Modal Logic of Transition Systems in the Topos of Trees

Colin Riba, Guilhem Jaber

Modal Logic of Transition Systems in the Topos of Trees

Colin Riba and Guilhem Jaber

ENS de Lyon, Université de Lyon, LIP*

Abstract. We investigate the adjunction between the category of transition systems (with not nec. bounded morphisms) and the topos of trees $\mathscr{S}$ from the perspective of the basic modal logic $\mathbf{K}$ on transition systems. More specifically, we show how the modal logic $\mathbf{K}$ over an object of $\mathscr{S}$ seen as a transition system can be lifted back as a subobject of $\mathscr{S}$. This relies on the usual mutually transpose formulations of modal satisfaction as either a map of transition system or as a map of Boolean algebras with operators. Moreover, thanks to the usual geometric morphism from the Boolean topos $\mathrm{Psh}(|\mathbb{N}_*|)$ of presheaves over the discrete category of natural numbers to $\mathscr{S}$, we can give a characterization of modal satisfaction within $\mathscr{S}$ as a map from formulae to total subobjects of $\mathscr{S}$ whose image in the Boolean topos $\mathrm{Psh}(|\mathbb{N}_*|)$ is an internal map of Boolean algebras.

1 Introduction

Modal logics over transition systems are widespread languages to express properties and reason about abstractions of executions of programs, in particular with automatic methods based on model-checking (see e.g. [3]). Most modal logics used in verification (LTL, CTL, CTL^*, the modal μ-calculus) are extensions of the basic modal logic $\mathbf{K}$, whose modalities simulate quantifications over the immediate successors of a state.

In this paper, we investigate the modal logic $\mathbf{K}$ over the objects of the *topos of trees* $\mathscr{S}$ [4], the presheaf category over the total order of natural numbers. The topos of trees is a versatile structure for semantics of programming languages. It is for instance a natural setting to formulate and reason about realizability models or logical relations (called Kripke logical relations) for ML-like languages with recursive and polymorphic types and higher-order references [4]. It can also be used to build denotational models, e.g. for the languages PCF [28] and FPC [25]. Moreover, $\mathscr{S}$ is a convenient model to program with and reason about coinductive types [7].

Each object X of $\mathscr{S}$, as a forest [17], can be seen as a transition system $X^{\oplus}$. We observe that there is an adjunction $(-)^{\oplus} \dashv (-)_{\circledast} : \mathbf{TS} \to \mathscr{S}$ (actually a coreflection[1]) where $\mathbf{TS}$ is the category of t.s.'s with not necessarily bounded

* UMR 5668 CNRS ENS Lyon UCBL INRIA

[1] The unit $\eta : \mathrm{Id}_{\mathscr{S}} \to (-)_{\circledast} \circ (-)^{\oplus}$ is a natural iso, or equivalently $(-)^{\oplus} : \mathscr{S} \to \mathbf{TS}$ is full and faithful (see e.g. [21, Thm. IV.3.1]).

morphisms. We show how via this adjunction, the modal logic of an object of $\mathscr{S}$ seen as a t.s. can be lifted back to $\mathscr{S}$. This adjunction moreover allows to see every t.s. with state set $|K|$ as a subobject of the internal type $\mathrm{Str}_{|K|}$ of streams over $|K|$ in $\mathscr{S}$.[2] Each type Str_S of $\mathscr{S}$, like every coinductive type for a polynomial functor, admits a representation as an object of $\mathscr{S}$ whose set of global sections (*i.e.* the set of $\mathscr{S}$-morphisms from $\mathbf{1}$ to that object) is the final coalgebra of that functor in **Sets** [24] (see also [7]). On the other hand, t.s.'s defined in $\mathscr{S}$ arise naturally e.g. with Kripke logical relations for languages with states, which require the representation of the heap as a transition system. Besides, in [8] a logic with an **S4** modality is used in external reasonings over Kripke logical relations.

A lot of the power of $\mathscr{S}$ comes from the presence of an endofunctor $\blacktriangleright$ equipped with a principle of guarded recursion [26]. The functor $\blacktriangleright$ induces a modality on the subobject classifier of $\mathscr{S}$ (*i.e.* its object of internal truth values) which satisfies a Löb rule, expressing a well-founded induction principle. In other words, the recursion principle of $\blacktriangleright$ on the one hand gives a great power to build complex structures in $\mathscr{S}$, but on the other hand makes its internal logic unsuitable for usual modal logics over descendent relations of transition systems, which are in general not well-founded.

Our approach rests on the well-established tradition of (co-)algebraic approaches to modal logics (see e.g. [5, 31, 12]). We start from the usual observation that the modal satisfaction relation $\Vdash$ over a Kripke model K leads to a **TS**-map from the t.s. part of K to the t.s. of ultrafilter over modal formulae modulo logical equivalence (*i.e.* over the Lindenbaum-Tarski algebra LT).[3] This gives, via the adjunction $(-)^{\oplus} \dashv (-)_{\circledast}$, a notion of modal satisfaction for an arbitrary object X of $\mathscr{S}$, which can moreover be expressed as a subobject of $X \times \Lambda$ for a suitable presheaf Λ representing modal formulae. We then follow the usual methodology (see e.g. [5, 31, 12]), turning $\Vdash$ the other way around as a map of Boolean algebras with operators (**BAO**) from LT to the powerset of states of the t.s. $X^{\oplus}$ induced by X. In order to express the Boolean connectives, we rely on the left-adjoint part of the geometric morphism from the Boolean topos of presheaves over the discrete category of natural numbers to $\mathscr{S}$.[4] On the other hand, the modality is captured thanks to the observation that the $\mathscr{S}$-map induced by $\Vdash$ from LT (with suitable presheaf structure) to the subobjects of X factors through the subobjects of X which are total in the sense of [4]. We moreover show that this completely captures the usual characterization of $\Vdash$ as inducing the unique **BAO** map from LT to the powerset algebra of a t.s.

The paper is organized as follows. The preliminary §2 introduces notations for the topos of trees, transition systems and modal satisfaction. Section 3 presents

[2] Transition systems are represented by their unfoldings as forests. Note that this is harmless for modal logics, since modal satisfaction is preserved under bisimulation.

[3] This **TS**-map is a coalgebra map iff the Kripke model K is modally saturated.

[4] This geometric morphism is an instance of a general pattern for categorical models of **S4** [9, 29, 19, 2]. However we consider **K** rather than **S4** (no reflexive-transitive closure is imposed on transitions).

the adjunction $(-)^{\oplus} \dashv (-)_{\circledast} : \mathbf{TS} \to \mathscr{S}$ and shows how the induced notion of modal satisfaction can be lifted back to $\mathscr{S}$. We then introduce some usual algebraic material in §4, that we use in §5 for our characterization of modal satisfaction. Finally, §6 presents some extensions of $\mathbf{K}$, namely an adaptation of [19, 2] to our setting, which gives $\mathbf{S4}$ for *reversed* (*i.e.* "past") modalities, and sketches an interpretation of the modal μ-calculus.

2 Preliminaries

2.1 The Topos of Trees

Let $\mathbb{N}_*$ be the set of strictly positive natural numbers. The *topos of trees* $\mathscr{S}$ is the category of presheaves over $(\mathbb{N}_*, \leq)$. Its objects are functors $X : \mathbb{N}_*^{\mathrm{op}} \to \mathbf{Sets}$, or equivalently families of sets $X(n)$ indexed over $\mathbb{N}_*$ and equipped with restriction maps $(-)\!\uparrow : X(n+1) \to X(n)$. The morphisms from X to Y are natural transformations $f : X \to Y$, equivalently families of functions[5] $f_n : X(n) \to Y(n)$ commuting with restriction: $f_{n+1}(x)\!\uparrow = f_n(x\!\uparrow)$ for $x \in X(n+1)$. If $k \leq n$, we write $x_{|k}$ for the restriction of $x \in X(n)$ into $X(k)$, so that $x_{|k} = x\!\uparrow^{n-k}$.

As any presheaf category, $\mathscr{S}$ is a topos, so it is in particular Cartesian closed. The product is given by $(X \times Y)(n) = X(n) \times Y(n)$ and the exponent presheaf Y^X at n is the set of all sequences $(f_\ell)_{\ell \leq n}$ of functions $f_\ell : X(\ell) \to Y(\ell)$ which are compatible with restriction (*i.e.* $(-)\!\uparrow \circ f_{\ell+1} = f_\ell \circ (-)\!\uparrow$), see e.g. [22, §I.6].

A subobject S of X, notation $S \hookrightarrow X$, is a family of subsets $S(n) \subseteq X(n)$ such that $x\!\uparrow \in S(n)$ whenever $x \in S(n+1)$. The subobject classifier of $\mathscr{S}$ is the object Ω with $\Omega(n) = \{0, \ldots, n\}$, and restriction $\Omega(n+1) \to \Omega(n)$ mapping k to $\min(k, n)$. The characteristic map $\chi_S : X \to \Omega$ of a subobject $S \hookrightarrow X$ takes $x \in X(n)$ to $\max\{m \leq n \mid x_{|m} \in S(m)\}$, with $\max \emptyset = 0$.

Remark 2.1. In $\mathscr{S}$, each object Ω^X is isomorphic to the object whose component at n is the set of sequences $(S_\ell)_{\ell \leq n}$ of sets $S_\ell \subseteq X(\ell)$ such that $x\!\uparrow \in S_\ell$ for all $x \in S_{\ell+1}$, and with restrictions taking $(S_1, \ldots, S_n, S_{n+1})$ to $(S_1, \ldots, S_n)$. $\square$

2.2 Transition Systems

A *transition system* $K = (|K|, \partial_K)$ is given by a set of states $|K|$, together with a transition function $\partial : |K| \to \mathcal{P}(|K|)$. We may write $x \to_K y$ for $y \in \partial_K(x)$. A *morphism* of transition systems $K \to K'$ is given by a function $h : |K| \to |K'|$ which respects transitions, in the sense that $h(y) \in \partial_{K'}(h(x))$ whenever $y \in \partial_K(x)$. We write $\mathbf{TS}$ for the category of transition systems and morphisms of transition systems.

As usual (see [31, 10]), transition systems are $\mathbf{Sets}$-coalgebras for the covariant powerset functor $\mathcal{P}$, and among the morphisms of transitions systems, we distinguish the *bounded morphisms*, *i.e.* the $\mathbf{TS}$-maps $h : K \to K'$ which are maps of coalgebras, that is s.t. $\partial_{K'} \circ h = \mathcal{P}(h) \circ \partial_K$. We write $\mathbf{Coalg}(\mathcal{P})$ for the category of transition systems and bounded morphisms.

[5] We write either f_C or $f(C)$ for the component at C of a natural transformation f.

2.3 Modal Logic

Assume given a set AP of *atomic propositions*, ranged over by p, q, etc. We take as *basic modal language* the formulae defined as follows:

$$\varphi, \psi \in \Lambda \quad ::= \quad p \quad | \quad \bot \quad | \quad \varphi \vee \psi \quad | \quad \neg\varphi \quad | \quad \Diamond\varphi$$

We use the following expected defined formulae

$$\varphi \wedge \psi \ := \ \neg(\neg\varphi \vee \neg\psi) \qquad \top \ := \ \neg\bot \qquad \varphi \to \psi \ := \ \neg\varphi \vee \psi \qquad \Box\varphi \ := \ \neg\Diamond\neg\varphi$$

Given a transition system K and a valuation $v : AP \to \mathbf{2}^{|K|}$, the modal satisfaction relation $s \Vdash^v \varphi$ is defined by induction on formulae as usual:

$$\begin{array}{ccc} s \Vdash^v p \ \text{ iff } \ s \in v(p) & s \nVdash^v \bot & s \Vdash^v \neg\varphi \ \text{ iff } \ s \nVdash^v \varphi \\ s \Vdash^v \varphi \vee \psi \ \text{ iff } \ s \Vdash^v \varphi \text{ or } s \Vdash^v \psi & & s \Vdash^v \Diamond\varphi \ \text{ iff } \ t \Vdash^v \varphi \text{ for some } t \in \partial(s) \end{array}$$

We write $s \Vdash \varphi$ for $s \Vdash^v \varphi$ when the valuation v is understood from the context.

Remark 2.2. Note that since the relation $\Vdash$ is a subset of $|K| \times \Lambda$, so we can see it equivalently as function $|K| \times \Lambda \to \mathbf{2}$, or as either of its transposes $|K| \to \mathbf{2}^{\Lambda}$ or $[\![-]\!]^v : \Lambda \to \mathbf{2}^{|K|}$. $\Box$

3 Modal Logic of Transition Systems in the Topos of Trees

This Section presents the adjunction $(-)^{\oplus} \dashv (-)_{\circledast} : \mathbf{TS} \to \mathscr{S}$ (§3.1), and briefly discuss how it specializes to $\mathbf{Coalg}(\mathcal{P})$ thanks to notion of *open* maps of [16, 18, 17] (§3.2). Then in §3.3 we explain how, for an object X of $\mathscr{S}$, the modal satisfaction relation induced by $(-)^{\oplus} \dashv (-)_{\circledast} : \mathbf{TS} \to \mathscr{S}$ leads to a subobject of $X \times \mathsf{\Lambda}$ in $\mathscr{S}$, for a suitable presheaf $\mathsf{\Lambda}$.

3.1 An Adjunction Between TS and $\mathscr{S}$

There is a functor $(-)_{\circledast} : \mathbf{TS} \to \mathscr{S}$ from transition systems to the topos of trees. On objects, $(-)_{\circledast}$ takes K to the presheaf $K_{\circledast}$ with

$$K_{\circledast}(n) \quad := \quad \{(s_1, \dots, s_n) \mid \forall i < n.\ s_{i+1} \in \partial(s_i)\} \quad \subseteq \quad |K|^n$$

and restriction maps $K_{\circledast}(n{+}1) \to K_{\circledast}(n)$ taking $(s_1, \dots, s_n, s_{n+1})$ to $(s_1, \dots, s_n)$. On maps, $(-)_{\circledast}$ takes $h : K \to K'$ to the natural transformation $h_{\circledast}$ with component at n mapping $(s_1, \dots, s_n)$ to $(h(s_1), \dots, h(s_n))$.

Remark 3.1. The functor $(-)_{\circledast}$ is faithful, but not full. A $\mathscr{S}$-map $f : K_{\circledast} \to K'_{\circledast}$ must be such that $f_n(s_1, \dots, s_n) = (t_1, \dots, t_n)$ whenever $f_{n+1}(s_1, \dots, s_{n+1}) = (t_1, \dots, t_{n+1})$. It is thus determined by a family of functions $K_{\circledast}(n) \to |K|$, which are in general not induced by functions $|K| \to |K|$. $\Box$

Remark 3.2. Note that $K_{\circledast}$ is a subobject of the streams $\mathrm{Str}_{|K|}$ over $|K|$, where following [4, Ex. 2.1], $\mathrm{Str}_{|K|}(n) = |K|^n$ and $(s_1, \ldots, s_n, s_{n+1})\!\uparrow = (s_1, \ldots, s_n)$. More generally, its was shown in [24, Thm. 2] (see also [7, Ex. 2.4.(i)]) that for each polynomial functor $T : \mathbf{Sets} \to \mathbf{Sets}$, there is an object A_T of $\mathscr{S}$ such that the set of global sections $\mathscr{S}[\mathbf{1}, A_T]$ is a terminal coalgebra for T. $\square$

It is known since [17] that the objects of $\mathscr{S}$ can be seen as transition systems. The functor $(-)^{\oplus} : \mathscr{S} \to \mathbf{TS}$ takes a presheaf X to the transition system $X^{\oplus}$ with states $|X^{\oplus}| := \coprod_{n>0} X(n)$ and transitions given by[6] $y \in \partial_{X^{\oplus}}(x)$ iff $x = y\!\uparrow$. On maps, $(-)^{\oplus}$ takes $f : X \to Y$ to the function $f^{\oplus} : \coprod_n X(n) \to \coprod_n Y(n)$ with $f^{\oplus}(n, x) := f(n)(x)$. Hence, if $y \in \partial_{X^{\oplus}}(x)$, then we have $y\!\uparrow = x$, so that $f^{\oplus}(n+1, y) \in \partial_{X^{\oplus}}(f^{\oplus}(n, x))$.

