\delta_n} \vdash (t'')^{\delta_n} : (A'')^{\delta_n}$ in $f_n(m)$. This image is in $\mathcal{L}^*_{lp}|_{lp}$ by construction. By construction of $d_n$, the last rule in the IN-deduction of this TND-sequent in $d^n$ is a lambda or polynomial lambda rule, and there is no lambda or polynomial lambda rule applied to $(\mathbf{M}'')^{\delta_n} \vdash (t'')^{\delta_n} : (A'')^{\delta_n}$ after $f_n(m)$ in $d^n$. Since $(t'')^{\delta_n} \in \mathcal{L}^*_{lp}|_{lp}$, it follows that it must be a proof variable, a proof polynomial corresponding to a normal-form lambda term typed by an intuitionistic axiom, or one of the following: $\lambda x.\ \mathbb{U}_t(x)$, $\lambda x.\ \mathbb{B}_t(x)$, $\lambda x.\ \mathbb{S}_t(x)$, $\lambda x.\lambda y.\ \mathbb{P}_{t,t'}(x, y)$. Since $(\mathbf{M}'')^{\delta_n} \neq (t'')^{\delta_n} : (A'')^{\delta_n}$ by the specification of the case, the TND-sequent $(\mathbf{M}'')^{\delta_n} \vdash (t'')^{\delta_n} : (A'')^{\delta_n}$ must have been constructed by a non-trivial IN-deduction (rather than being present in an axiom);

it follows that $(t'')^{\delta_n}$ is not a proof variable. Moreover, by inspection of the rules, for any non-trivial IN-deduction of a TND-sequent of the form $(\mathbf{M}'')^{\delta_n} \vdash (t'')^{\delta_n} : (A'')^{\delta_n}$ with $(t'')^{\delta_n}$ a proof polynomial corresponding to a normal-form lambda term typed by an intuitionistic axiom or one of $\lambda x.\ \mathbb{U}_t(x)$, $\lambda x.\ \mathbb{B}_t(x)$, $\lambda x.\ \mathbb{S}_t(x)$, $\lambda x.\lambda y.\ \mathbb{P}_{t,t'}(x,y)$, $(\mathbf{M}'')^{\delta_n}$ is empty. Hence $\mathbf{M}'' \vdash t'' : A''$ has the form $\vdash c : A''$ for some constant $c$ and some axiom $A''$ of **Ilp**.

Form the tree $d'$ by adding a sequence of nodes above each leaf $m$ in $d_n$ that is not an axiom, each of whose labels remove one TND-sequent that is not a TND-axiom with respect to its predecessor node, and each of which is related to its predecessor node by the rule $ci$. The leaves of $d'$ are all axioms, and each edge is labelled by a (correctly applied) rule of **Gilp**; hence $d'$ is a derivation. By construction, the image of conclusion of $d'$ under $\delta_n$, $(G')^{\delta_n} \mid (\Sigma')^{\delta_n}$ is $G \mid \Sigma$. Hence $d'$ is the required derivation in **Gilp**, and any realizing translation extending $\delta_n$ is the required $\delta$.

□

This theorem shows that **Gilp**$^*$ is conservative with respect to **Gilp**, in the sense that it does not prove anything more than it: for any proof sequent in the image of the language of **Gilp** which is derivable in **Gilp**$^*$, there is a corresponding proof sequent derivable in **Gilp**, namely one which is the pre-image of the initial proof sequent under an appropriate realizing translation.

Taken together with Theorems 6.7 and 7.10, the results of this section give the precise sense in which **Gilp**$^*$ represents the framework where the problem of the lack of subformula property for the logic of proofs is solved. Any derivation in the standard language of the logic of proofs (i.e. in **Gilp**) can be translated into a derivation in the extended language $\mathcal{L}^*_{lp}$ (i.e. a derivation in **Gilp**$^*$) (see Theorem 8.4). With respect to the image of the language $\mathcal{L}_{lp}$, the switch into **Gilp**$^*$ does not allow to prove more than what we could prove in **Gilp** (see Theorem 8.5). However, it does allow one to eliminate all cuts – in both the sequent and the TND-sequent part of the calculus (Theorems 6.7 and 7.10) – and thus obtain a derivation satisfying the subformula property. Since proof polynomials encode certain parts of the derivation in which they are derived, and these parts may precisely be what is modified when TND-cuts are removed, the reduced derivation has a conclusion that is different, but related to (it is a reduct of) the original one. The conclusion of the reduced derivation will typically not be in the image of the language of the logic of proofs, so the derivation itself will not be translatable back into the standard language of the logic of proofs. This can be seen as a confirmation of the intuition with which we started (Section 5): the subformula property cannot be obtained if one restricts oneself to the language of the logic of proofs, but it can be obtained by enriching the language appropriately.

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