Proposition 3.3. *$(-)^{\oplus}$ is left adjoint to $(-)_{\circledast}$. The unit has component $\eta_X : X \to (X^{\oplus})_{\circledast}$ taking $x \in X(n)$ to $(x_{|1}, \ldots, x_{|n-1}, x) \in (X^{\oplus})_{\circledast}(n)$.*

Remark 3.4. The adjunction $(-)^{\oplus} \dashv (-)_{\circledast} : \mathbf{TS} \to \mathscr{S}$ is a coreflection (the unit is a natural iso, equivalently $(-)^{\oplus}$ is full and faithful ([21, Thm. IV.3.1])). $\square$

Remark 3.5. As usual with presheaves, an object X of $\mathscr{S}$ can be represented by its *category of elements* $\int X$. The objects of $\int X$ are pairs (n, x) where $n > 0$ and $x \in X(n)$, and there is an arrow from (n, x) to (k, y) iff $n \leq k$ and $y_{|k} = x$.

Note that $\int X$ is a partial order. Moreover, the set of objects of $\int X$ is exactly $\coprod_n X(n)$, so that $\int X$ is the reflexive transitive closure of $(|X^{\oplus}|, \to_{X^{\oplus}})$. $\square$

3.2 Open Maps and Coalgebra Morphisms

The notion of open map, introduced in [16], provides categorical formulations of bisimulation [18]. The case of t.s.'s and $\mathscr{S}$ is already discussed in [17].

Definition 3.6. *A $\mathscr{S}$-map $f : X \to Y$ is* open *if for every $n > 0$ and every $x \in X(n)$, $y' \in Y(n+1)$ such that $y'\!\uparrow = f_n(x)$, there is some $x' \in X(n+1)$ such that $f_{n+1}(x') = y'$ and $x'\!\uparrow = x$.*

Note that open maps compose, and that the identity is open. Write $\mathcal{O}(\mathscr{S})$ for the lluf subcategory of open maps. The following result builds on the known correspondence between open maps and coalgebra morphisms.

Proposition 3.7. *The adjunction $(-)^{\oplus} \dashv (-)_{\circledast}$ of Prop. 3.3 restricts to an adjunction between $\mathbf{Coalg}(\mathcal{P})$ and $\mathcal{O}(\mathscr{S})$.*

Remark 3.8. In particular, for a transition system K the transition system $K_{\circledast}^{\oplus}$ is bisimilar to K. Note that the states of K are duplicated in $K_{\circledast}^{\oplus}$, so that this does not extend well to valuations $v : AP \to \mathbf{2}^{|K|}$. $\square$

[6] $\coprod_{n>0} X(n)$ is the coproduct of the $X(n)$ for $n > 0$. Formally, its elements are pairs (n, x) with $x \in X(n)$, but we write x for (n, x) whenever convenient.

3.3 Representation of Modal Satisfaction in the Topos of Trees

The functor $(-)^{\oplus} : \mathscr{S} \to \mathbf{TS}$ allows to define a modal satisfaction relation on objects of $\mathscr{S}$ seen as transitions systems. Note that this assumes, for an object X of $\mathscr{S}$, valuations of atomic propositions to be of the form $v : AP \to \mathbf{2}^{\coprod_n X(n)}$. They are thus not constrained by the restriction maps of $\mathscr{S}$.

Definition 3.9. *Given an object X of $\mathscr{S}$ and a valuation $v : AP \to \mathbf{2}^{|X^{\oplus}|}$, the relations $(\Vdash^v_n)_n \subseteq \prod_n (X(n) \times \Lambda)$ are defined by*

$$x \Vdash^v_n \varphi \qquad \textit{iff} \qquad (n, x) \Vdash^v \varphi \ \textit{in}\ X^{\oplus}$$

Given a transition system K and a valuation $v : AP \to \mathbf{2}^{|K|}$, the relation $\Vdash \subseteq |K| \times \Lambda$ can be seen as a function $|K| \times \Lambda \to \mathbf{2}$. We shall now see that for an object X of $\mathscr{S}$, we can actually represent the relations $(\Vdash_n)_n \subseteq \prod_n (X(n) \times \Lambda)$ as a subobject of $X \times \mathsf{\Lambda}$ in $\mathscr{S}$, for a suitable presheaf $\mathsf{\Lambda}$. This relies on a simple observation underlying the usual construction of ultrafilter frames in modal logic (see e.g. [5, §2.5]).

Recall from Rem. 2.2 that given a transition system K and a valuation $v : AP \to \mathbf{2}^{|K|}$, the relation $\Vdash^v \subseteq |K| \times \Lambda$ can be seen as a function

$$\begin{array}{rcl} |K| & \longrightarrow & \mathbf{2}^{\Lambda} \\ s & \longmapsto & \{\varphi \mid s \Vdash \varphi\} \end{array} \tag{1}$$

Note that if $t \in \partial(s)$, then $s \Vdash \Diamond\varphi$ whenever $t \Vdash \varphi$. In other words, if we equip $\mathbf{2}^{\Lambda}$ with the transition function

$$\begin{array}{rcrcl} \partial_{\mathbf{2}^{\Lambda}} & : & \mathbf{2}^{\Lambda} & \longrightarrow & \mathcal{P}(\mathbf{2}^{\Lambda}) \\ & & F & \longmapsto & \{G \mid \forall \varphi\, (\varphi \in G \Longrightarrow \Diamond\varphi \in F)\} \end{array} \tag{2}$$

then the function (1) becomes a **TS**-map from K to $(\mathbf{2}^{\Lambda}, \partial_{\mathbf{2}^{\Lambda}})$. We now devise an object $\mathsf{\Lambda}$ of $\mathscr{S}$ such that for any subobject $S \hookrightarrow \mathsf{\Lambda}$ we have

$$S(n+1) \in \partial_{\mathbf{2}^{\Lambda}}(S(n)) \qquad \text{for all } n > 0 \tag{3}$$

Definition 3.10. *Define the object $\mathsf{\Lambda}$ of $\mathscr{S}$ as $\mathsf{\Lambda}(n) := \Lambda$ for each $n > 0$, and with restriction maps $(-)\!\uparrow : \mathsf{\Lambda}(n+1) \to \mathsf{\Lambda}(n)$ taking φ to $\Diamond\varphi$.*

We indeed obtain (3) since the shape of subobjects in $\mathscr{S}$ imposes that for $S \hookrightarrow \mathsf{\Lambda}$, for all $n > 0$ we have $\varphi\!\uparrow = \Diamond\varphi \in S(n)$ whenever $\varphi \in S(n+1)$. Moreover, given an object X of $\mathscr{S}$ and a valuation v, for all $(x, \varphi) \in X(n+1) \times \mathsf{\Lambda}(n+1)$ such that $x \Vdash^v_{n+1} \varphi$, we have $x\!\uparrow \Vdash^v_n \Diamond\varphi$. We thus have shown the following.

Proposition 3.11. *Given X and v as above, the family of relations $(\Vdash^v_n)_n$ is a subobject of $X \times \mathsf{\Lambda}$ in $\mathscr{S}$.*

Definition 3.12. *Given X and v as above, we write $\kappa^v_{\Lambda} : X \times \mathsf{\Lambda} \to \Omega$ for the classifying map in $\mathscr{S}$ of the family of relations $(\Vdash^v_n)_n \hookrightarrow X \times \mathsf{\Lambda}$.*

Remark 3.13. As sanity check, note that $\Omega^{\Lambda} \simeq (\mathbf{2}^{\Lambda}, \partial_{\mathbf{2}^{\Lambda}})_{\circledast}$ in $\mathscr{S}$, so that the object Λ of $\mathscr{S}$ indeed represents in $\mathscr{S}$ the transition system $(\mathbf{2}^{\Lambda}, \partial_{\mathbf{2}^{\Lambda}})$. Moreover, the image under the adjunction $(-)^{\oplus} \dashv (-)_{\circledast}$ of the map $X \to (\mathbf{2}^{\Lambda}, \partial_{\mathbf{2}^{\Lambda}})_{\circledast}$ induced by κ^{v}_{Λ} is the **TS**-map $X^{\oplus} \to (\mathbf{2}^{\Lambda}, \partial_{\mathbf{2}^{\Lambda}})$ induced by the the transpose of $\Vdash^{v} \subseteq |X^{\oplus}| \times \Lambda$. $\square$

Remark 3.14. The topos of trees is equipped with a full and faithful endofunctor $\blacktriangleright$, which takes an object X to the presheaf defined as $(\blacktriangleright X)(1) := \mathbf{1}$ and $(\blacktriangleright X)(n+1) := X(n)$. Moreover, the action of $\blacktriangleright$ on subobjects can be represented by a modality $\triangleright$ on Ω [4, §2.2 & Thm. 2.7].

However, the modality $\Diamond$ does not seem to be easily interpretable via $\triangleright$, because intuitively $\blacktriangleright$ goes “in the wrong direction”. Besides, $\triangleright$ satisfies a Löb rule [4, Thm. 2.7], expressing well-founded induction principle, while on the other hand $\Diamond$ quantify over the descendents of a state in transition systems with possibly infinite descending paths.

The functor $\blacktriangleright$ has left-exact left-adjoint $\blacktriangleleft$ (so that $\blacktriangleleft \dashv \blacktriangleright$ is a geometric morphism), defined as $(\blacktriangleleft X)(n) = X(n+1)$ [4, §2.1 & §6.1]. However, it is not clear to us how the action of $\blacktriangleleft$ can be internalized on subobjects. Moreover, we do not know how to use $\blacktriangleleft \dashv \blacktriangleright$ for our purposes since $\blacktriangleleft(\blacktriangleright X)(n+1) = \blacktriangleright(\blacktriangleleft X)(n+1) = X(n+1)$ for all $n > 0$ (see also §6). $\square$

4 Elements of Algebraic Perspectives

So far, given an object X of $\mathscr{S}$ and a valuation $v : AP \to \mathbf{2}^{|X^{\oplus}|}$, we have seen that the modal satisfaction relations $\Vdash^{v}$ on the t.s. $X^{\oplus}$ can be expressed as a subobject of $X \times \Lambda$ in $\mathscr{S}$, for a suitable presheaf Λ.

We are going to see how we can *characterize* this subobject, or more precisely its classifying map $\kappa^{v}_{\Lambda} : X \times \Lambda \to \Omega$. This Section discusses some known algebraic tools on which we rely for this, and §5 presents the characterization itself. We recall the setting of Boolean algebras with operators in §4.1, and following a known construction, we present in §4.2 a functor from $\mathscr{S}$ to a Boolean topos.

4.1 Boolean Algebras with Operators

Given a t.s. K and a valuation $v : AP \to \mathbf{2}^{|K|}$, we have recalled in §3.3 how we can equip $\mathbf{2}^{\Lambda}$ with a transition relation so that the transpose (1) of $\Vdash^{v} \subseteq |K| \times \Lambda$ is a **TS**-map. We now discuss some known structure on this map.

Each set $\{\varphi \mid s \Vdash^{v} \varphi\}$, for s a state of K, is a complete consistent theory, that is a set of modal formulae F such that (1) F does not contain falsity $\bot$, (2) F is closed under logical consequence, and (3) F is maximal with these two properties (so that $\varphi \notin F$ implies $\neg\varphi \in F$). Such sets of formulae are best understood as ultrafilters on Boolean algebras, so we first recall the usual *Lindenbaum-Tarski* Boolean algebra over the basic modal language (see e.g. [5, §4.1]).

Definition 4.1. Lindenbaum-Tarski algebra LT *is the quotient of* Λ *with the notion of logical equivalence* $\equiv_{\mathbf{K}}$ *induced by the following rules*[7]*:*

$$\frac{\varphi \text{ propositional tautology}}{\vdash \varphi} \qquad \frac{\vdash \varphi}{\vdash \varphi[\psi/p]}\ (p \in AP) \qquad \frac{\vdash \psi \qquad \vdash \psi \to \varphi}{\vdash \varphi}$$

$$\frac{}{\vdash \Box(p \to q) \to (\Box p \to \Box q)} \qquad \frac{\vdash \varphi}{\vdash \Box\varphi}$$

LT is of course a Boolean algebra (see e.g. [5, Chap. 5]). We often leave implicit the function $\Lambda \to LT$ taking a formula $\varphi \in \Lambda$ to its $\equiv_{\mathbf{K}}$-class in $[\varphi]_{\mathbf{K}} \in LT$, and write $\varphi \in LT$ for $[\varphi]_{\mathbf{K}} \in LT$.

Continuing our discussion, if we look at the transpose (1) of $\Vdash^v$ as function $|K| \to \mathbf{2}^{LT}$, then for every state $s \in |K|$, its image at s is an ultrafilter on LT. Recall that an *ultrafilter* on a Boolean algebra B is a set $F \subseteq B$ such that (1) $\bot \notin F$, (2) $\top \in F$, (3) if $a \in F$ and $a \leq b$ then $b \in F$, (4) if $a, b \in F$ then $a \wedge b \in F$, and (5) if $a \notin F$ then $\neg a \in F$. Write $\mathfrak{Uf}(B)$ for the set of ultrafilter on B. The function (1) induces a **TS**-map $K \to \mathfrak{Uf}(LT)$, where $\partial_{\mathfrak{Uf}(LT)}$ is the restriction of the function $\mathbf{2}^{LT} \to \mathcal{P}(\mathbf{2}^{LT})$ induced by $\partial_{\mathbf{2}^{\Lambda}}$ (see e.g. [5, Def. 5.40]).

We are now going to see how we can characterize the function modal satisfaction in this algebraic setting. An interesting thing to notice is that the notion of coalgebra map does not seem to directly help us here, since as discussed in Rem. 4.2 below, for a given valuation v, the **TS**-map $K \to \mathbf{2}^{\mathfrak{Uf}(LT)}$ is map of coalgebras exactly when the model (K, v) is *modally-saturated*.

Remark 4.2. Following the usual terminology, a (Kripke) *model* is a transition system K together with a valuation of atomic propositions $v : AP \to \mathbf{2}^{|K|}$. We say that a model (K, v) is *modally saturated* (see e.g.[5, §2.5]) whenever for all state $s \in |K|$ and all (possibly infinite) set of formulae Φ, if $s \Vdash^v \Diamond \bigwedge \Psi$ for all *finite* $\Psi \subseteq \Phi$, then there is $t \in \partial(s)$ s.t. $t \Vdash^v \varphi$ for all $\varphi \in \Phi$. It is well-known that modally saturated models satisfy the Hennessy-Milner property (see e.g. [5, §2.5]), and moreover that ultrafilter frames as well as image finite models are modally saturated.

We note the following property. Given a t.s. K and a valuation $v : AP \to \mathbf{2}^{|K|}$, the **TS**-map $K \to \mathfrak{Uf}(LT)$ induced by $\Vdash^v$ is a map of coalgebras if and only if the model (K, v) is modally saturated. $\Box$

Moreover, Rem. 4.3 below tells us that we may equally well see modal satisfaction either as a **TS**-map $K \to \mathfrak{Uf}(LT)$ or as a Boolean algebra map $LT \to \mathbf{2}^{|K|}$.

Remark 4.3. Consider a Boolean algebra B, a set S and a function $f : S \times B \to \mathbf{2}$ together with its transposes $f^{\mathrm{t}} : S \to \mathbf{2}^B$ and ${}^{\mathrm{t}}f : B \to \mathbf{2}^S$. Then ${}^{\mathrm{t}}f$ is a map of Boolean algebras iff for all $s \in S$, the set $f^{\mathrm{t}}(s) \subseteq B$ is an ultrafilter. $\Box$

Hence modal satisfaction $\Vdash^v$ leads to a Boolean algebra map $LT \to \mathbf{2}^{|K|}$, which moreover extends $v : AP \to \mathbf{2}^{|K|}$ in the sense that the following diagram

[7] So that $\varphi \equiv_{\mathbf{K}} \psi$ if and only if ($\vdash \varphi \to \psi$ and $\vdash \psi \to \varphi$).

commutes:

$$\begin{tikzcd} LT \arrow[r] & \mathbf{2}^{|K|} \\ AP \arrow[u] \arrow[ur, "v"'] & \end{tikzcd} \qquad (4)$$

This, however, does not completely characterize modal satisfaction since the property of being a Boolean algebra map specifies the behavior of the Boolean connectives, but not of the modalities. In algebraic modal logic, modalities are handled by Boolean algebras equipped with certain *operators*.

Definition 4.4. *A* Boolean algebra with operator *(***BAO***) is a Boolean algebra B with a function $\Diamond_B : B \to B$ s.t. $\Diamond_B \bot = \bot$ and $\Diamond_B(a \vee b) = \Diamond_B a \vee \Diamond_B b$.*

A map of **BAO***'s form $(B, \Diamond_B)$ to $(B', \Diamond_{B'})$ is a map of Boolean algebras $f : B \to B'$ which preserves the operators in the sense that $f \circ \Diamond_B = \Diamond_{B'} \circ f$.*

Remark 4.5. We have taken in Def. 4.4 the notion of **BAO** used in [5], but it is also customary to work with the equivalent notion of a Boolean algebra equipped with a function $\Box_B$ which commutes over $\top$ and $\wedge$. □

LT equipped with the operator $\Diamond$ is a **BAO** (see e.g. [5, Lem. 4.6 & Thm. 5.33]). Moreover, the notion of **BAO** completely characterizes modal satisfaction once the powerset Boolean algebra $\mathbf{2}^{|K|}$ is equipped with the operator $\Diamond_K$ taking a set $A \subseteq |K|$ to the set $\Diamond_K(A)$ of all one-step predecessors of A, *i.e.* the set of all $s \in |K|$ such that $t \in \partial_K(s)$ for some $t \in A$ (see e.g. [5, §5.2]).

Proposition 4.6. *Given a t.s. K and a valuation $v : AP \to \mathbf{2}^{|K|}$, the function $LT \to \mathbf{2}^{|K|}$ induced by $\Vdash^v$ is the unique* **BAO** *map extending v (see (4)).*

4.2 Mapping $\mathscr{S}$ to a Boolean Topos

We now present some categorical machinery that will help us expressing the characterization of modal satisfaction given by Prop. 4.6 in our setting. We follow the usual approach when representing algebraic structures in categories.

Remark 4.7 (Algebraic Structures in Categories). We say that an object C of a category $\mathbb{C}$ with finite limits has a given algebraic structure if it is equipped with maps for each operation of the algebraic structure, which moreover satisfy diagrams corresponding to the equations of the algebraic structure. See e.g. [21, pp. 2-5] for the example of monoids. For Boolean algebras we follow the equational presentation induced by [14, §I.1]. □

First, we devise an object LT of $\mathscr{S}$ in the same way as the object Λ in Def. 3.10.

Definition 4.8. *Define the object LT of $\mathscr{S}$ as $\mathsf{LT}(n) := LT$ for each $n > 0$, and with restriction maps $(-)\!\uparrow : \mathsf{LT}(n+1) \to \mathsf{LT}(n)$ taking $[\varphi]_{\mathbf{K}}$ to $[\Diamond\varphi]_{\mathbf{K}}$.*

The restriction maps of the object LT are well-defined since $\Diamond$ preserves logical equivalence (see e.g. [5, Lem. 4.6]). Given an object X of $\mathscr{S}$ and a valuation v, for each $n > 0$ we lift $\Vdash^v_n \subseteq X(n) \times \Lambda$ to $\Vdash^v_n \subseteq X(n) \times LT$ in the obvious way, and Prop. 3.11 trivially extends.

Definition 4.9. *Given X and v as above, we write $\kappa^v : X \times \mathsf{LT} \to \Omega$ for the classifying map in $\mathscr{S}$ of the family of relations $(\Vdash_n^v)_n \hookrightarrow X \times \mathsf{LT}$.*

It is easy to see that since $\Diamond$ is an operator on LT, the $\mathscr{S}$-object LT is equipped in $\mathscr{S}$ with the structure of an internal $\vee$-semilattice. It is however *not* a Boolean algebra, because the Boolean connectives $\wedge, \top, \neg$ do not commute with $\Diamond$. In other words, the restriction maps of LT, which on the one hand allows us to see modal satisfaction as a subobject in $\mathscr{S}$, on the other hand prevent us from seeing LT as a Boolean algebra. We shall thus look to a version of LT "without restriction maps".

As simple way to "remove restriction maps" from an object of $\mathscr{S}$ is to send it to $\mathrm{Psh}(|\mathbb{N}_*|)$, the category of $\mathbb{N}_*$-indexed families of sets. The functor $\iota^* : \mathscr{S} \to \mathrm{Psh}(|\mathbb{N}_*|)$ just maps X to the family $X^* := (X_n)_n$. We shall use the following properties of ι^*, which are standard categorical material (see e.g. [15, §A.4]).

Lemma 4.10. *The functor ι^* is faithful and preserves limits. It induces for each object X of $\mathscr{S}$ an injective map of lattices $\Delta_X : \mathrm{Sub}_{\mathscr{S}}(X) \to \mathrm{Sub}_{\mathrm{Psh}(|\mathbb{N}_*|)}(X^*)$.*

Lemma 4.11. *The object $\mathsf{LT}^* = \iota^*(\mathsf{LT})$ is a Boolean algebra in $\mathrm{Psh}(|\mathbb{N}_*|)$.*

The topos $\mathrm{Psh}(|\mathbb{N}_*|)$ is *Boolean* since its subobject classifier $\mathbf{2}$ is an internal Boolean algebra. In particular, for any object X of $\mathscr{S}$, the object $\mathbf{2}^{X^*}$ of $\mathrm{Psh}(|\mathbb{N}_*|)$ is a Boolean algebra, whose connectives $(\wedge, \vee, \top, \bot, \neg)$ are given by the pointwise set-theoretic operations $(\cap, \cup, X(n), \emptyset, X(n) \setminus -)$. We will use this fact to characterize the behavior of the transpose $\mathsf{LT} \to \Omega^X$ on Boolean connectives.

5 Characterization of the Modal Satisfaction Relation

Using the material of §4.2, we can now give an adaptation of Prop. 4.6 to our context. Let us first set some concepts and notations.

Notation 5.1 (Valuations of Atomic Propositions). *We let AP be the object of $\mathrm{Psh}(|\mathbb{N}_*|)$ with constant value AP. Moreover, we write $\mathsf{AP} \to \mathsf{LT}^*$ for the extension with $\iota^*(\mathsf{\Lambda} \to \mathsf{LT})$ of the inclusion map $\mathsf{AP} \hookrightarrow \mathsf{\Lambda}^*$.*

Up to now, valuations of atomic propositions for an object X of $\mathscr{S}$ were seen as usual **Sets** *functions $v : AP \to \mathbf{2}^{|X^{\oplus}|}$, that is as functions from AP to $\coprod_{n>0}(X(n)) \to \mathbf{2}$. But such functions are in bijection with $\prod_{n>0}(AP \to \mathbf{2}^{X(n)})$, that is with $\mathrm{Psh}(|\mathbb{N}_*|)$-maps $\mathsf{AP} \to \mathbf{2}^{X^*}$. From now on, we assume valuations to be maps $\mathsf{AP} \to \mathbf{2}^{X^*}$.*

We thus arrive at the first characteristic property of modal satisfaction in $\mathscr{S}$, namely that it induces a map of Boolean algebras in $\mathrm{Psh}(|\mathbb{N}_*|)$. Given an object X of $\mathscr{S}$ and a valuation v, we write $\Delta(\kappa^v) : X^* \times \mathsf{LT}^* \to \mathbf{2}$ for the image in $\mathrm{Psh}(|\mathbb{N}_*|)$ of $\kappa^v : X \times \mathsf{LT} \to \Omega$ induced by the map $\Delta_{X \times \mathsf{LT}}$ of Lem. 4.10.

Proposition 5.2. *Given X and v as above, the transpose ${}^{\mathrm{t}}\Delta(\kappa^v) : \mathsf{LT}^* \to \mathbf{2}^{X^*}$ of $\Delta(\kappa^v) : X^* \times \mathsf{LT}^* \to \mathbf{2}$ is a map of Boolean algebras in $\mathrm{Psh}(|\mathbb{N}_*|)$, which moreover extends v, in the sense that*

$$\begin{tikzcd} \mathsf{LT}^* \arrow[r, "{}^{\mathrm{t}}\Delta(\kappa^v)"] & \mathbf{2}^{X^*} \\ \mathsf{AP} \arrow[u] \arrow[ur, "v"'] & \end{tikzcd}$$

It remains to characterize the behavior of the map $\kappa^v : X \times \mathsf{LT} \to \Omega$ on modalities. We rely on notion of *total subobjects* of $\mathscr{S}$ (see [4, Def. 2.6]).

Definition 5.3. *An object of $\mathscr{S}$ is* total *if all its restriction maps are surjective.*

Recall from §4.1 that usual modal satisfaction $LT \to \mathbf{2}^{X^\oplus}$ is map of **BAO** when $\mathbf{2}^{X^\oplus}$ is equipped with the $\Diamond_{\mathbf{2}^{X^\oplus}}$ takes a set of states S to the set of all states which have a one-step successor in S, that is the set of all x such that there is $y \in \partial_{X^\oplus}(x)$ with $y \in S$. Actually, for every subobject $S \hookrightarrow X$, we do have $\Diamond_{X^\oplus} S(n+1) \subseteq S(n)$ for each n. The converse inclusion corresponds to the surjectivity of restriction maps.

Lemma 5.4. *Consider an object X of $\mathscr{S}$, some $n > 0$ and subsets $S \subseteq X(n)$, $S' \subseteq X(n+1)$ which are compatible with restriction ($x{\uparrow} \in S$ whenever $x \in S'$). Then $\Diamond_{X^\oplus} S' \subseteq S$. Moreover, we have $\Diamond_{X^\oplus} S' = S$ iff the restriction of $(-){\uparrow} : X(n+1) \to X(n)$ to $S' \to S$ is surjective.*

Given an object X of $\mathscr{S}$, consider the family of sets $(\mathrm{TotSub}_X(n))_{n>0}$, where $\mathrm{TotSub}_X(n)$ is the set of all sequences $(S_\ell)_{\ell \leq n}$ of sets $S_\ell \subseteq X(\ell)$ such that the restrictions of $(-){\uparrow} : X(\ell+1) \to X(\ell)$ to $S(\ell+1) \to S(\ell)$ are surjective. We equip TotSub_X with restriction maps taking $(S_1, \ldots, S_n, S_{n+1})$ to $(S_1, \ldots, S_n)$, so that TotSub_X induces a subobject of Ω^X via Rem. 2.1. This leads to the second characteristic property of κ^v, which specifies its behavior on modalities.

Proposition 5.5. *The map ${}^{\mathrm{t}}\kappa^v : \mathsf{LT} \to \Omega^X$ factors as $\mathsf{LT} \to \mathrm{TotSub}_X \hookrightarrow \Omega^X$.*

We can now give our characterization of κ^v: it is the unique map satisfying Prop. 5.2 and Prop. 5.5.

Theorem 5.6. *Given $v : \mathsf{AP} \to \mathbf{2}^{X^*}$, the map $\kappa^v : X \times \mathsf{LT} \to \Omega$ is the unique map which satisfies all the following conditions:*

(i) ${}^{\mathrm{t}}\Delta(\kappa^v) : \mathsf{LT}^* \to \mathbf{2}^{X^*}$ *is a map of Boolean algebras which extends v, and*
(ii) ${}^{\mathrm{t}}\kappa^v : \mathsf{LT} \to \Omega^X$ *factors as* $\mathsf{LT} \to \mathrm{TotSub}_X \hookrightarrow \Omega^X$.

6 Extensions

We now discuss two extensions of **K** in our framework. The first one is the logic **S4** with reversed modalities, which corresponds to the direct adaptation of [2, 19] to our case. The second is a brief discussion on fixpoints. We begin with some standard categorical material extending §4.2.

6.1 A Geometric Morphism

The functor $\iota^* : \mathscr{S} \to \mathrm{Psh}(|\mathbb{N}_*|)$ of §4.2 takes $X : \mathbb{N}_*^{\mathrm{op}} \to \mathbf{Sets}$ to its pre-composition with $\iota : |\mathbb{N}_*| \hookrightarrow \mathbb{N}_*$. As usual (see e.g. [15, Ex. A.4.1.4]), ι^* has a right adjoint ι_* given by right Kan extensions along ι. Explicitly, $\iota_*(A)(n) = \prod_{1 \leq k \leq n} A(k)$ for $A \in \mathrm{Psh}(|\mathbb{N}_*|)$, and restrictions $\iota_*(A)(n+1) \to \iota_*(A)(n)$ map $(a_1, \ldots, a_n, a_{n+1})$ to $(a_1, \ldots, a_n)$. Since ι^* preserves finite limits [15, Ex. A.4.1.4], we have a geometric morphism $\iota^* \dashv \iota_* : \mathrm{Psh}(|\mathbb{N}_*|) \to \mathscr{S}$. This geometric morphism is an instance of a widespread construction in categorical approaches to **S4** (see e.g. [29, 2, 19]).

Lemma 6.1. *Consider a geometric morphism $f = f^* \dashv f_* : \mathcal{F} \to \mathcal{E}$. The inverse image functor f^* induces for each object A of $\mathcal{E}$ an (external) homomorphism of subobjects lattices $\Delta_A : \mathrm{Sub}_{\mathcal{E}}(A) \to \mathrm{Sub}_{\mathcal{F}}(f^*A)$, which moreover has an (external) right adjoint of posets $\Gamma_A : \mathrm{Sub}_{\mathcal{F}}(f^*A) \to \mathrm{Sub}_{\mathcal{E}}(A)$.*

Proposition 6.2. *In the case of $\iota^* \dashv \iota_* : \mathrm{Psh}(|\mathbb{N}_*|) \to \mathscr{S}$, given an object X of $\mathscr{S}$, the map $\Gamma_X : \mathrm{Sub}_{\mathrm{Psh}(|\mathbb{N}_*|)}(X^*) \to \mathrm{Sub}_{\mathscr{S}}(X)$ takes a subobject $A \hookrightarrow X^*$ in $\mathrm{Psh}(|\mathbb{N}_*|)$ to $\Gamma_X(A) \hookrightarrow X$ where $\Gamma_X(A)(n) = \{x \in X(n) \mid \forall k \leq n.\ x_{|k} \in A(k)\}$. Moreover, the composite $\Gamma_X \circ \Delta_X$ is the identity on $\mathrm{Sub}_{\mathscr{S}}(X)$.*

In the case of $\iota^* \dashv \iota_*$, the crucial construction of [2, 19] specializes to the following, where we write $\mathbf{2}_*$ for $\iota_*(\mathbf{2})$.

Lemma 6.3. *The map $\lambda : \Omega \to \mathbf{2}_*$, taking $k \in \Omega(n)$ to $(1^k, 0^{n-k})$ is a map of internal lattices. It is an internal left-adjoint to $\tau : \mathbf{2}_* \to \Omega$, the classifying map of $\iota_*(\mathrm{t}) : \iota_*(\mathbf{1}) \to \mathbf{2}_*$ in $\mathscr{S}$. Moreover, $\tau \circ \lambda = \mathrm{id}_\Omega$.*

In particular, $\lambda \circ \tau$ induces a left-exact comonad on $\mathbf{2}_*$.

6.2 Accommodating an S4 Reverse Modality

As a sanity check, we verify here that an obvious adaption of [9, 29, 2, 19] is compatible with our setting, and gives an **S4** reverse modality, that is a reflexive-transitive modality $\boxminus^*$ quantifying over the *predecessors* of a state. Consider formulae $\Lambda_{\boxminus^*}$ extending Λ with the clause $\boxminus^*\varphi \in \Lambda_{\boxminus^*}$ whenever $\varphi \in \Lambda_{\boxminus^*}$. Given a t.s. K and a valuation v, write $\to_K^*$ for the reflexive-transitive closure of $\to_K$. The relation $\Vdash^v_{\boxminus^*}$ extends $\Vdash^v$ with the clause

$$s \Vdash^v_{\boxminus^*} \boxminus^*\varphi \qquad \text{iff} \qquad \forall t \in |K|\,(t \to_K^* s \quad \Longrightarrow \quad t \Vdash^v_{\boxminus^*} \varphi)$$

Following Rem. 2.2, we look at the relation $\Vdash^v_{\boxminus^*} \subseteq |K| \times \Lambda_{\boxminus^*}$ as a function $[\![-]\!]^v_{\boxminus^*} : \Lambda_{\boxminus^*} \to \mathbf{2}^{|K|}$. In the case of an object X of trees, we therefore get $[\![-]\!]^v_{\boxminus^*} : \Lambda_{\boxminus^*} \to \mathbf{2}^{|X^\oplus|}$. We now extend the approach of §3.3. Define $(\Vdash^v_{\boxminus^*,n})_n \subseteq \prod_n (X \times \Lambda_{\boxminus^*})$ as

$$x \Vdash^v_{\boxminus^*,n} \varphi \qquad \text{iff} \qquad (n, x) \Vdash^v_{\boxminus^*} \varphi \qquad (\text{iff} \quad (n, x) \in [\![\varphi]\!]^v_{\boxminus^*})$$

Defining the object $\mathsf{\Lambda}_{\boxminus^*}$ of $\mathscr{S}$ similarly as $\mathsf{\Lambda}$ in Def. 3.10, Prop. 3.11 easily extends.

Lemma 6.4. *Given X and v, the family of relations $(\Vdash^{v}_{\boxminus^*,n})_n$ is a subobject of $X \times \Lambda_{\boxminus^*}$ in $\mathscr{S}$.*

Write $\kappa^{v}_{\Lambda_{\boxminus^*}} : X \times \Lambda_{\boxminus^*} \to \Omega$ for the classifying map of $(\Vdash^{v}_{\boxminus^*,n})_n$. Consider now the image $\Delta(\kappa^{v}_{\Lambda_{\boxminus^*}}) : X^* \times \Lambda_{\boxminus^*} \to \mathbf{2}$ of $\kappa^{v}_{\Lambda_{\boxminus^*}}$ induced by the map $\Delta_{\Lambda_{\boxminus^*} \times X}$ of Lem. 4.10. Note that $\Delta(\kappa^{v}_{\Lambda_{\boxminus^*}})$ is a function $\prod_n (X(n) \times \Lambda_{\boxminus^*} \to \mathbf{2})$, and moreover

$$\prod_n (X(n) \times \Lambda_{\boxminus^*} \to \mathbf{2}) \quad \simeq \quad \Lambda_{\boxminus^*} \to (\coprod_n X(n)) \to \mathbf{2} \tag{5}$$

Lemma 6.5. *The function $\Lambda_{\boxminus^*} \to (\coprod_n X(n)) \to \mathbf{2}$ induced by (5) coincides with the interpretation function $[\![-]\!]^{v}_{\boxminus^*} : \Lambda_{\boxminus^*} \to \mathbf{2}^{|X^{\oplus}|}$.*

In other words, the interpretation $[\![\varphi]\!]^{v}_{\boxminus^*}$ of a formula $\varphi \in \Lambda_{\boxminus^*}$ is a subobject of X^*, that we still write $[\![\varphi]\!]^{v}_{\boxminus^*}$. This interpretation is lifted to

$$X \quad \xrightarrow{\eta_X} \quad X_* \quad \xrightarrow{\iota_*([\![\varphi]\!]^{v}_{\boxminus^*})} \quad \mathbf{2}_* \tag{6}$$

where we have written X_* for $\iota_*(\iota^* X)$.

Proposition 6.6. *The interpretation of $\boxminus^* \varphi$ as the map $X \to \mathbf{2}_*$ of (6) is the image under the comonad $\lambda \circ \tau$ of the interpretation of φ as a map $X \to \mathbf{2}_*$.*

Remark 6.7. The comparison with [2, 19] can be pushed one step further. The approach of [2, 19] in particular includes the case of the (surjective) geometric morphism $\jmath^* \dashv \jmath_* : \mathbf{Sets}^{|K|} \to \mathbf{Sets}^{K}$ where K is a transition system seen as a preorder, and where $\jmath^*$ is precomposition with $\jmath : |K| \to K$ and $\jmath_*$ is given by right Kan extensions. When restricted to propositional logic over Kripke structures, a modal formula over K is interpreted as a map $\mathbf{1} \to \mathbf{2}_K$ where $\mathbf{2}_K$ is the image by $\jmath_*$ of the (Boolean) subobject classifier $\mathbf{2}$ of $\mathbf{Sets}^{|K|}$.

Recall from e.g. [22, Ex. III.8] that given a presheaf category $\mathrm{Psh}(\mathbb{C})$, the slice category $\mathrm{Psh}(\mathbb{C})/P$ is equivalent to the category of presheaves over $\int P$. The part $\tilde{-} : \mathrm{Psh}(\int P) \to \mathrm{Psh}(\mathbb{C})/P$ of the equivalence takes $S : (\int P)^{\mathrm{op}} \to \mathbf{Sets}$ to $(T, \theta : T \to P)$, where $T \in \mathrm{Psh}(\mathbb{C})$ is given by $T(C) = \coprod_{a \in P(C)} S(C, a)$, and $\theta_C : T(C) \to P(C)$ is the first projection (taking $(a, -)$ to $a \in P(C)$). Hence, if $\int P$ is equivalent to the opposite of the reflexive-transitive closure of a transition system K, then the interpretation of formulae as $\mathbf{Sets}^{K}$-maps $\mathbf{1} \to \mathbf{2}_K$ of [2, 19] induces an interpretation of formulae as $\mathrm{Psh}(\mathbb{C})/P$-maps from $\mathbf{1}$ to $\tilde{\mathbf{2}}_K \to P$, that is as $\mathrm{Psh}(\mathbb{C})$-maps $P \to \tilde{\mathbf{2}}_K$ since the $\mathbf{1}$ of $\mathrm{Psh}(\mathbb{C})/P$ is $\mathrm{id}_P : P \to P$.

In the case of $\mathscr{S}$ and $K^{\mathrm{op}} = \int X$, recall from Rem. 3.5 that K^{op} is the reflexive-transitive closure of $(|X^{\oplus}|, \to_{X^{\oplus}})$. Note that $\mathbf{2}_K(n, x) = \prod_{(\ell, y) \to (n, x)} \mathbf{2}$. But $\int X$-maps $(\ell, y) \to (n, x)$ for fixed (n, x) are completely determined by ℓ since we must have $y = x_{|\ell}$. We thus have $\mathbf{2}_K(n, x) \simeq \mathbf{2}^n = \mathbf{2}_*(n)$. It follows that $\tilde{-} : \mathbf{Sets}^{K} \to \mathscr{S}/X$ takes $\mathbf{2}_K$ to the first projection $X \times \mathbf{2}_* \to X$, so that a formula $\mathbf{1} \to \mathbf{2}_K$ in $\mathbf{Sets}^{K}$ is taken to a map $K \to K \times \mathbf{2}_*$ whose composite with the first projection is the identity, that is to a $\mathscr{S}$-map $\mathbf{1} \to \mathbf{2}_*$. In the case of a formula $\varphi \in \Lambda_{\boxminus^*}$ *with no occurrence of* $\Diamond$, it follows from the correctness of [2, 19] ([2, Prop. 4.9]) that the induced map $X \to \mathbf{2}_*$ coincides with (6). $\square$

6.3 Extension to the Modal μ-Calculus

The modal μ-calculus [20] extends the basic modal language (§2.3) with fixpoints of monotone formulae. Its formulae $\varphi, \psi, \ldots \in \Lambda_\mu$ are obtained by extending Λ with the clause $\mu p.\varphi \in \Lambda_\mu$ whenever $\varphi \in \Lambda_\mu$ and p only occurs under an even number of negations in φ.[8] Given a transition system K and a valuation $v : AP \to \mathbf{2}^{|K|}$, the modal satisfaction relation $\Vdash^v \subseteq |K| \times \Lambda$ of §2.3, seen as a function $[\![-]\!]^v : \Lambda \to \mathbf{2}^{|K|}$ (see Rem. 2.2) is extended to $[\![-]\!]^v_\mu : \Lambda_\mu \to \mathbf{2}^{|K|}$ with

$$[\![\mu p.\varphi]\!]^v_\mu \quad := \quad \bigcap\{[\![\varphi]\!]^{v[A/p]}_\mu \mid A \subseteq |K| \;\&\; [\![\varphi]\!]^{v[A/p]}_\mu \subseteq A\}$$

We now proceed similarly as in §6.2 above. For X an object of $\mathscr{S}$, we define the relations $(\Vdash^v_{\mu,n})_n \subseteq \prod_n (X \times \Lambda_\mu)$ as $x \Vdash^v_{\mu,n} \varphi$ iff $(n,x) \Vdash^v_\mu \varphi$. We let $\mathsf{\Lambda}_\mu$ be the object of $\mathscr{S}$ with the obvious adaptation of $\mathsf{\Lambda}$, and we get the expected extension of Prop. 3.11. Similarly as in §6.2 above, writing $\kappa^v_{\mathsf{\Lambda}_\mu} : X \times \mathsf{\Lambda}_\mu \to \Omega$ for the classifying map of $(\Vdash^v_{\mu,n})_n$, we consider the function $\Lambda_\mu \to |X^\oplus| \to \mathbf{2}$ induced by $\Delta(\kappa^v_{\mathsf{\Lambda}_\mu}) : X^* \times \mathsf{\Lambda}^*_\mu \to \mathbf{2}$. Lemma 6.5 is trivially adapted.

Lemma 6.8. *The function $\Lambda_\mu \to (\coprod_n X(n)) \to \mathbf{2}$ induced by $\Delta(\kappa^v_{\mathsf{\Lambda}_\mu}) : X^* \times \mathsf{\Lambda}_\mu \to \mathbf{2}$ coincides with $[\![-]\!]^v_\mu : \Lambda_\mu \to \mathbf{2}^{|X^\oplus|}$.*

We can say the following on the interpretation $[\![\varphi]\!]^v_\mu$ seen as a subobject of X^*.

Proposition 6.9. *Given a formula φ monotone in the propositional variable α, the interpretation $[\![\mu\alpha.\varphi]\!]^v_\mu$ coincides with the following subobject of X^*:*

$$\bigwedge\{A \mid A \in \mathrm{Sub}_{\mathrm{Psh}(|\mathbb{N}_*|)}(X^*) \;\&\; [\![\varphi]\!]^{v[A/\alpha]}_\mu \leq_{\mathrm{Sub}_{\mathrm{Psh}(|\mathbb{N}_*|)}(X^*)} A\} \tag{7}$$

Remark 6.10. Since Γ_X preserves limits, (7) leads in $\mathrm{Sub}_{\mathscr{S}}(X)$ to

$$\bigwedge\{\Gamma_X A \mid A \in \mathrm{Sub}_{\mathrm{Psh}(|\mathbb{N}_*|)}(X^*) \;\&\; [\![\varphi]\!]^{v[A/\alpha]}_\mu \leq_{\mathrm{Sub}_{\mathrm{Psh}(|\mathbb{N}_*|)}(X^*)} A\}$$

□

7 Conclusion

In this paper, we noticed the existence of a coreflection $(-)^\oplus \dashv (-)_\circledast : \mathbf{TS} \to \mathscr{S}$,[9] allowing to see every object X as t.s. $X^\oplus$, and moreover to lift the modal logic $\mathbf{K}$ over $X^\oplus$ as a subobject of $X \times \mathsf{\Lambda}$ in $\mathscr{S}$, for a suitable presheaf $\mathsf{\Lambda}$. We have shown how the usual (co)-algebraic characterization of modal satisfaction as a morphism of **BAO**'s can expressed using the left adjoint part of the usual geometric morphism $\mathrm{Psh}(|\mathbb{N}_*|) \to \mathscr{S}$ for Boolean connectives, and the notion of total subobject in $\mathscr{S}$ (in the sense of [4]) for the modality. Furthermore, we have sketched how the usual categorical approaches to **S4** instantiate with $\mathrm{Psh}(|\mathbb{N}_*|) \to \mathscr{S}$ to give an **S4** reverse (*i.e.* past) modality, and we briefly presented the computation of fixpoints of the modal μ-calculus [20].

[8] In this Section, we see atomic propositions as propositional variables.

[9] Reminiscent from the coreflection of synchronization trees in labeled transition systems [32, §4].

Further Works. First, among the general motivations of this work are the formulation of logics for verification directly in (fragments of) semantic models, whether they be denotational or based on realizability (or logical relations). One track to explore is the application of higher-order model checking [27] (with the hope that denotational models allow to interpret more structure than Böhm trees, *i.e.* fully expanded syntactic trees), but we also aim at devising syntactic type/proofs system lifted back from $\mathscr{S}$ (in the spirit of e.g. [7]).

A second (more exploratory) direction is to look at generalizations. Following Rem. 3.2, the coreflection $(-)^{\oplus} \dashv (-)_{\circledast}$ allows to see every object X of $\mathscr{S}$ as a subobject of the internal streams of $\mathscr{S}$ over $|X^{\oplus}|$. On the other hand, the carriers of final coalgebras for polynomial functors are represented in $\mathscr{S}$ as the sets of global sections of the corresponding coinductive types. This might suggest to look for generalizations to polynomial functors (see e.g. [12]). But our approach is also reminiscent from [13] (if we see $X^{\oplus}_{\circledast} \hookrightarrow \mathrm{Str}_{|X^{\oplus}|}$ as representing X by its traces), and [13, 11] might also present interesting hints.

An other direction concerns completeness, in particular for settings where algebraic approaches are available, e.g. normal modal logics (see e.g. [5, Chap. 4]), finitary Kripke polynomial functors (see e.g. [12]), or flat fixpoints [30] (a family of subsystems of the modal μ-calculus which in particular encompasses CTL). In a more prospective direction, it might be interesting to know if there is a connection with usual categorical completeness results (see e.g. [23]). Besides, our way to represent transition systems and modal logic in $\mathscr{S}$ (in particular with the usual cover modality ∇ in mind (see e.g. [30])), when formulated in first-order logic, is reminiscent from the construction of syntactic sites in topos theory (see e.g. [22, §X.5]).

Finally, we have shown in §6.3 how the fixpoints of the modal μ-calculus act on objects of $\mathscr{S}$. But further work is still required in order to see what could be the corresponding extension of Thm. 5.6. A possibly more accessible step would be to focus on flat fixpoints [30], because they are equipped with simpler algebraic structure than the full modal μ-calculus, which seems to require a full Stone-type approach [1] (see also [6] for modal logics over Vietoris polynomial functors).

References

[1] S. Ambler, M. Kwiatkowska, and N. Measor. Duality and the completeness of the modal μ-calculus. *Theoretical Computer Science*, 151(1):3 – 27, 1995. Selected Papers of the Workshop on Topology and Completion in Semantics. 15

[2] S. Awodey, K. Kishida, and H.-C. Kotzsch. Topos Semantics for Higher-Order Modal Logic. arXiv:1403.0020, February 2014. 2, 3, 11, 12, 13, 29, 30

[3] C. Baier and J.-P. Katoen. *Principles of Model Checking*. MIT Press, 2008. 1

[4] L. Birkedal, R. E. Møgelberg, J. Schwinghammer, and K. Støvring. First steps in synthetic guarded domain theory: step-indexing in the topos of trees. *Logical Methods in Computer Science*, 8(4), 2012. 1, 2, 5, 7, 11, 14

[5] P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic*. Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 2002. 2, 6, 7, 8, 9, 15, 21

[6] Kupke C., A. Kurz, and Y. Venema. Stone coalgebras. *Theor. Comput. Sci.*, 327(1-2):109–134, 2004. 15

[7] R. Clouston, A. Bizjak, H. Bugge Grathwohl, and L. Birkedal. The Guarded Lambda-Calculus: Programming and Reasoning with Guarded Recursion for Coinductive Types. *Logical Methods in Computer Science*, 12(3), 2016. 1, 2, 5, 15

[8] D. Dreyer, G. Neis, A. Rossberg, and L. Birkedal. A Relational Modal Logic for Higher-order Stateful ADTs. In *Proceedings POPL'10*, pages 185–198. ACM, 2010. 2

[9] S. Ghilardi and G. C. Meloni. Modal and Tense Predicate Logic: Models in Presheaves and Categorical Conceptualization. In F. Borceux, editor, *Categorical Algebra and its Applications*, LNM, pages 130–142. Springer Berlin Heidelberg, 1988. 2, 12

[10] V. Goranko and M. Otto. Model Theory of Modal Logic. In P. Blackburn, J. van Benthem, and F. Wolter, editors, *Handbook of Modal Logic*, volume 3 of *Studies in Logic and Practical Reasoning*, chapter 5, pages 249–329. Elsevier, 2007. 3

[11] I. Hasuo, B. Jacobs, and A. Sokolova. Generic trace semantics via coinduction. *Logical Methods in Computer Science*, 3(4), 2007. 15

[12] B. Jacobs. Many-Sorted Coalgebraic Modal Logic: a Model-theoretic Study. *ITA*, 35(1):31–59, 2001. 2, 15

[13] B. Jacobs. Trace Semantics for Coalgebras. In *Proceedings of CMCS'04*, Electronic Notes in Theoretical Computer Science. Elsevier, 2004. 15

[14] P.T. Johnstone. *Stone Spaces*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1986. 9

[15] P.T. Johnstone. *Sketches of an Elephant: A Topos Theory Compendium*. Oxford Logic Guides. Clarendon Press, 2002. 10, 12, 27, 28

[16] A. Joyal and Moerdijk. I. A completeness theorem for open maps. *Annals of Pure and Applied Logic*, 70(1):51–86, 1994. 4, 5, 19

[17] A. Joyal and I. Moerdijk. *Algebraic Set Theory*. Lecture note series. Cambridge University Press, 1995. 1, 4, 5, 19

[18] A. Joyal, M. Nielsen, and G. Winskel. Bisimulation from Open Maps. *Inf. Comput.*, 127(2):164–185, 1996. 4, 5, 19

[19] H.-C. Kotzsch. *Topos Semantics for Higher-Order Modal Logic*. PhD thesis, Ludwig-Maximilians-Universitt Mnchen, 2016. 2, 3, 11, 12, 13

[20] D. Kozen. Results on the propositional μ-calculus. *Theoretical Computer Science*, 27(3):333 – 354, 1983. Special Issue Ninth International Colloquium on Automata, Languages and Programming (ICALP) Aarhus, Summer 1982. 14

[21] S. Mac Lane. *Categories for the Working Mathematician*. Springer, 2nd edition, 1998. 1, 5, 9, 18

[22] S. Mac Lane and I. Moerdijk. *Sheaves in geometry and logic: A first introduction to topos theory*. Springer, 1992. 3, 13, 15, 25, 26, 28, 30

[23] M. Makkai and G.E Reyes. Completeness results for intuitionistic and modal logic in a categorical setting. *Annals of Pure and Applied Logic*, 72(1):25 – 101, 1995. 15

[24] R. E. Møgelberg. A type theory for productive coprogramming via guarded recursion. In *Proceedings of CSL-LICS 2014*, CSL-LICS '14. ACM, 2014. 2, 5

[25] R. E. Møgelberg and M. Paviotti. Denotational Semantics of Recursive Types in Synthetic Guarded Domain Theory. In *Proceedings of LICS'16*, pages 317–326. ACM, 2016. 1

[26] H Nakano. A Modality for Recursion. In *Proceedings of LICS'00*, pages 255–266. IEEE Computer Society, 2000. 2

[27] C.-H. L. Ong. On Model-Checking Trees Generated by Higher-Order Recursion Schemes. In *Proceedings of LICS 2006*, pages 81–90. IEEE Computer Society, 2006. 15

[28] M. Paviotti, R. E. Møgelberg, and L. Birkedal. A Model of PCF in Guarded Type Theory. *Electron. Notes Theor. Comput. Sci.*, 319(C):333–349, December 2015. 1

[29] G. E. Reyes and H. Zolfaghari. Topos-theoretic approaches to modality. In *Category Theory: Proceedings of the International Conference held in Como, Italy, July 22–28, 1990*, pages 359–378. Springer Berlin Heidelberg, 1991. 2, 12

[30] L. Santocanale and Y. Venema. Completeness for flat modal fixpoint logics. *Ann. Pure Appl. Logic*, 162(1):55–82, 2010. 15

[31] Y. Venema. Algebras and coalgebras. In P. Blackburn, J. van Benthem, and F. Wolter, editors, *Handbook of Modal Logic*, volume 3 of *Studies in Logic and Practical Reasoning*, chapter 6, pages 331–426. Elsevier, 2007. 2, 3

[32] G. Winskel and M. Nielsen. Models for Concurrency. In *Handbook of Logic in Computer Science*, volume 4, pages 1–148. Oxford University Press, 1995. 14

A Proofs of §2 (Preliminaries)

A.1 Proofs of §2.1 (The Topos of Trees)

Remark A.1 (Rem. 2.1). In $\mathscr{S}$, each object Ω^X is isomorphic to the object whose component at n is the set of sequences $(S_\ell)_{\ell \leq n}$ of sets $S_\ell \subseteq X(\ell)$ such that $x{\uparrow} \in S_\ell$ for all $x \in S_{\ell+1}$, and with restrictions taking $(S_1, \ldots, S_n, S_{n+1})$ to $(S_1, \ldots, S_n)$. $\square$

Proof. First note that $(\Omega^X)(n)$ is the set of sequences $(\chi_\ell)_{\ell \leq n}$ such that

$$\begin{tikzcd} X(\ell+1) \arrow[r] \arrow[d, "\chi_{\ell+1}"'] & X(\ell) \arrow[d, "\chi_\ell"] \\ \Omega(\ell+1) \arrow[r] & \Omega(\ell) \end{tikzcd}$$

(that is $\chi_\ell(x{\uparrow}) = \min(\ell, \chi_{\ell+1}(x))$).

For each $n > 0$, let g_n take $(\chi_1, \ldots, \chi_n)$ to $(S_1, \ldots, S_n)$ where S_ℓ is the set of all $x \in X(\ell)$ such that $\chi_\ell(x) = \ell$.

Each map g_n is surjective, since any $(S_1, \ldots, S_n)$ is the image of $(\chi_1, \ldots, \chi_n)$ where $\chi_\ell(x) = \max\{k \leq \ell \mid x_{|k} \in S_k\}$. Finally, each g_n is injective since $g_n(\chi_1, \ldots, \chi_n) = (S_1, \ldots, S_n)$ implies that $\chi_\ell(x) = \max\{k \leq \ell \mid x_{|k} \in S_k\}$, so $(\chi_1, \ldots, \chi_n)$ is completely determined by $(S_1, \ldots, S_n)$. It follows that each g_n is a bijection, and therefore that g is an iso. $\square$

B Proofs of §3 (Modal Logic of Transition Systems in the Topos of Trees)

B.1 Proofs of §3.1 (An Adjunction Between TS and $\mathscr{S}$)

Proposition B.1 (Prop. 3.3). *$(-)^\oplus$ is left adjoint to $(-)_\circledast$. The unit has component $\eta_X : X \to (X^\oplus)_\circledast$ taking $x \in X(n)$ to $(x_{|1}, \ldots, x_{|n-1}, x) \in (X^\oplus)_\circledast(n)$.*

Proof. Following [21, Thm. IV.1.2.(ii)], we show that for any $\mathscr{S}$-map $f : X \to K_\circledast$ there is a unique **TS**-map $h : X^\oplus \to K$ such that $f = h_\circledast \circ \eta_X$.

Consider $f : X \to K_\circledast$, so that $f_n(y{\uparrow}) = (s_1, \ldots, s_n)$ whenever $f_{n+1}(y) = (s_1, \ldots, s_n, s_{n+1})$. Define $h : \coprod_{n>0} X(n) \to |K|$ as $h(n, x) := s_n$ whenever $f_n(x) = (s_1, \ldots, s_n)$. We first show that h is a **TS**-map. Assume that $x = y{\uparrow} \in X(n)$ and let $f_{n+1}(y) = (s_1, \ldots, s_n, s_{n+1})$. We have $f_{n+1}(y){\uparrow} = f_n(x)$ and we get $h(n+1, y) \in \partial_K(h(n, x))$ by definition of $K_\circledast(n+1)$. We now show that $f = h_\circledast \circ \eta_X$. Given $x \in X(n)$, writting $f_n(x) = (s_1, \ldots, s_n)$, we have

$$\begin{array}{rcl} h_\circledast(n)(\eta_X(n)(x)) & = & h_\circledast(n)(x_{|1}, \ldots, x_{|n-1}, x) \\ & = & (h(x_{|1}), \ldots, h(x_{|n-1}), h(x)) \\ & = & (s_1, \ldots, s_n) \end{array}$$

It remains to show that h is unique. So let $g \in \mathbf{TS}[X^{\oplus}, K]$ such that $f = g_{\circledast} \circ \eta_X$. Given $x \in X(n)$ with $f_n(x) = (s_1, \dots, s_n)$, we have

$$g_{\circledast}(n)(\eta_X(n)(x)) \quad = \quad (g(x_{|1}), \dots, g(x_{|n-1}), g(x)) \quad = \quad (s_1, \dots, s_n)$$

so that $g(x) = h(x)$. $\square$

Remark B.2 (Rem. 3.4). The adjunction $(-)^{\oplus} \dashv (-)_{\circledast}$ is a coreflection (*i.e.* the unit η is a natural isomorphism). $\square$

Proof. Fix an object X of $\mathscr{S}$. First, each $\eta_X(n)$ is obviously injective. On the other hand, note that $(s_1, \dots, s_n) \in X^{\oplus}_{\circledast}(n)$ iff for all $\ell < n$ we have $s_{\ell+1} \in \partial_{X^{\oplus}}(s_\ell)$, that is iff $s_\ell = s{\uparrow}_{\ell+1}$. It follows that $s_\ell = (s_n)|\ell$, so that $(s_1, \dots, s_n)$ is of the form $(x_{|1}, \dots, x_{|n-1}, x)$ for some $x \in X(n)$, so that $\eta_X(n)$ is surjective. $\square$

B.2 Proofs of §3.2 (Open Maps and Coalgebra Morphisms)

Following [16], in a category with pullbacks we say that a commutative diagram

$$\begin{tikzcd} Q \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & C \end{tikzcd}$$

is a *quasi-pullback* if the canonical map $Q \to B \times_C B$ is an epi.

Still following [16] (but see also [17, §III.1] and [18, Lem. 17]), in presheaf topos $\mathrm{Psh}(\mathbb{C})$, we say that a map $f : P \to Q$ is *open* if for any $k \in \mathbb{C}[C, D]$, the following commutative diagram is a quasi-pullback:

$$\begin{tikzcd} P(D) \arrow[r, "P(k)"] \arrow[d, "f_D"'] & P(C) \arrow[d, "f_C"] \\ Q(D) \arrow[r, "Q(k)"'] & Q(C) \end{tikzcd}$$

Note that this means that for any $q \in Q(D)$, $p \in P(C)$ such that $Q(k)(q) = f_C(p)$, there is some $p' \in P(D)$ s.t. $P(k)(p') = p$ and $f_D(p') = q$.

In the case of the topos of trees, this leads to Def. 3.6, that we recall here.

Definition B.3 (Def. 3.6). *A $\mathscr{S}$-map $f : X \to Y$ is* open *if for every $n > 0$ and every $x \in X(n)$, $y' \in Y(n+1)$ s.t. $y'{\uparrow} = f_n(x)$, there is some $x' \in X(n+1)$ s.t. $f_{n+1}(x') = y'$ and $x'{\uparrow} = x$.*

Proposition B.4 (Prop. 3.7). *The adjunction $(-)^{\oplus} \dashv (-)_{\circledast}$ of Prop. 3.3 restricts to an adjunction between $\mathbf{Coalg}(\mathcal{P})$ and $\mathcal{O}(\mathscr{S})$.*

Proof. We first check that $(-)^{\oplus}$ (resp. $(-)_{\circledast}$) restricts to a functor $\mathcal{O}(\mathscr{S}) \to \mathbf{Coalg}(\mathcal{P})$ (resp. $\mathbf{Coalg}(\mathcal{P}) \to \mathcal{O}(\mathscr{S})$).

- Assume first that $f : X \to Y$ is an open map in $\mathscr{S}$. We check that $f^{\oplus}$ is a map of coalgebras $X^{\oplus} \to Y^{\oplus}$.
  Since $f^{\oplus}$ is **TS**-map, we just have to check the "back condition", namely that for every $(n, x) \in |X^{\oplus}|$ and every $(n+1, y) \in \partial_{Y^{\oplus}}(f^{\oplus}(n, x))$, there is some $(n+1, x') \in \partial_{X^{\oplus}}(n, x)$ s.t. $f^{\oplus}(n+1, x') = (n+1, y)$. But note that we have $x \in X(n)$ and $y \in Y(n+1)$ with $y{\uparrow} = f_n(x)$, so that since f is open, there is some $x' \in X(n+1)$ s.t. $x'{\uparrow} = x$ and $f_{n+1}(x') = y$, that is $(n+1, x') \in \partial_{X^{\oplus}}(n, x)$ and $f^{\oplus}(n+1, x') = (n+1, y)$.
- Conversely, assume that h is a $\mathbf{Coalg}(\mathcal{P})$ map $K \to L$. We check that $h_{\circledast} : K_{\circledast} \to L_{\circledast}$ is open. So consider $n > 0$, $(s_1, \ldots, s_n) \in K_{\circledast}(n)$ and $(t_1, \ldots, t_n, t_{n+1}) \in L_{\circledast}(n+1)$ such that

  $$h_{\circledast}(n)(s_1, \ldots, s_n) \quad = \quad (t_1, \ldots, t_n, t_{n+1}){\uparrow}$$

  Note that this implies $(t_1, \ldots, t_n) = (s_1, \ldots, s_n)$ and $t_{n+1} \in \partial_L(t_n)$. Now, since h is a coalgebra map, there is some $s_{n+1} \in \partial_K(s_n)$ such that $h(s_{n+1}) = t_{n+1}$, and we are done since this implies

  $$h_{\circledast}(n{+}1)(s_1, \ldots, s_{n+1})) = (t_1, \ldots, t_{n+1}) \qquad \text{and} \qquad (s_1, \ldots, s_{n+1}){\uparrow} = (s_1, \ldots, s_n)$$

Moreover, it is easy to see that the unit $\eta_X : X \to X^{\oplus}_{\circledast}$ is open. Indeed, given $x \in X(n)$ and $(s_1, \ldots, s_n, s_{n+1}) \in X^{\oplus}_{\circledast}(n+1)$, if $\eta_X(n)(x) = (s_1, \ldots, s_n)$, the we must have

$$(s_1, \ldots, s_n) = (x_{|1}, \ldots, x) \qquad \text{and} \qquad s_{n+1}{\uparrow} = x$$

It follows that $\eta_X(n+1)(s_{n+1}) = (s_1, \ldots, s_{n+1})$ and we are done.

Finally, we have to check that the universal property shown in Prop. 3.3 (Prop. B.1) restricts to $\mathcal{O}(\mathscr{S})$ and $\mathbf{Coalg}(\mathcal{P})$, namely that $h : X^{\oplus} \to K$ is a coalgebra map whenever $f : X \to K_{\circledast}$ is open. This amounts to show that given $(n, x) \in |X^{\oplus}|$ and $s_{n+1} \in \partial_K(s_n)$ (for $(s_1, \ldots, s_n) = f_n(x)$), there is some $(n+1, x') \in |X^{\oplus}|$ such that $x{\uparrow}' = x$ and $f_{n+1}(x') = (s_1, \ldots, s_n, s_{n+1})$, and this directly follows from f being an open map. □

B.3 Proofs of §3.3 (Representation of Modal Satisfaction in the Topos of Trees)

Remark B.5 (Rem. 3.13). As sanity check, note that $\Omega^{\Lambda} \simeq (\mathbf{2}^{\Lambda}, \partial_{\mathbf{2}^{\Lambda}})_{\circledast}$ in $\mathscr{S}$, so that the object Λ of $\mathscr{S}$ indeed represents the transition system $(\mathbf{2}^{\Lambda}, \partial_{\mathbf{2}^{\Lambda}})$. Moreover, the image under the adjunction $(-)^{\oplus} \dashv (-)_{\circledast}$ of the map $X \to (\mathbf{2}^{\Lambda}, \partial_{\mathbf{2}^{\Lambda}})_{\circledast}$ induced by κ^v_{Λ} is the **TS**-map $X^{\oplus} \to (\mathbf{2}^{\Lambda}, \partial_{\mathbf{2}^{\Lambda}})$ induced by the the tranpose of $\Vdash^v \subseteq |X^{\oplus}| \times \Lambda$. □

Proof. Recall the iso g from proof of Rem. A.1 (Rem. 2.1).

First note that we actually have $g : \Omega^{\Lambda} \to (\mathbf{2}^{\Lambda}, \partial_{\mathbf{2}^{\Lambda}})_{\circledast}$, since given $(S_1, \ldots, S_n)$ in the image of g_n, we have $\varphi \in S_{\ell+1}$ whenever $\Diamond\varphi \in S_{\ell}$, so that $S_{\ell+1} \in \partial_{\mathbf{2}^{\Lambda}}(S_{\ell})$. So the first part follows from Rem. A.1 (Rem. 2.1).

As for the second part of the statment, the image under $(-)^{\oplus} \dashv (-)_{\circledast}$ of a $\mathscr{S}$-map $f : X \to K_{\circledast}$ is the composite $X^{\oplus} \xrightarrow{f^{\oplus}} K^{\oplus}_{\circledast} \xrightarrow{\epsilon_K} K$, where ϵ_K, the co-unit of $(-)^{\oplus} \dashv (-)_{\circledast}$ takes $(k, (s_1, \dots, s_n)) \in |K^{\oplus}_{\circledast}|$ to $s_n \in |K|$. Consider now the composite

$$X^{\oplus} \quad \xrightarrow{\kappa^v} \quad (\Omega^{\wedge})^{\oplus} \quad \xrightarrow{g^{\oplus}} \quad (\mathbf{2}^{\Lambda})^{\oplus}_{\circledast} \quad \xrightarrow{\epsilon} \quad \mathbf{2}^{\Lambda}$$

It takes $(k, x) \in |X^{\oplus}|$ to the set of all formulae φ such that $x \Vdash^v_k \varphi$, that is such that $(x, k) \Vdash^v \varphi$. $\square$

C Proofs of §4 (Elements of Algebraic Perspectives)

C.1 Proofs of §4.1 (Boolean Algebras with Operators)

Remark C.1 (Rem. 4.2). Following the usual terminology, a (Kripke) *model* is a transition system K together with a valuation of atomic propositions $v : AP \to \mathbf{2}^{|K|}$. We say that a model (K, v) is *modaly saturated* (see e.g.[5, §2.5]) whenever for all state $s \in |K|$ and all (possibly infinite) set of formulae Φ, if $s \Vdash^v \Diamond \bigwedge \Psi$ for all *finite* $\Psi \subseteq \Phi$, then there is $t \in \partial(s)$ s.t. $t \Vdash^v \varphi$ for all $\varphi \in \Phi$. It is well-known that modaly saturated models satisfy the Hennessy-Milner property, and moreover that ultrafilter frames as well as image finite models are modaly saturated.

We note the following property. Given a t.s. K and a valuation $v : AP \to \mathbf{2}^{|K|}$, the **TS**-map $K \to \mathfrak{Uf}(LT)$ is a map of coalgebras if and only if the model (K, v) is modaly saturated. $\square$

The proof of Rem. C.1 (Rem. 4.2) relies on the following well-known fact.

Lemma C.2. *Given a* **BAO** (B, g)*, letting* $f(a) := \neg g(\neg a)$*, and* $F \in \mathfrak{Uf}(B)$*, if* $G_0 \subseteq B$ *is such that the set*

$$H \quad := \quad G_0 \cup \{a \mid f(a) \in F\}$$

has the finite meet property, then G_0 *is included in an ultrafilter* $G \in \partial_{\mathfrak{Uf}(B)}(F)$*.*

Proof (of Lem. C.2). First, we note that

– $G \in \partial_{\mathfrak{Uf}(B)}(B)$ iff $a \in G$ whenever $f(a) \in F$.

Proof. If $G \in \partial_{\mathfrak{Uf}(B)}(F)$ and $f(a) \in F$, we have $g(\neg a) \notin F$ so that $\neg a \notin G$ and $a \in G$. Conversely, given $a \in G$, we have $\neg a \notin G$ so that $f(\neg a) = \neg g(a) \notin F$ and $g(a) \in F$. $\square$

Returning to our property, it follows from Zorn's Lemma that H is contained in an ultrafilter G, and we have $G \in \partial_{\mathfrak{Uf}(B)}(F)$ because $a \in G$ whenever $f(a) \in F$. $\square$

Proof (of Rem. C.1). Write $\theta : K \to \mathfrak{Uf}(LT)$ for the **TS**-map induced by $\Vdash^v$. Note that since θ is a map of **TS**, we only have to consider the back property of bisimulation.

Assume first that (K, v) is modaly saturated. Let $s \in |K|$ and $F := \theta(s)$, and consider $G \in \partial_{\mathfrak{Uf}(LT)}(F)$. We have to show that for some $t \in \partial(s)$, G is exactly the set of all φ s.t. $t \Vdash^v \varphi$. But for every finite $\Psi \subseteq G$, we have $\bigwedge \Phi \in G$ since G is an ultrafilter, so that $s \Vdash^v \Diamond \bigwedge \Psi$ by definition of $\partial_{\mathfrak{Uf}(LT)}$, and by modal saturation we get some $t \in \partial(s)$ s.t. $t \Vdash^v \varphi$ for all $\varphi \in G$. Since G is an ultrafilter, we have $\varphi \notin G$ iff $\neg\varphi \in G$, so that $t \Vdash^v \varphi$ iff $\varphi \in G$.

Conversely, assume that θ is a map of coalgebras. Consider a state $s \in |K|$ and a set of formulae $\Phi \subseteq \Lambda$ such that $s \Vdash^v \Diamond \bigwedge \Psi$ for all finite $\Psi \subseteq \Phi$. We are going to show that Φ is contained in an ultrafilter $G \in \partial_{\mathfrak{Uf}(LT)}(F)$ for $F := \theta(s)$. As soon as this holds we are done since θ being a map of coalgebras implies $G = \theta(t)$ for some $t \in \partial(s)$. We invoke Lem. C.2, so that we have to show that the following set H has the finite meet property

$$H \quad := \quad \Phi \cup \{\psi \mid \Box\psi \in F\}$$

But note that the finite meet property holds for Φ (since for any finite $\Psi \subseteq \Phi$ we have $s \Vdash^v \Diamond \bigwedge \Psi$) as well as for the set of all ψ s.t. $\Box\psi \in F$. So we are left with showing that $\varphi \wedge \psi \not\equiv \bot$ for $\varphi \in \Phi$ and $\Box\psi \in F$. But this follows from the fact that $\varphi \in \Phi$ implies $t \Vdash^v \varphi$ for some $t \in \partial(s)$, which must also force ψ since $s \Vdash^v \Box\psi$. ☐

Remark C.3 (Rem. 4.3). Consider a Boolean algebra B, a set S and a function

$$f \quad : \quad S \times B \quad \longrightarrow \quad \mathbf{2}$$

together with its transposes $f^{\mathrm{t}} : S \to \mathbf{2}^B$ and ${}^{\mathrm{t}}f : B \to \mathbf{2}^S$. Then ${}^{\mathrm{t}}f$ is a map of Boolean algebras iff for all $s \in S$, the set $f^{\mathrm{t}}(s) \subseteq B$ is an ultrafilter. ☐

Proof. Assume first that ${}^{\mathrm{t}}f : B \to \mathbf{2}^S$ is a map of Boolan algebra, and consider $s \in S$. First we have $\bot \notin f^{\mathrm{t}}(s)$, since otherwise we would have $s \in {}^{\mathrm{t}}f(\bot)$, contradicting ${}^{\mathrm{t}}f(\bot) = \emptyset$. Moreover, we have $\top \in f^{\mathrm{t}}(s)$ since ${}^{\mathrm{t}}f(\top) = S$. Consider now $a, b \in B$. If $a \in f^{\mathrm{t}}(s)$ and $a \leq b$, then since ${}^{\mathrm{t}}f(a) \subseteq {}^{\mathrm{t}}f(b)$, we have $s \in {}^{\mathrm{t}}f(b)$, so that $b \in f^{\mathrm{t}}(s)$. Also, if $a, b \in f^{\mathrm{t}}(s)$, then we have $s \in {}^{\mathrm{t}}f(a) \cap {}^{\mathrm{t}}f(b)$, so that $s \in {}^{\mathrm{t}}f(a \wedge b)$ and $a \wedge b \in f^{\mathrm{t}}(s)$. Finally, if $a \notin f^{\mathrm{t}}(s)$, we have $s \in {}^{\mathrm{t}}f(\neg a)$ so that $\neg a \in f^{\mathrm{t}}(s)$.

Conversely, assume that each $f^{\mathrm{t}}(s)$ is an ultrafilter. We have ${}^{\mathrm{t}}f(\bot) = \emptyset$ since $\bot \notin f^{\mathrm{t}}(s)$ for all $s \in S$. Moreover, we have ${}^{\mathrm{t}}f(\top) = S$ since $\top \in f^{\mathrm{t}}(s)$ for all $s \in S$. Consider now $a, b \in B$. We have ${}^{\mathrm{t}}f(a \wedge b) = {}^{\mathrm{t}}f(a) \cap {}^{\mathrm{t}}f(b)$ since ${}^{\mathrm{t}}f(a \wedge b) \subseteq {}^{\mathrm{t}}f(a) \cap {}^{\mathrm{t}}f(b)$ and since $s \in {}^{\mathrm{t}}f(a) \cap {}^{\mathrm{t}}f(b)$ implies $a, b \in f^{\mathrm{t}}(s)$, so that $a \wedge b \in f^{\mathrm{t}}(s)$ and $s \in {}^{\mathrm{t}}f(a \wedge b)$. Finally, ${}^{\mathrm{t}}f(\neg a) = S \setminus {}^{\mathrm{t}}f(a)$ since $a \notin f^{\mathrm{t}}(s)$ implies $\neg a \in f^{\mathrm{t}}(s)$ by assumption on $f^{\mathrm{t}}(s)$, and $a \in f^{\mathrm{t}}(s)$ implies $\neg a \notin f^{\mathrm{t}}(s)$, since $a, \neg a \in f^{\mathrm{t}}(s)$ would lead to $\bot \in f^{\mathrm{t}}(s)$. ☐

C.2 Proofs of §4.2 (Mapping $\mathscr{S}$ to a Boolean Topos)

The proof of Lem. 4.10 is defered to §E.

Lemma C.4 (Lem. 4.11). *The object* $\mathsf{LT}^* = \iota^*(\mathsf{LT})$ *is a Boolean algebra in* $\mathrm{Psh}(|\mathbb{N}_*|)$.

Proof. This follows from the fact that each component $\mathsf{LT}^*(n) = \mathsf{LT}$ is a Boolean algebra. $\square$

D Proofs of §5 (Characterization of the Modal Satisfaction Relation)

Proposition D.1 (Prop. 5.2). *Given* X *and* v*, the transpose* ${}^{\mathrm{t}}\Delta(\kappa^v) : \mathsf{LT}^* \to \mathbf{2}^{X^*}$ *of* $\Delta(\kappa^v) : X^* \times \mathsf{LT}^* \to \mathbf{2}$ *is a map of Boolean algebras in* $\mathrm{Psh}(|\mathbb{N}_*|)$*, which moreover extends* v*, in the sense that*

$$
\begin{tikzcd}
\mathsf{LT}^* \arrow[r, "{}^{\mathrm{t}}\Delta(\kappa^v)"] & \mathbf{2}^{X^*} \\
\mathsf{AP} \arrow[u] \arrow[ur, "v"'] &
\end{tikzcd}
$$

Proof. We have to show that for each $n > 0$, ${}^{\mathrm{t}}\Delta(\kappa^v)_n$ takes $[p]^n_{\mathbf{K}}$ to $v_n(p)$ for $p \in AP$ and satisfy the equations:

$$
\begin{array}{rcl}
{}^{\mathrm{t}}\Delta(\kappa^v)_n(\bot) & = & \emptyset \\
{}^{\mathrm{t}}\Delta(\kappa^v)_n(\boldsymbol{\varphi} \vee \boldsymbol{\psi}) & = & {}^{\mathrm{t}}\Delta(\kappa^v)_n(\boldsymbol{\varphi}) \cup {}^{\mathrm{t}}\Delta(\kappa^v)_n(\boldsymbol{\psi}) \\
{}^{\mathrm{t}}\Delta(\kappa^v)_n(\neg\boldsymbol{\varphi}) & = & \prod_{\ell \leq n} X(\ell) \setminus {}^{\mathrm{t}}\Delta(\kappa^v)_n(\boldsymbol{\varphi})
\end{array}
$$

where $\boldsymbol{\varphi}$ stands for sequence $([\varphi_1]_{\mathbf{K}}, \ldots, [\varphi_n]_{\mathbf{K}})$ (and similarly for $\boldsymbol{\psi}$), $\cup$ is pointwise union and $\setminus$ pointwise set difference. Note that the equations for $\top$ and $\wedge$ follow, since they are defined connectives in Λ.

Then result follows from the fact that by definition of Δ (a.k.a. ι^*), we have $(x_1, \ldots, x_n) \in {}^{\mathrm{t}}\Delta(\kappa^v)_n(\boldsymbol{\varphi})$ iff $(\ell, x_\ell) \Vdash^v \varphi_\ell$ for all $\ell \leq n$. $\square$

Lemma D.2 (Lem. 5.4). *Consider an object* X *of* $\mathscr{S}$*, some* $n > 0$ *and subsets* $S \subseteq X(n)$*,* $S' \subseteq X(n{+}1)$ *which are compatible with restriction (*$x{\uparrow} \in S$ *whenever* $x \in S'$*). Then* $\Diamond_{X\oplus} S' \subseteq S$*. Moreover, we have* $\Diamond_{X\oplus} S' = S$ *iff the restriction of* $(-){\uparrow} : X(n+1) \to X(n)$ *to* $S' \to S$ *is surjective.*

Proof. If $x \in \Diamond_{X\oplus} S$, then for some $y \in \partial_{X\oplus}(x)$ we have $y \in S(n{+}1)$. But $y \in S$ implies $y{\uparrow} \in S(n)$, and $y{\uparrow} = x$ since $y \in \partial_{X\oplus}(x)$.

As for the second point, if $(-){\uparrow} : S' \to S$ is surjective, then for all $x \in S$ there is some $y \in S'$ with $y{\uparrow} = x$, so that $y \in \partial_{X\oplus}(x)$ and $x \in \Diamond_{X\oplus}(S')$. Conversely, if $S \subseteq \Diamond_{X\oplus}(S')$ then for all $x \in S$ there is some $y \in S'$ such that $y{\uparrow} = x$. $\square$

Proposition D.3 (Prop. 5.5). *The map* ${}^{\mathrm{t}}\kappa^v : \mathsf{LT} \to \Omega^X$ *factors as* $\mathsf{LT} \to \mathrm{TotSub}_X \hookrightarrow \Omega^X$.

Proof. Recall the iso g from proof of Rem. A.1 (Rem. 2.1).

Given $n > 0$ and a formula φ, let $(S_1, \ldots, S_n) := g_n(({}^{\mathrm{t}}\kappa^v)_n([\varphi]_{\mathbf{K}}))$ and note that for all $\ell \leq n$ we have $(S_1, \ldots, S_\ell) = g_\ell(({}^{\mathrm{t}}\kappa^v)_\ell([\Diamond^{n-\ell}\varphi]_{\mathbf{K}}))$.

Then for all $\ell < n$ and all $x \in S(\ell)$, we have $x \Vdash^v_\ell \Diamond^{n-\ell}\varphi$, that is $(\ell, x) \Vdash^v \Diamond^{n-\ell}\varphi$. Since $n - \ell > 0$, there is some $y \in X(\ell+1)$ such that $y{\uparrow} = x$ and $(\ell+1, y) \Vdash^v \Diamond^{n-\ell-1}\varphi$. But this implies $y \in S_{\ell+1}$ and we are done. $\square$

Theorem D.4 (Thm. 5.6). *Given $v : \mathsf{AP} \to \mathbf{2}^{X^*}$, the map $\kappa^v : X \times \mathsf{LT} \to \Omega$ is the unique map which satisfies all the following conditions:*

(i) ${}^{\mathrm{t}}\Delta(\kappa^v) : \mathsf{LT}^* \to \mathbf{2}^{X^*}$ *is a map of Boolean algebras which extends v.*
(ii) ${}^{\mathrm{t}}\kappa^v : \mathsf{LT} \to \Omega^X$ *factors as* $\mathsf{LT} \to \mathrm{TotSub}_X \hookrightarrow \Omega^X$.

Proof. It follows from Prop. 5.2 and Prop. 5.5 that κ^v satisfy (i) and (ii). As for uniqueness, consider two maps $f, g : X \times \mathsf{LT} \to \Omega$ which both satisfy (i) and (ii). We show by induction on formulae $\varphi \in \Lambda$ that for all $n > 0$ and all $x \in X(n)$, we have $f_n(\varphi, x) = g_n(\varphi, x)$.

In the base case φ is an atomic proposition p. Then by (i) for all $n > 0$ we have ${}^{\mathrm{t}}\Delta(f)_n([p]_{\mathbf{K}}) = v_n(p) = {}^{\mathrm{t}}\Delta(g)_n([p]_{\mathbf{K}})$, and we conclude by injectivity of Δ (Lem. 4.10). The Boolean connectives are dealt with similarly, using the I.H. and the fact that we must have e.g. ${}^{\mathrm{t}}\Delta(f)_n([\varphi \vee \psi]_{\mathbf{K}}) = {}^{\mathrm{t}}\Delta(f)_n([\varphi]_{\mathbf{K}}) \cup {}^{\mathrm{t}}\Delta(f)_n([\psi]_{\mathbf{K}})$.

It remains to deal with $\Diamond\varphi$. Fix $n > 0$. By (ii), let $(S_1, \ldots, S_{n+1})$ and $(T_1, \ldots, T_{n+1})$ be the respective images of ${}^{\mathrm{t}}f_{n+1}([\varphi]_{\mathbf{K}})$ and ${}^{\mathrm{t}}g_{n+1}([\varphi]_{\mathbf{K}})$. By I.H. we have $S_{n+1} = T_{n+1}$. But this implies $(S_1, \ldots, S_n) = (T_1, \ldots, T_n)$ since for all for all $\ell \leq n$ the restrictions of $(-){\uparrow} : X(\ell+1) \to X(\ell)$ to $S(\ell+1) \to S(\ell)$ and $T(\ell+1) \to T(\ell)$ are both surjective. It thus follows that ${}^{\mathrm{t}}f_n([\Diamond\varphi]_{\mathbf{K}}) = {}^{\mathrm{t}}g_n([\Diamond\varphi]_{\mathbf{K}})$. $\square$

E Proofs of §6.1 (A Geometric Morphism)

We spell out Lem. 6.1 in details.

Lemma E.1. *Consider a geometric morphism $f = f^* \dashv f_* : \mathcal{F} \to \mathcal{E}$.*

(a) The inverse image functor f^* induces for each object A of $\mathcal{E}$, an homomorphism of subobjects lattices

$$\Delta_A \quad : \quad \mathrm{Sub}_{\mathcal{E}}(A) \quad \longrightarrow \quad \mathrm{Sub}_{\mathcal{F}}(f^*A)$$

Moreover, the action of Δ_A on classifying maps takes $\chi_U : A \to \Omega_{\mathcal{E}}$ to

$$f^*A \quad \xrightarrow{f^*(\chi_U)} \quad f^*(\Omega_{\mathcal{E}}) \quad \xrightarrow{\rho} \quad \Omega_{\mathcal{F}}$$

where ρ is the classifying map of $f^*(\mathrm{t}) \rightarrowtail f^*(\Omega_{\mathcal{E}})$.
(b) For each object A of $\mathcal{E}$, Δ_A has an (external) right adjoint (of posets)

$$\Gamma_A \quad : \quad \mathrm{Sub}_{\mathcal{F}}(f^*A) \quad \longrightarrow \quad \mathrm{Sub}_{\mathcal{E}}(A)$$

which takes $v : V \rightarrowtail f^*(A)$ to $\Gamma_A(v) : \Gamma_A(V) \rightarrowtail A$ given by the pullback

$$\begin{tikzcd} \Gamma_A(V) \arrow[r] \arrow[d, tail, "\Gamma_A(v)"'] \arrow[dr, phantom, "\lrcorner", very near start] & f_*V \arrow[d, tail, "f_*(v)"] \\ A \arrow[r, "\eta_A"'] & f_*f^*A \end{tikzcd}$$

Moreover, the action of Γ_A on classifying maps takes $\chi_V : f^*A \to \Omega_{\mathcal{E}}$ to

$$A \quad \xrightarrow{\eta_A} \quad f_*f^*A \quad \xrightarrow{f_*(\chi_V)} \quad f_*(\Omega_{\mathcal{F}}) \quad \xrightarrow{\tau} \quad \Omega_{\mathcal{E}}$$

where τ is the classifying map of $f_*(\mathrm{t}) \rightarrowtail f_*(\Omega_{\mathcal{F}})$.

(c) Writing $\lambda : \Omega_{\mathcal{E}} \to f_*\Omega_{\mathcal{F}}$ for the transpose of $\rho : f^*\Omega_{\mathcal{F}} \to \Omega_{\mathcal{E}}$ along $f^* \dashv f_*$, we have an internal adjunction of posets $\lambda \dashv \tau : \Omega_{\mathcal{E}} \to f_*\Omega_{\mathcal{F}}$.

Proof. (a) Recall from [22, Prop. IV.6.4] that in subobjects lattices, meets are computed by pullbacks and joins by coproducts and images. So the result follows form the fact that f^* preserves colimits and finite limits.
As for the second point, consider for a subobject $u : U \rightarrowtail A$, the diagram

$$\begin{tikzcd} f^*(U) \arrow[r] \arrow[d, tail, "f^*(u)"'] \arrow[dr, phantom, "\lrcorner", very near start] & f^*(\mathbf{1}) \arrow[r] \arrow[d, tail, "f^*(\mathrm{t})"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{1} \arrow[d, tail, "\mathrm{t}"] \\ f^*(A) \arrow[r, "f^*(\chi_U)"'] & f^*(\Omega_{\mathcal{E}}) \arrow[r, "\rho"'] & \Omega_{\mathcal{F}} \end{tikzcd}$$

Since the outer rectangle is a pullback, it follows that $\rho \circ f^*(\chi_U)$ classifies $f^*(U)$, and therefore that $\chi_{f^*(U)} = \rho \circ f^*(\chi_U)$.

(b) We have to show that given subobjects $u : U \rightarrowtail A$ and $v : V \rightarrowtail f^*A$, we have

$$\Delta_A U \leq V \qquad \text{iff} \qquad U \leq \Gamma_A V$$

– Assume first that $\Delta_A U \leq V$, say with $h : f^*U \to V$ such that $f^*u = v \circ h$. Consider the composite map

$$U \quad \xrightarrow{\eta_U} \quad f_*f^*U \quad \xrightarrow{f_*h} \quad f_*f^*V \quad \xrightarrow{f_*v} \quad f_*f^*A$$

Since $f^*u = v \circ h$ and using the naturality of η we have:

$$\begin{array}{rcl} f_*v \circ f_*h \circ \eta_U & = & f_*(v \circ h) \circ \eta_U \\ & = & f_*f^*u \circ \eta_U \\ & = & \eta_A \circ u \end{array}$$

Then the pullback defining Γ_A gives a (unique) map $U \to \Gamma_A(V)$ such that

$$U \xrightarrow{u} X \quad = \quad U \longrightarrow \Gamma_A(V) \xrightarrow{\Gamma_A(v)} X$$

and we are done.

– Assume now that $U \leq \Gamma_A V$, say with $k : U \to \Gamma_A V$ such that $u = \Gamma_A(v) \circ k$. We will show that $\Delta_A U \leq V$ with

$$f^*U \xrightarrow{f^*k} f^*\Gamma_A V \longrightarrow f^*f_*V \xrightarrow{\epsilon_V} V$$

that is

$$f^*U \xrightarrow{f^*u} f^*A \quad = \quad f^*U \xrightarrow{f^*k} f^*\Gamma_A V \longrightarrow f^*f_*V \xrightarrow{\epsilon_V} V \xrightarrow{v} f^*A$$

First, by naturality of ϵ, we have

$$f^*f_*V \xrightarrow{\epsilon_V} V \xrightarrow{v} f^*A \quad = \quad f^*f_*V \xrightarrow{f^*f_*v} f^*f_*f^*A \xrightarrow{\epsilon_{f^*A}} f^*A$$

Thanks to the diagram defining Γ_A, we then have

$$f^*\Gamma_A V \longrightarrow f^*f_*V \xrightarrow{f^*f_*v} f^*f_*f^*A \quad = \quad f^*\Gamma_A V \xrightarrow{f^*\Gamma_A v} f^*A \xrightarrow{f^*\eta_A} f^*f_*f^*A$$

The triangular equality

$$f^*A \xrightarrow{f^*\eta_A} f^*f_*f^*A \xrightarrow{\epsilon_{f^*A}} f^*A \quad = \quad f^*A \xrightarrow{\mathrm{id}_{f^*A}} f^*A$$

implies

$$f^*U \xrightarrow{f^*k} f^*\Gamma_A V \longrightarrow f^*f_*V \xrightarrow{\epsilon_V} V \xrightarrow{v} f^*A \quad = \quad f^*U \xrightarrow{f^*k} f^*\Gamma_A V \xrightarrow{f^*\Gamma_A v} f^*A$$

and we are done since $u = \Gamma_A(v) \circ k$.

The second point on τ follows from the pasting of pullback diagrams

$$\begin{tikzcd}
\Gamma_A(V) \arrow[r] \arrow[d, tail, "\Gamma_A(v)"'] \arrow[dr, phantom, "\lrcorner", very near start] & f_*(V) \arrow[r] \arrow[d, tail, "f_*(v)"'] \arrow[dr, phantom, "\lrcorner", very near start] & f_*(\mathbf{1}) \arrow[r] \arrow[d, tail, "f_*(\mathrm{t})"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{1} \arrow[d, tail, "\mathrm{t}"] \\
A \arrow[r, "\eta_A"'] & f_*f^*A \arrow[r, "f_*(\chi_V)"'] & f_*(\Omega_{\mathcal{F}}) \arrow[r, "\tau"'] & \Omega_{\mathcal{E}}
\end{tikzcd}$$

(c) First note that

$$\Omega_{\mathcal{E}} \xrightarrow{\lambda} f_*\Omega_{\mathcal{F}} \quad = \quad \Omega_{\mathcal{E}} \xrightarrow{\eta_{\Omega_{\mathcal{E}}}} f_*f^*\Omega_{\mathcal{E}} \xrightarrow{f_*\rho} f_*\Omega_{\mathcal{F}}$$

We will show $\mathrm{id}_{\Omega_{\mathcal{E}}} \leq \tau \circ \lambda$ and $\lambda \circ \tau \leq \mathrm{id}_{f_*\Omega_{\mathcal{F}}}$. We refer to [22, §IV.9, pp. 205-206] for the precise notion of internal adjunction between partial orders (see also [22, Prop. IX.6.4]).

First, given $U \in \mathrm{Sub}_{\mathcal{E}}(A)$, using the naturality of η, a simple digram chasing shows that $\Gamma_A \Delta_A U$ is classified by

$$A \xrightarrow{\chi_U} \Omega_{\mathcal{E}} \xrightarrow{\lambda} f_*\Omega_{\mathcal{F}} \xrightarrow{\tau} \Omega_{\mathcal{E}}$$

But since $U \leq \Gamma_A \Delta_A U$, this implies $\mathrm{id}_{\Omega_{\mathcal{E}}} \leq \tau \circ \lambda$.

Conversely, assume given $V \in \mathrm{Sub}_{\mathcal{F}}(f^*A)$. Note that $\Delta_A \Gamma_A V$ is classified by

$$f^*A \xrightarrow{f^*\eta_A} f^*f_*f^*A \xrightarrow{f^*f_*\chi_V} f^*f_*\Omega_{\mathcal{F}} \xrightarrow{f^*\tau} f^*\Omega_{\mathcal{E}} \xrightarrow{\rho} \Omega_{\mathcal{F}}$$

We now apply f_* on the above composite, use the equality $f_*f^*\eta_A = \eta_{f_*f^*A}$ and apply the naturality of η twice to obtain

$$f_*\chi_{\Delta_A \Gamma_A V} \quad = \quad f_*f^*A \xrightarrow{f_*\chi_V} f_*\Omega_{\mathcal{F}} \xrightarrow{\tau} \Omega_{\mathcal{E}} \xrightarrow{\lambda} f_*\Omega_{\mathcal{F}}$$

But $\Delta_A \Gamma_A V \le V$ implies $f_*\Delta_A \Gamma_A V \le f_* V$ (since f_* preserves limits, hence monos), so that we must have $\lambda \circ \tau \le \mathrm{id}_{f_*\Omega_{\mathcal{F}}}$. $\square$

The following is a detailed version of Prop. 6.2.

Proposition E.2. *In the case of $\iota^* \dashv \iota_* : \mathrm{Psh}(|\mathbb{N}_*|) \to \mathscr{S}$:*

(a) The unit of the adjunction $\iota^* \dashv \iota_*$ has components $\eta_X \in \mathscr{S}[X, \iota_*\iota^*X]$ with

$$\eta_X(n) \quad : \quad s \in X(n) \quad \longmapsto \quad (s_{|1}, \dots, s_{|n-1}, s) \in \prod_{k \le n} X(k)$$

(b) Given an object X of $\mathscr{S}$, the (external) map $\Gamma_X : \mathrm{Sub}_{\mathrm{Psh}(|\mathbb{N}_*|)}(\iota^*X) \to \mathrm{Sub}_{\mathscr{S}}(X)$ takes a subobject $A \hookrightarrow \iota^*X$ in $\mathrm{Psh}(|\mathbb{N}_*|)$ to $\Gamma_X(A) \hookrightarrow X$ where

$$\Gamma_X(A)(n) \quad = \quad \{s \in X(n) \mid \forall k \le n.\ s_{|k} \in A(k)\}$$

(c) The composite $\Gamma_X \circ \Delta_X$ is the identity on $\mathrm{Sub}_{\mathscr{S}}(X)$.
(d) The map $\tau : \mathbf{2}_* \to \Omega$ at $n > 0$ takes $(b_1, \dots, b_n) \in \mathbf{2}^n$ to $\max\{\ell \le n \mid b_1 = \dots = b_\ell = 1\}$.
(e) The map $\rho : \iota^*(\Omega) \to \mathbf{2}$ at $n > 0$ takes $n \le n$ to 1 and $k < n$ to 0. Since $\lambda = \iota_*(\rho) \circ \eta_\Omega$, the map $\lambda : \Omega \to \mathbf{2}_*$ at $n > 0$ takes $k \le n$ to the sequence $(1^k, 0^{n-k}) \in \mathbf{2}_*(n)$.

Proof.

(a) This follows from the general form of the unit for a geometric morphisms $f^* \dashv f_* : \mathbf{Sets}^{\mathbb{C}} \to \mathbf{Sets}^{\mathbb{D}}$ induced by a functor $f : \mathbb{C} \to \mathbb{D}$, where the functor $f^* : \mathbf{Sets}^{\mathbb{D}} \to \mathbf{Sets}^{\mathbb{C}}$ is given by precomposition with f, and its right-adjoint $f_* : \mathbf{Sets}^{\mathbb{C}} \to \mathbf{Sets}^{\mathbb{D}}$ is given by right Kan extensions along f. (see e.g. [15, Ex. A.4.1.4]).
Explicitelly, given $Q : \mathbb{C} \to \mathbf{Sets}$ and $D \in \mathrm{Obj}(\mathbb{C})$, $f_*(Q)(D)$ is the set of all $s \in \prod_{g : D \to f(C)} Q(C)$ s.t. for $k \in \mathbb{C}[C, C']$, if $f(k) \circ g = g'$ then $s_{g'} = Q(k)(s_g)$.
Then, the unit η_P at $P : \mathbb{D} \to \mathbf{Sets}$ has component at $D \in \mathrm{Obj}(\mathbb{D})$ the function $\eta_P(D) : P(D) \to f_*f^*P(D)$ which takes $x \in P(D)$ to $(Pg(x))_{g : D \to f(C)}$ (recall that $f^*P(C) = Pf(C)$ for $C \in \mathrm{Obj}(\mathbb{C})$, and that $f^*P(k) = Pf(k)$ for $k \in \mathbb{C}[C, C']$).
In the case of an object X of $\mathscr{S}$, $\eta_X : X \to X_*$ thus takes $x \in X(n)$ to the family $(x_{|\ell})_{\ell \le n}$ and we are done.

(b) Since limits in functor cateogries are comupted pointwise (see e.g. [22, pp. 22-23]), we should have

$$\mathit{\Gamma}_X(A)(n) \quad \simeq \quad \{(x,(s_1,\dots,s_n)) \in X(n) \times \prod_{\ell \leq n} A(\ell) \mid s_\ell = x_{|\ell}\}$$

This implies that for $(x,(s_1,\dots,s_n)) \in \mathit{\Gamma}_X(A)(n)$, $(s_1,\dots,s_n)$ is completelly determined by x, so that we can indeed take

$$\mathit{\Gamma}_X(A)(n) \quad = \quad \{x \in X(n) \mid x_\ell \in A(\ell) \text{ for all } \ell \leq n\}$$

(c) Let $A \hookrightarrow X$. Lemma E.1.(b) implies that $\mathit{\Gamma}_X \mathit{\Delta}_X(A) \subseteq A$. Conversely, assume that $x \in A(n)$. Then since $A \in \mathrm{Sub}_{\mathscr{S}}(X)$, we have $x_{|\ell} \in A(\ell)$ for all $\ell \leq n$, so that $x \in \mathit{\Gamma}_X(A)(n)$.
(d) By definition, τ_n takes $(b_1,\dots,b_n) \in \mathbf{2}^*$ to

$$\max\{\ell \leq n \mid (b_1,\dots,b_n)_\ell = 1^\ell\}$$

that is to

$$\max\{\ell \leq n \mid b_1 = \dots = b_\ell = 1\}$$

(e) The part on ρ follows directly from the definition of $\mathrm{t} : \mathbf{1} \to \mathit{\Omega}$ in $\mathscr{S}$ as $\mathrm{t}_n(1) = n$. As for the second part, given $k \in \mathit{\Omega}(n)$, we have

$$\eta_{\mathit{\Omega}}(n)(k) \quad = \quad (\min(k,1),\dots,\min(k,n-1),k)$$

so that $(\iota_*(\rho) \circ \eta_{\mathit{\Omega}})_n(k) = (b_1,\dots,b_n)$ where $b_\ell = 1$ iff $\min(k,\ell) = \ell$ iff $\ell \leq k$. ☐

Lemma E.3 (Lem. 4.10). *The functor ι^* is faithful and preserves limits. It induces for each object X of $\mathscr{S}$ an injective map of lattices $\mathit{\Delta}_X : \mathrm{Sub}_{\mathscr{S}}(X) \to \mathrm{Sub}_{\mathrm{Psh}(|\mathbb{N}_*|)}(X^*)$.*

Proof. Preservation of limits (which in particular implies that $\iota^* \dashv \iota_*$ is a geometric morphism) is given by [15, Ex. A.4.1.4]. It follows from [15, Lem. A.4.2.6 & Ex. A.4.2.7.(b)] that ι^* is faithful since ι is surjective on objects. Moreover, in the context of a geometric morphism $f^* \dashv f_* : \mathcal{F} \to \mathcal{E}$, by [22, Lem. VII.4.3] f^* is faithful iff $\mathit{\Delta}_A$ is an injective map of lattices, or equivalently iff f^* reflects inclusion of subobjects (or equivalently iff the unit of $f^* \dashv f_*$ is a mono by [15, Lem. A.4.2.6]). ☐

Lemma E.4 (Lem. 6.3). *The map $\lambda : \mathit{\Omega} \to \mathbf{2}_*$, taking $k \in \mathit{\Omega}(n)$ to $(1^k, 0^{n-k})$ is a map of internal lattices. It is an internal left-adjoint to $\tau : \mathbf{2}_* \to \mathit{\Omega}$, with moreover $\tau \circ \lambda = \mathrm{id}_{\mathit{\Omega}}$.*

Proof. The adjunction $\lambda \dashv \tau$ follows from Lem. E.1.(c) and Prop. E.2.(e). As for the last part, let $k \in \mathit{\Omega}(n)$. Then $\lambda_n(k) = (1^k, 0^{n-k})$ so that $(\tau \circ \lambda)_n(k) = k$ by Prop. E.2.(d). ☐

F Proofs of §6 (Extensions)

F.1 Proofs of §6.2 (Accommodating an S4 Reverse Modality)

Lemma F.1 (Lem. 6.5). *The function $\Lambda_{\boxminus^*} \to (\coprod_n X(n)) \to \mathbf{2}$ induced by (5) coincides with the interpretation function $[\![-]\!]^v_{\boxminus^*} : \Lambda_{\boxminus^*} \to \mathbf{2}^{|X^\oplus|}$.*

Proof. First, note that $\Delta(\kappa^v_{\Lambda_{\boxminus^*}})(n)$ takes (x, φ) to 1 iff $(n, x) \Vdash^v_{\boxminus^*} \varphi$. It follows that the induced function $\Lambda_{\boxminus^*} \to (\coprod_n X(n)) \to \mathbf{2}$ takes φ and (n, x) to 1 iff $(n, x) \in [\![\varphi]\!]^v_{\boxminus^*}$ and we are done. $\square$

Proposition F.2 (Prop. 6.6). *The interpretation of $\boxminus^*\varphi$ as the map $X \to \mathbf{2}_*$ of (6) is the image under the comonad $\lambda \circ \tau$ of the interpretation of φ as a map $X \to \mathbf{2}_*$.*

Proof. Indeed, for a fixed formula φ, $[\![\varphi]\!]^v_{\boxminus^*}$ in $\mathrm{Psh}(|\mathbb{N}_*|)$ at n takes $x \in X(n)$ to 1 iff $x \Vdash^v_{\boxminus^*,n} \varphi$. It follows that the image of $[\![\varphi]\!]^v_{\boxminus^*}$ via $X \to X_* \to \mathbf{2}_*$ takes $x \in X(n)$ to the tuple $(b_1, \dots, b_n) \in \mathbf{2}^n$ where $b_\ell = [\![\varphi]\!]^v_{\boxminus^*}(\ell)(x_{|\ell})$. The composite of $X \to X_* \to \mathbf{2}_*$ with $\boxminus^* : \mathbf{2}_* \to \Omega \to \mathbf{2}_*$ takes $x \in X(n)$ to $(1^k, 0^{n-k})$ where $k = \max\{\ell \leq n \mid b_1 = \dots = b_\ell = 1\}$. It follows that $X \to X_* \to \mathbf{2}_* \to \Omega \to \mathbf{2}_*$ takes $x \in X(n)$ to the tuple $(b_1, \dots, b_n) \in \mathbf{2}^*$ where $b_\ell = [\![\boxminus^*\varphi]\!]^v_{\boxminus^*}(\ell)(x_{|\ell})$. $\square$

Proof (of Rem. 6.7). Fix an object X of $\mathscr{S}$ and a valuation $v : AP \to \mathbf{2}^{|X^\oplus|}$ (reverting the convention of Not. 5.1). We consider the following formulae:

$$\varphi, \psi \in \Lambda^0_{\boxminus^*} \quad ::= \quad p \quad | \quad \bot \quad | \quad \varphi \vee \psi \quad | \quad \neg\varphi \quad | \quad \boxminus^*\varphi$$

where $p \in AP$. We also assume the usual defined formulae (see §2.3). This amounts to the following terms in the language of [2, §2] (where P refer to the type of propositions):

$$\frac{}{p : \mathsf{P}} \qquad \frac{\varphi : \mathsf{P} \qquad \psi : \mathsf{P}}{\varphi \vee \psi : \mathsf{P}} \qquad \frac{\varphi : \mathsf{P}}{\neg\varphi : \mathsf{P}} \qquad \frac{\varphi : \mathsf{P}}{\boxminus^*\varphi : \mathsf{P}}$$

We first consider the interpretation of formulae following the instance of [2] for the geometric morphism $\jmath^* \dashv \jmath_* : \mathbf{Sets}^{|K|} \to \mathbf{Sets}^K$ (see [2, Ex. 2.1.1]). Following [2, Def. 2.3], formulae $\varphi \in \Lambda^0_{\boxminus^*}$ are interpreted as maps $[\![\varphi]\!]^v_K : \mathbf{1} \to \mathbf{2}_K$ in $\mathbf{Sets}^K$ for $K^{\mathrm{op}} = \int X$. The interpretation $[\![-]\!]^v_K$ is defined by induction on formulae, with the following clauses from [2, Def. 2.3]:

$$\begin{array}{rcl}
[\![p]\!]^v_K & := & \mathbf{1} \xrightarrow{\jmath_*(v(p))} \mathbf{2}_K \\
[\![\bot]\!]^v_K & := & \mathbf{1} \xrightarrow{\mathsf{f}} \mathbf{2}_K \\
[\![\varphi \vee \psi]\!]_K & := & \mathbf{1} \xrightarrow{\langle [\![\varphi]\!]^v_K, [\![\psi]\!]^v_K \rangle} \mathbf{2}_K \times \mathbf{2}_K \xrightarrow{\vee} \mathbf{2}_K \\
[\![\neg\varphi]\!]_K & := & \mathbf{1} \xrightarrow{[\![\varphi]\!]^v_K} \mathbf{2}_K \xrightarrow{\neg} \mathbf{2}_K \\
[\![\boxminus^*\varphi]\!]^v_K & := & \mathbf{1} \xrightarrow{[\![\varphi]\!]^v_K} \mathbf{2}_K \xrightarrow{\lambda_K \circ \tau_K} \mathbf{2}_K
\end{array}$$

where

- $v(p) : \mathbf{1} \to \mathbf{2}$ is a shorthand for

$$v(p) \quad : \quad \left(\coprod_{n>0} X(n)\right) \to \mathbf{2} \quad \simeq \quad \prod_{(n,x)\in\mathrm{Obj}(K)} \mathbf{2} \quad \simeq \quad \mathbf{Sets}^{|K|}[\mathbf{1}, \mathbf{2}]$$

- $\neg : \mathbf{2}_K \to \mathbf{2}_K$ is the lift of the $\neg : \mathbf{2} \to \mathbf{2}$ of $\mathbf{Sets}^{|K|}$ to $\mathbf{Sets}^{K}$, so that $\neg$ at (n, x) takes $(b_1, \dots, b_n)$ to $(\neg b_1, \dots, \neg b_n)$.
- The maps λ_K and τ_K are given by Lem. E.1.

It then follows from [2, Prop. 4.9] that for $(n, x) \in \mathrm{Obj}(K) = \coprod_{n>0} X(n)$, we have $[\![\varphi]\!]^v_K(n, x) = (b_1, \dots, b_n)$ where $b_\ell = 1$ iff $(\ell, x_{|\ell}) \Vdash^v_{\boxminus *} \varphi$. It follows that $\widetilde{[\![\varphi]\!]^v_K}$ induces the $\mathscr{S}$-map $X \to \mathbf{2}_*$ which at n takes $x \in X(n)$ to $(b_1, \dots, b_n)$ where $b_\ell = 1$ iff $(\ell, x_{|\ell}) \Vdash^v_{\boxminus *} \varphi$, and we are done (reasoning as in the proof of Prop. F.2). $\square$

F.2 Proofs of §6.3 (Extension to the Modal μ-Calculus)

Proposition F.3 (Prop. 6.9). *Given a formula φ monotone in the propositional variable α, the interpretation $[\![\mu\alpha.\varphi]\!]^v_\mu$ conincides with the following subobject of X^*:*

$$\bigwedge\{A \mid A \in \mathrm{Sub}_{\mathrm{Psh}(|\mathbb{N}_*|)}(X^*) \ \&\ [\![\varphi]\!]^{v[A/\alpha]}_\mu \leq_{\mathrm{Sub}_{\mathrm{Psh}(|\mathbb{N}_*|)}(X^*)} A\}$$

Proof. Since subobjects of X^* in $\mathrm{Psh}(|\mathbb{N}_*|)$ conincide with subsets of $|X_\circledast| = \coprod_n X(n)$, and since g.l.b.'s (resp. inclusions) of subobjects in presheaf topos are pointwise intersections (resp. inclusions) (see e.g. [22, §III.8]), we are left with showing that for all $n > 0$, $[\![\mu\alpha.\varphi]\!]^v_\mu(n)$ conincides with

$$\bigwedge\{A(n) \mid A : X^* \to \mathbf{2} \ \&\ [\![\varphi]\!]^{v[A/\alpha]}_\mu \subseteq A\}$$

But $x \in [\![\mu\alpha.\varphi]\!]^v_\mu(n)$ iff $(n, x) \in A$ for all $A : |X^\oplus| \to \mathbf{2}$ s.t. $[\![\varphi]\!]^{v[A/\alpha]}_\mu \subseteq A$, that is iff $x \in A(n)$ for all $A : \prod_n (X(n) \to \mathbf{2})$ s.t. $[\![\varphi]\!]^{v[A/\alpha]}_\mu \subseteq A$. $\square$

Table of Contents